C^*-algebras, and the Gelfand-Naimark Theorem

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Todo list

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Chapter 1

Preliminaries

1.1 History of C^*-Algebras

The noncommutative nature of Werner Heisenberg's work in 1925 on a new quantum mechanics [10] led to Born and Jordan [2], together with Heisenberg [3], developing the matrix mechanics required to concisely summarise the new quantum mechanical model. From 1935-1943, John von Neumann, together with F.J. Murray, developed the theory of *rings of operators* acting on a Hilbert space [17, 18, 19, 25], in an attempt to establish a general framework for this matrix mechanics. These rings of operators are now considered part of the theory of *von Neumann algebras*, a subsection of C^*-algebra theory. Discussion of the seminal quantum mechanical works of Heisenberg can be found in [16], and similarly [23] gives a summary of the works of Jordan expanding on this.

In 1943 [9], Gelfand and Naimark established an abstract characterisation of C^*-algebras, free from dependence on the operators acting on a Hilbert space. The Gelfand-Naimark theorem, which we will be considering here at length, gives the link between these abstract C^*-algebras and the rings of operators previously studied. Used in the proof of the GN theorem is the Gelfand-Naimark-Segal construction, a pair of results relating cyclic $*$-representations of C^*-algebras to certain linear functionals on that algebra.

1.2 Background Mathematics and Resources.

The following is some mathematics which may prove useful throughout the project, with relevant resources; we will of course be making definitions as needed, this is for further background and related theory.

We will be assuming some familiarity with the following theory, giving some explanation as necessary:

- Rings, algebras and linear spaces.
- Normed spaces, inner product spaces, Banach and Hilbert spaces.
- Point-set topology.

A good broad background on all of these can be found in [24].

Some texts which cover C^*-algebras: Dixmier [4] presents a summary of the general theory up to that time (1977), with [5] focussing on reworking and developing the theory of von Neumann algebras. Sakai [22] gives a treatment of C^*- and von Neumann algebras from a more topological point of view. In [11, 12], the authors aim to make accessible the "vast recent research literature" in this subsection of functional analysis. Blackadar [1] gives a much faster, more encyclopaedic coverage of the theory of operator algebras, and covering more specialised material and applications.

1.3 Aims

The aims for this project are:

- Give a good background understanding on C^*-algebras, including topological and geometric interpretation of results where possible.
- Consider the representation theory of C^*-algebras, using the Gelfand-Naimark-Segal construction as a starting point.
- Consider the commutative and general versions of the Gelfand-Naimark theorem, and understand their contents and proof.

most texts start from $\mathcal{B}(\mathcal{H})$ to justify the whole thing. we're algebraists, who don't need no justification. we jump right in at the deep (abstract) end.

rewrite this!

1.4 Constructions on Hilbert spaces

We need some Hilbert spaces constructions; particularly, the direct sum of a collection of Hilbert spaces and the direct sum of bounded operators on these Hilbert spaces. Given a finite collection $\{\mathcal{H}_1, \ldots, \mathcal{H}_n\}$ of Hilbert spaces, let $\mathcal{H}$ denote the set

$$\mathcal{H} := \{(x_1, \ldots, x_n) \mid x_i \in \mathcal{H}_i \text{ for } i = 1, \ldots, n\}.$$

Define addition and scalar multiplication coordinatewise, and given $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n) \in \mathcal{H}$, the equation

$$\langle x, y\rangle = \langle x_1, y_1\rangle + \cdots + \langle x_n, y_n\rangle$$

defines an inner product $\langle\cdot,\cdot\rangle$ on $\mathcal{H}$. The resulting norm $\|\cdot\|$ is given by

$$\|x\|^2 = \|x_1\|^2 + \cdots + \|x_n\|^2.$$

It is easy to show that $\mathcal{H}$ is a Hilbert space with these operations, and we call it the *direct sum* of the collection $\{\mathcal{H}_1, \ldots, \mathcal{H}_n\}$, denoted $\oplus\mathcal{H}_i$.

Similarly, we can construct a Hilbert space direct sum of an infinite collection $\{\mathcal{H}_i \mid i \in I\}$ of Hilbert spaces. Let $\mathcal{H}$ be the set

$$\mathcal{H} := \{x = (x_i) \mid x_i \in \mathcal{H}_i \text{ for each } i \text{ and } \sum_{i\in I} \|x_i\|^2 < \infty\}.$$

Given $x = (x_i)$ and $y = (y_i) \in \mathcal{H}$, we have that

$$\begin{aligned}\Big(\sum_{i\in I} \|x_i + y_i\|^2\Big)^{1/2} &\le \Big(\sum_{i\in I} (\|x_i\| + \|y_i\|)^2\|\Big)^{1/2} \\ &\le \Big(\sum_{i\in I} \|x_i\|^2\Big)^{1/2} + \Big(\sum_{i\in I} \|y_i\|^2\Big)^{1/2} \\ &< \infty.\end{aligned}$$

Hence, the sequence (x_i+y_i) is in $\mathcal{H}$. Thus we can define addition and scalar multiplication coordinatewise on $\mathcal{H}$. We also have

$$\begin{aligned}\sum_{i\in I} |\langle x, y\rangle| &\le \sum_{i\in I} \|x_i\|\|y_i\| \\ &\le \Big(\sum_{i\in I} \|x_i\|^2\Big)^{1/2}\Big(\sum_{i\in I} \|y_i\|^2\Big)^{1/2} \\ &< \infty,\end{aligned}$$

so that we can define an inner product $\langle\cdot,\cdot\rangle$, with induced norm $\|\cdot\|$, on $\mathcal{H}$ by

$$\langle x, y\rangle := \sum_{i\in I} |\langle x, y\rangle|, \qquad\qquad \|x\| = \Big(\sum_{i\in I} \|x_i\|^2\Big)^{1/2}.$$

To see that $\mathcal{H}$ is complete with respect to $\|\cdot\|$, suppose that $(x^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}$, where for each n, $x^n = (x_i^n)_{i\in I}$. Then given any positive ϵ there exists a positive integer N such that

$$\|x^m - x^n\| < \epsilon \text{ for all } m, n \ge N,$$

that is,

$$\sum_{i\in I} \|x_i^m - x_i^n\|^2 < \epsilon^2 \text{ for all } m, n \ge N. \tag{1.1}$$

Hence for each $i \in I$,

$$\|x_i^m - x_i^n\| < \epsilon \text{ for all } m, n \ge N$$

so that $(x_i^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}_i$, having a limit $x_i \in \mathcal{H}_i$. For any finite subset $J \subset I$, it follows from 1.1 that

$$\sum_{j\in J} \|x_j^m - x_j^n\|^2 < \epsilon^2 \text{ for all } m, n \ge N,$$

and letting m tend to infinity,

$$\sum_{j\in J} \|x_j - x_j^n\|^2 < \epsilon^2 \text{ for all } n \ge N. \tag{1.2}$$

This holds for any finite subset J, so

$$\sum_{i\in I} \|x_i - x_i^n\|^2 < \epsilon^2 \text{ for all } n \ge N,$$

and so $(x_i - x_i^n)$ and x_i^n are in $\mathcal{H}$ for $n \geq N$. Then (x_i) is in $\mathcal{H}$ and by 1.2, x^n converges to (x_i) as n tends to infinity. We conclude that $\mathcal{H}$ is complete and therefore a Hilbert space. Just like in the finite case, we call $\mathcal{H}$ the *direct sum* of the collection $\{\mathcal{H}_i \mid i \in I\}$ of Hilbert spaces, denoted $\oplus\mathcal{H}_i$.

Suppose now we have a (finite or infinite) collection of bounded operators $\{T_i \in \mathcal{B}(\mathcal{H}_i) \mid i \in I\}$ such that

$$\sup_{i \in I} \|T_i\| < \infty$$

(note that this is automatically true for finite I). For $x = (x_i)$ in $\oplus\mathcal{H}_i$, define an element Tx in $\oplus\mathcal{H}_i$ by $Tx = (T_i x_i)$. Then $T : \mathcal{H} \to \mathcal{H} : x \mapsto Tx$ is a bounded linear operator, called the *direct sum* of the collection $\{T_i \in \mathcal{B}(\mathcal{H}_i) \mid i \in I\}$, denoted $\oplus T_i$. For S_i, T_i in $\mathcal{B}(\mathcal{H}_i)$, α, β in $\mathbb{C}$, we have

$$\begin{aligned}
(\oplus T_i)^* &= \oplus T_i^*, \\
\oplus(\alpha S_i + \beta T_i) &= \alpha \oplus S_i + \beta \oplus T_i, \\
\oplus(S_i T_i) &= \oplus S_i \oplus T_i, \\
\| \oplus T_i\| &= \sup_{i \in I} \|T_i\|.
\end{aligned}$$

Chapter 2

C^*-algebras

We begin this chapter with some definitions and an example, then give some results we will need later and finish with a fundamental example.

Definition 1. A *Banach algebra* is a complex Banach space $(A, \|\cdot\|)$ which forms an algebra, such that

$$\|ab\| \leq \|a\|\|b\| \text{ for all } a, b \in A.$$

A $*$-*algebra* is an algebra A with an *involution* map $a \mapsto a^*$ on A such that, for all $a, b \in A$ and for $\alpha \in \mathbb{C}$,

(i) $a^{**} = (a^*)^* = a$,

(ii) $(\alpha a + b)^* = \overline{\alpha} a^* + b^*$,

(iii) $(ab)^* = b^* a^*$.

The element a^* is referred to as the *adjoint* of a.
A C^*-*algebra* is a Banach algebra $(A, \|\cdot\|)$ with involution map $a \mapsto a^*$ making it a $*$-algebra, with the condition that

$$\|a^* a\| = \|a\|^2 \text{ for all } a \in A.$$

This condition is known as the C^* *axiom*.

Here we consider complex C^*-algebras. The theory of real C^*-algebras has advanced....

remark on work in real C^*-algebras?

Unless specified otherwise, by an *ideal* of a Banach algebra, we mean a two-sided ideal. Given a subset S of a C^*-algebra A, let $C^*(S)$ denote the C^*-*subalgebra of* A *generated by* S, which is the smallest S^*-subalgebra of A containing S.

cauchy-schwarz, C* subalgebra generated by a set

2.1 $C(X)$ - an example.

Given a locally compact Hausdorff space X, let $C(X)$ be the algebra of continuous functions $f : X \to \mathbb{C}$, with addition and multiplication defined pointwise. Define $\|\cdot\|$ on $C(X)$ by

$$\|f\| := \sup_{x\in X} |f(x)|,$$

that is the norm inherited from the Banach space $\ell^2(X,\mathbb{C})$.

example of $C(X)$? could define states and stuff preemptively

2.2 Unitization

If a C^*-algebra A contains an identity element $\mathbb{1}$ such that $a \cdot \mathbb{1} = a = \mathbb{1} \cdot a$ for all $a \in A$, call $\mathbb{1}$ the *unit* in A, and A is then a *unital* C^*-algebra.

Proposition 1. *Any non-unital C^*-algebra A can be isometrically embedded in a unital C^*-algebra $\tilde{A}$ as a maximal ideal.*

tidy up proof

Proof. Let $\tilde{A} = A \oplus \mathbb{C}$ with pointwise addition, and define

$$\begin{aligned}(a,\lambda)(b,\mu) &:= (ab + \lambda b + \mu a, \lambda\mu),\\ (a,\lambda)^* &:= (a^*, \overline{\lambda}),\\ \|(a,\lambda)\| &:= \sup_{\|b\|=1} \|ab + \lambda b\|.\end{aligned}$$

Then $\tilde{A}$ is a $*$-algebra. The norm $\|(a,\lambda)\|$ is... Thus $\tilde{A}$ is a Banach $*$-algebra with unit $(0,1)$. By design, A is a maximal ideal of codimension 1. The embedding $a \mapsto (a,\lambda)$ is isometric as justify norm

$$\|a\| = \|a \cdot \frac{a}{\|a\|}\| \leq \|(a,0)\| \leq \sup_{\|b\|=1} \|ab\| \leq \|a\|.$$

It remains to verify the C^*-axiom:

$$\begin{aligned}\|(a,\lambda)\|^2 &= \sup_{\|b\|=1} \|ab + \lambda b\|^2\\ &= \sup_{\|b\|=1} \|b^*a^*ab + \lambda b^*a^*b + \overline{\lambda} b^*ab + |\lambda|^2 b^*b\\ &\leq \sup_{\|b\|=1} \|a^*ab + \lambda a^*b + \overline{\lambda} ab + |\lambda|^2 b\|\\ &= \|(a^*a + \lambda a^* + \overline{\lambda} a, |\lambda|^2)\\ &= \|(a,\lambda)^*(a,\lambda)\|\\ &\leq \|(a,\lambda)^*\|\|(a,\lambda)\|.\end{aligned}$$

By symmetry of $*$, $\|(a,\lambda)^*\| = \|(a,\lambda)\|$. Hence, the above inequality becomes equality and we have that

$$\|(a,\lambda)^*(a,\lambda)\| = \|(a,\lambda)\|^2.$$

$\square$

In light of this result, we take all C^*-algebras from here to be unital unless specified otherwise. For the results we will consider, we can simply consider the unital case. However, there are important circumstances in advanced theory in which one needs to relax the unital condition.

2.3 the spectrum

state without proof results?

Given an element $a \in A$ of a C^*-algebra, define its spectrum $\mathrm{sp}(a)$:

$$\mathrm{sp}(a) := \{\lambda \in \mathbb{C} \mid a - \lambda\mathbb{1} \text{ is not invertible in } A\}.$$

AAAAAAAAAAH I NEED SO MUCH SPECTRAL THEORY AND I DON'T KNOW ANY

2.4 more definitions

An element $a \in A$ of a C^*-algebra is called normal?

- *self-adjoint* if $a^* = a$;
- *unitary* if $aa^* = a^*a = \mathbb{1}$;
- *positive* if it is self-adjoint and $\mathrm{sp}(a) \subseteq \mathbb{R}^+$.

Denote the set of self-adjoint elements in A by A_{sa}, and the subset of positive elements in A_{sa} by A^+. The set of positive elements A_{sa} forms a *partially ordered (real) vector space*, with *positive cone* A^+. That is to say, all $a, b \in A^+$ satisfy

(i) $a, -a \in A^+$ implies $a = 0$,

(ii) $\alpha a \in A^+$ for all $\alpha \in \mathbb{R}^+$,

(iii) $a + b \in A^+$.

The unit $\mathbb{1}$ is positive, and for any $a \in A_{sa}$ we have $-\|a\|\mathbb{1} \leq a \leq \|a\|\mathbb{1}$. With commuting justify. elements $a, b \in A_{sa}$, we have $(ab)^* = b^*a^* = ba = ab$, so ab is self-adjoint. Since a, b, ab have the same spectrum in A as in the Abelian C^*-subalgebra $C^*(\mathbb{1}, a, b)$, by our spectral theory we have state in 'spectrum' - 4.1.5

$$\mathrm{sp}(ab) \subseteq \mathrm{sp}(a)\mathrm{sp}(b).$$

Definition 2. Given Banach $*$-algebras A and B, a map $\varphi : A \to B$ is a $*$-*homomorphism* if it is an algebra homomorphism for which $\varphi(a^*) = \varphi(a)^*$ for all $a \in A$. If A and B are both unital algebras and a homomorphism φ maps $\mathbb{1}_A$ to $\mathbb{1}_B$, say φ is a *unital* homomorphism. If a $*$-homomorphism φ is one-to-one, call it a $*$-*isomorphism*. is unital necessary?

Proposition 2. *Suppose A and B are C^*-algebras and $\varphi : A \to B$ is a $*$-homomorphism. Then $\|\varphi(a)\| \leq \|a\|$ for all $a \in A$. If φ is a $*$-isomporphism, then $\|\varphi(a)\| = \|a\|$ for all $a \in A$.*

Proof. □

Definition 3. A *linear functional* on a C^*-algebra A is a linear operator $\rho : A \to \mathbb{C}$. A linear functional ρ is *positive* if $\rho(a) \geq 0$ for all $a \in A^+$. A *state* on A is a positive linear functional ρ such that $\|\rho\| = 1$ and $\rho(a) \geq 0$ for all positive elements $a \in A^+$. Denote by $\mathscr{S}(A)$ the set of all states on A. An extreme point of $\mathscr{S}(A)$ is called a *pure* state on A, and the set of pure states on A is denoted by $\mathscr{P}(A)$.

Recall that an *extreme* point x of a convex subset X of a topological space is one for which an expression

$$x = \alpha x_1 + (1 - \alpha)x_2,$$

for $0 \leq \alpha \leq 1$ and $x_1, x_2 \in X$, implies that $x_1 = x = x_2$ – for example, the vertices of a polygon embedded in $\mathbb{R}^2$ are extreme points of that polygon.

geometric interpretation - for $A = \mathbb{C}$, self adjoint elements are real numbers and the positive cone is $\mathbb{R}^+$. explain $\geq$ notation

It is a simple exercise, using the fact that $A^+ \subseteq A_{sa}$, to verify that a linear functional ρ is a pure state on A if and only if the restriction $\rho|_{A_{sa}}$ is a pure state on A_{sa}. Every pure state on A_{sa} extends to a pure state on A. We will need the following few results on pure states later. justify (4.3.1

Proposition 3. *A state ρ on A_{sa} is pure if and only if, for all positive linear functionals τ on A_{sa} such that $0 \leq \tau \leq \rho$, we have $\tau = \lambda\rho$ for some $\lambda \in \mathbb{R}$.*

Proof. (Adapted from 3.4.6). Suppose that $\tau = \lambda\rho$ for all $0 \leq \tau \leq \rho$, and suppose we can write $\rho = \alpha\rho_1 + (1 - \alpha)\rho_2$ for some $0 \leq \alpha \leq 1$ and some $\rho_1, \rho_2 \in \mathscr{S}(A_{sa})$. Then $0 \leq \alpha\rho_1 \leq \rho$, so $\alpha\rho_0 = \lambda\rho$. Then $\rho_1(\mathbb{1}) = 1 = \rho(\mathbb{1})$, so $\alpha = \lambda$ so $\rho_0 = \rho$. Similarly, we can show that $\rho_2 = \rho$, and so we conclude that ρ is pure.

Conversely, suppose that ρ is a pure state and $0 \leq \tau \leq \rho$. Applying this to $\mathbb{1}$, we get $0 \leq \tau(\mathbb{1}) \leq \rho(\mathbb{1}) = 1$. Let $\lambda = \tau(\mathbb{1})$. If $\lambda = 0$, then for any $a \in A_{sa}$, applying τ to $-\|a\|\mathbb{1} \leq a \leq \|a\|\mathbb{1}$ gives

$$0 = -\|a\|\lambda = \tau(-\|a\|\mathbb{1}) \leq \tau(a) \leq \tau(\|a\|\mathbb{1}) = \|a\|\lambda = 0,$$

so $\tau = 0 = \lambda\rho$. A similar argument shows that $\lambda = 1$ implies $\tau - \rho = 0$ so that $\tau = \rho = \lambda\rho$. If $0 \leq \lambda \leq 1$, we can write $\rho = \lambda\rho_1 + (1 - \lambda)\rho_2$ for $\rho_1 = \lambda^{-1}\tau$ and $\rho_2 = (1 - \lambda)^{-1}(\rho - \tau)$. ρ is pure so $\tau = \lambda\rho_1 = \lambda\rho$.

□

Proposition 4. *The set of pure states on an Abelian C^*-algebra A is precisely the set*

$$\{\rho : A \to \mathbb{C} \mid \rho(ab) = \rho(a)\rho(b) \text{ for all } a, b \in A\}$$

of multiplicative linear functionals on A.

Proof.(Adapted from K&R, 4.4.1). Suppose ρ is a pure state on A. To show that $\rho(ab) = \rho(a)\rho(b)$ for $a, b, \in A$, we restrict attention to the case where $0 \leq b \leq \mathbb{1}$. Linearity gives us the general case. In this case, for $h \in A^+$ we have that $0 \leq hb \leq h$, so $0 \leq \rho(hb) \leq \rho(h)$. show

Hence $\rho_b(a) := \rho(ab)$ for $a \in A$ defines a positive linear functional on A with $\rho_b \leq \rho$. The restriction $\rho|_{A_{sa}}$ is a pure state on A_{sa} and $\rho_b|_{A_{sa}} \leq \rho|_{A_{sa}}$, and it follows from Proposition 3 that $\rho_b|_{A_{sa}} = \alpha\rho|_{A_{sa}}$ for some $\alpha \in \mathbb{R}^+$. Hence $\rho_b = \alpha\rho$ and so for $a \in A$:

$$\rho(ab) = \rho_b(a) = \alpha\rho(a) = \alpha\rho(\mathbb{1})\rho(a) = \rho_b(\mathbb{1})\rho(a) = \rho(b)\rho(a)$$

Conversely, suppose ρ is a multiplicative linear functional. Suppose we can write $\rho = \alpha\rho_1 + \beta\rho_2$ for states ρ_1, ρ_2 on A and $\alpha, \beta > 0$ such that $\alpha + \beta = 1$. For $c \in A_{sa}$, by the Cauchy-Schwarz inequality we have for $j = 1, 2$:

show a state.

$$\big(\rho_j(c)\big)^2 = \big(\rho_j(\mathbb{1}c)\big)^2 \leq \rho_j(\mathbb{1})\rho(c^2) = \rho(c^2).$$

Then:

$$\begin{aligned}
0 &= \rho(c^2) - \rho(c)^2 \\
&= \alpha\rho_1(c^2) + \beta\rho_2(c^2) - \big(\alpha\rho_1(c) + \beta\rho_2(c)\big)^2 \\
&\geq \alpha(\alpha+\beta)\rho_1(c)^2 + \beta(\alpha+\beta)\rho_2(c)^2 - \big(\alpha\rho_1(c) + \beta\rho_2(c)\big)^2 \\
&= \alpha\beta\big(\rho_1(c) - \rho_2(c)\big)^2.
\end{aligned}$$

Hence $\rho_1(c) = \rho_2(c)$, for all $c \in A_{sa}$, so $\rho_1 = \rho_2$ and we conclude that ρ is a pure state. $\square$

define weak* topology on P(S)

2.5 $\mathcal{B}(\mathcal{H})$ - an example.

This section concerns the fundamental example of a C^*-algebra - the set $\mathcal{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$. Here we will demonstrate that $\mathcal{B}(\mathcal{H})$ is a C^*-algebra and give some basic results.

Claim. *$\mathcal{B}(\mathcal{H})$ is a C^*-algebra with the operator norm*

$$\|T\| := \sup_{\|x\|=1} \|Tx\|$$

and involution taking T to its adjoint map T^*. The identity map $I : x \mapsto x$ is a unit for $\mathcal{B}(\mathcal{H})$

Proof. $\|\cdot\|$ is a norm on $\mathcal{B}(\mathcal{H})$. Let $\{T_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{B}(\mathcal{H})$. Then for any positive ϵ, there is a positive integer N such that

$$\|T_m - T_n\| < \epsilon \text{ for all } m, n \geq N.$$

Applying $T_m - T_n$ to $x \in \mathcal{H}$, we have

$$\|T_m x - T_n x\| \leq \|T_m - T_n\|\|x\| < \epsilon\|x\|, \tag{2.1}$$

so $\{T_n x\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}$, converging to an element in $\mathcal{H}$. Define a linear operator $T : \mathcal{H} \to \mathcal{H}$ by

$$Tx := \lim_{n\to\infty} T_n x \text{ for } x \in \mathcal{H}.$$

Taking limits as m tends to infinity in equation (2.1), we obtain

$$\|Tx - T_n x\| < \epsilon\|x\| \text{ for all } n \geq N,$$

and so we have that $T - T_n$ (and hence $T = (T - T_n) + T_n$) is a bounded operator and

$$\|T - T_n\| < \epsilon \text{ for all } n \geq N.$$

We conclude that $T_n \to T$, and so $\mathcal{B}(\mathcal{H})$ is complete.

Since boundedness is equivalent to continuity on $\mathcal{H}$, given $S, T \in \mathcal{B}(\mathcal{H})$, the operator $ST : \mathcal{H} \to \mathcal{H}; x \mapsto (S \circ T)(x)$ is bounded on $\mathcal{H}$. Given $x \in \mathcal{H}$ and $\lambda \in \mathbb{C}$,

$$\begin{aligned}((\lambda S)T)(x) &= ((\lambda S) \circ T)(x) \\ &= \lambda S(Tx) \\ &= \lambda(S \circ T)(x) \\ &= \lambda ST(x),\end{aligned}$$

so that $(\lambda S)T = \lambda ST$ in $\mathcal{B}(\mathcal{H})$, whence $\mathcal{B}(\mathcal{H})$ is an algebra. We have

$$\begin{aligned}\|ST\| &= \sup_{\|x\|=1} \|STx\| \\ &= \sup_{\|x\|=1} \|S(Tx)\| \\ &\leq \|S\| \sup_{\|x\|=1} \|Tx\| \\ &= \|S\|\|T\|.\end{aligned}$$

To see that * is an involution, use the fact that the adjoint operator is unique for each operator and the following equalities.

(i)
$$\begin{aligned}\langle(\alpha T + S)^* x, y\rangle &= \langle x, \alpha T + Sy\rangle \\ &= \overline{\alpha}\langle x, Ty\rangle + \langle x, Sy\rangle \\ &= \overline{\alpha}\langle T^* x, y\rangle + \langle S^* x, y\rangle \\ &= \langle(\overline{\alpha}T^* + S^*)x, y\rangle.\end{aligned}$$

(ii)
$$\begin{aligned}\langle(T^*)^* x, y\rangle &= \langle x, T^* y\rangle \\ &= \overline{\langle T^* y, x\rangle} \\ &= \overline{\langle y, Tx\rangle} \\ &= \langle Tx, y\rangle.\end{aligned}$$

(iii)
$$\begin{aligned}\langle(ST)^* x, y\rangle &= \langle x, STy\rangle \\ &= \langle S^* x, Ty\rangle \\ &= \langle T^* S^* x, y\rangle.\end{aligned}$$

It remains to demonstrate the C^*-axiom on $\mathcal{B}(\mathcal{H})$. For all $x \in \mathcal{H}$, we have

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^* Tx, x\rangle \leq \|T^* T\|\|x\|^2,$$

so that

$$\|T\|^2 \leq \|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2.$$

It is clear that I is a unit. Hence, the claim. $\square$

example of $\mathcal{B}(\mathcal{H})$ at end of section to retro-motivate notation. discuss nomenclature (state etc) coming from QM

Chapter 3

Representations of C^*-algebras

to include all representation theory, including CGN, GNS and GN

3.1 Abelian C^*-algebras

Let A be an Abelian C^*-algebra. For a in A, define a complex-valued function $\hat{a}$ on $\mathscr{P}(A)$ by $\hat{a}(\rho) := \rho(a)$. The *weak***-topology* is the coarsest topology on $\mathscr{P}(A)$ for which all of the maps $\hat{a}$ are continuous, so that $\hat{a} \in C(\mathscr{P}(A))$ for all $a \in A$. The map

why should this exist?

$$\Gamma : A \to C(\mathscr{P}(A)) : a \mapsto \hat{a}$$

is called the *Gelfand representation* of A. For $a, b \in A$, $\alpha, \beta \in \mathbb{C}$ and $\rho \in \mathscr{P}(A)$:

$$\begin{aligned} (\widehat{\alpha a + \beta b})(\rho) &= \rho(\alpha a + \beta b) = \alpha\rho(a) + \beta\rho(b) = \alpha\hat{a}(\rho) + \beta\hat{b}(\rho), \\ \widehat{a^*}(\rho) &= \rho(a^*) = \overline{\rho(a)} = \overline{\hat{a}(\rho)}. \end{aligned}$$

Since by Proposition 4,

$$\mathscr{P}(A) = \{\rho \in \mathscr{S}(A) \mid \rho(ab) = \rho(a)\rho(b)\},$$

we have that

$$\widehat{(ab)}(\rho) = \rho(ab) = \rho(a)\rho(b) = \hat{a}(\rho)\hat{b}(\rho).$$

Hence the Gelfand representation is a $*$-homomorphism. The following theorem gives us that it is in fact an isometric $*$-isomorphism.

Theorem 1 (Gelfand-Naimark, commutative)**.** *Every commutative C^*-algebra A is $*$-isomorphic to $C(X)$, the algebra of continuous functions on a compact Hausdorff space X.*

Proof. □

The previous result generalises to not-necessarily-unital Abelian C^*-algebras as follows:

Theorem. *Every commutative C^*-algebra A is $*$-isomorphic to $C_0(X)$, the algebra of continuous functions on a locally compact Hausdorff space X which vanish at infinity.*

The proof – not dissimilar from the proof above – can be found in [reference]. The unitization of this algebra corresponds to the one-point compactificaton of X.

3.2 The Gelfand-Naimark Theorem

Definition 4. Given a C^*-algebra A, a *representation of A on a Hilbert space $\mathcal{H}$* is a $*$-homomorphism $\varphi : A \to \mathcal{B}(\mathcal{H})$. An isomorphic representation is called *faithful*. If there exists an element $x \in \mathcal{H}$ such that the set $\{\varphi(a) \mid a \in A\}$ is everywhere-dense in $\mathcal{H}$, say that φ is a *cyclic* representation, with *cyclic vector* x.

The following theorem is proved using the Gelfand-Naimark-Segal (or GNS) construction.

Theorem 2. *If ρ is a state on a C^*-algebra A, then there exists a cyclic representation π_ρ of A on a Hilbert space H_ρ, with unit cyclic vector x_ρ, such that*

$$\rho(a) = \langle \pi_\rho(a) x_\rho, x_\rho \rangle, \qquad \forall a \in A.$$

Proof. We will construct from ρ the space $\mathcal{H}_\rho$, representation π_ρ, and vector x_ρ, and demonstrate the required properties.

Consider the *left kernel* of ρ:

$$L_\rho := \{t \in A \mid \rho(t^* t) = 0\}.$$

For $a, b \in A$, define $\langle a, b \rangle_0 := \rho(b^* a)$. Then $L_\rho = \{t \in A \mid \langle t, t \rangle_0 = 0\}$, and $\langle \cdot, \cdot \rangle_0$ satisfies

(i) Linearity in 1st argument: for $a, b \in A$, $\alpha, \beta \in \mathbb{C}$:

$$\begin{aligned} \langle \alpha a + \beta b, c \rangle_0 &= \rho(c^*(\alpha a + \beta b) \\ &= \rho(\alpha c^* a + \beta c^* b) \\ &= \alpha \rho(c^* a) + \beta \rho(c^* b) \\ &= \alpha \langle a, c \rangle_0 + \beta \langle b, c \rangle_0. \end{aligned}$$

(ii) Conjugate symmetric: for $a, b \in A$:

$$\begin{aligned} \langle b, a \rangle_0 &= \rho(a^* b) \\ &= \rho((b^* a)^*) \\ &= \overline{\rho(b^* a)} \\ &= \overline{\langle a, b \rangle_0}. \end{aligned}$$

(iii) Positive semi-definite. why?

Note that $\langle \cdot, \cdot \rangle$ is not necessarily positive definite on A – the left kernel is exactly where this fails.

To see that L_ρ is a linear subspace of A, consider

$$L := \{t \in A \mid \langle t, a \rangle_0 = 0, \ \forall a \in A\} \subseteq L_\rho.$$

For $t \in L_\rho$, by Cauchy-Schwarz we have

$$|\langle t, a\rangle_0|^2 \le \langle t, t\rangle_0 \langle a, a\rangle_0, \quad \forall a \in A;$$

that is,

$$\langle t, a\rangle_0 = 0, \quad \forall a \in A,$$

so $t \in L$ and $L_\rho = L$. Now, for $a, b \in L$, $\alpha \in \mathbb{C}$ and $c \in A$:

$$\langle \alpha a + b, c\rangle_0 = \alpha\langle a, c\rangle_0 + \langle b, c\rangle_0 = 0,$$

so $\alpha a + b \in L$; also, $\langle 0, c\rangle_0 = 0$ so $0 \in L$. Hence, $L(= L_\rho)$ is a linear subspace of A.

For $s \in A$, $t \in L_\rho$, by the Cauchy Schwarz inequality [ref] we have

$$\begin{aligned}|\rho(s^*t)|^2 &= |\langle t, s\rangle_0|^2 \\ &\le \langle t, t\rangle_0 \cdot \langle s, s\rangle_0 \\ &= \rho(t^*t) \cdot \rho(s^*s) \\ &= 0,\end{aligned}$$

so $\rho(s^*t) = 0$. Letting $s = a^*at$ for $a \in A$, then

$$\begin{aligned}\rho((at)^*at) &= \rho(at^*a^*at) \\ &= \rho((a^*at)^*t) \\ &= \rho(s^*t) \\ &= 0,\end{aligned}$$

so that $at \in L_\rho$, for all $a \in A$ and $t \in L_\rho$; we conclude that L_ρ is a left ideal in A. Closure of L_ρ follows from the fact that it is the preimage in A of $\{0\}$ under the continuous map $t \mapsto \rho(t^*t)$.

why do we need closure?

Consider now $V_\rho := A/L_\rho$, with $\langle \cdot, \cdot \rangle$ defined by

$$\langle a + L_\rho, b + L_\rho\rangle := \langle a, b\rangle_0, \qquad \text{for} \ \ a + L_\rho, b + L_\rho \in V_\rho.$$

It follows from properties $i), ii)$ and $iii)$ of $\langle \cdot, \cdot\rangle_0$ that $\langle \cdot, \cdot \rangle$ is an inner product on V_ρ – with

$$\begin{aligned}\langle a + L^\rho, a + L^\rho\rangle = 0 &\iff \langle a, a\rangle = 0 \\ &\iff a \in L_\rho \\ &\iff a + L_\rho = 0 + L_\rho\end{aligned}$$

giving positive definiteness. The completion of V_ρ with respect to $\langle \cdot, \cdot \rangle$ is a Hilbert space - this is the Hilbert space $\mathcal{H}_\rho$ we're looking for.

Now we fix $a \in A$, and consider the map

$$\pi_a : V_\rho \to V_\rho; b + L_\rho \mapsto ab + L_\rho.$$

Let $b_1, b_2 \in A$ be such that $b_1 + L_\rho = b_2 + L_\rho$. Then:

$$\begin{aligned}
&\implies b_1 - b_2 \in L_\rho \\
&\implies a(b_1 - b_2) \in L_\rho \\
&\implies ab_1 - ab_2 \in L_\rho \\
&\implies ab_1 + L_\rho = ab_2 + L_\rho \\
&\implies \pi_a(b_1 + L_\rho) = \pi_a(b_2 + L_\rho).
\end{aligned}$$

Hence π_a defines a linear operator on V_ρ.

For $b + L_\rho \in V_\rho$:

$$\begin{aligned}
\|a\|^2 \cdot \|b + L_\rho\| - \|\pi_a(b + L_\rho)\| &= \|a\|^2 \cdot \|b + L_\rho\| - \|ab + L_\rho\| \\
&= \|a\|^2 \cdot \langle b + L_\rho, b + L_\rho\rangle - \langle ab + L_\rho, ab + L_\rho\rangle \\
&= \|a\|^2 \cdot \rho(b^*b) - \rho((ab)^*ab) \\
&= \rho(\|a\|^2 b^*b - b^*a^*ab) \\
&= \rho(b^*(\|a\|^2 \mathbb{1} - a^*a)b) \\
&\geq 0.
\end{aligned}$$

Thus π_a is a bounded operator, with $\|\pi_a\| \leq \|a\|$. By continuity, π_a extends to a bounded operator on $\mathcal{H}_\rho$ – say $\pi_\rho(a) : \mathcal{H}_\rho \to \mathcal{H}_\rho$ such that

$$\pi_\rho(a)(v) = \pi_a(v)$$

for $v \in V_\rho$. Then $\pi_\rho(a) \in \mathcal{B}(\mathcal{H}_\rho)$ for each $a \in A$, so π_ρ defines a map $A \to \mathcal{B}(\mathcal{H}_\rho)$ such that $a \mapsto \pi_\rho(a)$. This will be our representation.

Now, for $a, b \in A$, $c + L_\rho \in V_\rho$ and $\alpha \in \mathbb{C}$:

$$\begin{aligned}
\pi_{\alpha a + b}(c + L_\rho) &= (\alpha a + b)(c + L_\rho) \\
&= (\alpha ac + L_\rho) + (bc + L_\rho) \\
&= \alpha\pi_a(c + L_\rho) + \pi_b(c + L_\rho),
\end{aligned}$$

so that $\pi_{\alpha a+b} = \alpha\pi_a + \pi_b$ on V_ρ.

For $a, b \in A$ and $c + L_\rho \in V_\rho$:

$$\begin{aligned}
\pi_{ab}(c + L_\rho) &= abc + L_\rho \\
&= \pi_a(bc + L_\rho) \\
&= \pi_a(\pi_b(c + L_\rho)) \\
&= (\pi_a \cdot \pi_b)(c + L_\rho),
\end{aligned}$$

so that $\pi_{ab} = \pi_a \cdot \pi_b$ on V_ρ.

For $a \in A$ and $b + L_\rho,\ c + L_\rho \in V_\rho$:

$$\begin{aligned}
\langle b + L_\rho, \pi_a^*(c + L_\rho)\rangle &= \langle \pi_a(b + L\rho), c + L_\rho\rangle \\
&= \langle ab + Lr, c + L_\rho\rangle \\
&= \rho(c^*ab) \\
&= \rho((a^*c)^*b) \\
&= \langle b + L_\rho, a^*c + L_\rho\rangle \\
&= \langle b + L_\rho, \pi_{a^*}(c + L_\rho)\rangle,
\end{aligned}$$

so that $\pi_a^* = \pi_{a^*}$ on V_ρ.

$V_\rho \subset \mathcal{H}_\rho$ is a dense subset, so the three properties above hold on $\mathcal{H}_\rho$ by continuity of π_ρ. Hence, $\pi_\rho : A \to \mathcal{B}(\mathcal{H}_\rho)$ is a representation of A. As to the unit vector, consider $x_\rho := \mathbb{1} + L_\rho \in V_\rho$. Then for $a \in A$,

$$\begin{aligned}\langle \pi_\rho(a)x_\rho, x_\rho\rangle &= \langle \pi_a(\mathbb{1} + L_\rho), \mathbb{1} + L_\rho\rangle \\ &= \langle a + L_\rho \mathbb{1} + L_\rho\rangle \\ &= \rho(a);\end{aligned}$$

in particular, $\langle x_\rho, x_\rho\rangle = \rho(\mathbb{1}) = 1$, so x_ρ is a unit vector in $\mathcal{H}_\rho$. □

example of this construction on C(X)? may just be a short explanation of how B(H) and C(X) link together. can then talk about noncommutative topology!

Theorem 3 (Gelfand-Naimark)**.** *Every C^*-algebra has a faithful representation.*

Proof. Let A be a C^*-algebra. Suppose we have a collection $\{\varphi_i \mid i \in I\}$ of representations of A on Hilbert spaces $\{\mathcal{H}_i \mid i \in I\}$. For a in A, we have $\|\varphi_i(a)\| \leq \|a\|$ (as each φ_i is a $*$-homomorphism), so we have a bounded operator $\oplus\varphi_i(a)$ on $\oplus\mathcal{H}_i$ by section 1.4. By the properties of $\oplus\varphi_i(a)$ stated therein, the map

$$\varphi : A \to \mathcal{B}(\oplus\mathcal{H}_i) : a \mapsto \oplus\varphi_i(a)$$

is a $*$-homomorphism, and so a representation of A on $\oplus\mathcal{H}_i$. Call φ the *direct sum representation* of A on $\oplus\mathcal{H}_i$, denoted $\oplus\varphi_i$.

With $\mathscr{S}_0$ any collection of states on A containing $\mathscr{P}(A)$, let φ be the direct sum of the collection $\{\pi_\rho \mid \rho \in \mathscr{S}_0\}$ of representations as constructed by the GNS construction.

Now for any $a \neq 0$ in A, there is a pure state on A such that $\rho(a) \neq 0$. By the GNS construction, $\rho(a) = \langle \pi_\rho(a)x_\rho, x_\rho\rangle$, so $\pi_\rho \neq 0$.

□

further topics: von neumann algebras (formal defn), K-theory, group C* algebras, amenable algebras,

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