C*-ALGEBRAS, AND THE GELFAND-NAIMARK THEOREM

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1. Preliminaries

we will assume knowledge on...
brief(er than asst 3) history

2. Basics

2.1. Definitions.

Definition 1 (C*-algebra)**.**

define C*algebras, states, representations, the weak* topology

3. Representations of C*-algebras

to include all representation theory, including GNS, CGN and GN

Theorem 1 (Gelfand- Naimark-Segal)**.** *If ρ is a state on a C^*-algebra A, then there exists a cyclic representation π_ρ of A on a Hilbert space H_ρ, with unit cyclic vector x_ρ, such that*

$$\rho(a) \; = \; \langle \pi_\rho(a) x_\rho, x_\rho \rangle, \; \forall a \in A.$$

Proof. We will construct from ρ the space $\mathcal{H}_\rho$, representation π_ρ, and vector x_ρ, and demonstrate the required properties.

Consider the *left kernel* of ρ:

$$L_\rho := \{t \in A | \rho(t^*t) = 0\}.$$

For $a, b \in A$, define $\langle a, b\rangle_0 := \rho(b^*a)$. Then $L_\rho = \{t \in A | \langle t, t\rangle_0 = 0\}$, and $\langle \cdot, \cdot \rangle_0$ satisfies

(i) Linearity in 1st argument: for $a, b \in A$, $\alpha, \beta \in \mathbb{C}$:

$$\begin{aligned} \langle \alpha a + \beta b, c\rangle_0 &= \rho(c^*(\alpha a + \beta b) \\ &= \rho(\alpha c^* a + \beta c^* b) \\ &= \alpha\rho(c^*a) + \beta\rho(c^*b) \\ &= \alpha\langle a, c\rangle_0 + \beta\langle b, c\rangle_0. \end{aligned}$$

(ii) Conjugate symmetric: for $a, b \in A$:

$$\begin{aligned} \langle b, a\rangle_0 &= \rho(a^*b) \\ &= \rho((b^*a)^*) \\ &= \overline{\rho(b^*a)} \\ &= \overline{\langle a, b\rangle_0}. \end{aligned}$$

(iii) Positive semi-definite. why?

Note that $\langle\cdot,\cdot\rangle$ is not necessarily positive definite on A – L_ρ is exactly where this fails.

L_ρ is a linear subspace of A: Consider

$$L := \{t \in A | \langle t, a\rangle_0 = 0,\ \forall a \in A\} \subseteq L_\rho.$$

For $t \in L_\rho$, by Cauchy-Schwarz we have

$$|\langle t, a\rangle_0|^2 \leq \langle t, t\rangle_0 \langle a, a\rangle_0,\ \forall a \in A;$$

that is,

$$\langle t, a\rangle_0 = 0,\ \forall a \in A,$$

so $t \in L$ and $L_\rho = L$.

Now, for $a, b \in L$, $\alpha \in \mathbb{C}$ and $c \in A$:

$$\langle \alpha a + b, c\rangle_0 = \alpha\langle a, c\rangle_0 + \langle b, c\rangle_0 = 0,$$

so $\alpha a + b \in L$; also, $\langle 0, c\rangle_0 = 0$ so $0 \in L$. Hence, $L (= L_\rho)$ is a linear subspace of A.

For $s \in A$, $t \in L_\rho$, by the Cauchy Schwarz inequality [ref] we have

$$\begin{aligned} |\rho(s^*t)|^2 &= |\langle t, s\rangle_0|^2 \\ &\leq \langle t, t\rangle_0 \cdot \langle s, s\rangle_0 \\ &= \rho(t^*t) \cdot \rho(s^*s) \\ &= 0, \end{aligned}$$

so $\rho(s^*t) = 0$. Letting $s = a^*at$ for $a \in A$, then

$$\begin{aligned} \rho((at)^*at) &= \rho(at^*a^*at) \\ &= \rho((a^*at)^*t) \\ &= \rho(s^*t) \\ &= 0, \end{aligned}$$

so that $at \in L_\rho$, for all $a \in A$ and $t \in L_\rho$; we conclude that L_ρ is a left ideal in A. L_ρ is the preimage in A of $\{0\}$ under the continuous map $t \mapsto \rho(t^*t)$, so is closed.

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Consider now $V_\rho := A/L_\rho$, with $\langle\cdot,\cdot\rangle$ defined by

$$\langle a + L_\rho, b + L_\rho\rangle := \langle a, b\rangle_0,\ \text{for } a + L_\rho, b + L_\rho \in V_\rho.$$

It follows from properties $i), ii)$ and $iii)$ of $\langle\cdot,\cdot\rangle_0$ that $\langle\cdot,\cdot\rangle$ is an inner product on V_ρ - with

$$\begin{aligned} \langle a + L^\rho, a + L^\rho\rangle = 0 &\iff \langle a, a\rangle = 0 \\ &\iff a \in L_\rho \\ &\iff a + L_\rho = 0 + L_\rho \end{aligned}$$

giving positive definiteness. The completion of V_ρ with respect to $\langle\cdot,\cdot\rangle$ is a Hilbert space - this is the Hilbert space $\mathcal{H}_\rho$ we're looking for.

Now we fix $a \in A$, and consider the map

$$\pi_a : V_\rho \to V_\rho; b + L_\rho \mapsto ab + L_\rho.$$

Let $b_1, b_2 \in A$ be such that $b_1 + L_\rho = b_2 + L_\rho$. Then:

$$\begin{aligned}
&\implies b_1 - b_2 \in L_\rho \\
&\implies a(b_1 - b_2) \in L_\rho \\
&\implies ab_1 - ab_2 \in L_\rho \\
&\implies ab_1 + L_\rho = ab_2 + L_\rho \\
&\implies \pi_a(b_1 + L_\rho) = \pi_a(b_2 + L_\rho).
\end{aligned}$$

Hence π_a defines a linear operator on V_ρ.

For $b + L_\rho \in V_\rho$:

$$\begin{aligned}
\|a\|^2 \cdot \|b + L_\rho\| - \|\pi_a(b + L_\rho)\| &= \|a\|^2 \cdot \|b + L_\rho\| - \|ab + L_\rho\| \\
&= \|a\|^2 \cdot \langle b + L_\rho, b + L_\rho\rangle - \langle ab + L_\rho, ab + L_\rho\rangle \\
&= \|a\|^2 \cdot \rho(b^*b) - \rho((ab)^*ab) \\
&= \rho(\|a\|^2 b^*b - b^*a^*ab) \\
&= \rho(b^*(\|a\|^2 \mathbb{1} - a^*a)b) \\
&\geq 0.
\end{aligned}$$

Thus π_a is a bounded operator, with $\|\pi_a\| \leq \|a\|$. By continuity, π_a extends to a bounded operator on $\mathcal{H}_\rho$ – say $\pi_\rho(a) : \mathcal{H}_\rho \to \mathcal{H}_\rho$ such that

$$\pi_\rho(a)(v) = \pi_a(v)$$

for $v \in V_\rho$. Then $\pi_\rho(a) \in \mathcal{B}(\mathcal{H}_\rho)$ for each $a \in A$, so π_ρ defines a map $A \to \mathcal{B}(\mathcal{H}_\rho)$ such that $a \mapsto \pi_\rho(a)$. This will be our representation.

Now, for $a, b \in A$, $c + L_\rho \in V_\rho$ and $\alpha \in \mathbb{C}$:

$$\begin{aligned}
\pi_{\alpha a + b}(c + L_\rho) &= (\alpha a + b)(c + L_\rho) \\
&= (\alpha ac + L_\rho) + (bc + L_\rho) \\
&= \alpha \pi_a
\end{aligned}$$

$\square$