Physics via Geometric Algebra

An alternative approach to physical theories

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July, 2015

To my wife and my son, Grecia and Adrian, for teaching me to live when everything is changing and to live changing everything around.

It is the harmony of the diverse parts, their symmetry, their happy balance; in a word it is all that introduces order, all that gives unity, that permits us to see clearly and to comprehend at once both the ensemble and the details.
Henri Poincare

Preface

Geometric Algebra is a mathematical theory with a long history. One can track its root back to 1840's, with the early works of Grassmann and Hamilton. Their ideas were ahead of time, and it was only 40 years after that W. Clifford put together this ideas in what we now call Clifford Algebras, but his early death and the vector system proposed by Gibbs made this remain as abstract ideas for pure mathematicians. In 1960's D. Hestenes used this ideas with a geometrical interpretation to explain Space-Time Lorentz transformation, whence the name Geometric Algebra.

Time has passed and we are 40 years since then, and a question remains open: Is Geometric Algebra a unified language for physics? I don't want to give an answer, because there are plenty of books and articles that have partially done that. The problem I had with them is that, with a few exceptions, they try to convince you that geometric algebra is a better tool than complex numbers, matrix algebra, tensors, differential forms, etc. and forget to actually do physics and give a hands on calculation.

I am in the process of writing this book, and my plan for this is the following: Write a mathematical introduction to the subject. From axioms to some really nice results like the isomorphism with the most important structures in mathematics. One feature of Geometric Algebra that seems very attractive is its coordinate-free formulations, but it is useful to know how to actually compute something when a basis is chosen. Once this is done, then I will continue to the physics.

In the physics section is it not only about showing the nice things you can do with geometric algebra, like reducing the Maxwell's equation to a single one, but deriving the most general consequences of this equation and, again, using some numbers to actually get something more than just expressions. Sometimes a graph is all that is needed to make sense of symbols, and sometimes relate a real situation is better. It's the "equations out of equations" chain that makes hard to follow the reasoning, and is that is exactly what I want to fill with this book. So here we go, and as I write the book I will come here to explain what has been done and what is missing.

Contents

Part I

Mathematical theory

Chapter 1

Introduction

Geometric algebra is a geometric interpretation of Clifford algebras. To build an axiomatic study of them, is useful to recover some basic definitions.

1.1 Axioms and definitions

We begin with the definitions[2] of an algebra and a subalgebra:

Definition 1.1.1 *An algebra over a commutative ring R, or R-algebra, is an R-module A with a multiplication that is bilinear ($a(b+c)) = ab + ac$, and $(ra)b = a(rb) = r(ab)$ for all $a, b, c \in A$ and $r \in R$), associative ($a(bc) = (ab)c$ for all $a, b, c \in A$), and has an identity element 1 ($1a = a = a1$ for all $a \in A$).*

Definition 1.1.2 *A subalgebra of an R-algebra A is a subset S of A that is both a subring of A and a submodule of A.*

An example of an algebra is easily found on complex numbers, that is an algebra over $\mathbb{R}$. It is also possible to see that every subalgebra of the R-algebra is itself an R-algebra. This basic ideas are fundamental in the study of Geometric algebra because there are several distint objects in it, and sometimes will be useful to restring our attention to a specific subalgebra. It's also necessary to define it as a graded algebra for reasons that will be obvious later.

Definition 1.1.3 *A graded R-algebra is an R-algebra A with submodules $(A_n)_{n\geq 0}$ such that $A = \bigoplus_{n\geq 0} A_n, 1 \in A_0$, and $A_m A_n \subseteq A_{m+n}$ for all $m, n \geq 0$.*

Definition 1.1.4 *A graded submodule (subring, subalgebra, two-sided ideal) of a graded R-algebra $A = \bigoplus_{n\geq 0} A_n$ is a submodule (subring, subalgebra, two-sided ideal) S of A such that $S = \bigoplus_{n\geq 0}(A_n \bigcap S)$.*

The definition I used before are going to make the following reasoning clearer, but I have not yet defined what is a geometric algebra. I will use the definitions in [1], [3] as start point.

Definition 1.1.5 *A **geometric algebra** $\mathcal{G}$ is a graded algebra that obey the following axioms:*

Axiom 1: *$\mathcal{G}$ contains a field $\mathcal{G}_0$ of characteristic zero which includes 0 and 1.*

Axiom 2: *$\mathcal{G}$ contains a subspace $\mathcal{G}_1$ equipped with a non-degenerate, symmetric bilinear form B. This means that*

$$\textit{for all } y \in \mathcal{G}_1, B(x,y) = 0 \textit{ implies } x = 0$$

Axiom 3: *The square of every vector a in $\mathcal{G}_1$ is a scalar*

$$a^2 = B(a,a) * 1 = B(a,a) \in \mathcal{G}_0$$

Axiom 4: *If $\mathcal{G}_0 = \mathcal{G}_1$, then $\mathcal{G} = \mathcal{G}_0$. Otherwise for each integer $r \geq 2$, $\mathcal{G}_r$ is spanned by all products of r mutually anti-commuting elements of $\mathcal{G}_1$. The anti-commuting means that $ab = -ba$*

Bibliography

[1] E. Chisolm. Geometric Algebra. *ArXiv e-prints*, May 2012.

[2] Pierre A. Grillet. *Abstract algebra/ Pierre Antoine Grillet.* Graduate texts in mathematics: v. 242. New York : Springer, c2007.

[3] Kristopher N Kilpatrick. *The geometry in geometric algebra*. PhD thesis, 2014.