C^*-Algebras, and the Gelfand-Naimark Theorems

Luke Armitage

University of York

June 8, 2017

Definitions

A C^*-**algebra** A is a Banach algebra with norm $\|\cdot\|$ and an involution map $a \mapsto a^*$ satisfying the following:

1. $a^{**} = a$
2. $(\alpha a + b)^* = \bar{\alpha} a^* + b^*$
3. $(ab)^* = b^* a^*$
4. $\|a^* a\| = \|a\|^2 \quad$ (C^* axiom)

Definitions

Spectrum of $a \in A$ is

$$\sigma(a) = \{\lambda \in \mathbb{C} \mid a - \lambda 1 \text{ is not invertible}\}.$$

Spectral radius of $a \in A$ is

$$r(a) = \sup_{\lambda \in \sigma(a)} |\lambda|.$$

Say that $a \in A$ is **positive** if $a^* = a$ and $\sigma(a) \subset \mathbb{R}$.
A **state** is a linear map $\rho : A \to \mathbb{C}$ such that $\rho(a) \geq 0$ for all positive $a \in A$, and $\rho(1) = 1$.
The **state space**, $\mathcal{S}(A)$, is a convex subset of the dual space of A.
Call the extreme points of the state space **pure states**.

Definitions

A $*$-**homomorphism**: algebra homomorphism such that $\varphi(a^*) = \varphi(a)^*$.

$*$-**isomorphism**: bijective $*$-homomorphism.

A **representation** of A on a Hilbert Space $\mathcal{H}$ is a $*$-homomorphism $A \to \mathcal{B}(\mathcal{H})$.

A bijective representation is called **faithful**.

Examples

Cool Asides

Uniqueness of norm: $\|a\|^2 = \|a^*a\| = r(a^*a)$. Requires spectral theory. The spectral radius of a normal element is equal to its norm. From this, and the C* axiom, we get that the norm of each element is given by the spectral radius, which is defined in terms of the spectrum which does not use the norm.
$*$**-homomorphisms are continuous:** homomorphisms do not increase norm, so are bounded and hence continuous.
isomorphisms are isometric. again uses spectral theory, this time to show that spectral radius is not increased / is preserved.

Gelfand-Naimark Theorems

Theorem

Every Abelian C^*-algebra A is $*$-isomorphic to $C(\mathscr{P}(A))$, the algebra of continuous functions on the compact Hausdorff space $\mathscr{P}(A)$ of pure states on A.

Gelfand-Naimark Theorems

Theorem

Every Abelian C^*-algebra A is $*$-isomorphic to $C(\mathscr{P}(A))$, the algebra of continuous functions on the compact Hausdorff space $\mathscr{P}(A)$ of pure states on A.

Theorem

Every C^*-algebra has a faithful representation.

The Gelfand-Naimark-Segal Construction

Used to prove the GN theorem.
Given a state on a C* algebra, we can construct a Hilbert space and a representation on that space. Given a and b in A, define $\langle a, b \rangle = \rho(b^* a)$. This is a semi-inner product – basically an inner product, but there exist $a \neq 0$ such that $\langle a, a \rangle = 0$. However, if we consider the quotient vector space of A by the collection of such elements, this space completes to a Hilbert space with $\langle \cdot, \cdot \rangle$ as the inner product.

References