C^*-algebras, and the Gelfand-Naimark Theorems

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Chapter 1

Introduction

1.1 History of The Study of Operator Algebras

The noncommutative nature of Werner Heisenberg's work in 1925 on a new quantum mechanics [11] led to Born and Jordan [2], together with Heisenberg [3], developing the matrix mechanics required to concisely summarise the new quantum mechanical model. From 1935-1943, John von Neumann, together with F.J. Murray, developed the theory of *rings of operators* acting on a Hilbert space [18, 19, 20, 27], in an attempt to establish a general framework for this matrix mechanics. These rings of operators are now considered part of the theory of *von Neumann algebras*, a subsection of C^*-algebra theory. Discussion of the seminal quantum mechanical works of Heisenberg can be found in [17], and similarly [25] gives a summary of the works of Jordan expanding on this.

In 1943 [10], Gelfand and Naimark established an abstract characterisation of C^*-algebras, free from dependence on the operators acting on a Hilbert space. The Gelfand-Naimark theorem, which we will be considering here at length, gives the link between these abstract C^*-algebras and the rings of operators previously studied. Used in the proof of the GN theorem is the Gelfand-Naimark-Segal construction, a pair of results relating cyclic $*$-representations of C^*-algebras to certain linear functionals on that algebra.

1.2 Background Mathematics and Resources.

The following is some mathematics which may prove useful throughout the project, with relevant resources; we will of course be making definitions as needed, this is for further background and related theory.

We will be assuming some familiarity with the following theory, giving some explanation as necessary:

- Rings, algebras and linear spaces.
- Normed spaces, inner product spaces, Banach and Hilbert spaces.
- Point-set topology.

A good broad background on all of these can be found in [26].

Some texts which cover C^*-algebras: Dixmier [5] presents a summary of the general theory up to that time (1977), with [6] focussing on reworking and developing the theory of von Neumann algebras. Sakai [24] gives a treatment of C^*- and von Neumann algebras from a more topological point of view. In [12, 13], the authors aim to make accessible the “vast recent research literature” in this subsection of functional analysis. Blackadar [1] gives a much faster, more encyclopaedic coverage of the theory of operator algebras, and covering more specialised material and applications.

move 'background mathematics' to 'definitions and results'

1.3 Summary of project

The aims for this project are:

- Give a good background understanding on C^*-algebras, including topological and geometric interpretation of results where possible.
- Consider the representation theory of C^*-algebras, using the Gelfand-Naimark-Segal construction as a starting point.
- Consider the commutative and general versions of the Gelfand-Naimark theorem, and understand their contents and proof.

rewrite as a summary of project chapter by chapter.

Chapter 2

some definitions and results

The following notation applies throughout: $\mathbb{K}$ denotes a field. $\mathbb{R}^+$ denotes the non-negative reals. An ideal of an algebra is, unless otherwise stated, taken to mean a two-sided ideal.

A *partial order* on a set X is a relation $\leq$ which satisfies the conditions

- $x \leq x$ for all x in X; (reflexivity)
- for all x and y in X, if $x \leq y$ and $y \leq x$, then $x = y$; (antisymmetry)
- for all x, y and z in X, if $x \leq y$ and $y \leq z$, then $x \leq z$. (transitivity)

2.1 Normed linear spaces

Given a linear space X over a field $\mathbb{K}$, a *linear functional* on X is a linear map $\rho : X \to \mathbb{K}$. The set X^* of linear functionals on X is itself a linear space, called the *dual space* of X. A linear functional ρ is called *multiplicative* if $\rho(xy) = \rho(x)\rho(y)$ for all x, y in X.

A normed linear space $(V, \|\cdot\|)$ is a *Banach space* if it is complete with respect to $\|\cdot\|$, in the sense that all Cauchy sequences in V converge with respect to the norm. An inner product space $(V, \langle\cdot,\cdot\rangle)$ is a *Hilbert space* if it is a Banach space with respect to the norm induced by $\langle\cdot,\cdot\rangle$.

Theorem (Hahn-Banach Extension theorem)**.** *If ρ_0 is a bounded linear functional on a subspace X_0 of a normed linear space X, then there is a bounded linear functional ρ on X such that $\|\rho\| = \|\rho_0\|$ and $\rho = \rho_0$ on X_0.*

Proof. Can be found in [12, Theorem 1.6.1, p. 44] $\square$

2.2 Constructions on Hilbert spaces

We need some Hilbert spaces constructions; particularly, the direct sum of a collection of Hilbert spaces and the direct sum of bounded operators on these Hilbert spaces. Given a finite collection $\{\mathcal{H}_1, \dots, \mathcal{H}_n\}$ of Hilbert spaces, let $\mathcal{H}$ denote the set

$$\mathcal{H} = \{(x_1, \dots, x_n) \mid x_i \in \mathcal{H}_i \text{ for } i = 1, \dots, n\}.$$

Define addition and scalar multiplication coordinatewise, and given $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n) \in \mathcal{H}$, the equation

$$\langle x, y \rangle = \langle x_1, y_1 \rangle + \dots + \langle x_n, y_n \rangle$$

defines an inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{H}$. The resulting norm $\| \cdot \|$ is given by

$$\|x\|^2 = \|x_1\|^2 + \dots + \|x_n\|^2.$$

It is easy to show that $\mathcal{H}$ is a Hilbert space with these operations, and we call it the *direct sum* of the collection $\{\mathcal{H}_1, \dots, \mathcal{H}_n\}$, denoted $\oplus \mathcal{H}_i$.

Similarly, we can construct a Hilbert space direct sum of an infinite collection $\{\mathcal{H}_i \mid i \in I\}$ of Hilbert spaces. Let $\mathcal{H}$ be the set

$$\mathcal{H} = \{(x_i) \mid x_i \in \mathcal{H}_i \text{ for each } i \text{ and } \sum_{i \in I} \|x_i\|^2 < \infty\}.$$

Given $x = (x_i)$ and $y = (y_i) \in \mathcal{H}$, we have that

$$\begin{aligned}
\Big(\sum_{i \in I} \|x_i + y_i\|^2 \Big)^{1/2} &\leq \Big(\sum_{i \in I} (\|x_i\| + \|y_i\|)^2 \| \Big)^{1/2} \\
&\leq \Big(\sum_{i \in I} \|x_i\|^2 \Big)^{1/2} + \Big(\sum_{i \in I} \|y_i\|^2 \Big)^{1/2} \\
&< \infty.
\end{aligned}$$

Hence, the sequence $(x_i + y_i)$ is in $\mathcal{H}$, and we can define addition and scalar multiplication coordinatewise on $\mathcal{H}$:

$$(x_i) + (y_i) = (x_i + y_i) \qquad\qquad \alpha(x_i) = (\alpha x_i).$$

We also have

$$\begin{aligned}
\sum_{i \in I} |\langle x, y \rangle| &\leq \sum_{i \in I} \|x_i\| \|y_i\| \\
&\leq \Big(\sum_{i \in I} \|x_i\|^2 \Big)^{1/2} \Big(\sum_{i \in I} \|y_i\|^2 \Big)^{1/2} \\
&< \infty,
\end{aligned}$$

so that we can define an inner product $\langle \cdot, \cdot \rangle$, with induced norm $\| \cdot \|$, on $\mathcal{H}$ by

$$\langle x, y \rangle = \sum_{i \in I} |\langle x, y \rangle|, \qquad \|x\| = \Big(\sum_{i \in I} \|x_i\|^2 \Big)^{1/2}.$$

To see that $\mathcal{H}$ is complete with respect to $\| \cdot \|$, suppose that $(x^n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}$, where $x^n = (x_i^n)_{i \in I}$ for each n. Then given any positive ϵ, there exists a positive integer N such that

$$\|x^m - x^n\| < \epsilon \text{ for all } m, n \geq N,$$

that is,

$$\sum_{i \in I} \|x_i^m - x_i^n\|^2 < \epsilon^2 \text{ for all } m, n \geq N. \tag{2.1}$$

Hence for each $i \in I$,

$$\|x_i^m - x_i^n\| < \epsilon \text{ for all } m, n \geq N$$

so that $(x_i^n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}_i$, having a limit $x_i \in \mathcal{H}_i$. For any finite subset $J \subset I$, it follows from (2.1) that

$$\sum_{j \in J} \|x_j^m - x_j^n\|^2 < \epsilon^2 \text{ for all } m, n \geq N,$$

and letting m tend to infinity,

$$\sum_{j \in J} \|x_j - x_j^n\|^2 < \epsilon^2 \text{ for all } n \geq N. \tag{2.2}$$

This holds for any finite subset J, so

$$\sum_{i \in I} \|x_i - x_i^n\|^2 < \epsilon^2 \text{ for all } n \geq N,$$

and so $(x_i - x_i^n)$ and x_i^n are in $\mathcal{H}$ for $n \geq N$. Then (x_i) is in $\mathcal{H}$ and by (2.2), x^n converges to (x_i) as n tends to infinity. We conclude that $\mathcal{H}$ is complete and therefore a Hilbert space. Just like in the finite case, we call $\mathcal{H}$ the *direct sum* of the collection $\{\mathcal{H}_i \mid i \in I\}$ of Hilbert spaces, denoted $\oplus \mathcal{H}_i$.

Suppose now we have a (finite or infinite) collection of bounded operators $\{T_i \in \mathcal{B}(\mathcal{H}_i) \mid i \in I\}$ such that

$$\sup_{i \in I} \|T_i\| < \infty$$

(by convention this is true when I is a finite set). For $x = (x_i)$ in $\oplus\mathcal{H}_i$, define an element Tx in $\oplus\mathcal{H}_i$ by $Tx = (T_i x_i)$. Then $T : \mathcal{H} \to \mathcal{H} : x \mapsto Tx$ is a bounded linear operator, called the *direct sum* of the collection $\{T_i \in \mathcal{B}(\mathcal{H}_i) \mid i \in I\}$, denoted $\oplus T_i$. For S_i, T_i in $\mathcal{B}(\mathcal{H}_i)$, α, β in $\mathbb{C}$, we have

$$\begin{aligned}(\oplus T_i)^* &= \oplus T_i^*,\\ \oplus(\alpha S_i + \beta T_i) &= \alpha \oplus S_i + \beta \oplus T_i,\\ \oplus(S_i T_i) &= \oplus S_i \oplus T_i,\\ \| \oplus T_i\| &= \sup_{i\in I} \|T_i\|.\end{aligned}$$

2.3 Topological stuff

define: hausdorff/compact/locally compact/denseness/uniform continuity

A *topological vector space* is a vector space V over $\mathbb{K}$, together with a topology on V such that the vector space operations $V \times V \to V : (x, y) \mapsto x + y$ and $\mathbb{K} \times V \to V : (\lambda, x) \mapsto \lambda x$ are continuous. Recall that an *extreme point* of a convex subset X_0 of a topological vector space is a point x for which an expression

$$x = \alpha x_1 + (1 - \alpha)x_2,$$

for $0 \leq \alpha \leq 1$ and $x_1, x_2 \in X_0$, implies that $x_1 = x = x_2$. The convex set X_0 is then equal to the set of all linear combinations

$$\alpha_1 x_1 + \cdots + \alpha_n x_n$$

of its extreme points, where $\alpha_1, \ldots, \alpha_n$ are positive scalars summing to 1. For example, for a polygon embedded in $\mathbb{R}^2$, the vertices of a polygon are its extreme points, and every point within the polygon can be written as a linear combination of the vertices.

Definition ([23, 3.14])**.** Let X be a topological vector space (over a field $\mathbb{K}$), with dual space X^*. Every x in X induces a linear functional f_x on X^* defined by $f_x(\rho) = \rho(x)$. The *weak* topology* on X^* is the topology generated by the sets

$$\{f_x^{-1}(V) \mid x \in X, V \subseteq \mathbb{K} \text{ open}\}.$$

This is the weakest topology on X such that each functional f_x is continuous, and the collection $\{f_x \mid x \in X\}$ separate the points of X^*, so the weak* topology is Hausdorff.

Theorem (The Banach-Alaoglu theorem [23]). *If V is a neighbourhood of* 0 *in a topological vector space X, and*

$$K = \{\rho \in X^* \mid |\rho(x)| \geq 1 \textit{ for all } x \in V\},$$

then K is compact in the weak topology.*

Chapter 3

C^*-algebras

We begin this chapter with some definitions and an example, then give some results we will need later and finish with a fundamental example.

Definition 1. A *Banach algebra* is a complex Banach space A with norm $\|\cdot\|$ which forms an algebra, such that

$$\|ab\| \leq \|a\|\|b\| \text{ for all } a, b \in A.$$

A $*$-*algebra* is an algebra A with an *involution* map $a \mapsto a^*$ on A such that, for all $a, b \in A$ and for $\alpha \in \mathbb{C}$,

(i) $a^{**} = (a^*)^* = a$,

(ii) $(\alpha a + b)^* = \overline{\alpha} a^* + b^*$,

(iii) $(ab)^* = b^* a^*$.

The element a^* is referred to as the *adjoint* of a.
A C^*-*algebra* is a Banach algebra A with involution $a \mapsto a^*$ making it a $*$-algebra, with the condition that

$$\|a^* a\| = \|a\|^2 \text{ for all } a \in A.$$

This condition is known as the C^* *axiom*.
A subalgebra B of a C^*-algebra A is a C^*-*subalgebra* if it is closed under the adjoint and complete with respect to the norm; equivalently, if B is itself a C^*-algebra.

An element $a \in A$ of a C^*-algebra is called

- *normal* if $a^* a = a a^*$;

- *self-adjoint* if $a^* = a$.

The set of self-adjoint elements A_{sa} forms a vector space over $\mathbb{R}$. For a in A, let $h = \frac{1}{2}(a + a^*)$ and $k = \frac{i}{2}(a - a^*)$. Then h and k are self-adjoint and we can write $a = h + ik$ – call h and k the *real* and *imaginary* part of a, respectively.

3.1 Unitization

If a C^*-algebra A contains an identity element $\mathbb{1}$ such that $a\mathbb{1} = a = \mathbb{1}a$ for all $a \in A$, call $\mathbb{1}$ the *unit* in A, and A is then a *unital* C^*-algebra.

Proposition ([4, I.1.3])**.** *Any non-unital C^*-algebra A can be isometrically embedded in a unital C^*-algebra $\tilde{A}$.*

Proof. Let $\tilde{A} = A \oplus \mathbb{C}$ with pointwise addition, and define

$$\begin{aligned}(a, \lambda)(b, \mu) &= (ab + \lambda b + \mu a, \lambda\mu),\\ (a, \lambda)^* &= (a^*, \overline{\lambda}).\end{aligned}$$

Then $\tilde{A}$ is a $*$-algebra. Consider (a, λ) as an operator acting on A via $b \mapsto ab + \lambda b$ for b in A. It is easy to see that this operator is linear, and

$$\begin{aligned}\sup_{\|b\|=1} \|ab + \lambda b\| &\leq \sup_{\|b\|=1} (\|ab\| + \lambda\|b\|)\\ &= \sup_{\|b\|=1} \|ab\| + \lambda < \infty\end{aligned}$$

so that the operator (a, λ) is bounded. It follows that

$$\|(a, \lambda)\| = \sup_{\|b\|=1} \|ab + \lambda b\|$$

is a norm on $\tilde{A}$, and so $\tilde{A}$ is a Banach $*$-algebra, with unit $(0, 1)$. The embedding $a \mapsto (a, \lambda)$ is isometric because

$$\|a\| = \|a \cdot \frac{a}{\|a\|}\| \leq \|(a, 0)\| \leq \sup_{\|b\|=1} \|ab\| \leq \|a\|.$$

It remains to verify the C^*-axiom:

$$\begin{aligned}
\|(a,\lambda)\|^2 &= \sup_{\|b\|=1} \|ab+\lambda b\|^2 \\
&= \sup_{\|b\|=1} \|b^*a^*ab+\lambda b^*a^*b+\overline{\lambda}b^*ab+|\lambda|^2b^*b \\
&\leq \sup_{\|b\|=1} \|a^*ab+\lambda a^*b+\overline{\lambda}ab+|\lambda|^2b\| \\
&= \|(a^*a+\lambda a^*+\overline{\lambda}a, |\lambda|^2) \\
&= \|(a,\lambda)^*(a,\lambda)\| \\
&\leq \|(a,\lambda)^*\|\|(a,\lambda)\|.
\end{aligned}$$

By symmetry of $*$, $\|(a,\lambda)^*\| = \|(a,\lambda)\|$. Hence, the above inequality becomes equality and we have that

$$\|(a,\lambda)^*(a,\lambda)\| = \|(a,\lambda)\|^2,$$

and $\tilde{A}$ is a C^*-algebra. □

In light of this result, we take all C^*-algebras from here to be unital unless specified otherwise. For the results we will consider, we can simply consider the unital case. However, there are circumstances in advanced theory in which one needs to relax the unital condition, where properties do not pass from a C^*-algebra to its maximal ideals, so it is by no means a universal assumption that a C^*-algebra has a unit element.

3.2 $C(X)$ - an example.

Given a compact Hausdorff space X, let $C(X)$ be the algebra of continuous functions $f : X \to \mathbb{C}$, with addition and multiplication defined pointwise. Define $\|\cdot\|$ on $C(X)$ by

$$\|f\| = \sup_{x\in X} |f(x)|,$$

and let $\overline{f}$ be the function $\overline{f}(x) = \overline{f(x)}$ for x in X. We will demonstrate that $C(X)$ is an Abelian C^*-algebra and use it to prototype the theory we will build up in later sections. Later, in Section 4.1, we will show that $C(X)$ is essentially *the* Abelian C^*-algebra.

Claim. *The set $C(X)$ of continuous complex valued functions on a compact Hausdorff space X is an Abelian C^*-algebra.*

Proof. We take $C(X)$ as an algebra with operations defined pointwise and norm defined as above. Multiplication in $C(X)$ is pointwise, so analogous to multiplication in $\mathbb{C}$, hence commutative. Showing that $C(X)$ is a Banach space with this norm is a simple exercise. We will demonstrate that $C(X)$ is a Banach algebra, that $f \mapsto \overline{f}$ is an involution, and that the norm satisfies the C^* axiom. Given f and g in $C(X)$, we have

$$\begin{aligned}\|fg\| &= \sup_{x\in X} |(fg)(x)| \\ &= \sup_{x\in X} |f(x)g(x)| \\ &\leq \sup_{x\in X} |f(x)||g(x)| \\ &= \|f\|\|g\|.\end{aligned}$$

Since $f \mapsto \overline{f}$ is simply pointwise complex conjugation, and the algebraic operations are defined pointwise, it is clear that it is an involution. For the C^* axiom: given f in $C(X)$,

$$\begin{aligned}\|\overline{f}f\| &= \sup_{x\in X} |(\overline{f}f)(x)| \\ &= \sup_{x\in X} |\overline{f(x)}f(x)| \\ &= \sup_{x\in X} |f(x)|^2 \\ &= \|f\|^2.\end{aligned}$$

The constant function $\mathbb{1} : X \to \mathbb{C} : x \mapsto 1$ is the unit in X, with $\|\mathbb{1}\| = 1$. $\square$

Suppose f is a self-adjoint element of $C(X)$ – that is to say, $\overline{f} = f$. Then $\overline{f(x)} = f(x)$ for all x in X, so f is real-valued on X. The set of all self-adjoints in $C(X)$ is the algebra $C(X, \mathbb{R})$ of all real valued functions on X. Call f *positive* if $f(x)$ is positive for all x in X. A linear functional ρ on $C(X)$ is *positive* if $\rho(f) \geq 0$ for all positive elements f, and a *state* on $C(X)$ if ρ is positive and $\rho(\mathbb{1}) = 1$. The *state space* $\mathscr{S}$, the set of all states on $C(X)$, is a convex set, and extreme points of $\mathscr{S}$ are called *pure states* on $C(X)$.

Given x in X, define ρ_x by $\rho_x : C(X) \to \mathbb{C} : f \mapsto f(x)$. If f is positive, then $\rho_x(f) \geq 0$, and $\rho_x(\mathbb{1}) = \mathbb{1}(x) = 1$, so ρ_x is a state for all x. We can also see that these functionals are multiplicative, and it can be shown (see [12, Corollary 3.4.2]) that every non-zero multiplicative functional on $C(X)$ arises in this way, as 'evaluation' at some point of X, and the general fact that the non-zero multiplicative linear functionals are exactly the pure states on a C^*-algebra (which we will show in Proposition 5) shows that these are in fact all of the pure states on $C(X)$.

We finish this section by remarking that if we take X to be a noncompact, *locally* compact space, then we are required to take $C(X)$ to be the set of continuous complex valued functions which *vanish at infinity* in some sense[1]. With this restriction, we lose the constant function $\mathbb{1}$, so $C(X)$ is no longer unital. It turns out that the unitization of $C(X)$ corresponds to the continuous functions on the one-point compactification of X – see [reference].

3.3 The Spectrum

write spectral theory section.

We will just state the necessary definitions and results here, and defer proofs to sources.

Definition 2. Given an element a of a Banach algebra A, define its spectrum $\sigma_A(a)$:

$$\sigma_A(a) = \{\lambda \in \mathbb{C} \mid a - \lambda\mathbb{1} \text{ is not invertible in } A\},$$

where a is *invertible in* A if there exists an element b such that $ab = \mathbb{1} = ba$. Say that $\lambda \in \sigma(a)$ is a *spectral value* of a in A. If it is clear from context where we are taking the spectrum, then we write $\sigma(a)$ for $\sigma_A(a)$.

Theorem ([12, 3.2.3])**.** *If a is an element of a Banach algebra A then $\sigma(a)$ is a non-empty closed subset of the closed disk in $\mathbb{C}$ with center 0 and radius $\|a\|$.*

The *spectral radius*, $r(a)$, of a is the radius of the smallest disk in $\mathbb{C}$ containing $\sigma(a)$; that is,

$$r(a) = \sup_{\lambda \in \sigma(a)} |\lambda|.$$

By the above theorem, $r(a) \leq \|a\|$.

Theorem (Spectral radius formula [12, 3.3.3])**.** *The spectral radius of an element a of a Banach algebra is given by the formula*

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}.$$

[1]We say $f : X \to \mathbb{C}$ *vanishes at infinity* if for all positive ϵ there exists a compact subset K of X such that $|f(x)| < \epsilon$ for all x outside of K.

Lemma 1 ([12, 4.1.1(i)])**.** *If a is a normal element of in a C^*-algebra, then $r(a) = \|a\|$.*

Given a self-adjoint element a, we denote by $C(\sigma(a))$ the C^*-algebra of continuous, complex valued functions on the spectrum of a. There is a unique continuous mapping $f \mapsto f(a)$ into A such that

continuous functional calculus.

Theorem (Spectral mapping theorem [12, 4.1.6])**.** *If a is a self adjoint element of a C^*-algebra A, and $f \in C(\sigma(a))$, then*

$$\sigma(f(a)) = \{f(t) \mid t \in \sigma(a)\}.$$

Proposition 1 ([12, 4.2.3(i)])**.** *Given f in $C(\sigma(a))$, $f(a)$ is positive if and only if $f(t) \geq 0$ for all t in $\sigma(a)$.*

Theorem ([12, 4.1.5])**.** *If B is a C^*-subalgebra of a C^*-algebra A, then $\sigma_B(b) = \sigma_A(b)$ for all b in B.*

With commuting, self-adjoint elements $a, b \in A_{sa}$, we have $(ab)^* = b^* a^* = ba = ab$, so ab is self-adjoint. Since each of a, b, ab have the same spectrum in A as in the Abelian C^*-subalgebra $C^*(\mathbb{1}, a, b)$, we have

$$\begin{aligned} \sigma(ab) &\subseteq \sigma(a)\sigma(b), \text{ and} \\ r(ab) &\leq r(a)r(b). \end{aligned}$$

this is 3.2.10/3.3.4. used in 4.1.1

3.4 Further Definitions

An element a of a C^*-algebra A is *positive* if it is self-adjoint and $\sigma(a) \subseteq \mathbb{R}^+$. Denote the set of positive elements in A by A^+. Then there is a partial order $\leq$ on A_{sa} defined by

$$a \leq b \iff b - a \text{ is positive.}$$

The set of positive elements form a *positive cone* in A_{sa}, which means that

(i) $a \in A^+$ and $-a \in A^+$ implies that $a = 0$,

(ii) $\alpha a + b \in A^+$ for all $a, b \in A^+$ and $\alpha \in \mathbb{R}^+$.

The unit $\mathbb{1}$ is positive (with $\sigma(\mathbb{1}) = \{1\}$).

Lemma 2 ([12, 4.2.3(ii)])**.** *For a self-adjoint element a in a C^*-algebra A,*

$$-\|a\|\mathbb{1} \leq a \leq \|a\|\mathbb{1}.$$

Proof. Let f in $C(\sigma(a))$ defined by $f(t) = \|a\| \pm t$, f takes non-negative values on $\sigma(a)$, so by Proposition 1, $f(a)$ is positive; that is, $\|a\|\mathbb{1} \pm a$ is positive. ☐

Definition 3. Given Banach $*$-algebras A and B, a map $\varphi : A \to B$ is a $*$-*homomorphism* if it is an algebra homomorphism for which $\varphi(a^*) = \varphi(a)^*$ for all $a \in A$. If a $*$-homomorphism is one-to-one, call it a $*$-*isomorphism*.

Proposition 2 ([12, 4.1.8])**.** *Suppose A and B are C^*-algebras and $\varphi : A \to B$ is a $*$-homomorphism. Then $\|\varphi(a)\| \leq \|a\|$ for all $a \in A$. If φ is a $*$-isomorphism, then $\|\varphi(a)\| = \|a\|$ for all $a \in A$.*

Proof. ☐

prove

Definition 4. A linear functional ρ on a C^*-algebra A is *positive* if $\rho(a) \geq 0$ for all $a \in A^+$. A *state* on A is a positive linear functional ρ such that $\rho(\mathbb{1}) = 1$. Denote by $\mathscr{S}(A)$ the set of all states on A.

states are real on self-adjoint elements

We can extend the partial order notation to linear functionals. Given linear functionals ρ, τ on A, define $\leq$ by

$$\rho \leq \tau \iff \tau - \rho \text{ is positive.}$$

Some of the concepts here may be very familiar to the reader as extensions of the theory of complex numbers – $\mathbb{C}$ is in fact a C^*-algebra itself. We know from complex analysis that Cauchy sequences converge in $\mathbb{C}$ and

geometric interpretation - for $A = \mathbb{C}$, self adjoint elements are real numbers and the positive cone is $\mathbb{R}^+$.

Proposition 3 ([12, 4.3.2])**.** *A linear functional ρ on a C^*-algebra A is positive if and only if ρ is bounded and $\|\rho\| = \rho(\mathbb{1})$.*

Proof. rework; explain further.

Suppose that ρ is positive. With a in A, let α be a scalar of modulus 1 such that $\alpha\rho(a) \geq 0$, and let h be the real part of a. Since h is self adjoint, (by 4.2.3(ii)) we have $h \leq \|h\|\mathbb{1} \leq \|a\|\mathbb{1}$. Thus, $\|a\|\mathbb{1} - h$ is positive and

why does this exist?

$$\rho(\|a\|\mathbb{1} - h) = \|a\|\rho(\mathbb{1}) - \rho(h) \geq 0.$$

Therefore,

$$|\rho(a)| = \rho(\alpha a) = \overline{\rho(\alpha a)} = \rho(\overline{\alpha} a^*) = \rho(\tfrac{1}{2}(\alpha a + \overline{\alpha} a^*)) = \rho(h) \leq \rho(\mathbb{1})\|a\|,$$

so ρ is bounded and $\|\rho\| \leq \rho(\mathbb{1})$. We also have $\|\rho\| = \sup\{\rho(a) \mid \|a\| = 1\} \geq \rho(\mathbb{1})$, so $\|\rho\| = \rho(\mathbb{1})$.

Conversely, suppose ρ is bounded and $\|\rho\| = \rho(\mathbb{1})$ – we can assume without loss that $\rho(\mathbb{1}) = 1$. With a a positive element of A, let $\rho(a) = \alpha + i\beta$ for real α, β. Then ρ is positive if and only if $\alpha \geq 0$ and $\beta = 0$. For small positive s,

$$\sigma(\mathbb{1} - sa) = \{1 - st \mid t \in \sigma(a) \subseteq \mathbb{R}^+\} \subseteq [0, 1],$$

so $\|\mathbb{1} - sa\| = r(\mathbb{1} - sa) \leq 1$. Hence

$$1 - s\alpha \leq |1 - s(\alpha + i\beta)| = |\rho(\mathbb{1} - sa)| \leq 1,$$

so $\alpha \geq 0$. With b_n in A defined by $b_n = a + (in\beta - \alpha)\mathbb{1}$ for each positive integer n,

$$\begin{aligned} \|b_n\|^2 = \|b_n^* b_n\| &= \|(a - \alpha\mathbb{1})^2 + n^2\beta^2\mathbb{1}\| \\ &\leq \|a - \alpha\mathbb{1}\|^2 + n^2\beta^2. \end{aligned}$$

Hence for all positive integers n, we have

$$\begin{aligned} (n^2 + 2n + 1)\beta = |\rho(b_n)|^2 & \\ &\leq \|a - \beta\mathbb{1}\|^2 + n^2\beta^2, \end{aligned}$$

so that $\beta = 0$. We conclude that ρ is positive. $\square$

Proposition (Cauchy-Schwarz inequality, [12, 4.3.1])**.** *If ρ is a positive linear functional on C^*-algebra A, then for all a and b in A,*

$$|\rho(b^* a)|^2 \leq \rho(a^* a)\rho(b^* b).$$

prove?

Lemma 3 ([12, 4.3.3])**.** *Let A be a C^*-algebra. For any a in A and $\alpha \in \sigma(a)$, there exists a state ρ on A such that $\rho(a) = \alpha$.*

Proof. For all complex numbers β and γ, $\alpha\beta + \gamma$ is a spectral value for the element $\beta a + \gamma\mathbb{1}$ of A, so

$$|\alpha\beta + \gamma| \leq r(\beta a + \gamma\mathbb{1}) = \|\beta a + \gamma\mathbb{1}\|.$$

Hence the equation $\rho_0(s) = \alpha\beta + \gamma$ defines a bounded linear functional ρ_0 on the linear subspace $B = \{\beta a + \gamma\mathbb{1} \mid \beta, \gamma \in \mathbb{C}\}$ of A, with $\rho_0(a) = \alpha$ and $\rho_0(\mathbb{1}) = 1 = \|\rho_0\|$. By the Hahn-Banach theorem, ρ_0 extends to a bounded linear functional ρ on A, with $\|\rho\| = 1$, such that $\rho = \rho_0$ on the subspace B. In particular, $\rho(\mathbb{1}) = 1 = \|\rho\|$ so ρ is positive by the previous result, and $\rho(a) = \alpha$. $\square$

Lemma 4 ([12, 4.3.4,(i)]). *Let A be a C^*-algebra. If $\rho(a) = 0$ for all states ρ on A, then $a = 0$.*

Proof. Suppose first that a is self-adjoint, and $\rho(a) = 0$ for all states ρ. By the previous result, $\sigma(a) = \{0\}$, so $\|a\| = r(a) = 0$. Hence $a = 0$.
Now write $a = h+ik$, for h and k the real and imaginary part of a respectively. Then

$$0 = \rho(a) = \rho(h) + i\rho(k),$$

and as h, k are self-adjoint, $\rho(h)$ and $\rho(k)$ are real and we must have $\rho(h) = 0 = \rho(k)$. By previous statement, $h = 0 = k$, and we conclude that $a = 0$. $\square$

Definition 5. An extreme point of $\mathscr{S}(A)$ is called a *pure* state on A, and the set of pure states on A is denoted by $\mathscr{P}(A)$.

The state space $\mathscr{S}(A)$ is then the weak*-closure of the set of convex linear combinations of pure states. It is a simple exercise, using the fact that $A^+ \subseteq A_{sa}$, to verify that a linear functional ρ is a pure state on A if and only if the restriction $\rho|_{A_{sa}}$ is a pure state on A_{sa}. Every pure state on A_{sa} extends to a pure state on A with the same norm, by the Hahn-Banach theorem. We will need the following few results on pure states later.

Lemma 5 ([12, 4.3.8,(i)]). *Let A be a C^*-algebra. If $\rho(a) = 0$ for all* pure *states ρ on A, then $a = 0$.*

Proof. Since every state is the limit of a sequence of linear combinations of pure states, if $\rho(a) = 0$ for all pure states ρ on A, then $\rho(a) = 0$ for *all* states ρ on A. The result follows immediately from Lemma 4. $\square$

Proposition 4 ([12, 3.4.6]). *A state ρ on A_{sa} is pure if and only if, for all positive linear functionals τ on A_{sa} such that $0 \leq \tau \leq \rho$, we have $\tau = \lambda\rho$ for some $\lambda \in \mathbb{R}$.*

Proof. Suppose that $\tau = \lambda\rho$ for all $0 \leq \tau \leq \rho$, and suppose we can write $\rho = \alpha\rho_1 + (1-\alpha)\rho_2$ for some $0 \leq \alpha \leq 1$ and some $\rho_1, \rho_2 \in \mathscr{S}(A_{sa})$. Then $0 \leq \alpha\rho_1 \leq \rho$, so $\alpha\rho_0 = \lambda\rho$. Then $\rho_1(\mathbb{1}) = 1 = \rho(\mathbb{1})$, so $\alpha = \lambda$ and $\rho_0 = \rho$. Similarly, we can show that $\rho_2 = \rho$, and so we conclude that ρ is a pure state.

Conversely, suppose that ρ is a pure state and $0 \leq \tau \leq \rho$. Applying this to $\mathbb{1}$, we get $0 \leq \tau(\mathbb{1}) \leq \rho(\mathbb{1}) = 1$. Let $\lambda = \tau(\mathbb{1})$. We work case-by-case:
If $\lambda = 0$, then for any $a \in A_{sa}$, applying τ to $-\|a\|\mathbb{1} \leq a \leq \|a\|\mathbb{1}$ (from Lemma 2) gives

$$0 = -\|a\|\lambda = \tau(-\|a\|\mathbb{1}) \leq \tau(a) \leq \tau(\|a\|\mathbb{1}) = \|a\|\lambda = 0,$$

so $\tau = 0 = \lambda\rho$.
A similar argument shows that $\lambda = 1$ implies $\tau - \rho = 0$, so that $\tau = \rho = \lambda\rho$. If $0 \leq \lambda \leq 1$, we can write $\rho = \lambda\rho_1 + (1-\lambda)\rho_2$ for $\rho_1 = \lambda^{-1}\tau$ and $\rho_2 = (1-\lambda)^{-1}(\rho - \tau)$. ρ is pure so $\tau = \lambda\rho_1 = \lambda\rho$.

□

Recall that a linear functional ρ on a C^*-algebra A is *multiplicative* if $\rho(ab) = \rho(a)\rho(b)$ for all a, b in A. The set of all non-zero multiplicative linear functionals on A is called the *maximal ideal space* of A, denoted $\mathcal{M}_A$. This name hints at the fact that the kernel of each of these functionals is a maximal ideal of A, and all maximal ideals of A arise in this way. [4, Theorem I.2.5]

Proposition 5 ([12, 4.4.1])**.** *The set of pure states on an Abelian C^*-algebra A is precisely the maximal ideal space of A.*

Proof. Suppose ρ is a pure state on A. To show that $\rho(ab) = \rho(a)\rho(b)$ for $a, b, \in A$, we restrict attention to the case where $0 \leq b \leq \mathbb{1}$. Linearity gives us the general case. In this case, for $h \in A^+$ we have that $0 \leq hb \leq h$, so $0 \leq \rho(hb) \leq \rho(h)$. Hence $\rho_b(a) = \rho(ab)$ for $a \in A$ defines a positive linear functional on A with $\rho_b \leq \rho$. The restriction $\rho|_{A_{sa}}$ is a pure state on A_{sa} and $\rho_b|_{A_{sa}} \leq \rho|_{A_{sa}}$, and it follows from Proposition 4 that $\rho_b|_{A_{sa}} = \alpha\rho|_{A_{sa}}$ for some $\alpha \in \mathbb{R}^+$. Hence $\rho_b = \alpha\rho$ and so for $a \in A$: show

$$\rho(ab) = \rho_b(a) = \alpha\rho(a) = \alpha\rho(\mathbb{1})\rho(a) = \rho_b(\mathbb{1})\rho(a) = \rho(b)\rho(a)$$

Conversely, suppose ρ is a multiplicative linear functional.

show a state.

Suppose we can write $\rho = \alpha\rho_1 + \beta\rho_2$ for states ρ_1, ρ_2 on A and $\alpha, \beta > 0$ such that $\alpha + \beta = 1$. For $c \in A_{sa}$, by the Cauchy-Schwarz inequality we have for $j = 1, 2$:

$$\big(\rho_j(c)\big)^2 = \big(\rho_j(\mathbb{1}c)\big)^2 \leq \rho_j(\mathbb{1})\rho(c^2) = \rho(c^2).$$

Then:

$$\begin{aligned}
0 &= \rho(c^2) - \rho(c)^2 \\
&= \alpha\rho_1(c^2) + \beta\rho_2(c^2) - \big(\alpha\rho_1(c) + \beta\rho_2(c)\big)^2 \\
&\geq \alpha(\alpha+\beta)\rho_1(c)^2 + \beta(\alpha+\beta)\rho_2(c)^2 - \big(\alpha\rho_1(c) + \beta\rho_2(c)\big)^2 \\
&= \alpha\beta\big(\rho_1(c) - \rho_2(c)\big)^2.
\end{aligned}$$

Hence $\rho_1(c) = \rho_2(c)$, for all $c \in A_{sa}$, so $\rho_1 = \rho_2$ and we conclude that ρ is a pure state. □

discuss weak* topology on P(S)

3.5 $\mathcal{B}(\mathcal{H})$ - an example.

This section concerns the fundamental example of a C^*-algebra - the set $\mathcal{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$. Here we will demonstrate that $\mathcal{B}(\mathcal{H})$ is a C^*-algebra and give some basic results.

Claim. *$\mathcal{B}(\mathcal{H})$ is a C^*-algebra with the operator norm*

$$\|T\| = \sup_{\|x\|=1} \|Tx\|$$

and involution taking T to its adjoint map T^*. The identity map $I : x \mapsto x$ is a unit for $\mathcal{B}(\mathcal{H})$

Proof. $\|\cdot\|$ is a norm on $\mathcal{B}(\mathcal{H})$. Let $\{T_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{B}(\mathcal{H})$. Then for any positive ϵ, there is a positive integer N such that

$$\|T_m - T_n\| < \epsilon \text{ for all } m, n \geq N.$$

Applying $T_m - T_n$ to $x \in \mathcal{H}$, we have

$$\|T_m x - T_n x\| \leq \|T_m - T_n\|\|x\| < \epsilon\|x\|, \tag{3.1}$$

so $\{T_n x\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}$, converging to an element in $\mathcal{H}$. Define a linear operator $T : \mathcal{H} \to \mathcal{H}$ by

$$Tx = \lim_{n\to\infty} T_n x \text{ for } x \in \mathcal{H}.$$

Taking limits as m tends to infinity in equation (3.1), we obtain

$$\|Tx - T_n x\| < \epsilon\|x\| \text{ for all } n \geq N,$$

and so we have that $T - T_n$ (and hence $T = (T - T_n) + T_n$) is a bounded operator and

$$\|T - T_n\| < \epsilon \text{ for all } n \geq N.$$

We conclude that $T_n \to T$, and so $\mathcal{B}(\mathcal{H})$ is complete.

Since boundedness is equivalent to continuity on $\mathcal{H}$, given $S, T \in \mathcal{B}(\mathcal{H})$, the operator $ST : \mathcal{H} \to \mathcal{H}; x \mapsto (S \circ T)(x)$ is bounded on $\mathcal{H}$. Given $x \in \mathcal{H}$ and $\lambda \in \mathbb{C}$,

$$\begin{aligned}
((\lambda S)T)(x) &= ((\lambda S) \circ T)(x) \\
&= \lambda S(Tx) \\
&= \lambda(S \circ T)(x) \\
&= \lambda ST(x),
\end{aligned}$$

so that $(\lambda S)T = \lambda ST$ in $\mathcal{B}(\mathcal{H})$. We have

$$\begin{aligned}\|ST\| &= \sup_{\|x\|=1} \|STx\| \\ &= \sup_{\|x\|=1} \|S(Tx)\| \\ &\leq \|S\| \sup_{\|x\|=1} \|Tx\| \\ &= \|S\|\|T\|.\end{aligned}$$

We conclude that $\mathcal{B}(\mathcal{H})$ is a Banach algebra.

To see that * is an involution, use the fact that the adjoint operator is unique for each operator and the following equalities.

(i)
$$\begin{aligned}\langle(\alpha T + S)^* x, y\rangle &= \langle x, \alpha T + Sy\rangle \\ &= \overline{\alpha}\langle x, Ty\rangle + \langle x, Sy\rangle \\ &= \overline{\alpha}\langle T^* x, y\rangle + \langle S^* x, y\rangle \\ &= \langle(\overline{\alpha}T^* + S^*)x, y\rangle.\end{aligned}$$

(ii)
$$\begin{aligned}\langle(T^*)^* x, y\rangle &= \langle x, T^* y\rangle \\ &= \overline{\langle T^* y, x\rangle} \\ &= \overline{\langle y, Tx\rangle} \\ &= \langle Tx, y\rangle.\end{aligned}$$

(iii)
$$\begin{aligned}\langle(ST)^* x, y\rangle &= \langle x, STy\rangle \\ &= \langle S^* x, Ty\rangle \\ &= \langle T^* S^* x, y\rangle.\end{aligned}$$

It remains to demonstrate the C^*-axiom on $\mathcal{B}(\mathcal{H})$. For all $x \in \mathcal{H}$, we have

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle \leq \|T^*T\|\|x\|^2,$$

so that

$$\|T\|^2 \leq \|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2.$$

It is clear that I is a unit. Hence, the claim. □

example of $\mathcal{B}(\mathcal{H})$ at end of section to retro-motivate notation. define compact operators and discuss Calkin algebra? discuss nomenclature (state etc) coming from QM

Chapter 4

Representations of C^*-algebras

4.1 Abelian C^*-algebras

rewrite

Let A be an Abelian C^*-algebra. For a in A, define a complex-valued function $\hat{a}$ on $\mathscr{P}(A)$ by $\hat{a}(\rho) = \rho(a)$. The weak*-topology is the weakest topology on $\mathscr{P}(A)$ for which all of the maps $\hat{a}$ are continuous, so that $\hat{a} \in C(\mathscr{P}(A))$ for all $a \in A$. The map

$$\Gamma : A \to C(\mathscr{P}(A)) : a \mapsto \hat{a}$$

is called the *Gelfand transform* of A [4]. For $a, b \in A$, $\alpha, \beta \in \mathbb{C}$ and $\rho \in \mathscr{P}(A)$:

$$\begin{aligned}(\widehat{\alpha a + \beta b})(\rho) = \rho(\alpha a + \beta b) = \alpha\rho(a) + \beta\rho(b) = \alpha\hat{a}(\rho) + \beta\hat{b}(\rho),\\ \widehat{a^*}(\rho) = \rho(a^*) = \overline{\rho(a)} = \overline{\hat{a}(\rho)}.\end{aligned}$$

Since by Proposition 5,

$$\mathscr{P}(A) = \{\rho \in \mathscr{S}(A) \mid \rho(ab) = \rho(a)\rho(b)\},$$

we have that

$$\widehat{(ab)}(\rho) = \rho(ab) = \rho(a)\rho(b) = \hat{a}(\rho)\hat{b}(\rho).$$

Hence the Gelfand transform is a $*$-homomorphism. The following theorem gives us that it is in fact a $*$-isomorphism.

Theorem 1 (Gelfand-Naimark, commutative [12, 4.4.3])**.** *Every Abelian C^*-algebra A is $*$-isomorphic to $C(X)$, the algebra of continuous functions on a compact Hausdorff space X.*

Proof. □

prove this.

The previous result generalises to not-necessarily-unital Abelian C^*-algebras as follows:

Theorem ([4, I.2.7])**.** *Every Abelian C^*-algebra A is $*$-isomorphic to $C_0(X)$, the algebra of continuous functions on a locally compact Hausdorff space X which vanish at infinity.*

Proof. citation needed

□

The unitization of this algebra then corresponds to the one-point compactificaton of X.

4.2 The Gelfand-Naimark Theorem

Definition 6. Given a C^*-algebra A, a *representation of A on a Hilbert space $\mathcal{H}$* is a $*$-homomorphism $\varphi : A \to \mathcal{B}(\mathcal{H})$. A $*$-isomorphic representation is called *faithful*. If there exists an element $x \in \mathcal{H}$ such that the set $\{\varphi(a) \mid a \in A\}$ is dense in $\mathcal{H}$, say that φ is a *cyclic* representation, with *cyclic vector* x.

The construction used in the proof of the following theorem is known as the Gelfand-Naimark-Segal (GNS) construction.

Theorem 2 ([12, 4.5.2])**.** *If ρ is a state on a C^*-algebra A, then there exists a cyclic representation π_ρ of A on a Hilbert space H_ρ, with unit cyclic vector x_ρ, such that*

$$\rho(a) = \langle \pi_\rho(a)x_\rho, x_\rho \rangle, \qquad \forall a \in A.$$

Proof. We will construct from ρ the space $\mathcal{H}_\rho$, representation π_ρ, and vector x_ρ, and demonstrate the required properties.

Consider the *left kernel* of ρ:

$$L_\rho = \{t \in A \mid \rho(t^*t) = 0\}.$$

For $a, b \in A$, define $\langle a, b\rangle_0 = \rho(b^*a)$, then $\langle \cdot, \cdot \rangle_0$ satisfies

(i) Linearity in 1st argument: for $a, b \in A$, $\alpha, \beta \in \mathbb{C}$:

$$\begin{aligned}
\langle \alpha a + \beta b, c\rangle_0 &= \rho(c^*(\alpha a + \beta b)\\
&= \rho(\alpha c^* a + \beta c^* b)\\
&= \alpha\rho(c^*a) + \beta\rho(c^*b)\\
&= \alpha\langle a, c\rangle_0 + \beta\langle b, c\rangle_0.
\end{aligned}$$

(ii) Conjugate symmetric: for $a, b \in A$:

$$\begin{aligned}\langle b, a\rangle_0 &= \rho(a^*b)\\ &= \rho((b^*a)^*)\\ &= \overline{\rho(b^*a)}\\ &= \overline{\langle a, b\rangle_0}.\end{aligned}$$

(iii) Positive semi-definite: for $a \in A$:

$$= \langle a, a\rangle_0 = \rho(a^*a) \geq 0,$$

since a^*a is positive for all a.

show: 4.2.5/6

Note that $\langle\cdot,\cdot\rangle$ is not necessarily positive definite on A – the left kernel is exactly where this fails.

To see that L_ρ is a linear subspace of A, consider

$$L = \{t \in A \mid \langle t, a\rangle_0 = 0,\ \forall a \in A\} \subseteq L_\rho.$$

For $t \in L_\rho$, by the Cauchy-Schwarz inequality we have

$$|\langle t, a\rangle_0|^2 \leq \langle t, t\rangle_0 \langle a, a\rangle_0, \quad \forall a \in A;$$

that is,

$$\langle t, a\rangle_0 = 0, \quad \forall a \in A,$$

so $t \in L$ and $L_\rho = L$. Now, for $a, b \in L$, $\alpha \in \mathbb{C}$ and $c \in A$:

$$\langle \alpha a + b, c\rangle_0 = \alpha\langle a, c\rangle_0 + \langle b, c\rangle_0 = 0,$$

so $\alpha a + b \in L$; also, $\langle 0, c\rangle_0 = 0$ so $0 \in L$. Hence, L_ρ is a linear subspace of A.

For $s \in A$, $t \in L_\rho$, by the Cauchy Schwarz inequality we have

$$\begin{aligned}|\rho(s^*t)|^2 &= |\langle t, s\rangle_0|^2\\ &\leq \langle t, t\rangle_0 \cdot \langle s, s\rangle_0\\ &= \rho(t^*t)\cdot\rho(s^*s)\\ &= 0,\end{aligned}$$

so $\rho(s^*t) = 0$. Letting $s = a^*at$ for $a \in A$, then

$$\begin{aligned}\rho((at)^*at) &= \rho(at^*a^*at)\\ &= \rho((a^*at)^*t)\\ &= \rho(s^*t)\\ &= 0,\end{aligned}$$

so that $at \in L_\rho$, for all $a \in A$ and $t \in L_\rho$; we conclude that L_ρ is a left ideal in A. Closure of L_ρ follows from the fact that it is the preimage in A of $\{0\}$ under the continuous map $t \mapsto \rho(t^*t)$.

why do we need closure?

Consider now the quotient space $V_\rho = A/L_\rho$, with $\langle \cdot, \cdot \rangle$ defined by

$$\langle a + L_\rho, b + L_\rho \rangle = \langle a, b \rangle_0, \qquad \text{for} \quad a + L_\rho, b + L_\rho \in V_\rho.$$

It follows from properties $i)$, $ii)$ and $iii)$ of $\langle \cdot, \cdot \rangle_0$ that $\langle \cdot, \cdot \rangle$ is an inner product on V_ρ – with

$$\begin{aligned} \langle a + L^\rho, a + L^\rho \rangle = 0 &\iff \langle a, a \rangle_0 = 0 \\ &\iff a \in L_\rho \\ &\iff a + L_\rho = 0 + L_\rho \end{aligned}$$

giving positive definiteness. The completion of V_ρ with respect to the induced norm $\| \cdot \|$ is a Hilbert space - this is the Hilbert space $\mathcal{H}_\rho$ we're looking for.

Now we fix $a \in A$, and consider the map

$$\pi_a : V_\rho \to V_\rho; b + L_\rho \mapsto ab + L_\rho.$$

Let $b_1, b_2 \in A$ be such that $b_1 + L_\rho = b_2 + L_\rho$. Then:

$$\begin{aligned} &\implies b_1 - b_2 \in L_\rho \\ &\implies a(b_1 - b_2) \in L_\rho \\ &\implies ab_1 - ab_2 \in L_\rho \\ &\implies ab_1 + L_\rho = ab_2 + L_\rho \\ &\implies \pi_a(b_1 + L_\rho) = \pi_a(b_2 + L_\rho). \end{aligned}$$

Hence π_a defines a linear operator on V_ρ. For $b + L_\rho \in V_\rho$:

$$\begin{aligned} \|a\|^2 \cdot \|b + L_\rho\| - \|\pi_a(b + L_\rho)\| &= \|a\|^2 \cdot \|b + L_\rho\| - \|ab + L_\rho\| \\ &= \|a\|^2 \cdot \langle b + L_\rho, b + L_\rho \rangle - \langle ab + L_\rho, ab + L_\rho \rangle \\ &= \|a\|^2 \cdot \rho(b^*b) - \rho((ab)^*ab) \\ &= \rho(\|a\|^2 b^*b - b^*a^*ab) \\ &= \rho(b^*(\|a\|^2 \mathbb{1} - a^*a)b) \\ &\geq 0. \end{aligned}$$

Thus π_a is a bounded operator on V_ρ, with $\|\pi_a\| \leq \|a\|$. By continuity, π_a extends to a bounded operator $\pi_\rho(a) : \mathcal{H}_\rho \to \mathcal{H}_\rho$, such that

$$\pi_\rho(a)(v) = \pi_a(v)$$

for $v \in V_\rho$. Then $\pi_\rho(a) \in \mathcal{B}(\mathcal{H}_\rho)$ for each $a \in A$, so π_ρ defines a map $A \to \mathcal{B}(\mathcal{H}_\rho)$ such that $a \mapsto \pi_\rho(a)$. This will be our representation.

Now, for $a, b \in A$, $c + L_\rho \in V_\rho$ and $\alpha \in \mathbb{C}$:

$$\begin{aligned}\pi_{\alpha a+b}(c + L_\rho) &= (\alpha a + b)(c + L_\rho)\\ &= (\alpha ac + L_\rho) + (bc + L_\rho)\\ &= \alpha\pi_a(c + L_\rho) + \pi_b(c + L_\rho),\end{aligned}$$

so that $\pi_{\alpha a+b} = \alpha\pi_a + \pi_b$ on V_ρ.
For $a, b \in A$ and $c + L_\rho \in V_\rho$:

$$\begin{aligned}\pi_{ab}(c + L_\rho) &= abc + L_\rho\\ &= \pi_a(bc + L_\rho)\\ &= \pi_a(\pi_b(c + L_\rho))\\ &= (\pi_a \cdot \pi_b)(c + L_\rho),\end{aligned}$$

so that $\pi_{ab} = \pi_a \cdot \pi_b$ on V_ρ.
For $a \in A$ and $b + L_\rho,\ c + L_\rho \in V_\rho$:

$$\begin{aligned}\langle b + L_\rho, \pi_a^*(c + L_\rho)\rangle &= \langle \pi_a(b + L_\rho), c + L_\rho\rangle\\ &= \langle ab + Lr, c + L_\rho\rangle\\ &= \rho(c^*ab)\\ &= \rho((a^*c)^*b)\\ &= \langle b + L_\rho, a^*c + L_\rho\rangle\\ &= \langle b + L_\rho, \pi_{a^*}(c + L_\rho),\end{aligned}$$

so that $\pi_a^* = \pi_{a^*}$ on V_ρ.
$V_\rho \subset \mathcal{H}_\rho$ is a dense subset, so the three properties above hold on $\mathcal{H}_\rho$ by continuity of π_ρ. Hence, $\pi_\rho : A \to \mathcal{B}(\mathcal{H}_\rho)$ is a representation of A. As to the unit vector, consider $x_\rho = \mathbb{1} + L_\rho \in V_\rho$. Then for $a \in A$,

$$\begin{aligned}\langle \pi_\rho(a)x_\rho, x_\rho\rangle &= \langle \pi_a(\mathbb{1} + L_\rho), \mathbb{1} + L_\rho\rangle\\ &= \langle a + L_\rho\mathbb{1} + L_\rho\rangle\\ &= \rho(a);\end{aligned}$$

and in particular, $\langle x_\rho, x_\rho\rangle = \rho(\mathbb{1}) = 1$, so x_ρ is a unit vector in $\mathcal{H}_\rho$. $\square$

example of construction on C(X)? short explanation of how B(H) and C(X) link together. can then talk about noncommutative topology!

Theorem 3 (Gelfand-Naimark, [12, 4.5.6]). *Every C^*-algebra has a faithful representation.*

Proof. Let A be a C^*-algebra. Suppose we have a collection $\{\varphi_i \mid i \in I\}$ of representations of A on Hilbert spaces $\{\mathcal{H}_i \mid i \in I\}$. For a in A, we have $\|\varphi_i(a)\| \leq \|a\|$ (by Proposition 2, as each φ_i is a $*$-homomorphism), so we have a bounded operator $\oplus\varphi_i(a)$ on $\oplus\mathcal{H}_i$ by section 2.2. By the properties of $\oplus\varphi_i(a)$ stated therein, the map

$$\varphi : A \to \mathcal{B}(\oplus\mathcal{H}_i) : a \mapsto \oplus\varphi_i(a)$$

is a $*$-homomorphism, and so is a representation of A on $\oplus\mathcal{H}_i$. Call φ the *direct sum* of the collection $\{\varphi_i \mid i \in I\}$, denoted $\oplus\varphi_i$.

With $\mathscr{S}_0$ any collection of states on A containing $\mathscr{P}(A)$, let φ be the direct sum of the collection $\{\pi_\rho \mid \rho \in \mathscr{S}_0\}$ of representations as constructed by the GNS construction. We will show that φ is a faithful representation.

Given a in A, if $\varphi(a) = 0$ then $\pi_\rho(a) = 0$ for all pure states ρ on A. But then, since $\rho(a) = \langle \pi_\rho(a)x_\rho, x_\rho \rangle$ by the GNS construction, we have $\rho(a) = 0$, and then by Lemma 5, $a = 0$. Thus, φ is one-to-one and hence a faithful representation of A. $\square$

remark on different reps given by different $\mathscr{S}_0$s. the universal rep, the reduced atomic rep

give reference to further topics: von neumann algebras (formal defn), K-theory, group C* algebras, amenable algebras,

references!!!!

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