Investments

Fixed Income Securities

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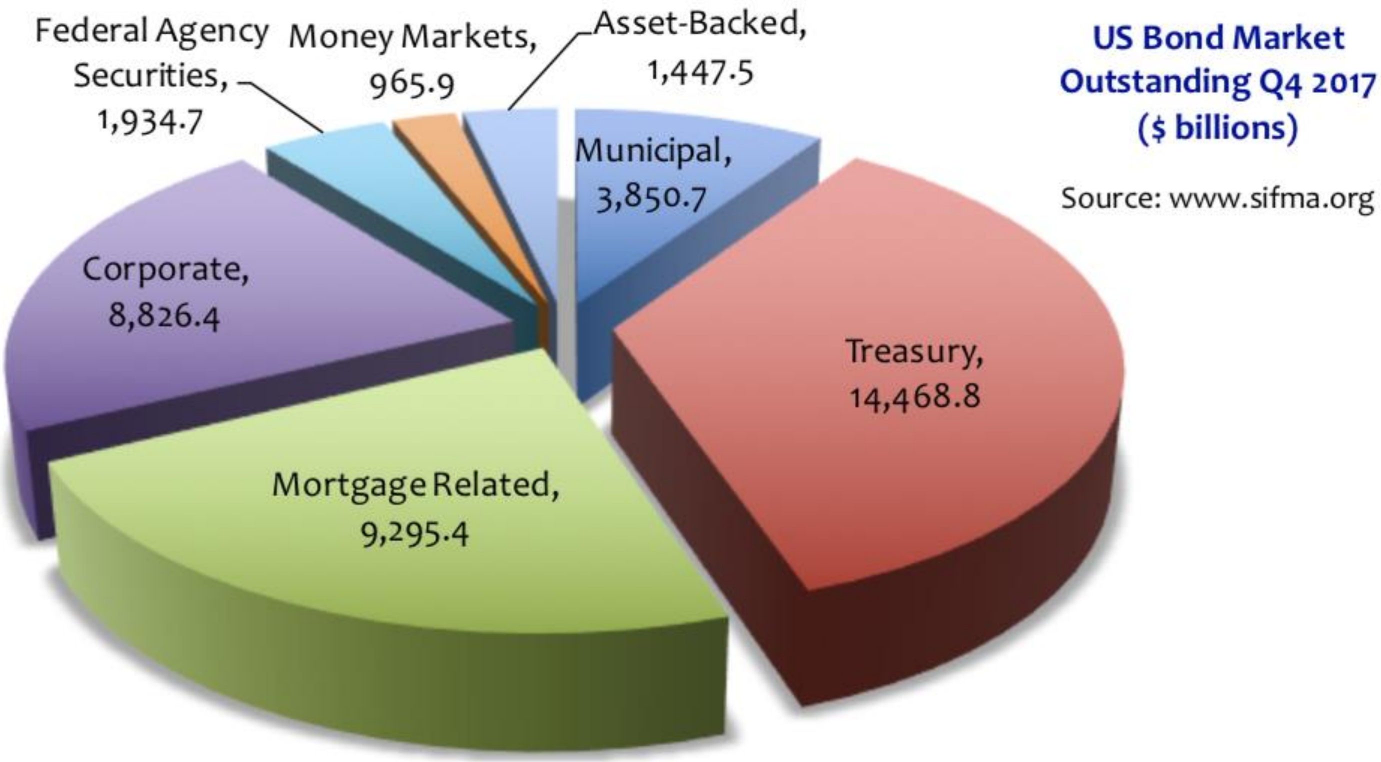
Outline

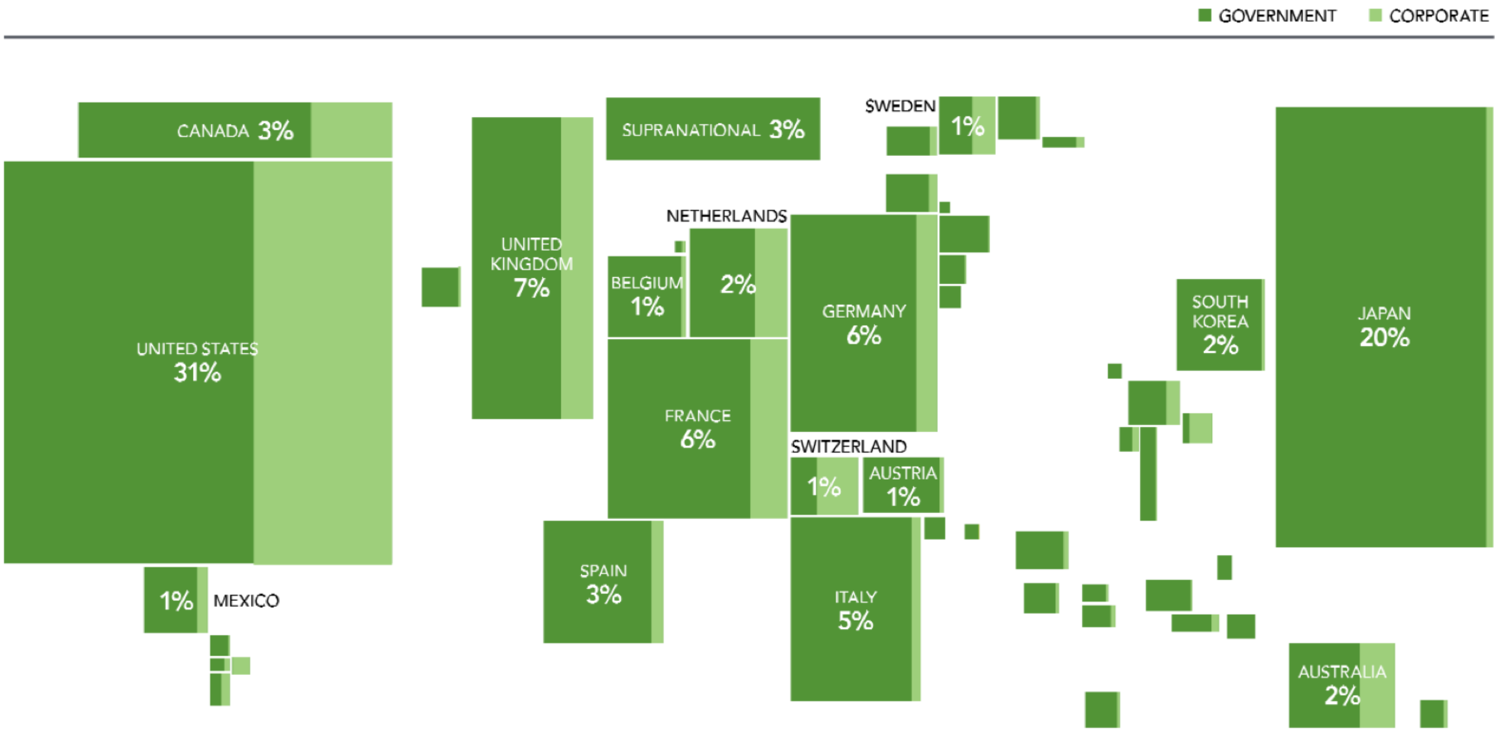
- What are bonds? What kind of bonds are traded?
- Bond Pricing.
- Bond Yields.
- Zero-coupon bonds.
- The Term Structure of Interest Rates.
- Forward Rates.
- Interest Rate Risk and Duration.
- Default Risk and Bond Ratings.

What is a bond?

A form of borrowing by governments, corporates and others



Global Bond Markets



Data is from Barclays Global Aggregate Ex-Securitized Bond Index. Many nations not displayed. Total may not equal 100% due to rounding.
For educational purposes; should not be used as investment advice. Barclays data provided by Barclays Bank PLC.

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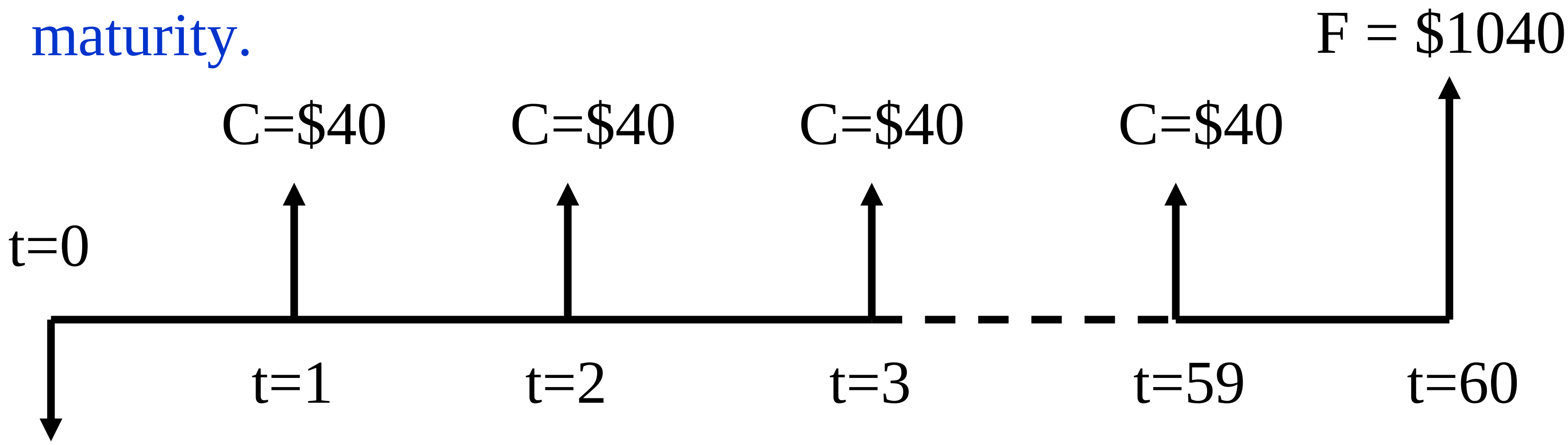
What is a bond?

- A **bond** is a financial security that promises the holder a set of pre-specified payments (fixed amounts or based on a formula) at known future dates
- Example: a 3-yr bond with notional 1,000 and 2% annual coupon
  - **Issuer**: the borrower (e.g., Governments, Corporations, etc.).
  - **Maturity Date** or **Redemption Date**: date of final cash flow, here 3yrs from now
  - **Principal value** (a.k.a par value or face value): cash flow at maturity, here 1,000
  - **Coupon Rate** (stated as APR): total annual cash flows expressed as a percentage of par value, here due at the end of each of the next three years
  - Other features: embedded options, covenants etc

Promised Cashflows	t=0	t=1	t=2	t=3
Coupon		20	20	20
Principal				1000
Total		20	20	1020

Example: Coupon Bonds

- **Example:** bond with par value of \$1000, annual coupon rate of 8%, and maturity of 30 years.
- The bondholder is entitled to semiannual payments of \$40 $\left(=\frac{0.08}{2}\times 1000\right)$ for the life of the bond, plus \$1000 par value at maturity.



Example Coupon Bond

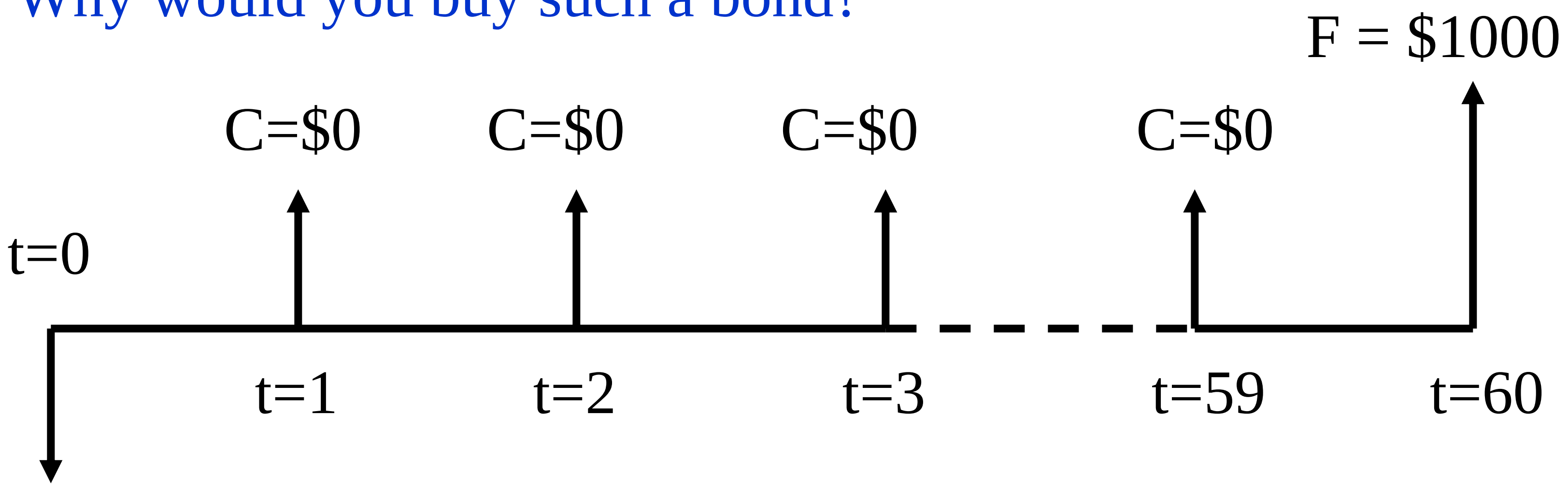
- 20-year bond issued by US state of S.C. January 1867.
- Face value of $1000
- Interest rate of 6% paid semiannually—each of the coupons could be turned in for $30 every 6 months.

Treasury Note

<HELP> for explanation. Govt GP

Example: Zero-Coupon Bonds

- Zero-coupon bonds (or discount bonds) make no coupon payments (zero-coupon rate). Investors receive par value at maturity but no interest until then.
- Why would you buy such a bond?

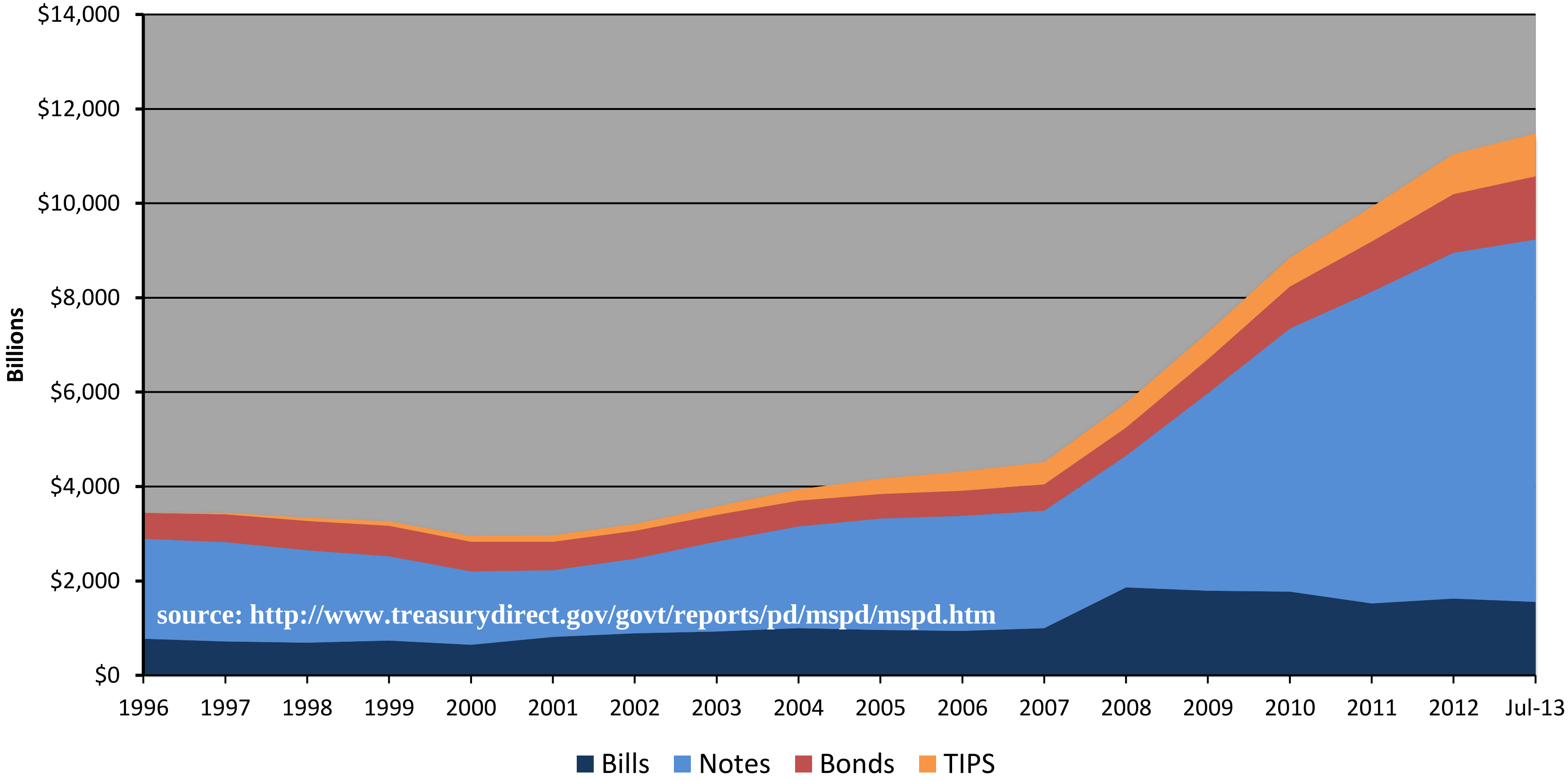


US Government Securities ("Treasuries")

- Mostly fixed-rate nominal bonds:
  - **T-Bills**: original maturity of 1 year or less (typically 1m, 3m, 6m and 1yr) discount (ie zero coupon) securities
  - **T-Notes**: original maturity 1 to 10 years (typically 2, 3, 5, 7 and 10 yrs) coupon-paying (semi-annual)
  - **T-Bonds**: original maturity of more than 10 years (typically 30 yrs) coupon-paying (semi-annual)
- Inflation-protected (**TIPS**)
  - Original maturity of 5, 10 and 30 years
  - Coupon-paying (semi annual), coupon and principal adjusted for inflation
- Other (Floating Rate Notes, STRIPS etc)

Outstanding U.S. Treasury Debt

- July 2013: $11,464 Billion held by public



More Recently

TABLE I -- SUMMARY OF TREASURY SECURITIES OUTSTANDING, JULY 31, 2019

(Millions of dollars)

	Amount Outstanding		
	Debt Held By the Public	Intragovernmental Holdings	Totals
Marketable:			
Bills	2,205,307	590	2,205,897
Notes	9,636,337	5,885	9,642,221
Bonds	2,282,191	8,418	2,290,610
Treasury Inflation-Protected Securities	1,432,181	355	1,432,536
Floating Rate Notes [20]	388,058	9	388,067
Federal Financing Bank [1]	0	8,809	8,809
Total Marketable [a]	**15,944,073**	**24,067** [2]	**15,968,140**
Nonmarketable:			
Domestic Series	29,995	0	29,995
Foreign Series	264	0	264
State and Local Government Series	43,643	0	43,643
United States Savings Securities	153,029	0	153,029
Government Account Series	38,352	5,787,117	5,825,469
Other	1,827	0	1,827
Total Nonmarketable [b]	**267,112**	**5,787,117**	**6,054,228**
Total Public Debt Outstanding	**16,211,185**	**5,811,184**	**22,022,369**

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BOND PRICING

Bond Pricing

- How do we value bonds with **guaranteed** cash flows?
  - Assuming cashflows are default-free then we should price them using interest rates/discount factors for riskless borrowing/lending at various maturities
  - In theory, all bonds must be priced by the **same** set of riskfree interest rates/discount factors otherwise there is an **arbitrage opportunity**

Bond Pricing

- Assuming cashflows are guaranteed (default-free) then value bonds using risk-free interest rates/discount factors
  - Eg the value of the 3yr, 1,000 notional, 2% annual coupon bond is given by:

$$P = \frac{20}{(1+r_1)} + \frac{20}{(1+r_2)^2} + \frac{1020}{(1+r_3)^3}$$

$$= \quad 20 \cdot d_1 \quad + \quad 20 \cdot d_2 \quad + \quad 1020 \cdot d_3$$

where:

r_t denotes the **spot interest rate** for maturity *t*: ie the rate for borrowing and/or lending between today and *t* (here, compounded once annually)

d_t denotes the **discount factor** for maturity *t*

Bond Pricing

- To value a security we discount its cash-flows by the appropriate discount rate:

Bond value = Present value of coupons+
Present value of par value

- Using discount factor notation

$$Price = \sum_t \delta_t \, C_t$$

  - δ_t is the discount factor, C_t is the cash flow at time t

Bond Pricing

- **Another example**: 8% coupon, 30-year maturity bond, par value of $1,000, semi-annual coupon payments.
- Suppose the discount rate is 4% per six-month period. The value of the bond:

$$Price = \sum_{t=1}^{60} \frac{1}{(1+0.04)^t} \times 40 + \frac{1}{(1+0.04)^{60}} \times 1{,}000 = 1{,}000$$

- In this example the coupon rate equals the discount rate, and the bond price equals par value.
- What if the discount/interest rate is higher?

Bond Price at different interest rates

Market Interest/Discount Rate	Bond Price
4%	$1,695.22
6%	$1,276.76
8%	$1,000.00
10%	$810.71
12%	$676.77

- Price (= PV(payments)) falls as interest rates rise.
  - Crucial general rule in bond valuation.

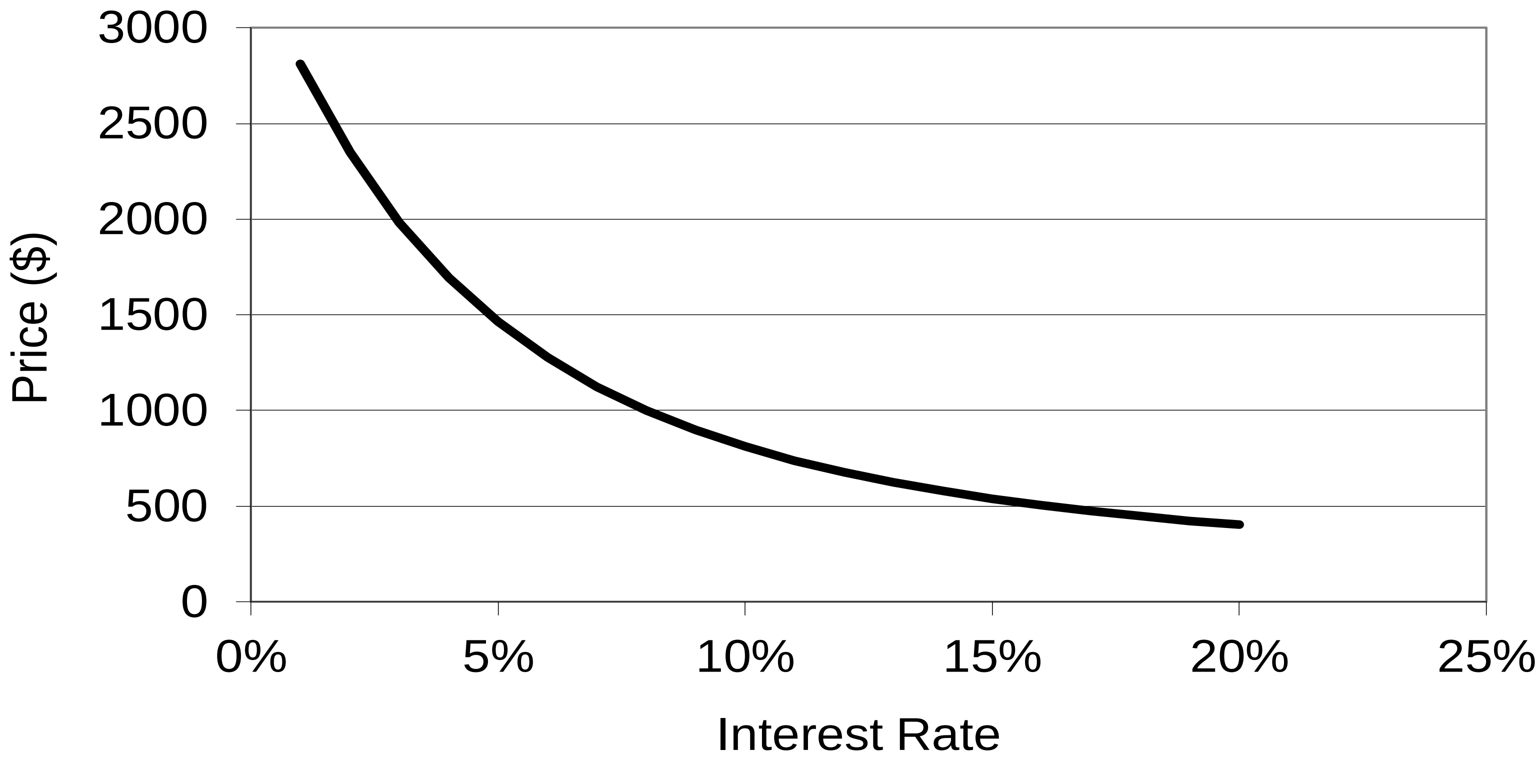
Bond Pricing Terminology

- P_T > Face Value: bond sells at a ***premium***
- P_T < Face Value : bond sells at a ***discount***
- P_T = 100: bond sells at ***par***

- **Intuition (for bond at discount):** Since the coupon is below the market yield, investors will demand more than their initial investment at maturity (i.e., they'll pay less than face because the coupons are low)

Bond Pricing

- Price of the 30-year, 8% coupon bond for a range of interest rates:



Bond Pricing: Price vs. r graph

- Negative slope means price falls as *r* rises.
  - If you buy a bond at par for 8% and market interest rates rise, then you suffer a loss.
  - Intuition is you tied up your money earning 8% when alternative investments now earn higher returns.
- The (convex) shape means an increase in interest rates results in a price decline that is smaller than the price increase resulting from an interest rate fall of equal magnitude.
  - From prior example, compare rate changes of +/- 2%
  - This property is called “convexity”.

Treasury Market Conventions

- For Treasuries, coupons and compounding are, by convention, semi-annual
- Discount factor for cash flows in *t* years:

$$\delta_t = \frac{1}{\left(1+\frac{r_t}{2}\right)^{2t}}$$

- Where the spot rate r_t is the ***annual*** interest rate with semi-annual compounding (APR) for borrowing between now and *t* years
  - For now, assume the spot rates r_t are known.
  - We have thus far assumed constant discount rates, but in reality we often have different discount rates, and hence spot rates, for each time period.

Semi-annual Bond Pricing Recipe

1. Write down cash-flows (C_t) from coupon payments (*C/2*) and par value (*F*)
2. Get spot rates and compute discount factors
3. Multiply and add up via:

$$P_T = \sum_{t=1}^{2T} \delta_{\frac{t}{2}} \frac{C}{2} + \delta_T F = \sum_{t=1}^{2T} \delta_{\frac{t}{2}} C_{\frac{t}{2}} = \sum_{t=1}^{2T} \frac{1}{\left(1 + \frac{r_t}{2}\right)^t} C_{\frac{t}{2}}$$

[t = indexed cash flow, t = 1, 2, 3 ... 2T]

Ex: 1-year, $100 par, 5% Coupon Bond

1. Get cash flows:

Time	6-M (Coupon)	1-year (Coupon)	1-year (Par)
Cashflow	$2.50	$2.50	$100

2. Assume $r_{.5}$=0.04 and r_1=0.03: Discount rates are

$$\delta_{.5} = \left(1 + \frac{r_{.5}}{2}\right)^{-1} = (1 + 0.02)^{-1} = 0.980$$

$$\delta_1 = \left(1 + \frac{r_1}{2}\right)^{-2} = (1 + 0.015)^{-2} = 0.971$$

3. Multiply and add up:

$$P_1 = \delta_{.5} C_{.5} + \delta_1 C_1 = 0.98 \times 2.5 + 0.971 \times 102.5 = \$101.94$$

BOND YIELDS

Comparing bonds with different characteristics

- Consider three bonds:
  - Bond 1: T=2, C=2%, P=90.92
  - Bond 2: T=2, C=8%, P=102.26
  - Bond 3: T=2, C=0%, P=87.14

- A natural metric is to compare the expected returns
  - a.k.a. computing the "yield"

Yield-To-Maturity (YTM)

- The **yield-to-maturity** of a bond is the **single discount rate** that sets the PV of a bond's (promised) cashflows equal to the observed price
  - Eg consider the 3 year, 1,000 notional, 2% annual coupon bond. If the bond trades at 985.72 then YTM is the number y such that:

$$985.72 = \frac{20}{(1+y)} + \frac{20}{(1+y)^2} + \frac{1020}{(1+y)^3} \quad \Rightarrow \quad y = 2.5\%$$

Note that (as before, when the bond price was the cashflows discounted at a single interest rate):

- Bond price and yield are inversely related
- Bond price is convex in the yield
- Bond trades above/at/below par if coupon rate is above/at/below the yield

Yield-To-Maturity (YTM)

- For a semi-annual bond:

$$P_T = \sum_{t=1}^{2T} \frac{\frac{C}{2}}{\left(1 + \frac{y_T}{2}\right)^t} + \frac{F}{\left(1 + \frac{y_T}{2}\right)^{2T}}$$

- y_T is the ***IRR of the bond*** (discount rate that equates the PV of the cash flows with the market price)

Back to example

- Consider Bond 1: T=2, C=2%, P=90.92
- Yield solves:

$$90.92 = \frac{1}{\left(1+\frac{y}{2}\right)^1} + \frac{1}{\left(1+\frac{y}{2}\right)^2} + \frac{1}{\left(1+\frac{y}{2}\right)^3} + \frac{101}{\left(1+\frac{y}{2}\right)^4}$$

- Which gives y=6.941% (annualized)

Calculator:	PV = -90.92, PMT = 1, FV =100, T = 4
	I = ? = 3.47 → YTM = 2 x 3.47 = 6.94%

Relationship between P, y, and C

$$P_T = \sum_{t=1}^{2T} \frac{\frac{C}{2}}{\left(1+\frac{y_T}{2}\right)^t} + \frac{100}{\left(1+\frac{y_T}{2}\right)^{2T}}$$

- If y_T ↑ then P_T ↓
  - prices and yields move in opposite directions.
- If C ↑ then P_T ↑
  - prices and coupons move in the same direction
- What about T ↑(y, C held constant)?

Government Bond Yields

BONDS: BENCHMARK GOVERNMENT

	Red Date	Coupon	Bid Price	Bid Yield	Day chg yield	Wk chg yield	Month chg yld	Year chg yld
Australia	11/20	1.75	99.21	2.11	0.01	0.03	0.09	0.12
	11/28	2.75	100.42	2.70	0.04	0.01	0.06	-0.09
Austria		-	-	-	-	-	-	-
	10/19	0.25	100.84	-0.44	0.02	0.03	0.10	0.08
Belgium	12/20	1.80	105.15	-0.40	-0.02	0.02	0.07	-0.01
	03/28	5.50	145.13	0.66	0.06	0.10	0.11	-0.05
Canada	02/20	1.25	98.87	2.03	-0.02	0.02	0.14	-
	06/28	2.00	96.73	2.37	0.03	0.11	0.20	0.33
Denmark	11/20	0.25	101.62	-0.45	0.02	0.06	0.10	-0.03
	11/27	0.50	100.67	0.43	-0.02	0.08	0.10	-0.14
Finland	09/20	0.38	101.81	-0.47	0.02	0.05	0.10	-0.02
	09/27	0.50	99.25	0.59	0.05	0.09	0.10	-
France	11/20	0.25	101.60	-0.44	0.03	0.05	0.09	-0.04
	05/23	1.75	108.63	-0.05	-0.01	0.07	0.13	-0.07
	05/28	0.75	100.27	0.72	-0.01	0.08	0.12	-
Germany	10/20	0.25	101.78	-0.56	-0.02	0.04	0.11	0.01
	08/23	2.00	110.85	-0.15	0.04	0.09	0.13	-0.04
	08/27	0.50	101.19	0.37	0.05	0.10	0.12	-0.12
	08/48	1.25	103.50	1.11	-0.02	0.05	0.10	-
United Kingdom	01/21	1.50	101.62	0.84	-0.01	0.04	0.12	0.45
	07/23	0.75	98.27	1.11	-0.01	0.07	0.11	0.34
	12/27	4.25	124.97	1.39	0.06	0.12	0.11	0.18
	07/47	1.50	92.91	1.82	0.05	0.11	0.08	-0.02
United States	11/19	1.75	98.93	2.57	0.01	0.01	0.09	-
	10/22	2.00	96.69	2.83	-0.03	-0.01	0.09	-
	11/27	2.25	94.05	2.99	-0.02	0.01	0.12	-
	11/47	2.75	92.84	3.13	0.00	0.03	0.14	-

Interactive Data Pricing and Reference Data LLC, an ICE Data Services company.

Source: Financial Times, 4 Aug 2018

Government Bond Yields

US Treasury yields fall as investors shelter in safe haven bonds

Alexandra Gibbs | Thomas Franck

Published 5:05 AM ET Wed, 9 Aug 2017 | **Updated 1:48 PM ET Wed, 9 Aug 2017**

CNBC

U.S. Treasury yields fell on Wednesday, as investors monitored tensions between the U.S. and North Korea.

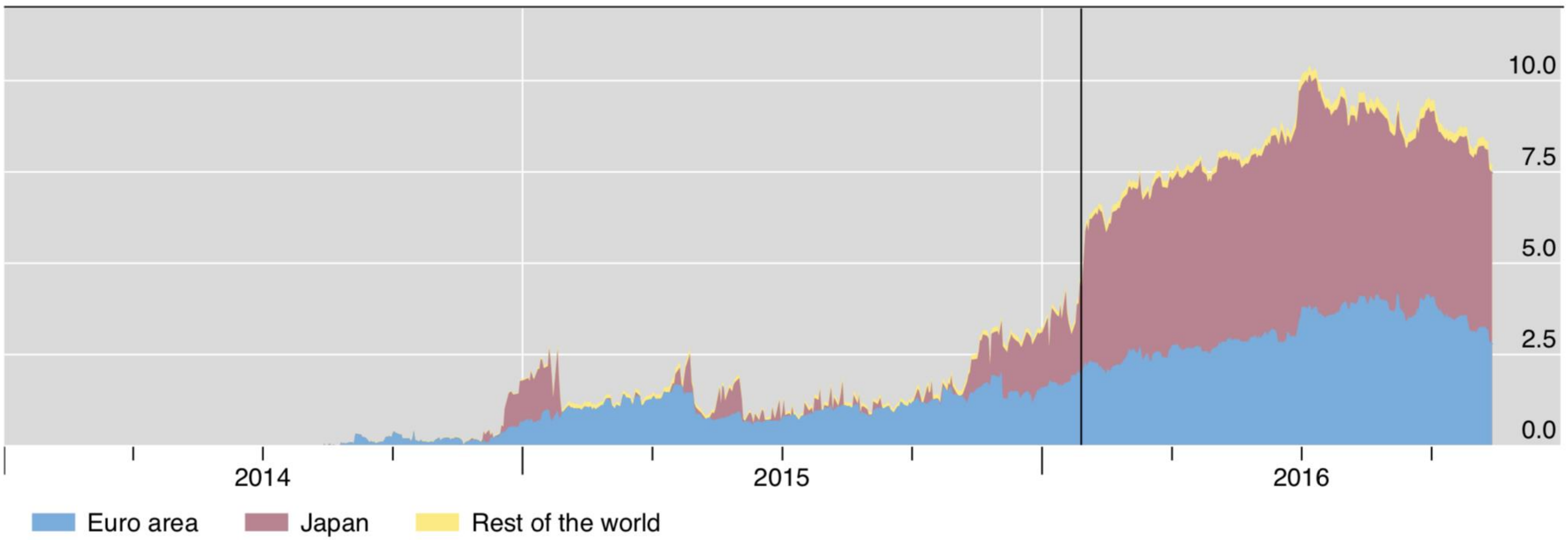
The yield on the benchmark 10-year Treasury note hit a six-week low before trading at 2.246 percent at 1:44 p.m. ET, while the yield on the 30-year Treasury bond was lower at 2.824 percent. Bond yields move inversely to prices.

Government Bond Yields

Stock of government bonds with negative yields[1]

In trillions of US dollars

Graph 1



Yields of the 3 Bonds

- The yields of the three bonds are:
  - Bond 1: C=2%, P=90.92, y= 6.941%
  - Bond 2: C=8%, P=102.26, y= 6.773%
  - Bond 3: C=0%, P=87.14, y= 7.003%

- Questions:
  1. Should we always invest in the bond with the highest yield?
  2. How should we interpret the yield?

Interpreting the “yield”

- Yield is the expected rate of return on a bond ***if***
  1. Coupons are paid
  2. You re-invest coupons at y
  3. You hold the bond to maturity

- Is this realistic?

- Either way, yields provide a useful benchmark for comparison

Interpreting the "yield"

- For coupon paying bonds held to maturity, returns depend on the rate at which you reinvest coupons
- The **YTM** on a coupon paying bond is equal to the **compound return if held to maturity if the coupons can be reinvested at the original yield**
  - Eg a 3yr 2% annual coupon bond with face value 1,000 trading at 985.72
    - Yield:

$$985.72 = \frac{20}{(1+y)} + \frac{20}{(1+y)^2} + \frac{1020}{(1+y)^3} \quad \Rightarrow \quad y = 2.5\%$$

    - Rearrange (multiply both sides by $(1 + y)^3$):

$$\underbrace{985.72 \cdot (1+y)^3}_{\text{Value of investing 985.72 at the yield, compounded for 3 years}} = \underbrace{20 \cdot (1+y)^2}_{\text{Reinvest t=1 coupon for 2 years at y}} + \underbrace{20 \cdot (1+y)}_{\text{Reinvest t=2 coupon for 1 year at y}} + \underbrace{1020}_{\text{Cashflows in year 3}}$$

Interpreting the "yield"

- If the coupons are reinvested at lower/higher rates than the YTM, then the realised return will end up lower/higher than the YTM:

	Value at t=3 if coupons reinvested to maturity at		
	1%	**2.5%**	**5%**
t=1 coupon reinvested for 2 years	$20 \times (1.01)^2 =$ 20.402	$20 \times (1.025)^2 =$ 21.0125	$20 \times (1.05)^2 =$ 22.05
t=2 coupon reinvested for 1 year	20 x 1.01 = 20.2	20 x 1.025 = 20.5	20 x 1.05 = 21
t=3 coupon plus principal	1020	1020	1020
Total at t=3	1060.602	1061.5125	1063.05
Compound Return, R	2.47%	**2.50% = YTM**	2.55%

Note: $\frac{\text{Total at } t = 3}{985.72} = (1 + R)^3$

Bonds Returns and Investment Horizon

- So far we have looked at compound returns if a bond is held to maturity
- If the **investment horizon differs from the bond maturity** then, in addition:
  - If the investment horizon is longer than the original bond maturity, then all bonds (coupon paying or not) will face **reinvestment risk** on the principal
  - If the investment horizon is shorter than the bond maturity, then all bonds (coupon paying or not) will face "**price risk**" in that the value of the bonds at the end of the horizon will not be known ex-ante (as they depend on interest rates at that future date)

Bonds **Returns**

- The **total return** earned on the bond over a period, sometimes called the
  holding period return (HPR), consists of:
  - capital gain/loss
  - return due to cash income (coupon)

- The simplest HPR to calculate is that between coupon dates in which case:

$$HPR_t = \frac{P_t - P_{t-1} + C}{P_{t-1}}$$

- Where
  - P_t is the ending price
  - P_{t-1} is the initial price
  - C is the coupon income received at the end of the period

Bonds Returns

- **Example**: a 3 year, 2% annual coupon bond with face value of 1,000
  - Assume the initial price is 985.72 (yield = 2.5%).
  - Let's assume one year later (after the first coupon of 20 has been paid), that the bond has a price of 1019.70 (yield = 1%).
  - What is the holding period return if you buy the bond today for 985.72 and sell it one year later for 1019.70?

P_0 = 985.72
P_1 = 1019.70
C = 20

$$HPR_1 = \frac{P_1 - P_0 + C}{P_0} = 5.48\%$$

Bonds **Returns and Yields**

- **Example**: a 3 year, 2% annual coupon bond with face value of 1,000
  - Initial price of 985.72, implies an **original yield of 2.5%:**

$$P_0 = 985.72 = \frac{20}{1+y_0} + \frac{20}{(1+y_0)^2} + \frac{1020}{(1+y_0)^3} \quad \Rightarrow \quad y_0 = 2.5\%$$

  - Price one year later (after first coupon) of 1019.70, implies a **new yield of 1%**:

$$P_1 = 1019.70 = \frac{20}{1+y_1} + \frac{1020}{(1+y_1)^2} \quad \Rightarrow \quad y_1 = 1\%$$

  - Note that:

HPR (5.48%) exceeds initial yield (2.5%)
New yield (1%) lower than initial yield (2.5%)

Bonds Returns and Yields

- In general, relationship between **HPR** and **change in yields**:

HPR > initial yield	new yield is less than initial yield
HPR = initial yield	new yield equals initial yield
HPR < initial yield	new yield is greater than old yield

- Mathematically true, intuition?

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ZERO-COUPON BONDS

Zero-Coupon Bonds (ZCB)

- Recall that ZCB make just one payment at maturity.
  - The price of the zero coupon bond is given by discounting the face value at the yield or equivalently at the interest rate for the bond maturity.
- Therefore, the YTM of a ZCB is the risk-free discount rate.
  - Eg a 3 year zero coupon bond with face value of 1,000, trading at 921.84

$$921.84 = \frac{1000}{(1+y)^3} = \frac{1000}{(1+r_3)^3} \quad \Rightarrow \quad y = r_3 = 2.75\%$$

- Moreover, a zero coupon bond earns the yield as a compound return if held to maturity
  - Eg return if invest 921.84 in the 3 yr zero coupon bond to get 1000 in 3 years' time

$$1,000 = 921.84 \cdot (1+R)^3 \quad \Rightarrow \quad R = 2.75\%$$

Price of a Zero is the discount factor

- Simplest bonds:
  - Cash flows are just the face value *F* (i.e., no coupons)
  - PV formula: $Z_t = \delta_t C_t = \frac{1}{\left(1+\frac{r_t}{2}\right)^{2t}} \times F$
  - *t* = years in the future
- Since $\frac{Z_t}{F} = \delta_t$, zeros give discount factors

The YTM of a Zero is the spot rate

- What is the difference between yields on zeros (F=par) and spot rates? No difference
  - Yield on T-year zero bond: $$Z_T = \frac{1}{\left(1 + \frac{y_T}{2}\right)^{2T}} F$$
  - Zero expiring in T-years $$Z_T = \frac{1}{\left(1 + \frac{r_T}{2}\right)^{2T}} F$$
- ***Yields on zeros (y_T) are spot rates (r_T)***!

Zero-Coupon Bonds (ZCB)

- The following Table gives prices and YTM (annual compounding) of a ZCB with a par value of $1,000:

Time to Maturity	Price	YTM
1	$925.93	8.000%
2	$841.75	8.995%
3	$758.33	9.660%
4	$683.18	9.993%

- Recall that: $841.75 = 1{,}000/(1+YTM_2)^2$ or YTM_2 = 8.995%. This is the (annual) spot rate that prevails today for a period of two years.
- Question: What is the PV today of receiving $25M in two years?

Zero-Coupon Bonds (ZCB)

- Question: What is the PV today of receiving $25M in two years? There are two alternatives:
  1. Divide the price of a two-year ZCB by its principal value to obtain the discount factor = 841.75/1000 = 0.84175. Multiply the cash-flow by the discount factor to obtain the PV:

     PV= 0.84175*$25M = $21.044M

  2. Discount the cash-flow using the appropriate discount rate:

     PV= $25M/(1.08995)^2 = $21.044M

Zero-Coupon Bonds (ZCB)

- The prices (and YTM) of zero coupon bonds is also important because it allows us to price (value) coupon bonds.

- What is the price of a 7% coupon (annual coupon payments), 4 years to maturity, $100 face value bond? Use the information in the previous table.

Zero-Coupon Bonds (ZCB)

- What is the price of a 7% coupon (annual coupon payments), 4 years to maturity, $100 face value bond?

Maturity	Zeros	Coupon	Principal	PV
1y	0.92593	7.00	0	6.482
2y	0.84175	7.00	0	5.892
3y	0.75833	7.00	0	5.308
4y	0.68318	7.00	100	73.100
				90.782

Pricing Zeroes vs. Coupon Bonds

- Consider 1-year, \$100 par, 5% coupon bond

Time	6-Month	1-Year (Coupon + par)
Cashflow	\$2.50	\$102.50

- Same cash flows as portfolio with a 6-Month \$2.50 par zero and a 1-Year \$102.50 par zero
- **Implication:** the price of the coupon bond is equal to the price of a portfolio of zeroes with the right maturities and par values

Example of zero/coupon arbitrage

- Consider 1-year, 5% coupon bond
  - $r_{.5}$=0.04 and r_1=0.03 ($Z_{.5}$=0.980 and Z_1=0.971)
  - The price of the 1-year bond should be $101.9438

Time	6-Month	1-year (Coupon + par)
Cash flow	$2.50	$102.50

- Suppose that the “market” price of 1-year bond is 100 and you can trade in the zeroes at the above prices

Arbitrage!

- What should you do?
  - Buy the "cheap" 1-year coupon bond and replicate the cash flows by trading in the zeroes
  - Sell 2.5 zeroes maturing in ½ year and 102.5 zeroes maturing in 1-year

Action	Cost today:	Cash flow at t=1/2	Cash flow at t=1
Buy 1 Coupon bond	-100	2.5	102.5
Sell 2.5 zeroes	2.45	-2.50	
Sell 102.5 zeroes	99.49		-102.50
Net	1.94	0	0

- Arbitrage: 1.94 with no risk!

Realistic?

- No, but it provides the general approach to ***relative value*** or ***arbitrage pricing*** of assets

- We price assets assuming there is no arbitrage
  - Stat-arb, merger-arb, etc. aren't "pure" arbitrages

More realistic arbitrage example: 3 Coupon Bonds

- Bond Triples
  - On February 15, 2001 there were three bonds maturing on the same day: August 15, 2003.

Bond Triplet	Coupon	Original Maturity	Price (15-Feb-01)
Bond 1	5.25%	3 years	100.84375
Bond 2	5.75%	10 years	102.020
Bond 3	11.125%	20 years	114.375

- Arbitrage opportunity?

Bond Triplet	Coupon	Original Maturity	Bond Price (2-Feb-01)
Bond 1	5.25%	3 years	100.84375
Bond 2	5.75%	10 years	102.020
Bond 3	11.125%	20 years	114.375

- Idea: Replicate Cashflows of Bond 2 via a portfolio composed of Bond 1 and Bond 3
  - Let c_1, c_2, and c_3 be the coupon rates of the three bonds.
  - Let F_1 and F_3 be the face amount of bond 1 and bond 3 needed to replicate a unit face amount of the second bond:
  - In Math: To replicate cashflows of Bond 2 we need:

1. To match principal payment at maturity: $F_1 + F_3 = 1$
2. and match coupon payments: $c_1 F_1 + c_3 F_3 = c_2$

Bond Triplet	Coupon	Original Maturity	Bond Price (2-Feb-01)
Bond 1	5.25%	3 years	100.84375
Bond 2	5.75%	10 years	102.020
Bond 3	11.125%	20 years	114.375

- Idea: Replicate Cashflows of Bond 2 via a Portfolio composed of Bond 1 and Bond 3
  - Solution: $F_1 = \frac{C_2 - C_3}{C_1 - C_3} = 91.49\%$ & $F_3 = \frac{C_1 - C_2}{C_1 - C_3} = 8.51\%$ ($C_1 = 5.25\%, C_2 = 5.75\%, C_3 = 11.125\%$)
  - In words: A portfolio with 91.49% of its face value in the 5.25s and 8.51% of its value in the 11.125s will replicate one unit face value of the 5.75s.
  - What's the price of the portfolio? $P_1 F_1 + P_3 F_3 = 101.995$
  - What's the price difference? 0.025 cheaper
  - How would you exploit this price difference?

Outline

- What are bonds? What kind of bonds are traded? ✓
- Bond Pricing. ✓
- Bond Yields. ✓
- Zero-coupon bonds. ✓
- The Term Structure and Forward Rates.
- Interest Rate Risk and Duration.
- Default Risk and Bond Ratings.

The Term Structure of Interest Rates

- The term structure of interest rates is the set of yields to maturity, at a given time, on zero-coupon bonds of different maturities.

- The (pure) yield curve is a plot of the term structure.

- “Spot” curves across countries?

Issues with G-7 countries

- A couple of practical issues in G-7 countries
  - Are they riskless? (What is the credit rating of Japan?)
- When do countries pay coupons?
  - UK, US, Japan, Australia, Switzerland, Canada: semi-annual coupons
  - Germany, France, and Euroland: annual coupons
- What do current spot rates look like?
  - Plot of spot rates as a function of maturity (T) is known as the **yield curve** or the **term structure of interest rates**.

Rate (Mid Conventional %)
25
20
15
10
5
0
I156 EUR Greece Sovereign Curve 08/28/12 21:39:53
I22 GBP United Kingdom Sovereign Curve 08/28/12 21:39:53
I18 JPY Japan Sovereign Curve 08/28/12 21:39:53
I25 US Treasury Actives Curve 08/28/12 21:39:53
-5
-10
-15
-20
Spread
I22 - I156 (08/28/12)
I18 - I156 (08/28/12)
I25 - I156 (08/28/12)
1M
1Y
...
5Y
...
9Y
12Y
15Y
20Y
25Y
30Y
40Y
50Y
Tenor

Constructing yield curves

- Thus far, we have assumed spot rates are available
  - e.g., via zeroes
- In practice, only 3 and 6-month zeroes are available
  - 1-year zeroes used to be available
  - No zeroes after 1 year.
- How do we construct zeros/spot rates beyond six months?

Could we use Treasury STRIPS?

- **S**eparate **T**rading of **R**egistered **I**nterest and **P**rincipal of **S**ecurities (since Feb. 85)
  - STRIPS let investors trade coupons and principal separately
  - STRIPS can give you the zero yield beyond the first year

- However:
  - STRIPS are not as liquid as the regular bonds

U.S. Treasury Strips

Monday, September 09, 2013

U.S. zero-coupon STRIPS allow investors to hold the interest and principal components of eligible Treasury notes and bonds as separate securities. STRIPS offer no interest payment; investors receive payment only at maturity. Quotes are as of 3 p.m. Eastern time based on transactions of $1 million or more. Yields calculated on the ask quote.

Maturity	Bid	Asked	Chg	Asked yield
Treasury Bond, Stripped Principal				
2015 Feb 15	99.563	99.577	0.051	0.30
2015 Aug 15	99.143	99.162	0.038	0.44
2015 Nov 15	98.866	98.887	0.057	0.51
2016 Feb 15	98.443	98.466	0.092	0.64
2016 May 15	98.068	98.094	0.109	0.72
2016 Aug 15	97.476	97.504	0.148	0.87
2016 Nov 15	96.872	96.902	0.185	0.99
2017 May 15	95.551	95.586	0.210	1.23
2017 May 15	95.569	95.604	0.144	1.23
2017 Aug 15	94.950	94.987	0.219	1.31
2018 May 15	92.736	92.779	0.329	1.61
2018 Nov 15	91.063	91.110	0.309	1.81
2019 Feb 15	90.251	90.300	0.303	1.89
2019 Aug 15	88.473	88.525	0.295	2.07
2020 Feb 15	86.719	86.774	0.339	2.22
2020 May 15	85.842	85.898	0.340	2.29
2020 Aug 15	84.901	84.959	0.348	2.37
2021 Feb 15	83.001	83.061	0.297	2.51

What to do?

- In practice, spot rates are obtained from the coupon bond prices through a procedure known as bootstrapping
  - We always have the first spot rate, $r_{.5}$, from the 6-month T-bill.
  - Suppose we have a 1-year coupon bond (e.g., a two year bond that is one year old). Can we solve for the next spot rate from a coupon bond expiring in 1 year?

$$P(1) = \frac{\frac{C}{2}}{\left(1 + \frac{r_{.5}}{2}\right)} + \frac{F + \frac{C}{2}}{\left(1 + \frac{r_1}{2}\right)^2}$$

Bootstrapping the Yield Curve

- Suppose $r_{.5}$=.08 and there is a $100 par 8.5% bond maturing in 1 year whose price is $100.19

- The cash flows for this bond are
  - 4.25 at time 1 (6 months)
  - 104.25 at time 2 (1 year)

- What is r_1?

$$100.19 = \frac{4.25}{\left(1 + \frac{0.08}{2}\right)^1} + \frac{104.25}{\left(1 + \frac{r_1}{2}\right)^2}$$

Bootstrapped Spot Rates

- Solve for r_1:

$$100.19 = \frac{4.25}{\left(1 + \frac{0.08}{2}\right)^1} + \frac{104.25}{\left(1 + \frac{0.083}{2}\right)^2}$$

- This implies that $r_1 = 0.083$.
- Can continue with this procedure all the way out the yield curve.
  - Get a 1.5 year bond and solve for $r_{1.5}$.

Example: Pricing bonds from term structure

- Price a T-note with exactly 2 years to maturity and 5.5% coupon from the current spot curve.
  - $r_{.5}$=1.90%, r_1=2.1%, $r_{1.5}$=2.25% and r_2 =2.4%

Time	6 Months	1Year	1 ½ Years	2 Years
Cashflow	\$2.75	\$2.75	\$2.75	\$102.75

  - Price each coupon as a zero, then add them up

The Mechanics are simple ...

$$\delta_t = \frac{1}{\left(1+\frac{r_t}{2}\right)^{2t}} \qquad P_T = \sum_{t=1}^{2T} \delta_{\frac{t}{2}} \frac{C}{2} + \delta_T F = \sum_{t=1}^{2T} \delta_{\frac{t}{2}} C_{\frac{t}{2}}$$

Time	Cash Flow	Spot rate	Discount factor	Present Value
0.5	2.75	1.90%	0.9906	2.7241
1	2.75	2.10%	0.9793	2.6931
1.5	2.75	2.25%	0.9670	2.6592
2	102.75	2.40%	0.9534	97.9625
			Price	106.0390

“The Carry Trade”

- Suppose you knew that the shape of the U.S. yield curve is going to stay the same
- Consider the following trade:
  - Borrow at the short end: Federal Funds @2%
  - Buy long-dated Treasuries, Mortgages or AAA’s @ 4-6%
- Profit? If nothing moves, you get about 2-4% profit per dollar.

- What happens when rates change?

Where are we?

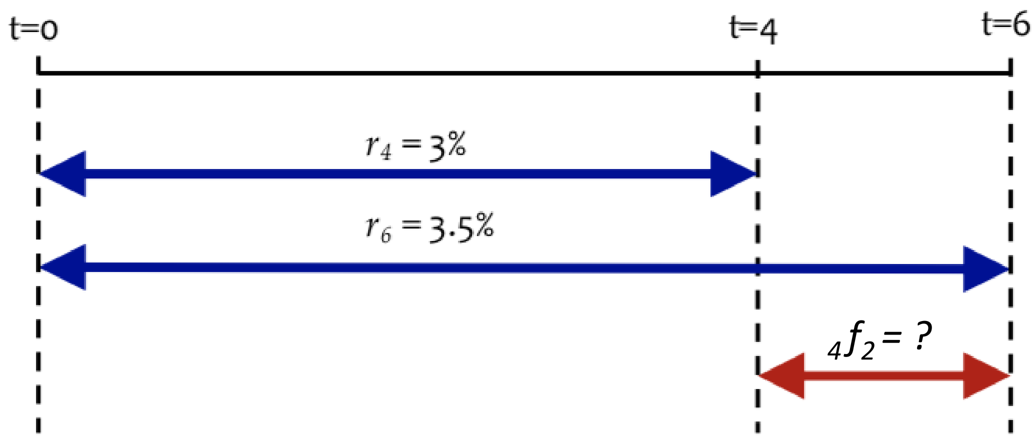
- We now know the basics of pricing Treasury securities

- Can we lock a rate of an investment which starts in 4 years from now and lasts for 6?

- The answer is “forward rates”.

Forward Rates

- Let r_t be the **spot** interest rate between today and a future date *t*
- Then the **forward rate** $_xf_y$ is the interest rate per annum between **future dates** $t=x$ and $t=x+y$ **that can be locked in today:**
  - Eg spot rates for t=4 and t=6 years are 3% and 3.5% pa, what is the forward rate that can be locked in for borrowing/lending between t=4 and t=6?



Forward Rates

- Example: spot rates for *t*=4 and *t*=6 years are 3% and 3.5% p.a., what is the forward rate that can be locked in between t=4 and t=6?
  - Consider zero coupon bonds maturing at *t*=4 and *t*=6, priced at:

$$P_4 = \frac{1000}{(1.03)^4} = 888.5 \qquad P_6 = \frac{1000}{(1.035)^6} = 813.5$$

  - Cashflows on zero coupon bonds:

	t=0	t=4	t=6
Buy 4yr ZCB	**-888.5**	**1000**	**0**
Buy 6yr ZCB	**-813.5**	**0**	**1000**

Forward Rates

- Consider the following strategy that locks in the forward rate:

	t=0	t=4	t=6
Short 1 4yr ZCB	**888.5**	**-1000**	**0**
Buy P_4/P_6=1.09219 of 6yr ZCB	**-888.5**	**0**	**1092.19**
Net	**0**	**-1000**	**1092.19**

- Net: invest 1000 at t=4, receive 1092 at t=6, implying a forward rate of:

$$1000\,(1 + {}_4f_2)^2 = 1092.19 \qquad {}_4f_2 = 4.51\%$$

- And, substituting back that P_4/P_6=1.09219 we get the relationship between spot and forward rates

$$(1 + r_4)^4 \cdot (1 + {}_4f_2)^2 = (1 + r_6)^6$$

ie investing at 4yr spot rate for four years and then at the 2yr forward rate starting year 4 is equivalent to investing at the 6yr spot rate for 6 years

Forward Rates

- Note that the relationship between spot and forward rates has to hold by **no arbitrage** arguments
- Relationship between spot and forward rates and **shape** of the zero coupon yield curve:
  - If zero yield curve is flat, forward rates equal the spot rate
  - If zero yield curve is upward (downward) sloping then forward rates exceed (are below) spot rates
  - Eg previous example:

$$r_4 = 3.0\%$$

$$r_6 = 3.5\%$$

$${}_4f_2 = 4.51\%$$

$${}_4f_2 > r_6 > r_4$$

Forward Rates

- You are given the following spot rates, annual compounding:

Year	Spot rate
1	5%
2	5.4%
3	5.7%
4	5.9%
5	6%
6	4.8%

- Why would you not believe the six year spot rate? How could you make a risk-less profit with this term structure?

Forward Rates

- Let us compute the forward rates

Year	Spot rate	ZCB price	Forward Rate
1	5.0%	$1/1.050 = 0.9523810$	
2	5.4%	$1/1.054^2 = 0.900158$	$0.952381/0.900158 - 1 =$ 5.80%
3	5.7%	$1/1.057^3 = 0.846789$	$0.900158/0.846789 - 1 =$ 6.30%
4	5.9%	$1/1.059^4 = 0.79509$	$0.846879/0.795090 - 1 =$ 6.51%
5	6.0%	$1/1.060^5 = 0.747258$	$0.795090/0.747258 - 1 =$ 6.40%
6	4.8%	$1/1.048^6 = 0.754801$	$0.747258/0.754801 - 1 =$ -0.99%

- The last forward rate is negative. This is an arbitrage opportunity:
  1. Borrow $1Billion at 4.8% for 6 years.
  2. Invest $1Billion at 6% for 5 years and keep the cash in the last year.

Forward Rates

- Debt to be repaid = $\$1B*(1.048)^6=\$1.32485B$

  Revenues from investment = $\$1B*(1.06)^5=\$1.33823B$

- The total profits is $0.01338B$ or $\$13\ M$.

- This is one of the most typical arbitrage trade of investment banks.

Generalizing Forward Rate Notation

- The rate on a loan beginning in t periods and lasting for t' periods
  - "in t periods for t' periods forward rate"
- Use ${}_{t}f_{t'}$ to denote the forward rate on a loan
  - Starts *t* years from now
  - Lasts for *t'* years
- Examples:
  - ${}_{1}f_{0.5}$: rate on a 6-month loan, 1 year from now
  - ${}_{5}f_{0.5}$: rate on a 6-month loan, 5 years from now
  - ${}_{0.5}f_{2.5}$: rate on a 2.5-year loan, 6 months from now
  - ${}_{7}f_{3}$: rate on a 3-year loan, 7 years from now

The generalized case

- We can obtain any forward rate ${}_tf_{t'}$ by solving a simple equation:

$$\left(1+\frac{r_{t+t'}}{m}\right)^{m(t+t')} = \left(1+\frac{r_t}{m}\right)^{mt} \times \left(1+\frac{{}_tf_{t'}}{m}\right)^{mt'}$$

Where *m* is the number of compounding periods per year.

Example: T-bills

- On 09/09/13:
  - 6 month: $r_{.5}$=0.04%
  - 1 year: r_1=0.12%
- What is the "in 6M for 6M" forward rate?
  - Solve the following equation:

$$\left(1+\frac{0.0012}{2}\right)^2=\left(1+\frac{0.0004}{2}\right)\left(1+\frac{{}_{.5}f_{.5}}{2}\right)$$

- Simplifying, we have

$${}_{.5}f_{.5}=2\left[\frac{\left(1+\frac{0.0012}{2}\right)^2}{\left(1+\frac{0.0004}{2}\right)}-1\right]=0.002=0.2\%$$

Summary of what we have done.

- Bonds are priced by computing the ***present value*** of all their cash flows at appropriate spot rates
- These discount rates can be obtained through ***zeros***
- Plotting the zeros, or spot rates, against time to maturity gives the ***term structure of interest rates***
- ***Forward interest rates*** are arbitrage-free rates that allow locking in future rates

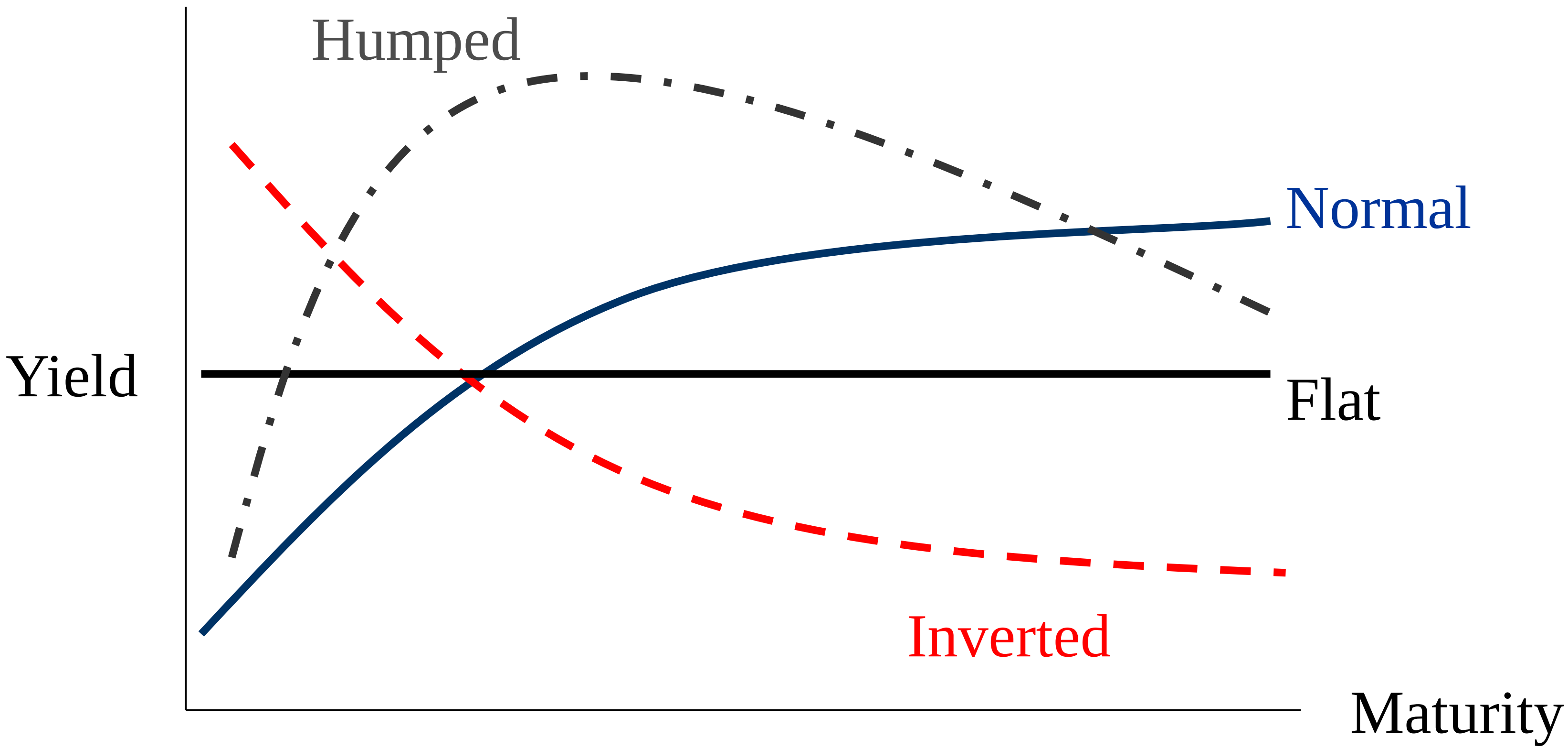
Facts about the Term Structure

- The term structure can have many shapes
  - Inverted yield curves thought to be a predictor of recessions (Harvey 1986).

- Spot rates, both for short and long maturities, move substantially over time

What about the Shape?

- What might explain different shapes?



Shape of the Yield Curve

- **Short end** determined by monetary policy (eg target Fed Funds rate in US)
- Remainder depends on several factors, primarily:
  - Market's expectations of future rates
  - Risk premium
  - Supply and demand
- **"Theories" of the term structure** propose explanations based (mostly) on **one** of these factors
- **Empirical evidence** indicates **all three** play a role

Role of Expectations

- If future spot rates were known **with certainty** then, by no arbitrage the forward rate for a future period is the future spot rate in that period
- In the **pure expectations hypothesis** the **forward rate is the expected future spot rate**
  - therefore if spot curve is upward sloping (implying forward rates are higher than spot rates), you expect future spot rates to be higher than today
  - inconsistent with yield curves that are mostly upward sloping
- Evidence: forward rates tend to **overstate** future spot rates
- Expectations do matter: all else equal, yield curve steepens (flattens) if expected future spot rates go up (down)

Role of Risk Premium

- Investors in longer in longer maturity bonds need to earn a risk premium to compensate them for greater interest rate risk
- All else equal, this should result in an **upward sloping** yield curve
- Historical excess returns by **maturity:**

Role of Supply/Demand

- **Preferred habitat** or **market segmentation** hypothesis:
  - Curve is shaped by supply and demand at various maturities
  - This can lead to any shape of curve – eg if most investors prefer short-run investing then this could also explain the traditional upward sloping shape of the yield curve
  - Commonly accepted that high demand from pension funds and insurance companies for long dated bonds depresses yields at the long end

Shape & Dynamics of Yield Curve

- In summary, all components matter:
  - Market's expectations of future rates
  - Bond risk premia
  - Supply and demand

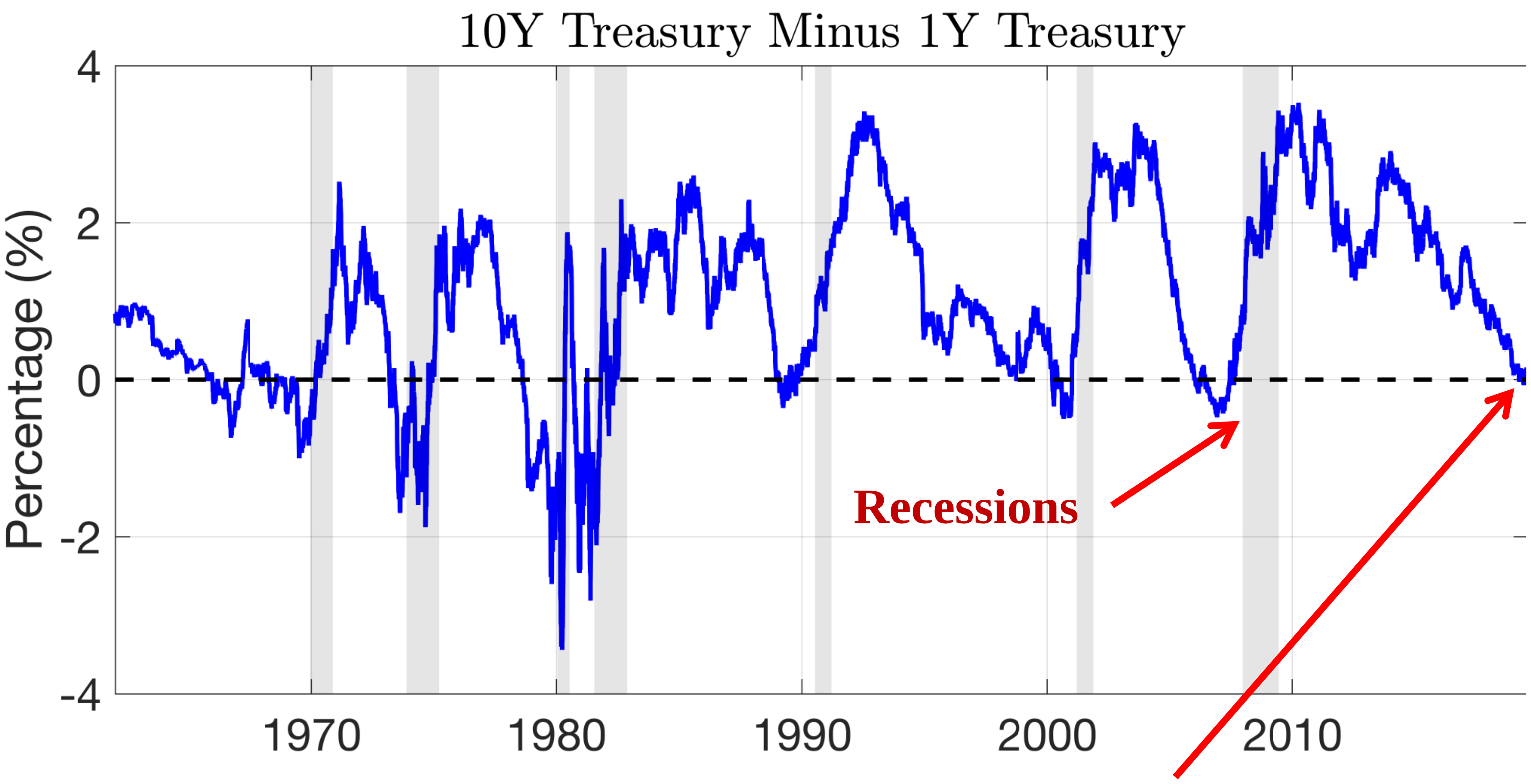
- Exact drivers of (changes in) the shape of the yield curve hard to pin down
  - Eg curve steepening can reflect either an increase in market's expectations of future rates or an increase in risk premia, or both. . .

Slope of the Yield Curve

- What we observe:
  - Usually **upward sloping**
  - **Yields on short maturity bonds are more volatile** than yields on long- maturity bonds
  - **Negative slope** tends to **precede recessions**

- Focus on the slope of the yield curve
  - Slope=long yield-short yield

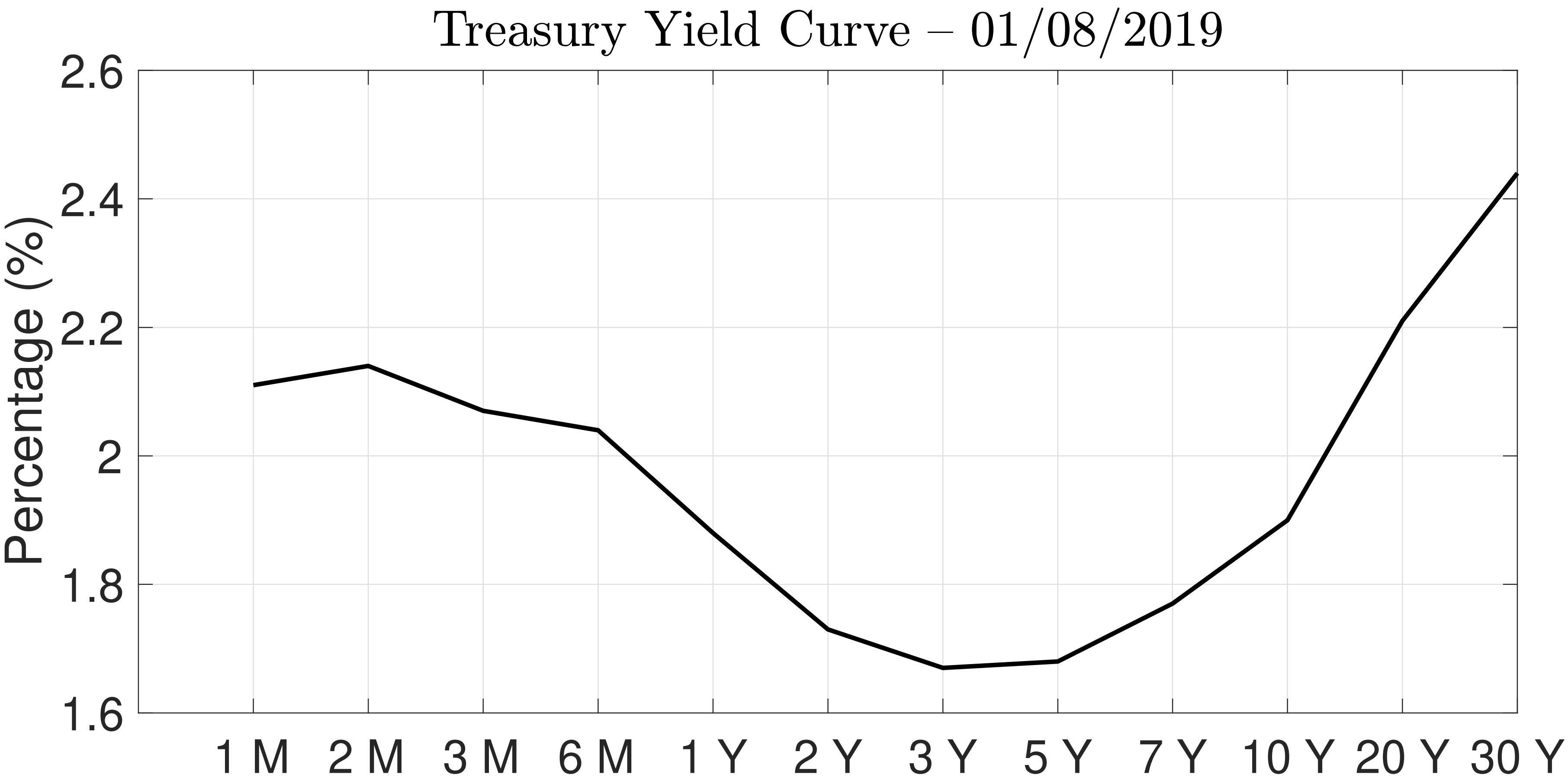
Time Series of the Slope (10Y-1Y)



What does an inverted yield curve mean?


FINANCIAL TIMES
US yield curve sends strongest recession warning since 2007
IT'S OFFICIAL: THE YIELD CURVE IS TRIGGERED. DOES A RECESSION LOOM ON THE HORIZON?
Bloomberg
Markets
Yield Curve Blares Loudest U.S. Recession Warning Since 2007
The ‘yield curve’ is one of the most accurate predictors of a future recession – and it’s flashing warning signs
July 12, 2019 1.00pm BST

Recent Yield Curve



Concept Check: Expectations Hypothesis

- Assume the Expectations Hypothesis is true.

  Say the yield on a 6-month zero is 5% and the yield on a 1-year zero is 7%.

  1. Do you earn more from rolling-over 6-month bonds or from buying a 1-year bond?
  2. What is the expected return on the 1-year bond over the first 6 months?

Rollover vs. buying 1-year bond

- Under the expectations hypothesis the forward rate is the expected future short rate ($\bar{r}$):

$$\left(1+\frac{r_{0.5}}{2}\right)\left(1+\frac{{}_{0.5}f_{0.5}}{2}\right)=\left(1+\frac{r_1}{2}\right)^2$$

$$\left(1+\frac{5\%}{2}\right)\left(1+\frac{{}_{0.5}f_{0.5}}{2}\right)=\left(1+\frac{7\%}{2}\right)^2$$

$$\Rightarrow {}_{0.5}f_{0.5}=9.02\%=\bar{r}$$

Invest in 1-year vs. rollover

$$\text{year strategy}=\left(1+\frac{r_1}{2}\right)^2=(1.035)^2 \qquad\qquad =1.0712$$

$$\text{rollover}=\left(1+\frac{r_{0.5}}{2}\right)\left(1+\frac{\bar{r}}{2}\right)=1.025\times 1.0451=1.0712$$

Expected 6m return of the 1-year bond

$$\text{current price} = \frac{100}{1.035^2} = 93.3511$$

$$\text{expected price} = \frac{100}{1 + \bar{r}/2} = \frac{100}{1.0451} = 95.6848$$

$$6\text{−month return} = \frac{\text{expected price}}{\text{current price}} - 1 = \frac{95.6848}{93.3511} - 1$$

$$= 0.025$$

Outline

- What are bonds? What kind of bonds are traded? ✔
- Bond Pricing. ✔
- Bond Yields. ✔
- Zero-coupon bonds. ✔
- The Term Structure and Forward Rates. ✔
- Interest Rate Risk and Duration.
- Default Risk and Bond Ratings.

Interest Rate Risk and Duration

- We saw that the bond prices depend on interest rates.
  - The returns of all fixed income securities are highly correlated, because they all depend on the interest rate.

- How do interest rate changes affect my portfolio?
  - Portfolio managers and firms need to know how the value of their bond portfolio responds to changes in interest rates and then manage this sensitivity.

Interest Rate Risk: Maturity

- How does bond price sensitivity to interest rates depend on bond **maturity**?
  - Suppose the yield curve is flat at 8%. Consider two 8% annual coupon bonds. Bond #1 has 15 years to maturity, bond #2 has 30 years. Both sell for $1,000.
  - Now let's see what happens if the yield curve moves up by 1% (or 100 basis points):

    Bond #1 will now sell for $919.39 (a 8.1% drop in value),
    Bond #2 will now sell for $897.26 (a 10.3% drop in value).

Other things equal, the longer the time to maturity, the more sensitive is the price of a bond to changes in the interest rate.

- A rare exception to this general rule: very deep discount bonds.

Interest Rate Risk: coupon rates

- How does bond price sensitivity to interest rates depend on coupon rates?
  - Suppose the yield curve is flat at 8%. Consider two 15 year bonds. Bond #1 has an 8% annual coupon rate and sells for $1,000, bond #2 has a 3% annual coupon rate and sells for $572.03.
  - Now let's see what happens if the yield curve moves up by 1% (or 100 basis points):

    Bond #1 will now sell for $919.39 (a 8% drop in value)

    Bond #2 will now sell for $516.36 (a 9.7% drop in value).
- Other things equal, the lower the coupon rate, the more sensitive is the price of a bond to changes in the interest rate.
  - Long maturity pure discount bonds are most sensitive to changes in the interest rate.

Duration: Overview

- Duration: measuring and managing interest rate risk
  - Duration of an **individual bond**
  - Duration of a **portfolio**
  - **Duration-neutral strategies** (hedging and speculation)

Duration

- The duration of a bond provides a measure of the sensitivity of a bond to interest rate changes.

- We focus on two measures:
  - D_{mac}: *Macaulay duration* (time duration)
    - What is the average present value-weighted maturity of the bond's cash flows
  - D_{mod}: *Modified Duration*
    - What is the % change in a bond's price when interest rates change by a given amount?
- Another popular measure that we will not emphasize
  - DV01: *Dollar value of basis point change* (dollar duration)
    - By how many \$ does the value of the bond change when interest rates change by 0.01% (1 bp)

Uses of Duration

1. General quantification of risk in a portfolio of interest rate sensitive securities.

2. "Immunization" of a portfolio against certain types of movements.
   - Hedging interest rate and curve risk.
     - Important Assumptions: flat curve and parallel shifts
     - There are ways to relax these assumptions, but it's beyond the scope of this class.
   - **Note**: not a tool to hedge against default risk

Macaulay Duration

- Because of differences in coupon rates, maturity is an ambiguous measure for bonds making many payments.
- Macaulay (1856) suggested that duration be computed as the weighted average of the times to each coupon or principal payment made by the bond.
- The weight applied to each payment is the proportion of the total value of the bond accounted for by that payment.

The weight w_t associated with the cash flow paid at time t (CF_t) with **annual compounding** would be:

$$w_t = \frac{CF_t/(1+y)^t}{\text{Bond Price}}, \text{ and } D_{mac} = \sum_{t=1}^{T} t \times w_t$$

D_{mac} for semi-annual bonds

- Under this interpretation, bonds with higher duration give cash flows farther into the future, and thus, are riskier

- Formula:

$$D_{mac}(T) = \frac{1}{P_T}\left[\sum_{t=1}^{2T} \frac{t}{2} \frac{\frac{C}{2}}{\left(1+\frac{y_T}{2}\right)^t} + T \frac{F}{\left(1+\frac{y_T}{2}\right)^{2T}}\right]$$

- Notice we have time to cash flows (t/2 or T) weighted by present value of the cash flow received.

D_{mac} Example

- Compute the duration of an 8% coupon (semiannual coupon payments) and a zero-coupon bond, each with two years to maturity. Assume that the yield to maturity on each bond is 10%, or 5% per half year.

8% Coupon Bond

(1) Time (Years)	(2) CF	(3) Discounted CF	(4) Weight	(1)*(4)
0.5	$40	$38.095	0.0395	0.0198
1.0	$40	$36.281	0.0376	0.0376
1.5	$40	$34.553	0.0358	0.0537
2.0	$1,040	$855.611	0.8871	1.7742
Sum		$964.540	1.0000	1.8853 = D_{mac}

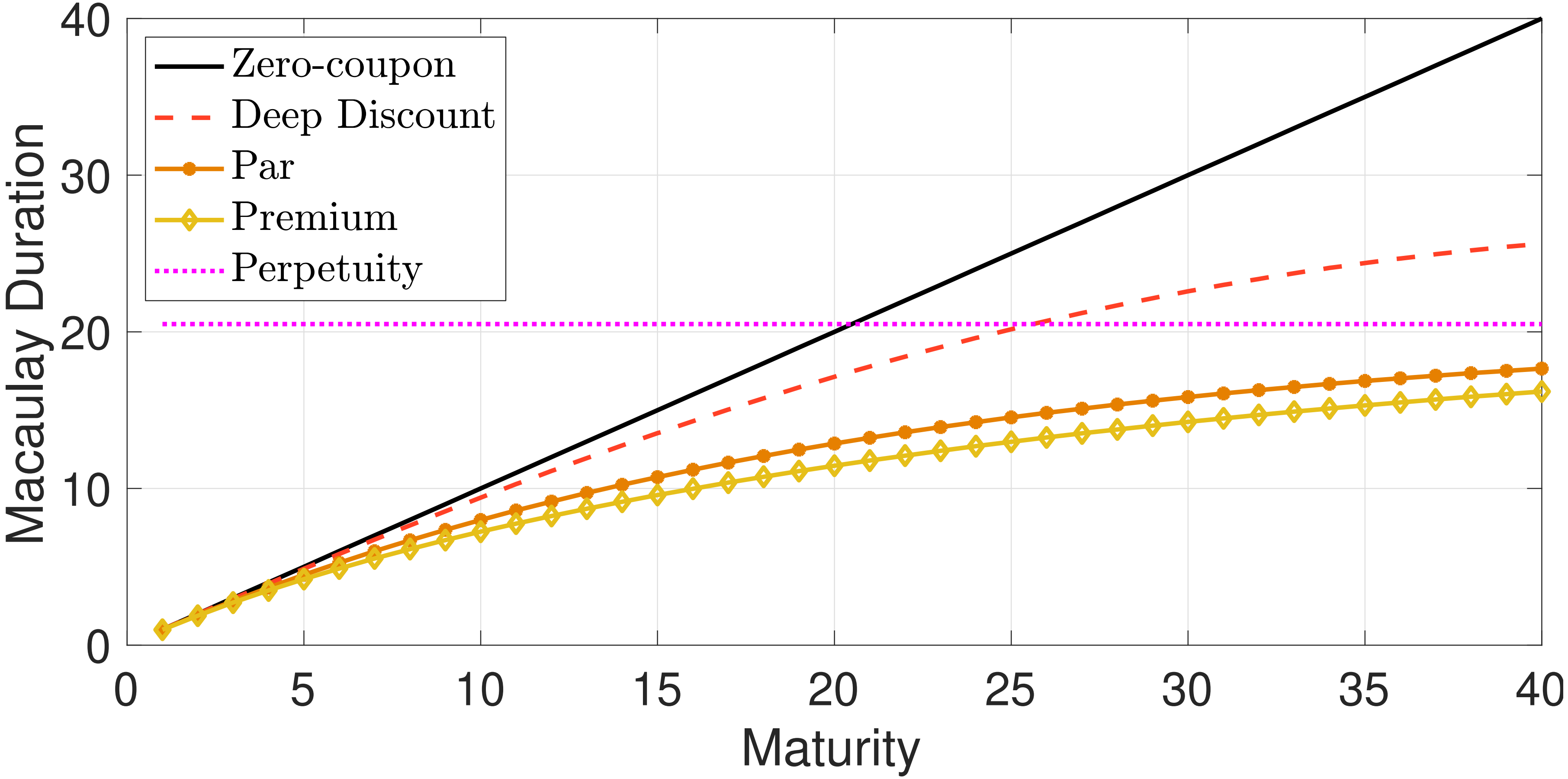
Macauley Duration for Zeros

- Zero-coupon Bond

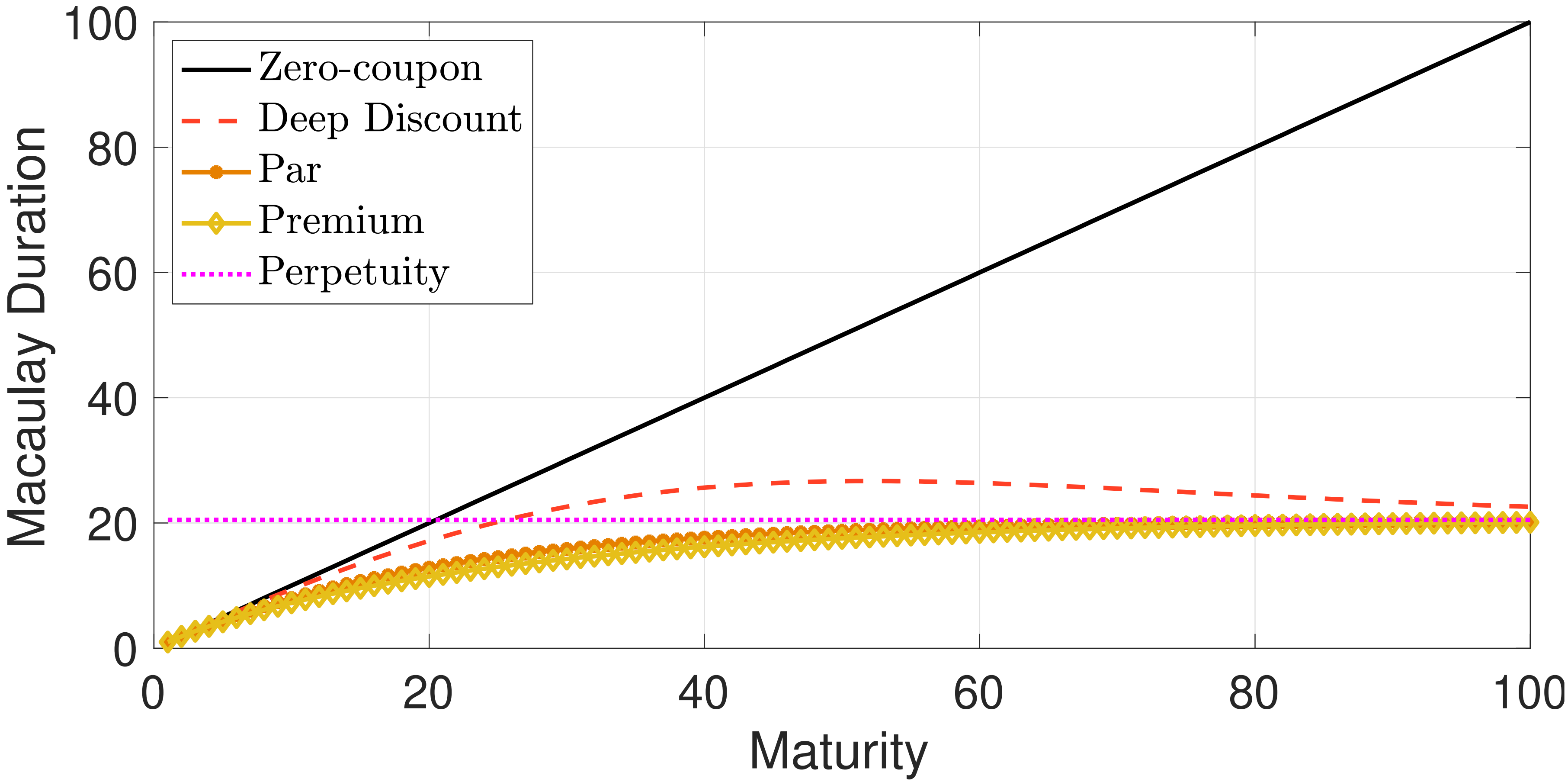
Time (Years)	CF	Discounted CF	Weight	(1)×(4)
0.5- 1.5	$0	$0	0.0	0.0
2.0	$1,000	$822.70	1.0	2.0
Sum		$822.70	1.0	2.0 = D_{mac}

- The Macauley duration of a zero-coupon bond is exactly equal to its time to maturity.

Coupon and Macaulay Duration



Coupon and Macaulay Duration (longer maturities)



How to measure sensitivity?

- What about $\frac{\delta P}{\delta y}$?

$$P = \sum_{t=1}^{2T} \frac{C}{2}\left(1+\frac{y}{2}\right)^{-t} + F\left(1+\frac{y}{2}\right)^{-2T}$$

$$\frac{\delta P}{\delta y} = \sum_{t=1}^{2T} -t\frac{C}{2}\left(1+\frac{y}{2}\right)^{-t-1}\left(\frac{1}{2}\right) + (-2T)F\left(1+\frac{y}{2}\right)^{-2T-1}\left(\frac{1}{2}\right)$$

$$= \frac{-1}{1+\frac{y}{2}}\left[\sum_{t=1}^{2T} \frac{t}{2}\frac{C}{2}\left(1+\frac{y}{2}\right)^{-t} + (T)F\left(1+\frac{y}{2}\right)^{-2T}\right]$$

- Look familiar? The term in brackets is $D_{mac} \times P$.

Modified Duration

- When interest rates change, the proportional change in a bond's price is related to the change in its yield to maturity, *y*, according to modified duration

- Rearrange D_{mac} equation showing price interpretation:

$$\frac{\Delta P}{P} \approx -\underbrace{\left[\frac{D_{mac}}{1+\frac{y}{m}}\right]}_{\text{This term in brackets is sometimes referred to as Modified Duration}} \cdot \Delta y$$

Modified Duration Example

- How much would the price of a 2 year zero decrease if the yield curve shifted from a flat 10% to a flat 11% (with annual compounding)?

  Answer:

  – $V_{10\%} = \frac{\$1,000}{1.10^2} = \$826.45, \ V_{11\%} = \frac{\$1,000}{1.11^2} = \$811.62$

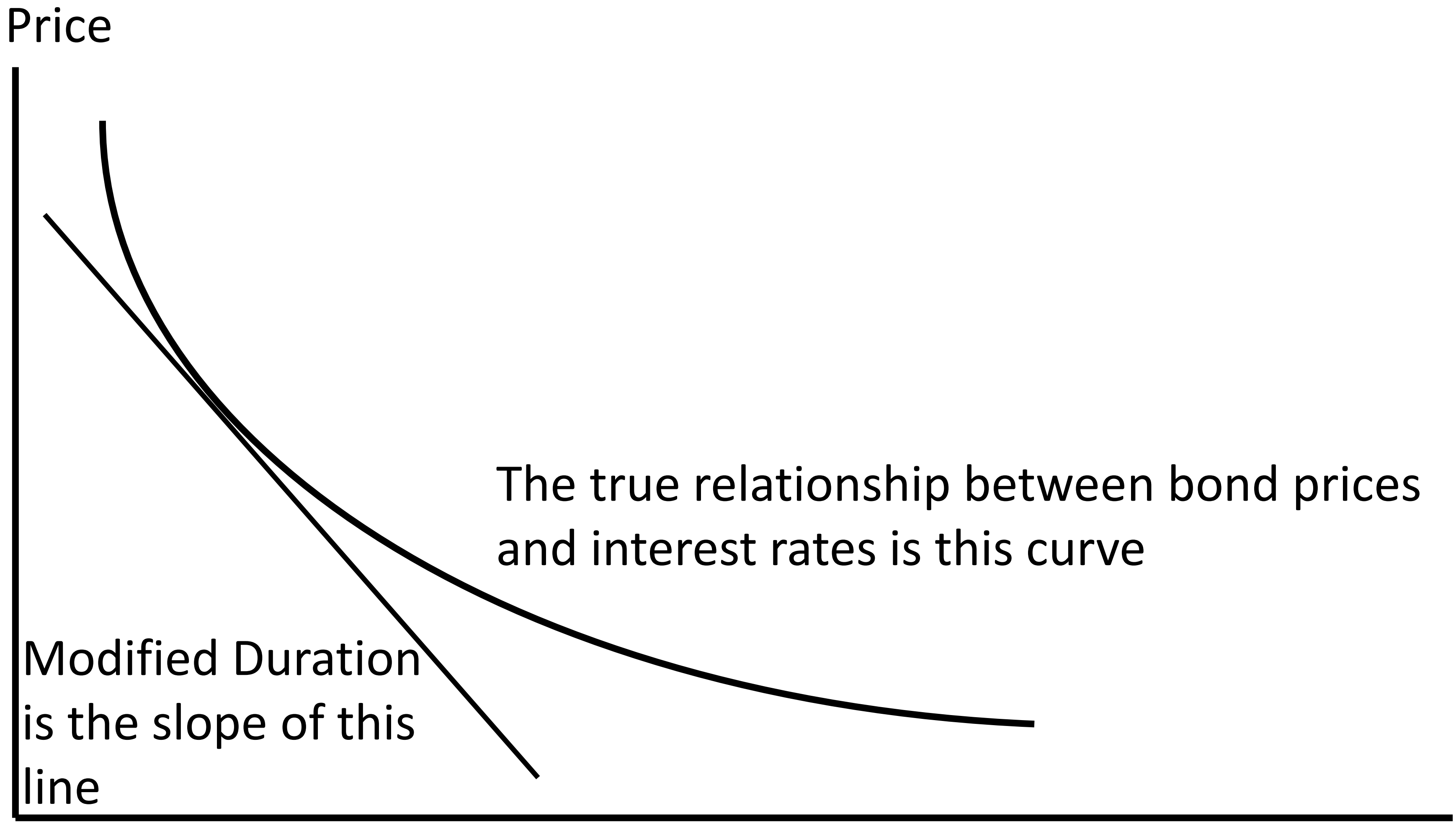
  – $D/(1 + y) = 2/1.1$,

  – The price goes down by approximately $0.01 \times 2/1.1 = 1.82\%$ or \$15.02

  – Duration estimate: $\$826.45 - \$15.02 = \$811.43$

- Why approximately? Because duration is exact only locally as can be understood from the following figure.

Interest Rate Risk and Duration



Properties of Modified Duration

- Modified Duration is a function of **coupon**, **yield** and **maturity:**

Modified Durations of a 6-yr bond:

yield	coupon 0%	5%	10%	20%
4%	5.77	5.14	4.77	4.34
8%	5.56	4.88	**4.49**	4.06
12%	5.36	4.63	4.23	3.79

- Maturity and yield held constant, duration is higher, the lower the coupon
- Maturity and coupon held constant, duration is higher, the lower the yield
- Holding coupon and yield constant, a bond's duration generally increases with maturity (always the case for bonds selling at par or at a premium to par)

Duration of a Portfolio

- Consider a **portfolio** of n_A units of bond A and n_B units of bond B with value:

$$\Pi = \eta_A \cdot P_A + \eta_B \cdot P_B$$

- Define/derive **durations of the portfolio**:

$$\text{Modified Duration} = \frac{n_A P_A}{n_A P_A + n_B P_B} D_{mod}^{A} + \frac{n_B P_B}{n_A P_A + n_B P_B} D_{mod}^{B}$$
$$= \omega_A \times D_{mod}^{A} + \omega_B \times D_{mod}^{B}$$

where ω_i is the portfolio weight (percentage of total \$) of bond I

- Portfolio durations have the standard interpretations assuming **all bond yields move up and down together** (parallel shifts of the yield curve) **by the same (small) amount.**

Duration-Neutral Strategies

- General idea of **duration-neutral strategies**:
  - Set up a position so that it is duration neutral ie has zero duration
  - Therefore, you will be hedged against small parallel changes in yields
  - If yields move in a different manner, then the hedge will be less effective (could either make or lose money)

- Duration neutral positions can be used in different contexts:
  1. **Hedging:** primary objective is to hedge interest rate risk of position
  2. **Speculation:** set up a position so that it is unaffected by small parallel changes in yields with a primary objective of taking a view on a different type of yield shift (eg steepening or flattening of the yield curve)

Hedging – Immunisation

- How to invest assets to fund fixed income(-like) liabilities while hedging or minimising interest rate risk
- Of great importance to banks (ALM), defined benefit pension funds (LDI), insurance companies
- Cleanest solution is to invest in zero coupon bonds that exactly match the liabilities ("cash-matching")
  - By construction PV(assets) = PV(liabilities), regardless of how interest rates move
  - In practice may be hard to implement

Hedging – Immunisation

- Instead, **duration hedge** ie invest in a portfolio of bonds today such that:
  - PV(assets) = PV(liabilities)
  - ($ or Modified) Duration of assets = ($ or Modified) Duration of liabilities
- This will "immunise" the liabilities against small parallel moves in yields
- For larger moves, the hedge will not be as effective and will need to be "reset"
- Note that there are other issues with this approach eg reinvestment risk etc

Immunisation Example

- The XYZ Pension Fund has an obligation to pay $1m in 10 years
- Assume that all interest rates/yields are 5% pa
- What positions in the two following bonds should XYZ invest in so that:

PV (assets) = PV (liabilities)

Duration (assets) = Duration (liabilities)

	Mat.	Coupon (Annual)	Price.	$Duration.	$D_{MOD.}$	D_{MAC}
Bond A	8	4%	935.37	6,201.94	6.6305	6.9620
Bond B	20	3%	750.76	10,348.87	13.7846	14.4738
Liability	10	-	613,913	5,846,792.89	9.5238	10

Immunisation Example

- To be immunised, solve either for **number of bonds** such that:

PV (assets) = PV (liabilities)	$n_A 935.37 + n_B 750.76 = 613{,}913$
\$D(Assets) = \$D(liabilities)	$n_A 6{,}201.94 + n_B 10{,}348.89 = 5{,}846{,}729.89$

Solution: $n_A = 390.89$ *and* $n_B = 330.71$

- Or for **portfolio weights** such that:

$$w_A + w_B = 1$$

$$w_A 6.6305 + w_B 13.7846 = 9.5238$$

Solution: $w_A = 59.6\%$ *and* $w_B = 40.4\%$

Immunisation Example

- How does the duration hedge perform?

Yields	0.04	0.05	0.06
PV(liab)	675,564.17	613,913.25	558,394.78
Bond A			
• Price	1,000.00	935.37	875.80
• Number	390.89	390.89	390.89
• Total Value	390,893.79	365,629.49	342,346.39
Bond B			
• Price	864.10	750.76	655.90
• Number	330.71	330.71	330.71
• Total Value	285,766.95	248,283.77	216,914.62
PV(assets)	676,660.73	613,913.25	559,261.01
PV(assets) - PV (liab)	**1,096.56**	**0**	**866.23**

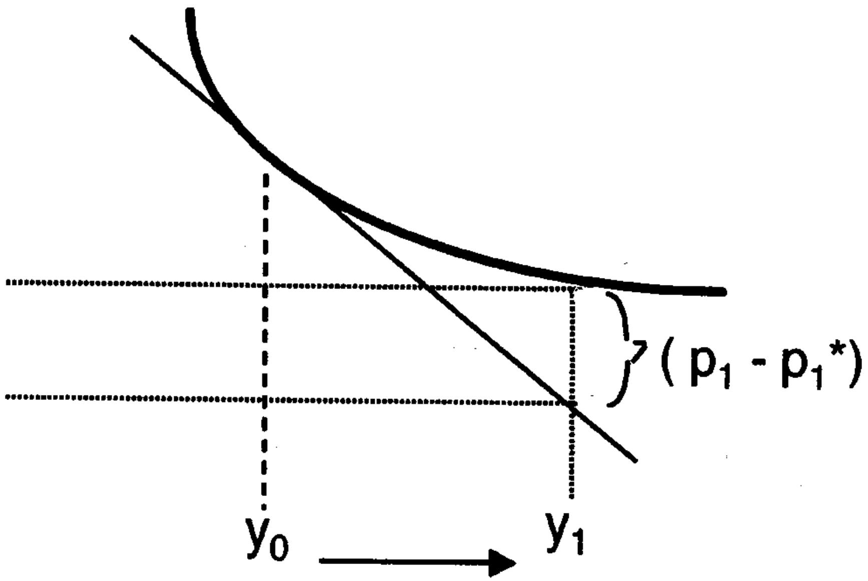
Problems with Duration

1. Duration estimate assumes a parallel shift in the *entire* yield curve.
   - Short end is more volatile than the long end of curve
   - Parallel shift explains 66% of T-Bill yield variance

2. Duration is a ***straight line estimate to a curved (convex) price-yield function***

Problems with Duration

- Duration over-estimates the change in the bond price following an increase in yield from y_0 to y_1
- Error gets bigger as Δy increases



- What if the yield decreases?

Duration formula summary

Assuming m compounding periods per year:

$$D_{mac} = \sum_{t=1}^{T\times m} \frac{t}{m} \times w_t \text{, where } w_t = \frac{CF_t / \left(1+\frac{y}{m}\right)^t}{\text{Bond Price}}$$

$$D_{mod} = \frac{D_{mac}}{1+\frac{y}{m}}$$

$$DV01 = D_{mac} \times P \times \frac{0.0001}{1+\frac{y}{m}}$$

Summary

- Bonds are priced by computing the ***present value*** of all their cash flows at appropriate spot rates
- These discount rates can be obtained through ***zeros***
- Plotting the zeros, or spot rates, against time to maturity gives the ***term structure of interest rates***
- ***Forward interest rates*** are arbitrage-free rates that allow locking in future rates
- Interest rate risk is measured by a bond's ***duration***

Key Concepts

- Coupon Bonds
- Zeros
- Bond Pricing
- Treasury market convention
- Spot rates
- Yield to Maturity
- Yield Curve
- Convexity
- Bootstrapping
- Duration
- Dollar value of basis point change
- Portfolio Immunization
- Forward rates

Outline

- What are bonds? What kind of bonds are traded? ✓
- Bond Pricing. ✓
- Bond Yields. ✓
- Zero-coupon bonds. ✓
- The Term Structure of Interest Rates & Forward Rates. ✓
- Interest Rate Risk and Duration. ✓
- Default Risk and Bond Ratings.

Bonds and Credit Risk – Overview

- Introduction to credit risk and default
- Measures of credit risk
- Understanding yields and credit spreads of risky bonds
- Credit risk and duration

Fixed Income? Credit Risk & Default

- Bonds promise the holder pre-specified cashflows, but in practice the issuer (sovereign or corporate) may default

- A default typically sets in motion either:
  - **Liquidation** (corporates only)
  - **Restructuring/reorganisation**

- In default, the investor tipically gets some positive **recovery value** (less than face value)

Types of Credit Risk

- There are two main types of credit risk:
  - **credit default risk** is the risk of financial loss due to an issuer default
  - **credit spread risk** is the risk of financial loss due to changes in the level of an issuer's credit spreads/creditworthiness

- The key determinants of both types of credit risk are:
  - the **probability of default**
  - the **severity of default** – loss or recovery in default

Measures of Credit Risk

- **Traditional measures of credit risk in capital markets:**
  - Credit Ratings
  - Credit Spreads
  - Default Rates
  - Recovery Rates

Credit Ratings

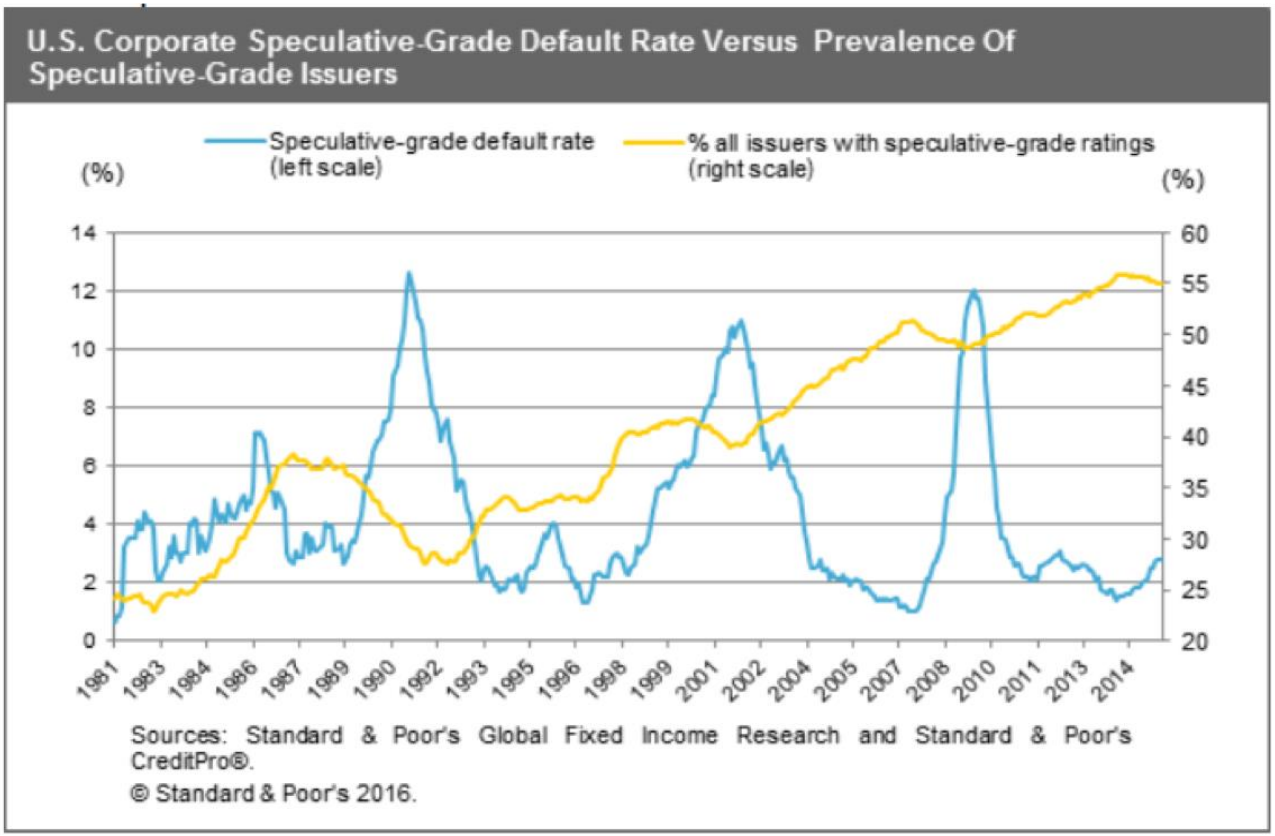
Credit Risk	Moody's*	Standard & Poor's	Fitch†
Investment Grade			
Highest quality	Aaa	AAA	AAA
High quality (very strong)	Aa	AA	AA
Upper medium grade (strong)	A	A	A
Medium grade	Baa	BBB	BBB
Below Investment Grade			
Lower medium grade (somewhat speculative)	Ba	BB	BB
Low grade (speculative)	B	B	B
Poor quality (may default)	Caa	CCC	CCC
Most speculative	Ca	CC	CC
No interest being paid or bankruptcy petition filed	C	C	C
In default	C	D	D

Source: Moody's, Standard & Poor's, Fitch

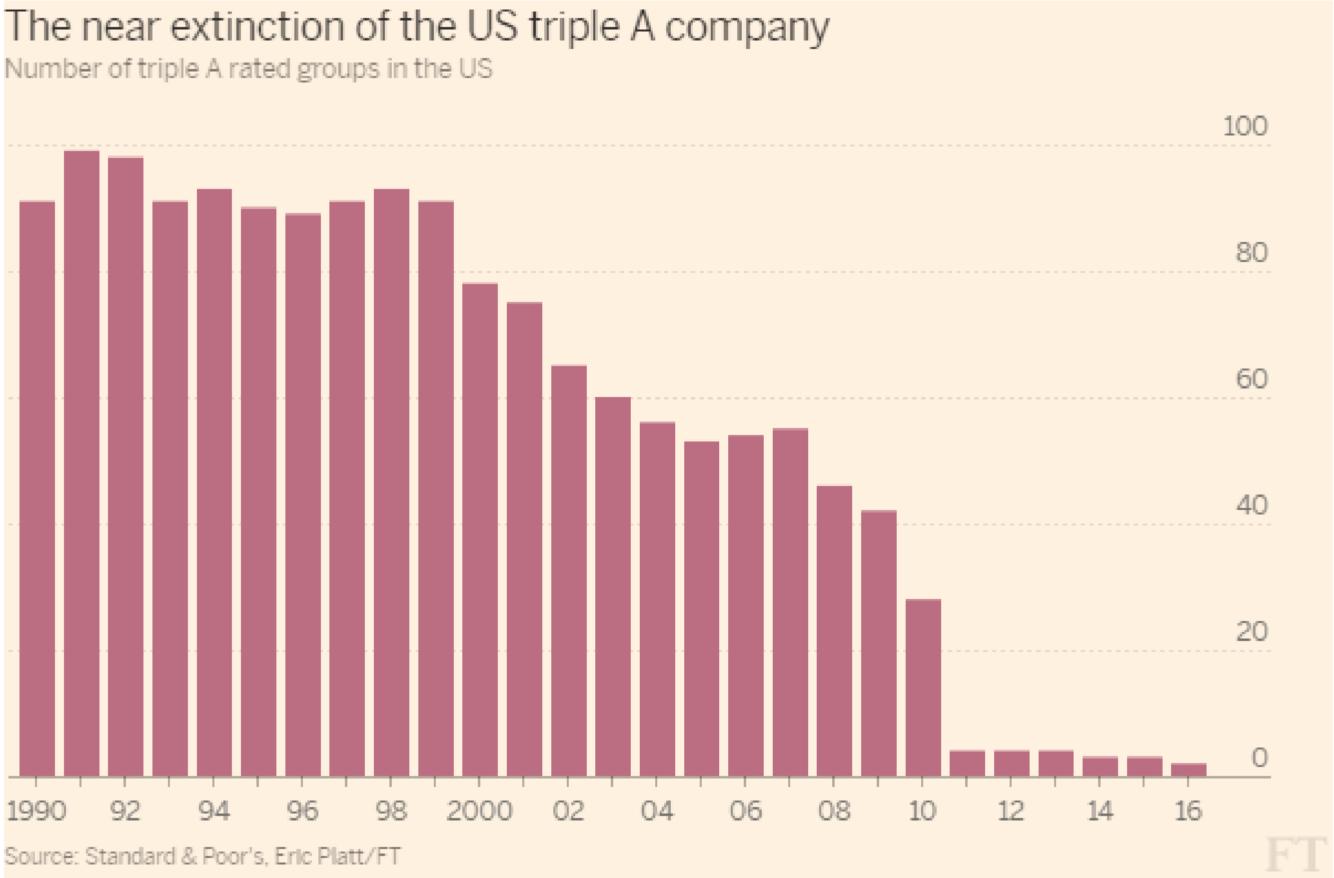
*The ratings from Aa to Ca by Moodys may be modified by the addition of a 1, 2 or 3 to show relative standing within the category, e.g., Baa2.

†The ratings from AA to CC by Standard & Poor's and Fitch may be modified by the addition of plus (+) or minus (-) sign to show relative standing within the category, e.g., A-.

Credit Ratings



Credit Ratings



Determinants of Bond Ratings

- Determinants of bond safety:
  - Coverage ratios: ratios of earnings to fixed costs.
  - Leverage ratio: debt-to-equity ratio.
  - Liquidity ratios: current assets/current liabilities and current assets excluding inventories/current liabilities.
  - Profitability ratios: return on assets.
  - Cash flow-to-debt ratio.

Bond Ratings and Financial Ratios

Median ratio data by bond rating: there is significant variation across industry and over time

Ratio	AAA	AA	A	BBB	BB	B	CCC
EBIT interest cover *	21.4	10.1	6.1	3.7	2.1	0.8	0.1
return on capital %	34.9	21.7	19.4	13.6	11.6	6.6	1
Gross profit margin %	27	22.1	18.6	15.4	15.9	11.9	11.9
Total debt/capital %	22.9	37.7	42.5	48.2	62.6	74.8	87.7
* Earnings before interst and tax divided by interest							

Measures of Credit Risk

- **Traditional Measures of Credit Risk**
  - ✓ *Credit Ratings*
  - **Credit Spreads**
  - Default Rates
  - Recovery Rates

Credit Spreads

Standard credit spread measures for corporate bonds:

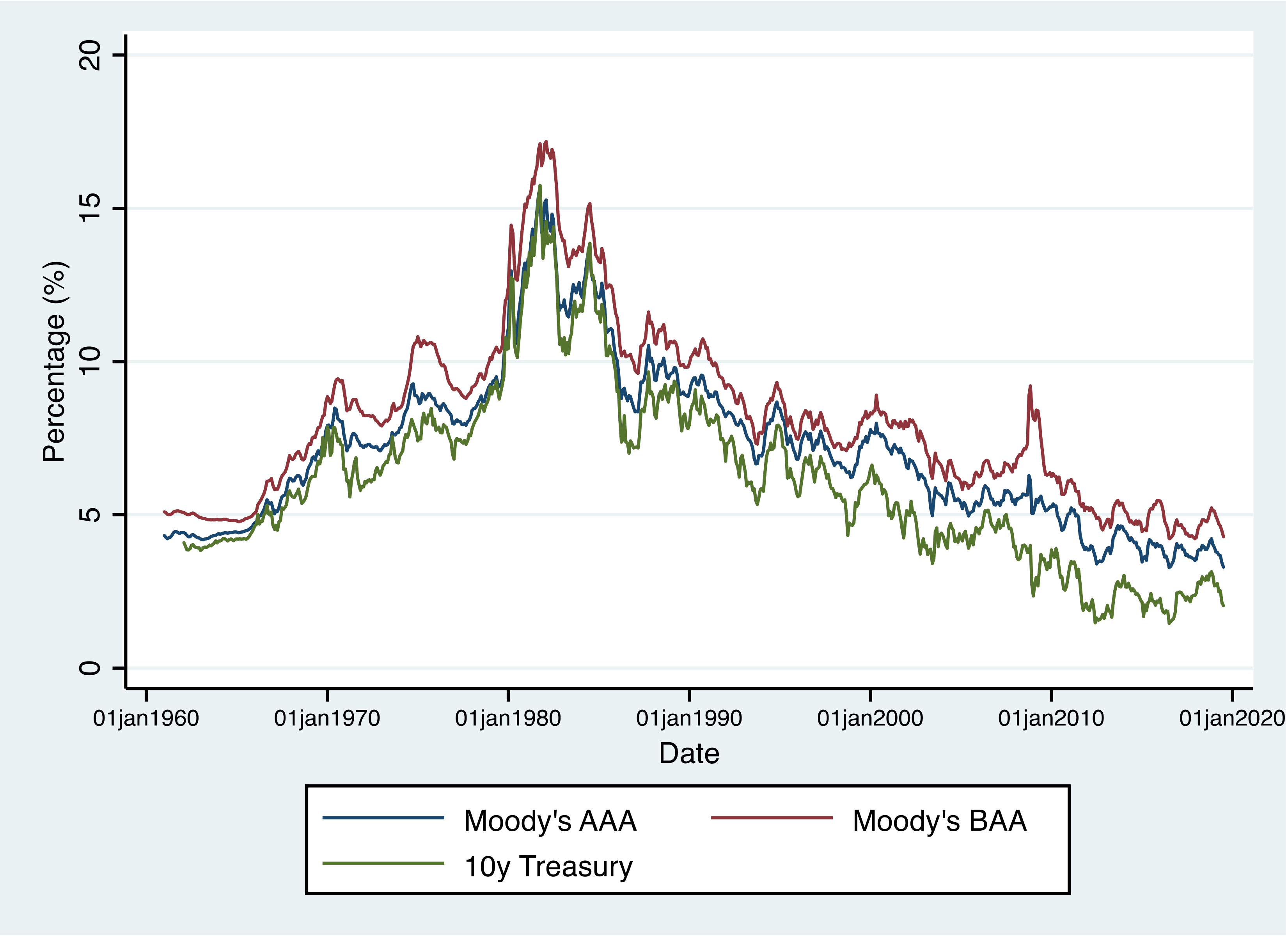
1. **Difference in Yield-to-Maturity** of the corporate bond and an equivalent maturity government bond

3yr **Government Bond** with 2.5% annual coupon, trading at 100.925

$$100.925 = \frac{2.5}{(1+y)} + \frac{2.5}{(1+y)^2} + \frac{102.5}{(1+y)^3} \Rightarrow y = 2.1785\%$$

3yr **Corporate Bond** with 4.5% annual coupon, trading at 99.325

$$99.325 = \frac{4.5}{(1+y)} + \frac{4.5}{(1+y)^2} + \frac{104.5}{(1+y)^3} \Rightarrow y = 4.7467\%, \; cs = 2.5682\%$$



Percentage (%)
0
5
10
15
20
01jan1960
01jan1970
01jan1980
01jan1990
01jan2000
01jan2010
01jan2020
Date
Moody's AAA
Moody's BAA
10y Treasury

Credit Spreads

Standard credit spread measures for corporate bonds:

2. Z-spread: the flat spread one needs to add to the riskfree zero coupon yield curve, so that the PV of the bond cash flows (using the adjusted yield curve) equals the market price of the bond

3yr **Government Bond** with 2.5% annual coupon, trading at 100.925
Spot interest rates for 1, 2 and 3 years: 1%, 1.5% and 2.2% pa

$$100.925 = \frac{2.5}{1.01} + \frac{2.5}{1.015^2} + \frac{102.5}{1.022^3}$$

3yr **Corporate Bond** with 4.5% annual coupon, trading at 99.325

$$99.325 = \frac{4.5}{(1.01 + z)} + \frac{4.5}{(1.015 + z)^2} + \frac{104.5}{(1.022 + z)^3} \Rightarrow z = 2.5853\%$$

Credit Spreads

Standard credit spread measures for corporate bonds:

3. Option Adjusted Spread (OAS): the flat spread one needs to add to the riskfree zero coupon yield curve **in a pricing model that accounts for embedded options** so that the PV of the bond cash flows equals the market price of the bond.

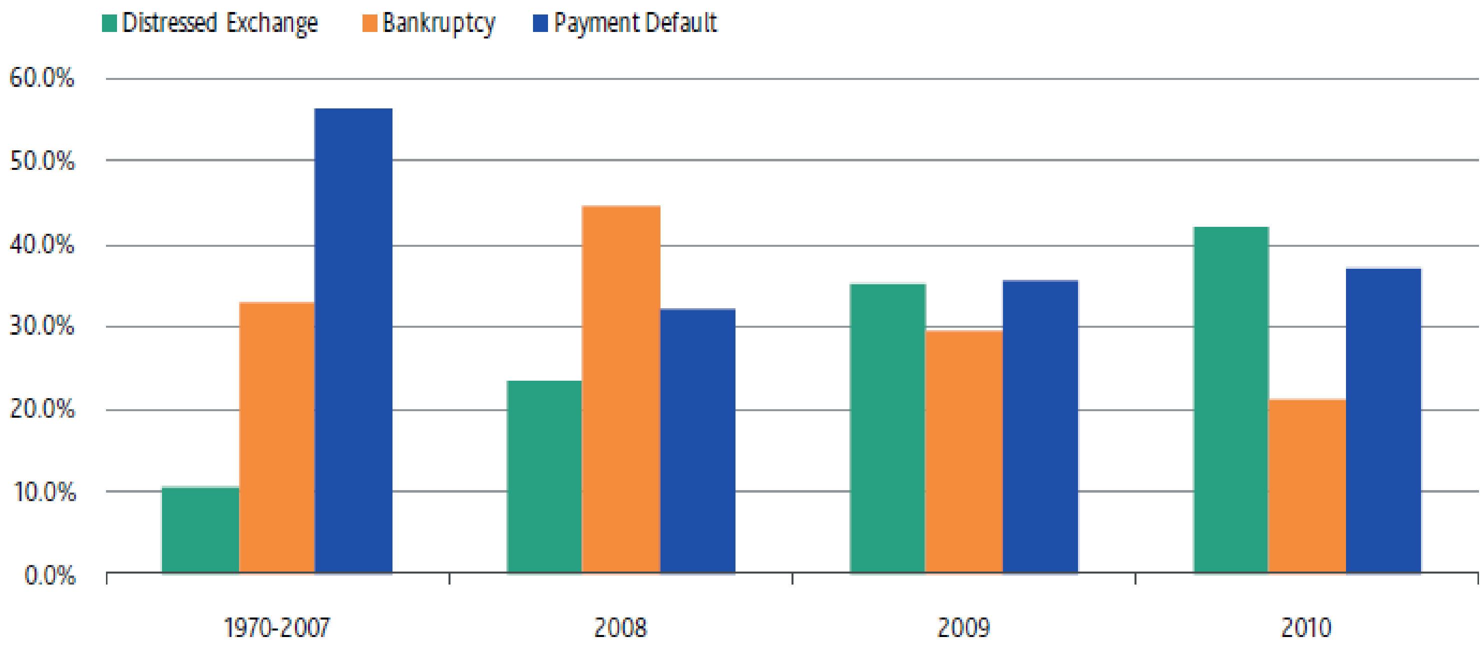
Measures of Credit Risk

- **Traditional Measures of Credit Risk**
  - ✓ *Credit Ratings*
  - ✓ *Credit Spreads*
  - **Default Rates**
  - **Recovery Rates**

Defaults – by Default Type

EXHIBIT 2

Distressed Exchanges Remained Active

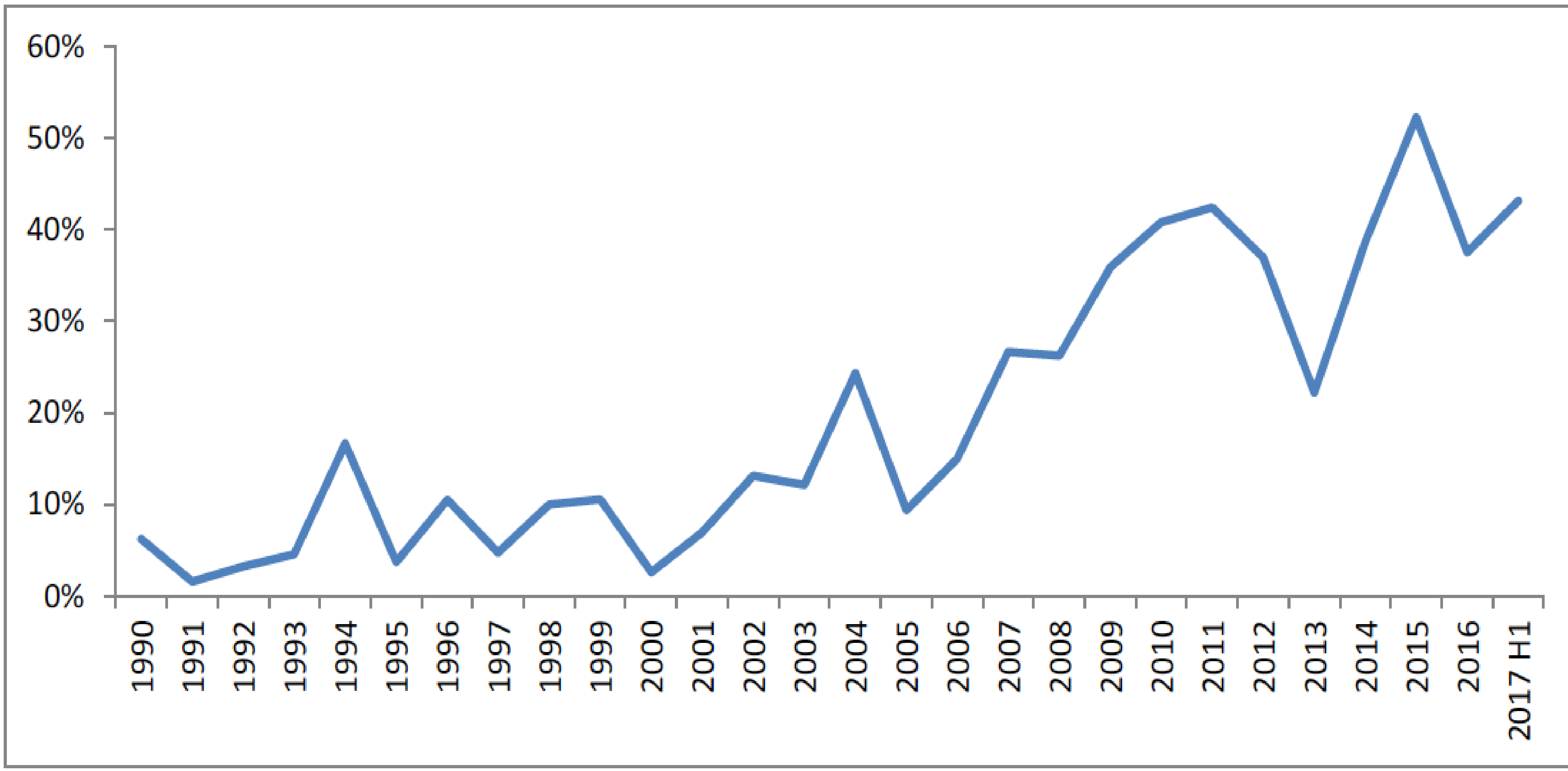


Moody's definition of a distressed exchange:

- the issuer offers bondholders a new package of securities that amount to a diminished financial obligation (such as debt or preferred stock, or debt with a lower coupon or par amount)
- the exchange had the apparent purpose of helping the borrower avoid default

Distressed Exchanges

Share of distressed exchanges on the rise

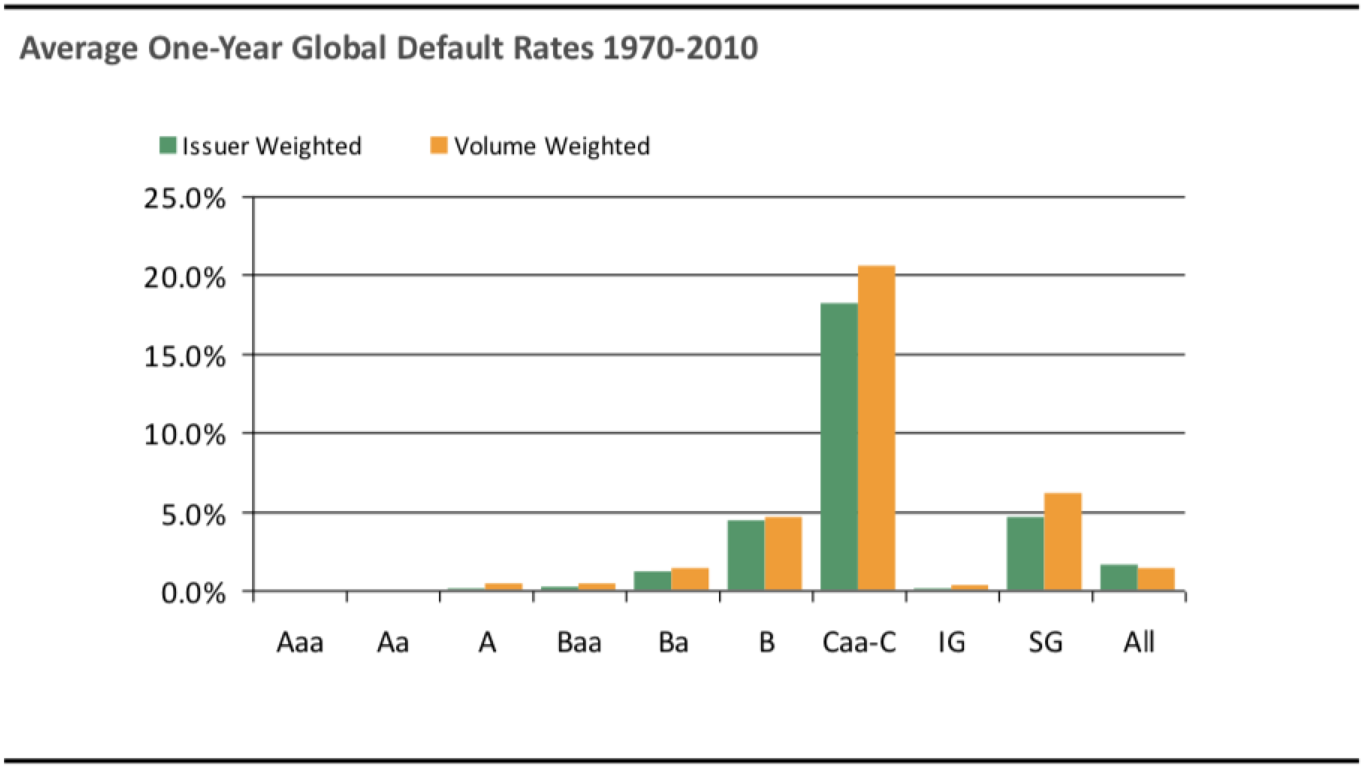


Source: Moody's Investors Service

Default Rates

- Historical **default rates** are typically measured as follows:
  - group issuers by rating category at a point in time
  - determine percentage of **issuers** (**issuer-weighted** default rate) or **face value of debt** (**volume-weighted** default rate) that defaulted
  - within 1 yr (**annual** default rate) or within ***n*** years (*n* >1: **cumulative** default rate)

Annual Default Rates

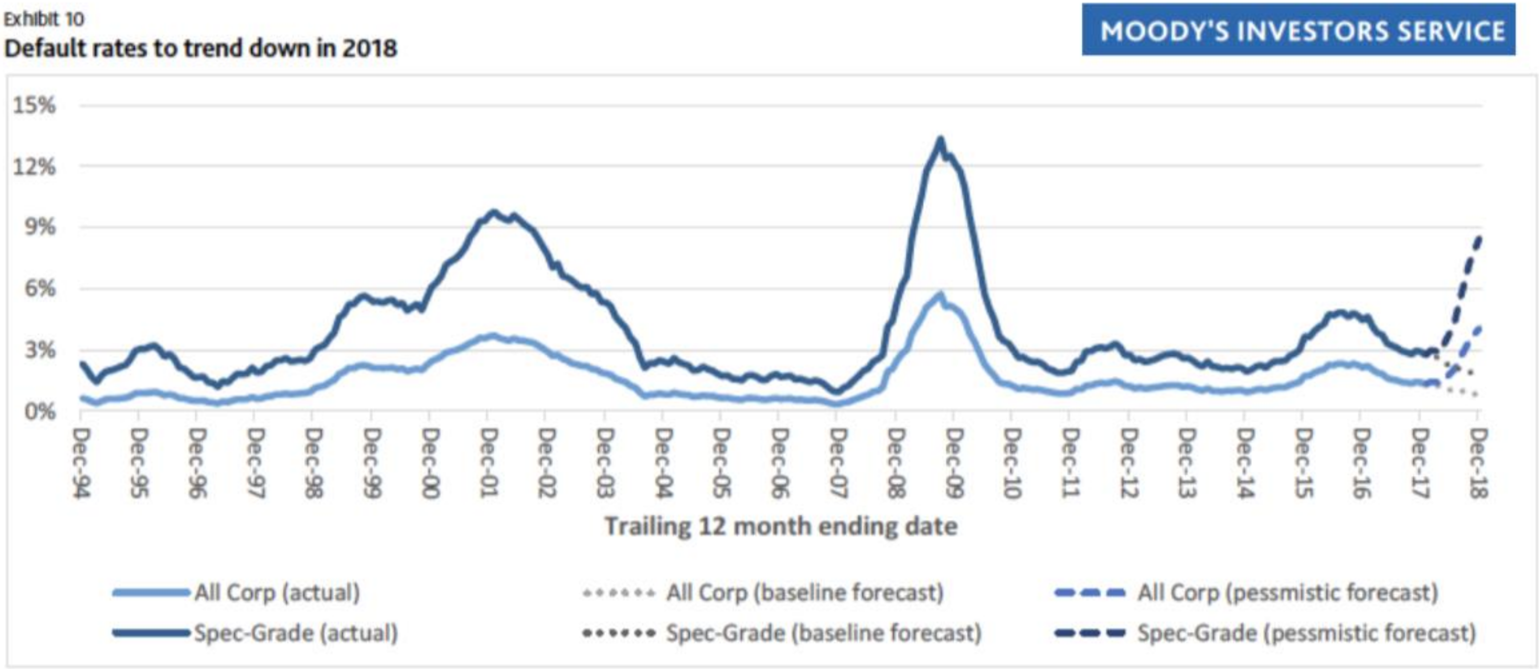


Source: Moody's

Default Rates

Exhibit 10

Default rates to trend down in 2018



Recovery Rates

- Recovery rates are typically measured by **seniority** and **security**:

Exhibit 7

Average corporate debt recovery rates measured by trading prices

Panel A - Recoveries	Issuer-weighted recoveries			Volume-weighted recoveries		
Priority Position	2017	2016	1983-2017	2017	2016	1983-2017
1st Lien Bank Loan	69.04%	75.05%	67.07%	74.73%	77.95%	63.74%
2nd Lien Bank Loan	17.87%	22.50%	30.38%	30.29%	22.50%	27.73%
Sr. Unsecured Bank Loan	9.00%	n.a.	45.87%	9.00%	n.a.	40.21%
1st Lien Bond	62.43%	48.72%	53.62%	66.21%	40.89%	53.80%
2nd Lien Bond	52.75%	34.07%	45.18%	36.61%	35.82%	43.63%
Sr. Unsecured Bond	53.85%	31.45%	37.74%	39.79%	27.10%	33.48%
Sr. Subordinated Bond	38.00%	36.72%	31.10%	50.62%	56.10%	26.34%
Subordinated Bond	74.38%	24.50%	32.05%	76.37%	24.50%	27.55%
Jr. Subordinated Bond	17.50%	0.63%	22.79%	4.84%	0.63%	13.97%

Source: Moody's

Bonds and Credit Risk – Overview

- Introduction to credit risk and default
- Measures of credit risk
- **Understanding yields and credit spreads of risky bonds**
- Credit risk and duration

Understanding Yields of Risky Bonds

- For a bond with **credit risk** is the bond's yield:
  - A **cost of borrowing? Yes,** in the usual sense
  - An **interest rate? No**
  - A (compound) rate of return of if you buy and hold the bond to maturity?

Understanding Yields of Risky Bonds

- Recall the simple YTM-based spread calculations:

3yr **Government Bond** with 2.5% annual coupon, trading at 100.925

$$100.925 = \frac{2.5}{(1+y)} + \frac{2.5}{(1+y)^2} + \frac{102.5}{(1+y)^3} \Rightarrow y = 2.1785\%$$

3yr **Corporate Bond** with 4.5% annual coupon, trading at 99.325

$$99.325 = \frac{4.5}{(1+y)} + \frac{4.5}{(1+y)^2} + \frac{104.5}{(1+y)^3} \Rightarrow y = 4.7467\%,\ cs = 2.5682\%$$

- Is the YTM on a ***government or corporate bond*** the return you earn on the bond by holding it to maturity?

Understanding Yields of Risky Bonds

- For a riskfree bond, the yield is the return you earn by buying and holding the bond to maturity if and only if you can reinvest the coupons at the original yield
- **For a risky bond, the yield is calculated from the promised cashflows,** therefore it is the return you earn by buying and holding the bond to maturity if and only if:
  - you can reinvest the coupons at the original yield
  - ***and* the bond does not default**
- Therefore the yield on the risky bond is a best case (no default) scenario return, not even the expected return!
- **Problem:** the yield on the risky bond is calculated from the **promised cash flows only**, completely ignoring the probability and severity of default. . . .

Understanding Credit Spreads

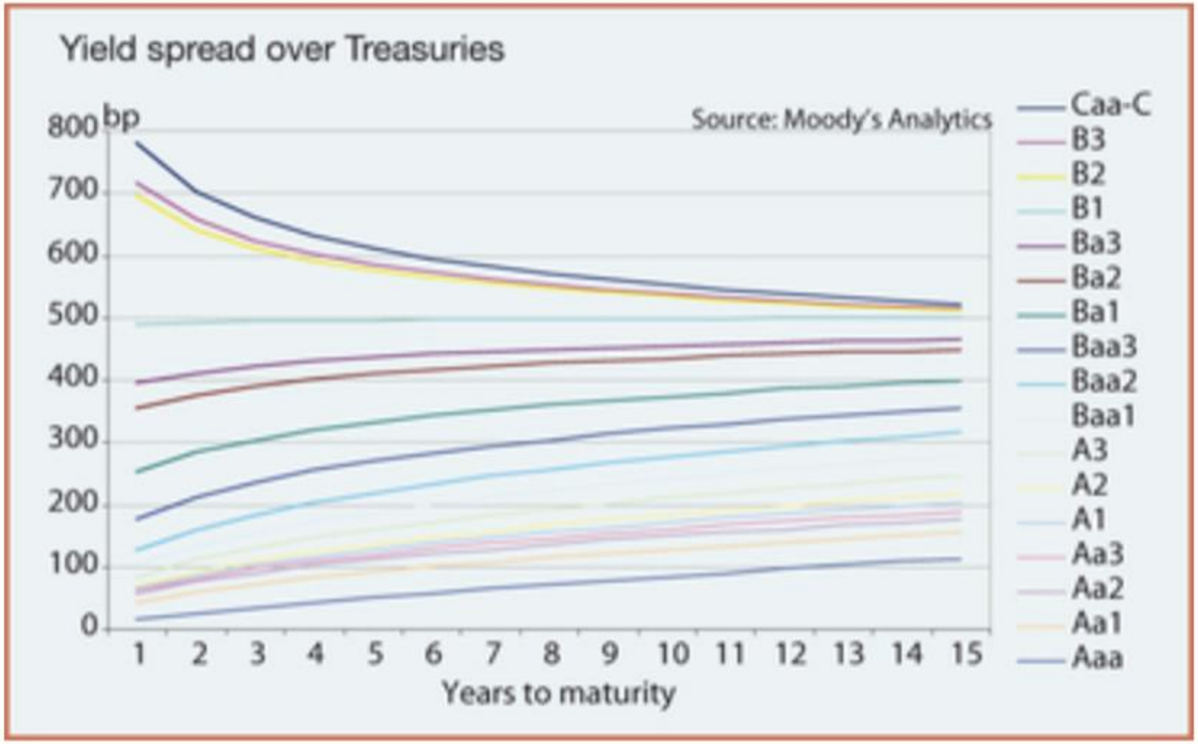
- Does this mean that credit spreads (calculated as difference of YTMs) are not particularly useful as measures of credit risk for bonds that are quite risky?

- **Example:** Consider two bonds from the same issuer, same seniority, both with a 5% coupon, one with 1 year to maturity, the other with 10 years to maturity:
  - YTM on 1yr bond = 162.5% pa
  - YTM on 10yr bond = 18.68% pa

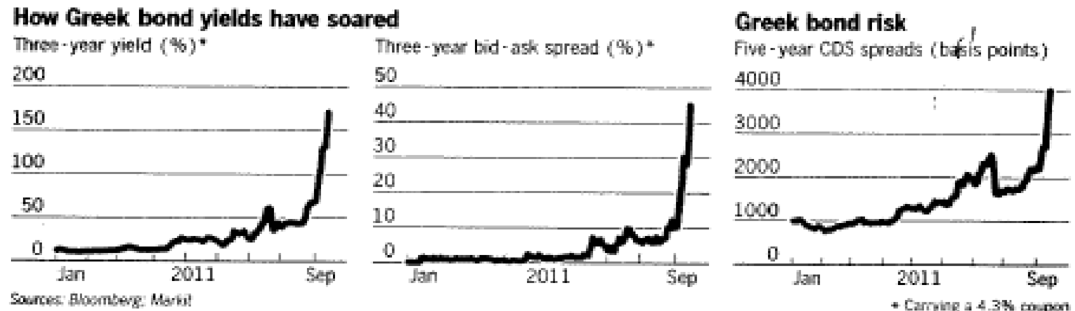
Assume the government yield curve is essentially flat.

Is the 1yr bond much riskier than the 10yr bond?

Credit spreads by maturity and rating as of mid-January 2011



What happened to Greece?



Greek yields off the scale as 3-year bond hits 172%

By David Oakley in London

Greek bond markets have gone off the scale. As investor concerns over a potential debt default by Athens mount, the country's debt has entered territory previously uncharted by a European sovereign.

Yields have risen by 150 percentage points on some bonds in the space of three months and volumes have slumped, almost to nothing on some days.

A trader at one big bank said: "Yield levels are unprecedented. Even other severely distressed

One three-year bond, which was trading at 20 per cent in June, is trading at a yield of 172 per cent, with a bid-offer spread – the difference between what a bank is offering to buy and sell the bonds at – of 47 percentage points.

As a comparison Belize, which investors warn is close to default, has 18-year bonds trading at 15 per cent, while Venezuela, shunned by mainstream investors, has nine-year bonds trading at 14 per cent.

Bankers say volumes have fallen below a daily average of

For example, three-year bonds that mature in March next year are quoted at a price of 53 per cent of par, a heavy discount that suggests Greece may default soon.

In the credit default swaps market, which offers investors a kind of protection against default, prices for Greece have jumped to levels that many traders would have considered fanciful only a few months ago.

Greek CDSs have risen above 4,000 basis points – more than double levels at the start of August, according to Markit.

Yields of Distressed Bonds

- As bonds of an issuer become increasingly distressed, their price will be determined mainly by what bondholders receive in default ie recovery rate
- Bonds representing the same claim (eg that rank "pari passu" for a corporate) should start trading at similar levels
- For the same (low) price today, the YTM – calculated from promised cashflows only – on a short maturity bond has to be higher than that on a long maturity bond
- Therefore, as YTM not very informative, distressed bonds "trade on price"

Bonds and Credit Risk – Overview

- Introduction to credit risk and default
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- **Credit risk and duration**

Credit Risk and Duration

- Recall again the simple YTM-based spread calculations:

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$$100.925 = \frac{2.5}{(1+y)} + \frac{2.5}{(1+y)^2} + \frac{102.5}{(1+y)^3} \Rightarrow y = 2.1785\%$$

3yr **Corporate Bond** with 4.5% annual coupon, trading at 99.325

$$99.325 = \frac{4.5}{(1+y)} + \frac{4.5}{(1+y)^2} + \frac{104.5}{(1+y)^3} \Rightarrow y = 4.7467\%, cs = 2.5682\%$$

- Modified duration (calculated the usual way wrt bond's own yield) = 2.746
- Percentage change in bond price for *x* bps (small) change in yield is -2.746 *x* %

Credit Risk and Duration

- Modified duration of 2.746 is the sensitivity with respect to the bond's yield, y
- But the corporate bond's yield is made up of two components:

$$y = g + s$$

here: 4.75% = 2.18% + 2.57%

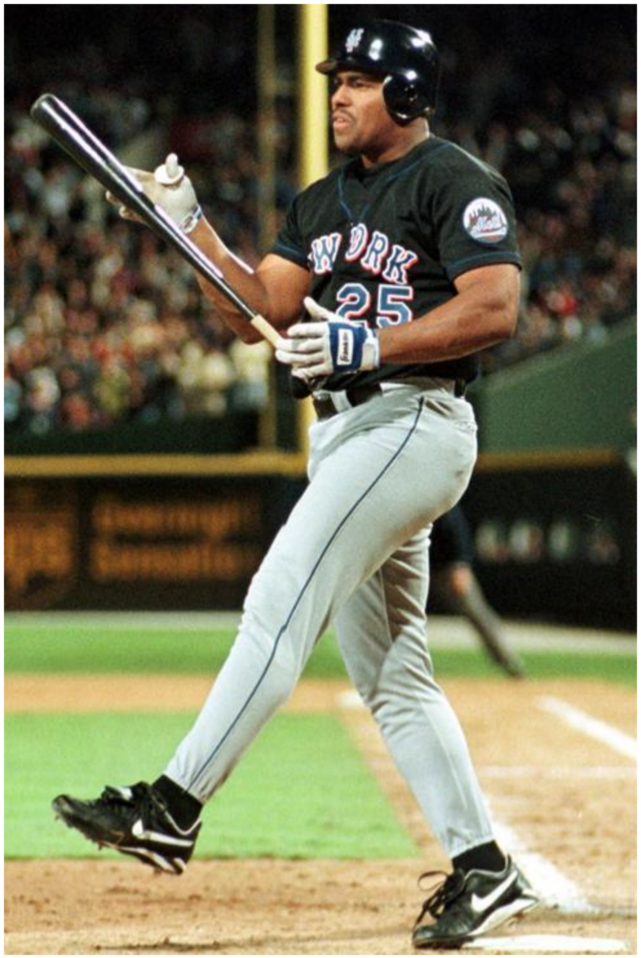
  - the yield on the government bond of the same maturity, g
  - the credit spread on the corporate bond, s

- Therefore **the standard modified duration of 2.746** is both the:
  - **Interest rate duration**: the sensitivity of the corporate bond to changes in government bond yields (all else equal)
  - **Spread duration:** the sensitivity of the corporate bond to changes in credit spreads (all else equal)

Bonds with Credit Risk – Summary

- **Measures** of credit risk: **ratings, spreads, default and recovery rates**
- **Understanding yields and credit spreads of risky bonds**:
  - Yields are calculated from **promised** cash flows
  - **Yields overstate** expected returns
  - If corporate bonds are **trading at distressed levels**, credit spreads are a **poor measure of (relative) credit risk**
- **Two (modified) durations – interest rate and spread duration**:
  - Sensitivity to **riskfree yield** and to **credit spread**
  - **Same expression/value and equal to normal (modified) duration wrt bond's yield**

Bobby Bonilla contract



Bonilla agreed to the Mets deferring the \$5.9m owed him for the summer 2000 in exchange for 25 annual payments of \$1,193,248.20 beginning July, 2011.

They used an 8% discount rate. Prime rate was 8.5%, long-term government bonds yielded ~6.5%.

"This is like a gift," Bonilla said. "I'm ecstatic now. Hey, I'm doing the macarena.“ Chicago Tribune, 5/21/2000.

Contracts have same PV @ 8%.

$$\$5.9m = \frac{1{,}193{,}248.20}{1.08^{10}}\left(\frac{1}{.08} - \frac{1}{.08(1.08^{25})}\right)$$

Risk of the new contract

- The new contract has a lot more interest rate risk.
  - Imagine interest rates go down by 3% (to 5%).

$$\$10.32\ m = \frac{1{,}193{,}248.20}{1.05^{10}}\left(\frac{1}{.05} - \frac{1}{.05(1.05^{25})}\right)$$

  - The value of the contract almost doubles.

- Imagine you are the CFO of the Mets.
  - Are you worried about this risk?
  - How can you hedge this risk?

Cash-flow matching

- We can hedge this exactly by purchasing zeros with face value of \$1.19m maturing each year from 2011 to 2025. The risk-free rate was ~6.5%, so the cost is (note Treasuries so semi-annual compounding):

$$\sum_{t=1}^{25} \frac{1{,}193{,}248.20}{\left(1+\frac{0.065}{2}\right)^{20+2t}} = \$7.6m$$

- This portfolio replicates the payouts to Bonilla exactly, and requires no rebalancing.
- Question: Where do we get a 35-year risk-free zero coupon bond?

Immunization

- The exact hedge illustrates the intuition, but has downsides.
  - Must purchase 25 bonds.
  - You cannot buy risk-free bonds with 31, 32, … year maturities.
- Alternatives
  - "Bullet" hedge: Buy 1 bond with the same duration as the liability
  - Portfolio hedge: Buy any two bonds, say 5y and 10y zeros.
- Steps.
  1. Find duration of the liability to hedge
  2. Find duration of hedging instruments
  3. Solve system of equations for weights in hedging instruments.

Duration of the liability

- We need D_{mod} or DV01 to hedge. The first step to getting these is calculating the D_{mac} for the liability:
  - Recall, we weight the time (t) to each cash flow by the present value of the cash flow as a % of the total value.

$$\sum_{t=1}^{25}(10+t)\times w = D_{mac}$$

$$\sum_{t=1}^{25}(10+t)\left(\frac{1{,}193{,}248.20}{1.0325^{2\times(10+t)}}/7{,}602{,}942.22\right) = 19.81$$

  - This implies

$$D_{mod} = \frac{19.81}{1.0325} = 19.18$$

$$DV01 = \frac{19.81}{1.0325}\times\frac{\$7.6m}{10{,}000} = \$14{,}585.60$$

Are we approximating $\Delta P \ / \ \Delta y$?

- The liability is $7,602,942.22 at 6.5%. What about at +/- 0.1%?

6.4% semi = 1.032^2-1= 6.502% compound rate:

$$\frac{1{,}193{,}248.2}{1.06502^{10}} \times \left(\frac{1}{0.06502} - \frac{1}{0.06502(1.06502)^{25}}\right) = \$7{,}750{,}409.60$$

6.6% semi = 1.033^2-1= 6.709% compound rate:

$$\frac{1{,}193{,}248.2}{1.06709^{10}} \times \left(\frac{1}{0.06709} - \frac{1}{0.06709(1.06709)^{25}}\right) = \$7{,}458{,}671.55$$

Compare to approximations

$$\frac{\%\Delta P}{\Delta y}\Delta y \approx D_{mod} \times 0.1\% = 19.18 \times 0.1\% = 0.01918$$

$$\frac{\Delta P}{\Delta y}\Delta y \approx DV01 \times \Delta y \text{ in bps} = \$14{,}585.60 \times 10 \text{ bps}{=}\$145{,}856$$

Using D_{mod} we should be +/- 1.918%

$$\$7{,}602{,}942.22 \times (1 \pm 0.01918) = \\ \$7{,}457{,}086.25/\$7{,}748{,}798.19$$

Using DV01 we should be +/- $145,855.97

$$\$7{,}602{,}942.22 \pm \$145{,}855.97 = \\ \$7{,}457{,}086.25/\$7{,}748{,}798.19$$

Due to convexity, we overestimate the price decline for +0.1% and underestimate the price increase for -0.1%

Duration of the hedging instruments

- Simple: Buy a 20-year zero, you're done . . . if available.
- More complex: with a 5y and 10y zero, with $y_5 = 6.5\%$, $y_{10} = 6.58\%$.

$$D_{mod_5} = \frac{5}{1 + \frac{y_5}{2}} = 4.84$$

$$DV01_5 = D_{mod_5} \times \frac{PV}{10{,}000} = 4.84 \times \left(\frac{\$100}{1.0325^{10}}\right) / 10{,}000 = 0.035$$

$$D_{mod_{10}} = \frac{10}{1 + \frac{y_{10}}{2}} = 9.68$$

$$DV01_{10} = D_{mod_{10}} \times \frac{PV}{10{,}000} = 9.68 \times \left(\frac{\$100}{1.0329^{20}}\right) / 10{,}000 = 0.051$$

Finding the hedge portfolio

We could use either D_{mod} or DV01. Here we'll use D_{mod}.

We want gain/loss on hedge to offset the liability:

$$\left(\% \frac{\Delta P_{5y}}{\Delta y}\right) V_{5y} + \left(\% \frac{\Delta P_{10y}}{\Delta y}\right) V_{10y} = \left(\% \frac{\Delta \mathrm{P}_{\mathrm{Bonilla}}}{\Delta y}\right) \mathrm{V}_{\mathrm{Bonilla}}$$

We also want the value of the hedge portfolio to be the same as for the liability:

$$V_{5y} + V_{10y} = V_{Bonilla}$$

Plugging in D_{mod} and the liability value, we have two equations and two unknowns: V_{5y} and V_{10y}.

Solving the system

$$4.8426 \times V_{5y} + 9.6815 \times V_{10y} = 19.1841 \times \$7{,}602{,}942.22$$
$$V_{5y} + V_{10y} = \$7{,}602{,}942.22$$

The solution is

- Short \$14,930,607.94 of the 5y (i.e., negative position)

  That is equivalent to $\frac{-\$14{,}930{,}607.94(1.0325)^{10}}{\$100} = -205{,}578.69 \text{ bonds}$

- Long \$22,533,550.16 of the 10y.

  That is equivalent to $\frac{\$22{,}533{,}550.16(1.0329)^{20}}{\$100} = 430{,}521.82 \text{bonds}$

Check solution

- What is the value of the hedge portfolio if there is parallel downward shift in yields of 0.1%?

$$-\frac{\$20{,}557{,}869.02}{\left(1+\frac{6.5\%-0.1\%}{2}\right)^{10}}+\frac{\$43{,}052{,}182.10}{\left(1+\frac{6.58\%-0.1\%}{2}\right)^{20}}$$
$$=\$7{,}749{,}716.98$$

Pretty close to actual value of

$$\frac{1{,}193{,}248.2}{1.06502^{10}}\times\left(\frac{1}{0.06502}-\frac{1}{0.06502(1.06502)^{25}}\right)$$
$$=\$7{,}750{,}409.60$$