C*-ALGEBRAS, AND THE GELFAND-NAIMARK THEOREM

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1. Introduction

1.1. **History of C*-Algebras.** The noncommutative nature of W. Heisenberg's work on a new quantum mechanics [7] lead to Born and Jordan [2], together with Heisenberg [3], developing the matrix mechanics required to concisely summarise the new quantum mechanical model. From 1930-1943, J. von Neumann, together with F. J. Murray, developed the theory of *rings of operators* acting on a Hilbert space [14], [15], [17], [16], in an attempt to establish a general framework for this matrix mechanics. These rings of operators are now considered part of the theory of *von Neumann algebras*, a subsection of C*-algebra theory. Discussion of the seminal quantum mechanical works of Heisenberg can be found in [10], and likewise in [12], for the works of Jordan expanding on this.

In [6], Gelfand and Naimark established an abstract characterisation of C*-algebras, free from dependence on the operators acting on a Hilbert space. The Gelfand-Naimark theorem, which we will be considering here at length, gives the link between these abstract C*-algebras and the rings of operators previously studied. Used in the proof of the G-N theorem is the Gelfand-Naimark-Segal construction, a pair of results relating cyclic $*$-representations of C*-algebras to certain linear functionals on that algebra.

1.2. **Background Mathematics and Resources.** The following is some mathematics which may prove useful throughout the project, with relevant resources; we will of course be making definitions as needed, this is just for further background and related theory.

We will be assuming some familiarity with the following theory, giving some explanation as necessary as we go:

- Rings, algebras and linear spaces.
- Normed spaces, inner product spaces, Banach and Hilbert spaces.
- Point-set topology.

A good broad background on all of these can be found in [13].

Some texts which cover C*-algebras: Dixmier [4] presents a summary of the general theory up to that time, with [5] focussing on reworking and developing the theory of von Neumann algebras. Sakai [11] gives a treatment of C*- and von Neumann algebras from a more topological point of view. In [8], [9], the authors aim to make accessible the "vast recent research literature" in this subsection of functional analysis. Blackadar [1] gives a much faster, more encyclopaedic coverage of the theory of operator algebras, and covering more specialised material and applications.

1.3. **C*-Algebras.**

We can define C*-algebras in two ways - either as an algebra of bounded linear operators acting on a Hilbert space satisfying two conditions, or abstractly as a normed algebra together with an involution map satisfying four axioms.

To get to the abstract definition of a C*-algebra, we need to know about the following.

- Banach spaces.
- Banach algebras.
- $*$-algebras.
- Banach $*$-algebras.

Definition 1. A *C*-algebra* is a Banach $*$-algebra $(A, \|\cdot\|)$ with the condition that

$$\|a^*a\| = \|a\|^2 \text{ for all } a \in A.$$

This condition is known as the *C* axiom.* The study of C*-algebras started with the consideration of matrix mechanics by H

Theorem (Gelfand-Naimark, Commutative)**.** *Every commutative, unital C*-algebra A is isometrically $*$-isomorphic to the algebra of continuous functions on the algebra of characters on A.*

The theorem first appeared in [gelfand-naimark] in a form relating 'normed rings' and a closed subrings of the set of bounded operators on a Hilbert space.

1.4. **Aims.**

The aims for this project are, provisionally:

- Take the Gelfand-Naimark theorem, and understand its contents and proof.
- Consider the representation theory of C*-algebras, using the Gelfand-Naimark-Segal construction as a starting point.

1.5. **Further research and applications.**

- Give an overview of areas in which operator algebra theory can be taken (for example, Cuntz algebras and operator K-theory, von Neumann algebras and Factors, abstract harmonic analysis).

References

[1] Blackadar, B., *Operator Algebras: Theory of C*-Algebras and von Neumann Algebras.* Encyclopaedia of Mathematical Sciences, 122. Operator Algebras and Non-commutative Geometry, III. Springer-Verlag, Berlin, 2006.

[2] Born, M. & Jordan, P., *Zur Quantenmechanik.* Z. Physik (1925) 34: 858.

[3] Born, M.; Heisenberg, W. & Jordan, P., *Zur Quantenmechanik. II.* Z. Physik (1926) 35: 557.

[4] Dixmier, J., *C-algebras.* Holland Mathematical Library, Vol. 15. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977

[5] Dixmier, J., *von Neumann algebras.* North-Holland Publishing Co., Amsterdam-New York, 1981.

[6] Gelfand, I. & Neumark, M., *On the imbedding of normed rings into the ring of operators in Hilbert space.* Rec. Math. [Mat. Sbornik] N.S. 12(54), (1943), pp. 197-213.

[7] Heisenberg, W., *Über quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen.* Z. Physik (1925) 33: 879.

[8] Kadison, R. V. & Ringrose, J. R., *Fundamentals of the theory of operator algebras: Vol. I. Elementary theory.* Pure and Applied Mathematics, 100. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, 1983.

[9] Kadison, R. V. & Ringrose, J. R., *Fundamentals of the theory of operator algebras: Vol. II. Advanced theory.* Pure and Applied Mathematics, 100. Academic Press, Inc., Orlando, FL, 1986.

[10] MacKinnon, E., *Heisenberg, Models, and the Rise of Matrix Mechanics.* Hist. Stud. Phys. Sci., Vol. 8 (1977), pp. 137–188

[11] Sakai, S., *C-algebras and W-algebras.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 60. Springer-Verlag, New York-Heidelberg, 1971.

[12] Schroer, B., *Pascual Jordan, Glory and Demise and his legacy in contemporary local quantum physics.* (2003) Unpublished manuscript.

[13] Simmons, G., *Introduction to Topology and Modern Analysis.* Robert E. Krieger Publishing Co., Inc., Melbourne, Fl., (1983).

[14] Murray, F. J. & von Neumann, J., *On rings of operators.* Ann. of Math. (2) 37 (1936), no. 1, pp. 116-229.

[15] Murray, F. J. & von Neumann, J., *On rings of operators. II.* Trans. Amer. Math. Soc. 41 (1937), no. 2, pp. 208-248.

[16] Murray, F. J. & von Neumann, J., *On rings of operators. IV.* Ann. of Math. (2) 44, (1943), pp. 716-808.

[17] von Neumann, J., *On rings of operators. III.* Ann. of Math. (2) 41, (1940), pp. 94-161.