

C^* -algebras, and the Gelfand-Naimark Theorems

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Chapter 1

Introduction

abstractify!

1.1 History of The Study of Operator Algebras

The noncommutative nature of Werner Heisenberg's work in 1925 on a new quantum mechanics [8] led to Born and Jordan [2], together with Heisenberg [3], developing the matrix mechanics required to concisely summarise the new quantum mechanical model. From 1935-1943, John von Neumann, together with F.J. Murray, developed the theory of *rings of operators* acting on a Hilbert space [12, 13, 14, 20], in an attempt to establish a general framework for this matrix mechanics. These rings of operators are now considered part of the theory of *von Neumann algebras*, a subsection of C^* -algebra theory. Discussion of Heisenberg's discoveries in quantum mechanics can be found in [11], and similarly [17] gives a summary of the works of Jordan expanding on this.

cut down

In 1943, Gelfand and Naimark [7] established an abstract characterisation of C^* -algebras, free from dependence on the operators acting on a Hilbert space. The Gelfand-Naimark theorem, which we will be considering here at length, gives the link between these abstract C^* -algebras and the rings of operators previously studied. Used in the proof of the GN theorem is the Gelfand-Naimark-Segal construction, a technique which, given a certain linear function on a C^* -algebra, yields a representation of that C^* -algebra as an algebra of operators on a Hilbert space.

Throughout this work we will be making extensive use of (the first volume of) '*Fundamentals of the Theory of Operator Algebras*', by Kadison and Ringrose; the authors aim to make accessible the "vast recent research literature", by first introducing and exploring the prototypical examples and developing the abstract theory later.

Some other texts which cover C^* -algebras:

- Dixmier [5] presents a summary of the general theory up to that time (1977), with [6] focussing on reworking and developing the theory of von Neumann algebras;
- Sakai [16] gives a treatment of C^* - and von Neumann algebras from a more topological point of view;

- Davidson [4] gives a quick overview of the theory and uses that as a baseline to explore many classes of examples;
- Blackadar [1] gives a concise, encyclopaedic coverage of the theory of operator algebras, and covers more specialised material and applications.

1.2 An Overview

After this introductory chapter, we give some definitions and results which will be used throughout, and establish notation. Then the definition of a C^* -algebra is given, with some discussion of algebras with a multiplicative identity, followed by an exploration of an example with some previews of concepts later defined. We move on to discussing a powerful tool in this area, and indeed all of functional analysis, which is the concept of the spectrum of an element; then we make some more definitions and discuss how the topological and algebraic structures of C^* -algebras are related. We finish that chapter with another example, with a remark on what makes the abstract study of C^* -algebras applicable to quantum mechanics. The final chapter contains the statements and proofs of the Gelfand-Naimark theorems, and a discussion of how they relate to each other.

1.3 Further topics

Topics in C^* -algebra theory which follow on from this include the classification of different types of C^* -algebras, such as *approximately-finite dimensional* C^* -algebras, which are made up in a certain way from finite dimensional C^* -algebras. Another area of research is K -theory, which for C^* -algebras refers to the study of groups related to the structure of projective elements.

Chapter 2

Background Mathematics and Resources

The following is some mathematics which may prove useful throughout the project, with relevant resources; we will of course be making definitions as needed.

We will be assuming some familiarity with the following theory:

- Rings, algebras and linear spaces.
- Normed spaces, inner product spaces, Banach and Hilbert spaces.
- Point-set topology.

A good broad background on all of these can be found in [18].

The following notation applies throughout: \mathbb{K} denotes a field. \mathbb{R}^+ denotes the non-negative reals. An ideal of an algebra is, unless otherwise stated, taken to mean a two-sided ideal.

2.1 Linear spaces

A *partial order* on a set X is a relation \leq which satisfies the conditions

- $x \leq x$ for all x in X ; (reflexivity)
- for all x and y in X , if $x \leq y$ and $y \leq x$, then $x = y$; (antisymmetry)
- for all x, y and z in X , if $x \leq y$ and $y \leq z$, then $x \leq z$. (transitivity)

A *positive cone* in a real vector space V is a subset V^+ that is closed under addition and scaling by positive scalars, and if $v, -v$ are in V^+ , then $v = 0$. A *partially ordered vector space* is a real vector space V with a positive cone V^+ – the partial order on V given by

$$x \leq y \iff y - x \in V^+,$$

for x, y in V . A familiar example is the non-negative real numbers, which form a positive cone in \mathbb{R} .

Given a linear space X over a field \mathbb{K} , a *linear functional* on X is a linear map $\rho : X \rightarrow \mathbb{K}$. The set X^* of linear functionals on X is itself a linear space, called the *dual space* of X . A linear functional ρ is called *multiplicative* if $\rho(xy) = \rho(x)\rho(y)$ for all x, y in X .

A normed linear space $(V, \|\cdot\|)$ is a *Banach space* if it is complete with respect to $\|\cdot\|$, in the sense that all Cauchy sequences in V converge with respect to the norm. An inner product space $(V, \langle \cdot, \cdot \rangle)$ is a *Hilbert space* if it is a Banach space with respect to the norm induced by $\langle \cdot, \cdot \rangle$.

Theorem (Hahn-Banach Extension theorem). *If ρ_0 is a bounded linear functional on a subspace X_0 of a normed linear space X , then there is a bounded linear functional ρ on X such that $\|\rho\| = \|\rho_0\|$ and $\rho = \rho_0$ on X_0 .*

Proof. Can be found in [9, Theorem 1.6.1, p. 44] □

2.2 Constructions on Hilbert spaces

We need some Hilbert spaces constructions; particularly, the direct sum of a collection of Hilbert spaces and the direct sum of bounded operators on these Hilbert spaces. Given a finite collection $\{\mathcal{H}_1, \dots, \mathcal{H}_n\}$ of Hilbert spaces, let \mathcal{H} denote the set

$$\mathcal{H} = \{(x_1, \dots, x_n) \mid x_i \in \mathcal{H}_i \text{ for } i = 1, \dots, n\}.$$

Define addition and scalar multiplication coordinatewise, and given $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n) \in \mathcal{H}$, the equation

$$\langle x, y \rangle = \langle x_1, y_1 \rangle + \dots + \langle x_n, y_n \rangle$$

defines an inner product $\langle \cdot, \cdot \rangle$ on \mathcal{H} . The resulting norm $\|\cdot\|$ is given by

$$\|x\|^2 = \|x_1\|^2 + \dots + \|x_n\|^2.$$

It is easy to show that \mathcal{H} is a Hilbert space with these operations, and we call it the *direct sum* of the collection $\{\mathcal{H}_1, \dots, \mathcal{H}_n\}$, denoted $\oplus \mathcal{H}_i$.

Similarly, we can construct a Hilbert space direct sum of an infinite collection $\{\mathcal{H}_i \mid i \in I\}$ of Hilbert spaces. Let \mathcal{H} be the set

$$\mathcal{H} = \{(x_i) \mid x_i \in \mathcal{H}_i \text{ for each } i \text{ and } \sum_{i \in I} \|x_i\|^2 < \infty\}.$$

Given $x = (x_i)$ and $y = (y_i) \in \mathcal{H}$, we have that

$$\begin{aligned} \left(\sum_{i \in I} \|x_i + y_i\|^2 \right)^{1/2} &\leq \left(\sum_{i \in I} (\|x_i\| + \|y_i\|)^2 \right)^{1/2} \\ &\leq \left(\sum_{i \in I} \|x_i\|^2 \right)^{1/2} + \left(\sum_{i \in I} \|y_i\|^2 \right)^{1/2} \\ &< \infty. \end{aligned}$$

Hence, the sequence $(x_i + y_i)$ is in \mathcal{H} , and we can define addition and scalar multiplication coordinatewise on \mathcal{H} :

$$(x_i) + (y_i) = (x_i + y_i) \quad \alpha(x_i) = (\alpha x_i).$$

We also have

$$\begin{aligned} \sum_{i \in I} |\langle x, y \rangle| &\leq \sum_{i \in I} \|x_i\| \|y_i\| \\ &\leq \left(\sum_{i \in I} \|x_i\|^2 \right)^{1/2} \left(\sum_{i \in I} \|y_i\|^2 \right)^{1/2} \\ &< \infty, \end{aligned}$$

so that we can define an inner product $\langle \cdot, \cdot \rangle$, with induced norm $\|\cdot\|$, on \mathcal{H} by

$$\langle x, y \rangle = \sum_{i \in I} |\langle x, y \rangle|, \quad \|x\| = \left(\sum_{i \in I} \|x_i\|^2 \right)^{1/2}.$$

To see that \mathcal{H} is complete with respect to $\|\cdot\|$, suppose that $(x^n)_{n \in \mathbb{N}}$ is a Cauchy sequence in \mathcal{H} , where $x^n = (x_i^n)_{i \in I}$ for each n . Then given any positive ϵ , there exists a positive integer N such that

$$\|x^m - x^n\| < \epsilon \text{ for all } m, n \geq N,$$

that is,

$$\sum_{i \in I} \|x_i^m - x_i^n\|^2 < \epsilon^2 \text{ for all } m, n \geq N. \quad (2.1)$$

Hence for each $i \in I$,

$$\|x_i^m - x_i^n\| < \epsilon \text{ for all } m, n \geq N$$

so that $(x_i^n)_{n \in \mathbb{N}}$ is a Cauchy sequence in \mathcal{H}_i , having a limit $x_i \in \mathcal{H}_i$. For any finite subset $J \subset I$, it follows from (2.1) that

$$\sum_{j \in J} \|x_j^m - x_j^n\|^2 < \epsilon^2 \text{ for all } m, n \geq N,$$

and letting m tend to infinity,

$$\sum_{j \in J} \|x_j - x_j^n\|^2 < \epsilon^2 \text{ for all } n \geq N. \quad (2.2)$$

This holds for any finite subset J , so

$$\sum_{i \in I} \|x_i - x_i^n\|^2 < \epsilon^2 \text{ for all } n \geq N,$$

and so $(x_i - x_i^n)$ and x_i^n are in \mathcal{H} for $n \geq N$. Then (x_i) is in \mathcal{H} and by (2.2), x^n converges to (x_i) as n tends to infinity. We conclude that \mathcal{H} is complete and therefore a Hilbert

space. Just like in the finite case, we call \mathcal{H} the *direct sum* of the collection $\{\mathcal{H}_i \mid i \in I\}$ of Hilbert spaces, denoted $\oplus \mathcal{H}_i$.

Suppose now we have a (finite or infinite) collection of bounded operators $\{T_i \in \mathcal{B}(\mathcal{H}_i) \mid i \in I\}$ such that

$$\sup_{i \in I} \|T_i\| < \infty$$

(by convention this is true when I is a finite set). For $x = (x_i)$ in $\oplus \mathcal{H}_i$, define an element Tx in $\oplus \mathcal{H}_i$ by $Tx = (T_i x_i)$. Then $T : \mathcal{H} \rightarrow \mathcal{H} : x \mapsto Tx$ is a bounded linear operator, called the *direct sum* of the collection $\{T_i \in \mathcal{B}(\mathcal{H}_i) \mid i \in I\}$, denoted $\oplus T_i$. For S_i, T_i in $\mathcal{B}(\mathcal{H}_i)$, α, β in \mathbb{C} , we have

$$\begin{aligned} (\oplus T_i)^* &= \oplus T_i^*, \\ \oplus(\alpha S_i + \beta T_i) &= \alpha \oplus S_i + \beta \oplus T_i, \\ \oplus(S_i T_i) &= \oplus S_i \oplus T_i, \\ \|\oplus T_i\| &= \sup_{i \in I} \|T_i\|. \end{aligned}$$

2.3 Topological stuff

By a *topological space* we mean a set X endowed with a collection \mathcal{T} (a *topology*) of subsets of X such that \mathcal{T} is closed under finite intersections and arbitrary unions, and contains both X and the empty set. A *neighbourhood* of a point x is a subset of X which contains an open set which itself contains x . An *open cover* of X is a collection \mathcal{C} of open sets whose union contains X , and a subset of \mathcal{C} which still covers X is called a *subcover*. A topological space X is said to be:

- *Hausdorff* if, for every distinct pair of points x, y of X , there is a neighbourhood of x which is disjoint from some neighbourhood of y ;
- *compact* if every open cover of X has a finite subcover;
- *locally compact* if every point of x has a compact neighbourhood.

Most of the topology we use will be compact and Hausdorff, and locally compact and Hausdorff at the least. A subset Y of X is said to be *dense* in X if every non-empty open set in X contains an element of Y – equivalently, if X is the largest closed set containing Y . The topology *generated by a collection* \mathcal{B} is the collection \mathcal{T} containing \mathcal{B} and all finite intersections and arbitrary unions over \mathcal{B} . It is the weakest topology on X containing \mathcal{B} , in the sense that any topology containing \mathcal{B} contains all of \mathcal{T} .

A *topological vector space* is a vector space V over \mathbb{K} , together with a topology on V such that the vector space operations $V \times V \rightarrow V : (x, y) \mapsto x + y$ and $\mathbb{K} \times V \rightarrow V : (\lambda, x) \mapsto \lambda x$ are continuous. Recall that an *extreme point* of a convex subset X_0 of a topological vector space is a point x for which an expression

$$x = \alpha x_1 + (1 - \alpha)x_2,$$

for $0 \leq \alpha \leq 1$ and $x_1, x_2 \in X_0$, implies that $x_1 = x = x_2$. The convex set X_0 is then equal to the set of all linear combinations

$$\alpha_1 x_1 + \cdots + \alpha_n x_n$$

of its extreme points, where $\alpha_1, \dots, \alpha_n$ are positive scalars summing to 1. For example, for a polygon embedded in \mathbb{R}^2 , the vertices of a polygon are its extreme points, and every point within the polygon can be written as a linear combination of the vertices.

Definition ([15, 3.14]). Let X be a topological vector space (over a field \mathbb{K}), with dual space X^* . Every x in X induces a linear functional f_x on X^* defined by $f_x(\rho) = \rho(x)$. The *weak** topology on X^* is the topology generated by the sets

$$\{f_x^{-1}(V) \mid x \in X, V \subseteq \mathbb{K} \text{ open}\}.$$

This is the weakest topology on X^* such that each functional f_x is continuous, and the collection $\{f_x \mid x \in X\}$ separate the points of X^* , so the weak* topology is Hausdorff.

Theorem (The Banach-Alaoglu theorem [15]). *If V is a neighbourhood of 0 in a topological vector space X , and*

$$K = \{\rho \in X^* \mid |\rho(x)| \leq 1 \text{ for all } x \in V\},$$

then K is compact in the weak topology.*

Chapter 3

C^* -algebras

We begin this chapter with some definitions and an important example, then give some results we will need later and finish with another fundamental example.

3.1 Definition

Definition 1. A *Banach algebra* is a complex Banach space A , with norm $\|\cdot\|$, which forms an algebra, such that

$$\|ab\| \leq \|a\|\|b\| \text{ for all } a, b \in A.$$

A $*$ -algebra is an algebra A with an *involution* map $a \mapsto a^*$ on A such that, for all $a, b \in A$ and for $\alpha \in \mathbb{C}$,

- (i) $a^{**} = (a^*)^* = a$,
- (ii) $(\alpha a + b)^* = \bar{\alpha}a^* + b^*$,
- (iii) $(ab)^* = b^*a^*$.

The element a^* is referred to as the *adjoint* of a .

A C^* -algebra is a Banach algebra A with involution $a \mapsto a^*$ making it a $*$ -algebra, with the condition that

$$\|a^*a\| = \|a\|^2 \text{ for all } a \in A.$$

This condition is known as the C^* axiom.

A subalgebra B of a C^* -algebra A is a C^* -subalgebra if it is closed under the adjoint and complete with respect to the norm; equivalently, if B is itself a C^* -algebra.

An element $a \in A$ of a C^* -algebra is called

- *normal* if $a^*a = aa^*$;
- *self-adjoint* if $a^* = a$;
- *unitary* if $a^*a = aa^* = \mathbb{1}$.

The set of self-adjoint elements A_{sa} forms a vector space over \mathbb{R} . Note that an element being self-adjoint, or unitary, implies that it is normal.

As a simple example, we can consider \mathbb{C} as a C^* -algebra: We know from complex analysis that Cauchy sequences converge in \mathbb{C} and that complex conjugation is an involutive operation. In this way we can interpret self-adjoint elements of a C^* -algebra A as ‘reals’ – in fact, for a in A , let $h = \frac{1}{2}(a + a^*)$ and $k = \frac{i}{2}(a - a^*)$. Then h and k are self-adjoint and we can write $a = h + ik$ – call h and k the *real* and *imaginary* part of a , respectively.

3.2 Unitization

If a C^* -algebra A contains a multiplicative identity element $\mathbb{1}$, such that $a\mathbb{1} = a = \mathbb{1}a$ for all $a \in A$, call $\mathbb{1}$ the *unit* in A , and A is then a *unital* C^* -algebra.

Proposition ([4, I.1.3]). *Any non-unital C^* -algebra A can be isometrically embedded in a unital C^* -algebra \tilde{A} .*

Proof. Let $\tilde{A} = A \oplus \mathbb{C}$ with pointwise addition, and define

$$\begin{aligned}(a, \lambda)(b, \mu) &= (ab + \lambda b + \mu a, \lambda\mu), \\ (a, \lambda)^* &= (a^*, \bar{\lambda}).\end{aligned}$$

Then \tilde{A} is a $*$ -algebra. Consider (a, λ) as an operator acting on A via $b \mapsto ab + \lambda b$ for b in A . It is easy to see that this operator is linear, and

$$\begin{aligned}\sup_{\|b\|=1} \|ab + \lambda b\| &\leq \sup_{\|b\|=1} (\|ab\| + \lambda\|b\|) \\ &= \sup_{\|b\|=1} \|ab\| + \lambda < \infty\end{aligned}$$

so that the operator (a, λ) is bounded. It follows that

$$\|(a, \lambda)\| = \sup_{\|b\|=1} \|ab + \lambda b\|$$

is a norm on \tilde{A} , and so \tilde{A} is a Banach $*$ -algebra, with unit $(0, 1)$. The embedding $a \mapsto (a, 0)$ is isometric because

$$\|a\| = \|a \cdot \frac{a}{\|a\|}\| \leq \|(a, 0)\| \leq \sup_{\|b\|=1} \|ab\| \leq \|a\|.$$

It remains to verify the C^* -axiom:

$$\begin{aligned}\|(a, \lambda)\|^2 &= \sup_{\|b\|=1} \|ab + \lambda b\|^2 \\ &= \sup_{\|b\|=1} \|b^* a^* ab + \lambda b^* a^* b + \bar{\lambda} b^* ab + |\lambda|^2 b^* b\| \\ &\leq \sup_{\|b\|=1} \|a^* ab + \lambda a^* b + \bar{\lambda} ab + |\lambda|^2 b\| \\ &= \|(a^* a + \lambda a^* + \bar{\lambda} a, |\lambda|^2)\| \\ &= \|(a, \lambda)^*(a, \lambda)\| \\ &\leq \|(a, \lambda)^*\| \|(a, \lambda)\|.\end{aligned}$$

By symmetry of $*$, $\|(a, \lambda)^*\| = \|(a, \lambda)\|$. Hence, the above inequality becomes equality and we have that

$$\|(a, \lambda)^*(a, \lambda)\| = \|(a, \lambda)\|^2,$$

and \tilde{A} is a C^* -algebra. \square

In light of this result, we take all C^* -algebras from here to be unital unless specified otherwise. For the results we will consider, we can simply consider the unital case. However, there are circumstances in advanced theory in which one needs to relax the unital condition, where properties do not pass from a C^* -algebra to its maximal ideals, so it is by no means a universal assumption that a C^* -algebra has a unit element.

3.3 $C(X)$ - an example

Given a compact Hausdorff space X , let $C(X)$ be the algebra of continuous functions $f : X \rightarrow \mathbb{C}$, with addition and multiplication defined pointwise. Define $\|\cdot\|$ on $C(X)$ by

$$\|f\| = \sup_{x \in X} |f(x)|,$$

and let \bar{f} be the function $\bar{f}(x) = \overline{f(x)}$ for x in X . We will demonstrate that $C(X)$ is an Abelian C^* -algebra and preview some of the theory we will build up in later sections. Later, in Section 4.1, we will show that $C(X)$ is essentially *the* Abelian C^* -algebra.

Claim. *The set $C(X)$, of continuous complex valued functions on a compact Hausdorff space X , is an Abelian C^* -algebra.*

Proof. We take $C(X)$ as an algebra with operations defined pointwise and norm defined as above. Multiplication in $C(X)$ is pointwise, so analogous to multiplication in \mathbb{C} , hence commutative. Showing that $C(X)$ is a Banach space with this norm is a simple exercise. We will demonstrate that $C(X)$ is a Banach algebra, that $f \mapsto \bar{f}$ is an involution, and that the norm satisfies the C^* axiom. Given f and g in $C(X)$, we have

$$\begin{aligned} \|fg\| &= \sup_{x \in X} |(fg)(x)| \\ &= \sup_{x \in X} |f(x)g(x)| \\ &\leq \sup_{x \in X} |f(x)||g(x)| \\ &= \|f\|\|g\|. \end{aligned}$$

Since $f \mapsto \bar{f}$ is simply pointwise complex conjugation, and the algebraic operations are defined pointwise, it is clear that it is an involution. For the C^* axiom: given f in $C(X)$,

$$\begin{aligned} \|\bar{f}f\| &= \sup_{x \in X} |(\bar{f}f)(x)| \\ &= \sup_{x \in X} |\overline{f(x)}f(x)| \\ &= \sup_{x \in X} |f(x)|^2 \\ &= \|f\|^2. \end{aligned}$$

The constant function $\mathbb{1} : X \rightarrow \mathbb{C} : x \mapsto 1$ is the unit in $C(X)$, with $\|\mathbb{1}\| = 1$. \square

Suppose that f is a self-adjoint element of $C(X)$ – that is to say, $\overline{f} = f$. Then $\overline{f(x)} = f(x)$ for all x in X , so f is real-valued on X . The set of all self-adjoints in $C(X)$ is the algebra $C(X, \mathbb{R})$ of all real valued functions on X . Call f *positive* if $f(x)$ is positive for all x in X . A linear functional ρ on $C(X)$ is *positive* if $\rho(f) \geq 0$ for all positive elements f , and a *state* on $C(X)$ if ρ is positive and $\rho(1) = 1$. The *state space* \mathcal{S} , the set of all states on $C(X)$, is a convex set, and extreme points of \mathcal{S} are called *pure states* on $C(X)$.

Given x in X , define ρ_x by $\rho_x : C(X) \rightarrow \mathbb{C} : f \mapsto f(x)$. If f is positive, then $\rho_x(f) \geq 0$, and $\rho_x(1) = 1(x) = 1$, so ρ_x is a state for all x . We can also see that these functionals are multiplicative, and it can be shown (see [9, Corollary 3.4.2]) that every non-zero multiplicative functional on $C(X)$ arises in this way, as ‘evaluation’ at some point of X , and the general fact that the non-zero multiplicative linear functionals are exactly the pure states on a C^* -algebra (which we will show in Proposition 5) shows that these are in fact all of the pure states on $C(X)$.

We finish this section by remarking that if we take X to be a non-compact, *locally* compact Hausdorff space, then we are required to take $C(X)$ to be the set of continuous complex valued functions which *vanish at infinity* in some sense¹. This restriction ensures that the supremum used to define the norm on $C(X)$ exists – however, we lose the constant function 1 , so $C(X)$ is no longer unital. It turns out that the unitization of $C(X)$ corresponds to the continuous functions on the one-point compactification of X – see [1, II.1.2.2].

3.4 The Spectrum

We will just state the necessary definitions and results here, and defer proofs to sources.

Definition 2. Given an element a of a Banach algebra A , define its spectrum $\sigma_A(a)$:

$$\sigma_A(a) = \{\lambda \in \mathbb{C} \mid a - \lambda 1 \text{ is not invertible in } A\},$$

where a is *invertible in* A if there exists an element b such that $ab = 1 = ba$. Say that $\lambda \in \sigma(a)$ is a *spectral value* of a in A . If it is clear from context where we are taking the spectrum, then we write $\sigma(a)$ for $\sigma_A(a)$.

Theorem ([9, 3.2.3]). *If a is an element of a Banach algebra A then $\sigma(a)$ is a non-empty closed subset of the closed disk in \mathbb{C} with center 0 and radius $\|a\|$.*

The *spectral radius*, $r(a)$, of a is the radius of the smallest disk in \mathbb{C} containing $\sigma(a)$; that is,

$$r(a) = \sup_{\lambda \in \sigma(a)} |\lambda|.$$

By the above theorem, $r(a) \leq \|a\|$.

¹We say $f : X \rightarrow \mathbb{C}$ *vanishes at infinity* if for all positive ϵ there exists a compact subset K of X such that $|f(x)| < \epsilon$ for all x outside of K .

Theorem (Spectral radius formula [9, 3.3.3]). *The spectral radius of an element a of a Banach algebra is given by the formula*

$$r(a) = \lim_{n \rightarrow \infty} \|a^n\|^{1/n}.$$

Lemma 1 ([9, 4.1.1(i)]). *If a is a normal element of a C^* -algebra, then $r(a) = \|a\|$.*

From this it can be shown that the norm on a C^* algebra A is unique, in the sense that only one norm on A can satisfy the C^* -axiom. [1, II.1.6.5]

Theorem (Continuous functional calculus, [9, 4.1.3]). *Given a self-adjoint element a , we denote by $C(\sigma(a))$ the C^* -algebra of continuous, complex valued functions on the spectrum of a . There is a unique continuous mapping $C(\sigma(a)) : f \mapsto f(a)$ such that:*

(i) *if f is a polynomial, $f(z) = \alpha_1 + \alpha_2 z + \cdots + \alpha_m z^m$, then*

$$f(a) = \alpha_1 \mathbb{1} + \alpha_2 a + \cdots + \alpha_m a^m;$$

(ii) $\|f(a)\| = \|f\|$;

(iii) $(\alpha f + \beta g)(a) = \alpha f(a) + \beta g(a)$;

(iv) $fg(a) = f(a)g(a)$;

(v) $\overline{f}(a) = (f(a))^*$;

(vi) $f(a)$ is normal;

(vii) $f(a)b = bf(a)$ for all b in A such that $ab = ba$.

Moreover, each f in $C(\sigma(a))$ is the limit of a sequence of polynomials over \mathbb{C} .

Theorem (Spectral mapping theorem, [9, 4.1.6]). *If a is a self adjoint element of a C^* -algebra A , and $f \in C(\sigma(a))$, then*

$$\sigma(f(a)) = \{f(t) \mid t \in \sigma(a)\}.$$

Proposition 1 ([9, 4.2.3(i)]). *Given f in $C(\sigma(a))$, $f(a)$ is positive if and only if $f(t) \geq 0$ for all t in $\sigma(a)$.*

Theorem ([9, 4.1.5]). *If B is a C^* -subalgebra of a C^* -algebra A , and b is an element of B , then*

$$\sigma_B(b) = \sigma_A(b).$$

As an aside, this theorem allows us to make the convention that the spectrum of an element a in a non-unital C^* -algebra A is defined to be the spectrum of a in the unitization \tilde{A} of A .

3.5 Further Definitions and results

In this section we will set out some further definitions, some of which we have already seen in the context of $C(X)$, and prove some cool results.

Definition 3. An element a of a C^* -algebra A is *positive* if it is self-adjoint and $\sigma(a) \subseteq \mathbb{R}^+$. Denote the set of positive elements in A by A^+ . Then A_{sa} is a partially ordered vector space with \leq given by

$$a \leq b \iff b - a \text{ is positive}$$

for a, b in A_{sa} . The set of positive elements form a positive cone in A_{sa} , which means that

- (i) $\alpha a + b \in A^+$ for all $a, b \in A^+$ and $\alpha \in \mathbb{R}^+$,
- (ii) $a \in A^+$ and $-a \in A^+$ implies that $a = 0$.

The unit 1 is positive (with $\sigma(1) = \{1\}$).

Lemma 2 ([9, 4.2.3(ii)]). *For a self-adjoint element a in a C^* -algebra A ,*

$$-\|a\|1 \leq a \leq \|a\|1.$$

Proof. Let f in $C(\sigma(a))$ defined by $f(t) = \|a\| \pm t$, f takes non-negative values on $\sigma(a)$, so by Proposition 1, $f(a)$ is positive; that is, $\|a\|1 \pm a$ is positive. \square

Definition 4. Given Banach $*$ -algebras A and B , a map $\varphi : A \rightarrow B$ is a *$*$ -homomorphism* if it is an algebra homomorphism for which $\varphi(a^*) = \varphi(a)^*$ for all $a \in A$. If a $*$ -homomorphism is one-to-one, call it a *$*$ -isomorphism*.

Proposition 2 ([9, 4.1.8]). *Suppose A and B are C^* -algebras and $\varphi : A \rightarrow B$ is a $*$ -homomorphism. Then $\|\varphi(a)\| \leq \|a\|$ for all $a \in A$. If φ is a $*$ -isomorphism, then $\|\varphi(a)\| = \|a\|$ for all $a \in A$.*

Proof. With a in A , if α is not a spectral value for a , $\alpha 1_A - a$ has an inverse s in A . Since $\varphi(1_A) = 1_B$, $\alpha 1_B - \varphi(a)$ has inverse $\varphi(s)$ in B , so α is not a spectral value of $\varphi(a)$; hence $\sigma(\varphi(a)) \subseteq \sigma(a)$, and in particular, $r(\varphi(a)) \leq r(a)$. By Lemma 1, since a^*a is normal, we have

$$\begin{aligned} \|a\|^2 &= \|a^*a\| = r(a^*a); \text{ and} \\ \|\varphi(a)\|^2 &= \|\varphi(a)^*\varphi(a)\| = \|\varphi(a^*a)\| = r(\varphi(a^*a)). \end{aligned}$$

so $\|\varphi(a)\| \leq \|a\|$.

If a is a self-adjoint element of A and $f \in C(\sigma(a))$, then $\varphi(f(a)) = f(\varphi(a))$. This follows from the final assertion of the continuous functional calculus.

Now suppose φ is a $*$ -isomorphism, and for a self-adjoint a in A , suppose $\sigma(\varphi(a)) \subsetneq \sigma(a)$. Then there is a non-zero f in $C(\sigma(a))$ whose restriction to $\sigma(\varphi(a))$ is 0. From the previous paragraph, however, $\varphi(f(a)) = f(\varphi(a)) = 0$, so that $f(a) \neq 0$ is in the kernel of φ ; thus contradicting the assumption that φ is one-to-one, and we conclude that $\sigma(\varphi(a)) = \sigma(a)$. A similar argument to the one in the first half of this proof shows that $\|\varphi(a)\| = \|a\|$ and completes the proof. \square

This result shows that $*$ -homomorphisms between C^* -algebras are automatically continuous, and $*$ -isomorphisms are isometries, demonstrating how the algebraic and topological structure of these spaces is tied together by the C^* axiom.

Definition 5. A linear functional ρ on a C^* -algebra A is *positive* if $\rho(a) \geq 0$ for all $a \in A^+$. A *state* on A is a positive linear functional ρ such that $\rho(\mathbb{1}) = 1$. Denote the set of all states on A by $\mathcal{S}(A)$. The equation $\rho^*(a) = \overline{\rho(a^*)}$ defines a functional ρ , and call ρ *Hermitian* if $\rho = \rho^*$.

Hermitian functionals take real values on self-adjoint elements, and all positive linear functionals are Hermitian. We can extend the partial order notation to linear functionals. Given linear functionals ρ, τ on A , define \leq by

$$\rho \leq \tau \iff \tau - \rho \text{ is positive.}$$

When we take \mathbb{C} as a C^* -algebra, the positive elements are exactly the positive reals, and the linear functionals are given by multiplication by a complex number. Positive linear functionals correspond again to the positive reals, and the only state on \mathbb{C} is ‘multiplication by 1’. Extending the analogy between \mathbb{C} and a C^* -algebra A , we can even show that a positive element has a unique positive square root, and that a^*a is positive for all a in A . (See [9, 4.2.5 and 4.2.6]).

Proposition 3 ([9, 4.3.2]). *A linear functional ρ on a C^* -algebra A is positive if and only if ρ is bounded and $\|\rho\| = \rho(\mathbb{1})$.*

Proof. Suppose that ρ is positive. With a in A , let α be a scalar of modulus 1 such that $\alpha\rho(a) \geq 0$, and let h be the real part of a . Since h is self adjoint, (by 4.2.3(ii)) we have $h \leq \|h\|\mathbb{1} \leq \|a\|\mathbb{1}$. Thus, $\|a\|\mathbb{1} - h$ is positive and

$$\rho(\|a\|\mathbb{1} - h) = \|a\|\rho(\mathbb{1}) - \rho(h) \geq 0.$$

Therefore,

$$|\rho(a)| = \rho(\alpha a) = \overline{\rho(\alpha a)} = \rho(\overline{\alpha a^*}) = \rho(\tfrac{1}{2}(\alpha a + \overline{\alpha} a^*)) = \rho(h) \leq \rho(\mathbb{1})\|a\|,$$

so ρ is bounded and $\|\rho\| \leq \rho(\mathbb{1})$. We also have $\|\rho\| = \sup\{\rho(a) \mid \|a\| = 1\} \geq \rho(\mathbb{1})$, so $\|\rho\| = \rho(\mathbb{1})$.

Conversely, suppose ρ is bounded and $\|\rho\| = \rho(\mathbb{1})$ – we can assume without loss that $\rho(\mathbb{1}) = 1$. With a a positive element of A , let $\rho(a) = \alpha + i\beta$ for real α, β . Then ρ is positive if and only if $\alpha \geq 0$ and $\beta = 0$. For small positive s ,

$$\sigma(\mathbb{1} - sa) = \{1 - st \mid t \in \sigma(a) \subseteq \mathbb{R}^+\} \subseteq [0, 1],$$

so $\|\mathbb{1} - sa\| = r(\mathbb{1} - sa) \leq 1$. Hence

$$1 - s\alpha \leq |1 - s(\alpha + i\beta)| = |\rho(\mathbb{1} - sa)| \leq 1,$$

so $\alpha \geq 0$. With b_n in A defined by $b_n = a + (in\beta - \alpha)\mathbb{1}$ for each positive integer n ,

$$\begin{aligned} \|b_n\|^2 &= \|b_n^* b_n\| = \|(a - \alpha\mathbb{1})^2 + n^2\beta^2\mathbb{1}\| \\ &\leq \|a - \alpha\mathbb{1}\|^2 + n^2\beta^2. \end{aligned}$$

Hence for all positive integers n , we have

$$\begin{aligned}(n^2 + 2n + 1)\beta &= |\rho(b_n)|^2 \\ &\leq \|a - \beta\mathbb{1}\|^2 + n^2\beta^2,\end{aligned}$$

so that $\beta = 0$. We conclude that ρ is positive. \square

The following result follows easily from the Cauchy-Schwarz inequality for inner products.

Proposition (Cauchy-Schwarz inequality, [9, 4.3.1]). *If ρ is a positive linear functional on C^* -algebra A , then for all a and b in A ,*

$$|\rho(b^*a)|^2 \leq \rho(a^*a)\rho(b^*b).$$

Lemma 3 ([9, 4.3.3]). *Let A be a C^* -algebra. For any a in A and $\alpha \in \sigma(a)$, there exists a state ρ on A such that $\rho(a) = \alpha$.*

Proof. For all complex numbers β and γ , $\alpha\beta + \gamma$ is a spectral value for the element $\beta a + \gamma\mathbb{1}$ of A , so

$$|\alpha\beta + \gamma| \leq r(\beta a + \gamma\mathbb{1}) = \|\beta a + \gamma\mathbb{1}\|.$$

Hence the equation $\rho_0(s) = \alpha\beta + \gamma$ defines a bounded linear functional ρ_0 on the linear subspace $B = \{\beta a + \gamma\mathbb{1} \mid \beta, \gamma \in \mathbb{C}\}$ of A , with $\rho_0(a) = \alpha$ and $\rho_0(\mathbb{1}) = 1 = \|\rho_0\|$. By the Hahn-Banach theorem, ρ_0 extends to a bounded linear functional ρ on A , with $\|\rho\| = 1$, such that $\rho = \rho_0$ on the subspace B . In particular, $\rho(\mathbb{1}) = 1 = \|\rho\|$ so ρ is positive by the previous result, and $\rho(a) = \alpha$. \square

Lemma 4 ([9, 4.3.4,(i)]). *Let A be a C^* -algebra. If $\rho(a) = 0$ for all states ρ on A , then $a = 0$.*

Proof. Suppose first that a is self-adjoint, and $\rho(a) = 0$ for all states ρ . By the previous result, $\sigma(a) = \{0\}$, so $\|a\| = r(a) = 0$ (using Lemma 1). Hence $a = 0$.

Now write $a = h + ik$, for h and k the real and imaginary part of a respectively. Then

$$0 = \rho(a) = \rho(h) + i\rho(k),$$

and as h, k are self-adjoint, $\rho(h)$ and $\rho(k)$ are real and we must have $\rho(h) = 0 = \rho(k)$. By previous statement, $h = 0 = k$, and we conclude that $a = 0$. \square

Definition 6. An extreme point of $\mathcal{S}(A)$ is called a *pure state* on A , and the set of pure states on A is denoted by $\mathcal{P}(A)$.

The state space $\mathcal{S}(A)$ is then the weak*-closure of the set of convex linear combinations of pure states. It is a simple exercise, using the fact that $A^+ \subseteq A_{sa}$, to verify that a linear functional ρ is a pure state on A if and only if the restriction $\rho|_{A_{sa}}$ is a pure state on A_{sa} . Conversely, every pure state on A_{sa} extends to a pure state on A with the same norm, by the Hahn-Banach theorem.

Lemma 5 ([9, 4.3.8,(i)]). *Let a be an element of a C^* -algebra A . If $\rho(a) = 0$ for all pure states ρ on A , then $a = 0$.*

Proof. Since every state is the limit of a sequence of linear combinations of pure states, if $\rho(a) = 0$ for all pure states ρ on A , then $\rho(a) = 0$ for *all* states ρ on A . The result follows immediately from Lemma 4. \square

Lemma 6 ([9, 4.3.8,(i)]). *If a is a normal element of a C^* -algebra A , there is a pure state ρ_0 on A such that $|\rho_0(a)| = \|a\|$.*

Proof. By Lemma ??, there is a scalar γ and a state τ on A such that $\tau(a) = \gamma$ and $|\gamma| = \|a\|$. Let \hat{t} be the continuous linear functional on A^* that takes ρ to $\rho(t)$, and let α be a complex number such that $|\alpha| = 1$ and $\tau(\alpha a) = |\gamma| = \|a\|$. From ??, there is a ρ_0 in $\mathcal{P}(A)$ such that

$$\begin{aligned} \|a\| &\geq |\rho_0(a)| \geq \operatorname{Re} \widehat{\alpha a}(\rho_0) \\ &\geq \sup_{\rho \in \mathcal{S}(A)} \operatorname{Re} \widehat{\alpha a}(\rho) \\ &\geq \operatorname{Re} \widehat{\alpha a}(\tau) = \operatorname{Re} \tau(\alpha a) = \|a\| \end{aligned}$$

\square

finish by adding referred to lemmata

Proposition 4 ([9, 3.4.6]). *A state ρ on A_{sa} is pure if and only if, for all positive linear functionals τ on A_{sa} such that $0 \leq \tau \leq \rho$, we have $\tau = \lambda\rho$ for some $\lambda \in \mathbb{R}$.*

Proof. Suppose that $\tau = \lambda\rho$ for all $0 \leq \tau \leq \rho$, and suppose we can write $\rho = \alpha\rho_1 + (1 - \alpha)\rho_2$ for some $0 \leq \alpha \leq 1$ and some $\rho_1, \rho_2 \in \mathcal{S}(A_{sa})$. Then $0 \leq \alpha\rho_1 \leq \rho$, so $\alpha\rho_1 = \lambda\rho$. Then $\rho_1(\mathbb{1}) = 1 = \rho(\mathbb{1})$, so $\alpha = \lambda$ and $\rho_1 = \rho$. Similarly, we can show that $\rho_2 = \rho$, and so we conclude that ρ is a pure state.

Conversely, suppose that ρ is a pure state and $0 \leq \tau \leq \rho$. Applying this to $\mathbb{1}$, we get $0 \leq \tau(\mathbb{1}) \leq \rho(\mathbb{1}) = 1$. Let $\lambda = \tau(\mathbb{1})$. We work case-by-case:

If $\lambda = 0$, then for any $a \in A_{sa}$, applying τ to $-\|a\|\mathbb{1} \leq a \leq \|a\|\mathbb{1}$ (from Lemma 2) gives

$$0 = -\|a\|\lambda = \tau(-\|a\|\mathbb{1}) \leq \tau(a) \leq \tau(\|a\|\mathbb{1}) = \|a\|\lambda = 0,$$

so $\tau = 0 = \lambda\rho$.

A similar argument shows that $\lambda = 1$ implies $\tau - \rho = 0$, so that $\tau = \rho = \lambda\rho$.

If $0 < \lambda < 1$, we can write $\rho = \lambda\rho_1 + (1 - \lambda)\rho_2$ for $\rho_1 = \lambda^{-1}\tau$ and $\rho_2 = (1 - \lambda)^{-1}(\rho - \tau)$. ρ is pure so $\tau = \lambda\rho_1 = \lambda\rho$. \square

Recall that a linear functional ρ on a C^* -algebra A is *multiplicative* if $\rho(ab) = \rho(a)\rho(b)$ for all a, b in A . The set of all non-zero multiplicative linear functionals on A is called the *maximal ideal space* of A , denoted \mathcal{M}_A . This name hints at the fact that the kernel of each of these functionals is a maximal ideal of A , and all maximal ideals of A arise in this way.[4, Theorem I.2.5]

Proposition 5 ([9, 4.4.1]). *The set of pure states on an Abelian C^* -algebra A is precisely the maximal ideal space of A .*

Proof. Suppose ρ is a pure state on A . To show that $\rho(ab) = \rho(a)\rho(b)$ for $a, b \in A$, we restrict attention to the case where $0 \leq b \leq \mathbb{1}$. Linearity gives us the general case. In this case, for $h \in A^+$ we have that $0 \leq hb \leq h$, so $0 \leq \rho(hb) \leq \rho(h)$. Hence $\rho_b(a) = \rho(ab)$ for $a \in A$ defines a positive linear functional on A with $\rho_b \leq \rho$. The restriction $\rho|_{A_{sa}}$ is a pure state on A_{sa} and $\rho_b|_{A_{sa}} \leq \rho|_{A_{sa}}$, and it follows from Proposition 4 that $\rho_b|_{A_{sa}} = \alpha\rho|_{A_{sa}}$ for some $\alpha \in \mathbb{R}^+$. Hence $\rho_b = \alpha\rho$ and so for $a \in A$:

$$\rho(ab) = \rho_b(a) = \alpha\rho(a) = \alpha\rho(\mathbb{1})\rho(a) = \rho_b(\mathbb{1})\rho(a) = \rho(b)\rho(a)$$

Conversely, suppose ρ is a multiplicative linear functional. By Lemma 7, ρ is bounded and $\|\rho\| = \rho(\mathbb{1}) = 1$, so by Proposition 3, ρ is a state. Suppose we can write $\rho = \alpha\rho_1 + \beta\rho_2$ for states ρ_1, ρ_2 on A and $\alpha, \beta > 0$ such that $\alpha + \beta = 1$. For $c \in A_{sa}$, by the Cauchy-Schwarz inequality we have for $j = 1, 2$:

$$(\rho_j(c))^2 = (\rho_j(\mathbb{1}c))^2 \leq \rho_j(\mathbb{1})\rho_j(c^2) = \rho_j(c^2).$$

Then:

$$\begin{aligned} 0 &= \rho(c^2) - \rho(c)^2 \\ &= \alpha\rho_1(c^2) + \beta\rho_2(c^2) - (\alpha\rho_1(c) + \beta\rho_2(c))^2 \\ &\geq \alpha(\alpha + \beta)\rho_1(c)^2 + \beta(\alpha + \beta)\rho_2(c)^2 - (\alpha\rho_1(c) + \beta\rho_2(c))^2 \\ &= \alpha\beta(\rho_1(c) - \rho_2(c))^2. \end{aligned}$$

Hence $\rho_1(c) = \rho_2(c)$, for all $c \in A_{sa}$, so $\rho_1 = \rho_2 = \rho$ and we conclude that ρ is a pure state. \square

The set $\mathcal{P}(A)$ of pure states on A is a subset of the dual space A^* , and when we take A^* with the weak* topology, we get a compact Hausdorff space, which is the object of study in the Gelfand-Naimark theorem for commutative algebras.

Lemma 7 ([9, 3.2.20]). *The maximal ideal space of an Abelian Banach algebra A form a compact subset of the unit ball of the dual space A^* , relative to the weak* topology.*

Proof. Follows from the Banach-Alaoglu theorem. \square

3.6 $\mathcal{B}(\mathcal{H})$ - an example

This section concerns the fundamental example of a C^* -algebra - the set $\mathcal{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space \mathcal{H} , otherwise known as *operator algebras*. Here we will demonstrate that $\mathcal{B}(\mathcal{H})$ is a C^* -algebra and discuss how the theory of C^* -algebras relates to quantum mechanics, the very subject which started the study of these objects, explaining a good deal of the terminology used here.

Claim. $\mathcal{B}(\mathcal{H})$ is a C^* -algebra with the operator norm

$$\|T\| = \sup_{\|x\|=1} \|Tx\|$$

and involution taking T to its adjoint map T^* . The identity map $I : x \mapsto x$ is a unit for $\mathcal{B}(\mathcal{H})$

Proof. It is an easy exercise to show that $\|\cdot\|$ is a norm on $\mathcal{B}(\mathcal{H})$. Let $\{T_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in $\mathcal{B}(\mathcal{H})$. Then for any positive ϵ , there is a positive integer N such that

$$\|T_m - T_n\| < \epsilon \text{ for all } m, n \geq N.$$

Applying $T_m - T_n$ to $x \in \mathcal{H}$, we have

$$\|T_m x - T_n x\| \leq \|T_m - T_n\| \|x\| < \epsilon \|x\|, \quad (3.1)$$

so $\{T_n x\}_{n \in \mathbb{N}}$ is a Cauchy sequence in \mathcal{H} , converging to an element in \mathcal{H} . Define a linear operator $T : \mathcal{H} \rightarrow \mathcal{H}$ by

$$Tx = \lim_{n \rightarrow \infty} T_n x \text{ for } x \in \mathcal{H}.$$

Taking limits as m tends to infinity in equation (3.1), we obtain

$$\|Tx - T_n x\| < \epsilon \|x\| \text{ for all } n \geq N,$$

and so we have that $T - T_n$ (and hence $T = (T - T_n) + T_n$) is a bounded operator and

$$\|T - T_n\| < \epsilon \text{ for all } n \geq N.$$

We conclude that $T_n \rightarrow T$, and so $\mathcal{B}(\mathcal{H})$ is complete.

Since boundedness is equivalent to continuity on \mathcal{H} , given $S, T \in \mathcal{B}(\mathcal{H})$, the operator $ST : \mathcal{H} \rightarrow \mathcal{H} : x \mapsto (S \circ T)(x)$ is bounded on \mathcal{H} . Given $x \in \mathcal{H}$ and $\lambda \in \mathbb{C}$,

$$\begin{aligned} ((\lambda S)T)(x) &= ((\lambda S) \circ T)(x) \\ &= \lambda S(Tx) \\ &= \lambda(S \circ T)(x) \\ &= \lambda ST(x), \end{aligned}$$

so that $(\lambda S)T = \lambda ST$ in $\mathcal{B}(\mathcal{H})$. We have

$$\begin{aligned} \|ST\| &= \sup_{\|x\|=1} \|STx\| \\ &= \sup_{\|x\|=1} \|S(Tx)\| \\ &\leq \|S\| \sup_{\|x\|=1} \|Tx\| \\ &= \|S\| \|T\|. \end{aligned}$$

We conclude that $\mathcal{B}(\mathcal{H})$ is a Banach algebra.

To see that $*$ is an involution, use the fact that the adjoint operator is unique for each operator and the following equalities.

$$\begin{aligned} \text{(i)} \quad \langle (\alpha T + S)^* x, y \rangle &= \langle x, \alpha T + Sy \rangle \\ &= \bar{\alpha} \langle x, Ty \rangle + \langle x, Sy \rangle \\ &= \bar{\alpha} \langle T^* x, y \rangle + \langle S^* x, y \rangle \\ &= \langle (\bar{\alpha} T^* + S^*) x, y \rangle. \end{aligned}$$

$$\begin{aligned}
\text{(ii)} \quad \langle (T^*)^* x, y \rangle &= \langle x, T^* y \rangle \\
&= \overline{\langle T^* y, x \rangle} \\
&= \overline{\langle y, Tx \rangle} \\
&= \langle Tx, y \rangle.
\end{aligned}$$

$$\begin{aligned}
\text{(iii)} \quad \langle (ST)^* x, y \rangle &= \langle x, STy \rangle \\
&= \langle S^* x, Ty \rangle \\
&= \langle T^* S^* x, y \rangle.
\end{aligned}$$

It remains to demonstrate the C^* -axiom on $\mathcal{B}(\mathcal{H})$. For all $x \in \mathcal{H}$, we have

$$\|Tx\|^2 = \langle Tx, Tx \rangle = \langle T^*Tx, x \rangle \leq \|T^*T\| \|x\|^2,$$

so that

$$\|T\|^2 \leq \|T^*T\| \leq \|T^*\| \|T\| = \|T\|^2.$$

It is clear that I is a unit. Hence, the claim. \square

The terminology for normal, positive and unitary elements of a C^* -algebra comes from the corresponding concepts when studying operator algebras. Paul Dirac's *Principles of Quantum Mechanics* and von Neumann's *Mathematical Foundations of Quantum Mechanics* jointly established a formulation of Quantum Mechanics in terms of Hilbert spaces, and operators thereon. To summarise briefly, a physical quantity in a quantum mechanical system has associated with it a self-adjoint operator T on a Hilbert space \mathcal{H} , while the state of the system corresponds to a unit vector φ in \mathcal{H} . Then the expected outcome of measuring the quantity corresponding to T is given by evaluating $\langle \varphi, T\varphi \rangle$. In the context of C^* -algebras, these translate, in turn, to a self adjoint element a of a C^* -algebra A , a normalised positive linear functional ρ (that is, a state) on A , and evaluation of $\rho(a)$.

After stating his postulate, which he refers to as '**P.**', on the expectation value of a system of observables $\mathcal{R}, \dots, \mathcal{R}_\ell$ with corresponding operators R_1, \dots, R_ℓ , von Neumann makes a comment on a consequence in the model of the non-commutativity of operators on a Hilbert space: "In the case of noncommuting R_1, \dots, R_ℓ , [...] **P.** gave no information regarding the probability relations of $\mathcal{R}, \dots, \mathcal{R}_\ell$. In this case, **P.** could be used only determine the probability distribution of each of these quantities by itself, without consideration of the others." ([19, p. 211])

Chapter 4

Representations of C^* -algebras

4.1 Abelian C^* -algebras

Let A be an Abelian C^* -algebra. For a in A , define a complex-valued function \hat{a} on $\mathcal{P}(A)$ by $\hat{a}(\rho) = \rho(a)$. The weak*-topology is the weakest topology on $\mathcal{P}(A)$ for which all of the maps \hat{a} are continuous, so that $\hat{a} \in C(\mathcal{P}(A))$ for all $a \in A$. The map

$$\Gamma : A \rightarrow C(\mathcal{P}(A)) : a \mapsto \hat{a}$$

is called the *Gelfand transform* on A [4]. For $a, b \in A$, $\alpha, \beta \in \mathbb{C}$ and $\rho \in \mathcal{P}(A)$:

$$\begin{aligned} \widehat{(\alpha a + \beta b)}(\rho) &= \rho(\alpha a + \beta b) = \alpha \rho(a) + \beta \rho(b) = \alpha \hat{a}(\rho) + \beta \hat{b}(\rho), \\ \widehat{a^*}(\rho) &= \rho(a^*) = \overline{\rho(a)} = \overline{\hat{a}(\rho)}. \end{aligned}$$

Since pure states are multiplicative by Proposition 5, we have that

$$\widehat{(ab)}(\rho) = \rho(ab) = \rho(a)\rho(b) = \hat{a}(\rho)\hat{b}(\rho).$$

Hence the Gelfand transform is a $*$ -homomorphism. The following theorem gives us that it is in fact a $*$ -isomorphism.

Theorem 1 (Gelfand-Naimark, commutative [9, 4.4.3]). *Every Abelian C^* -algebra A is $*$ -isomorphic to $C(\mathcal{P}(A))$, the algebra of continuous functions on the compact Hausdorff space $\mathcal{P}(A)$ of pure states on A .*

Proof. Since A is Abelian, every a in A is normal, so by Lemma 6 there is a pure state ρ_0 on A such that $|\rho_0(a)| = \|a\|$. From this,

$$\|a\| = |\hat{a}(\rho_0)| \leq \sup_{\rho \in \mathcal{P}(A)} |\hat{a}(\rho)| \leq \|a\|,$$

so

$$\|a\| = \sup_{\rho \in \mathcal{P}(A)} |\hat{a}(\rho)| = \|\hat{a}\|.$$

Given any pure state ρ , $\hat{\mathbb{1}}(\rho) = \rho(\mathbb{1}) = 1$ (where $\mathbb{1}$ is the identity of A), so $\hat{\mathbb{1}}$ is the constant function equal to 1 on $\mathcal{P}(A)$ and thus the unit in $C(\mathcal{P}(A))$. \square

finish proof by showing one-to-one

comment on noncommutative topology

4.2 The Gelfand-Naimark Theorem

Definition 7. Given a C^* -algebra A , a *representation* of A on a Hilbert space \mathcal{H} is a $*$ -homomorphism $\varphi : A \rightarrow \mathcal{B}(\mathcal{H})$. A $*$ -isomorphic representation is called *faithful*. If there exists an element $x \in \mathcal{H}$ such that the set $\{\varphi(a)x \mid a \in A\}$ is dense in \mathcal{H} , say that φ is a *cyclic* representation, with *cyclic vector* x .

The construction used in the proof of the following theorem is known as the Gelfand-Naimark-Segal (GNS) construction.

Theorem 2 ([9, 4.5.2]). *If ρ is a state on a C^* -algebra A , then there exists a cyclic representation π_ρ of A on a Hilbert space H_ρ , with unit cyclic vector x_ρ , such that*

$$\rho(a) = \langle \pi_\rho(a)x_\rho, x_\rho \rangle, \quad \forall a \in A.$$

Proof. We will construct from ρ the space \mathcal{H}_ρ , representation π_ρ , and vector x_ρ , and demonstrate the required properties.

Consider the *left kernel* of ρ :

$$L_\rho = \{a \in A \mid \rho(a^*a) = 0\}.$$

For $a, b \in A$, define $\langle a, b \rangle_0 = \rho(b^*a)$, then $\langle \cdot, \cdot \rangle_0$ satisfies

(i) Linearity in 1st argument: for $a, b \in A$, $\alpha, \beta \in \mathbb{C}$:

$$\begin{aligned} \langle \alpha a + \beta b, c \rangle_0 &= \rho(c^*(\alpha a + \beta b)) \\ &= \rho(\alpha c^*a + \beta c^*b) \\ &= \alpha \rho(c^*a) + \beta \rho(c^*b) \\ &= \alpha \langle a, c \rangle_0 + \beta \langle b, c \rangle_0. \end{aligned}$$

(ii) Conjugate symmetric: for $a, b \in A$:

$$\begin{aligned} \langle b, a \rangle_0 &= \rho(a^*b) \\ &= \rho((b^*a)^*) \\ &= \overline{\rho(b^*a)} \\ &= \overline{\langle a, b \rangle_0}. \end{aligned}$$

(iii) Positive semi-definite: for $a \in A$:

$$= \langle a, a \rangle_0 = \rho(a^*a) \geq 0,$$

since a^*a is positive for all a .

Note that $\langle \cdot, \cdot \rangle$ is not necessarily positive definite on A – the left kernel is exactly where this fails.

To see that L_ρ is a linear subspace of A , consider

$$L = \{t \in A \mid \langle t, a \rangle_0 = 0, \forall a \in A\} \subseteq L_\rho.$$

For $t \in L_\rho$, by the Cauchy-Schwarz inequality we have

$$|\langle t, a \rangle_0|^2 \leq \langle t, t \rangle_0 \langle a, a \rangle_0, \quad \forall a \in A;$$

that is,

$$\langle t, a \rangle_0 = 0, \quad \forall a \in A,$$

so $t \in L$ and $L_\rho = L$. Now, for $a, b \in L$, $\alpha \in \mathbb{C}$ and $c \in A$:

$$\langle \alpha a + b, c \rangle_0 = \alpha \langle a, c \rangle_0 + \langle b, c \rangle_0 = 0,$$

so $\alpha a + b \in L$; also, $\langle 0, c \rangle_0 = 0$ so $0 \in L$. Hence, L_ρ is a linear subspace of A .

For $s \in A$, $t \in L_\rho$, by the Cauchy Schwarz inequality we have

$$\begin{aligned} |\rho(s^*t)|^2 &= |\langle t, s \rangle_0|^2 \\ &\leq \langle t, t \rangle_0 \cdot \langle s, s \rangle_0 \\ &= \rho(t^*t) \cdot \rho(s^*s) \\ &= 0, \end{aligned}$$

so $\rho(s^*t) = 0$. Letting $s = a^*at$ for $a \in A$, then

$$\begin{aligned} \rho((at)^*at) &= \rho(at^*a^*at) \\ &= \rho((a^*at)^*t) \\ &= \rho(s^*t) \\ &= 0, \end{aligned}$$

so that $at \in L_\rho$, for all $a \in A$ and $t \in L_\rho$; we conclude that L_ρ is a left ideal in A . L_ρ is closed as the preimage in A of $\{0\}$ under the continuous map $a \mapsto \rho(a^*a)$.

Consider now the quotient space $V_\rho = A/L_\rho$, with $\langle \cdot, \cdot \rangle$ defined by

$$\langle a + L_\rho, b + L_\rho \rangle = \langle a, b \rangle_0, \quad \text{for } a + L_\rho, b + L_\rho \in V_\rho.$$

It follows from properties *i*), *ii*) and *iii*) of $\langle \cdot, \cdot \rangle_0$ that $\langle \cdot, \cdot \rangle$ is an inner product on V_ρ – with

$$\begin{aligned} \langle a + L_\rho, a + L_\rho \rangle = 0 &\iff \langle a, a \rangle_0 = 0 \\ &\iff a \in L_\rho \\ &\iff a + L_\rho = 0 + L_\rho \end{aligned}$$

giving positive definiteness. The completion of V_ρ , with respect to the norm $\|\cdot\|$ induced by $\langle \cdot, \cdot \rangle$, is a Hilbert space – this is the Hilbert space \mathcal{H}_ρ we're looking for.

Now we fix $a \in A$, and consider the map

$$\pi_a : V_\rho \rightarrow V_\rho; b + L_\rho \mapsto ab + L_\rho.$$

Let $b_1, b_2 \in A$ be such that $b_1 + L_\rho = b_2 + L_\rho$. Then:

$$\begin{aligned} &b_1 - b_2 \in L_\rho \\ \implies &a(b_1 - b_2) \in L_\rho \\ \implies &ab_1 - ab_2 \in L_\rho \\ \implies &ab_1 + L_\rho = ab_2 + L_\rho \\ \implies &\pi_a(b_1 + L_\rho) = \pi_a(b_2 + L_\rho). \end{aligned}$$

Hence π_a defines a linear operator on V_ρ . For $b + L_\rho \in V_\rho$:

$$\begin{aligned}
\|a\|^2 \cdot \|b + L_\rho\| - \|\pi_a(b + L_\rho)\| &= \|a\|^2 \cdot \|b + L_\rho\| - \|ab + L_\rho\| \\
&= \|a\|^2 \cdot \langle b + L_\rho, b + L_\rho \rangle - \langle ab + L_\rho, ab + L_\rho \rangle \\
&= \|a\|^2 \cdot \rho(b^*b) - \rho((ab)^*ab) \\
&= \rho(\|a\|^2 b^*b - b^*a^*ab) \\
&= \rho(b^*(\|a\|^2 \mathbb{1} - a^*a)b) \\
&\geq 0.
\end{aligned}$$

Thus π_a is a bounded operator on V_ρ , with $\|\pi_a\| \leq \|a\|$. By continuity, π_a extends to a bounded operator $\pi_\rho(a) : \mathcal{H}_\rho \rightarrow \mathcal{H}_\rho$, such that

$$\pi_\rho(a)(v) = \pi_a(v)$$

for $v \in V_\rho$. Then $\pi_\rho(a) \in \mathcal{B}(\mathcal{H}_\rho)$ for each $a \in A$, so π_ρ defines a map $A \rightarrow \mathcal{B}(\mathcal{H}_\rho)$ such that $a \mapsto \pi_\rho(a)$. This will be our representation.

Now, for $a, b \in A$, $c + L_\rho \in V_\rho$ and $\alpha \in \mathbb{C}$:

$$\begin{aligned}
\pi_{\alpha a + b}(c + L_\rho) &= (\alpha a + b)(c + L_\rho) \\
&= (\alpha ac + L_\rho) + (bc + L_\rho) \\
&= \alpha \pi_a(c + L_\rho) + \pi_b(c + L_\rho),
\end{aligned}$$

so that $\pi_{\alpha a + b} = \alpha \pi_a + \pi_b$ on V_ρ .

For $a, b \in A$ and $c + L_\rho \in V_\rho$:

$$\begin{aligned}
\pi_{ab}(c + L_\rho) &= abc + L_\rho \\
&= \pi_a(bc + L_\rho) \\
&= \pi_a(\pi_b(c + L_\rho)) \\
&= (\pi_a \cdot \pi_b)(c + L_\rho),
\end{aligned}$$

so that $\pi_{ab} = \pi_a \cdot \pi_b$ on V_ρ .

For $a \in A$ and $b + L_\rho, c + L_\rho \in V_\rho$:

$$\begin{aligned}
\langle b + L_\rho, \pi_a^*(c + L_\rho) \rangle &= \langle \pi_a(b + L_\rho), c + L_\rho \rangle \\
&= \langle ab + L_\rho, c + L_\rho \rangle \\
&= \rho(c^*ab) \\
&= \rho((a^*c)^*b) \\
&= \langle b + L_\rho, a^*c + L_\rho \rangle \\
&= \langle b + L_\rho, \pi_{a^*}(c + L_\rho) \rangle,
\end{aligned}$$

so that $\pi_a^* = \pi_{a^*}$ on V_ρ .

The subset V_ρ is dense in \mathcal{H}_ρ , so the three properties above hold on \mathcal{H}_ρ by continuity of π_ρ . Hence, $\pi_\rho : A \rightarrow \mathcal{B}(\mathcal{H}_\rho)$ is a representation of A . As to the unit vector, consider $x_\rho = \mathbb{1} + L_\rho \in V_\rho$. Then for $a \in A$,

$$\begin{aligned}
\langle \pi_\rho(a)x_\rho, x_\rho \rangle &= \langle \pi_a(\mathbb{1} + L_\rho), \mathbb{1} + L_\rho \rangle \\
&= \langle a + L_\rho, \mathbb{1} + L_\rho \rangle \\
&= \rho(a);
\end{aligned}$$

and in particular, $\langle x_\rho, x_\rho \rangle = \rho(\mathbb{1}) = 1$, so x_ρ is a unit vector in \mathcal{H}_ρ . The set $\{\pi_\rho(a)x_\rho \mid a \in A\}$ is the dense subset V_ρ of \mathcal{H}_ρ , so x_ρ is a cyclic vector for ρ . \square

We know from Section 3.3 that all functionals of the form $\rho_x(f) = f(x)$, for x in X , are pure states on the C^* -algebra $C(X)$. So, what happens if we apply the GNS construction to ρ_x ?

Fix x in X . If $\rho_x(\bar{f}f) = 0$, then $(\bar{f}f)(x) = 0$ and $f(x) = 0$. Then the left kernel L_{ρ_x} reduces to the kernel (in the usual linear sense) of ρ_x :

$$L_{\rho_x} = \{f \in C(X) \mid f(x) = 0\} = \ker(\rho_x).$$

The kernel of a non-zero linear functional is a proper ideal, with codimension 1, so when we take the quotient we find that

$$V_{\rho_x} = C(X)/L_{\rho_x} \cong \mathbb{C},$$

which we identify with \mathbb{C} in the natural way: $f + L_{\rho_x} \mapsto f(x)$. Then for g in $C(X)$, the map

$$\pi_g : \mathbb{C} \rightarrow \mathbb{C} : z \mapsto g(x)z$$

is a linear operator on \mathbb{C} , and the map

$$\pi_{\rho_x} : C(X) \rightarrow \mathcal{B}(\mathbb{C}) : g \mapsto \pi_g$$

is a representation of $C(X)$ on \mathbb{C} . The unit in $C(X)$ is the constant function $\mathbb{1}$, which in V_{ρ_x} is identified with $\mathbb{1}(x) = 1$; so we have that 1 is a cyclic vector for the representation π_{ρ_x} .

We finish this section with the big result; that every C^* -algebra can be isometrically embedded as a C^* -subalgebra of $\mathcal{B}(\mathcal{H})$, the C^* -algebra of bounded linear operators on a Hilbert space \mathcal{H} . First, we need another definition.

Definition 8. Let A be a C^* -algebra. Suppose we have a collection $\{\varphi_i \mid i \in I\}$ of representations of A on Hilbert spaces $\{\mathcal{H}_i \mid i \in I\}$. For a in A , we have $\|\varphi_i(a)\| \leq \|a\|$ (by Proposition 2, as each φ_i is a $*$ -homomorphism), so we have a bounded operator $\oplus \varphi_i(a)$ on $\oplus \mathcal{H}_i$ by section 2.2. By the properties of $\oplus \varphi_i(a)$ stated therein, the map

$$\varphi : A \rightarrow \mathcal{B}(\oplus \mathcal{H}_i) : a \mapsto \oplus \varphi_i(a)$$

is a $*$ -homomorphism, and so is a representation of A on $\oplus \mathcal{H}_i$. Call φ the *direct sum* of the collection $\{\varphi_i \mid i \in I\}$, denoted $\oplus \varphi_i$.

Theorem 3 (Gelfand-Naimark, [9, 4.5.6]). *Every C^* -algebra has a faithful representation.*

Proof. With \mathcal{S}_0 any collection of states on A containing $\mathcal{P}(A)$, let φ be the direct sum of the collection $\{\pi_\rho \mid \rho \in \mathcal{S}_0\}$ of representations as constructed by the GNS construction. We will show that φ is a faithful representation.

Given a in A , if $\varphi(a) = 0$ then $\pi_\rho(a) = 0$ for all pure states ρ on A . But then, since $\rho(a) = \langle \pi_\rho(a)x_\rho, x_\rho \rangle$ by the GNS construction, we have $\rho(a) = 0$, and then by Lemma 5, $a = 0$. Thus, φ is one-to-one and hence a faithful representation of A . \square

Note that the proof of the Gelfand-Naimark theorem depends on a choice of the set \mathcal{S}_0 of states on A , with the only restriction being that it contains the set of pure states $\mathcal{P}(A)$. If we take the direct sum over the representations associated with all of $\mathcal{S}(A)$, then φ is called *the universal representation* of A .

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