

# $C^*$ -ALGEBRAS, AND THE GELFAND-NAIMARK THEOREM

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## 1. PRELIMINARIES

we will assume knowledge on... (algebra homos map 1 to 1,)  
brief(er than asst 3) history

## 2. DEFINITIONS

**Definition 1.** A *Banach algebra* is a complex Banach space  $(A, \|\cdot\|)$  which forms an algebra, such that

$$\|ab\| \leq \|a\|\|b\| \text{ for all } a, b \in A.$$

A *\*-algebra* is an algebra  $A$  with an *involution* map  $a \mapsto a^*$  on  $A$  such that, for all  $a, b \in A$  and for  $\alpha \in \mathbb{C}$ ,

- (i)  $a^{**} = (a^*)^* = a$ ,
- (ii)  $(\alpha a + b)^* = \overline{\alpha}a^* + b^*$ ,

$$(iii) \ (ab)^* = b^*a^*.$$

The element  $a^*$  is referred to as the *adjoint* of  $a$ .

A  $C^*$ -algebra is a Banach algebra  $(A, \|\cdot\|)$  with involution map  $a \mapsto a^*$ , with the condition

$$\|a^*a\| = \|a\|^2 \text{ for all } a \in A.$$

This condition is known as the  $C^*$  axiom.

Here we consider complex  $C^*$ -algebras. The theory of real  $C^*$ -algebras has advanced....

remark on work in real  $C^*$ -algebras

Unless specified otherwise, by an *ideal* of a Banach algebra, we mean a two-sided ideal. A  $*$ -ideal in a Banach  $*$ -algebra is a  $*$ -closed ideal. For  $C^*$ -algebras, it turns out that any ideal is automatically a  $*$ -ideal.

**2.1. Unitization.** If a  $C^*$ -algebra  $A$  contains an identity element  $\mathbb{1}$  such that  $a \cdot \mathbb{1} = a = \mathbb{1} \cdot a$  for all  $a \in A$ , call  $\mathbb{1}$  the *unit* in  $A$ , and  $A$  is then a *unital*  $C^*$ -algebra.

**Proposition 1.** *Any non-unital  $C^*$ -algebra  $A$  can be isometrically embedded in a unital  $C^*$ -algebra  $\tilde{A}$  as a maximal ideal.*

tidy up proof

*Proof.* Let  $\tilde{A} = A \oplus \mathbb{C}$  with pointwise addition, and define

$$\begin{aligned} (a, \lambda)(b, \mu) &:= (ab + \lambda b + \mu a, \lambda\mu), \\ (a, \lambda)^* &:= (a^*, \bar{\lambda}), \\ \|(a, \lambda)\| &:= \sup_{\|b\|=1} \|ab + \lambda b\|. \end{aligned}$$

Then  $\tilde{A}$  is a  $*$ -algebra. The norm  $\|(a, \lambda)\|$  is the norm in  $\mathcal{B}(A)$  of left-multiplication by  $a$  on something iunno??? Thus  $\tilde{A}$  is a Banach  $*$ -algebra with unit  $(0, 1)$ . By design,  $A$  is a maximal ideal of codimension 1. The embedding  $a \mapsto (a, 0)$  is isometric as

$$\|a\| = \|a \cdot \frac{a}{\|a\|}\| \leq \|(a, 0)\| \leq \sup_{\|b\|=1} \|ab\| \leq \|a\|.$$

It remains to verify the  $C^*$ -axiom:

$$\begin{aligned}
 \|(a, \lambda)\|^2 &= \sup_{\|b\|=1} \|ab + \lambda b\|^2 \\
 &= \sup_{\|b\|=1} \|b^*a^*ab + \lambda b^*a^*b + \bar{\lambda}b^*ab + |\lambda|^2b^*b\| \\
 &\leq \sup_{\|b\|=1} \|a^*ab + \lambda a^*b + \bar{\lambda}ab + |\lambda|^2b\| \\
 &= \|(a^*a + \lambda a^* + \bar{\lambda}a, |\lambda|^2)\| \\
 &= \|(a, \lambda)^*(a, \lambda)\| \\
 &\leq \|(a, \lambda)^*\| \|(a, \lambda)\|.
 \end{aligned}$$

By symmetry of  $*$ ,  $\|(a, \lambda)^*\| = \|(a, \lambda)\|$ . Hence, the above inequality becomes equality and we have that

$$\|(a, \lambda)^*(a, \lambda)\| = \|(a, \lambda)\|^2.$$

□

In light of this result, we take all  $C^*$ -algebras from here to be unital. For the results we will consider, we can simply consider the unital case. However, there are important circumstances in which we need to relax the unital condition,

does the following theory carry from a unital  $C^*$ alg to any non-unital ideal? a remark

**2.2. the spectrum.** Given an element  $a \in A$  of a  $C^*$ -algebra, define its spectrum  $\text{sp}(a)$ :

$$\text{spec}(a) := \{\lambda \in \mathbb{C} \mid a - \lambda 1 \text{ is not invertible in } A\}.$$

**2.3. more definitions.** An element  $a \in A$  of a  $C^*$ -algebra is called

- *self-adjoint* if  $a^* = a$ ;
- *unitary* if  $aa^* = a^*a = 1$ ;
- *positive* if it is self-adjoint and  $\text{sp}(a) \subseteq \mathbb{R}^+$ .

Denote the set of positive elements in  $A$  by  $A^+$ .

Given Banach  $*$ -algebras  $A$  and  $B$ , a map  $\varphi : A \rightarrow B$  is a  *$*$ -homomorphism* if it is an algebra homomorphism for which  $\varphi(a^*) = \varphi(a)^*$  for all  $a \in A$ . If a  $*$ -homomorphism  $\varphi$  is one-to-one, call it a  *$*$ -isomorphism*.

norm preservation theorem here?

A *linear functional* on a  $C^*$ -algebra  $A$  is a linear operator  $\rho : A \rightarrow \mathbb{C}$ . A *state* on  $A$  is a linear functional  $\rho$  such that  $\|\rho\| = 1$  and  $\rho(a) \geq 0$  for all positive elements  $a \in A^+$ .

normal(?), the weak\* topology(?), cauchy-schwarz

is this right?  
what of  $\rho(1) = 1$ ?

**2.4.  $\mathcal{B}(\mathcal{H})$  - an example.** This section concerns the fundamental example of a  $C^*$ -algebra - the set  $\mathcal{B}(\mathcal{H})$  of bounded linear operators on a Hilbert space  $\mathcal{H}$ . Here we will demonstrate that  $\mathcal{B}(\mathcal{H})$  is a  $C^*$ -algebra and give some basic results.

example of  $\mathcal{B}(\mathcal{H})$  at end of section to retro-motivate notation.

### 3. REPRESENTATIONS OF $C^*$ -ALGEBRAS

to include all representation theory, including GNS, CGN and GN

**Definition 2.** Given a  $C^*$ -algebra  $A$ , a *representation of  $A$  on a Hilbert space  $\mathcal{H}$*  is a  $*$ -homomorphism  $\varphi : A \rightarrow \mathcal{B}(\mathcal{H})$ . An isomorphic representation is called *faithful*.

find correct name - Gelfand Representation theorem?

**Theorem 1** (Gelfand-Naimark for commutative algebras). *Every commutative  $C^*$ -algebra  $A$  is  $*$ -isomorphic to  $C(\mathcal{P}(A))$ , the continuous functions on the topological space of pure states on  $A$ .*

remark about nonunital commutative.

**Theorem 2** (Gelfand-Naimark-Segal). *If  $\rho$  is a state on a  $C^*$ -algebra  $A$ , then there exists a cyclic representation  $\pi_\rho$  of  $A$  on a Hilbert space  $H_\rho$ , with unit cyclic vector  $x_\rho$ , such that*

$$\rho(a) = \langle \pi_\rho(a)x_\rho, x_\rho \rangle, \quad \forall a \in A.$$

*Proof.* We will construct from  $\rho$  the space  $\mathcal{H}_\rho$ , representation  $\pi_\rho$ , and vector  $x_\rho$ , and demonstrate the required properties.

Consider the *left kernel* of  $\rho$ :

$$L_\rho := \{t \in A \mid \rho(t^*t) = 0\}.$$

For  $a, b \in A$ , define  $\langle a, b \rangle_0 := \rho(b^*a)$ . Then  $L_\rho = \{t \in A \mid \langle t, t \rangle_0 = 0\}$ , and  $\langle \cdot, \cdot \rangle_0$  satisfies

(i) Linearity in 1st argument: for  $a, b \in A$ ,  $\alpha, \beta \in \mathbb{C}$ :

$$\begin{aligned} \langle \alpha a + \beta b, c \rangle_0 &= \rho(c^*(\alpha a + \beta b)) \\ &= \rho(\alpha c^*a + \beta c^*b) \\ &= \alpha \rho(c^*a) + \beta \rho(c^*b) \\ &= \alpha \langle a, c \rangle_0 + \beta \langle b, c \rangle_0. \end{aligned}$$

(ii) Conjugate symmetric: for  $a, b \in A$ :

$$\begin{aligned} \langle b, a \rangle_0 &= \rho(a^*b) \\ &= \rho((b^*a)^*) \\ &= \overline{\rho(b^*a)} \\ &= \overline{\langle a, b \rangle_0}. \end{aligned}$$

why?

(iii) Positive semi-definite.

Note that  $\langle \cdot, \cdot \rangle$  is not necessarily positive definite on  $A - L_\rho$  is exactly where this fails.

$L_\rho$  is a linear subspace of  $A$ : Consider

sentence

$$L := \{t \in A \mid \langle t, a \rangle_0 = 0, \forall a \in A\} \subseteq L_\rho.$$

For  $t \in L_\rho$ , by Cauchy-Schwarz we have

$$|\langle t, a \rangle_0|^2 \leq \langle t, t \rangle_0 \langle a, a \rangle_0, \forall a \in A;$$

that is,

$$\langle t, a \rangle_0 = 0, \forall a \in A,$$

so  $t \in L$  and  $L_\rho = L$ .

Now, for  $a, b \in L$ ,  $\alpha \in \mathbb{C}$  and  $c \in A$ :

$$\langle \alpha a + b, c \rangle_0 = \alpha \langle a, c \rangle_0 + \langle b, c \rangle_0 = 0,$$

so  $\alpha a + b \in L$ ; also,  $\langle 0, c \rangle_0 = 0$  so  $0 \in L$ . Hence,  $L (= L_\rho)$  is a linear subspace of  $A$ .

For  $s \in A$ ,  $t \in L_\rho$ , by the Cauchy Schwarz inequality [ref] we have

$$\begin{aligned} |\rho(s^*t)|^2 &= |\langle t, s \rangle_0|^2 \\ &\leq \langle t, t \rangle_0 \cdot \langle s, s \rangle_0 \\ &= \rho(t^*t) \cdot \rho(s^*s) \\ &= 0, \end{aligned}$$

so  $\rho(s^*t) = 0$ . Letting  $s = a^*at$  for  $a \in A$ , then

$$\begin{aligned} \rho((at)^*at) &= \rho(at^*a^*at) \\ &= \rho((a^*at)^*t) \\ &= \rho(s^*t) \\ &= 0, \end{aligned}$$

so that  $at \in L_\rho$ , for all  $a \in A$  and  $t \in L_\rho$ ; we conclude that  $L_\rho$  is a left ideal in  $A$ .  $L_\rho$  is the preimage in  $A$  of  $\{0\}$  under the continuous map  $t \mapsto \rho(t^*t)$ , so is closed.

start sentence properly

Consider now  $V_\rho := A/L_\rho$ , with  $\langle \cdot, \cdot \rangle$  defined by

$$\langle a + L_\rho, b + L_\rho \rangle := \langle a, b \rangle_0, \text{ for } a + L_\rho, b + L_\rho \in V_\rho.$$

It follows from properties *i*), *ii*) and *iii*) of  $\langle \cdot, \cdot \rangle_0$  that  $\langle \cdot, \cdot \rangle$  is an inner product on  $V_\rho$  - with

$$\begin{aligned} \langle a + L_\rho, a + L_\rho \rangle = 0 &\iff \langle a, a \rangle_0 = 0 \\ &\iff a \in L_\rho \\ &\iff a + L_\rho = 0 + L_\rho \end{aligned}$$

giving positive definiteness. The completion of  $V_\rho$  with respect to  $\langle \cdot, \cdot \rangle$  is a Hilbert space - this is the Hilbert space  $\mathcal{H}_\rho$  we're looking for.

Now we fix  $a \in A$ , and consider the map

$$\pi_a : V_\rho \rightarrow V_\rho; b + L_\rho \mapsto ab + L_\rho.$$

Let  $b_1, b_2 \in A$  be such that  $b_1 + L_\rho = b_2 + L_\rho$ . Then:

$$\begin{aligned} \implies b_1 - b_2 &\in L_\rho \\ \implies a(b_1 - b_2) &\in L_\rho \\ \implies ab_1 - ab_2 &\in L_\rho \\ \implies ab_1 + L_\rho &= ab_2 + L_\rho \\ \implies \pi_a(b_1 + L_\rho) &= \pi_a(b_2 + L_\rho). \end{aligned}$$

Hence  $\pi_a$  defines a linear operator on  $V_\rho$ .

For  $b + L_\rho \in V_\rho$ :

$$\begin{aligned} \|a\|^2 \cdot \|b + L_\rho\| - \|\pi_a(b + L_\rho)\| &= \|a\|^2 \cdot \|b + L_\rho\| - \|ab + L_\rho\| \\ &= \|a\|^2 \cdot \langle b + L_\rho, b + L_\rho \rangle - \langle ab + L_\rho, ab + L_\rho \rangle \\ &= \|a\|^2 \cdot \rho(b^*b) - \rho((ab)^*ab) \\ &= \rho(\|a\|^2 b^*b - b^*a^*ab) \\ &= \rho(b^*(\|a\|^2 \mathbb{1} - a^*a)b) \\ &\geq 0. \end{aligned}$$

of what?

Thus  $\pi_a$  is a bounded operator, with  $\|\pi_a\| \leq \|a\|$ . By continuity,  $\pi_a$  extends to a bounded operator on  $\mathcal{H}_\rho$  – say  $\pi_\rho(a) : \mathcal{H}_\rho \rightarrow \mathcal{H}_\rho$  such that

$$\pi_\rho(a)(v) = \pi_a(v)$$

for  $v \in V_\rho$ . Then  $\pi_\rho(a) \in \mathcal{B}(\mathcal{H}_\rho)$  for each  $a \in A$ , so  $\pi_\rho$  defines a map  $A \rightarrow \mathcal{B}(\mathcal{H}_\rho)$  such that  $a \mapsto \pi_\rho(a)$ . This will be our representation.

Now, for  $a, b \in A$ ,  $c + L_\rho \in V_\rho$  and  $\alpha \in \mathbb{C}$ :

$$\begin{aligned} \pi_{\alpha a + b}(c + L_\rho) &= (\alpha a + b)(c + L_\rho) \\ &= (\alpha ac + L_\rho) + (bc + L_\rho) \\ &= \alpha \pi_a(c + L_\rho) + \pi_b(c + L_\rho), \end{aligned}$$

so that  $\pi_{\alpha a + b} = \alpha \pi_a + \pi_b$  on  $V_\rho$ .

For  $a, b \in A$  and  $c + L_\rho \in V_\rho$ :

$$\begin{aligned} \pi_{ab}(c + L_\rho) &= abc + L_\rho \\ &= \pi_a(bc + L_\rho) \\ &= \pi_a(\pi_b(c + L_\rho)) \\ &= (\pi_a \cdot \pi_b)(c + L_\rho), \end{aligned}$$

so that  $\pi_{ab} = \pi_a \cdot \pi_b$  on  $V_\rho$ .

For  $a \in A$  and  $b + L_\rho, c + L_\rho \in V_\rho$ :

$$\begin{aligned}
 \langle b + L_\rho, \pi_a^*(c + L_\rho) \rangle &= \langle \pi_a(b + L_\rho), c + L_\rho \rangle \\
 &= \langle ab + L_\rho, c + L_\rho \rangle \\
 &= \rho(c^*ab) \\
 &= \rho((a^*c)^*b) \\
 &= \langle b + L_\rho, a^*c + L_\rho \rangle \\
 &= \langle b + L_\rho, \pi_{a^*}(c + L_\rho) \rangle,
 \end{aligned}$$

so that  $\pi_a^* = \pi_{a^*}$  on  $V_\rho$ .

$V_\rho \subset \mathcal{H}_\rho$  is a dense subset, so the three properties above hold on  $\mathcal{H}_\rho$  by continuity. Hence,  $\pi_\rho : A \rightarrow \mathcal{B}(\mathcal{H}_\rho)$  is a representation of  $A$ . As to the unit vector, consider  $x_\rho := \mathbb{1} + L_\rho \in V_\rho$ . Then for  $a \in A$ ,

$$\begin{aligned}
 \langle \pi_\rho(a)x_\rho, x_\rho \rangle &= \langle \pi_a(\mathbb{1} + L_\rho), \mathbb{1} + L_\rho \rangle \\
 &= \langle a + L_\rho\mathbb{1} + L_\rho \rangle \\
 &= \rho(a);
 \end{aligned}$$

in particular,  $\langle x_\rho, x_\rho \rangle = \rho(\mathbb{1}) = 1$ , so  $x_\rho$  is a unit vector in  $\mathcal{H}_\rho$ .  $\square$

**Theorem 3** (Gelfand-Naimark). *Every C\*-algebra has a faithful representation.*

of what?