C^*-ALGEBRAS, AND THE GELFAND-NAIMARK THEOREM

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Todo list

1. Preliminaries

we will assume knowledge on... (algebra homos map 1 to 1,)
brief(er than asst 3) history
most texts start from $\mathcal{B}(\mathcal{H})$ to justify the whole thing. we're algebraists, who don't need no justification. we jump right in at the deep (abstract) end.

write this!

2. Definitions

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Definition 1. A *Banach algebra* is a complex Banach space $(A, \|\cdot\|)$ which forms an algebra, such that

$$\|ab\| \leq \|a\|\|b\| \text{ for all } a, b \in A.$$

A $*$-*algebra* is an algebra A with an *involution* map $a \mapsto a^*$ on A such that, for all $a, b \in A$ and for $\alpha \in \mathbb{C}$,

(i) $a^{**} = (a^*)^* = a$,
(ii) $(\alpha a + b)^* = \overline{\alpha} a^* + b^*$,
(iii) $(ab)^* = b^* a^*$.

The element a^* is referred to as the *adjoint* of a.
A C^*-*algebra* is a Banach algebra $(A, \|\cdot\|)$ with involution map $a \mapsto a^*$, with the condition

$$\|a^* a\| = \|a\|^2 \text{ for all } a \in A.$$

This condition is known as the C^* *axiom*.

Here we consider complex C^*-algebras. The theory of real C^*-algebras has advanced....

remark on work in real C^*-algebras

Unless specified otherwise, by an *ideal* of a Banach algebra, we mean a two-sided ideal. A $*$-ideal in a Banach $*$-algebra is a $*$-closed ideal. For C^*-algebras, it turns out that any ideal is automatically a $*$-ideal.

2.1. **Unitization.** If a C^*-algebra A contains an identity element $\mathbb{1}$ such that $a \cdot \mathbb{1} = a = \mathbb{1} \cdot a$ for all $a \in A$, call $\mathbb{1}$ the *unit* in A, and A is then a *unital* C^*-algebra.

Proposition 1. *Any non-unital C^*-algebra A can be isometrically embedded in a unital C^*-algebra $\tilde{A}$ as a maximal ideal.*

tidy up proof

Proof. Let $\tilde{A} = A \oplus \mathbb{C}$ with pointwise addition, and define

$$(a, \lambda)(b, \mu) := (ab + \lambda b + \mu a, \lambda\mu),$$
$$(a, \lambda)^* := (a^*, \overline{\lambda}),$$
$$\|(a, \lambda)\| := \sup_{\|b\|=1} \|ab + \lambda b\|.$$

Then $\tilde{A}$ is a $*$-algebra. The norm $\|(a, \lambda)\|$ is the norm in $\mathcal{B}(A)$ of left-multiplication by a on something iunno??? Thus $\tilde{A}$ is a Banach $*$-algebra with unit $(0, 1)$. By design, A is a maximal ideal of codimension 1. The embedding $a \mapsto (a, \lambda)$ is isometric as

$$\|a\| = \|a \cdot \frac{a}{\|a\|}\| \leq \|(a, 0)\| \leq \sup_{\|b\|=1} \|ab\| \leq \|a\|.$$

It remains to verify the C^*-axiom:

$$\begin{aligned}
\|(a,\lambda)\|^2 &= \sup_{\|b\|=1} \|ab+\lambda b\|^2 \\
&= \sup_{\|b\|=1} \|b^*a^*ab+\lambda b^*a^*b+\overline{\lambda}b^*ab+|\lambda|^2b^*b \\
&\leq \sup_{\|b\|=1} \|a^*ab+\lambda a^*b+\overline{\lambda}ab+|\lambda|^2b\| \\
&= \|(a^*a+\lambda a^*+\overline{\lambda}a, |\lambda|^2) \\
&= \|(a,\lambda)^*(a,\lambda)\| \\
&\leq \|(a,\lambda)^*\|\|(a,\lambda)\|.
\end{aligned}$$

By symmetry of $*$, $\|(a,\lambda)^*\| = \|(a,\lambda)\|$. Hence, the above inequality becomes equality and we have that

$$\|(a,\lambda)^*(a,\lambda)\| = \|(a,\lambda)\|^2.$$

□

In light of this result, we take all C^*-algebras from here to be unital. For the results we will consider, we can simply consider the unital case. However, there are important circumstances in which we need to relax the unital condition,

does the following theory carry from a unital C*alg to any non-unital ideal? a remark

2.2. **the spectrum.** Given an element $a \in A$ of a C^*-algebra, define its spectrum $\mathrm{sp}(a)$:

$$\mathrm{spec}(a) := \{\lambda \in \mathbb{C} \mid a - \lambda\mathbb{1} \text{ is not invertible in } A\}.$$

2.3. **more definitions.** An element $a \in A$ of a C^*-algebra is called

- *self-adjoint* if $a^* = a$;
- *unitary* if $aa^* = a^*a = \mathbb{1}$;
- *positive* if it is self-adjoint and $\mathrm{sp}(a) \subseteq \mathbb{R}^+$.

Denote the set of positive elements in A by A^+.

Given Banach $*$-algebras A and B, a map $\varphi : A \to B$ is a $*$-*homomorphism* if it is an algebra homomorphism for which $\varphi(a^*) = \varphi(a)^*$ for all $a \in A$. If a $*$-homomorphism φ is one-to-one, call it a $*$-*isomorphism*.

norm preservation theorem here?

A *linear functional* on a C^*-algebra A is a linear operator $\rho : A \to \mathbb{C}$. A *state* on A is a linear functional ρ such that $\|\rho\| = 1$ and $\rho(a) \geq 0$ for all positive elements $a \in A^+$.

is this right? what of $\rho(\mathbb{1}) = 1$?

normal(?), the weak* topology(?), cauchy-schwarz

2.4. **$\mathcal{B}(\mathcal{H})$ - an example.** This section concerns the fundamental example of a C^*-algebra - the set $\mathcal{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$. Here we will demonstrate that $\mathcal{B}(\mathcal{H})$ is a C^*-algebra and give some basic results.

Claim. $\mathcal{B}(\mathcal{H})$ is a Banach $*$-algebra with the operator norm

$$\|T\| := \sup_{\|x\|=1} \|Tx\|$$

and involution taking T to its adjoint map T^*.

Proof. $\|\cdot\|$ is a norm on $\mathcal{B}(\mathcal{H})$. Let $\{T_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{B}(\mathcal{H})$. Then for any positive ϵ, there is a positive integer N such that

$$\|T_m - T_n\| < \epsilon \text{ for all } m, n \geq N.$$

Applying $T_m - T_n$ to $x \in \mathcal{H}$, we have

$$\|T_m x - T_n x\| \leq \|T_m - T_n\|\|x\| < \epsilon\|x\|, \tag{1}$$

so $\{T_n x\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{H}$, converging to an element in $\mathcal{H}$. Define a linear operator $T : \mathcal{H} \to \mathcal{H}$ by

$$T = \lim_{n\to\infty} T_n x.$$

Taking limits as m tends to infinity in equation (1), we obtain

$$\|Tx - T_n x\| < \epsilon\|x\| \text{ for all } n \geq N,$$

and so we have that $T - T_n$ (and hence $T = (T - T_n) + T_n$) is a bounded operator and

$$\|T - T_n\| < \epsilon \text{ for all } n \geq N.$$

We conclude that $T_n \to T$, and so $\mathcal{B}(\mathcal{H})$ is complete.

Since boundedness is equivalent to continuity on $\mathcal{H}$, given $S, T \in \mathcal{B}(\mathcal{H})$, the operator $ST : \mathcal{H} \to \mathcal{H}; x \mapsto (S \circ T)(x)$ is bounded on $\mathcal{H}$. Given $x \in \mathcal{H}$ and $\lambda \in \mathbb{C}$,

$$\begin{aligned}
((\lambda S)T)(x) &= ((\lambda S) \circ T)(x) \\
&= \lambda S(Tx) \\
&= \lambda(S \circ T)(x) \\
&= \lambda ST(x),
\end{aligned}$$

so that $(\lambda S)T = \lambda ST$ in $\mathcal{B}(\mathcal{H})$, whence $\mathcal{B}(\mathcal{H})$ is an algebra. We have

$$\begin{aligned}
\|ST\| &= \sup_{\|x\|=1} \|STx\| \\
&= \sup_{\|x\|=1} \|S(Tx)\| \\
&\leq \|S\| \sup_{\|x\|=1} \|Tx\| \\
&= \|S\|\|T\|.
\end{aligned}$$

To see that * is an involution:

i)

$$\begin{aligned}\langle (T^*)^* x, y\rangle &= \langle x, T^* y\rangle \\ &= \langle Tx, y\rangle.\end{aligned}$$

ii)

$$\begin{aligned}\langle (\alpha T + S)^* x, y\rangle &= \langle x, \alpha T + Sy\rangle \\ &= \overline{\alpha}\langle x, Ty\rangle + \langle x, Sy\rangle \\ &= \overline{\alpha}\langle T^* x, y\rangle + \langle S^* x, y\rangle \\ &= \langle (\overline{\alpha} T^* + S^*)x, y\rangle\end{aligned}$$

Hence, the claim. □

example of $\mathcal{B}(\mathcal{H})$ at end of section to retro-motivate notation.

example of $C(X)$?

3. Representations of C*-algebras

to include all representation theory, including GNS, CGN and GN

Definition 2. Given a C^*-algebra A, a *representation of A on a Hilbert space $\mathcal{H}$* is a $*$-homomorphism $\varphi : A \to \mathcal{B}(\mathcal{H})$. An isomorphic representation is called *faithful.*

Theorem 1 (Gelfand-Naimark for commutative algebras)**.** *Every commutative C^*-algebra A is $*$-isomorphic to $C(\mathcal{P}(A))$, the continuous functions on the topological space of pure states on A.*

find correct name - Gelfand Representation theorem?

remark about nonunital commutative.

Theorem 2 (Gelfand-Naimark-Segal)**.** *If ρ is a state on a C^*-algebra A, then there exists a cyclic representation π_ρ of A on a Hilbert space H_ρ, with unit cyclic vector x_ρ, such that*

$$\rho(a) \;=\; \langle \pi_\rho(a)x_\rho, x_\rho\rangle,\ \forall a \in A.$$

Proof. We will construct from ρ the space $\mathcal{H}_\rho$, representation π_ρ, and vector x_ρ, and demonstrate the required properties.

Consider the *left kernel* of ρ:

$$L_\rho := \{t \in A \mid \rho(t^*t) = 0\}.$$

For $a, b \in A$, define $\langle a, b\rangle_0 := \rho(b^*a)$. Then $L_\rho = \{t \in A \mid \langle t, t\rangle_0 = 0\}$, and $\langle\cdot,\cdot\rangle_0$ satisfies

(i) Linearity in 1st argument: for $a, b \in A$, $\alpha, \beta \in \mathbb{C}$:

$$\begin{aligned}\langle \alpha a + \beta b, c\rangle_0 &= \rho(c^*(\alpha a + \beta b) \\ &= \rho(\alpha c^* a + \beta c^* b) \\ &= \alpha\rho(c^*a) + \beta\rho(c^*b) \\ &= \alpha\langle a, c\rangle_0 + \beta\langle b, c\rangle_0.\end{aligned}$$

(ii) Conjugate symmetric: for $a, b \in A$:

$$\begin{aligned}\langle b, a\rangle_0 &= \rho(a^*b)\\ &= \rho((b^*a)^*)\\ &= \overline{\rho(b^*a)}\\ &= \overline{\langle a, b\rangle_0}.\end{aligned}$$

why?

(iii) Positive semi-definite.

Note that $\langle\cdot,\cdot\rangle$ is not necessarily positive definite on $A - L_\rho$ is exactly where this fails.

sentence

L_ρ is a linear subspace of A: Consider

$$L := \{t \in A \mid \langle t, a\rangle_0 = 0,\ \forall a \in A\} \subseteq L_\rho.$$

For $t \in L_\rho$, by Cauchy-Schwarz we have

$$|\langle t, a\rangle_0|^2 \leq \langle t, t\rangle_0 \langle a, a\rangle_0,\ \forall a \in A;$$

that is,

$$\langle t, a\rangle_0 = 0,\ \forall a \in A,$$

so $t \in L$ and $L_\rho = L$.

Now, for $a, b \in L$, $\alpha \in \mathbb{C}$ and $c \in A$:

$$\langle \alpha a + b, c\rangle_0 = \alpha\langle a, c\rangle_0 + \langle b, c\rangle_0 = 0,$$

so $\alpha a + b \in L$; also, $\langle 0, c\rangle_0 = 0$ so $0 \in L$. Hence, $L(= L_\rho)$ is a linear subspace of A.

For $s \in A$, $t \in L_\rho$, by the Cauchy Schwarz inequality [ref] we have

$$\begin{aligned}|\rho(s^*t)|^2 &= |\langle t, s\rangle_0|^2\\ &\leq \langle t, t\rangle_0 \cdot \langle s, s\rangle_0\\ &= \rho(t^*t)\cdot\rho(s^*s)\\ &= 0,\end{aligned}$$

so $\rho(s^*t) = 0$. Letting $s = a^*at$ for $a \in A$, then

$$\begin{aligned}\rho((at)^*at) &= \rho(at^*a^*at)\\ &= \rho((a^*at)^*t)\\ &= \rho(s^*t)\\ &= 0,\end{aligned}$$

so that $at \in L_\rho$, for all $a \in A$ and $t \in L_\rho$; we conclude that L_ρ is a left ideal in A. L_ρ is the preimage in A of $\{0\}$ under the continuous map$t \mapsto \rho(t^*t)$, so is closed.

start sentence properly

Consider now $V_\rho := A/L_\rho$, with $\langle\cdot,\cdot\rangle$ defined by

$$\langle a + L_\rho, b + L_\rho\rangle := \langle a, b\rangle_0,\ \text{for } a + L_\rho, b + L_\rho \in V_\rho.$$

It follows from properties $i), ii)$ and $iii)$ of $\langle\cdot,\cdot\rangle_0$ that $\langle\cdot,\cdot\rangle$ is an inner product on V_ρ – with

$$\begin{aligned}\langle a+L^\rho, a+L^\rho\rangle = 0 &\iff \langle a,a\rangle = 0\\ &\iff a\in L_\rho\\ &\iff a+L_\rho = 0+L_\rho\end{aligned}$$

giving positive definiteness. The completion of V_ρ with respect to $\langle\cdot,\cdot\rangle$ is a Hilbert space - this is the Hilbert space $\mathcal{H}_\rho$ we're looking for.

Now we fix $a\in A$, and consider the map

$$\pi_a : V_\rho \to V_\rho; b+L_\rho \mapsto ab+L_\rho.$$

Let $b_1, b_2 \in A$ be such that $b_1+L_\rho = b_2+L_\rho$. Then:

$$\begin{aligned}&\implies b_1-b_2\in L_\rho\\ &\implies a(b_1-b_2)\in L_\rho\\ &\implies ab_1-ab_2\in L_\rho\\ &\implies ab_1+L_\rho = ab_2+L_\rho\\ &\implies \pi_a(b_1+L_\rho) = \pi_a(b_2+L_\rho).\end{aligned}$$

Hence π_a defines a linear operator on V_ρ.

For $b+L_\rho\in V_\rho$:

$$\begin{aligned}\|a\|^2\cdot\|b+L_\rho\| - \|\pi_a(b+L_\rho)\| &= \|a\|^2\cdot\|b+L_\rho\| - \|ab+L_\rho\|\\ &= \|a\|^2\cdot\langle b+L_\rho, b+L_\rho\rangle - \langle ab+L_\rho, ab+L_\rho\rangle\\ &= \|a\|^2\cdot\rho(b^*b) - \rho((ab)^*ab)\\ &= \rho(\|a\|^2 b^*b - b^*a^*ab)\\ &= \rho(b^*(\|a\|^2\mathbb{1} - a^*a)b)\\ &\geq 0.\end{aligned}$$

Thus π_a is a bounded operator, with $\|\pi_a\|\leq\|a\|$. By continuity, π_a extends to a bounded operator on $\mathcal{H}_\rho$ – say $\pi_\rho(a):\mathcal{H}_\rho\to\mathcal{H}_\rho$ such that cont of what?

$$\pi_\rho(a)(v) = \pi_a(v)$$

for $v\in V_\rho$. Then $\pi_\rho(a)\in\mathcal{B}(\mathcal{H}_\rho)$ for each $a\in A$, so π_ρ defines a map $A\to\mathcal{B}(\mathcal{H}_\rho)$ such that $a\mapsto\pi_\rho(a)$. This will be our representation.

Now, for $a,b\in A$, $c+L_\rho\in V_\rho$ and $\alpha\in\mathbb{C}$:

$$\begin{aligned}\pi_{\alpha a+b}(c+L_\rho) &= (\alpha a+b)(c+L_\rho)\\ &= (\alpha ac+L_\rho)+(bc+L_\rho)\\ &= \alpha\pi_a(c+L_\rho)+\pi_b(c+L_\rho),\end{aligned}$$

so that $\pi_{\alpha a+b} = \alpha\pi_a+\pi_b$ on V_ρ.

For $a, b \in A$ and $c + L_\rho \in V_\rho$:

$$\begin{aligned}\pi_{ab}(c + L_\rho) &= abc + L_\rho \\ &= \pi_a(bc + L_\rho) \\ &= \pi_a(\pi_b(c + L_\rho)) \\ &= (\pi_a \cdot \pi_b)(c + L_\rho),\end{aligned}$$

so that $\pi_{ab} = \pi_a \cdot \pi_b$ on V_ρ.

For $a \in A$ and $b + L_\rho,\ c + L_\rho \in V_\rho$:

$$\begin{aligned}\langle b + L_\rho, \pi_a^*(c + L_\rho)\rangle &= \langle \pi_a(b + L_\rho), c + L_\rho\rangle \\ &= \langle ab + Lr, c + L_\rho\rangle \\ &= \rho(c^*ab) \\ &= \rho((a^*c)^*b) \\ &= \langle b + L_\rho, a^*c + L_\rho\rangle \\ &= \langle b + L_\rho, \pi_{a^*}(c + L_\rho),\end{aligned}$$

so that $\pi_a^* = \pi_{a^*}$ on V_ρ.

cont of what?

$V_\rho \subset \mathcal{H}_\rho$ is a dense subset, so the three properties above hold on $\mathcal{H}_\rho$ by continuity. Hence, $\pi_\rho : A \to \mathcal{B}(\mathcal{H}_\rho)$ is a representation of A. As to the unit vector, consider $x_\rho := \mathbb{1} + L_\rho \in V_\rho$. Then for $a \in A$,

$$\begin{aligned}\langle \pi_\rho(a)x_\rho, x_\rho\rangle &= \langle \pi_a(\mathbb{1} + L_\rho), \mathbb{1} + L_\rho\rangle \\ &= \langle a + L_\rho \mathbb{1} + L_\rho\rangle \\ &= \rho(a);\end{aligned}$$

in particular, $\langle x_\rho, x_\rho\rangle = \rho(\mathbb{1}) = 1$, so x_ρ is a unit vector in $\mathcal{H}_\rho$. □

Theorem 3 (Gelfand-Naimark)**.** *Every C^*-algebra has a faithful representation.*

references!!!!