# $C^*$-Algebras, and the Gelfand-Naimark Theorems

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# Definitions

A $C^*$-**algebra** $A$ is a Banach algebra with norm $\| \cdot \|$ and an involution map $a \mapsto a^*$ satisfying the following:

1. $a^{**} = a$
2. $(\alpha a + b)^* = \bar{\alpha} a^* + b^*$
3. $(ab)^* = b^* a^*$
4. $\|a^* a\| = \|a\|^2 \quad$ ($C^*$ axiom)

# Definitions

spectrum, spectral radius, state, pure state, * isomorphism, representation, faithful representation,

# Examples

# Cool Asides

**Uniqueness of norm:** $\|a\|^2 = \|a^*a\| = r(a^*a)$. Requires spectral theory. The spectral radius of a normal element is equal to its norm. From this, and the C* axiom, we get that the norm of each element is given by the spectral radius, which is defined in terms of the spectrum which does not use the norm.

$*$**-homomorphisms are continuous:** homomorphisms do not increase norm, so are bounded and hence continuous.

isomorphisms are isometric. again uses spectral theory, this time to show that spectral radius is not increased / is preserved.

# Gelfand-Naimark Theorems

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**Theorem**

*Every $C^*$-algebra has a faithful representation.*

# The Gelfand-Naimark-Segal Construction

Used to prove the GN theorem.
Given a state on a C* algebra, we can construct a Hilbert space and a representation on that space. Given $a$ and $b$ in $A$, define $\langle a, b\rangle = \rho(b^*a)$. This is a semi-inner product – basically an inner product, but there exist $a \neq 0$ such that $\langle a, a\rangle = 0$. However, if we consider the quotient vector space of $A$ by the collection of such elements, this space completes to a Hilbert space with $\langle \cdot, \cdot\rangle$ as the inner product.

# References