C^*-ALGEBRAS, AND THE GELFAND-NAIMARK THEOREM

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Todo list

1. Preliminaries

List of stuff we're gonna go right ahead an assume:

- Familiarity with algebras, Banach spaces, Hilbert spaces and other guff.

we will assume knowledge on... (algebra homos map 1 to 1,hausdorff/compact spaces)
brief(er than asst 3) history
most texts start from $\mathcal{B}(\mathcal{H})$ to justify the whole thing. we're algebraists, who don't need no justification. we jump right in at the deep (abstract) end.

write this!

2. Definitions

Definition 1. A *Banach algebra* is a complex Banach space $(A, \|\cdot\|)$ which forms an algebra, such that

$$\|ab\| \leq \|a\|\|b\| \text{ for all } a, b \in A.$$

A $*$*-algebra* is an algebra A with an *involution* map $a \mapsto a^*$ on A such that, for all $a, b \in A$ and for $\alpha \in \mathbb{C}$,

(i) $a^{**} = (a^*)^* = a$,
(ii) $(\alpha a + b)^* = \overline{\alpha} a^* + b^*$,
(iii) $(ab)^* = b^* a^*$.

The element a^* is referred to as the *adjoint* of a.
A C^**-algebra* is a Banach algebra $(A, \|\cdot\|)$ with involution map $a \mapsto a^*$ making it a $*$-algebra, with the condition that

$$\|a^* a\| = \|a\|^2 \text{ for all } a \in A.$$

This condition is known as the *C^* axiom*. There is a weaker, but ultimately equivalent, axiom called the *B^* axiom*.

state and prove

Here we consider complex C^*-algebras. The theory of real C^*-algebras has advanced....

remark on work in real C^*-algebras?

Unless specified otherwise, by an *ideal* of a Banach algebra, we mean a two-sided ideal. A $*$-ideal in a Banach $*$-algebra is a $*$-closed ideal. For C^*-algebras, it turns out that any ideal is automatically a $*$-ideal.

prove

cauchy-schwarz

2.1. Unitization.

If a C^*-algebra A contains an identity element $\mathbb{1}$ such that $a \cdot \mathbb{1} = a = \mathbb{1} \cdot a$ for all $a \in A$, call $\mathbb{1}$ the *unit* in A, and A is then a *unital* C^*-algebra.

Proposition 1. *Any non-unital C^*-algebra A can be isometrically embedded in a unital C^*-algebra $\tilde{A}$ as a maximal ideal.*

tidy up proof

Proof. Let $\tilde{A} = A \oplus \mathbb{C}$ with pointwise addition, and define

$$\begin{aligned}(a, \lambda)(b, \mu) &:= (ab + \lambda b + \mu a, \lambda\mu),\\ (a, \lambda)^* &:= (a^*, \overline{\lambda}),\\ \|(a, \lambda)\| &:= \sup_{\|b\|=1} \|ab + \lambda b\|.\end{aligned}$$

Then $\tilde{A}$ is a $*$-algebra. The norm $\|(a, \lambda)\|$ is the norm in $\mathcal{B}(A)$ of left-multiplication by a on something iunno??? Thus $\tilde{A}$ is a Banach $*$-algebra with unit $(0, 1)$. By design, A is a maximal ideal of codimension 1. The embedding $a \mapsto (a, \lambda)$ is isometric as

$$\|a\| = \|a \cdot \frac{a}{\|a\|}\| \leq \|(a, 0)\| \leq \sup_{\|b\|=1} \|ab\| \leq \|a\|.$$

It remains to verify the C^*-axiom:

$$\begin{aligned}\|(a, \lambda)\|^2 &= \sup_{\|b\|=1} \|ab + \lambda b\|^2\\ &= \sup_{\|b\|=1} \|b^*a^*ab + \lambda b^*a^*b + \overline{\lambda} b^*ab + |\lambda|^2 b^*b\\ &\leq \sup_{\|b\|=1} \|a^*ab + \lambda a^*b + \overline{\lambda} ab + |\lambda|^2 b\|\\ &= \|(a^*a + \lambda a^* + \overline{\lambda} a, |\lambda|^2)\\ &= \|(a, \lambda)^*(a, \lambda)\|\\ &\leq \|(a, \lambda)^*\|\|(a, \lambda)\|.\end{aligned}$$

By symmetry of $*$, $\|(a, \lambda)^*\| = \|(a, \lambda)\|$. Hence, the above inequality becomes equality and we have that

$$\|(a, \lambda)^*(a, \lambda)\| = \|(a, \lambda)\|^2.$$

$\square$

In light of this result, we take all C^*-algebras from here to be unital unless specified otherwise. For the results we will consider, we can simply consider the unital case. However, there are important circumstances in advanced theory in which we need to relax the unital condition.

2.2. **the spectrum.** Given an element $a \in A$ of a C^*-algebra, define its spectrum $\mathrm{sp}(a)$:

$$\mathrm{sp}(a) := \{\lambda \in \mathbb{C} \mid a - \lambda \mathbb{1} \text{ is not invertible in } A\}.$$

AAAAAAAAAAH I NEED SO MUCH SPECTRAL THEORY AND I DON'T KNOW ANY

2.3. **more definitions.** An element $a \in A$ of a C^*-algebra is called

normal?

- *self-adjoint* if $a^* = a$;
- *unitary* if $aa^* = a^*a = \mathbb{1}$;
- *positive* if it is self-adjoint and $\mathrm{sp}(a) \subseteq \mathbb{R}^+$.

Denote the set of self-adjoing elements in A by A_{sa}, and the subset of positive elements in A_{sa} by A^+.

order structure on A^+

Given Banach $*$-algebras A and B, a map $\varphi : A \to B$ is a $*$-*homomorphism* if it is an algebra homomorphism for which $\varphi(a^*) = \varphi(a)^*$ for all $a \in A$. If A and B are both unital algebras and a homomorphism φ maps $\mathbb{1}_A$ to $\mathbb{1}_B$, say φ is a *unital* homomorphism. If a $*$-homomorphism φ is one-to-one, call it a $*$-*isomorphism.*

Proposition 2. *Suppose A and B are C^*-algebras and $\varphi : A \to B$ is a $*$-homomorphism. Then $\|\varphi(a)\| \leq \|a\|$ for all $a \in A$. If φ is a $*$-isomporphism, then $\|\varphi(a)\| = \|a\|$ for all $a \in A$.*

Proof. □

norm preservation theorem here?

A *linear functional* on a C^*-algebra A is a linear operator $\rho : A \to \mathbb{C}$. A *multiplicative* linear functional ρ satisfies $\rho(ab) = \rho(a)\rho(b)$ for all $a, b \in A$. A *state* on A is a linear functional ρ such that $\|\rho\| = 1$ and $\rho(a) \geq 0$ for all positive elements $a \in A^+$. Denote by $\mathscr{S}(A)$ the set of all states on A. A (topologically) extreme point of $\mathscr{S}(A)$ is called a *pure* state of A, and the set of pure states on A is denoted by $\mathscr{P}(A)$.

Proposition 3. *The set of pure states on an Abelian C^*-algebra A is precisely the set of multiplicative linear functionals on A.*

Proof.(Adapted from K&R, 4.4.1). Suppose ρ is a pure state on A. To show that $\rho(ab) = \rho(a)\rho(b)$ for $a, b, \in A$, we restrict attention to the case where $0 \leq b \leq \mathbb{1}$. Linearity gives us the general case. In this case, for $h \in A^+$ we have that $0 \leq hb \leq h$, so $0 \leq \rho(hb) \leq \rho(h)$. Hence $\rho_b(a) := \rho(ab)$ for $a \in A$ defines a positive linear functional on A with $\rho_b \leq \rho$. The restriction $\rho|_{A_{sa}}$ is a pure state on A_{sa} and $\rho_b|_{A_{sa}} \leq \rho|_{A_{sa}}$, and it follows that $\rho_b|_{A_{sa}} = \alpha\rho|_{A_{sa}}$ for some $\alpha \in \mathbb{R}^+$. Hence $\rho_b = \alpha\rho$ and so for $a \in A$:

show

show

via what?

$$\rho(ab) = \rho_b(a) = \alpha\rho(a) = \alpha\rho(\mathbb{1})\rho(a) = \rho_b(\mathbb{1})\rho(a) = \rho(b)\rho(a)$$

□

pure states are precisely multiplicative linear functionals. define weak* topology on P(S)

2.4. **$\mathcal{B}(\mathcal{H})$ - an example.** This section concerns the fundamental example of a C^*-algebra - the set $\mathcal{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$. Here we will demonstrate that $\mathcal{B}(\mathcal{H})$ is a C^*-algebra and give some basic results.

Claim. *$\mathcal{B}(\mathcal{H})$ is a C^*-algebra with the operator norm*

$$\|T\| := \sup_{\|x\|=1} \|Tx\|$$

and involution taking T to its adjoint map T^*. The identity map $I : x \mapsto x$ is a unit for $\mathcal{B}(\mathcal{H})$

Proof. $\|\cdot\|$ is a norm on $\mathcal{B}(\mathcal{H})$. Let $\{T_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{B}(\mathcal{H})$. Then for any positive ϵ, there is a positive integer N such that

$$\|T_m - T_n\| < \epsilon \text{ for all } m, n \geq N.$$

Applying $T_m - T_n$ to $x \in \mathcal{H}$, we have

$$\|T_m x - T_n x\| \leq \|T_m - T_n\|\|x\| < \epsilon\|x\|, \tag{1}$$

so $\{T_n x\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}$, converging to an element in $\mathcal{H}$. Define a linear operator $T : \mathcal{H} \to \mathcal{H}$ by

$$Tx := \lim_{n\to\infty} T_n x \text{ for } x \in \mathcal{H}.$$

Taking limits as m tends to infinity in equation (1), we obtain

$$\|Tx - T_n x\| < \epsilon\|x\| \text{ for all } n \geq N,$$

and so we have that $T - T_n$ (and hence $T = (T - T_n) + T_n$) is a bounded operator and

$$\|T - T_n\| < \epsilon \text{ for all } n \geq N.$$

We conclude that $T_n \to T$, and so $\mathcal{B}(\mathcal{H})$ is complete.

Since boundedness is equivalent to continuity on $\mathcal{H}$, given $S, T \in \mathcal{B}(\mathcal{H})$, the operator $ST : \mathcal{H} \to \mathcal{H}; x \mapsto (S \circ T)(x)$ is bounded on $\mathcal{H}$. Given $x \in \mathcal{H}$ and $\lambda \in \mathbb{C}$,

$$\begin{aligned}
((\lambda S)T)(x) &= ((\lambda S) \circ T)(x) \\
&= \lambda S(Tx) \\
&= \lambda(S \circ T)(x) \\
&= \lambda ST(x),
\end{aligned}$$

so that $(\lambda S)T = \lambda ST$ in $\mathcal{B}(\mathcal{H})$, whence $\mathcal{B}(\mathcal{H})$ is an algebra. We have

$$\begin{aligned}
\|ST\| &= \sup_{\|x\|=1} \|STx\| \\
&= \sup_{\|x\|=1} \|S(Tx)\| \\
&\leq \|S\| \sup_{\|x\|=1} \|Tx\| \\
&= \|S\|\|T\|.
\end{aligned}$$

To see that * is an involution, use the fact that the adjoint operator is unique for each operator and the following equalities.

(i)
$$\begin{aligned}\langle(\alpha T+S)^*x,y\rangle &= \langle x,\alpha T+Sy\rangle\\ &= \overline{\alpha}\langle x,Ty\rangle+\langle x,Sy\rangle\\ &= \overline{\alpha}\langle T^*x,y\rangle+\langle S^*x,y\rangle\\ &= \langle(\overline{\alpha}T^*+S^*)x,y\rangle.\end{aligned}$$

(ii)
$$\begin{aligned}\langle(T^*)^*x,y\rangle &= \langle x,T^*y\rangle\\ &= \overline{\langle T^*y,x\rangle}\\ &= \overline{\langle y,Tx\rangle}\\ &= \langle Tx,y\rangle.\end{aligned}$$

(iii)
$$\begin{aligned}\langle(ST)^*x,y\rangle &= \langle x,STy\rangle\\ &= \langle S^*x,Ty\rangle\\ &= \langle T^*S^*x,y\rangle.\end{aligned}$$

It remains to demonstrate the C^*-axiom on $\mathcal{B}(\mathcal{H})$. For all $x\in\mathcal{H}$, we have

$$\|Tx\|^2=\langle Tx,Tx\rangle=\langle T^*Tx,x\rangle\leq\|T^*T\|\|x\|^2,$$

so that

$$\|T\|^2\leq\|T^*T\|\leq\|T^*\|\|T\|=\|T\|^2.$$

It is clear that I is a unit. Hence, the claim. □

example of $\mathcal{B}(\mathcal{H})$ at end of section to retro-motivate notation. discuss nomenclature (state etc) coming from QM

maybe move this to as early as we can?

2.5. $C(X)$ **- another example.** Given a locally compact Hausdorff space X, let $C(X)$ be the algebra of continuous functions $f:X\to\mathbb{C}$, with addition and multiplication defined pointwise. Define $\|\cdot\|$ on $C(X)$ by

$$\|f\|:=\sup_{x\in X}|f(x)|,$$

that is the norm inherited from the Banach space $\ell^2(X,\mathbb{C})$.

example of $C(X)$?

3. Representations of C*-algebras

to include all representation theory, including GNS, CGN and GN

Theorem 1 (Gelfand-Naimark, commutative)**.** *Every commutative C^*-algebra A is $*$-isomorphic to $C(X)$, the algebra of continuous functions a compact Hausdorff space X.*

Proof. Our compact topological space will be the set $\mathscr{P}(A)$ of pure states, endowed with the weak* topology as defined above. □

remark about nonunital commutative.

Definition 2. Given a C^*-algebra A, a *representation of A on a Hilbert space $\mathcal{H}$* is a $*$-homomorphism $\varphi : A \to \mathcal{B}(\mathcal{H})$. An isomorphic representation is called *faithful*. If there exists an element $x \in \mathcal{H}$ such that the set $\{\varphi(a) \mid a \in A\}$ is everywhere-dense in $\mathcal{H}$, say that φ is a *cyclic* representation, with *cyclic vector* x.

Theorem 2 (Gelfand-Naimark-Segal construction)**.** *If ρ is a state on a C^*-algebra A, then there exists a cyclic representation π_ρ of A on a Hilbert space H_ρ, with unit cyclic vector x_ρ, such that*

$$\rho(a) \;=\; \langle \pi_\rho(a)x_\rho, x_\rho\rangle, \;\; \forall a \in A.$$

Proof. We will construct from ρ the space $\mathcal{H}_\rho$, representation π_ρ, and vector x_ρ, and demonstrate the required properties.

Consider the *left kernel* of ρ:

$$L_\rho := \{t \in A \mid \rho(t^*t) = 0\}.$$

For $a, b \in A$, define $\langle a, b\rangle_0 := \rho(b^*a)$. Then $L_\rho = \{t \in A \mid \langle t, t\rangle_0 = 0\}$, and $\langle \cdot, \cdot\rangle_0$ satisfies

(i) Linearity in 1st argument: for $a, b \in A$, $\alpha, \beta \in \mathbb{C}$:

$$\begin{aligned}\langle \alpha a + \beta b, c\rangle_0 &= \rho(c^*(\alpha a + \beta b)\\ &= \rho(\alpha c^* a + \beta c^* b)\\ &= \alpha\rho(c^*a) + \beta\rho(c^*b)\\ &= \alpha\langle a, c\rangle_0 + \beta\langle b, c\rangle_0.\end{aligned}$$

(ii) Conjugate symmetric: for $a, b \in A$:

$$\begin{aligned}\langle b, a\rangle_0 &= \rho(a^*b)\\ &= \rho((b^*a)^*)\\ &= \overline{\rho(b^*a)}\\ &= \overline{\langle a, b\rangle}_0.\end{aligned}$$

(iii) Positive semi-definite. why?

Note that $\langle \cdot, \cdot\rangle$ is not necessarily positive definite on $A - L_\rho$ is exactly where this fails.

L_ρ is a linear subspace of A: Consider sentence

$$L := \{t \in A \mid \langle t, a\rangle_0 = 0, \;\forall a \in A\} \subseteq L_\rho.$$

For $t \in L_\rho$, by Cauchy-Schwarz we have

$$|\langle t, a\rangle_0|^2 \le \langle t, t\rangle_0\langle a, a\rangle_0, \;\; \forall a \in A;$$

that is,

$$\langle t, a\rangle_0 = 0, \;\; \forall a \in A,$$

so $t \in L$ and $L_\rho = L$.

Now, for $a, b \in L$, $\alpha \in \mathbb{C}$ and $c \in A$:

$$\langle \alpha a + b, c\rangle_0 = \alpha\langle a, c\rangle_0 + \langle b, c\rangle_0 = 0,$$

so $\alpha a + b \in L$; also, $\langle 0, c\rangle_0 = 0$ so $0 \in L$. Hence, $L(= L_\rho)$ is a linear subspace of A.

For $s \in A$, $t \in L_\rho$, by the Cauchy Schwarz inequality [ref] we have

$$\begin{aligned} |\rho(s^*t)|^2 &= |\langle t, s\rangle_0|^2 \\ &\leq \langle t, t\rangle_0 \cdot \langle s, s\rangle_0 \\ &= \rho(t^*t) \cdot \rho(s^*s) \\ &= 0, \end{aligned}$$

so $\rho(s^*t) = 0$. Letting $s = a^*at$ for $a \in A$, then

$$\begin{aligned} \rho((at)^*at) &= \rho(at^*a^*at) \\ &= \rho((a^*at)^*t) \\ &= \rho(s^*t) \\ &= 0, \end{aligned}$$

so that $at \in L_\rho$, for all $a \in A$ and $t \in L_\rho$; we conclude that L_ρ is a left ideal in A. L_ρ is the preimage in A of $\{0\}$ under the continuous map$t \mapsto \rho(t^*t)$, so is closed.

start sentence properly

Consider now $V_\rho := A/L_\rho$, with $\langle\cdot,\cdot\rangle$ defined by

$$\langle a + L_\rho, b + L_\rho\rangle := \langle a, b\rangle_0, \text{ for } a + L_\rho, b + L_\rho \in V_\rho.$$

It follows from properties $i), ii)$ and $iii)$ of $\langle\cdot,\cdot\rangle_0$ that $\langle\cdot,\cdot\rangle$ is an inner product on V_ρ – with

$$\begin{aligned} \langle a + L^\rho, a + L^\rho\rangle = 0 &\iff \langle a, a\rangle = 0 \\ &\iff a \in L_\rho \\ &\iff a + L_\rho = 0 + L_\rho \end{aligned}$$

giving positive definiteness. The completion of V_ρ with respect to $\langle\cdot,\cdot\rangle$ is a Hilbert space - this is the Hilbert space $\mathcal{H}_\rho$ we're looking for.

Now we fix $a \in A$, and consider the map

$$\pi_a : V_\rho \to V_\rho; b + L_\rho \mapsto ab + L_\rho.$$

Let $b_1, b_2 \in A$ be such that $b_1 + L_\rho = b_2 + L_\rho$. Then:

$$\begin{aligned} &\implies b_1 - b_2 \in L_\rho \\ &\implies a(b_1 - b_2) \in L_\rho \\ &\implies ab_1 - ab_2 \in L_\rho \\ &\implies ab_1 + L_\rho = ab_2 + L_\rho \\ &\implies \pi_a(b_1 + L_\rho) = \pi_a(b_2 + L_\rho). \end{aligned}$$

Hence π_a defines a linear operator on V_ρ.

For $b + L_\rho \in V_\rho$:

$$\begin{aligned}
\|a\|^2 \cdot \|b + L_\rho\| - \|\pi_a(b + L_\rho)\| &= \|a\|^2 \cdot \|b + L_\rho\| - \|ab + L_\rho\| \\
&= \|a\|^2 \cdot \langle b + L_\rho, b + L_\rho\rangle - \langle ab + L_\rho, ab + L_\rho\rangle \\
&= \|a\|^2 \cdot \rho(b^*b) - \rho((ab)^*ab) \\
&= \rho(\|a\|^2 b^*b - b^*a^*ab) \\
&= \rho(b^*(\|a\|^2\mathbb{1} - a^*a)b) \\
&\geq 0.
\end{aligned}$$

Thus π_a is a bounded operator, with $\|\pi_a\| \leq \|a\|$. By continuity, π_a cont of what? extends to a bounded operator on $\mathcal{H}_\rho$ – say $\pi_\rho(a) : \mathcal{H}_\rho \to \mathcal{H}_\rho$ such that

$$\pi_\rho(a)(v) = \pi_a(v)$$

for $v \in V_\rho$. Then $\pi_\rho(a) \in \mathcal{B}(\mathcal{H}_\rho)$ for each $a \in A$, so π_ρ defines a map $A \to \mathcal{B}(\mathcal{H}_\rho)$ such that $a \mapsto \pi_\rho(a)$. This will be our representation.

Now, for $a, b \in A$, $c + L_\rho \in V_\rho$ and $\alpha \in \mathbb{C}$:

$$\begin{aligned}
\pi_{\alpha a+b}(c + L_\rho) &= (\alpha a + b)(c + L_\rho) \\
&= (\alpha ac + L_\rho) + (bc + L_\rho) \\
&= \alpha\pi_a(c + L_\rho) + \pi_b(c + L_\rho),
\end{aligned}$$

so that $\pi_{\alpha a+b} = \alpha\pi_a + \pi_b$ on V_ρ.

For $a, b \in A$ and $c + L_\rho \in V_\rho$:

$$\begin{aligned}
\pi_{ab}(c + L_\rho) &= abc + L_\rho \\
&= \pi_a(bc + L_\rho) \\
&= \pi_a(\pi_b(c + L_\rho)) \\
&= (\pi_a \cdot \pi_b)(c + L_\rho),
\end{aligned}$$

so that $\pi_{ab} = \pi_a \cdot \pi_b$ on V_ρ.

For $a \in A$ and $b + L_\rho,\ c + L_\rho \in V_\rho$:

$$\begin{aligned}
\langle b + L_\rho, \pi_a^*(c + L_\rho)\rangle &= \langle \pi_a(b + L_\rho), c + L_\rho\rangle \\
&= \langle ab + Lr, c + L_\rho\rangle \\
&= \rho(c^*ab) \\
&= \rho((a^*c)^*b) \\
&= \langle b + L_\rho, a^*c + L_\rho\rangle \\
&= \langle b + L_\rho, \pi_{a^*}(c + L_\rho),
\end{aligned}$$

so that $\pi_a^* = \pi_{a^*}$ on V_ρ.

$V_\rho \subset \mathcal{H}_\rho$ is a dense subset, so the three properties above hold on $\mathcal{H}_\rho$ by continuity. Hence, $\pi_\rho : A \to \mathcal{B}(\mathcal{H}_\rho)$ is a representation of A. As to cont of what?

the unit vector, consider $x_\rho := \mathbb{1} + L_\rho \in V_\rho$. Then for $a \in A$,

$$\begin{aligned}\langle \pi_\rho(a)x_\rho, x_\rho\rangle &= \langle \pi_a(\mathbb{1} + L_\rho), \mathbb{1} + L_\rho\rangle \\ &= \langle a + L_\rho \mathbb{1} + L_\rho\rangle \\ &= \rho(a);\end{aligned}$$

in particular, $\langle x_\rho, x_\rho\rangle = \rho(\mathbb{1}) = 1$, so x_ρ is a unit vector in $\mathcal{H}_\rho$. □

example of this construction on C(X)? may just be a short explanation of how B(H) and C(X) link together. can then talk about noncommutative topology!

Theorem 3 (Gelfand-Naimark)**.** *Every C^*-algebra has a faithful representation.*

Proof. for this we just take the direct sum representation of the representations given from GNS by some set of states containing all pure states. □

further topics: K-theory, group C* algebras, amenable algebras, von neumann algebras,

references!!!!