

# 1 Perceptual Similarities Among Wallpaper Group 2 Exemplars

3 Peter J. Kohler<sup>1,2</sup>, Shivam Vedak<sup>3</sup>, and Rick O. Gilmore<sup>3</sup>

4 <sup>1</sup> York University, Department of Psychology, Toronto, ON M3J 1P3, Canada

5 <sup>2</sup> Centre for Vision Research, York University, Toronto, ON, M3J 1P3, Canada

6 <sup>3</sup>Department of Psychology, The Pennsylvania State University, Pennsylvania, USA

## 7 Abstract

8 Symmetries are abundant within the visual environment, and many animals species are  
9 sensitive to visual symmetries. Wallpaper groups constitute a class of 17 regular textures  
10 that each contain a distinct combination of the four fundamental symmetries, translation,  
11 reflection, rotation and glide reflection, and together represent the complete set of possible  
12 symmetries in two-dimensional images. Wallpapers are visually compelling and elicit responses  
13 in visual brain areas that precisely capture the symmetry content of each group in humans  
14 and other primates. Here we ask to what extent exemplars from the same wallpaper group are  
15 perceptually similar. We algorithmically produce a set of well-matched exemplars from 5 of the  
16 17 wallpaper groups and instructed participants to freely sort the exemplars from each group  
17 into as many subsets as they wished based on any criteria they saw appropriate.  $P_1$ , the simplest  
18 of the 17 groups, was consistently rated more self-similar than any other group, while the other  
19 four groups, although varying in symmetry content, were comparable in self-similarity. Our  
20 results suggest that except for the most extreme case ( $P_1$ ), perceived self-similarity of wallpaper  
21 groups is not directly tied to categories of symmetry based on group theory.

## 22 Introduction

23 Symmetry has been recognized as important for human visual perception since the late 19th cen-  
24 tury (Mach, 1959). In the two spatial dimensions relevant for images, symmetries can be combined  
25 in 17 distinct ways, *the wallpaper groups* (Fedorov, 1891; Polya, 1924; Liu et al., 2010). Wallpaper  
26 groups are different from stimuli typically used to probe the role of symmetry in visual perception  
27 in two ways: First, they contain combinations of the four fundamental symmetry types trans-  
28 lation, reflection, rotation and glide reflection, rather than just reflection or mirror symmetry,  
29 which have been the focus of most vision research. Second, the symmetries in wallpaper groups  
30 are repeated to tile the plane, rather than positioned at a single image location as is usually the case.  
31 These differences, and the fact that wallpaper groups together form the complete set of symme-  
32 tries possible in the two-dimensional image plane, make wallpapers an interesting stimulus set for  
33 studying perception of visual symmetries.

34 Brain imaging studies using functional MRI (Kohler  
 35 et al., 2016) and EEG (Kohler et al., 2018; Kohler and  
 36 Clarke, 2021) have shown that the human visual system  
 37 carries detailed and precise representations of the sym-  
 38 metries within the individual wallpaper groups. Func-  
 39 tional MRI evidence from macaque monkeys reveal sim-  
 40 ilar representations in analogous areas of the macaque  
 41 visual system (Audurier et al., 2021).

42 These representations, complex as they are, do not  
 43 appear to be readily available for driving conscious  
 44 behaviour: Humans have limited intuitive sense of  
 45 group membership for wallpaper group exemplars, as  
 46 evidenced by behavioral experiments showing that al-  
 47 though naïve observers can distinguish many of the wall-  
 48 paper groups (Landwehr, 2009), they tend to sort exem-  
 49 plars into fewer (4-12) sets than the number of wallpaper  
 50 groups, often placing exemplars from different wallpaper  
 51 groups in the same set (Clarke et al., 2011). Wallpaper  
 52 groups are nonetheless visually compelling, and anec-  
 53 dotally we have observed that exemplars from a given  
 54 group can be quite perceptually diverse. This observa-  
 55 tion inspired the current study. Here, we use behavioral  
 56 sorting, a common technique to study perceptual catego-  
 57 rization (Milton et al., 2008; Pothos et al., 2011), to probe  
 58 the perceptual self-similarity of different exemplars from  
 59 the same wallpaper group, and assess the extent to which  
 60 self-similarity varies across five groups.

61 We algorithmically generated 20 well-matched exem-  
 62 plars from each group (see Figures 1 and 2 for a selection  
 63 of the exemplars, and the **Materials and Methods** sec-  
 64 tion for details on how they were generated) and printed  
 65 them out on white cardstock. We then gave participants  
 66 the 20 cards with exemplars from each wallpaper group,  
 67 and asked them to freely sort them into as many sub-  
 68 sets as they wished based on any criteria they saw ap-  
 69 propriate. This approach allowed us to compare the five  
 70 wallpaper groups, both in terms of how many subsets  
 71 participants generated, and also in terms of the *Jaccard*  
 72 index, a summary statistic capturing the similarity across  
 73 exemplar pairs for each group. Within each group, we  
 74 were also able to identify exemplar pairs that were rated

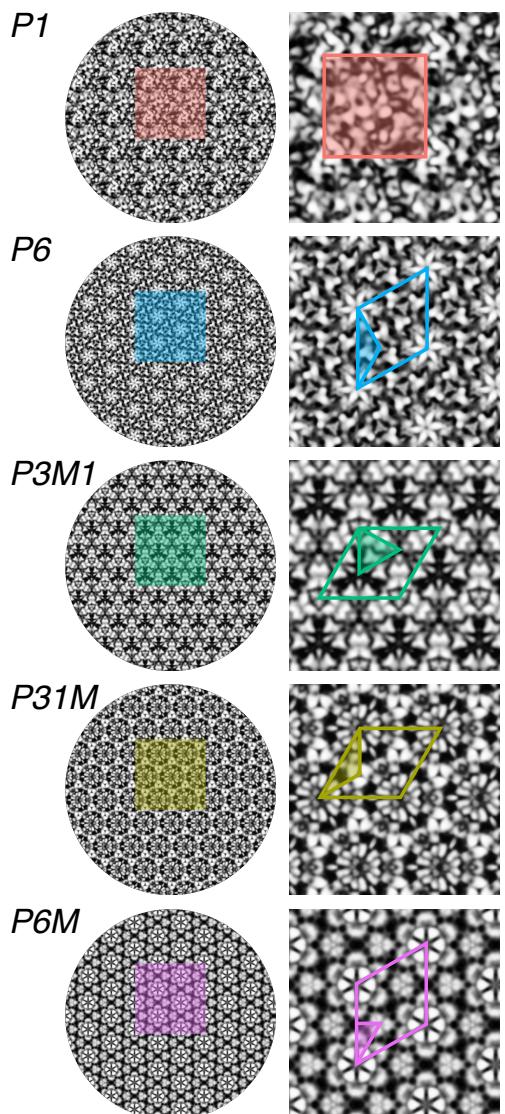


Figure 1: The fundamental region and lat-  
 tice structure of the five wallpaper groups  
 used in the study. The complete wallpa-  
 per is shown in the left-hand column with  
 a shaded region that is repeated and en-  
 larged in the right-hand column. The col-  
 ored outline in the enlarged region indi-  
 cates the repeating lattice for each group,  
 while the shaded area indicates the funda-  
 mental region (see text). For  $P_1$  the funda-  
 mental region covers the entire lattice.  
 Note that even though  $P_6$  and  $P_{31}M$  have  
 the same fundamental region and lattice  
 shapes, they differ in terms of the symme-  
 tries present within the lattice - most no-  
 tably,  $P_{31}M$  contains reflection symmetry  
 while  $P_6$  does not. The symmetry content  
 of each group is detailed on the wallpaper  
 group wikipedia page.

75 as highly similar and highly dissimilar. Our main conclusion is that  $P_1$  was systematically more  
 76 self-similar than the any other groups, while the other four other groups could not be distin-  
 77 guished on these measures. We also show that for all five groups, participants consistently group  
 78 certain pairs of exemplars together, although the number of consistent pairs varies among groups.  
 79 Our results open the door to further investigations into the factors that drive perceptual simi-  
 80 larity among wallpaper group exemplars, and indeed among exemplars from different classes of  
 81 structured patterns.

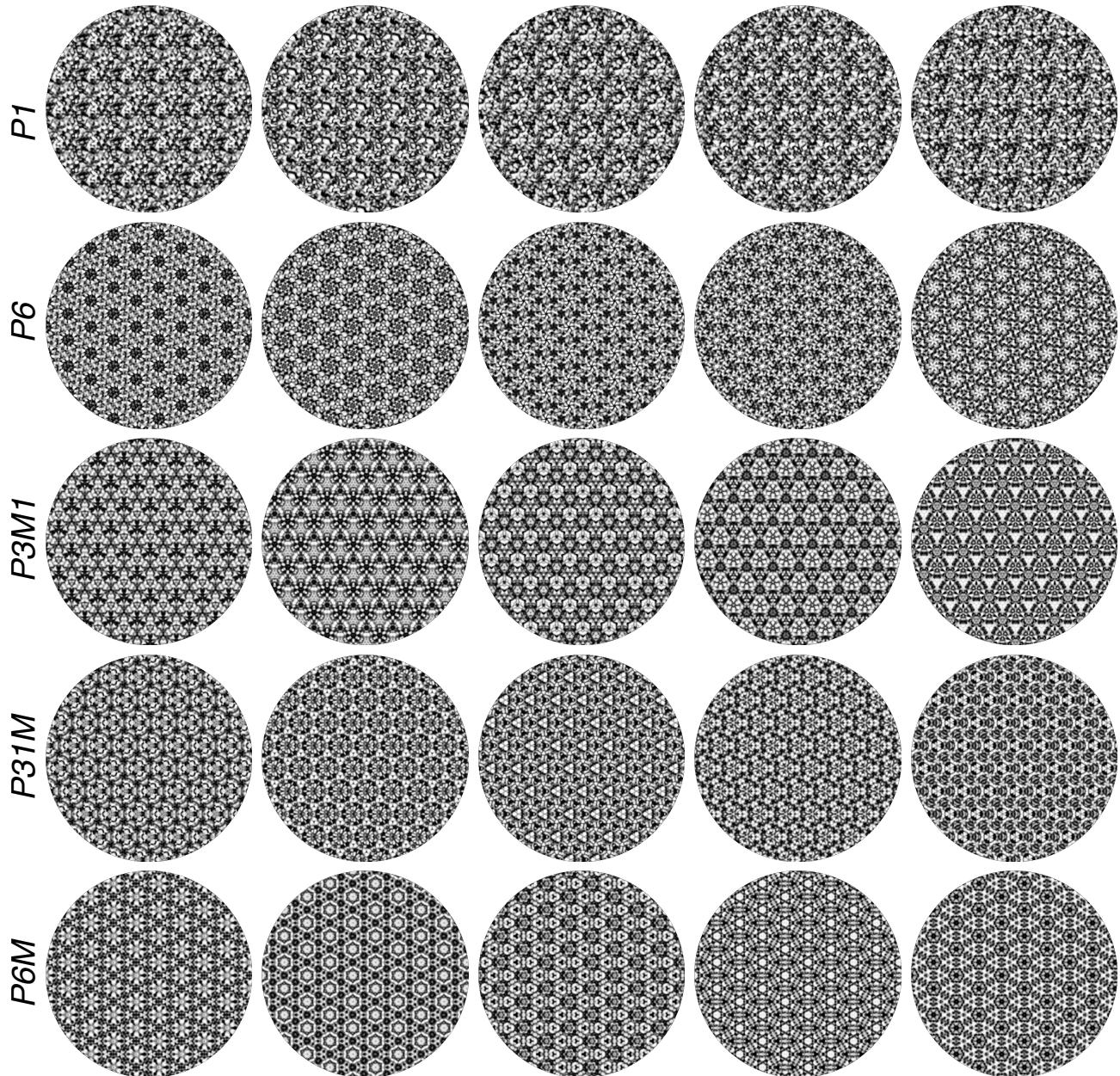


Figure 2: 5 of the 20 exemplars used for each group are shown to highlight the diversity among exemplars.

## 82 Results

83 Wallpaper group  $P_1$  was more self-similar than the other four groups. This was evident in the  
 84 number of sets generated for this group across participants, which was lower for  $P_1$  (median  
 85 = 3) than for the other groups (median = 4-5, see Figure 3). We confirmed this observation  
 86 statistically by running a repeated measures analysis of variance (ANOVA) with group as a fixed

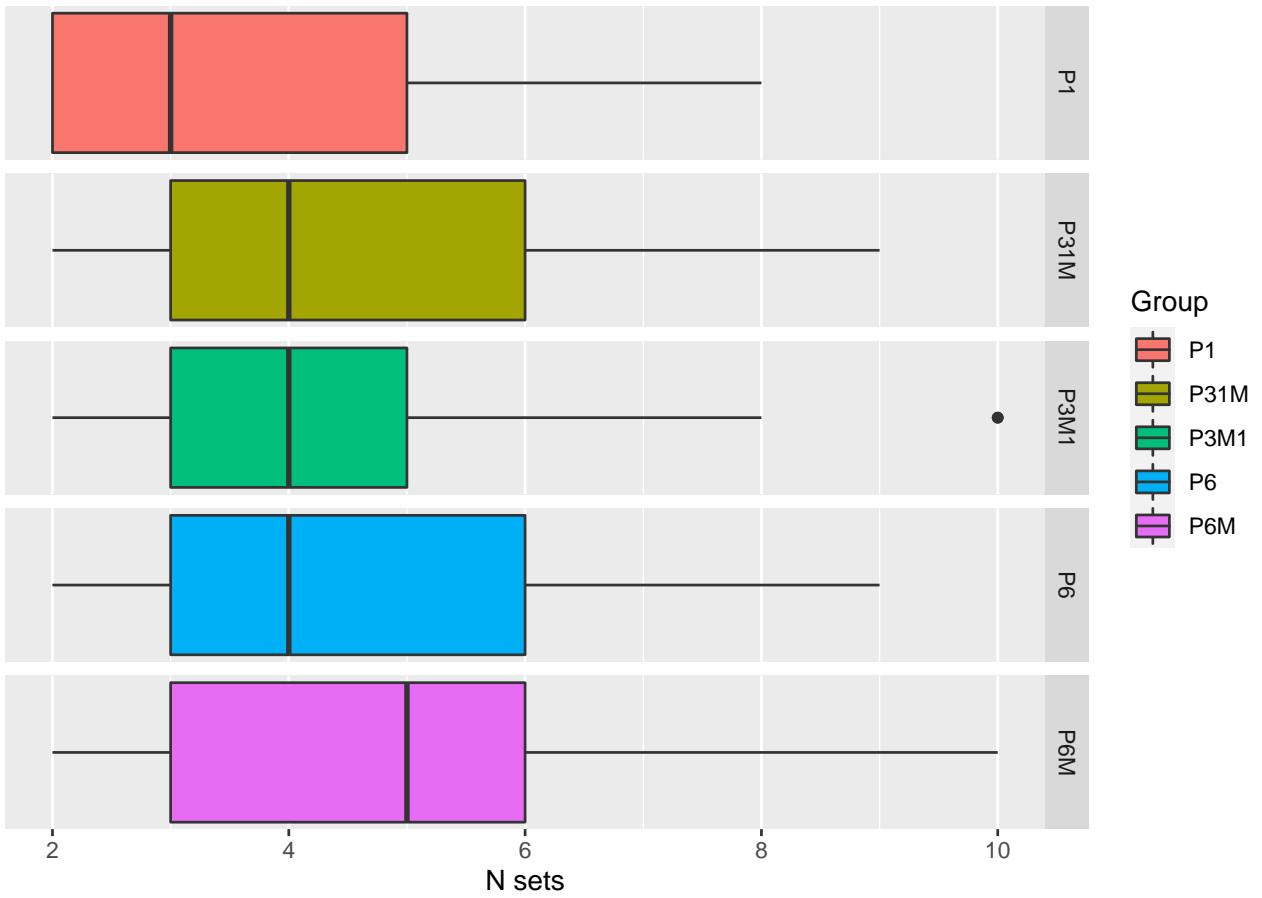


Figure 3: Boxplots showing the number of subsets generated by participants for each of the wallpaper groups. The lower box boundary is the 25th percentile. The dark line in the box is the median. The upper box boundary is the 75th percentile. The “whiskers” show -/+ the interquartile range \* 1.5.

factor and participant as a random factor, which revealed a significant effect of group ( $F(4,124) = 7.830, p < 0.0001$ ). Post-hoc pairwise *t*-tests showed that the mean number of sets was lower for *P<sub>1</sub>* than all other groups, but no other means differed. Next, we computed the Jaccard index (see Materials and Methods) across participants for every pairwise combination of exemplars in each group. This provides a measure of the similarity between exemplars within each group. *P<sub>1</sub>* had systematically higher Jaccard indices than the four other groups (see Figure 4), as confirmed by an ANOVA with wallpaper group as a factor. The analysis revealed a statistically significant effect of group ( $F(4, 495) = 20.178, p < 0.0001$ ). Post-hoc pairwise *t*-tests showed that *P<sub>1</sub>* had higher Jaccard indices than all other groups ( $p < 0.0001$ ). The fact that the group (*P<sub>1</sub>*) for which fewer subsets were generated also had higher Jaccard indices than the other groups illustrates the inherent link between the two measures. For wallpaper groups where the 20 exemplars are sorted into fewer subsets, each individual exemplar pair is more likely to be a member of the same subset, and less likely to be a member of distinct subsets. This in turn leads to higher Jaccard indices. Our pairwise *t*-tests also showed that *P<sub>31M</sub>* had lower Jaccard indices than *P<sub>6</sub>* ( $p = 0.037$ ). This effect is relatively weak, but may reflect real differences in how consistently exemplars were grouped together across participants. We will explore this idea more in depth shortly, but for now we can conclude that out of the five groups tested, *P<sub>1</sub>* is the only one that can be reliably differentiated based on our measures, being higher on self-similarity among the exemplars, and thus lower on

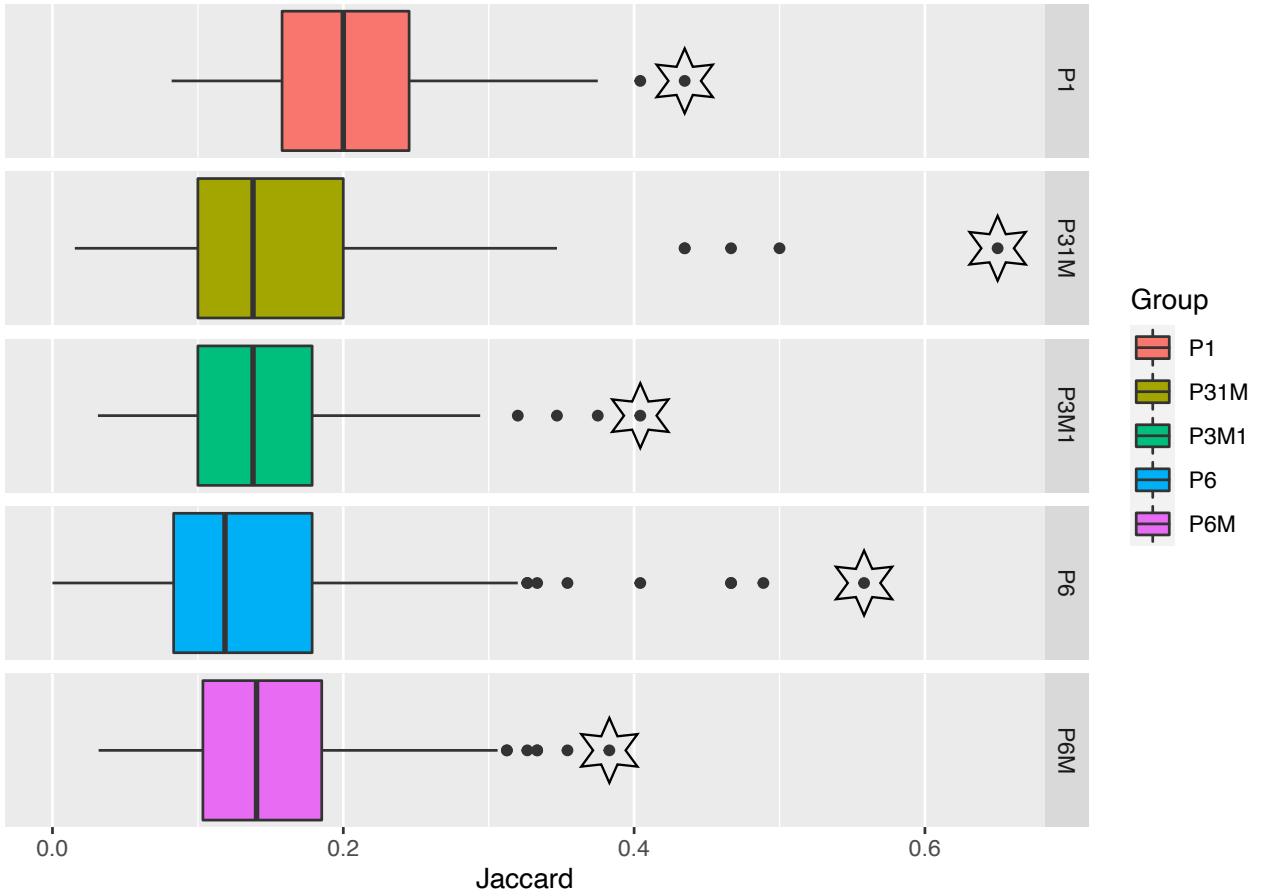


Figure 4: Boxplots showing Jaccard indices for every pairwise combination of exemplars in each of the wallpaper groups. Note that each data point here is the Jaccard index for a particular exemplar pair calculated across participants, unlike Figure 3 where each data point is a participant. The box boundary and whiskers follow the same logic as in Figure 3. The exemplar pairs with the highest Jaccard indices have been highlighted with stars. Those outlier pairs are explored further in Figure 6.

105 diversity among exemplars.

106 In order to quantify the extent to which exemplars were consistently grouped together, we  
 107 ran a permutation analysis in which exemplar labels were shuffled among the sets generated  
 108 for each participant (see Materials and Methods). This provides, for each group, the expected  
 109 distribution of Jaccard indices for every pairwise combination of exemplars, if exemplars were as-  
 110 signed randomly to subsets. And the analysis allows us to compute an empirical  $z$ -score that  
 111 expressed the extent to which a given pair of exemplars deviates from random assignment.  
 112 Because the random distribution is generated  
 113 by shuffling exemplars across the specific sets  
 114 generated by each participant for each group,  
 115 this  $z$ -score is independent of the number of  
 116 sets. If for a given group, none of the pairs de-  
 117 viate significantly from the random distribu-  
 118 tion, it would indicate that no exemplar pairs  
 119 were consistently grouped together across par-  
 120 ticipants. To estimate the extent to which this  
 121 is the case, we look at the distribution of  $z$ -

consistent pairings		
group	$p < 0.01$	$p < 0.0001$
$P_1$	6	1
$P_{31M}$	17	10
$P_{3M1}$	12	3
$P_6$	17	11
$P_{6M}$	15	4

Table 1: Number of consistent pairings at two different  $\alpha$ -levels for the five groups.

122 scores across the pairs for each group, as plotted  
 123 in Figure 5, and count the number of pairs for each group for which the  $p$ -value associated  
 124 with the threshold exceeds a given  $\alpha$  value. At a threshold of  $\alpha = 0.01$ , several pairs survive for all  
 125 groups, and even at a much more conservative criterion of  $\alpha = 0.0001$  most groups have more than  
 126 one pairing that survives (see Table 1). It is worth noting that the latter threshold ( $\alpha = 0.0001$ ) is  
 127 lower than the  $\alpha$  associated with a Bonferroni correction within group, given that there are 190  
 128 pairs per group:

$$\alpha = \frac{0.05}{190} = 0.0003$$

129 . So we conclude that for several exemplar pairs, participants are consistent in how they tend to  
 130 pair the exemplars. It is interesting to consider that this measure of consistency might provide  
 131 another way of differentiating wallpaper groups in terms of perceptual self-similarity. While  
 132 groups  $P_{31}M$ ,  $P_3M_1$ ,  $P_6$  and  $P_6M$  have comparable Jaccard scores (see Figure 4), they differ in the  
 133 number of consistent pairings, with  $P_{31}M$  and  $P_6$  producing more consistent pairs than the other  
 134 two (see Figure 5).

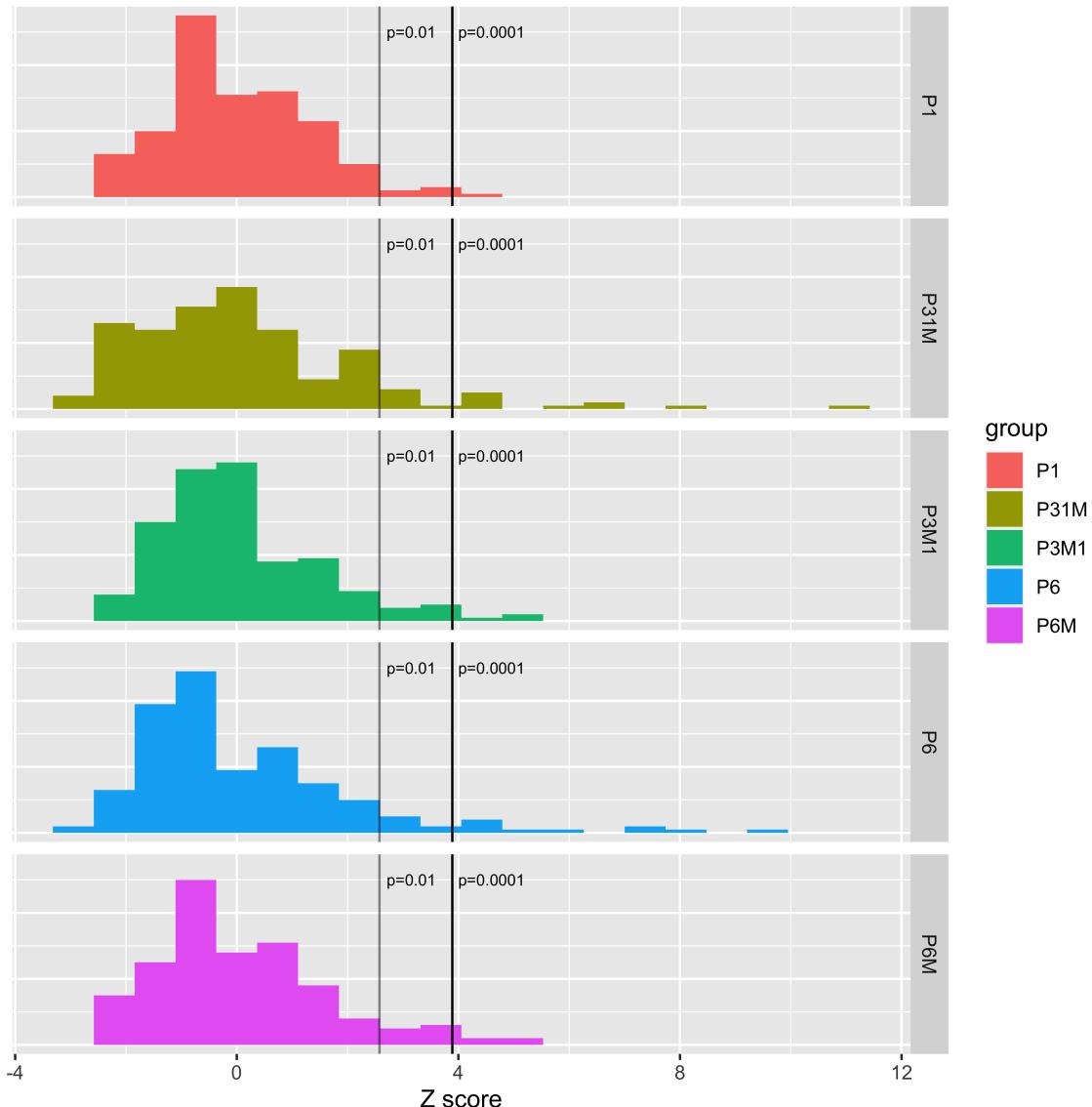


Figure 5: Distribution of  $z$ -scores across the 190 pairs in each of the five wallpaper groups. The two lines indicate the  $z$ -scores associated with  $\alpha$  of 0.01 and 0.0001, respectively.

135 The Jaccard indices also allow us to focus on exemplar pairs that have a high level of similarity  
136 relative to the rest of the pairs in the set. We do this by identifying outliers pairs from each  
137 group in term of Jaccard indices, as identified with stars in Figure 4. Because the Jaccard indices  
138 are computed across participants, these outliers are also among the pairs most consistently sorted  
139 together, as identified in Figure 5. For each exemplar in each outlier pair, we can visualize the  
140 pairwise similarity (as measured by the Jaccard index) to every other exemplar in the set (see  
141 Figure 6). That is, we can visualize portions of the network of perceived similarity within a set of  
142 exemplars. Future work could probe the extent to which networks of perceived similarity have  
143 similar structure across wallpaper groups and examine what perceptual features best account for  
144 participants' perceptions of exemplar similarity.

## 145 Discussion

146 Previous work has demonstrated that visual cortex of both humans and macaque monkeys car-  
147 rries highly detailed representations of the symmetries within wallpaper groups, as evidenced  
148 by systematic differences in the magnitude of the response elicited by different groups (Kohler  
149 et al., 2016; Kohler and Clarke, 2021; Audurier et al., 2021). This distinction between groups can  
150 also be observed in psychophysical threshold measurements (Kohler and Clarke, 2021), although  
151 observers may not have a strong awareness of the wallpaper group membership of individual  
152 exemplars (Clarke et al., 2011). In the current study, we explored a new piece of the story about  
153 how wallpaper groups are processed by the visual system, namely the issue of how self-similar  
154 different exemplars from the *same* wallpaper group appear to untrained observers. We tested this  
155 by asking participants to spontaneously sort 20 exemplars from each of five wallpaper groups into  
156 different subsets.

157 Our first finding concerns the number of subsets generated for each group. We find that  $P_1$  is  
158 divided into fewer subsets than the other four groups. This indicates that the limited complexity  
159 of this group, which contains only translation symmetry, has a direct effect of the number of dis-  
160 tinct subsets. The relationship between complexity / symmetry content and number of subsets  
161 produced is not straightforward, however, as indicated by the fact that  $P_{6M}$  is not consistently  
162 grouped into more subsets than  $P_6$ ,  $P_{2M_1}$  and  $P_{3_1M}$ , despite the fact that these other groups all  
163 contain fewer symmetries than  $P_{6M}$  - and generate weaker brain activity (Kohler and Clarke,  
164 2021). We speculate that this lack of further differentiation is a result of an upper limit on how  
165 additional complexity can influence perceptual self-similarity.

166 We also computed Jaccard indices that, for every possible exemplar pair, expresses the fre-  
167 quency of those two exemplars being grouped together. As described above, the average Jaccard  
168 index for a group is inherently linked to the number of subsets produced for that group, be-  
169 cause fewer subsets mean that exemplars are more likely to be made members of the same pair,  
170 and less likely to be made members of the same pair. It is therefore not surprising that we find  
171 the same general pattern for Jaccard indices and number of subsets, namely that  $P_1$  has higher  
172 indices than the other groups. The advantage of the Jaccard indices, however, is that they allow  
173 us to conduct a permutation analysis that quantifies the extent to which pairs of exemplars are

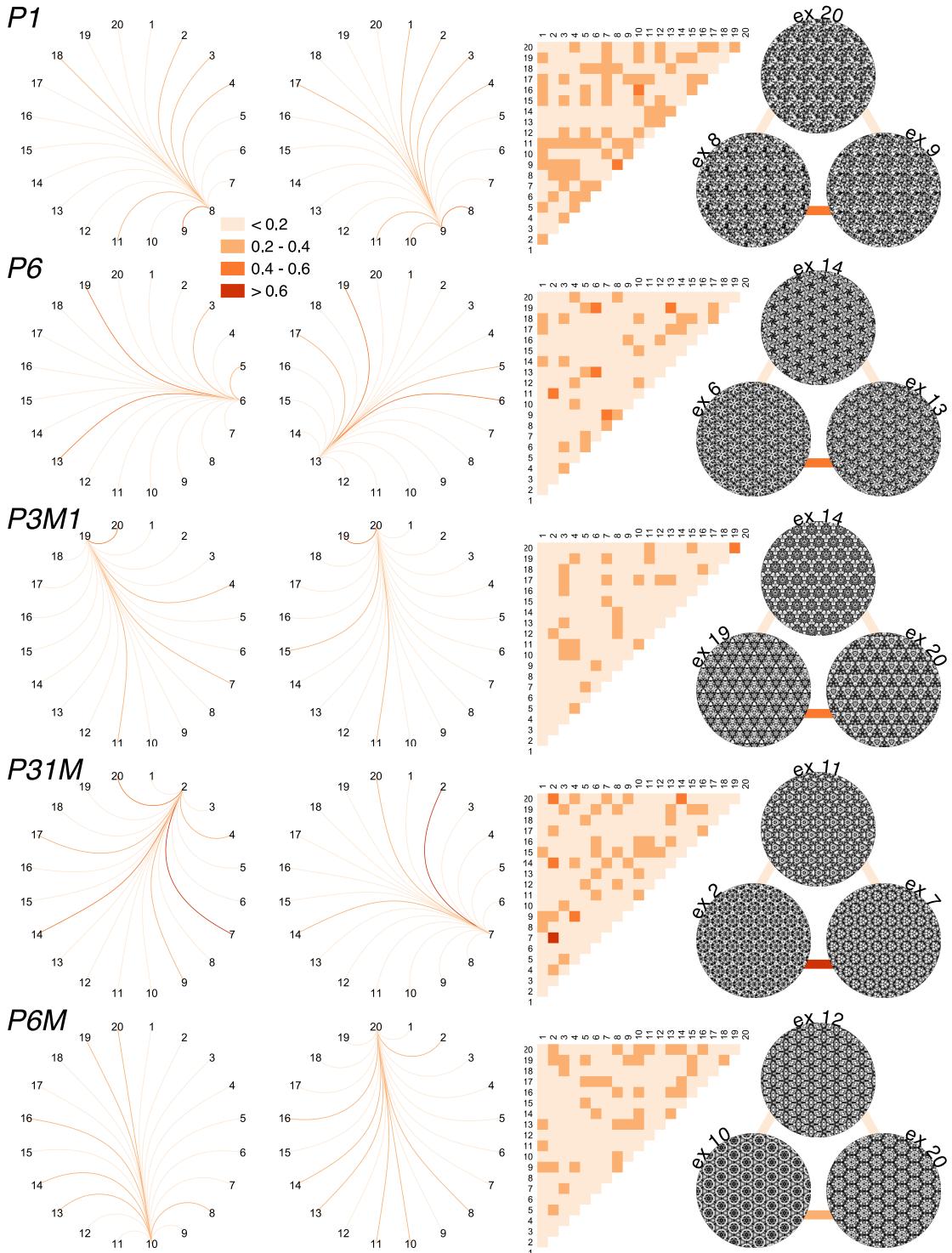


Figure 6: For each wallpaper group, we identified the two most self-similar exemplars, the same pair that is indicated by the right-most datapoint for each group in Figure 5. The two circular network plots are showing the pairwise similarities between those two exemplars and every other exemplar in the set. The pairwise similarities across all exemplars are plotted as a similarity matrix and on the rightmost side of the plot, the two most self-similar exemplars (bottom) are plotted with the exemplar that was least similar to both (top). The connecting lines between the exemplars indicate the similarity.

174 consistently grouped together across participants, independent of the number of sets produced  
 175 for a given group. It is important to note that consistency in the choice of which exemplars to  
 176 group into subsets is not an unavoidable consequence of our experimental design, and it does  
 177 not follow naturally from the results described so far. It would be perfectly possible for partic-

178 ipants to group the sets together, producing fewer subsets for  $P_1$  as observed, but exhibit no  
179 consistency across participants at all. That is not what we see, however. Even when setting a  
180 conservative threshold, all five groups produce one or more pairs that are consistently grouped  
181 together, demonstrating that the sorting of exemplars into subsets is not done randomly or in  
182 an arbitrary across the participants. Rather, different individuals agree to some extent on which  
183 exemplars belong together. Because our measure of consistency is independent of the number of  
184 subsets produced for a given group, it allows us to show that although  $P_1$  has the highest overall  
185 Jaccard indices (as a result of the fewer sets produced for this group), it in fact produces fewer  
186 consistent pairs than other groups (see Table 1).

187 In sum, we find consistencies in the way that untrained human observers sort wallpaper  
188 images. Observers sort exemplars with translational symmetry alone ( $P_1$ ) into smaller numbers  
189 of sets than exemplars with rotational or reflection symmetry. On average, pairs of  $P_1$  exemplars  
190 are sorted together more often than exemplars from other wallpaper groups. At the same time,  
191 some specific exemplar pairs from wallpaper groups with 3- or 6-fold rotational or reflection  
192 symmetry are sorted together substantially more often than predicted by chance.

193 Taken together, our results suggest that human observers are sensitive to the dimensions of  
194 2D symmetry (translation, rotation, and reflection) embedded in wallpaper exemplars. However,  
195 their behavior shows only weak evidence that group-theoretic measures of symmetry influence  
196 the perception of self-similarity. These results contribute to a small, but growing literature on  
197 the perception of visual aesthetics (Carneiro et al., 2012; Graham et al., 2010; Friedenberg,  
198 2012; Laine-Hernandez and Westman, 2008; Richards, 1972) where symmetry is one of many  
199 contributing factors.

## 200 Materials and Methods

### 201 Participants

202 33 participants (9 Male, 24 Female), ranging in age between 18 and 35 completed this study. All  
203 participants had self-reported 20/20 or corrected to 20/20 vision. We obtained written consent to  
204 participate from all participants under procedures approved by the Institutional Review Board  
205 of The Pennsylvania State University (#38536). The research was conducted according to the  
206 principles expressed in the Declaration of Helsinki. Participants include  $n=11$  collected and  
207 described in (Vedak, 2014), plus an additional group collected at a later date using the same  
208 protocol.

### 209 Stimulus Generation

210 Five wallpaper groups ( $P_1$ ,  $P_3M_1$ ,  $P_31M$ ,  $P_6$  and  $P_6M$ ) that has previously been shown to be high in  
211 self-similarity (Clarke et al., 2011), were selected. 20 exemplars from each of these five wallpaper  
212 groups were generated using a modified version of the methodology developed by Clarke and  
213 colleagues (Clarke et al., 2011) that we have described in detail elsewhere (Kohler et al., 2016).  
214 Briefly, exemplar patterns for each group were generated from random-noise textures, which  
215 were then repeated and transformed to cover the plane, according to the symmetry axes and

216 geometric lattice specific to each group. The use of noise textures as the starting point for  
217 stimulus generation allowed the creation of an almost infinite number of distinct exemplars of  
218 each wallpaper group. To make individual exemplars as similar as possible we replaced the power  
219 spectrum of each exemplar with the median across exemplars within a group. These images  
220 were printed onto white cardstock and cut into squares, allowing participants to manipulate the  
221 orientation of the images during the sorting tasks. Five exemplars from each group are shown  
222 (in reduced size) in Figure 2.

## 223 Procedure

224 Participants were presented with the 20 exemplars of a single wallpaper group (i.e. P<sub>1</sub>, P<sub>3</sub>M<sub>1</sub>,  
225 P<sub>3</sub>M, P<sub>6</sub>, P<sub>6</sub>M) and instructed to sort them into subsets by placing them into piles. Participants  
226 were advised to sort the exemplars into as many piles as they deemed necessary based on  
227 whatever criteria they desired. There were no time constraints placed on this sorting task, and  
228 the participants were allowed to move exemplars between piles until they were satisfied with  
229 their classification. This method was then repeated for the remaining four wallpaper groups for  
230 each participant, with group presentation order randomized between participants. These tasks  
231 were carried out on a large table with sufficient space to randomly lay out all twenty exemplars  
232 of each set, illuminated by normal overhead room lighting. Upon completion of each sorting  
233 task, participants were asked to verbalize which features they used to sort the exemplars. After  
234 completion of all five sorting tasks, participants were asked which if they had a distinct method  
235 for sorting the images, and if any wallpaper group was particular easy or difficult to sort.

## 236 Generating the Jaccard Index

237 The data was prepared for analysis by creating one binary variable for each subset created by  
238 each participant within a sorting task. Then, each exemplar was assigned a value of one (1) if  
239 it was included in a subset, or a value zero (0) if it was not. Next, the similarity of each pair of  
240 exemplars within a sorting task was calculated using the Jaccard index, a measure of similarity  
241 and diversity for binary data. This index is calculated by the equation

$$J = \frac{x}{x + y + z}$$

242 with x representing the number of subsets that contained both exemplars, and y and z the number  
243 of subsets that contain only one exemplar of the pair (Capra, 2005), across participants. Thus,  
244 the Jaccard index is the ratio of the number of subsets containing both exemplars of a pair to the  
245 number of subsets containing at least one of the exemplars of a pair, thereby excluding subsets  
246 with joint absences.

## 247 Permutation Analysis

248 The permutation analysis involved generating a randomized dataset, as follows. For each partic-  
249 ipant and wallpaper group, we randomized which specific exemplars were sorted together. This  
250 retained the basic structure of each participants' sorting data—the number of subsets created—  
251 but randomized the relationship between specific wallpaper exemplars that were sorted together

252 across the participants. We then created 1,000 such permuted datasets, and calculated the Jac-  
253 card index for each exemplar pair within each group for each of the permuted datasets. This  
254 permitted the calculation of an *empirical* Jaccard index based on the permuted data from which  
255 distributional statistics like  $z$  could be calculated. The *observed* Jaccard indices for each exemplar  
256 pair were then compared to the empirically-derived reference distribution to determine which  
257 exemplar pairs were sorted together more frequently than chance would predict.

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