C^*-Algebras, and the Gelfand-Naimark Theorems

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Definitions

Banach Algebra

A **Banach algebra** is a complete normed algebra A such that

$$\|ab\| \leq \|a\| \cdot \|b\| \quad \forall a, b \in A.$$

Definitions

C^*-Algebra

A C^*-**algebra** A is a Banach algebra with involution map $a \mapsto a^*$ satisfying the following:

1. $a^{**} = a$
2. $(\alpha a + b)^* = \bar{\alpha} a^* + b^*$
3. $(ab)^* = b^* a^*$
4. $\|a^* a\| = \|a\|^2$ (C^* axiom)

Examples

Continuous linear functionals on a compact, Hausdorff space.

Bounded operators on a Hilbert space, $\mathcal{B}(\mathcal{H})$.

Ideal of compact operators, $\mathcal{K}(\mathcal{H})$.

Calkin algebra, the quotient algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$.

Definitions

Spectrum

Spectrum of $a \in A$ is

$$\sigma(a) = \{\lambda \in \mathbb{C} \mid a - \lambda 1 \text{ is not invertible}\} .$$

Spectral radius of $a \in A$ is

$$r(a) = \sup_{\lambda \in \sigma(a)} |\lambda| .$$

Say that $a \in A$ is **positive** if $a^* = a$ and $\sigma(a) \subset \mathbb{R}$.

Definitions

States

A **state** is a linear map $\rho : A \to \mathbb{C}$ such that $\rho(a) \geq 0$ for all positive $a \in A$, and $\rho(1) = 1$.

The **state space**, $\mathcal{S}(A)$, is a convex subset of the dual space of A. Call the extreme points of the state space **pure states**.

Definitions

Maps between C^*-algebras

A $*$-**homomorphism** is an algebra homomorphism such that $\varphi(a^*) = \varphi(a)^*$.

A $*$-**isomorphism** is a bijective $*$-homomorphism.

Cool Results

Uniqueness of norm

$\|a\|^2 = \|a^*a\| = r(a^*a)$. The spectral radius of a normal element is equal to its norm. From this, and the C* axiom, we get that the norm of each element is given by the spectral radius, which is defined in terms of the spectrum which is not defined in terms of the norm.

Cool Results

$*$-homomorphisms are continuous

homomorphisms do not increase norm, so are bounded and hence continuous. isomorphisms are isometric. again uses spectral theory, this time to show that spectral radius is not increased / is preserved.

Definitions

Representation

A **representation** of A on a Hilbert Space $\mathcal{H}$ is a $*$-homomorphism $A \to \mathcal{B}(\mathcal{H})$.

A bijective representation is called **faithful**.

Gelfand-Naimark Theorem

Theorem

Every C^*-algebra has a faithful representation.

Proof.

Uses the Gelfand-Naimark-Segal construction.

Gelfand-Naimark Theorem

Proof: Gelfand-Naimark-Segal Construction

Let $L = \{a \in A \mid \rho(a^*a) = 0\}$.

$\mathcal{H}$ is the Hilbert space completion of A/L.

Define operators

$$\pi_a : A/L \to A/L : b + L \mapsto ab + L,$$

and extend to $\pi_a : \mathcal{H} \to \mathcal{H}$.

Representation is given by

$$\pi : A \to \mathcal{B}(\mathcal{H}) : a \mapsto \pi_a.$$

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Gelfand-Naimark Theorem

Proof: Direct Sum

Proof concludes by taking 'direct sum' representation over the representations given by doing GNS construction to a subset of state space containing all pure states. This gives a faithful representation.

Gelfand-Naimark Theorem

For Commutative C^*-algebras

Theorem

Every Abelian C^*-algebra A is $*$-isomorphic to $C(\mathscr{P}(A))$, the algebra of continuous functions on the compact Hausdorff space $\mathscr{P}(A)$ containing all pure states on A.

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References – Any Questions?

Kadison, R.V. and Ringrose, J.R., 1983. *Fundamentals of the Theory of Operator Algebras, Vol. I. Elementary Theory*. Springer.

Blackadar, B., 2006. *Algebras: Theory of C^*-Algebras and Von Neumann Algebras* (Vol. 122). Springer Science & Business Media.

My project report can be downloaded from `goo.gl/Qv1zas`.