C*-ALGEBRAS, AND THE GELFAND-NAIMARK THEOREM

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1. Preliminaries

we will assume knowledge on...
brief(er than asst 3) history

2. Basics

2.1. **Definitions.**

Definition 1 (C*-algebra)**.**

define C*algebras, states, representations, the weak* topology

3. Representations of C*-algebras

to include all representation theory, including GNS, CGN and GN

Theorem 1 (Gelfand- Naimark-Segal)**.** *If ρ is a state on a C^*-algebra A, then there exists a cyclic representation π_ρ of A on a Hilbert space H_ρ, with unit cyclic vector x_ρ, such that*

$$\rho(a) \;=\; \langle \pi_\rho(a)x_\rho, x_\rho\rangle, \; \forall a \in A.$$

Proof. We will construct from ρ the space H_rho, representation π_ρ, and vector x_ρ, and demonstrate the required properties.

Consider the *left kernel* of ρ:

$$L_\rho := \{t \in A | \rho(t^*t) = 0\}.$$

For $a, b \in A$, define $\langle a, b\rangle_0 := \rho(b^*a)$. Then $L_\rho = \{t \in A | \langle t, t\rangle_0 = 0\}$, and $\langle \cdot, \cdot\rangle_0$ satisfies

(i) Linearity in 1st argument: for $a, b \in A$, $\alpha, \beta \in \mathbb{C}$:

$$\begin{aligned}\langle \alpha a + \beta b, c\rangle_0 &= \rho(c^*(\alpha a + \beta b)\\ &= \rho(\alpha c^* a + \beta c^* b)\\ &= \alpha\rho(c^*a) + \beta\rho(c^*b)\\ &= \alpha\langle a, c\rangle_0 + \beta\langle b, c\rangle_0.\end{aligned}$$

(ii) Conjugate symmetric: for $a, b \in A$:

$$\begin{aligned}\langle b, a\rangle_0 &= \rho(a^*b)\\ &= \rho((b^*a)^*)\\ &= \overline{\rho(b^*a)}\\ &= \overline{\langle a, b\rangle_0}.\end{aligned}$$

why?

(iii) Positive semi-definite.

Note that $\langle\cdot,\cdot\rangle$ is not necessarily positive definite on A – L_ρ is exactly where this fails.

L_ρ is a linear subspace of A: Consider

$$L := \{t \in A | \langle t, a\rangle_0 = 0 \forall a \in A\} \subseteq L_\rho.$$

For $t \in L_\rho$, by Cauchy-Schwarz we have

$$|\langle t, a\rangle_0|^2 \leq \langle t, t\rangle_0 \langle a, a\rangle_0, \ \forall a \in A;$$

that is,

$$\langle t, a\rangle_0 = 0, \ \forall a \in A,$$

so $t \in L$ and $L_\rho = L$. $\square$