C^*-Algebras, and the Gelfand-Naimark Theorems

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Definitions

A C^*-**algebra** A is a Banach algebra with norm $\| \cdot \|$ and an involution map $a \mapsto a^*$ satisfying the following:

1. $a^{**} = a$
2. $(\alpha a + b)^* = \bar{\alpha} a^* + b^*$
3. $(ab)^* = b^* a^*$
4. $\|a^* a\| = \|a\|^2 \quad$ (C^* axiom)

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Definitions

A **state** is a positive linear functional $\rho : A \to \mathbb{K}$ such that $\rho(a) \geq 0$ for all positive $a \in A$.
spectrum, spectral radius, state, pure state, * homo/isomorphism, representation, faithful representation,

Examples

Cool Asides

Uniqueness of norm: $\|a\|^2 = \|a^*a\| = r(a^*a)$. Requires spectral theory. The spectral radius of a normal element is equal to its norm. From this, and the C* axiom, we get that the norm of each element is given by the spectral radius, which is defined in terms of the spectrum which does not use the norm.

$*$**-homomorphisms are continuous:** homomorphisms do not increase norm, so are bounded and hence continuous.

isomorphisms are isometric. again uses spectral theory, this time to show that spectral radius is not increased / is preserved.

Gelfand-Naimark Theorems

Theorem

Every Abelian C^*-algebra A is $*$-isomorphic to $C(\mathscr{P}(A))$, the algebra of continuous functions on the compact Hausdorff space $\mathscr{P}(A)$ of pure states on A.

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Theorem

Every C^*-algebra has a faithful representation.

The Gelfand-Naimark-Segal Construction

Used to prove the GN theorem.
Given a state on a C* algebra, we can construct a Hilbert space and a representation on that space. Given a and b in A, define $\langle a, b \rangle = \rho(b^* a)$. This is a semi-inner product – basically an inner product, but there exist $a \neq 0$ such that $\langle a, a \rangle = 0$. However, if we consider the quotient vector space of A by the collection of such elements, this space completes to a Hilbert space with $\langle \cdot, \cdot \rangle$ as the inner product.

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References