C^*-Algebras, and the Gelfand-Naimark Theorems

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A **Banach algebra** is a complete normed algebra A such that

$$\|ab\| \leq \|a\| \cdot \|b\| \quad \forall a, b \in A.$$

Definitions

C^*-algebra

A C^*-**algebra** A is a Banach algebra with **adjoint** map $a \mapsto a^*$ on A, satisfying the following:

1. $a^{**} = a$
2. $(\alpha a + b)^* = \bar{\alpha} a^* + b^*$
3. $(ab)^* = b^* a^*$
4. $\|a^* a\| = \|a\|^2 \quad$ (C^* axiom)

We assume here that C^*-algebras have an identity, denoted 1.

Examples

Continuous complex-valued functions on a compact Hausdorff space.

Bounded operators on a Hilbert space, $\mathcal{B}(\mathcal{H})$.

Ideal of compact operators, $\mathcal{K}(\mathcal{H})$.

Calkin algebra, the quotient algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$.

Definitions

Spectrum

Spectrum of $a \in A$ is

$$\sigma(a) = \{\lambda \in \mathbb{C} \mid a - \lambda 1 \text{ is not invertible}\}.$$

Spectral radius of $a \in A$ is

$$r(a) = \sup_{\lambda \in \sigma(a)} |\lambda|.$$

Definitions

Normal elements

Say that $a \in A$ is **self-adjoint** if $a^* = a$, and **normal** if $a^* a = a a^*$.

Say that a self-adjoint a is **positive** if $\sigma(a) \subset \mathbb{R}^+$.

Definitions

States

A **state** is a linear map $\rho : A \to \mathbb{C}$ such that $\rho(a) \geq 0$ for all positive $a \in A$, and $\rho(1) = 1$.

The **state space**, $\mathcal{S}(A)$, is a convex subset of the dual space of A. Call the extreme points of the state space **pure states**.

Definitions

Maps between C^*-algebras

A $*$-**homomorphism** is an algebra homomorphism $\varphi : A \to B$ such that $\varphi(a^*) = \varphi(a)^*$, for all $a \in A$.

A $*$-**isomorphism** is a bijective $*$-homomorphism.

Cool Results

Uniqueness of norm

The norm of each element is given by the spectral radius, so the norm is unique.

$$\|a\|^2 = \|a^*a\| = r(a^*a)$$
$$\|a\| = r(a^*a)^{1/2}$$

Cool Results

$*$-homomorphisms are continuous

Let $\varphi : A \to B$ be a $*$-homomorphism, then

$$\|\varphi(a)\| \leq \|a\|$$

for all $a \in A$.

Equality if φ is a $*$-isomorphism.

Definitions

Representation

A **representation** of A on a Hilbert Space $\mathcal{H}$ is a $*$-homomorphism $A \to \mathcal{B}(\mathcal{H})$.

A bijective representation is called **faithful**.

Gelfand-Naimark Theorem

Theorem

Every C^*-algebra has a faithful representation.

Proof.

Uses the Gelfand-Naimark-Segal construction.

Gelfand-Naimark Theorem

Proof: Gelfand-Naimark-Segal Construction

Let $L = \{a \in A \mid \rho(a^*a) = 0\}$.

$\mathcal{H}$ is the Hilbert space completion of A/L.

Define operators

$$\pi_a : A/L \to A/L : b + L \mapsto ab + L,$$

and extend to $\pi_a : \mathcal{H} \to \mathcal{H}$.

Representation is given by

$$\pi : A \to \mathcal{B}(\mathcal{H}) : a \mapsto \pi_a.$$

Gelfand-Naimark Theorem

Proof: Direct Sum

Proof concludes by taking 'direct sum' representation over the representations given by doing GNS construction to a subset of state space containing all pure states. This gives a faithful representation.

Gelfand-Naimark Theorem

For Commutative C^*-algebras

Theorem

Every commutative C^*-algebra A is $*$-isomorphic to $C(\mathscr{P}(A))$, the algebra of continuous functions on the compact Hausdorff space $\mathscr{P}(A)$ containing all pure states on A.

References – Any Questions?

Kadison, R.V. and Ringrose, J.R., 1983. *Fundamentals of the Theory of Operator Algebras, Vol. I. Elementary Theory*. Springer.

Blackadar, B., 2006. *Algebras: Theory of C^*-Algebras and Von Neumann Algebras* (Vol. 122). Springer Science & Business Media.

My project report can be downloaded from `goo.gl/Qv1zas`.