C*-ALGEBRAS, AND THE GELFAND-NAIMARK THEOREM

LUKE ARMITAGE

1. Introduction

1.1. **History of C*-Algebras.** The noncommutative nature of W. Heisenberg's work on a new quantum mechanics [6] lead to Born and Jordan [1], together with Heisenberg [2], developing the matrix mechanics required to concisely summarise the new quantum mechanical model. From 1930-1943, J. von Neumann, together with F. J. Murray, developed the theory of *rings of operators* acting on a Hilbert space [11], [12], [14], [13], in an attempt to establish a general framework for this matrix mechanics. These rings of operators are now considered part of the theory of *von Neumann algebras*, a subsection of C*-algebra theory. Discussion of the seminal quantum mechanical works of Heisenberg can be found in [7], and likewise in [9], for the works of Jordan expanding on this.

In [5], Gelfand and Naimark established an abstract characterisation of C*-algebras, free from dependence on the operators acting on a Hilbert space. The Gelfand-Naimark theorem, which we will be considering here at length, gives the link between these abstract C*-algebras and the rings of operators previously studied. Used in the proof of the G-N theorem is the Gelfand-Naimark-Segal construction, a pair of results relating cyclic $*$-representations of C*-algebras to certain linear functionals on that algebra.

1.2. **Background Mathematics and Resources.** The following is some mathematics which may prove useful throughout the project, with relevant resources; we will of course be making definitions as needed, this is just for further background and related theory.

We will be assuming some familiarity with the following theory, giving some explanation as necessary as we go:

- Rings, algebras and linear spaces.
- Normed spaces, inner product spaces, Banach and Hilbert spaces.
- Point-set topology.

A good broad background on all of these can be found in [10].

Some texts which cover C*-algebras: [3] is a summary of the general theory up to that time, with [4] focussing on reworking and developing the theory of von Neumann algebras. Sakai [8] gives a treatment of C*- and von Neumann algebras from a more topological point of view.

1.3. **C*-Algebras.** A brief overview of what a C*-algebra is.

- banach space
- banach algebra
- *-algebra
- C* algebra
- representations

Definition 1. A *C*-algebra* is a Banach algebra $(A, \|\cdot\|)$ with involution $* : A \to A$ with the condition that

$$\|x^*x\| = \|x\|^2 \text{ for all } x \in A.$$

This condition is known as the *C* axiom.* The study of C*-algebras started with the consideration of matrix mechanics by H

Theorem (Gelfand-Naimark, Commutative)**.** *Every commutative, unital C*-algebra A is isometrically ∗-isomorphic to the algebra of continuous functions on the algebra of characters on A.*

The theorem first appeared in [gelfand-naimark] in a form relating 'normed rings' and a closed subrings of the set of bounded operators on a Hilbert space.

1.4. **Aims.** The aims for my project are, provisionally:

- Take the Gelfand-Naimark theorem, and understand its contents and proof.
- Consider the representation theory of C*-algebras, using the Gelfand-Naimark-Segal construction as a starting point.
- (as 'further reading') Give an overview of areas in which operator algebra theory can be taken (for example, Cuntz algebras and operator K-theory, von Neumann algebras and Factors, abstract harmonic analysis).

References

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