C*-ALGEBRAS, AND
THE GELFAND-NAIMARK THEOREM

LUKE ARMITAGE

1. Preliminaries

we will assume knowledge on...
brief(er than asst 3) history

2. Basics

2.1. Definitions.

Definition 1 (C*-algebra).

define C*algebras, states, representations, the weak* topology

3. Representations of C*-algebras

to include all representation theory, including GNS, CGN and GN

Theorem 1 (Gelfand- Naimark-Segal). *If ρ is a state on a C*-algebra A, then there exists a cyclic representation π_ρ of A on a Hilbert space H_ρ, with unit cyclic vector x_ρ, such that*

$$\rho(a) \;=\; \langle \pi_\rho(a)x_\rho, x_\rho \rangle, \; \forall a \in A.$$

Proof. We will construct from ρ the space $\mathcal{H}_\rho$, representation π_ρ, and vector x_ρ, and demonstrate the required properties.

Consider the *left kernel* of ρ:

$$L_\rho := \{t \in A | \rho(t^*t) = 0\}.$$

For $a, b \in A$, define $\langle a, b \rangle_0 := \rho(b^*a)$. Then $L_\rho = \{t \in A | \langle t, t \rangle_0 = 0\}$, and $\langle \cdot, \cdot \rangle_0$ satisfies

   (i) Linearity in 1st argument: for $a, b \in A$, $\alpha, \beta \in \mathbb{C}$:

$$\begin{aligned}
\langle \alpha a + \beta b, c \rangle_0 &= \rho(c^*(\alpha a + \beta b) \\
&= \rho(\alpha c^* a + \beta c^* b) \\
&= \alpha \rho(c^* a) + \beta \rho(c^* b) \\
&= \alpha \langle a, c \rangle_0 + \beta \langle b, c \rangle_0.
\end{aligned}$$

(ii) Conjugate symmetric: for $a, b \in A$:

$$\langle b, a \rangle_0 = \rho(a^*b)$$
$$= \rho((b^*a)^*)$$
$$= \overline{\rho(b^*a)}$$
$$= \overline{\langle a, b \rangle_0}.$$

why?

(iii) Positive semi-definite.

Note that $\langle \cdot, \cdot \rangle$ is not necessarily positive definite on A – L_ρ is exactly where this fails.

sentence

L_ρ is a linear subspace of A: Consider

$$L := \{t \in A | \langle t, a \rangle_0 = 0, \ \forall a \in A\} \subseteq L_\rho.$$

For $t \in L_\rho$, by Cauchy-Schwarz we have

$$|\langle t, a \rangle_0|^2 \leq \langle t, t \rangle_0 \langle a, a \rangle_0, \ \forall a \in A;$$

that is,

$$\langle t, a \rangle_0 = 0, \ \forall a \in A,$$

so $t \in L$ and $L_\rho = L$.

Now, for $a, b \in L$, $\alpha \in \mathbb{C}$ and $c \in A$:

$$\langle \alpha a + b, c \rangle_0 = \alpha \langle a, c \rangle_0 + \langle b, c \rangle_0 = 0,$$

so $\alpha a + b \in L$; also, $\langle 0, c \rangle_0 = 0$ so $0 \in L$. Hence, $L(= L_\rho)$ is a linear subspace of A.

For $s \in A$, $t \in L_\rho$, by the Cauchy Schwarz inequality [ref] we have

$$|\rho(s^*t)|^2 = |\langle t, s \rangle_0|^2$$
$$\leq \langle t, t \rangle_0 \cdot \langle s, s \rangle_0$$
$$= \rho(t^*t) \cdot \rho(s^*s)$$
$$= 0,$$

so $\rho(s^*t) = 0$. Letting $s = a^*at$ for $a \in A$, then

$$\rho((at)^*at) = \rho(at^*a^*at)$$
$$= \rho((a^*at)^*t)$$
$$= \rho(s^*t)$$
$$= 0,$$

so that $at \in L_\rho$, for all $a \in A$ and $t \in L_\rho$; we conclude that L_ρ is a left ideal in A.

start sentence
properly

L_ρ is the preimage in A of $\{0\}$ under the continuous map $t \mapsto \rho(t^*t)$, so is closed.

Consider now $V_\rho := A/L_\rho$, with $\langle \cdot, \cdot \rangle$ defined by

$$\langle a + L_\rho, b + L_\rho \rangle := \langle a, b \rangle_0, \ \text{for } a + L_\rho, b + L_\rho \in V_\rho.$$

It follows from properties $i), ii)$ and $iii)$ of $\langle \cdot, \cdot \rangle_0$ that $\langle \cdot, \cdot \rangle$ is an inner product on V_ρ - with

$$\langle a + L^\rho, a + L^\rho \rangle = 0 \iff \langle a, a \rangle = 0$$
$$\iff a \in L_\rho$$
$$\iff a + L_\rho = 0 + L_\rho$$

giving positive definiteness. The completion of V_ρ with respect to $\langle \cdot, \cdot \rangle$ is a Hilbert space - this is the Hilbert space $\mathcal{H}_\rho$ we're looking for.

Now we fix $a \in A$, and consider the map

$$\pi_a : V_\rho \to V_\rho; b + L_\rho \mapsto ab + L_\rho.$$

Let $b_1, b_2 \in A$ be such that $b_1 + L_\rho = b_2 + L_\rho$. Then:

$$\implies b_1 - b_2 \in L_\rho$$
$$\implies a(b_1 - b_2) \in L_\rho$$
$$\implies ab_1 - ab_2 \in L_\rho$$
$$\implies ab_1 + L_\rho = ab_2 + L_\rho$$
$$\implies \pi_a(b_1 + L_\rho) = \pi_a(b_2 + L_\rho).$$

Hence π_a defines a linear operator on V_ρ.

For $b + L_\rho \in V_\rho$:

$$\|a\|^2 \cdot \|b + L_\rho\| - \|\pi_a(b + L_\rho)\| = \|a\|^2 \cdot \|b + L_\rho\| - \|ab + L_\rho\|$$
$$= \|a\|^2 \cdot \langle b + L_\rho, b + L_\rho \rangle - \langle ab + L_\rho, ab + L_\rho \rangle$$
$$= \|a\|^2 \cdot \rho(b^*b) - \rho((ab)^*ab)$$
$$= \rho(\|a\|^2 b^*b - b^*a^*ab)$$
$$= \rho(b^*(\|a\|^2 \mathbb{1} - a^*a)b)$$
$$\geq 0.$$

Thus π_a is a bounded operator, with $\|\pi_a\| \leq \|a\|$. By continuity, π_a [of what?] extends to a bounded operator on $\mathcal{H}_\rho$ – say $\pi_\rho(a) : \mathcal{H}_\rho \to \mathcal{H}_\rho$ such that

$$\pi_\rho(a)(v) = \pi_a(v)$$

for $v \in V_\rho$. Then $\pi_\rho(a) \in \mathcal{B}(\mathcal{H}_\rho)$ for each $a \in A$, so π_ρ defines a map $A \to \mathcal{B}(\mathcal{H}_\rho)$ such that $a \mapsto \pi_\rho(a)$. This will be our representation.

Now, for $a, b \in A$, $c + L_\rho \in V_\rho$ and $\alpha \in \mathbb{C}$:

$$\pi_{\alpha a + b}(c + L_\rho) = (\alpha a + b)(c + L_\rho)$$
$$= (\alpha ac + L_\rho) + (bc + L_\rho)$$
$$= \alpha \pi_a(c + L_\rho) + \pi_b(c + L_\rho),$$

so that $\pi_{\alpha a + b} = \alpha \pi_a + \pi_b$ on V_ρ.

For $a, b \in A$ and $c + L_\rho \in V_\rho$:

$$\begin{aligned}
\pi_{ab}(c + L_\rho) &= abc + L_\rho \\
&= \pi_a(bc + L_\rho) \\
&= \pi_a(\pi_b(c + L_\rho)) \\
&= (\pi_a \cdot \pi_b)(c + L_\rho),
\end{aligned}$$

so that $\pi_{ab} = \pi_a \cdot \pi_b$ on V_ρ.

For $a \in A$ and $b + L_\rho, \ c + L_\rho \in V_\rho$:

$$\begin{aligned}
\langle b + L_\rho, \pi_a^*(c + L_\rho) \rangle &= \langle \pi_a(b + L_\rho), c + L_\rho \rangle \\
&= \langle ab + Lr, c + L_\rho \rangle \\
&= \rho(c^* ab) \\
&= \rho((a^* c)^* b) \\
&= \langle b + L_\rho, a^* c + L_\rho \rangle \\
&= \langle b + L_\rho, \pi_{a^*}(c + L_\rho),
\end{aligned}$$

so that $\pi_a^* = \pi_{a^*}$ on V_ρ.

$V_\rho \subset \mathcal{H}_\rho$ is a dense subset, so the three properties above hold on $\mathcal{H}_\rho$ by continuity. Hence, $\pi_\rho : A \to \mathcal{B}(\mathcal{H}_\rho)$ is a representation of A. As to the unit vector, consider $x_\rho := \mathbb{1} + L_\rho \in V_\rho$. Then for $a \in A$,

$$\begin{aligned}
\langle \pi_\rho(a) x_\rho, x_\rho \rangle &= \langle \pi_a(\mathbb{1} + L_\rho), \mathbb{1} + L_\rho \rangle \\
&= \langle a + L_\rho \mathbb{1} + L_\rho \rangle \\
&= \rho(a);
\end{aligned}$$

in particular, $\langle x_\rho, x_\rho \rangle = \rho(\mathbb{1}) = 1$, so x_ρ is a unit vector in $\mathcal{H}_\rho$. $\square$