C*-ALGEBRAS, THE GELFAND-NAIMARK THEOREM, AND [OTHER THING]

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1. Introduction

1.1. **History of C*-Algebras.** The noncommutative nature of W. Heisenberg's work on (finish sentence) [4] lead to Born and Jordan [1], together with Heisenberg [2], developing the matrix mechanics required to concisely convey the new quantum mechanical model. From 1930-1943, J. von Neumann, together with F. J. Murray, developed the theory of *rings of operators* acting on a Hilbert space [7], [8], [10], [9], in an attempt to establish a general framework for this matrix mechanics. These rings of operators are now considered part of the theory of *von Neumann algebras*, a subsection of C*-algebra theory. Discussion of the seminal quantum mechanical works of Heisenberg can be found in [5], and likewise in [6], for the works of Jordan expanding on this.

In [3], Gelfand and Naimark established an abstract characterisation of C*-algebras, free from dependence on the operators acting on a Hilbert space. The Gelfand-Naimark theorem, which we will be considering here at length, gives the link between these abstract C*-algebras and the rings of operators previously studied. Used in the proof of the G-N theorem is the Gelfand-Naimark-Segal construction, a pair of results relating cyclic $*$-representations of C*-algebras to certain linear functionals on that algebra.

1.2. **C*-Algebras.** A brief overview of what a C*-algebra is.

- banach space
- banach algebra
- *-algebra
- C* algebra
- representations

Definition 1. A *C*-algebra* is a Banach algebra $(A, \|\cdot\|)$ with involution $* : A \to A$ with the condition that

$$\|x^*x\| = \|x\|^2 \text{ for all } x \in A.$$

This condition is known as the *C* axiom.* The study of C*-algebras started with the consideration of matrix mechanics by H

Theorem (Gelfand-Naimark, Commutative)**.** *Every commutative, unital C^*-algebra A is isometrically $*$-isomorphic to the algebra of continuous functions on the algebra of characters on A.*

The theorem first appeared in [gelfand-naimark] in a form relating 'normed rings' and a closed subrings of the set of bounded operators on a Hilbert space.

1.3. **Aims.** The aims for my project are, provisionally:

- Take the Gelfand-Naimark theorem, and understand its contents and proof.
- Consider the representation theory of C*-algebras, using the Gelfand-Naimark-Segal construction as a starting point.
- (as 'further reading') Give an overview of areas in which operator algebra theory can be taken (for example, Cuntz algebras and operator K-theory, von Neumann algebras and Factors, abstract harmonic analysis).

References

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[5] MacKinnon, E., *Heisenberg, Models, and the Rise of Matrix Mechanics.* Hist. Stud. Phys. Sci., Vol. 8 (1977), pp. 137–188
[6] Schroer, B., *Pascual Jordan, Glory and Demise and his legacy in contemporary local quantum physics.* (no idea where published)
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[9] Murray, F. J. & von Neumann, J., *On rings of operators. IV.* Ann. of Math. (2) 44, (1943), pp. 716-808.
[10] von Neumann, J., *On rings of operators. III.* Ann. of Math. (2) 41, (1940), pp. 94-161.