C^*-Algebras, and the Gelfand-Naimark Theorems

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Abstract

The collection of bounded linear operators from a Hilbert space to itself forms an algebra, complete with respect to the operator norm and possessing an involution map taking an operator to its adjoint operator. These operations interact in a very strong way, leading to a linking of the algebraic and topological structures of such an operator algebra. These algebras are the prototype for a C^*-algebra, which we define as a complete normed algebra with an involution such that the involution and norm interact in the same way as in an operator algebra. In this report, we define an abstract C^*-algebra and discuss the standard examples – operators on a Hilbert space and continuous functions on a compact Hausdorff space. We discuss some of the algebraic and topological structure of these spaces, and go on to prove the Gelfand-Naimark theorems. The Gelfand-Naimark theorem says that any C^*-algebra is the same as an operator algebra on some Hilbert space, and the commutative Gelfand-Naimark theorem states that any commutative C^*-algebra is the same as an algebra of continuous functions on some compact Hausdorff space.

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Chapter 1

Introduction

1.1 History of the Study of Operator Algebras

The study of C^*-algebras started with the study of rings of operators acting on a Hilbert space, which was introduced in the 1930s as a framework for John von Neumann's 1932 formulation of quantum mechanics in [1]. These rings of operators are now considered part of the theory of *von Neumann algebras*, a subsection of C^*-algebra theory.

In 1943, Gelfand and Naimark [2] established an abstract characterisation of C^*-algebras, free from dependence on the operators acting on a Hilbert space. They proved the Gelfand-Naimark (GN) theorem, which we will be considering here, giving the link between these abstract C^*-algebras and the rings of operators previously studied. Used in the proof of the GN theorem is the Gelfand-Naimark-Segal construction, a technique which, given a certain kind of linear function on a C^*-algebra, yields a representation of that C^*-algebra as an algebra of operators on a Hilbert space.

1.2 Resources

Throughout this work we will be making extensive use of the first volume of *'Fundamentals of the Theory of Operator Algebras'* [3, 4], by Kadison and Ringrose; the authors aim to make accessible the "vast recent research literature", by first introducing and exploring the prototypical examples and developing the abstract theory later.

Some other texts which cover C^*-algebras:

- Dixmier [5] presents a summary of the general theory up to that time (1977), with [6] focussing on reworking and developing the theory of von Neumann algebras;
- Sakai [7] gives a treatment of C^*-algebras and von Neumann algebras from a more topological point of view;
- Davidson [8] gives a quick overview of the theory and uses that as a baseline to explore many classes of examples;
- Blackadar [9] gives a concise, encyclopaedic coverage of the theory of operator algebras, and covers more specialised material and applications.

1.3 Summary

The objective of this project is to give a good understanding of the definition of C^*-algebras, to examine their topological, algebraic and geometric properties, and to consider the Gelfand-Naimark theorems and the methods used to prove them.

After this introductory chapter, we establish notation, and recall some definitions and results which will be used throughout. [these include...]

In Chapter 3, we define a C^*-algebra, and some important elements of a C^*-algebra: normal elements and self-adjoint elements. A discussion of C^*-algebras with a multiplicative identity follows, and we show that while we don't require that an algebra have such an identity, we can always adjoin one. Next we introduce an example, in the form of the algebra of continuous functions on a topological space, introducing a few concepts which we define in the next two sections. These concepts are that of the spectrum of an element, positive elements and positive linear functionals, and states on a C^*-algebra. In Section 3.4, we introduce the continuous functional calculus and then show that the norm on a C^*-algebra is unique. In Section 3.5, we go on to prove that $*$-homomorphism are always continuous and that $*$-isomorphisms are isometric maps. We finish out the chapter with another example, the algebra of bounded linear operators on a Hilbert space, remarking on the ideas that link the study of abstract C^*-algebras to the field of Quantum Mechanics.

The final chapter contains the Gelfand-Naimark theorems we are interested in. The commutative Gelfand-Naimark theorem, found in Section 4.1, says that an Abelian C^*-algebra is in some way 'the same as' the algebra of continuous complex valued functions on a compact, Hausdorff topological space. After proving this result, we remark on how this leads to the idea of 'noncommutative topology'. Section 4.3 contains a constructive proof of the Gelfand-Naimark theorem concerning all C^*-algebras, showing that any C^*-algebra is 'the same as' the algebra of bounded operators on a Hilbert space.

1.4 Further Topics

Topics in C^*-algebra theory which follow on from this include the classification of different types of C^*-algebras, such as *approximately-finite dimensional* C^*-algebras, which are made up in a certain way from finite dimensional C^*-algebras.

A *von Neumann algebra* is a C^*-subalgebra $\mathcal{W}$ of $\mathcal{B}(\mathcal{H})$ which is unital and closed in the *weak operator topology*, the weakest topology such that for all T in $\mathcal{W}$, the maps $T \mapsto \langle Tx, y\rangle$ are continuous for all x, y in $\mathcal{H}$. This enforces an even stricter structure on $\mathcal{W}$, giving results such as the von Neumann Double Commutant theorem.

Another area of research is K-theory, which for C^*-algebras refers to the study of groups related to the structure of *projective elements*: elements x such that $x^2 = x = x^*$.

For more on all of these topics, see [6, 7, 8].

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Chapter 2

Preliminary Definitions and Results

This chapter contains outlines and recalls definitions and results from other areas of mathematics which will prove useful. We will be assuming some familiarity with the following theory:

- Rings, algebras and linear spaces;
- Normed spaces, inner product spaces, Banach and Hilbert spaces;
- Point-set topology.

A good broad background on all of these can be found in [10, 11].

The following notation applies throughout: $\mathbb{R}^+$ denotes the non-negative reals. $\operatorname{Re} z$ refers to the real part of the complex number z. An ideal of an algebra is, unless otherwise stated, taken to mean a two-sided ideal. An algebra A is described as *commutative* or *Abelian* if

$$ab = ba \text{ for all } a, b \in A.$$

2.1 Linear Spaces

A *partial order* on a set X is a relation $\leq$ which satisfies the conditions

- $x \leq x$ for all x in X; (reflexivity)
- for all x and y in X, if $x \leq y$ and $y \leq x$, then $x = y$; (antisymmetry)
- for all x, y and z in X, if $x \leq y$ and $y \leq z$, then $x \leq z$. (transitivity)

A *positive cone* in a real vector space V is a subset V^+ that is closed under addition and scaling by positive scalars, and if both of $v, -v$ are in V^+, then $v = 0$. A *partially ordered vector space* is a real vector space V with a positive cone V^+ – the partial order on V given by

$$x \leq y \quad \Longleftrightarrow \quad y - x \in V^+,$$

for x, y in V. A familiar example is the non-negative real numbers, which form a positive cone in $\mathbb{R}$.

Given a linear space X over a field $\mathbb{K}$, a *linear functional* on X is a linear map $\rho : X \to \mathbb{K}$. The set X^* of linear functionals on X is itself a linear space, called the *dual space* of X. We can endow the dual space with the operator norm $\|\cdot\|$ given by:

$$\sup_{x \in X} |f(x)| .$$

A linear functional ρ is called *multiplicative* if $\rho(xy) = \rho(x)\rho(y)$ for all x, y in X.

Theorem (Hahn-Banach Extension theorem)**.** *If ρ_0 is a bounded linear functional on a subspace X_0 of a normed linear space X, with dual space X^* having norm $\|\cdot\|_*$, then there is a bounded linear functional ρ on X such that $\|\rho\|_* = \|\rho_0\|_*$ and $\rho = \rho_0$ on X_0.*

Proof. Can be found in [3, Theorem 1.6.1, p. 44]. $\square$

2.2 Hilbert Spaces

A normed linear space $(V, \|\cdot\|)$ is a *Banach space* if it is complete with respect to $\|\cdot\|$, in the sense that all Cauchy sequences in V converge with respect to the norm. An inner product space $(V, \langle\cdot,\cdot\rangle)$ is a *Hilbert space* if it is a Banach space with respect to the norm induced by $\langle\cdot,\cdot\rangle$ (that is, $\|x\| = \langle x, x\rangle^{1/2}$ for x in X.

A linear map, or a *linear operator*, $T : \mathcal{H} \to \mathcal{H}$, from a Hilbert space to itself, is said to be *bounded* if there exists a scalar M such that

$$\|Tx\| \leq M \|x\| ,$$

for all x in $\mathcal{H}$. Bounded linear operators are automatically continuous – see [10, 1.32]. The collection $\mathcal{B}(\mathcal{H})$ of all bounded linear operators on a Hilbert space $\mathcal{H}$ is itself a Banach space with the operator norm

$$\|T\| = \sup_{\|x\|=1} \|Tx\| .$$

Every operator T in $\mathcal{B}(\mathcal{H})$ has a unique *adjoint* operator T^* in $\mathcal{B}(\mathcal{H})$, which is the map satisfying

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \text{ for all } x, y \in \mathcal{H}.$$

We will discuss $\mathcal{B}(\mathcal{H})$ in more detail in Section 3.6.

We need some Hilbert space constructions – particularly, the direct sum of a collection of Hilbert spaces and the direct sum of bounded operators on these Hilbert spaces. Given a finite collection $\{\mathcal{H}_1, \ldots, \mathcal{H}_n\}$ of Hilbert spaces, let $\mathcal{H}$ denote the set

$$\mathcal{H} = \{(x_1, \ldots, x_n) \mid x_i \in \mathcal{H}_i \text{ for } i = 1, \ldots, n\} .$$

Define addition and scalar multiplication coordinatewise, and given $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n) \in \mathcal{H}$, the equation

$$\langle x, y\rangle = \langle x_1, y_1\rangle + \cdots + \langle x_n, y_n\rangle$$

defines an inner product $\langle\cdot,\cdot\rangle$ on $\mathcal{H}$. The resulting norm $\|\cdot\|$ is given by

$$\|x\|^2 = \|x_1\|^2 + \cdots + \|x_n\|^2 .$$

It is easy to show that $\mathcal{H}$ is a Hilbert space with these operations, and we call it the *direct sum* of the collection $\{\mathcal{H}_1, \ldots, \mathcal{H}_n\}$, denoted by $\bigoplus \mathcal{H}_i$.

Similarly, we can construct a Hilbert space direct sum of an infinite collection $\{\mathcal{H}_i \mid i \in I\}$ of Hilbert spaces. Let $\mathcal{H}$ be the set

$$\mathcal{H} = \Big\{ (x_i) \mid x_i \in \mathcal{H}_i \text{ for each } i \text{ and } \sum_{i \in I} \|x_i\|^2 < \infty \Big\}.$$

Given $x = (x_i)$ and $y = (y_i) \in \mathcal{H}$, we have that

$$\begin{aligned}
\Big(\sum_{i \in I} \|x_i + y_i\|^2 \Big)^{1/2} &\le \Big(\sum_{i \in I} (\|x_i\| + \|y_i\|)^2 \Big)^{1/2} \\
&\le \Big(\sum_{i \in I} \|x_i\|^2 \Big)^{1/2} + \Big(\sum_{i \in I} \|y_i\|^2 \Big)^{1/2} \\
&< \infty.
\end{aligned}$$

Hence, the sequence $(x_i + y_i)$ is in $\mathcal{H}$, and we can define addition and scalar multiplication coordinatewise on $\mathcal{H}$:

$$(x_i) + (y_i) = (x_i + y_i) \qquad\qquad \alpha(x_i) = (\alpha x_i).$$

We also have

$$\begin{aligned}
\sum_{i \in I} |\langle x, y \rangle| &\le \sum_{i \in I} \|x_i\| \, \|y_i\| \\
&\le \Big(\sum_{i \in I} \|x_i\|^2 \Big)^{1/2} \Big(\sum_{i \in I} \|y_i\|^2 \Big)^{1/2} \\
&< \infty,
\end{aligned}$$

so that we can define an inner product $\langle \cdot, \cdot \rangle$, with induced norm $\|\cdot\|$, on $\mathcal{H}$ by

$$\langle x, y \rangle = \sum_{i \in I} |\langle x, y \rangle|, \qquad\qquad \|x\| = \Big(\sum_{i \in I} \|x_i\|^2 \Big)^{1/2}.$$

To see that $\mathcal{H}$ is complete with respect to $\|\cdot\|$, suppose that $(x^n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}$, where $x^n = (x_i^n)_{i \in I}$ for each n. Then given any positive ϵ, there exists a positive integer N such that

$$\|x^m - x^n\| < \epsilon \text{ for all } m, n \ge N,$$

that is,

$$\sum_{i \in I} \|x_i^m - x_i^n\|^2 < \epsilon^2 \text{ for all } m, n \ge N. \tag{2.1}$$

Hence for each $i \in I$,

$$\|x_i^m - x_i^n\| < \epsilon \text{ for all } m, n \ge N$$

so that $(x_i^n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}_i$, having a limit $x_i \in \mathcal{H}_i$. For any finite subset $J \subset I$, it follows from equation (2.1) that

$$\sum_{j \in J} \left\|x_j^m - x_j^n\right\|^2 < \epsilon^2 \text{ for all } m, n \ge N,$$

and letting m tend to infinity,

$$\sum_{j\in J} \|x_j - x_j^n\|^2 < \epsilon^2 \text{ for all } n \geq N. \tag{2.2}$$

This holds for any finite subset J, so

$$\sum_{i\in I} \|x_i - x_i^n\|^2 < \epsilon^2 \text{ for all } n \geq N,$$

and so $(x_i - x_i^n)$ and x_i^n are in $\mathcal{H}$ for $n \geq N$. Then (x_i) is in $\mathcal{H}$ and by equation (2.2), (x^n) converges to (x_i) as n tends to infinity. We conclude that $\mathcal{H}$ is complete and therefore a Hilbert space. Just like in the finite case, we call $\mathcal{H}$ the *direct sum* of the collection $\{\mathcal{H}_i \mid i \in I\}$ of Hilbert spaces, denoted by $\bigoplus \mathcal{H}_i$.

Suppose now we have a (finite or infinite) collection of bounded operators $\{T_i \in \mathcal{B}(\mathcal{H}_i) \mid i \in I\}$ such that

$$\sup_{i\in I} \|T_i\| < \infty$$

(by convention this is true when I is a finite set). For $x = (x_i)$ in $\bigoplus \mathcal{H}_i$, define an element Tx in $\bigoplus \mathcal{H}_i$ by $Tx = (T_i x_i)$. Then $T : \mathcal{H} \to \mathcal{H} : x \mapsto Tx$ is a bounded linear operator, called the *direct sum* of the collection $\{T_i \in \mathcal{B}(\mathcal{H}_i) \mid i \in I\}$, denoted by $\bigoplus T_i$. For S_i, T_i in $\mathcal{B}(\mathcal{H}_i)$, α in $\mathbb{C}$, we have

$$\begin{aligned}
(\bigoplus T_i)^* &= \bigoplus T_i^*, \\
\bigoplus(\alpha S_i + T_i) &= \alpha \bigoplus S_i + \bigoplus T_i, \\
\bigoplus(S_i T_i) &= \bigoplus S_i \bigoplus T_i, \\
\|\bigoplus T_i\| &= \sup_{i\in I} \|T_i\|.
\end{aligned}$$

2.3 Topological Spaces

By a *topological space* we mean a set X endowed with a collection $\mathcal{T}$ (a *topology*) of subsets of X such that $\mathcal{T}$ is closed under finite intersections and arbitrary unions, and contains both X and the empty set. Elements of $\mathcal{T}$ are referred to as *open sets*. A *neighbourhood* of a point x is a subset of X which contains an open set which itself contains x. An *open cover* of X is a collection $\mathcal{C}$ of open sets whose union contains X, and a subset of $\mathcal{C}$ which still covers X is called a *subcover*.

A topological space X is said to be:

- *Hausdorff* if, for every distinct pair of points x, y of X, there is a neighbourhood of x which is disjoint from some neighbourhood of y. In this case, say these neighbourhoods *separate* x and y;
- *compact* if every open cover of X has a finite subcover;
- *locally compact* if every point of x has a compact neighbourhood.

Most of the topologies we encounter will be compact and Hausdorff, and locally compact and Hausdorff at the least.

A subset Y of X is said to be *dense* in X if every non-empty open set in X contains an element of A – equivalently, if X is the largest closed set containing Y. The topology *generated by a collection* $\mathcal{B}$ is the collection $\mathcal{T}$ containing $\mathcal{B}$ and all finite intersections and arbitrary unions over $\mathcal{B}$. It is the weakest topology on X containing $\mathcal{B}$, in the sense that any topology containing $\mathcal{B}$ contains all of $\mathcal{T}$.

A *topological vector space* is a vector space V over a field $\mathbb{K}$ (where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. together with a topology on V such that the vector space operations $V \times V \to V : (x, y) \mapsto x + y$ and $\mathbb{K} \times V \to V : (\lambda, x) \mapsto \lambda x$ are continuous (with respect to $|\cdot|$). Recall that a subset X_0 of a vector space V is said to be *convex* if any point $x \in X_0$ can be written as

$$x = \alpha x_1 + \beta x_2, \tag{2.3}$$

for $\alpha, \beta \geq$ such that $\alpha + \beta = 1$, and for $x_1, x_2 \in X_0$. An *extreme point* of X_0 is a point $x \in X_0$ for which an expression of the form of (2.3) only holds for $x_1 = x = x_2$. The convex set X_0 is then equal to the set of all finite linear combinations

$$\alpha_1 x_1 + \cdots + \alpha_n x_n$$

of its extreme points, where $\alpha_1, \ldots, \alpha_n$ are positive scalars summing to 1. For example, for a polygon embedded in $\mathbb{R}^2$, the vertices of a polygon are its extreme points, and every point within the polygon can be written as a linear combination of the vertices.

Lemma 1 ([3, 1.4.4])**.** *If X is a non-empty, compact, convex subset of a locally convex space V, and ρ is a continuous linear functional on V, then there is an extreme point x_0 of X such that, for every x in X,*

$$\operatorname{Re} \rho(x) \leq \operatorname{Re} \rho(x_0).$$

Definition ([10, 3.14])**.** Let V be a topological vector space over a field $\mathbb{K}$, with dual space V^*. Every x in V induces a linear functional f_x on V^* defined by $f_x(\rho) = \rho(x)$. The *weak* topology* on V^* is the topology generated by the sets

$$\left\{ f_x^{-1}(X) \mid x \in V, X \subseteq \mathbb{K} \text{ open} \right\}.$$

This is the weakest topology on V^* such that each functional f_x is continuous, and the collection $\{f_x \mid x \in V\}$ separates the points of V^*, so the weak* topology is Hausdorff.

Theorem (The Banach-Alaoglu theorem [10])**.** *If X is a neighbourhood of 0 in a topological vector space V, and*

$$K = \{\rho \in V^* \mid |\rho(x)| \leq 1 \textit{ for all } x \in X\},$$

then K is compact in the weak topology.*

Chapter 3

C^*-Algebras

We begin this chapter with some definitions and an important example, the algebra of continuous functions on a topological space, then give some results we will need later and finish by continuing our other fundamental example, the algebra of bounded operators on a Hilbert space.

3.1 Definition

Definition 1. A *Banach algebra* is a complex Banach space A, with norm $\|\cdot\|$, which forms an algebra, such that

$$\|ab\| \leq \|a\| \cdot \|b\| \text{ for all } a, b \in A.$$

A $*$-*algebra* is an algebra A with an *involution* map $a \mapsto a^*$ on A such that, for all $a, b \in A$ and for $\alpha \in \mathbb{C}$,

(i) $a^{**} = (a^*)^* = a$;

(ii) $(\alpha a + b)^* = \overline{\alpha} a^* + b^*$;

(iii) $(ab)^* = b^* a^*$.

The element a^* is referred to as the *adjoint* of a.

A C^*-*algebra* is a Banach algebra A with involution $a \mapsto a^*$ making it a $*$-algebra, with the condition that

$$\|a^* a\| = \|a\|^2 \text{ for all } a \in A.$$

This condition is known as the C^* *axiom*.

A subalgebra B of a C^*-algebra A is a C^*-*subalgebra* if it is closed under the adjoint and complete with respect to the norm; equivalently, if B is itself a C^*-algebra.

An element $a \in A$ of a C^*-algebra is called

- *normal* if $a^* a = a a^*$;
- *self-adjoint* if $a^* = a$;
- *unitary* if $a^* a = a a^* = \mathbb{1}$,

where $\mathbb{1}$ is the identity element (should it exist) of A, as defined in Section 3.2.

The set of self-adjoint elements A_{sa} forms a vector space over $\mathbb{R}$. Note that an element being self-adjoint, or unitary, implies that it is normal.

As a simple example, we can consider $\mathbb{C}$ as a C^*-algebra: we know from complex analysis that Cauchy sequences converge in $\mathbb{C}$ and that complex conjugation is an involutive operation. With this in mind, we can interpret self-adjoint elements of a C^*-algebra A as 'reals' – in fact, for a in A, let $h = \frac{1}{2}(a + a^*)$ and $k = \frac{i}{2}(a - a^*)$. Then h and k are self-adjoint and we can write $a = h + ik$ – call h and k the *real* and *imaginary* part of a, respectively.

3.2 Unitization

If a C^*-algebra A contains a multiplicative identity element $\mathbb{1}$, such that $a\mathbb{1} = a = \mathbb{1}a$ for all $a \in A$, call $\mathbb{1}$ the *unit* in A, and A is then a *unital* C^*-algebra.

Proposition ([8, I.1.3])**.** *Any non-unital C^*-algebra A can be isometrically embedded in a unital C^*-algebra $\tilde{A}$.*

Proof. Let $\tilde{A} = A \bigoplus \mathbb{C} = \{(a, \lambda) \mid a \in A, \lambda \in \mathbb{C}\}$ with pointwise addition, and for $a, b \in A$ and $\lambda, \mu \in \mathbb{C}$, define

$$\begin{aligned}(a, \lambda)(b, \mu) &= (ab + \lambda b + \mu a, \lambda\mu),\\ (a, \lambda)^* &= (a^*, \overline{\lambda}).\end{aligned}$$

Then $\tilde{A}$ is a $*$-algebra. Consider (a, λ) as an operator acting on A via $b \mapsto ab + \lambda b$ for b in A. It is easy to see that this operator is linear, and

$$\begin{aligned}\sup_{\|b\|=1} \|ab + \lambda b\| \leq \sup_{\|b\|=1} (\|ab\| + \lambda \|b\|)\\ = \sup_{\|b\|=1} \|ab\| + \lambda < \infty\end{aligned}$$

so that the operator (a, λ) is bounded. It follows that

$$\|(a, \lambda)\| = \sup_{\|b\|=1} \|ab + \lambda b\|$$

defines a norm $\|\cdot\|$ on $\tilde{A}$, and so $\tilde{A}$ is a Banach $*$-algebra, with unit $(0, 1)$. The embedding $a \mapsto (a, \lambda)$ is isometric because

$$\|a\| = \left\| a \cdot \frac{a}{\|a\|} \right\| \leq \|(a, 0)\| \leq \sup_{\|b\|=1} \|ab\| \leq \|a\|.$$

It remains to verify the C^* axiom:

$$\begin{aligned}\|(a, \lambda)\|^2 &= \sup_{\|b\|=1} \|ab + \lambda b\|^2\\ &= \sup_{\|b\|=1} \left\|b^*a^*ab + \lambda b^*a^*b + \overline{\lambda}b^*ab + |\lambda|^2 b^*b\right\|\\ &\leq \sup_{\|b\|=1} \left\|a^*ab + \lambda a^*b + \overline{\lambda}ab + |\lambda|^2 b\right\|\\ &= \left\|(a^*a + \lambda a^* + \overline{\lambda}a, |\lambda|^2)\right\|\\ &= \|(a, \lambda)^*(a, \lambda)\|\\ &\leq \|(a, \lambda)^*\| \, \|(a, \lambda)\|.\end{aligned}$$

By symmetry of $*$, $\|(a,\lambda)^*\| = \|(a,\lambda)\|$. Hence, the above inequality becomes equality and we have that

$$\|(a,\lambda)^*(a,\lambda)\| = \|(a,\lambda)\|^2,$$

and $\tilde{A}$ is a C^*-algebra. $\square$

In light of this result, we take all C^*-algebras from here to be unital unless specified otherwise. For the results we will consider, we can simply take the unital case. However, there are circumstances in advanced theory in which one needs to relax the unital condition, where properties do not pass from a C^*-algebra to its maximal ideals, so it is by no means a universal assumption that a C^*-algebra has a unit element.

3.3 Continuous Functions on a Compact Hausdorff Space

Given a compact Hausdorff space X, let $C(X)$ be the algebra of continuous functions $f : X \to \mathbb{C}$, with addition and multiplication defined pointwise. Define $\|\cdot\|$ on $C(X)$ by

$$\|f\| = \sup_{x\in X} |f(x)|,$$

and let $\overline{f}$ be the function $\overline{f}(x) = \overline{f(x)}$ for x in X. We will demonstrate that $C(X)$ is an Abelian C^*-algebra and preview some of the theory we will build up in later sections. In Section 4.1 we will show that $C(X)$ is essentially *the* Abelian C^*-algebra.

Claim. *The set $C(X)$, of continuous complex valued functions on a compact Hausdorff space X, is an Abelian C^*-algebra.*

Proof. We take $C(X)$ as an algebra with operations defined pointwise and norm defined as above. Multiplication in $C(X)$ is pointwise, so analogous to multiplication in $\mathbb{C}$, hence commutative. Showing that $C(X)$ is a Banach space with this norm is a simple exercise. We will demonstrate that $C(X)$ is a Banach algebra, that $f \mapsto \overline{f}$ is an involution, and that the norm satisfies the C^* axiom.

Given f and g in $C(X)$, we have $|f(x)g(x)| = |f(x)|\,|g(x)|$ for all x in X, and so

$$\begin{aligned}\|fg\| &= \sup_{x\in X} |(fg)(x)| \\ &= \sup_{x\in X} |f(x)g(x)| \\ &= \sup_{x\in X} |f(x)| \cdot |g(x)| \\ &\leq \sup_{x\in X} \|f\| \cdot |g(x)| \\ &\leq \|f\| \cdot \|g\|.\end{aligned}$$

Since $f \mapsto \overline{f}$ is simply pointwise complex conjugation, and since the algebraic operations are defined pointwise, it is clear that it is an involution. For the C^* axiom: given f in $C(X)$,

$$\begin{aligned}\left\|\overline{f}f\right\| &= \sup_{x\in X} |(\overline{f}f)(x)| \\ &= \sup_{x\in X} |\overline{f(x)}f(x)| \\ &= \sup_{x\in X} |f(x)|^2 \\ &= \|f\|^2.\end{aligned}$$

The constant function $\mathbb{1} : X \to \mathbb{C} : x \mapsto 1$ is the unit in $C(X)$, with $\|\mathbb{1}\| = 1$. $\square$

Suppose that f is a self-adjoint element of $C(X)$ – that is to say, $\overline{f} = f$. Then $\overline{f(x)} = f(x)$ for all x in X, so f is real-valued on X. The set of all self-adjoints in $C(X)$ is the algebra $C(X, \mathbb{R})$ of all real valued continuous functions on X. Call f *positive* if $f(x)$ is positive for all x in X. A linear functional ρ on $C(X)$ is *positive* if $\rho(f) \geq 0$ for all positive elements f, and a *state* on $C(X)$ if ρ is positive and $\rho(\mathbb{1}) = 1$. The *state space* $\mathscr{S}$, the set of all states on $C(X)$, is a convex set, and extreme points of $\mathscr{S}$ are called *pure states* on $C(X)$.

Given x in X, define ρ_x by $\rho_x : C(X) \to \mathbb{C} : f \mapsto f(x)$. If f is positive, then $\rho_x(f) = f(x) \geq 0$, and $\rho_x(\mathbb{1}) = \mathbb{1}(x) = 1$, so ρ_x is a state for all x. We can also see that these functionals are multiplicative, and it can be shown (see [3, Corollary 3.4.2]) that every non-zero multiplicative functional on $C(X)$ arises in this way, as 'evaluation' at some point of X, and we will show later that the non-zero multiplicative linear functionals are exactly the pure states on a C^*-algebra (Proposition 5), so these states ρ_x are in fact all of the pure states on $C(X)$.

The following theorem is a result which is crucial in the proof of the commutative Gelfand-Naimark theorem. We state it without proof here as the proof is long and involved, and doesn't contribute to understanding. The proof, along with a discussion of other consequences of the theorem, can be found in Section 3.4 of [3].

Theorem (Stone-Weierstrass theorem [3, 3.4.15])**.** *Let A be a norm-closed subalgebra of $C(X)$, containing the constant function $\mathbb{1}$ and the conjugate function $\overline{f}$ for each f in A. If for each pair of distinct points $p \neq p'$ in X, there is an f in A such that $f(p) \neq f(p')$, then $A = C(X)$.*

The term 'norm-closed' means that the set is closed in the norm topology, meaning it contains all its limit points. If the condition in the statement of the theorem holds, say that A *separates the points of* X.

We finish this section by remarking that if we take X to be a non-compact, *locally* compact Hausdorff space, then we are required to take $C(X)$ to be the set of continuous complex valued functions which *vanish at infinity* in some sense[1]. This restriction ensures that the supremum used to define the norm on $C(X)$ exists – however, we lose the constant function $\mathbb{1}$, so $C(X)$ is no longer unital. It turns out that the unitization of $C(X)$ corresponds to the continuous functions on the one-point compactification of X – see [9, II.1.2.2].

3.4 The Spectrum

We will just state the necessary definitions and results here, proofs can be found in the relevant sources.

Definition 2. Given an element a of a Banach algebra A with unit $\mathbb{1}$, define its spectrum $\sigma_A(a)$:

$$\sigma_A(a) = \{\lambda \in \mathbb{C} \mid a - \lambda\mathbb{1} \text{ is not invertible in } A\},$$

where a is *invertible in A* if there exists an element $b \in A$ such that $ab = \mathbb{1} = ba$. Say that $\lambda \in \sigma(a)$ is a *spectral value* of a in A. If the space we're taking the spectrum within is clear, then we write $\sigma(a)$ for $\sigma_A(a)$.

Theorem ([3, 3.2.3])**.** *If a is an element of a Banach algebra A then $\sigma(a)$ is a non-empty closed subset of the closed disk in $\mathbb{C}$ with center 0 and radius $\|a\|$.*

[1]We say $f : X \to \mathbb{C}$ *vanishes at infinity* if for all positive ϵ there exists a compact subset K of X such that $|f(x)| < \epsilon$ for all x outside of K.

The *spectral radius*, $r(a)$, of a is the radius of the smallest disk in $\mathbb{C}$ containing $\sigma(a)$; that is,

$$r(a) = \sup_{\lambda \in \sigma(a)} |\lambda|.$$

By the above theorem, $r(a) \leq \|a\|$.

Theorem (Spectral radius formula [3, 3.3.3]). *The spectral radius of an element a of a Banach algebra is given by the formula*

$$r(a) = \lim_{n \to \infty} \|a^n\|^{1/n}.$$

Lemma 2 ([3, 4.1.1(i)]). *If a is a normal element of a C^*-algebra, then $r(a) = \|a\|$.*

Since a^*a is normal for each a in a C^*-algebra A, $\|a\| = r(a^*a)^{1/2}$. Thus the norm on a C^*-algebra A is unique, in the sense that only one norm on A can satisfy the C^* axiom. [9, II.1.6.5]

Theorem (Continuous functional calculus, [3, 4.1.3]). *Given a self-adjoint element a, we denote by $C(\sigma(a))$ the C^*-algebra of continuous, complex valued functions on the spectrum of a. There is a unique continuous mapping $C(\sigma(a)) \to \mathbb{C} : f \mapsto f(a)$ such that:*

(i) *if f is a polynomial, $f(z) = \alpha_1 + \alpha_2 z + \cdots + \alpha_m z^m$, then*

$$f(a) = \alpha_1 \mathbb{1} + \alpha_2 a + \cdots + \alpha_m a^m;$$

and for all f, g in $C(X)$ and α in $\mathbb{C}$:

(ii) $\|f(a)\| = \|f\|$;

(iii) $(\alpha f + g)(a) = \alpha f(a) + g(a)$;

(iv) $fg(a) = f(a)g(a)$;

(v) $\overline{f}(a) = (f(a))^*$;

(vi) $f(a)$ *is normal*;

(vii) $f(a)b = bf(a)$ *for all b in A such that $ab = ba$.*

Moreover, each f in $C(\sigma(a))$ is the limit of a sequence of polynomials over $\mathbb{C}$.

Theorem (Spectral mapping theorem, [3, 4.1.6]). *If a is a self adjoint element of a C^*-algebra A, and $f \in C(\sigma(a))$, then*

$$\sigma(f(a)) = f(\sigma(a)) = \{f(t) \mid t \in \sigma(a)\}.$$

Proposition 1 ([3, 4.2.3(i)]). *Given f in $C(\sigma(a))$, $f(t) \geq 0 \ \forall\ t \in \sigma(a)$ if and only if $f(a)$ is self-adjoint and $\sigma(f(a)) \subseteq \mathbb{R}^+$.*

Theorem ([3, 4.1.5]). *If B is a C^*-subalgebra of a C^*-algebra A, and b is an element of B, then*

$$\sigma_B(b) = \sigma_A(b).$$

As an aside, this theorem allows us to make the convention that the spectrum of an element a in a non-unital C^*-algebra A is defined to be the spectrum of a in the unitization $\tilde{A}$ of A.

3.5 Further Definitions and Results

In this section we will set out some further definitions, some of which we have already seen in the context of $C(X)$, and prove some cool results.

Definition 3. An element a of a C^*-algebra A is *positive* if it is self-adjoint and $\sigma(a) \subseteq \mathbb{R}^+$. Denote the set of positive elements in A by A^+. Then A_{sa} is a partially ordered vector space with $\leq$ given by

$$a \leq b \iff b - a \text{ is positive}$$

for a, b in A_{sa}. The set of positive elements form a positive cone in A_{sa}, which means that

(i) $\alpha a + b \in A^+$ for all $a, b \in A^+$ and $\alpha \in \mathbb{R}^+$;

(ii) $a \in A^+$ and $-a \in A^+$ implies that $a = 0$.

The unit $\mathbb{1}$ is positive (with $\sigma(\mathbb{1}) = \{1\}$).

Lemma 3 ([3, 4.2.3(ii)])**.** *For a self-adjoint element a in a C^*-algebra A with unit $\mathbb{1}$,*

$$-\|a\| \mathbb{1} \leq a \leq \|a\| \mathbb{1}.$$

Proof. Let f in $C(\sigma(a))$ defined by $f(t) = \|a\| \pm t$, f takes non-negative values on $\sigma(a)$, so by Proposition 1, $f(a)$ is positive; that is, $\|a\| \mathbb{1} \pm a$ is positive.

□

Definition 4. Given Banach $*$-algebras A and B, a map $\varphi : A \to B$ is a *$*$-homomorphism* if it is an algebra homomorphism for which $\varphi(a^*) = \varphi(a)^*$ for all $a \in A$. If a $*$-homomorphism is bijective, call it a *$*$-isomorphism*.

Proposition 2 ([3, 4.1.8])**.** *Suppose A and B are C^*-algebras and $\varphi : A \to B$ is a $*$-homomorphism. Then $\|\varphi(a)\| \leq \|a\|$ for all $a \in A$. If φ is a $*$-isomorphism, then $\|\varphi(a)\| = \|a\|$ for all $a \in A$.*

Proof. With a in A, if α is not a spectral value for a, then $\alpha\mathbb{1}_A - a$ has an inverse s in A. Since $\varphi(\mathbb{1}_A) = \mathbb{1}_b$, the element $\alpha\mathbb{1}_A - \varphi(a)$ has inverse $\varphi(s)$ in B, so α is not a spectral value of $\varphi(a)$; hence $\sigma(\varphi(a)) \subseteq \sigma(a)$, and in particular, $r(\varphi(a)) \leq r(a)$. By Lemma 2, since a^*a is normal, we have

$$\begin{aligned} \|a\|^2 &= \|a^*a\| = r(a^*a); \text{ and} \\ \|\varphi(a)\|^2 &= \|\varphi(a)^*\varphi(a)\| = \|\varphi(a^*a)\| = r(\varphi(a^*a)), \end{aligned}$$

so $\|\varphi(a)\| \leq \|a\|$.

If a is a self-adjoint element of A and $f \in C(\sigma(a))$, then $\varphi(f(a)) = f(\varphi(a))$. This follows from the final assertion of the continuous functional calculus.

Now suppose φ is a $*$-isomorphism, and for a self-adjoint a in A, suppose $\sigma(\varphi(a)) \subsetneq \sigma(a)$. Then there is a non-zero f in $C(\sigma(a))$ whose restriction to $\sigma(\varphi(a))$ is 0. From the previous paragraph, however, $\varphi(f(a)) = f(\varphi(a)) = 0$, so that $f(a) \neq 0$ is in the kernel of φ; thus contradicting the assumption that φ is one-to-one, and we conclude that $\sigma(\varphi(a)) = \sigma(a)$. A similar argument to the one in the first half of this proof shows that $\|\varphi(a)\| = \|a\|$ and completes the proof.

□

The previous result shows that $*$-homomorphisms between C^*-algebras are automatically bounded (and hence continuous), and $*$-isomorphisms are isometries, demonstrating how the algebraic and topological structure of these spaces is tied together by the C^* axiom.

Definition 5. A linear functional ρ on a C^*-algebra A is *positive* if $\rho(a) \geq 0$ for all $a \in A^+$. A *state* on A is a positive linear functional ρ such that $\rho(\mathbb{1}) = 1$. Denote the set of all states on A by $\mathscr{S}(A)$. Given a linear functional ρ on A, the equation $\rho^*(a) = \overline{\rho(a^*)}$ defines a functional ρ^*, and call ρ *Hermitian* if $\rho = \rho^*$.

Hermitian functionals take real values on self-adjoint elements, and all positive linear functionals are Hermitian. We can extend the partial order notation to linear functionals. Given linear functionals ρ, τ on A, define $\leq$ by

$$\rho \leq \tau \quad \Longleftrightarrow \quad \tau - \rho \text{ is positive.}$$

When we take $\mathbb{C}$ as a C^*-algebra, the positive elements are exactly the positive reals, and the linear functionals are given by multiplication by a complex number. Positive linear functionals correspond again to the positive reals, and the only state on $\mathbb{C}$ is 'multiplication by 1'. Extending the analogy between $\mathbb{C}$ and a C^*-algebra A, we can even show that a positive element has a unique positive square root, and that a^*a is positive for all a in A. (See [3, 4.2.5 and 4.2.6]).

Proposition 3 ([3, 4.3.2])**.** *A linear functional ρ on a C^*-algebra A is positive if and only if ρ is bounded and $\|\rho\| = \rho(\mathbb{1})$ (where $\|\cdot\|$ is the operator norm on A^*).*

Proof. Suppose that ρ is positive. With a in A, let α be a scalar of modulus 1 such that $\alpha\rho(a) \geq 0$, and let h be the real part of a. Since h is self adjoint, (by 4.2.3(ii)) we have $h \leq \|h\| \, \mathbb{1} \leq \|a\| \, \mathbb{1}$. Thus, $\|a\| \, \mathbb{1} - h$ is positive and

$$\rho(\|a\| \, \mathbb{1} - h) = \|a\| \, \rho(\mathbb{1}) - \rho(h) \geq 0.$$

Therefore,

$$|\rho(a)| = \rho(\alpha a) = \overline{\rho(\alpha a)} = \rho(\overline{\alpha} a^*) = \rho(\tfrac{1}{2}(\alpha a + \overline{\alpha} a^*)) = \rho(h) \leq \rho(\mathbb{1}) \, \|a\| \, ,$$

so ρ is bounded and $\|\rho\| \leq \rho(\mathbb{1})$. We also have $\|\rho\| = \sup \{\rho(a) \mid \|a| = 1\} \geq \rho(\mathbb{1})$, so $\|\rho\| = \rho(\mathbb{1})$.

Conversely, suppose ρ is bounded and $\|\rho\| = \rho(\mathbb{1})$ – we can assume without loss that $\rho(\mathbb{1}) = 1$. With a a positive element of A, let $\rho(a) = \alpha + i\beta$ for real α, β. Then ρ is positive if and only if $\alpha \geq 0$ and $\beta = 0$. For small positive $s \in \mathbb{R}^+$,

$$\sigma(\mathbb{1} - sa) = \left\{1 - st \mid t \in \sigma(a) \subseteq \mathbb{R}^+\right\} \subseteq [0, 1],$$

so $\|\mathbb{1} - sa\| = r(\mathbb{1} - sa) \leq 1$. Hence

$$1 - s\alpha \leq |1 - s(\alpha + i\beta)| = |\rho(\mathbb{1} - sa)| \leq 1,$$

so $\alpha \geq 0$. With b_n in A defined by $b_n = a + (in\beta - \alpha)\mathbb{1}$ for each positive integer n,

$$\begin{aligned} \|b_n\|^2 = \|b_n^* b_n\| &= \left\|(a - \alpha\mathbb{1})^2 + n^2\beta^2\mathbb{1}\right\| \\ &\leq \|a - \alpha\mathbb{1}\|^2 + n^2\beta^2. \end{aligned}$$

Hence for all positive integers n, we have

$$\begin{aligned} (n^2 + 2n + 1)\beta &= |\rho(b_n)|^2 \\ &\leq \|a - \beta\mathbb{1}\|^2 + n^2\beta^2, \end{aligned}$$

so that $\beta = 0$. We conclude that ρ is positive. □

The following result follows from the Cauchy-Schwarz inequality for inner products.

Proposition (Cauchy-Schwarz inequality, [3, 4.3.1]). *If ρ is a positive linear functional on C^*-algebra A, then for all a and b in A,*

$$|\rho(b^*a)|^2 \leq \rho(a^*a)\rho(b^*b).$$

Lemma 4 ([3, 4.3.3]). *Let A be a C^*-algebra. For any a in A and $\alpha \in \sigma(a)$, there exists a state ρ on A such that $\rho(a) = \alpha$.*

Proof. For all complex numbers β and γ, $\alpha\beta+\gamma$ is a spectral value for the element $\beta a+\gamma\mathbb{1}$ of A, so

$$|\alpha\beta + \gamma| \leq r(\beta a + \gamma\mathbb{1}) = \|\beta a + \gamma\mathbb{1}\| .$$

Hence the equation $\rho_0(s) = \alpha\beta + \gamma$ defines a bounded linear functional ρ_0 on the linear subspace $B = \{\beta a + \gamma\mathbb{1} \mid \beta, \gamma \in \mathbb{C}\}$ of A, with $\rho_0(a) = \alpha$ and $\rho_0(\mathbb{1}) = 1 = \|\rho_0\|$. By the Hahn-Banach theorem, ρ_0 extends to a bounded linear functional ρ on A, with $\|\rho\| = 1$, such that $\rho = \rho_0$ on the subspace B. In particular, $\rho(\mathbb{1}) = 1 = \|\rho\|$ so ρ is positive by the previous result, and $\rho(a) = \alpha$. □

Lemma 5 ([3, 4.3.4,(i)]). *Let A be a C^*-algebra. If $\rho(a) = 0$ for all states ρ on A, then $a = 0$.*

Proof. Suppose first that a is self-adjoint, and $\rho(a) = 0$ for all states ρ. By the previous result, $\sigma(a) = \{0\}$, so by Lemma 2, $\|a\| = r(a) = 0$. Hence $a = 0$.

Now write $a = h + ik$, for h and k the real and imaginary part of a respectively. Then

$$0 = \rho(a) = \rho(h) + i\rho(k),$$

and as h, k are self-adjoint, $\rho(h)$ and $\rho(k)$ are real and we must have $\rho(h) = 0 = \rho(k)$. By previous statement, $h = 0 = k$, and we conclude that $a = 0$. □

Lemma 6 ([3, 4.3.4,(iv)]). *If a is a normal element of a C^*-algebra A, there is a state ρ on A such that $|\rho(a)| = \|a\|$.*

Proof. By Lemma 2, $r(a) = \|a\|$, so $\sigma(a)$ contains a complex number α such that $|\alpha| = \|a\|$. By Lemma 4, $\alpha = \rho(a)$ for some state ρ, so

$$|\rho(a)| = |\alpha| = \|a\| .$$

□

Definition 6. An extreme point of $\mathscr{S}(A)$ is called a *pure* state on A, and the set of pure states on A is denoted by $\mathscr{P}(A)$.

The state space $\mathscr{S}(A)$ is then the closure, relative to the weak* topology, of the set of convex linear combinations of pure states. The set $\mathscr{P}(A)$ is not necessarily closed, and so we often consider its closure, which is called the *pure state space*, in the dual space A^*. It is a simple exercise, using the fact that $A^+ \subseteq A_{sa}$, to verify that a linear functional ρ is a pure state on A if and only if the restriction $\rho|_{A_{sa}}$ is a pure state on A_{sa}. Conversely, every pure state on A_{sa} extends to a pure state on A with the same norm, by the Hahn-Banach theorem (given in Section 2.1).

Lemma 7 ([3, 4.3.8,(i)]). *Let a be an element of a C^*-algebra A. If $\rho(a) = 0$ for all* pure *states ρ on A, then $a = 0$.*

Proof. Since every state is the limit of a sequence of linear combinations of pure states, if $\rho(a) = 0$ for all pure states ρ on A, then $\rho(a) = 0$ for *all* states ρ on A. The result then follows immediately from Lemma 5. $\square$

Lemma 8 ([3, 4.3.8,(iv)]). *If a is a normal element of a C^*-algebra A, there is a pure state ρ_0 on A such that $|\rho_0(a)| = \|a\|$.*

Proof. By Lemma 6, there is a scalar γ and a state τ on A such that $\tau(a) = \gamma$ and $|\gamma| = \|a\|$. Define $\hat{t} : A^* \to \mathbb{C} : \rho \mapsto \rho(t)$, and let α be a complex number such that $|\alpha| = 1$ and $\tau(\alpha a) = |c| = \|a\|$. From Lemma 1, there is a ρ_0 in $\mathscr{P}(A)$ such that

$$\begin{aligned}\|a\| &\geq |\rho_0(a)| \geq \operatorname{Re}\widehat{\alpha a}(\rho_0)\\ &\geq \sup_{\rho\in\mathscr{S}(A)} \operatorname{Re}\widehat{\alpha a}(\rho)\\ &\geq \operatorname{Re}\widehat{\alpha a}(\tau) = \operatorname{Re}\tau(\alpha a) = \|a\|\,,\end{aligned}$$

so that $|\rho_0(a)| = \|a\|$. $\square$

The following result gives a useful characterisation of pure states:

Proposition 4 ([3, 3.4.6]). *A state ρ on A_{sa} is pure if and only if, for all positive linear functionals τ on A_{sa} such that $0 \leq \tau \leq \rho$, we have $\tau = \lambda\rho$ for some $\lambda \in \mathbb{R}$.*

Proof. Suppose that $\tau = \lambda\rho$ for all $0 \leq \tau \leq \rho$, and suppose we can write $\rho = \alpha\rho_1 + (1-\alpha)\rho_2$ for some $0 \leq \alpha \leq 1$ and some $\rho_1, \rho_2 \in \mathscr{S}(A_{sa})$. Then $0 \leq \alpha\rho_1 \leq \rho$, so $\alpha\rho_0 = \lambda\rho$. Then $\rho_1(\mathbb{1}) = 1 = \rho(\mathbb{1})$, so $\alpha = \lambda$ and $\rho_0 = \rho$. Similarly, we can show that $\rho_2 = \rho$, and so we conclude that ρ is a pure state.

Conversely, suppose that ρ is a pure state and $0 \leq \tau \leq \rho$. Applying this to $\mathbb{1}$,we get $0 \leq \tau(\mathbb{1}) \leq \rho(\mathbb{1}) = 1$. Let $\lambda = \tau(\mathbb{1})$. We work case-by-case: If $\lambda = 0$, then for any $a \in A_{sa}$, applying τ to $-\|a\|\,\mathbb{1} \leq a \leq \|a\|\,\mathbb{1}$ (from Lemma 3) gives

$$\begin{aligned}0 &= -\|a\|\,\lambda\\ &= \tau(-\|a\|\,\mathbb{1})\\ &\leq \tau(a)\\ &\leq \tau(\|a\|\,\mathbb{1})\\ &= \|a\|\,\lambda = 0,\end{aligned}$$

so $\tau = 0 = \lambda\rho$.

A similar argument shows that $\lambda = 1$ implies $\tau - \rho = 0$, so that $\tau = \rho = \lambda\rho$. If $0 \leq \lambda \leq 1$, we can write $\rho = \lambda\rho_1 + (1-\lambda)\rho_2$ for $\rho_1 = \lambda^{-1}\tau$ and $\rho_2 = (1-\lambda)^{-1}(\rho - \tau)$. ρ is pure so $\tau = \lambda\rho_1 = \lambda\rho$.

$\square$

Recall that a linear functional ρ on a C^*-algebra A is *multiplicative* if $\rho(ab) = \rho(a)\rho(b)$ for all a, b in A. The set of all non-zero multiplicative linear functionals on A is known as the *maximal ideal space* of A, denoted by $\mathcal{M}_A$. This name hints at the fact that the kernel of each of these functionals is a maximal ideal of A, and all maximal ideals of A arise in this way. [8, Theorem I.2.5]

Proposition 5 ([3, 4.4.1]). *The set of pure states on an Abelian C^*-algebra A is precisely the maximal ideal space of A.*

Proof. Suppose ρ is a pure state on A. To show that $\rho(ab) = \rho(a)\rho(b)$ for $a, b, \in A$, we restrict attention to the case where $0 \leq b \leq \mathbb{1}$. Linearity gives us the general case. In this case, for $h \in A^+$ we have that $0 \leq hb \leq h$, so $0 \leq \rho(hb) \leq \rho(h)$. Hence $\rho_b(a) = \rho(ab)$ for $a \in A$ defines a positive linear functional on A with $\rho_b \leq \rho$. The restriction $\rho|_{A_{sa}}$ is a pure state on A_{sa} and $\rho_b|_{A_{sa}} \leq \rho|_{A_{sa}}$, and it follows from Proposition 4 that $\rho_b|_{A_{sa}} = \alpha\rho|_{A_{sa}}$ for some $\alpha \in \mathbb{R}^+$. Hence $\rho_b = \alpha\rho$ and so for $a \in A$:

$$\rho(ab) = \rho_b(a) = \alpha\rho(a) = \alpha\rho(\mathbb{1})\rho(a) = \rho_b(\mathbb{1})\rho(a) = \rho(b)\rho(a)$$

Conversely, suppose ρ is a multiplicative linear functional. By Lemma 9, ρ is bounded and $\|\rho\| = \rho(\mathbb{1}) = 1$, so by Proposition 3, ρ is a state. Suppose we can write $\rho = \alpha\rho_1 + \beta\rho_2$ for states ρ_1, ρ_2 on A and $\alpha, \beta > 0$ such that $\alpha + \beta = 1$. For $c \in A_{sa}$, by the Cauchy-Schwarz inequality we have for $j = 1, 2$:

$$\big(\rho_j(c)\big)^2 = \big(\rho_j(\mathbb{1}c)\big)^2 \leq \rho_j(\mathbb{1})\rho(c^2) = \rho(c^2).$$

Then:

$$\begin{aligned}
0 &= \rho(c^2) - \rho(c)^2 \\
&= \alpha\rho_1(c^2) + \beta\rho_2(c^2) - \big(\alpha\rho_1(c) + \beta\rho_2(c)\big)^2 \\
&\geq \alpha(\alpha+\beta)\rho_1(c)^2 + \beta(\alpha+\beta)\rho_2(c)^2 - \big(\alpha\rho_1(c) + \beta\rho_2(c)\big)^2 \\
&= \alpha\beta\big(\rho_1(c) - \rho_2(c)\big)^2.
\end{aligned}$$

Hence $\rho_1(c) = \rho_2(c)$, for all $c \in A_{sa}$, so $\rho_1 = \rho_2 = \rho$ and we conclude that ρ is a pure state. □

The set $\mathscr{P}(A)$ of pure states on A is a subset of the dual space A^*, and when we take A^* with the weak* topology, we get a compact Hausdorff space, which is the object of study in the Gelfand-Naimark theorem for commutative C^*-algebras.

Lemma 9 ([3, 3.2.20])**.** *The maximal ideal space of an Abelian Banach algebra A forms a compact subset of the unit ball of the dual space A^*, relative to the weak* topology.*

Proof. Follows from the Banach-Alaoglu theorem (given in Section 2.3). □

3.6 Bounded Operators on a Hilbert Space

This section concerns itself with the fundamental example of a C^*-algebra: the set $\mathcal{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$, otherwise known as an *operator algebra*. Here we will demonstrate that $\mathcal{B}(\mathcal{H})$ is a C^*-algebra and discuss how the theory of C^*-algebras relates to quantum mechanics, the very subject which started the study of these objects, which explains a good deal of the terminology used here.

Claim. *$\mathcal{B}(\mathcal{H})$ is a C^*-algebra with the operator norm*

$$\|T\| = \sup_{\|x\|=1} \|Tx\|,$$

and involution taking T to its adjoint map T^*. The identity map $I : x \mapsto x$ is a unit for $\mathcal{B}(\mathcal{H})$.

Proof. It is an easy exercise to show that $\|\cdot\|$ is a norm on $\mathcal{B}(\mathcal{H})$. Let $\{T_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{B}(\mathcal{H})$. Then for any positive ϵ, there is a positive integer N such that

$$\|T_m - T_n\| < \epsilon \text{ for all } m, n \geq N.$$

Applying $T_m - T_n$ to $x \in \mathcal{H}$, we have

$$\|T_m x - T_n x\| \leq \|T_m - T_n\| \, \|x\| < \epsilon \, \|x\| \, , \tag{3.1}$$

so $\{T_n x\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}$, converging to an element in $\mathcal{H}$. Define a linear operator $T : \mathcal{H} \to \mathcal{H}$ by

$$Tx = \lim_{n \to \infty} T_n x \text{ for } x \in \mathcal{H}.$$

Taking limits as m tends to infinity in equation (3.1), we obtain

$$\|Tx - T_n x\| < \epsilon \, \|x\| \text{ for all } n \geq N,$$

and so we have that $T - T_n$ (and hence $T = (T - T_n) + T_n$) is a bounded operator and

$$\|T - T_n\| < \epsilon \text{ for all } n \geq N.$$

We conclude that $T_n \to T$, and so $\mathcal{B}(\mathcal{H})$ is complete.

Since boundedness is equivalent to continuity on $\mathcal{H}$, given $S, T \in \mathcal{B}(\mathcal{H})$, the operator $ST : \mathcal{H} \to \mathcal{H} : x \mapsto (S \circ T)(x)$ is bounded on $\mathcal{H}$. Given $x \in \mathcal{H}$ and $\lambda \in \mathbb{C}$,

$$\begin{aligned} ((\lambda S)T)(x) &= ((\lambda S) \circ T)(x) \\ &= \lambda S(Tx) \\ &= \lambda (S \circ T)(x) \\ &= \lambda ST(x), \end{aligned}$$

so that $(\lambda S)T = \lambda ST$ in $\mathcal{B}(\mathcal{H})$. We have

$$\begin{aligned} \|ST\| &= \sup_{\|x\|=1} \|STx\| \\ &= \sup_{\|x\|=1} \|S(Tx)\| \\ &\leq \|S\| \sup_{\|x\|=1} \|Tx\| \\ &= \|S\| \, \|T\| \, . \end{aligned}$$

We conclude that $\mathcal{B}(\mathcal{H})$ is a Banach algebra.

To see that * is an involution, use the fact that the adjoint operator is unique for each operator and the following equalities.

(i)
$$\begin{aligned} \langle (\alpha T + S)^* x, y \rangle &= \langle x, \alpha T + Sy \rangle \\ &= \overline{\alpha} \langle x, Ty \rangle + \langle x, Sy \rangle \\ &= \overline{\alpha} \langle T^* x, y \rangle + \langle S^* x, y \rangle \\ &= \langle (\overline{\alpha} T^* + S^*) x, y \rangle. \end{aligned}$$

(ii)
$$\begin{aligned} \langle (T^*)^* x, y \rangle &= \langle x, T^* y \rangle \\ &= \overline{\langle T^* y, x \rangle} \\ &= \overline{\langle y, Tx \rangle} \\ &= \langle Tx, y \rangle. \end{aligned}$$

(iii)
$$\begin{aligned} \langle (ST)^* x, y \rangle &= \langle x, STy \rangle \\ &= \langle S^* x, Ty \rangle \\ &= \langle T^* S^* x, y \rangle. \end{aligned}$$

It remains to demonstrate the C^*-axiom on $\mathcal{B}(\mathcal{H})$. For all $x \in \mathcal{H}$, we have

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle \leq \|T^*T\| \, \|x\|^2 \,,$$

so that

$$\|T\|^2 \leq \|T^*T\| \leq \|T^*\| \, \|T\| = \|T\|^2 \,.$$

It is clear that I is a unit, and so $\mathcal{B}(\mathcal{H})$ is a unital C^*-algebra. □

A basic finite dimensional example of a Hilbert space is $\mathbb{C}^n$, and a linear map $\mathbb{C}^n \to \mathbb{C}^n$ can be represented, given a choice of basis of $\mathbb{C}^n$, as an element of the algebra $\mathrm{Mat}_n(\mathbb{C})$ of $n \times n$ complex matrices. Since $\mathbb{C}^n$ is finite dimensional, all linear maps on it are bounded so we can identify, after choosing a basis, $\mathcal{B}(\mathbb{C}^n)$ with $\mathrm{Mat}_n(\mathbb{C})$ – in this way, we see that $\mathrm{Mat}_n(\mathbb{C})$ is a C^*-algebra. The spectrum of a map in $\mathcal{B}(\mathbb{C}^n)$ corresponds to the set of eigenvalues of its matrix representation (recall from linear algebra that eigenvalues are basis independent), so a self-adjoint operator is positive if it has non-negative eigenvalues – that is, it has a positive-semidefinite Hermitian matrix representation.

We can consider $\mathrm{Mat}_n(\mathbb{C})$ as an n^2-dimensional complex vector space, with a basis $\{e_{ij} \mid i, j = 1, \ldots, n\}$. This there is a basis $\{f_{pq} \mid p, q = 1, \ldots, n\}$ of the dual of $\mathrm{Mat}_n(\mathbb{C})$, where f_{pq} is given by

$$f_{pq}(e_{ij}) = \delta_{ip}\delta_{jq}$$

extended linearly, where δ_{ij} is the Kronecker delta function. Hence any linear functional ρ can be written in the form

$$\rho = \sum_{i,j=1}^{n} \alpha_{ij} f_{ij}$$

where $\alpha_{ij} = \rho(e_{ij})$.

Writing the identity matrix I_n in $\mathrm{Mat}_n(\mathbb{C})$ as $I_n = \sum_{i,j=1}^{n} \delta_{ij}$, if a positive functional ρ is a state then it satisfies

$$1 = \rho(I_n) = \sum_{i,j=1}^{n} \delta_{ij}\rho(e_{ij}) = \sum_{i=1}^{n} \rho(e_{ii}).$$

Pure states on $\mathrm{Mat}_n(\mathbb{C})$ are those ρ for which $\rho(e_{ii}) = 1$ for exactly one $i \in \{1, \ldots, n\}$ and 0 for the rest.

The terminology for normal, positive and unitary elements of a C^*-algebra comes from the corresponding concepts when studying operator algebras. Paul Dirac's *'Principles of Quantum Mechanics'* [12] and von Neumann's *'Mathematical Foundations of Quantum Mechanics'* [1] jointly established a formulation of Quantum Mechanics in terms of Hilbert spaces, and operators thereon. To summarise briefly, a physical quantity in a quantum mechanical system has associated with it a self-adjoint operator T on a Hilbert space $\mathcal{H}$, while the state of the system corresponds to a unit vector φ in $\mathcal{H}$. Then the expected outcome of measuring the quantity corresponding to T is given by evaluating $\langle \varphi, T\varphi\rangle$. In the context of C^*-algebras, these translate, in turn, to a self adjoint element a of a C^*-algebra A, a normalised positive linear functional ρ (that is, a state) on A, and evaluation of $\rho(a)$.

After stating his postulate, which he refers to as '**P.**', on the expectation value of a system of observables $\mathscr{R}, \ldots, \mathscr{R}_\ell$ with corresponding operators $R_1, \ldots, R_\ell$, von Neumann makes a comment on a consequence in the model of the non-commutativity of operators on a Hilbert space: "In the case of noncommuting $R_1, \ldots, R_\ell$, [...] **P.** gave no information regarding the probability relations of $\mathscr{R}, \ldots, \mathscr{R}_\ell$. In this case, **P.** could be used only determine the probability distribution of each of these quantities by itself, without consideration of the others." ([1, Section III.3])

Chapter 4

Representations of C^*-Algebras

4.1 Abelian C^*-Algebras

Let A be an Abelian C^*-algebra. For a in A, define a complex-valued function $\hat{a}$ on $\mathscr{P}(A)$ by $\hat{a}(\rho) = \rho(a)$ – that is, *evaluation* at a. The weak*-topology is the weakest topology on $\mathscr{P}(A)$ for which all of the maps $\hat{a}$ are continuous, so that $\hat{a} \in C(\mathscr{P}(A))$ for all $a \in A$. The map

$$\Gamma : A \to C(\mathscr{P}(A)) : a \mapsto \hat{a}$$

is called the *Gelfand transform* on A [8]. For $a, b \in A$, $\alpha \in \mathbb{C}$ and $\rho \in \mathscr{P}(A)$:

$$\begin{aligned} (\widehat{\alpha a + b})(\rho) &= \rho(\alpha a + b) = \alpha\rho(a) + \rho(b) = \alpha\hat{a}(\rho) + \hat{b}(\rho), \\ \widehat{a^*}(\rho) &= \rho(a^*) = \overline{\rho(a)} = \overline{\hat{a}(\rho)}. \end{aligned}$$

Since pure states are multiplicative by Proposition 5, we have that

$$(\widehat{ab})(\rho) = \rho(ab) = \rho(a)\rho(b) = \hat{a}(\rho)\hat{b}(\rho).$$

Hence the Gelfand transform is a $*$-homomorphism, and the image $\Gamma(A)$ is a C^*-subalgebra of $C(\mathscr{P}(A))$. The following theorem gives us that Γ is in fact a $*$-isomorphism.

Theorem 1 (Gelfand-Naimark, commutative [3, 4.4.3])**.** *Every Abelian C^*-algebra A is $*$-isomorphic to $C(\mathscr{P}(A))$, the algebra of continuous functions on the compact Hausdorff space $\mathscr{P}(A)$ of pure states on A.*

Proof. Since A is Abelian, every a in A is normal, so by Lemma 8 there is a pure state ρ_0 on A such that $|\rho_0(a)| = \|a\|$. From this,

$$\|a\| = |\rho_0(a)| = |\hat{a}(\rho_0)| \le \sup_{\rho \in \mathscr{P}(A)} |\hat{a}(\rho)| \le \|a\|,$$

so

$$\|a\| = \sup_{\rho \in \mathscr{P}(A)} |\hat{a}(\rho)| = \|\hat{a}\|.$$

Hence Γ is isometric. Given any pure state ρ, $\hat{\mathbb{1}}(\rho) = \rho(\mathbb{1}) = 1$ (where $\mathbb{1}$ is the identity of A), so $\hat{\mathbb{1}}$ is the constant function equal to 1 on $\mathscr{P}(A)$ and thus the unit in $C(\mathscr{P}(A))$. Given distinct pure states ρ_1 and ρ_2 on A, we can choose a in A such that

$$\hat{a}(\rho_1) = \rho_1(a) \ne \rho_2(a) = \hat{a}(\rho_2),$$

so the image $\Gamma(A)$ separates points of $\mathscr{P}(A)$, and so $\Gamma(A) = C(\mathscr{P}(A))$, by the Stone-Weierstrass theorem as stated in Section 3.3.

□

This theorem allows us to think of the study of C^*-algebras in general as a sort of non-commutative topology; the 'topological space' corresponds to the pure state space on a C^*-algebra A, and the 'topology' corresponds to the preimages of the 'evaluation at a' maps on states in the pure state space.

4.2 The Gelfand-Naimark-Segal Construction

In this section, we show that corresponding to every state of a C^*-algebra, there is a $*$-homomorphism into the operator algebra $\mathcal{B}(\mathcal{H})$ on some Hilbert space $\mathcal{H}$.

Definition 7. Given a C^*-algebra A, a *representation of A on a Hilbert space $\mathcal{H}$* is a $*$-homomorphism $\varphi : A \to \mathcal{B}(\mathcal{H})$. A $*$-isomorphic representation is called *faithful.* If there exists an element $x \in \mathcal{H}$ such that the set $\{\varphi(a)x \mid a \in A\}$ is dense in $\mathcal{H}$, say that φ is a *cyclic* representation, with *cyclic vector x.*

The construction used in the proof of this theorem is known as the Gelfand-Naimark-Segal (GNS) construction.

Theorem 2 ([3, 4.5.2])**.** *If ρ is a state on a C^*-algebra A, then there exists a cyclic representation π_ρ of A on a Hilbert space H_ρ, with unit cyclic vector x_ρ, such that*

$$\rho(a) = \langle \pi_\rho(a)x_\rho, x_\rho\rangle, \text{ for all } a \in A.$$

Proof. We will construct from ρ the space $\mathcal{H}_\rho$, representation π_ρ, and vector x_ρ, and demonstrate the required properties.

Consider the *left kernel* of ρ:

$$L_\rho = \{a \in A \mid \rho(a^*a) = 0\}.$$

For $a, b \in A$, define $\langle a, b\rangle_0 = \rho(b^*a)$, then $\langle \cdot, \cdot\rangle_0$ satisfies the following properties:

(i) Linearity in 1st argument: for $a, b \in A$, $\alpha \in \mathbb{C}$:

$$\begin{aligned} \langle \alpha a + b, c\rangle_0 &= \rho(c^*(\alpha a + b)) \\ &= \rho(\alpha c^* a + c^* b) \\ &= \alpha\rho(c^*a) + \rho(c^*b) \\ &= \alpha\langle a, c\rangle_0 + \langle b, c\rangle_0. \end{aligned}$$

(ii) Conjugate symmetric: for $a, b \in A$:

$$\begin{aligned} \langle b, a\rangle_0 &= \rho(a^*b) \\ &= \rho((b^*a)^*) \\ &= \overline{\rho(b^*a)} \\ &= \overline{\langle a, b\rangle_0}. \end{aligned}$$

(iii) Positive semi-definite: for $a \in A$:

$$\langle a, a\rangle_0 = \rho(a^*a) \geq 0,$$

since a^*a is positive for all a.

Note that $\langle \cdot, \cdot \rangle$ is not necessarily positive definite on A – the left kernel is exactly where this fails.

To see that L_ρ is a linear subspace of A, consider the set

$$L = \{t \in A \mid \langle t, a \rangle_0 = 0, \ \forall a \in A\} \subseteq L_\rho.$$

For $t \in L_\rho$, by the Cauchy-Schwarz inequality we have

$$|\langle t, a \rangle_0|^2 \leq \langle t, t \rangle_0 \langle a, a \rangle_0, \ \forall a \in A;$$

that is,

$$\langle t, a \rangle_0 = 0, \ \forall a \in A,$$

so $t \in L$ and $L_\rho = L$. Now, for $a, b \in L$, $\alpha \in \mathbb{C}$ and $c \in A$:

$$\langle \alpha a + b, c \rangle_0 = \alpha \langle a, c \rangle_0 + \langle b, c \rangle_0 = 0,$$

so $\alpha a + b \in L$; also, $\langle 0, c \rangle_0 = 0$ so $0 \in L$. Hence, L_ρ is a linear subspace of A.

For $s \in A$, $t \in L_\rho$, by the Cauchy Schwarz inequality we have

$$\begin{aligned} |\rho(s^* t)|^2 &= |\langle t, s \rangle_0|^2 \\ &\leq \langle t, t \rangle_0 \cdot \langle s, s \rangle_0 \\ &= \rho(t^* t) \cdot \rho(s^* s) \\ &= 0, \end{aligned}$$

so $\rho(s^* t) = 0$. Letting $s = a^* a t$ for $a \in A$, then

$$\begin{aligned} \rho((at)^* at) &= \rho(a t^* a^* a t) \\ &= \rho((a^* a t)^* t) \\ &= \rho(s^* t) \\ &= 0, \end{aligned}$$

so that $at \in L_\rho$, for all $a \in A$ and $t \in L_\rho$; we conclude that L_ρ is a left ideal in A. L_ρ is closed as the preimage in A of $\{0\}$ under the continuous map $a \mapsto \rho(a^* a)$.

Consider now the quotient space $V_\rho = A/L_\rho$, with $\langle \cdot, \cdot \rangle$ defined by

$$\langle a + L_\rho, b + L_\rho \rangle = \langle a, b \rangle_0, \ \text{for } a + L_\rho, b + L_\rho \in V_\rho.$$

It follows from properties (i), (ii) and (iii) of $\langle \cdot, \cdot \rangle_0$ that $\langle \cdot, \cdot \rangle$ is an inner product on V_ρ, with

$$\begin{aligned} \langle a + L^\rho, a + L^\rho \rangle = 0 &\iff \langle a, a \rangle_0 = 0 \\ &\iff a \in L_\rho \\ &\iff a + L_\rho = 0 + L_\rho \end{aligned}$$

giving positive definiteness. The completion of V_ρ, with respect to the norm $\|\cdot\|$ induced by $\langle \cdot, \cdot \rangle$, is the Hilbert space $\mathcal{H}_\rho$ we're looking for.

Now we fix $a \in A$, and consider the map

$$\pi_a : V_\rho \to V_\rho; b + L_\rho \mapsto ab + L_\rho.$$

Let $b_1, b_2 \in A$ be such that $b_1 + L_\rho = b_2 + L_\rho$. Then:

$$\begin{aligned}
&b_1 - b_2 \in L_\rho \\
\implies\ &a(b_1 - b_2) \in L_\rho \\
\implies\ &ab_1 - ab_2 \in L_\rho \\
\implies\ &ab_1 + L_\rho = ab_2 + L_\rho \\
\implies\ &\pi_a(b_1 + L_\rho) = \pi_a(b_2 + L_\rho).
\end{aligned}$$

Hence π_a defines a linear operator on V_ρ. For $b + L_\rho \in V_\rho$:

$$\begin{aligned}
\|a\|^2 \cdot \|b + L_\rho\| - \|\pi_a(b + L_\rho)\| &= \|a\|^2 \cdot \|b + L_\rho\| - \|ab + L_\rho\| \\
&= \|a\|^2 \cdot \langle b + L_\rho, b + L_\rho\rangle - \langle ab + L_\rho, ab + L_\rho\rangle \\
&= \|a\|^2 \cdot \rho(b^*b) - \rho((ab)^*ab) \\
&= \rho(\|a\|^2 b^*b - b^*a^*ab) \\
&= \rho(b^*(\|a\|^2 \mathbb{1} - a^*a)b) \\
&\geq 0.
\end{aligned}$$

Thus π_a is a bounded operator on V_ρ, with $\|\pi_a\| \leq \|a\|$. By continuity, π_a extends to a bounded operator $\pi_\rho(a) : \mathcal{H}_\rho \to \mathcal{H}_\rho$, such that

$$\pi_\rho(a)(v) = \pi_a(v)$$

for $v \in V_\rho$. Then $\pi_\rho(a) \in \mathcal{B}(\mathcal{H}_\rho)$ for each $a \in A$, so π_ρ defines a map $A \to \mathcal{B}(\mathcal{H}_\rho)$ such that $a \mapsto \pi_\rho(a)$. This will be our representation.

Now, for $a, b \in A$, $c + L_\rho \in V_\rho$ and $\alpha \in \mathbb{C}$:

$$\begin{aligned}
\pi_{\alpha a+b}(c + L_\rho) &= (\alpha a + b)(c + L_\rho) \\
&= (\alpha ac + L_\rho) + (bc + L_\rho) \\
&= \alpha\pi_a(c + L_\rho) + \pi_b(c + L_\rho),
\end{aligned}$$

so that $\pi_{\alpha a+b} = \alpha\pi_a + \pi_b$ on V_ρ.

For $a, b \in A$ and $c + L_\rho \in V_\rho$:

$$\begin{aligned}
\pi_{ab}(c + L_\rho) &= abc + L_\rho \\
&= \pi_a(bc + L_\rho) \\
&= \pi_a(\pi_b(c + L_\rho)) \\
&= (\pi_a \cdot \pi_b)(c + L_\rho),
\end{aligned}$$

so that $\pi_{ab} = \pi_a \cdot \pi_b$ on V_ρ.

For $a \in A$ and $b + L_\rho,\ c + L_\rho \in V_\rho$:

$$\begin{aligned}
\langle b + L_\rho, \pi_a^*(c + L_\rho)\rangle &= \langle \pi_a(b + L_\rho), c + L_\rho\rangle \\
&= \langle ab + Lr, c + L_\rho\rangle \\
&= \rho(c^*ab) \\
&= \rho((a^*c)^*b) \\
&= \langle b + L_\rho, a^*c + L_\rho\rangle \\
&= \langle b + L_\rho, \pi_{a^*}(c + L_\rho)\rangle,
\end{aligned}$$

so that $\pi_a^* = \pi_{a^*}$ on V_ρ.

The subset V_ρ is dense in $\mathcal{H}_\rho$, so the three properties above hold on $\mathcal{H}_\rho$ by continuity of π_ρ. Hence, $\pi_\rho : A \to \mathcal{B}(\mathcal{H}_\rho)$ is a representation of A. As to the unit vector, consider $x_\rho = \mathbb{1} + L_\rho \in V_\rho$. Then for $a \in A$,

$$\begin{aligned}\langle \pi_\rho(a)x_\rho, x_\rho\rangle &= \langle \pi_a(\mathbb{1} + L_\rho), \mathbb{1} + L_\rho\rangle \\ &= \langle a + L_\rho, \mathbb{1} + L_\rho\rangle \\ &= \rho(a);\end{aligned}$$

and in particular, $\langle x_\rho, x_\rho\rangle = \rho(\mathbb{1}) = 1$, so x_ρ is a unit vector in $\mathcal{H}_\rho$. The set $\{\pi_\rho(a)x_\rho \mid a \in A\}$ is the dense subset V_ρ of $\mathcal{H}_\rho$, so x_ρ is a cyclic vector for ρ.

□

We know from Section 3.3 that all functionals of the form $\rho_x(f) = f(x)$, for x in X, are pure states on the C^*-algebra $C(X)$. So, what happens if we apply the GNS construction to one of these ρ_x?

Fix x in X. If $\rho_x(\overline{f}f) = 0$, then $(\overline{f}f)(x) = 0$ and $f(x) = 0$. Then the left kernel L_{ρ_x} reduces to the kernel (in the usual linear sense) of ρ_x:

$$L_{\rho_x} = \{f \in C(X) \mid f(x) = 0\} = \ker(\rho_x).$$

The kernel of a non-zero linear functional is a proper ideal, with codimension 1, so when we take the quotient we find that

$$V_{\rho_x} = C(X)/L_{\rho_x} \cong \mathbb{C},$$

which we identify with $\mathbb{C}$ in the natural way: $f + L_{\rho_x} \mapsto f(x)$. Then for g in $C(X)$, the map

$$\pi_g : \mathbb{C} \to \mathbb{C} : z \mapsto g(x)z$$

is a linear operator on $\mathbb{C}$, and the map

$$\pi_{\rho_x} : C(X) \to \mathcal{B}(\mathbb{C}) : g \mapsto \pi_g$$

is a representation of $C(X)$ on $\mathbb{C}$. The unit in $C(X)$ is the constant function $\mathbb{1}$, which in V_{ρ_x} is identified with $\mathbb{1}(x) = 1$; so we have that 1 is a cyclic vector for the representation π_{ρ_x}.

4.3 The Gelfand-Naimark Theorem

We finish this chapter with the big result; that every C^*-algebra can be isometrically embedded as a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$, the C^*-algebra of bounded linear operators on a Hilbert space $\mathcal{H}$. First, we need another definition.

Definition 8. Let A be a C^*-algebra. Suppose we have a collection $\{\varphi_i \mid i \in I\}$ of representations of A on Hilbert spaces $\{\mathcal{H}_i \mid i \in I\}$. For a in A, we have $\|\varphi_i(a)\| \leq \|a\|$ (by Proposition 2, as each φ_i is a $*$-homomorphism), so we have a bounded operator $\bigoplus \varphi_i(a)$ on $\bigoplus \mathcal{H}_i$ by Section 2.2. By the properties of $\bigoplus \varphi_i(a)$ stated therein, the map

$$\varphi : A \to \mathcal{B}(\bigoplus \mathcal{H}_i) : a \mapsto \bigoplus \varphi_i(a)$$

is a $*$-homomorphism, and so is a representation of A on $\bigoplus \mathcal{H}_i$. Call φ the *direct sum* of the collection $\{\varphi_i \mid i \in I\}$, denoted by $\bigoplus \varphi_i$.

Theorem 3 (Gelfand-Naimark, [3, 4.5.6])**.** *Every C^*-algebra has a faithful representation.*

Proof. With $\mathscr{S}_0$ any collection of states on A containing $\mathscr{P}(A)$, let φ be the direct sum of the collection $\{\pi_\rho \mid \rho \in \mathscr{S}_0\}$ of representations as constructed by the GNS construction. We will show that φ is a faithful representation.

Given a in A, if $\varphi(a) = 0$ then $\pi_\rho(a) = 0$ for all pure states ρ on A. But then, since $\rho(a) = \langle \pi_\rho(a)x_\rho, x_\rho \rangle$ by the GNS construction, we have $\rho(a) = 0$, and then by Lemma 7, $a = 0$. Thus, φ is one-to-one and hence a faithful representation of A.

□

Note that the proof of the Gelfand-Naimark theorem depends on a choice of the set $\mathscr{S}_0$ of states on A, with the only restriction being that it contains the set of pure states $\mathscr{P}(A)$. If we take the direct sum over the representations associated with all of $\mathscr{S}(A)$, then φ is called *the universal representation* of A.

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