Investments

Present Value Theory

Nicholas Hirschey

Assistant Professor of Finance at Nova SBE

Outline

- Discounting Basics
  - Time value of money
  - Present and Future values
  - Incorporating Risk
- Perpetuities and Annuities
  - Perpetuity (constant & growing)
  - Annuity (constant & growing)
- Compounding
  - Stated rates (e.g., APR) vs. compound interest (e.g., EAR).
  - Inflation

DISCOUNTING BASICS

Introduction

- The objective of a financial decision maker:

  Maximize value!

- Need to know:
  - How to make investment decisions
    - Invest only in positive **Net Present Value (NPV)** projects
  - How prices of financial assets are determined
    - Concepts of **Time Value of Money** and **Risk and Return Tradeoff**

Questions answered by discounting

Anything involving valuation of cash flows arriving in different periods.

- How much do I need to save for retirement?
- Should I take the job offer with the higher salary, or the one with a lower salary but an equity grant?
- My sister-in-law wants to sell her business. What is a fair price?
- What is this stock worth?

Present and Future Values

If I invest $1 today and earn a 10% return, how much will I have after T years?

Compounding Periods	Present Value	Calculation	Future Value
1	$1	$1(1.1)	$1.1
2	$1	$1(1.1)(1.1)	$1.21
3	$1	$1(1.1)(1.1)(1.1)	$1.33
4	$1	$1(1.1)(1.1)(1.1)(1.1)	$1.46
T	$1	$1(1.1)^T	$1(1.1)^T

In General

$FV = PV(1+r)^T$ or $PV = \frac{FV}{(1+r)^T}$

*Call $1/(1+r)^T$ the **Present Value Factor** or **Discount Factor**.

Where does the *r* come from?

- The *r* is called the discount rate.
  - a.k.a. required return or opportunity cost of capital
  - It is the rate of return investors require to invest in cash flow streams with similar risk.
  - Investors require higher returns to invest in riskier cash flow streams.
  - Makes you indifferent between these two options
    a. getting the present value in cash today
    b. waiting to get the actual cash flows when they arrive in the future
- Key concept: Time value of money
  - Other things equal, having one dollar today is worth more than having the same dollar next year.
  - A dollar is generally worth more today because:
    - We are impatient (e.g., psychology, possibility of death).
    - We are generally wealthier in the future, so we do not need the money as badly next year.
  - As a result, we typically have positive real discount rates.

$$FV = PV(1+r)^T$$

- **NOTE:** There are 4 inputs: PV, FV, r, and T.

  If you have 3 you can solve for the 4th.

- **Example: How long will it take to double my money if I invest at 8%? (i.e. PV=1, FV=2, r=8%, T=?)**

$$\begin{aligned} 2 &= 1 \times 1.08^T \\ 2 &= 1.08^T \\ \ln(2) &= \ln(1.08^T) = T \times \ln(1.08) \\ T &= \frac{\ln(2)}{\ln(1.08)} \quad = 9 \end{aligned}$$

APPL yearly *r* 2003-2013 (excl. div.)

- Solve for *r* in $FV = PV(1+r)^T$
  1. $\frac{FV}{PV} = (1+r)^T$
  2. $\left(\frac{FV}{PV}\right)^{\frac{1}{T}} - 1 = r$

- Yearly return on AAPL

$$\left(\frac{487.22}{23.12}\right)^{\frac{1}{10}} - 1 = 35.6\%$$

The excl. dividend return on SPY was 4.4%

- Buy AAPL instead of 20gb iPod?

$$FV = \$399(1+0.356)^{10} = \$8{,}386.29$$

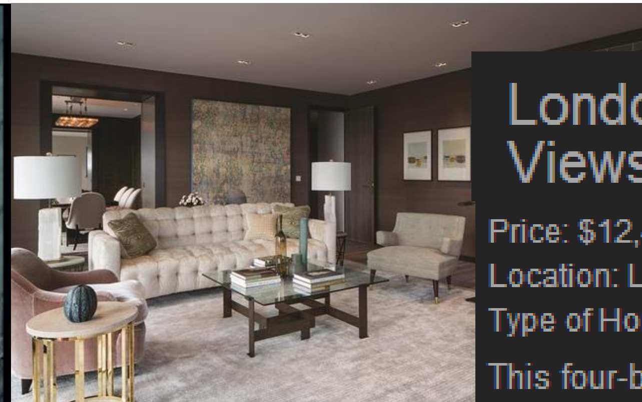
Image from wikipedia

Dealing with Fractional Periods

- If you invest \$10,000 in an account earning 8% per year, how much will you have after 33.5 years?

$$
\begin{aligned}
FV_{33.5} &= PV \times (1 + r)^T \\
&= \$10{,}000 \times (1.08)^{33.5} \\
&= \$131{,}733.37
\end{aligned}
$$

Comparing PV bids for a penthouse



London Penthouse With City Views

Price: $12,400,000

Location: London, United Kingdom

Type of Home: High Rise Home

This four-bedroom penthouse features two glass "sky rooms" overlooking the capital

Source: wsj.com

Comparing PV bids for a penthouse

- Bid 1: pay $11m today
- Bid 2: pay $11.5m 1 year from now

Assume Bid 2 has no risk of failure and the risk-free rate of return is 3%

- Step 1: Calculate present value of cash flows

PV(Bid 1) = $\$11m$

PV(Bid 2) = $\frac{C_1}{(1+r)^T} = \frac{\$11.5}{1+0.03} = \$11.17m$

- Step 2: Accept the bid with the highest PV

PV(Bid 2) = $11.17m > $11m = PV(Bid 1)

Risk and Present Values

- Higher risk investments require a higher rate of return.
- Higher required rates of return cause lower PV.

Assume Bid 2 (payment in 1 year) from prior example is risky and equally risky investments in capital markets offer returns of 8%. Now we discount Bid 2 with r = 8% instead of 3%.

PV(Bid 1) = $\$11m$

PV(Bid 2) = $\frac{C_1}{(1+r)^T} = \frac{\$11.5}{1+0.08} = \$10.65m$

Now the certain Bid 1 paying today is better

Discount Factor more generally

$$PV = DF_t \times C_t = \delta_t \times C_t = \frac{C_t}{(1+r)^t}$$

- Discount factors can be used to compute the present value of any cash flow.
- Replacing "1" with "t" allows the formula to be used for cash flows that exist at any point in time.

Cash flows in multiple periods?

- Present values can be added together to evaluate cash flows arriving in multiple periods.

$$PV = \frac{C_1}{(1+r)^1} + \frac{C_2}{(1+r)^2} + \cdots$$

or, equivalently

$$PV = \delta_1 C_1 + \delta_2 C_2 + \cdots$$

Valuing a grain bin "trade"

- 20k bu. grain bin for sale in Columbus, NE for $20k.
- You can sell the grain bin in Belize for $25k in 1 year.
- Transportation costs are $2,500 and it takes 6 months to ship the bin to Belize. Shipping costs are paid at the time of shipment.

Assume

1. Equally risky investments in capital markets offer returns of 5%, and
2. You can store the bin on your neighbor's property for $100 the first six months, payable in advance.

Period	Discount Factor	Cash Flow	Present Value ($\delta \times CF$)
0	1	-20,100	-20,100
0.5	$1/1.05^{0.5} = 0.976$	-2,500	-2,440
1	$1/1.05^{1} = 0.952$	25,000	23,800
		NPV =	$1,260

Source:http://www.flickr.com/photos/locosteve/4333124261/lightbox/

Summary of Discounting Basics

- We can value any cash flow stream using very simple present and future value formulas.

$$PV = \frac{FV_t}{(1+r)^t} = \delta_t FV_t$$

- You should be indifferent between
  a) Receiving a given cash flow stream, and
  b) That specific cash flow stream's present value in cash today, and
  c) Receiving cash at time *t* equal to that cash flow stream's future value at time *t*
- But with long cash flow streams, these calculations can become tedious … is there a solution for us lazy financiers?

PERPETUITIES

We have short-cuts for some common cash flow stream patterns

Sometimes computations become easier. This can be the case when cash flows are “perpetuities,” “growing perpetuities” or “annuities”.

Example: *Consider the case of the purchase of a security that generates cash flows equal to $3,000 forever. If the interest rate is 10%, what is the Present Value of the investment?*

time	0	1	2	3	...	forever
Cash Flows		+$3,000	+$3,000	+$3,000	...	+$3,000

The constant stream of cash flows equal to +$3,000 that start in period 1 and last for ever is a *perpetuity*.

Perpetuities

Definition: *"A Perpetuity is a financial Asset that promises to pay a fixed nominal amount C forever"*

time	0	1	2	3	...	forever
Cash Flows		+$3,000	+$3,000	+$3,000	...	+$3,000

Instead of calculating the PV with long calculations, we can use the short cut:

$$PV_0 = \frac{CF_1}{r} = \frac{\$3{,}000}{0.10}$$

$$= \$30{,}000$$

The derivation of this shortcut formula is in the appendix.

Bank of America Corporation Non (BAC-PL) - NYSE
1,081.29 Sep 3, 4:00PM EDT

6,000,000 Shares of 7.25% Non-Cumulative Perpetual Convertible Preferred Stock, Series L

Bank of America Corporation is offering 6,000,000 shares of 7.25% Non-Cumulative Perpetual Convertible Preferred Stock, Series L, $0.01 par value, with a liquidation preference of $1,000 per share (the "Preferred Stock").

- **Perpetuity Example:** Perpetual dividend of $72.50 each year. What is its price if investors require a 12% return?

$$PV_0 = \frac{\$72.50}{0.12} = \$604.17$$

- *If the preferred stock currently trades at $1,081.29, what is the implied required rate of return?*

$$\$1081.29 = \frac{\$72.5}{r} \Rightarrow r = \frac{\$72.5}{\$1081.29} = 0.067 = 6.7\%$$

Perpetuity example: delayed payment

- **Example:** What is the present value of a perpetuity that pays \$100 forever starting in period 6?

  Assume r = 1%.

Time	**0**	**1**	**2**	**…**	**6**	**…**	Forever
Cash-flow	0	0	0		C	…	C

$$PV_5 = \frac{C_6}{r} = \frac{\$100}{0.01} = \$10{,}000$$

$$PV_0 = \frac{1}{(1+0.01)^5} PV_5 = \frac{\$10{,}000}{(1.01)^5} = \$9{,}514.66$$

Growing Perpetuities

- **Example:** *Consider the case of the purchase of a security that will generate a cash flow of $3k this year, growing at 2% per year forever. If the interest rate is 10%, what is the Net Present Value of the investment?*

time	0	1	2	3	…	forever
Cash Flows		+\$3k	+\$3k (1.02)	+\$3k $(1.02)^2$		+\$3k $(1.02)^t$

- The stream of cash flows equal to +$3,000 starting in period 1 and growing at a constant rate forever is a *growing perpetuity*.

Growing Perpetuities

- A Growing Perpetuity is a financial Asset that gives the right to receive a cash flow growing at a rate equal to g forever.

$$PV = \frac{C}{r-g}$$

- Example:
  - Next Year's Cash Flow = \$100
  - Constant Expected Growth Rate = 10%
  - Cost of Capital = 15%

$$= \frac{\$100}{0.15 - 0.10} = 2000$$

Growing Perpetuities

time	0	1	2	3	...	forever
Cash Flows		+\$3k	+\$3k (1.02)	+\$3k $(1.02)^2$		+\$3k $(1.02)^t$

Instead of calculating the PV with long calculations, we can use the far simpler short cut:

$$PV = \frac{CF_1}{r-g} = \frac{\$3{,}000}{10\% - 2\%}$$
$$= \$37{,}500$$

ANNUITIES

Annuities

• An Annuity is a financial asset that gives the right to receive a constant cash-flow C for t periods starting one year from today.

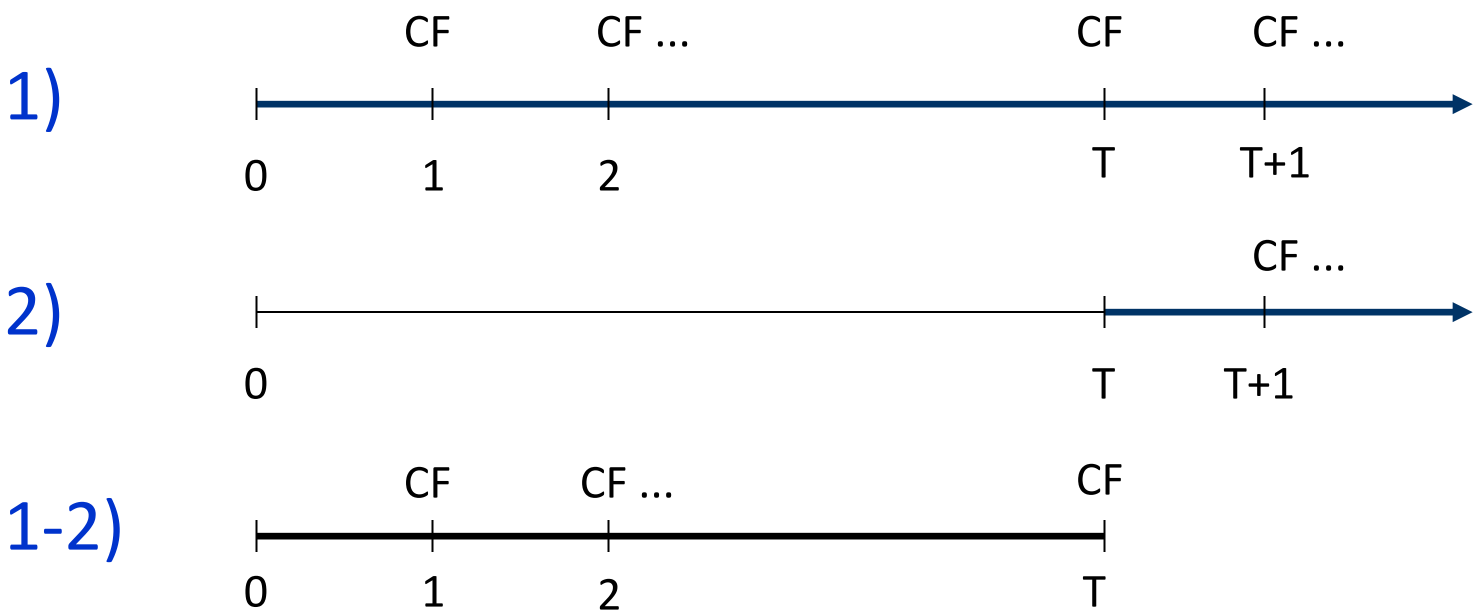
Yr 0	Yr 1	Yr 2	Yr 3	Yr 4	Yr 5	Yr 6	Yr 7
	C	C	C	C	C	C	0

$$PV_0 = C\left[\frac{1}{r} - \frac{1}{r(1+r)^t}\right]$$

EXAMPLE: *Consider a 20-year mortgage. The Annual Payment is equal to $100,000. If you were the Chief Financial Officer of Barclays and the interest rate were 20%, what is the maximum amount you would lend at the previous annual payment schedule?* You would lend at most the Present Value of the future Cash Flows from the mortgage payments. Using the shortcut the result can be easily obtained:

$$PV = 100,000\left[\frac{1}{0.2} - \frac{1}{0.2(1+0.2)^{20}}\right] = \$487,000$$

Where does the formula come from?



- An annuity is the ***difference*** between ***two perpetuities***.
  - The 1st perpetuity starts at t = 1
  - The 2nd starts at t = T+1

- So, PV of annuity must be the difference between the PV of the two perpetuities!

The 2nd perpetuity makes its first payment at t=T+1. So, CF/r is its value brought to t=T. We multiply it by $1/(1+r)^T$ to bring it back to t=0 (i.e. bring it back T years to the present.)

$$PV(1) = \frac{C}{r}$$

$$PV(2) = \frac{1}{(1+r)^T}\frac{C}{r}$$

$$PV(1) - PV(2) = \frac{C}{r} - \frac{1}{(1+r)^T}\frac{C}{r} = C\underbrace{\left(\frac{1}{r} - \frac{1}{r(1+r)^T}\right)}_{\boxed{\text{Discount Factor}}}$$

More Complicated Annuity Example

- **Example:** *You are offered a building for \$5M today (Year 0). After some extensive research on the housing market and on interest rates, your expectations are that you will be able to sell the building for \$7M 10 years from now (Year 10). If the Cost of Capital is 15%, what is the smallest fixed rent that does not make the deal unprofitable?*

time	0	1	2	3	4	5	6	7	8	9	10
sale											\$7Ml
rent	x	x	x	x	x	x	x	x	x	x	

$$PV_0 = \frac{x}{r}\left[1 - \frac{1}{(1+0.15)^9}\right]$$

Let *x* be the rent such that:

$$\$5mil = \frac{\$7mil}{1.15^{10}} + x + x\left[\frac{1}{0.15} - \frac{1}{0.15(1+0.15)^9}\right]$$

More Complicated Annuity Example

- Solving for x:

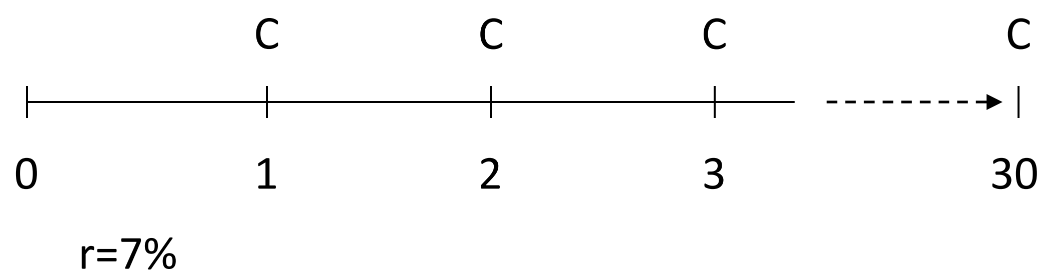
$$x = \left(\$5m - \frac{\$7m}{1.15^{10}}\right)/5.7716 = \$566{,}520$$

Note: In this example, rent was due at the beginning of the month. This is sometimes called an “Annuity Due”, but as you saw valuation took no special knowledge.

Future Value of Annuity Example

Future Value of Annuity Example

- How much do you need to put into your retirement account each year if you need \$2 million and plan to retire in 30 years? (assume a 7% rate of return)
- Equivalent to an annuity whose *future value* is \$2 million.
- **Step 1**: Organize information



- **Step 2:** Set up the problem

$$\$2\text{million} = \underbrace{\left[C\left(\frac{1}{.07} - \frac{1}{.07(1+.07)^{30}}\right)\right]}_{\boxed{\text{Present Value}}} \underbrace{(1+.07)^{30}}_{\boxed{\text{To FV}}}$$

- **Step 3:** Solve!

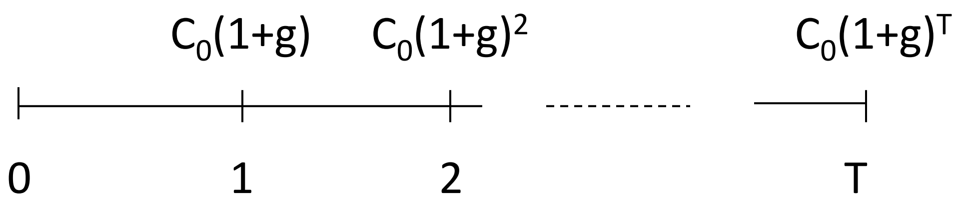
$$\$2\text{million} = C(94.46)$$

$$C = \frac{\$2\text{million}}{94.46} = \$21{,}172.98$$

Practice: What if we wait 5 years to start saving? (Answer: C = $31,621)

Uneven CF: Growing Annuities

- Formula can be derived from differencing two growing perpetuities.



$$\Rightarrow \mathrm{PV}_0 = \frac{\mathrm{C}_0(1+g)}{\mathrm{r}-\mathrm{g}} - \frac{\mathrm{C}_0(1+\mathrm{g})^{T+1}}{\mathrm{r}-\mathrm{g}} \times \frac{1}{(1+r)^T}$$

$$= \frac{C_0(1+g)}{r-g}\left(1-\frac{(1+g)^T}{(1+r)^T}\right) = \frac{C_1}{r-g}\left(1-\frac{(1+g)^T}{(1+r)^T}\right)$$

Formula Summary

For Single Cash Flows (lump sums)

$$PV = \frac{C_T}{(1+r)^T}$$

$$FV = PV(1+r)^T$$

Note: r = discount rate
T = # periods = # payments
g = growth rate

For Streams

$$PV_{Perp.} = \frac{C}{r-g}$$

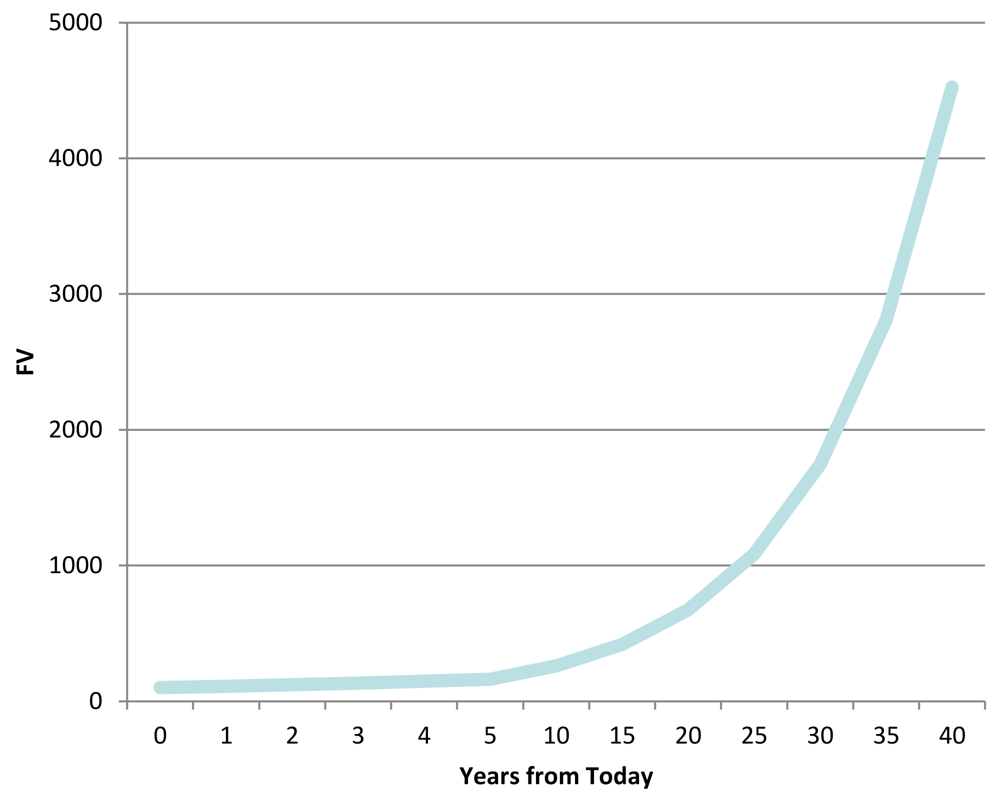
$$PV_{Ann.} = \frac{C}{r-g}\left(1 - \frac{(1+g)^T}{(1+r)^T}\right)$$

$$FV_{stream} = PV_{stream} \times (1+r)^T$$

COMPOUND INTEREST AND INFLATION

Effect of Time (T) on FV

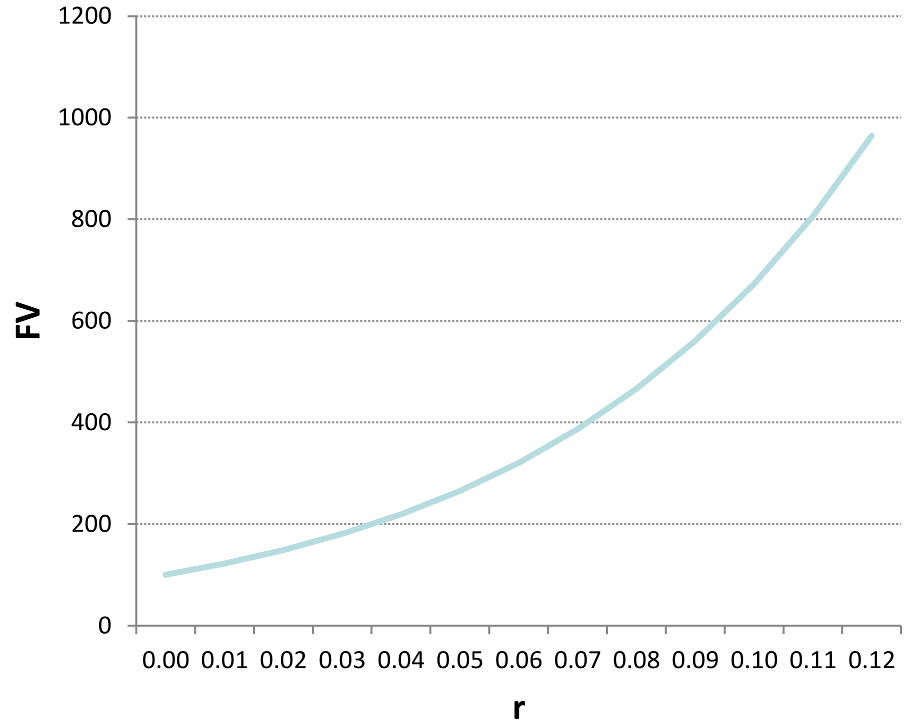
FV of $100 invested today		
Years from today	**Rate of return**	**FV**
T	r	$PV*(1+r)^T$
0	0.10	100
1	0.10	110
2	0.10	121
3	0.10	133
4	0.10	146
5	0.10	161
10	0.10	259
15	0.10	418
20	0.10	673
25	0.10	1083
30	0.10	1745
35	0.10	2810
40	0.10	4526



Exponential growth -> "the miracle of compound interest"

Effect of Return (r) on FV

FV of $100 invested today		
Years from today	**Rate of return**	**FV**
T	r	$PV*(1+r)^T$
20	0.00	100
20	0.01	122
20	0.02	149
20	0.03	181
20	0.04	219
20	0.05	265
20	0.06	321
20	0.07	387
20	0.08	466
20	0.09	560
20	0.10	673
20	0.11	806
20	0.12	965



Small differences in r -> big differences in FV

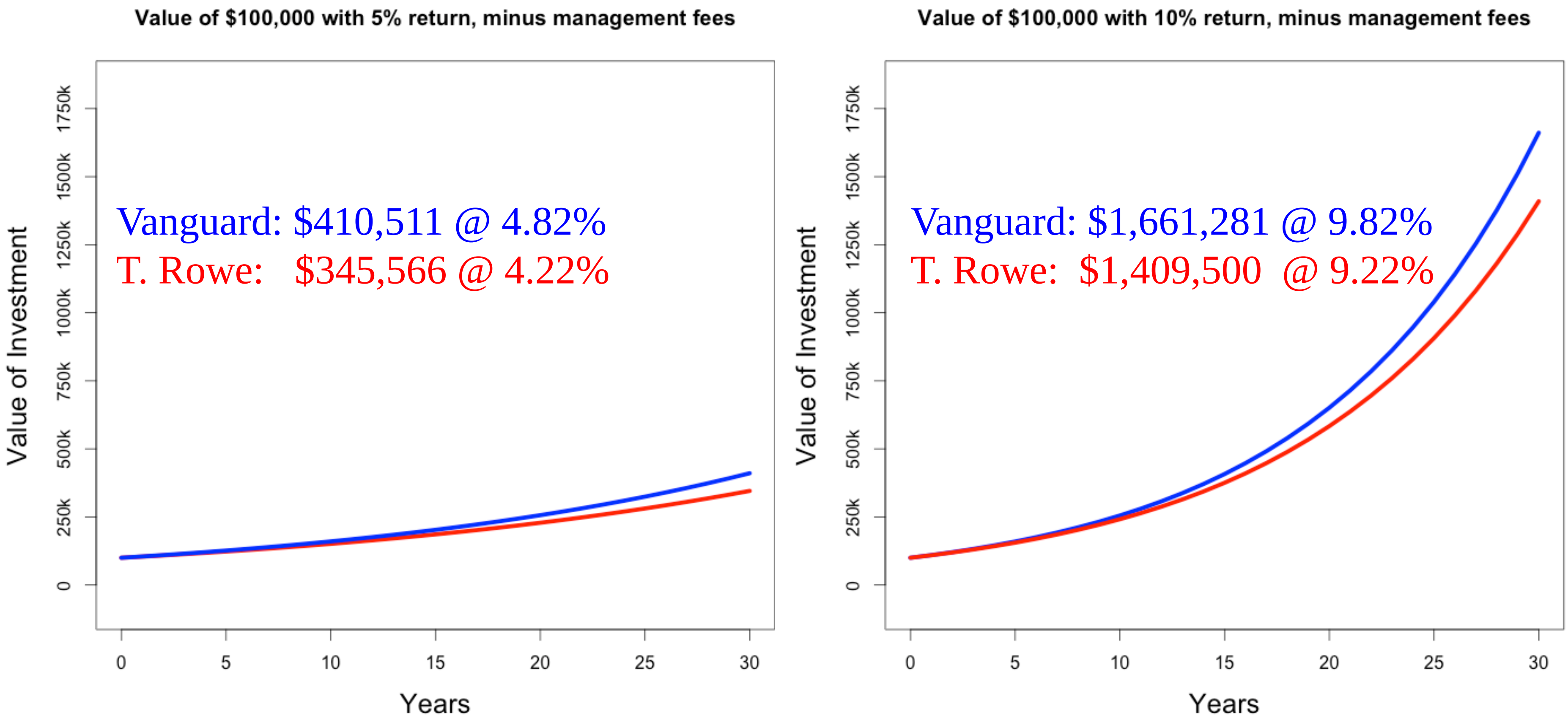
Consequences of small return differences

	Vanguard 2055 Retirement[2]	T. Rowe Price 2055 Retirement
Domestic Stock	63.6%	59.6%
International Stock	25.7%	29.5%
Domestic Bond	8.0%	6.0%
International Bond	2.0%	2.4%
Other	0.0%	2.6%
Expense Ratio	**0.18%**	**0.78%**

Near-Identical Composition, Different Expense Ratios

1. http://www3.troweprice.com/fb2/fbkweb/composition.do?ticker=TRRNX
2. https://personal.vanguard.com/us/funds/snapshot?FundId=1487&FundIntExt=INT

Consequences of small return differences



PV of Difference (@3%): $26,756

PV of Difference (@3%): $103,731

Compounding

- When we have discounted thus far, we have used compound interest. Common rates that are compound interest:
  - Discount rate; rate of return; required return; opportunity cost of capital
- Most *quoted* rates for financial instruments are ***Stated Annual Interest Rates (APR).***
  - Credit Cards (12% compounded monthly)
  - CD rates (5% with daily compounding)
  - Coupon rates on bonds (10% with semiannual compounding)
- Stated rates such as those above ignore compounding!

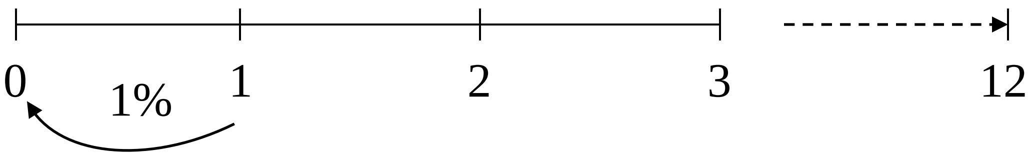
  $\Rightarrow$ cannot be used for PV or FV
- Must translate a stated rate (e.g., APR) into a compound interest rate. A compound interest rate at the annual horizon is often called an ***Effective Annual Rate (EAR).***

How do we know if we have an APR?

- As noted on the prior slide, if it's a financial product there's a good chance it is an APR.
- Look for key tells:
  - "This CD has an 8% **APR** with **monthly compounding**."
  - "This credit card charges 12% interest with **quarterly compounding**."
  - "This **semi-annual bond** has a yield of 8%."
- In almost all cases, if its an APR it will tell you. The only real exception is bonds, but it's easy enough to remember that bonds are quoted in APRs.

Translating APR into EAR

- A stated rate of 12% with monthly compounding means you pay 1% a month for twelve months!



- From the stated rate (APR), the only thing useful is the periodic rate:

$$r_{Periodic} = \frac{r_{APR}}{m} = \frac{.12}{12} = .01$$

# of compounding periods/year

Does 12% really mean 12%?

- If we deposit $1 today in a bank account that has a stated rate of 12% with monthly compounding, what will we have at the end of the year?

- FV = $1(1.01)^{12}$ = \$1.1268 => 12.68% (effective rate)

- **<u>So what's the rule?</u>**

$$\boxed{r_{EAR} = \left(1 + \frac{r_{APR}}{m}\right)^{m} - 1}$$

10% APR

Compounding	m	Formula	EAR
Annual	1	$\left(1+\frac{0.10}{1}\right)^{1}-1$	10.00%
Semiannual	2	$\left(1+\frac{0.10}{2}\right)^{2}-1$	10.25%
Monthly	12	$\left(1+\frac{0.10}{12}\right)^{12}-1$	10.47%
Daily	365	$\left(1+\frac{0.10}{365}\right)^{365}-1$	10.52%
Continuous	∞	$e^{.10}-1$	10.52%

e = exponential constant = 2.718...

General Approach

1. Cash flows define the time line
   – E.g., if cash flows come every 3-months then timeline is in terms of 3-month ticks
2. Convert to a discount rate that fits the tick interval of the timeline
   – If you are given a compound discount rate, just make sure it fits the tick interval. E.g., to convert annual compound rate to two-year rate use $1 + r_2 = (1 + r_1)^2$.
   – If you are given a stated rate (e.g., APR), first convert to a compound rate (e.g., EAR) and then make sure it matches the tick interval.
3. Now solve

Hard Practice Question (try on your own)

- The U.K Treasury announced it will offer a new bond to the market place and you have been asked to estimate its worth. The bond will make a total of 50 payments of £50, but these payments occur every three-years (i.e., this bond makes it's last payment in 150 years), and there is no face value. The way the bond works is that the payments don't actually go to the owner of the bond. Rather these payments go into a secret government controlled account that earns 7% per year. At maturity, the owner of the bond simply turns in the bond to get the balance of the secret account.

- ***What price should you recommend as a fair price given you think the market will require a 10% annual rate of return on this bond.***

Solution

- Answer: P_0 = £3.51

What is an Interest Rate?

1. Foregone consumption
   - All else equal, I would prefer to have it now rather than wait for a year
2. Inflation
   - Things will cost more in the future
3. Risk
   - How bad is the downside?

Inflation

- Def: Rate at which price levels are increasing.
- We should distinguish between real and nominal cash flows
- Rule: Always discount nominal cash flows with the nominal rate and real cash flow with the real rate.
  - Nominal Interest Rate: Rate at which money invested grows.
  - Real Interest Rate: Rate at which the purchasing power of an investment increases.

Inflation (cont'd)

$$1 + \text{real interest rate} = \frac{1 + \text{nominal interest rate}}{1 + \text{inflation rate}}$$

$$\text{Real interest rate} \approx \text{nominal interest rate - inflation rate}$$

- **Example:** If the interest rate on one year govt. bonds is 5.9% and the inflation rate is 3.3%, what is the real interest rate?

$$\text{real interest rate} = \frac{1.059}{1.033} - 1 = 0.025 = 2.5\%$$

Approximation: 5.9% - 3.3% = 2.6%

Summary

- Net Present Value analysis key to maximizing value
  - Financial securities
  - Real assets
- Calculate PV by matching cash flows to common patterns.
  - Lump sums
  - Annuities
  - Perpetuities

Key Concepts

- Present Value and Future Value
- Discount Factor
- Perpetuity, Annuity
- Annuity Due
- Growing Perpetuity
- Compounding
- Stated Annual Rate (APR)
- Effective Annual Rate (EAR)
- Inflation
- Real and Nominal Interest Rates
- Concept of Arbitrage

Appendix: Perpetuity Formula

- Consider the following geometric series:

$$(1) \quad x = \frac{1}{1+r} + \frac{1}{(1+r)^2} + \frac{1}{(1+r)^3} + \ldots + \frac{1}{(1+r)^{n-1}} + \frac{1}{(1+r)^n}$$

- Now, multiply both sides by the factor (1+r):

$$(1') \quad x\,(1+r) = 1 + \frac{1}{(1+r)^1} + \frac{1}{(1+r)^2} + \ldots + \frac{1}{(1+r)^{n-2}} + \frac{1}{(1+r)^{n-1}}$$

- Differencing the two series (1'- 1) yields:

$$(2) \quad x\,(1+r) - x = 1 - \frac{1}{(1+r)^n}$$

Perpetuity Formula Derivation (cont'd)

- Solving for x:

$$(2') \quad \mathrm{x\,r} = 1 - \frac{1}{(1+\mathrm{r})^{\mathrm{n}}} \quad \Rightarrow \quad \mathrm{x} = \frac{1}{\mathrm{r}}\left[1 - \frac{1}{(1+\mathrm{r})^{\mathrm{n}}}\right]$$

- In the limit (n→∞), for r > 0 it follows that ...

$$\lim_{n\to\infty}\left[1 - \frac{1}{(1+\mathrm{r})^{\mathrm{n}}}\right] = 1$$

- ... which leads to the well-known perpetuity formula:

$$(3) \quad \mathrm{x} = \frac{1}{\mathrm{r}} \qquad q.e.d.$$

What about Growing Perpetuities?

- Apply the same steps on the following series:

$$(1) \quad x = \frac{1(1+g)}{1+r} + \frac{1(1+g)^2}{(1+r)^2} + \frac{1(1+g)^3}{(1+r)^3} + \dots + \frac{1(1+g)^{n-1}}{(1+r)^{n-1}} + \frac{1(1+g)^n}{(1+r)^n}$$

- Multiply by (1+r)/(1+g)...

$$(1') \quad x\frac{(1+r)}{(1+g)} = 1 + \frac{1(1+g)}{1+r} + \frac{1(1+g)^2}{(1+r)^2} + \dots + \frac{1(1+g)^{n-2}}{(1+r)^{n-2}} + \frac{1(1+g)^{n-1}}{(1+r)^{n-1}}$$

- ...Differencing: $(1'\text{-}1) \quad x\frac{(1+r)}{(1+g)} - x = 1 - \frac{(1+g)^n}{(1+r)^n}$

- Which simplifies to:

$$x\left(\frac{1+r}{1+g}-1\right)=1-\frac{(1+g)^n}{(1+r)^n} \quad \text{or} \quad x\underbrace{\left(\frac{1+r}{1+g}-\frac{1+g}{1+g}\right)}_{\left(\frac{1+r-1-g}{1+g}\right)=\frac{r-g}{1+g}}=1-\underbrace{\left(\frac{1+g}{1+r}\right)^n}_{\boxed{\text{= 0 in the limit as long as } g<r}}$$

- Therefore, we have:

$$x\,\frac{r-g}{1+g}=1 \quad \text{or} \quad x=\frac{1+g}{r-g} \qquad q.e.d.$$