Investments

Basic Portfolio Statistics & Risk

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Outline

- Motivation for studying portfolio theory
- What is mean-variance analysis?
  - Basic definition
  - Calculating means and variances
- Variance as a measure of risk
  - Empirical support
  - Hedging intuition
  - Portfolio statistics (mean, variance, covariance)
- The mean-sd plot
  - Two-asset portfolio (gold and stocks)
  - Covariance as a measure of an individual investment's contribution towards a portfolio's risk

What do we learn from portfolio theory?

1. How to estimate risk premiums

- Thus far, all cash flows have been certain

$$\Rightarrow PV = \sum_{t=1}^{T} \frac{CF_t}{\left(1 + r_{f,t}\right)^t}$$

- What happens when cash flows are uncertain?
  - Include a ***risk premium*** in the discount rate

$$\Rightarrow PV = \sum_{t=1}^{T} \frac{E[\widetilde{CF}_t]}{\left(1 + r_{f,t} + r_{p,t}\right)^t}$$

2. How to identify our optimal portfolio

- How do I save for retirement?
- How do I hedge the risks I am exposed to (e.g., labor income)?

Mean-variance analysis

- What is it?
  - Portfolio theory developed by Harry Markowitz (1990 Nobel)
  - Explains how to achieve the highest **mean return (a.k.a. expected return)** for a given **variance of return.**
- Uses
  - Contribution of a stock to a portfolio's expected return and variance.
  - Contribution of an investment to a firm's expected earnings and volatility of earnings.
- Key insight
  - Portfolio variance determined by constituent stocks' covariances.
  - Diversification is "the only free lunch on Wall Street"

What is a return?

- Return (aka Holding Period Return):
  - It is the amount an investor earns in percentage terms over a specific period of time.
  - Calculated as price plus any cash income over the period, expressed as a percentage of the initial price.
  - Holding period return for AAPL for Aug 2016:

	29-Jul-2016	04-Aug-2016	31-Aug-2016
Price	103.65		106.10
Dividend		0.57	

$$r = \frac{(106.10 - 103.65) + 0.57}{103.65} = \underbrace{\frac{(106.10 - 103.65)}{103.65}}_{\text{capital gain or loss}} + \underbrace{\frac{0.57}{103.65}}_{\text{dividend yield}} = 2.91\%$$

Mean-Variance Analysis requires estimates of expected returns and variances

- We could estimate these values in two ways:
  1. From a “known distribution”
  2. From historical data.

- In practice, we do not know what the “true” distribution of returns is. So it is most common to estimate from historical data.

Means ($\bar{r}$) and variances (σ^2)

- Example calculation with a **known distribution**

Probability (p_i)	**1/3**	**1/2**	**1/6**
Return ($\tilde{r}_i$)	5	2	10

- The expectation of $\tilde{r}$ is denoted $E[\tilde{r}] = \bar{r}$

$$\bar{r} = E(\tilde{r}) = \sum_{i=1}^{N} p_i r_i = \frac{1}{3} \times 5 + \frac{1}{2} \times 2 + \frac{1}{6} \times 10 = 4.33$$

$$\sigma^2 = E[(\tilde{r} - \bar{r})^2] = \frac{1}{3} \times (5 - 4.33)^2 + \frac{1}{2} \times (2 - 4.33)^2 + \frac{1}{6} \times (10 - 4.33)^2 = 8.22$$

Means ($\bar{r}$) and variances (σ^2)

- Example calculation with **historical data** (S&P 500)

Year	2009	2010	2011
Return ($\tilde{r}_i$)	22.66	12.68	0.85

$$\bar{r} = E(\tilde{r}) = \frac{1}{N}(22.66 + 12.68 + 0.85) = 12.06$$

$$\sigma^2 = E[(\tilde{r} - \bar{r})^2] = \frac{(.2266 - .1206)^2 + (.1268 - .1206)^2 + (.0085 - .1206)^2}{3 - 1} = 0.0119$$

Note: We divide by N-1 (3-1) rather than N (3) when calculating variance because it is a better estimate of the expectation in small samples. A proof is beyond the scope of the class.

A bigger historical return example

- Mean (average monthly return):

$$\bar{r} = \frac{1}{N}\sum_{i=1}^{N} r_i = 0.08/12 = 0.0067 = 0.67\%$$

- Variance (average squared difference of observations from the mean):

$$\sigma^2 = \frac{1}{N-1}\sum_{i=1}^{N}(r_i - \bar{r})^2 = 0.001876$$

- Standard deviation (square root of the variance):

$$\sigma = \sqrt{\sigma^2} = \sqrt{0.001876} = 4.33\%$$

Month	**Return, r_t**	**$(r_t - \bar{r})^2$**
Jan	2.26%	0.0003
Feb	3.24%	0.0004
Mar	5.96%	0.0025
Apr	-3.90%	0.0019
May	4.30%	0.0011
Jun	-4.13%	0.0022
Jul	3.51%	0.0005
Aug	-5.09%	0.0035
Sep	3.31%	0.0002
Oct	3.61%	0.0004
Nov	1.63%	0.0000
Dec	-6.70%	0.0054
sum	**8.00%**	**0.0184**

Annualizing Monthly Values

- To annualize monthly returns:
  - multiply by 12: $\mathrm{r_A} = 12 \times 0.00667 = 8.00\%$

- To annualize monthly variance or standard deviation:
  - $\sigma^2(\text{annual}) = 12 \times \sigma^2(\text{monthly}) = 12 \times 0.001876 = 0.0225$
  - $\sigma(\text{annual}) = \sqrt{12} \times \sigma(\text{monthly}) = \sqrt{12} \times 0.0433 = 0.15 = 15\%$

Arithmetic vs. Geometric means

- When we estimate expected returns from historical data, we calculate what is called an Arithmetic mean:

$$E(\tilde{r}) = \frac{1}{N}\sum_{i=1}^{N} r_i$$

- From a statistical perspective, the arithmetic mean is the best estimate of the expected return. But, it is not the return an investor would earn.
  - Say stocks go up 100% this year and down 50% next.

$$\bar{r}_{arithmetic} = \frac{1}{2}(1 - 0.5) = 25\%$$

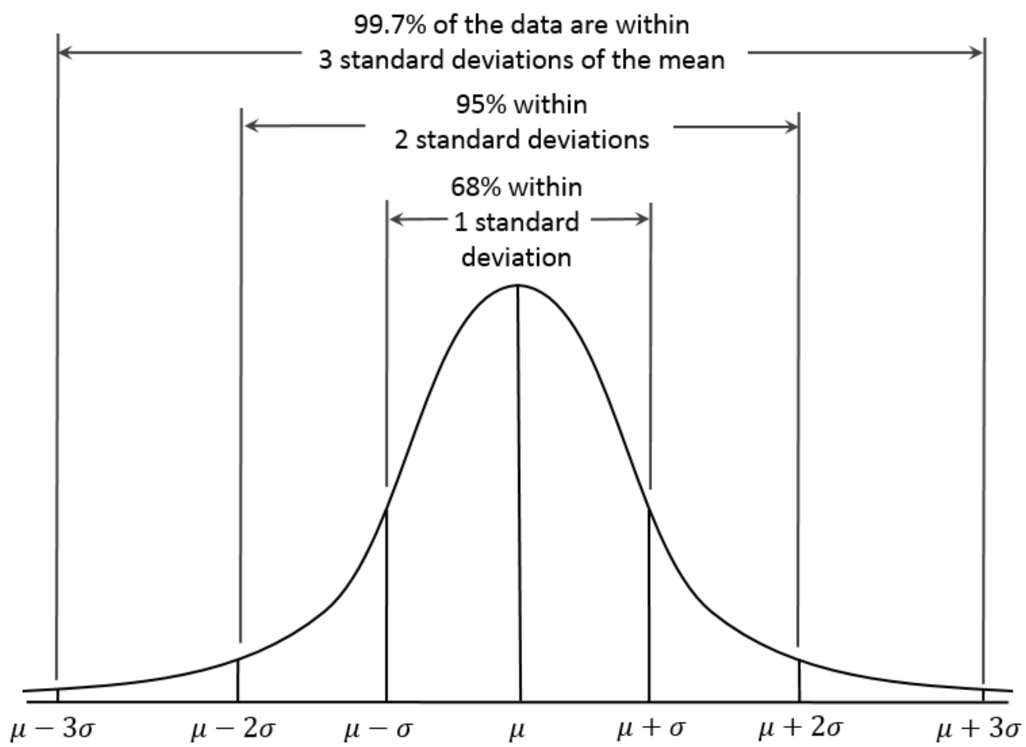
  - The annual return the investor actually earns is the Geometric mean return (or average buy-and-hold return)

$$\bar{r}_{geometric} = \left(\frac{P_T}{P_0}\right)^{1/T} - 1 = \big((1 + 1)(1 - 0.5)\big)^{1/2} - 1 = 0\%$$

What is risk?

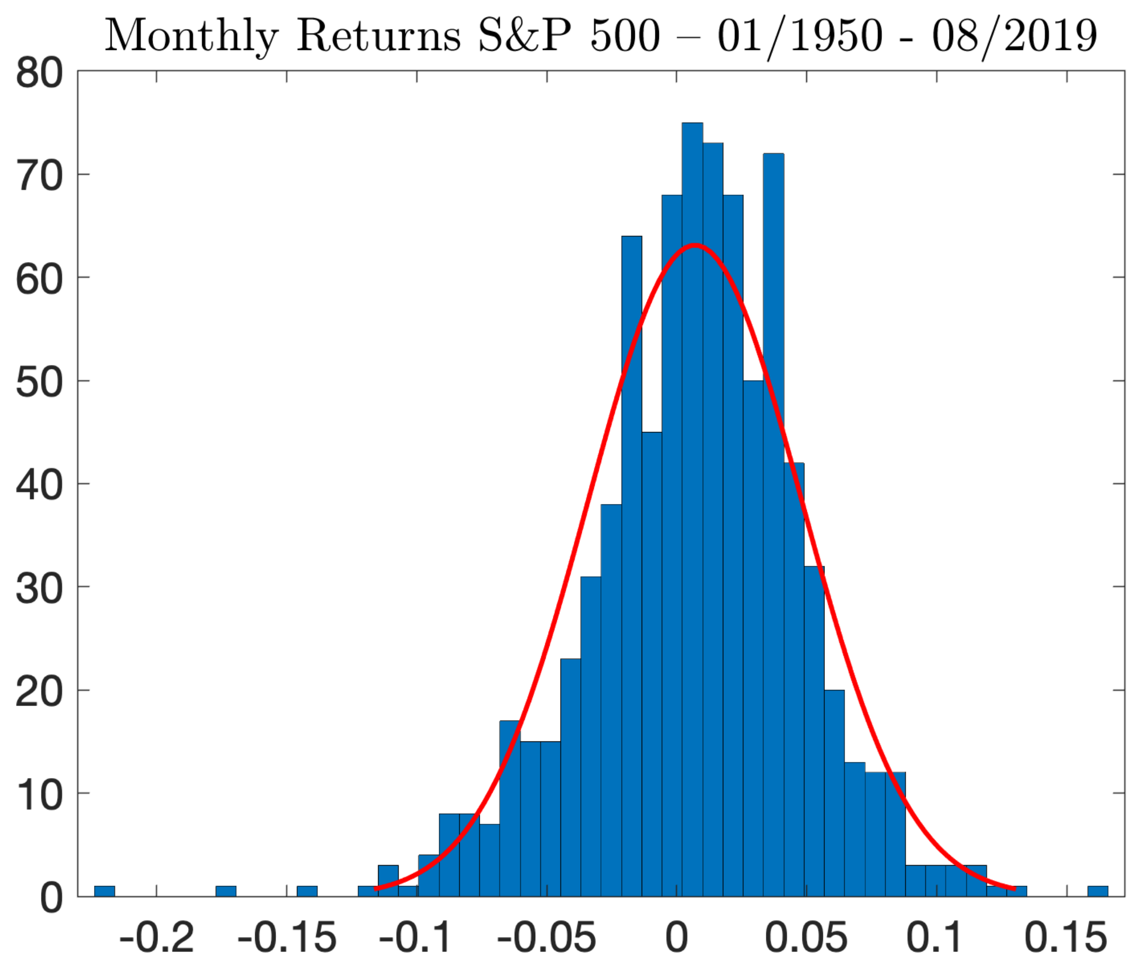
- We will focus on variance as a measure of risk
  - Basis for mean-variance analysis
    - i.e., the foundation for portfolio theory
  - Most common measure used in practice
- We will often work with the square-root of variance, called a standard deviation (σ).
  - In our S&P example, $\sigma = \sqrt{0.0119} = 10.9\%$
  - Also known as monthly *volatility* and is typically quoted as a percentage.
  - Means (roughly) that the value of the portfolio can readily move up or down 10.9% relative to the mean, so $12.06\% \pm 10.9\%$

Intuition for standard deviation (σ)



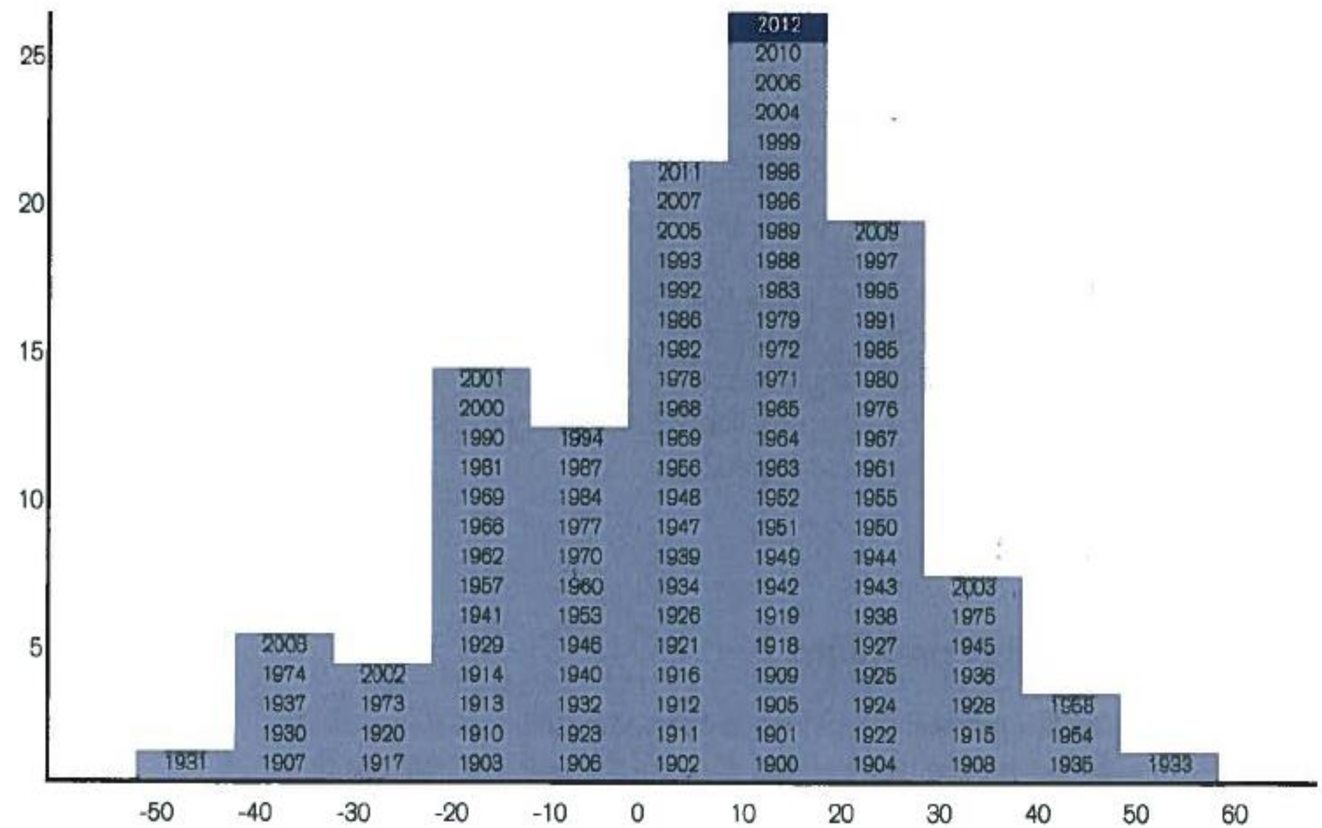
- If monthly σ is 10.9%%, roughly means returns can readily move up/down by 10.9% in any given month.

Monthly Stock Mkt Return ≈ Normal



Annual Stk Mkt Return ≈ Normal

Chart 9: Histogram of US equity risk premium relative to treasury bills, 1900–2012



Source: Elroy Dimson, Paul Marsh, and Mike Staunton, Triumph of the Optimists, Princeton University Press, 2002, and subsequent research.

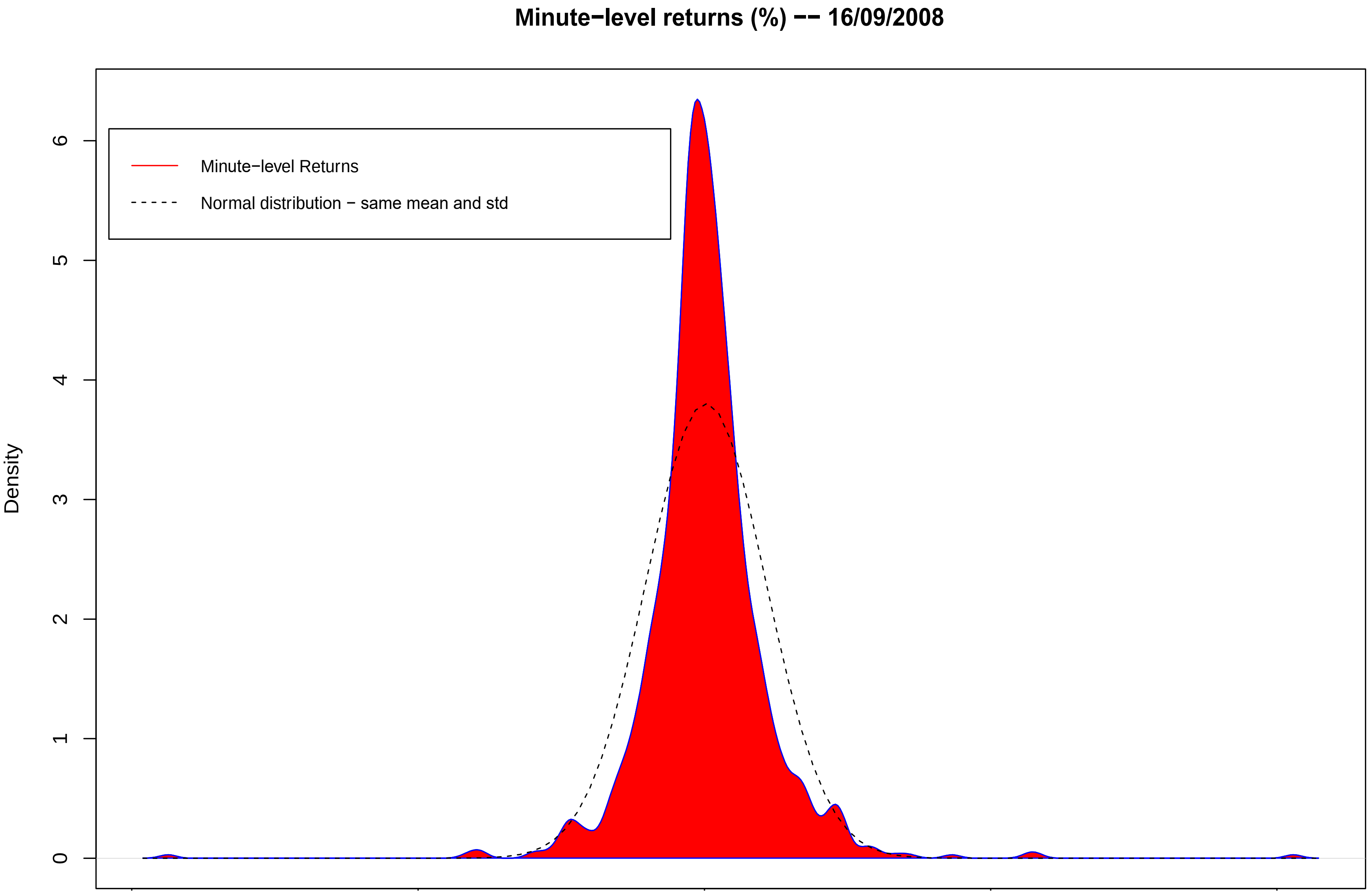
When are departures from normality important?

- At higher frequencies (e.g., daily or hourly), departures from normality are much more severe.
- Returns of highly levered assets, or assets with embedded leverage, are not normally distributed.

Assuming ≈ normality not such a problem with:

- Broadly diversified portfolios (e.g., entire stock market) held for longer investment horizons (e.g., annual).

Does it look familiar ?

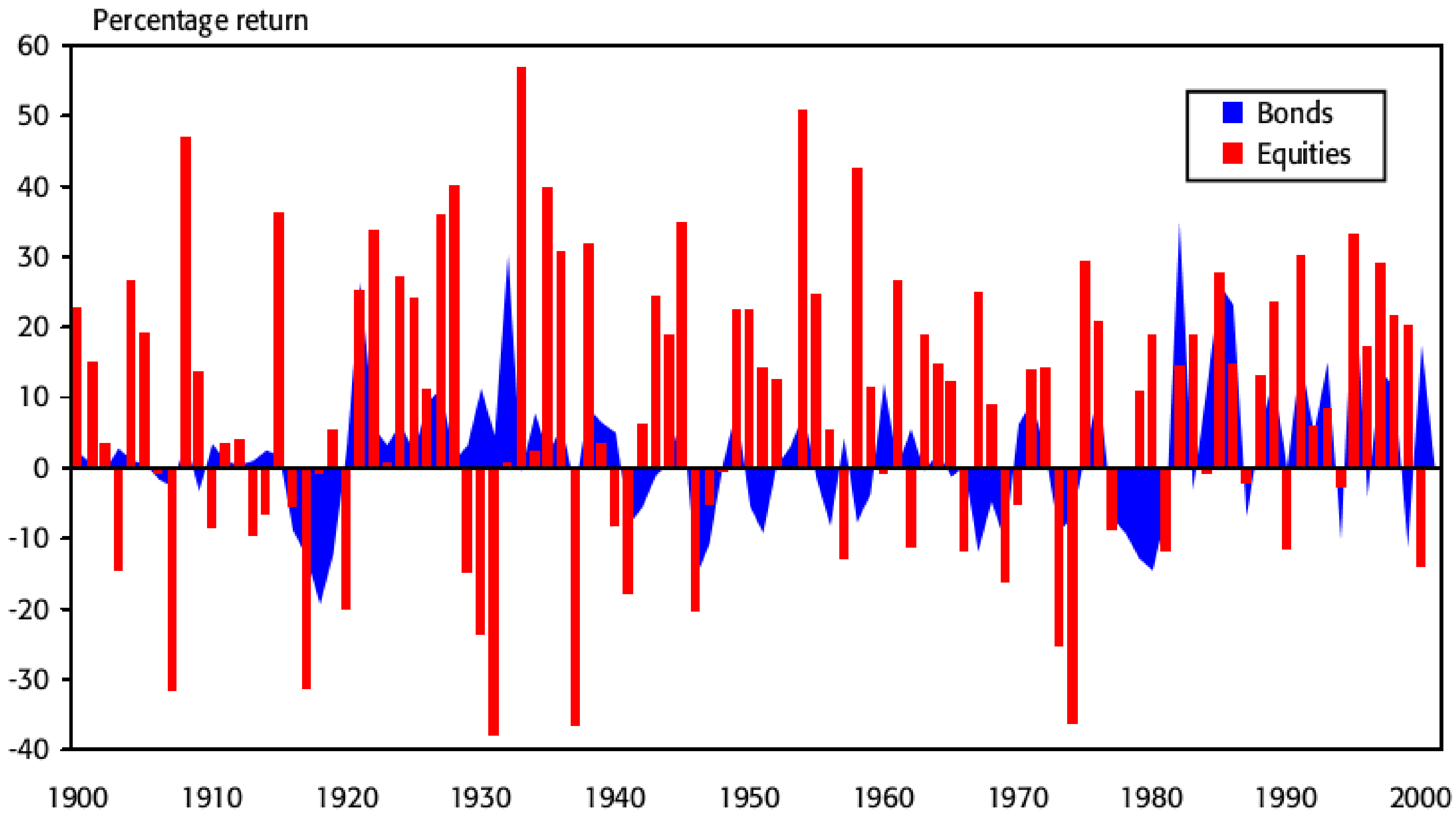


What is the evidence that variance is related to risk?

- When you look at well-diversified portfolios, such as different asset classes, more volatile portfolios tend to have higher returns.
  - There are some limitations to this, especially within asset classes, that we will discuss later.
- For example, bonds are less volatile than stocks, and they have lower average returns.

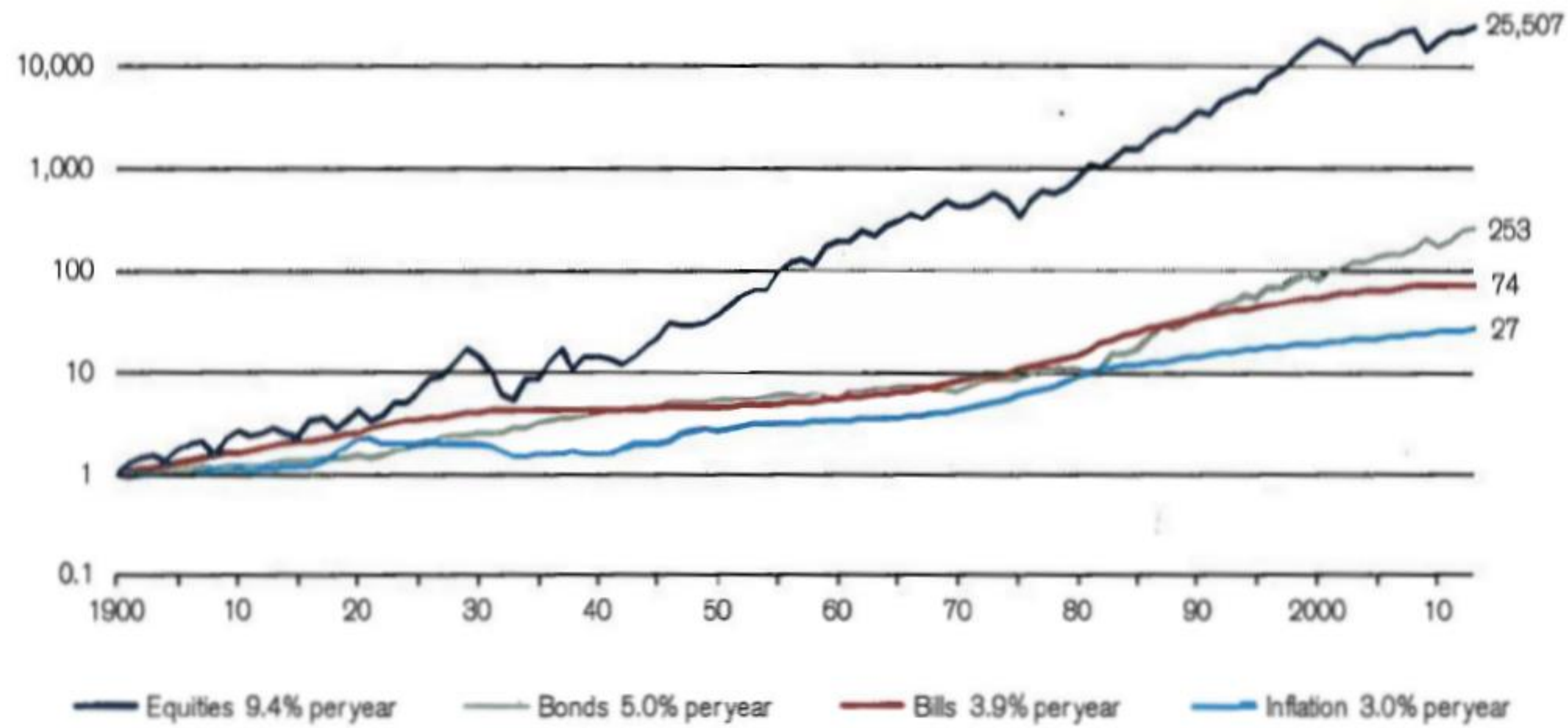
Risk and Return: Bonds are less volatile

Figure 4-8: Time series of annual real returns on US equities and bonds, 1900–2000



And Bonds have lower returns: Value of $1 invested at the start of 1900

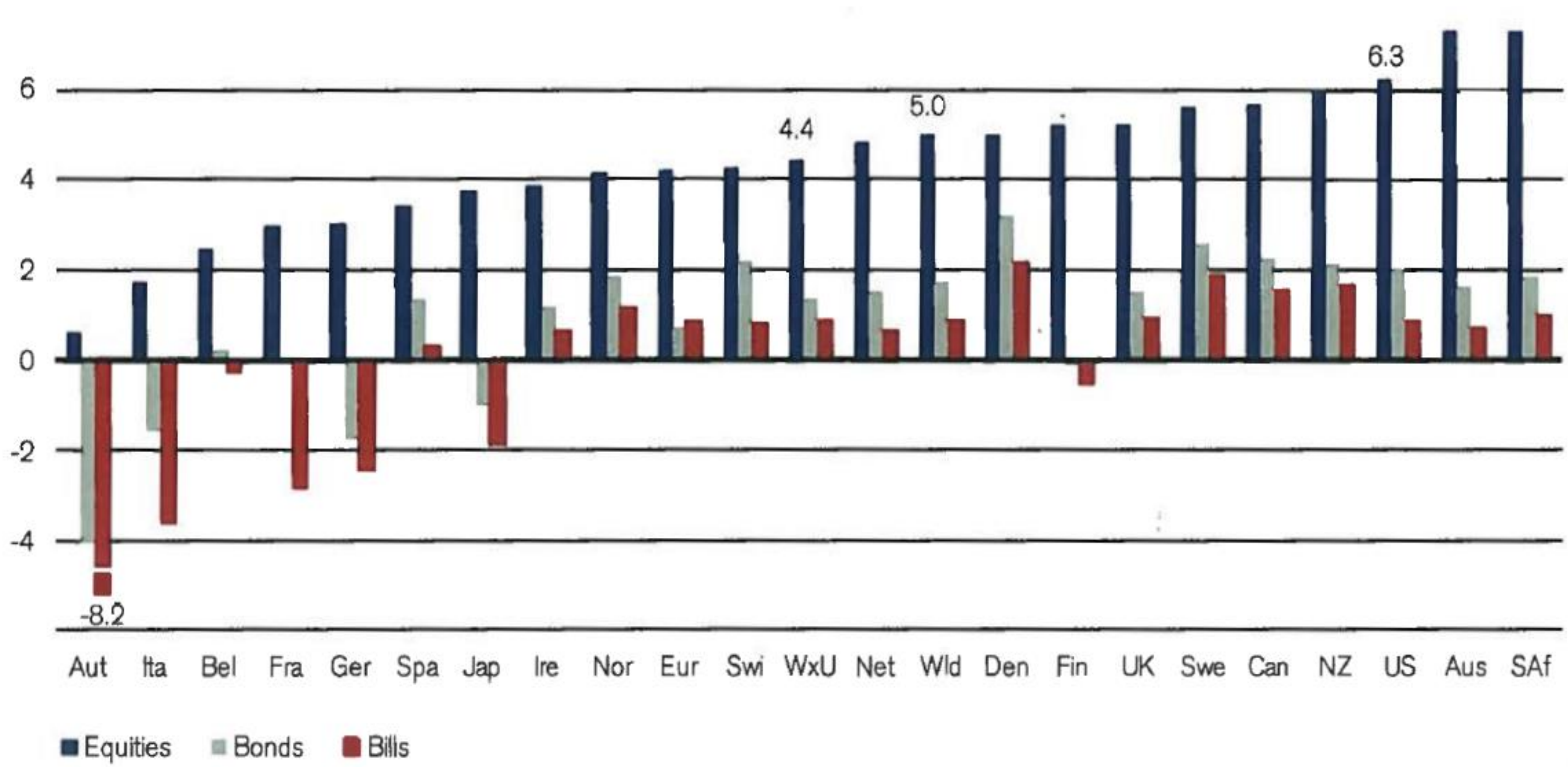
Chart 2: Cumulative returns on US asset classes in nominal terms, 1900–2012



Source: Elroy Dimson, Paul Marsh, and Mike Staunton, Triumph of the Optimists, Princeton University Press, 2002, and subsequent research

International Risk Premia

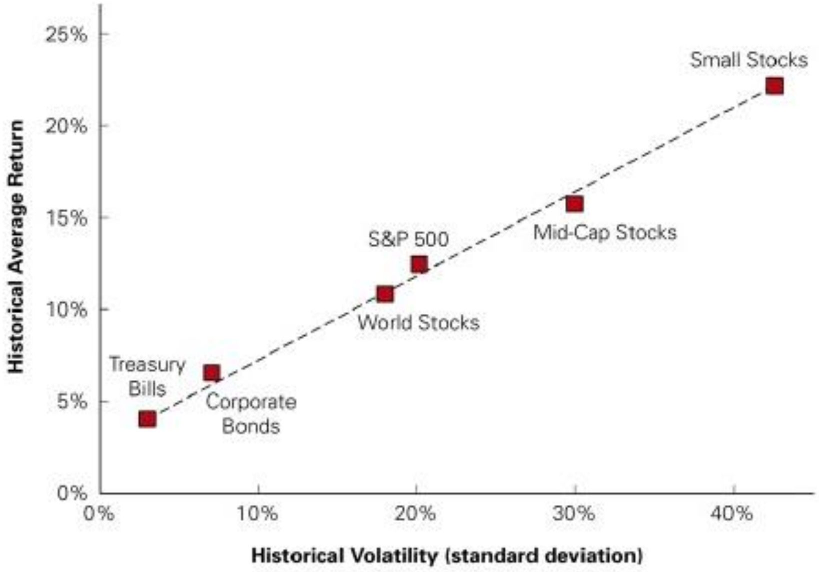
Chart 4: Real annualized returns (%) on equities versus bonds and bills internationally, 1900–2012



Source: Elroy Dimson, Paul Marsh, and Mike Staunton, Triumph of the Optimists, Princeton University Press, 2002, and subsequent research

Relationship Between Risk and Return

- For "broad" portfolios there seems to be a general increasing relationship between historical volatility and average return:



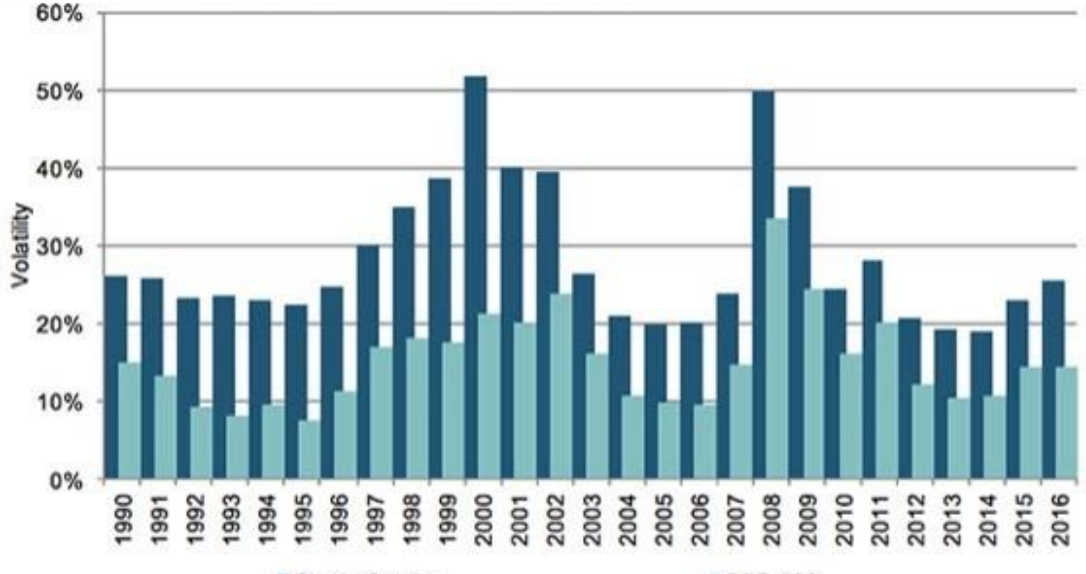
Source: CRSP, Morgan Stanley Capital International

Intuitively

- The required rate of return should depend on risk, because we are risk-averse.
- We generally view risks as dispersion around expected rates of return.
  - Suppose investments A and B have the same expected return.
  - If the returns of A are less variable than B, then investors generally prefer investment A.
- But this may not be the whole story.

Relationship Between Risk and Return

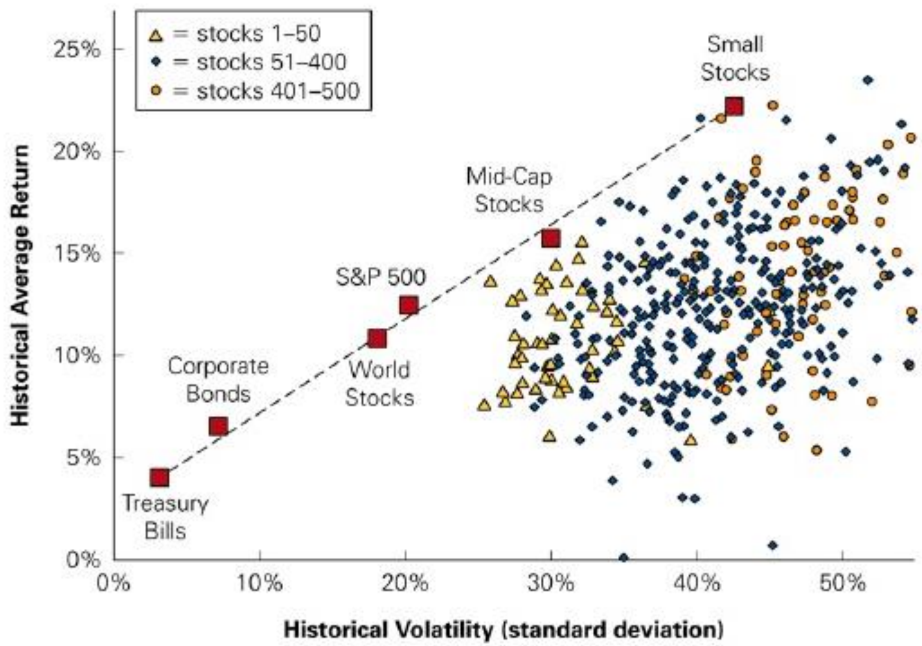
- Average volatility of S&P500 stocks versus index volatility:



Source: S&P Dow Jones Indices LLC. Data as of June 2016. Volatiity is measured by the standard deviation of total return for index and constituents. Past performance is no guarantee of future results. Chart is provided for illustrative purposes.

Relationship Between Risk and Return – Individual Stocks

- No precise relationship between individual stock volatility and returns.



Individual Shares versus the Stock Market: A Paradox?

Investment	Risk Premium	Variability
Stock Market	5%	20%
Typical Individual Share	5%	30-40%

- The risk premium for a typical individual share is not closely related to its volatility.
  - Standard deviation is NOT useful for measuring the risk of individual stocks.
- Is this a paradox? No. Individual stocks, in general, are not well diversified. The component of risk that can be diversified away is not priced by the market.
- What determines the risk premium is the contribution of the asset to the overall risk of the portfolio? Let us look at one example.

Portfolio Theory: A Two Asset Portfolio

- Example: Fire insurance
  - Asset 1: Your house, worth $100,000
  - Asset 2: Your insurance policy on the house
- Two states of the world:
  - State 1: Your house burns down and retains no value. The insurance policy pays out $100,000. The probability of state 1 is 10%.
  - State 2: Your house does not burn down and retains its full value. The insurance policy does not pay out.
- Question: What is the riskiness of each of these two assets (a) individually, (b) together if held in a portfolio?

Portfolio Theory: A Two Asset Portfolio

State	House Payoffs	Insurance Payoffs	Total Payoffs
1	0	100,000	100,000
2	100,000	0	100,000

- Compute expected payoff and risk (standard deviation)

$$\mathrm{E}(\text{Value House}) = 0.1 \times \$0 + 0.9 \times \$100{,}000 = \$90{,}000$$

$$StDev(\text{Value House}) =$$

$$= \sqrt{.1 \times (0 - 90{,}000)^2 + 0.9 \times (100{,}000 - 90{,}000)^2}$$

$$= \$30{,}000$$

Portfolio Theory: A Two Asset Portfolio

	House Only	Insurance Only	Portfolio of both
Expected Payoff	90,000	10,000	100,000
Risk	30,000	30,000	0

- Important: expected value is additive, but risk is not.
- Perfect negative correlation gives perfect insurance (hedging).
- This hedging intuition holds for financial markets too.
  - A risky asset has high returns when you are rich and low returns when you are poor.
  - A safe asset has high returns when you are poor and low returns when you are rich.

Another Example: Why Buy Gold?

Asset	Return	Variability
Gold	8.8%	20.8
S&P 500	12.8%	18.3

- Coefficient of correlation between S&P 500 and Gold $= -0.4$
- Gold has a lower return and a higher variability than the S&P 500
- Is Gold a bad investment?

Portfolio Theory: A Two Asset Portfolio.

- Investment strategy
  - a fraction of w_{Gold}% in gold, (100- w_{Gold})% in the S&P500.

w_{Gold}	Return	Variability	
100%	8.8%	20.8%	***Hold only gold***
80%	9.6%	15.5%	
60%	10.4%	11.7%	
45.4%	11.0%	10.7%	***Minimum variance portfolio***
40%	11.2%	10.8%	
20%	12.0%	13.5%	
0%	12.8%	18.3%	***Hold only S&P500***

Portfolio Theory: A Two Asset Portfolio

- Gold does not look good on its own.
  - Lower return than the stock market (8.8% vs. 12.8%)
  - Higher variability than the stock market (20.8% vs. 18.3%)

- BUT, adding it to a portfolio can reduce total risk
  - It acts as insurance against some forms of risk (inflation risk, … etc.)
  - Technically speaking, the low correlation coefficient (here negative) with the S&P 500 makes gold a reasonably good heding instrument.

Let us take a step back and derive the formulas to obtain the data in the previous table.

Calculating Portfolio Weights

- We are often interested in knowing how much of a portfolio is invested in a given position.
  - Stock j's weight in a portfolio is $w_j = \frac{\text{Dollars held in stock j}}{\text{Dollar value of the portfolio}}$
  - Weights must add up to 1
  - Long positions have positive weights
  - Short positions have negative weights

- Example positions
  - Long $1 million GOOG
  - Long $2 million AAPL
  - Short $500,000 T-bills.

$$w_{GOOG} = \frac{\$1{,}000{,}000}{\$1{,}000{,}000 + \$2{,}000{,}000 - \$500{,}000} = 0.40$$

$$w_{AAPL} = \frac{\$2{,}000{,}000}{\$1{,}000{,}000 + \$2{,}000{,}000 - \$500{,}000} = 0.80$$

$$w_{T-BILL} = \frac{-\$500{,}000}{\$1{,}000{,}000 + \$2{,}000{,}000 - \$500{,}000} = -0.20$$

Calculating Portfolio Returns

- Two methods to calculate a portfolio return ($\tilde{r}_p$)
  - Ratio method: $\tilde{r}_p = \frac{\text{Ending Portfolio Value+Distributions}}{\text{Beginning Portfolio Value}} - 1$
  - Portfolio-weighted average method: $\tilde{r}_p = \sum_{j=1}^{N} w_j \tilde{r}_j$
- Example
  - \$1,000,000 in IBM, return is 10%
  - \$3,000,000 in ATT, return is 5%
  - Ratio method

$$\frac{\$1{,}000{,}000 \times 1.1 + \$3{,}000{,}000 \times 1.05}{\$1{,}000{,}000 + \$3{,}000{,}000} - 1 = 6.25\%$$

  - Portfolio-weighted average

$$0.25 \times 10\% + 0.75 \times 5\% = 6.25\%$$

Covariances/Correlations

- Measures how much things tend to move together
  - A high covariance or correlation means variables are similar to each other

- Covariance between stock 1 returns ($\tilde{r}_1$) and stock 2 returns ($\tilde{r}_2$)
$$\sigma_{12} = E[(\tilde{r}_1 - \bar{r}_1)(\tilde{r}_2 - \bar{r}_2)]$$

- Correlation is like covariance rescaled to be between -1 and +1. Divide covariance of stock 1 and 2 by their respective standard deviations (σ_1 and σ_2).
$$\rho(\tilde{r}_1, \tilde{r}_2) = \rho_{12} = \frac{\sigma_{12}}{\sigma_1 \sigma_2} = \frac{cov(\tilde{r}_1, \tilde{r}_2)}{\sigma_1 \sigma_2} = cov\left(\frac{\tilde{r}_1}{\sigma_1}, \frac{\tilde{r}_2}{\sigma_2}\right)$$

  Easy to convert between correlation and covariance using above formula

- Steps for calculating covariance similar to that for variance.
  - Variance of stock 1 is $\sigma_1^2 = E[(\tilde{r}_1 - \bar{r}_1)^2] = E[(\tilde{r}_1 - \bar{r}_1)(\tilde{r}_1 - \bar{r}_1)]$
  - Covariance of stocks 1 and 2 is $\sigma_{12} = E[(\tilde{r}_1 - \bar{r}_1)(\tilde{r}_2 - \bar{r}_2)]$

Properties of Random Variables

- Adding up random variables X, Y, and Z with constants α and β:

$$E(X+Y)=E(X)+E(Y)$$

$$E(\alpha \cdot X+\beta \cdot Y)=\alpha \cdot E(X)+\beta \cdot E(Y)$$

$$Var(X+Y)=Var(X)+Var(Y)+2 \cdot Cov(X,Y)$$

$$Cov(\alpha X+\beta Y,Z)=Cov(\alpha X,Z)+Cov(\beta Y,Z)$$

- Other properties:

$$Var(\alpha \cdot X)=\alpha^2 \cdot Var(X)$$

$$Cov(\alpha \cdot X,\beta \cdot Y)=\alpha\beta \cdot Cov(X,Y)$$

$$Cov(X,X)=Var(X)$$

Ex: Factor model return correlations

- Consider the following model of returns:

$$r_i = \alpha + \beta_i r_m + \varepsilon_i$$
$$r_j = \alpha + \beta_j r_m + \varepsilon_j$$

  - in which α are constants and ε are noise with $E(\varepsilon) = 0$
  - Both stocks' returns depend on a common (random) factor r_m
  - Assume that the errors are <u>uncorrelated</u> with each other and with the common factor r_m (i.e., $Cov(\varepsilon_i,\varepsilon_j)=0$, $Cov(\varepsilon_i,r_m)=0$, $Cov(\varepsilon_j,r_m)=0$)

- From this, it follows that $$Cov(r_i, r_j) = \beta_i \beta_j Var(r_m)$$

Now we have the tools necessary to calculate portfolio variance as in the earlier Gold and S&P 500 example.

Variance of a stock portfolio

- Consider a two-stock portfolio with return $\tilde{r}_p$ containing stock 1 and stock 2. Returns of stock 1 and stock 2 ($\tilde{r}_1$ and $\tilde{r}_2$) have standard deviations σ_1 and σ_2, covariance σ_{12}, and correlation ρ_{12}.

$$\begin{aligned} var(\tilde{r}_p) &= var(w_1\tilde{r}_1 + w_2\tilde{r}_2) \\ &= E\left[\left(w_1\tilde{r}_1 + w_2\tilde{r}_2 - (w_1\bar{r}_1 + w_2\bar{r}_2)\right)^2\right] \\ &\ \ \vdots \\ &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12} \\ &\qquad\qquad \text{or, using } \rho \ \ldots \\ &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2 \end{aligned}$$

- For a portfolio of N stocks

$$\sigma_p^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij} = \sum_{i=1}^{N} w_i^2\sigma_i^2 + 2\sum_{j<i} w_i w_j \sigma_{ij}$$

Example: 2-stock portfolio variance

- Example with monthly AAPL and GOOG returns
  - $\sigma^2_{AAPL} = 0.011;\ \sigma^2_{GOOG} = 0.010; \sigma_{AAPL,GOOG} = 0.007$

  Assume we put equal amounts in both stocks

$$\begin{aligned} var(\tilde{R}) &= \ 0.5^2 \times 0.011 + 0.5^2 \times 0.01 + 2 \times 0.5^2 \times 0.007 \\ &= \ 0.00875 \end{aligned}$$

- Variance of portfolio < variance of either stock!

Recap of Portfolio mean/var formulas

- Portfolio weights

$$w_j = \frac{\text{Investment in Asset A}}{\text{Portfolio Value}}$$

- Portfolio return/variance of N-asset portfolio

$$\tilde{r}_p = \sum_{j=1}^{N} w_j \tilde{r}_j$$

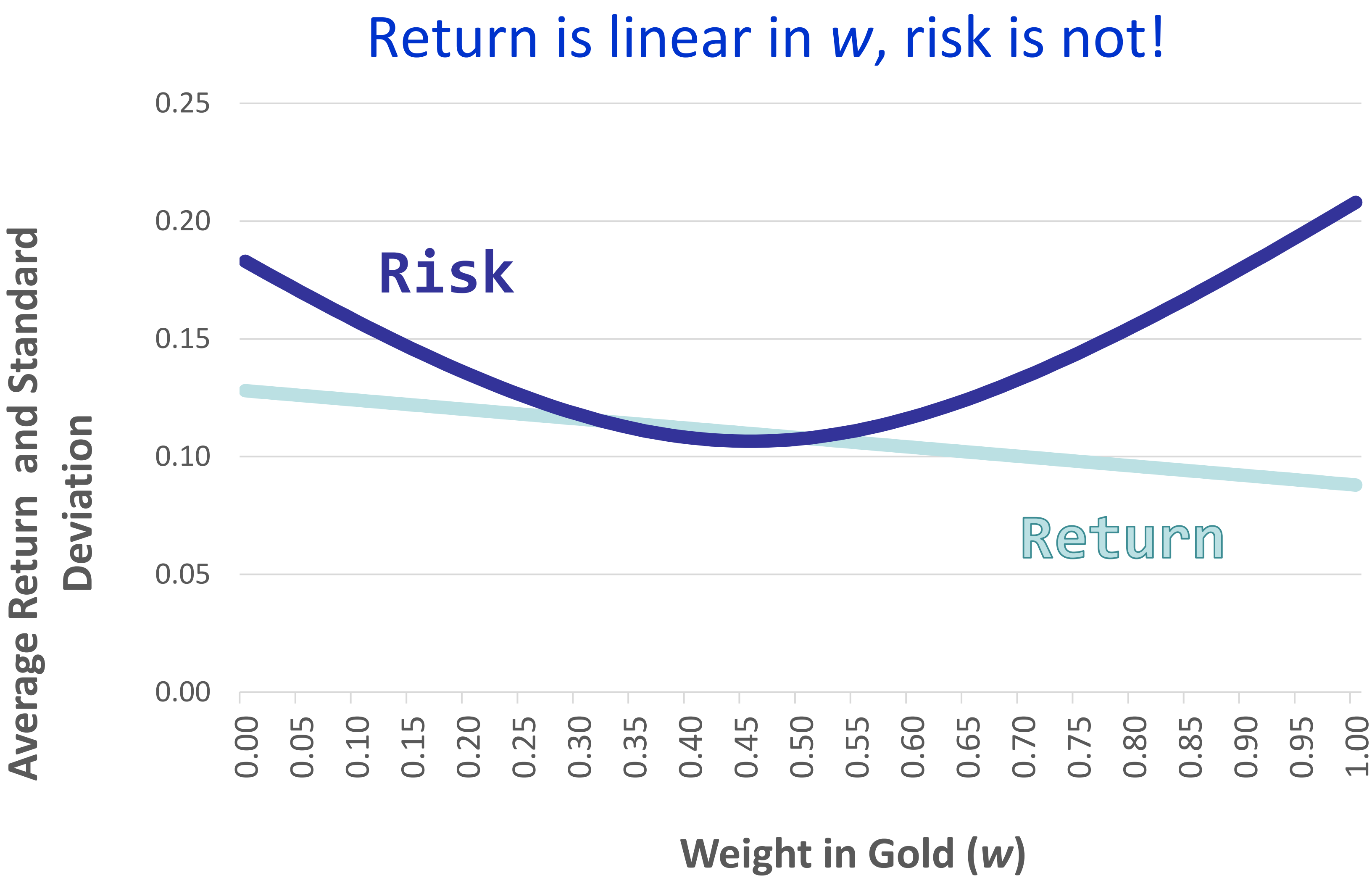
Varies linearly with weights

$$\sigma_p^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + 2 \sum_{j<i} w_i w_j \sigma_{ij}$$

Does not vary linearly with weights

- Question: How do we annualize these values?

Return & Risk of Stock+Gold Portfolios

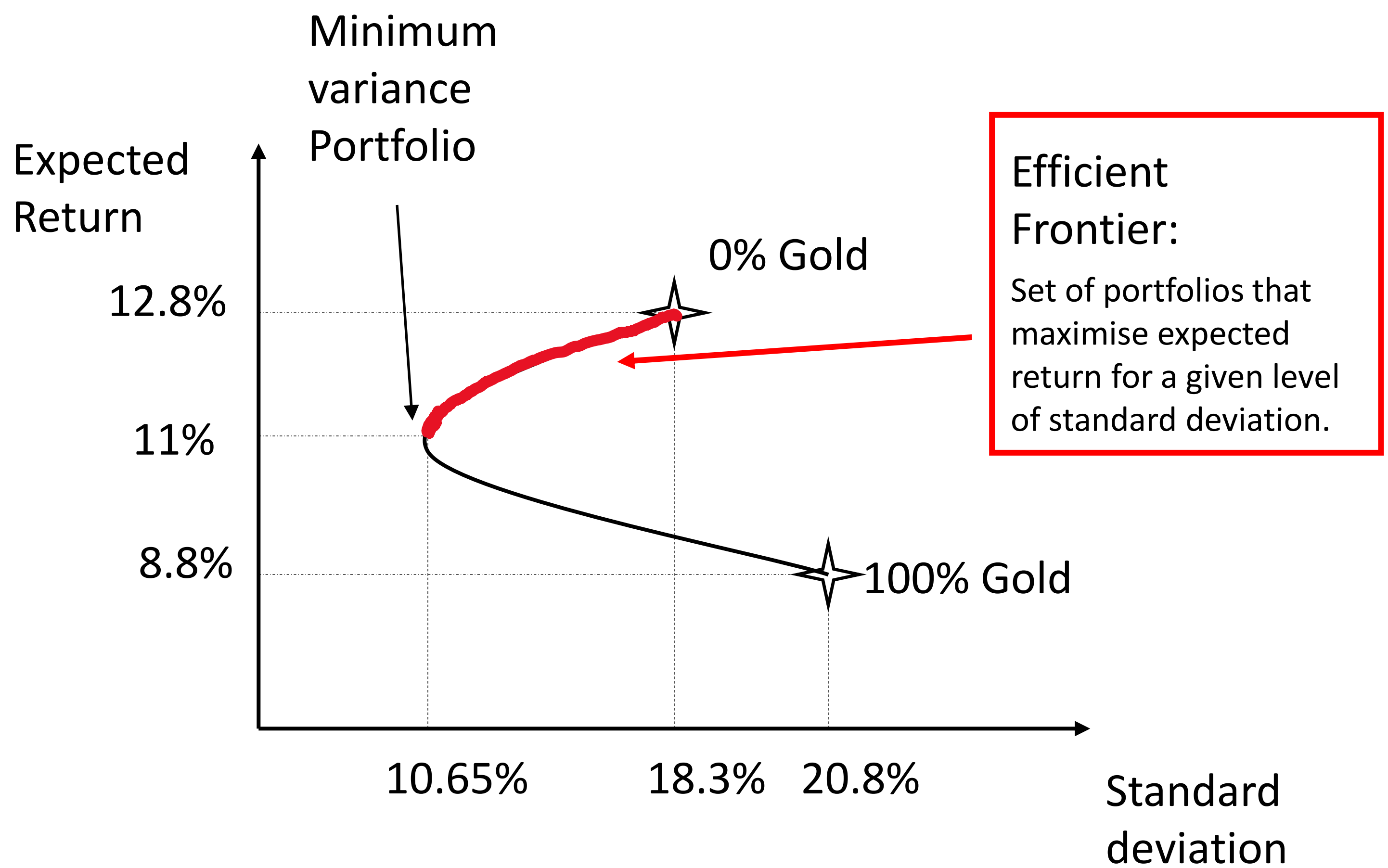


Portfolio Theory: A Two Asset Portfolio.

- The expected return of a portfolio varies linearly with the portfolio weights.
- The variance of the portfolio return does not vary linearly with the portfolio weights.
  - This is why diversification reduces risk.
  - The crucial parameter is the correlation coefficient.

- We usually plot the previous picture in the mean-standard deviation space.

Plotting Mean vs. Std. Dev.



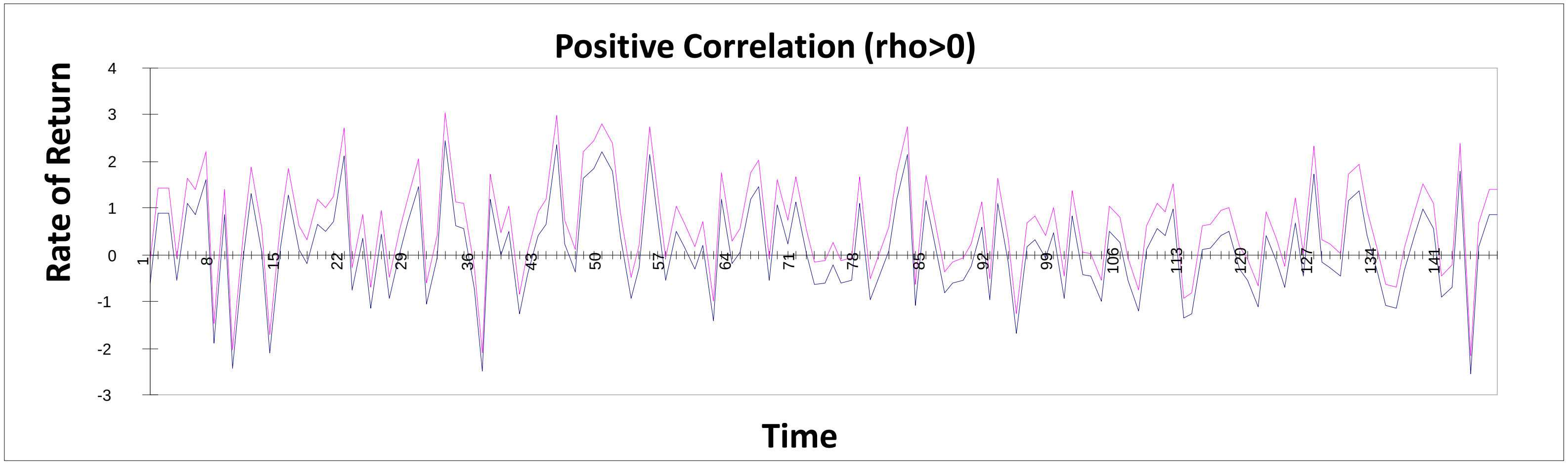
Interpreting the Mean-Std. Dev. Plot

- The efficient frontier—set of portfolios that maximise expected return for a given level of standard deviation.

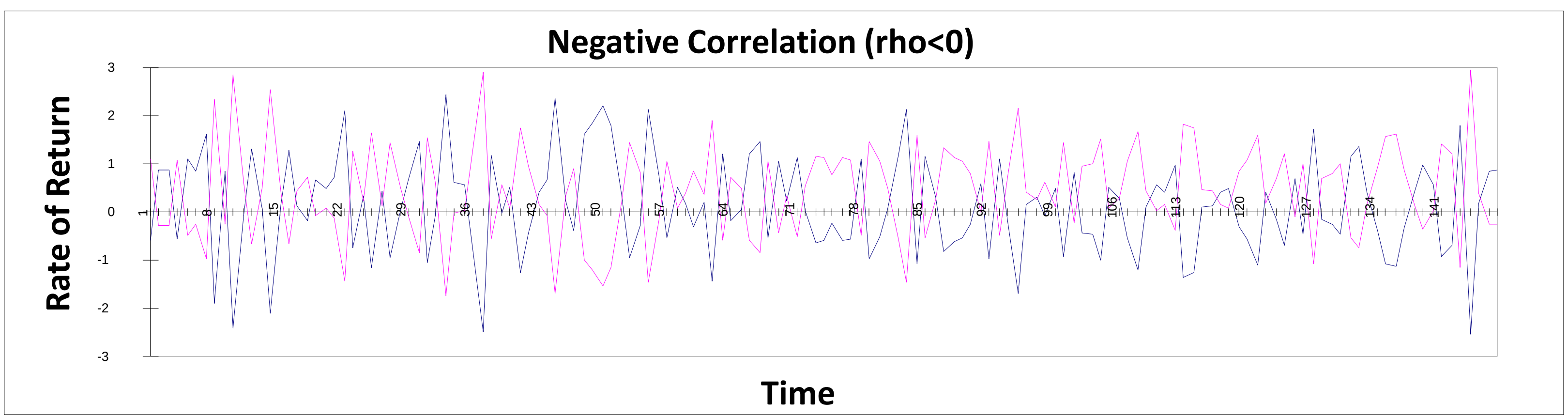
- Would you ever own Gold on its own? No.
  - Gold only is strictly dominated by the S&P500.
  - Similarly all portfolios between 100% Gold and the minimum variance portfolio are strictly dominated.

- Where is the magic?
  - The crucial parameter that is driving this diversification effect is the correlation between the two stocks.
- Let us look at what happens when the correlation is equal to +1 or to -1.

Positive correlation: *"If the correlation between two stocks is positive r > 0, when the first stock goes up, the second stock goes up as well."*

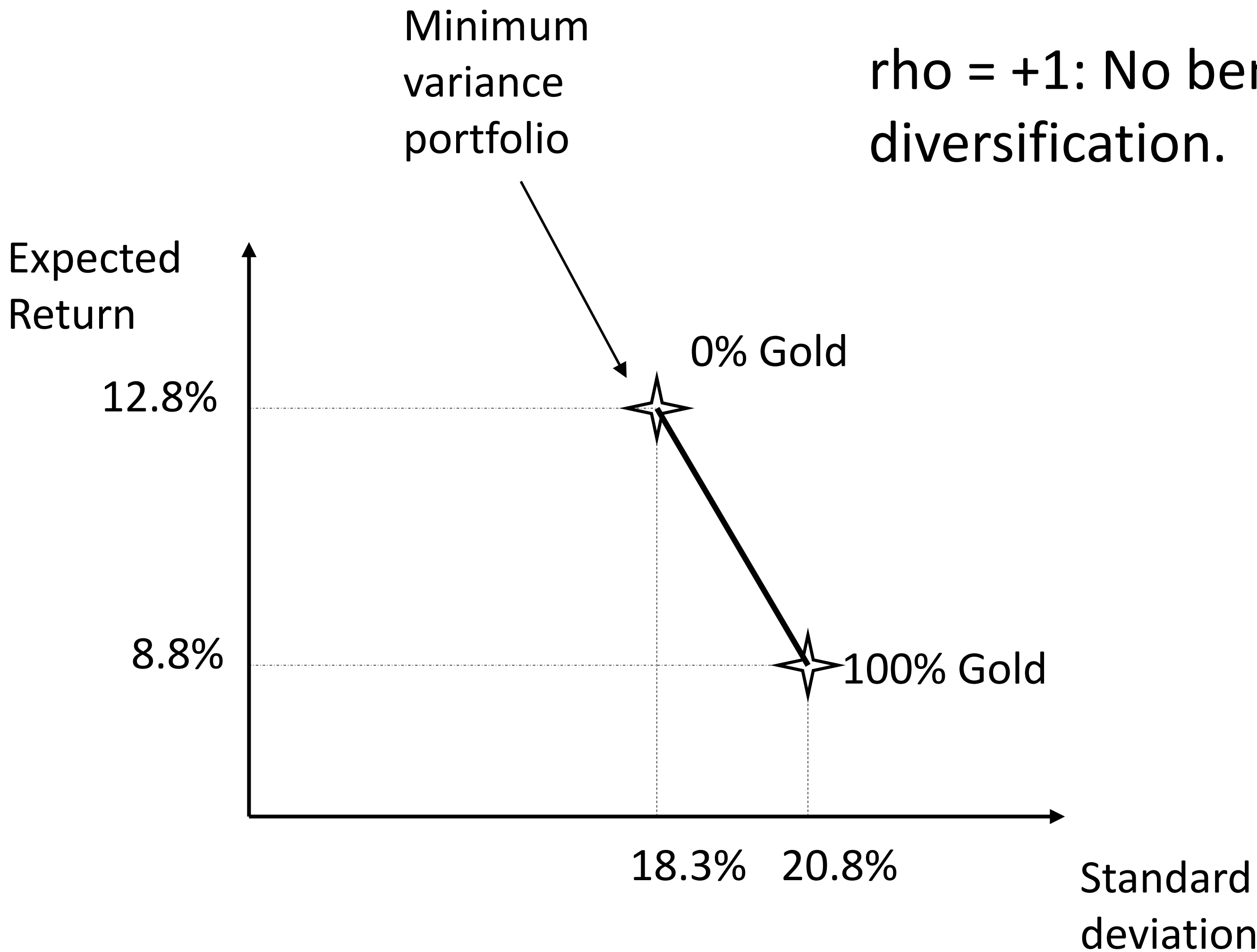


- **Negative correlation:** *"If the correlation between two stocks is negative r < 0, when the first stock suffers negative returns, the second stock gives a positive return."*
- The second stock works as a form of insurance. Gold is an *example*.

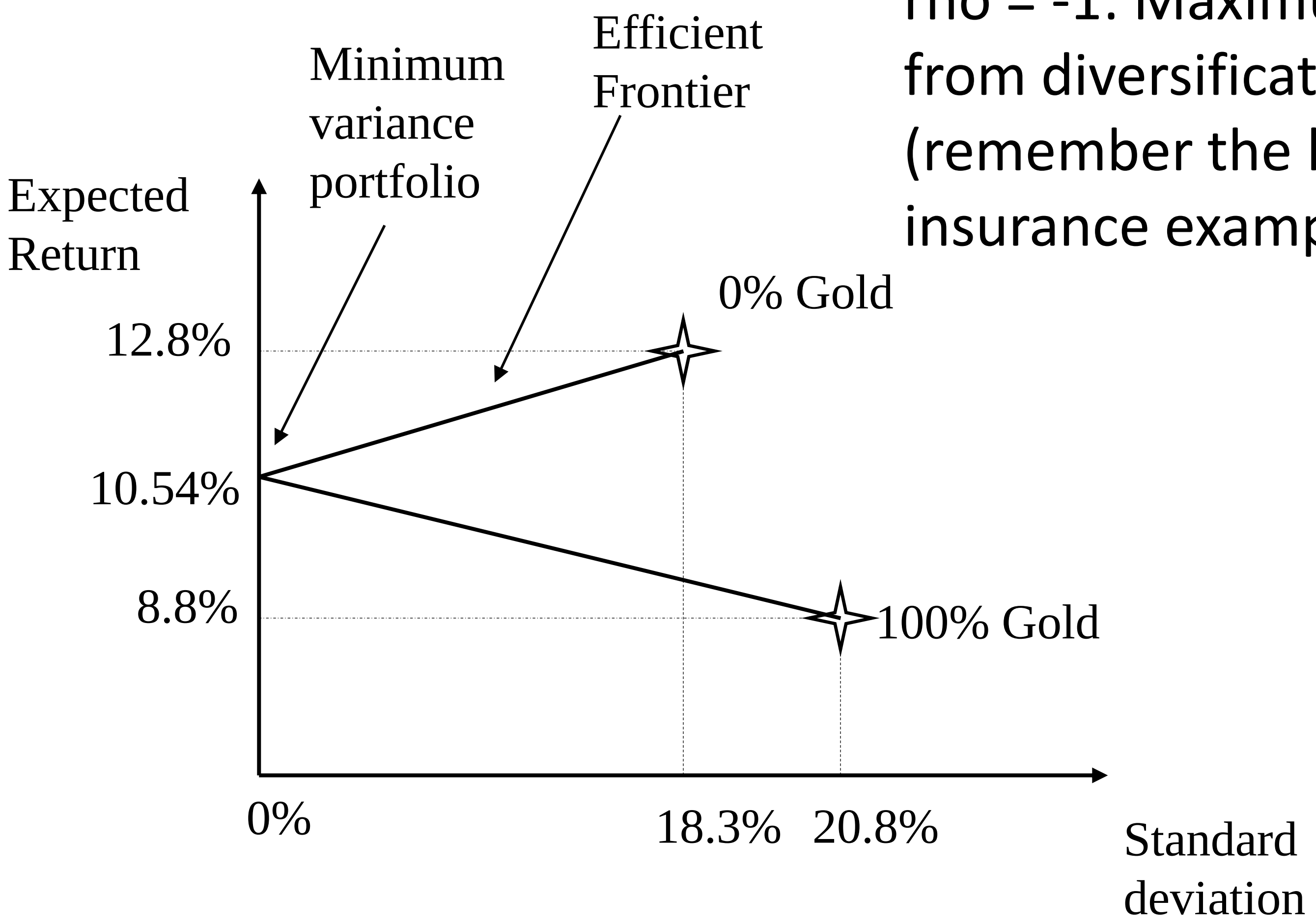


Case of perfect positive correlation

rho = +1: No benefits from diversification.

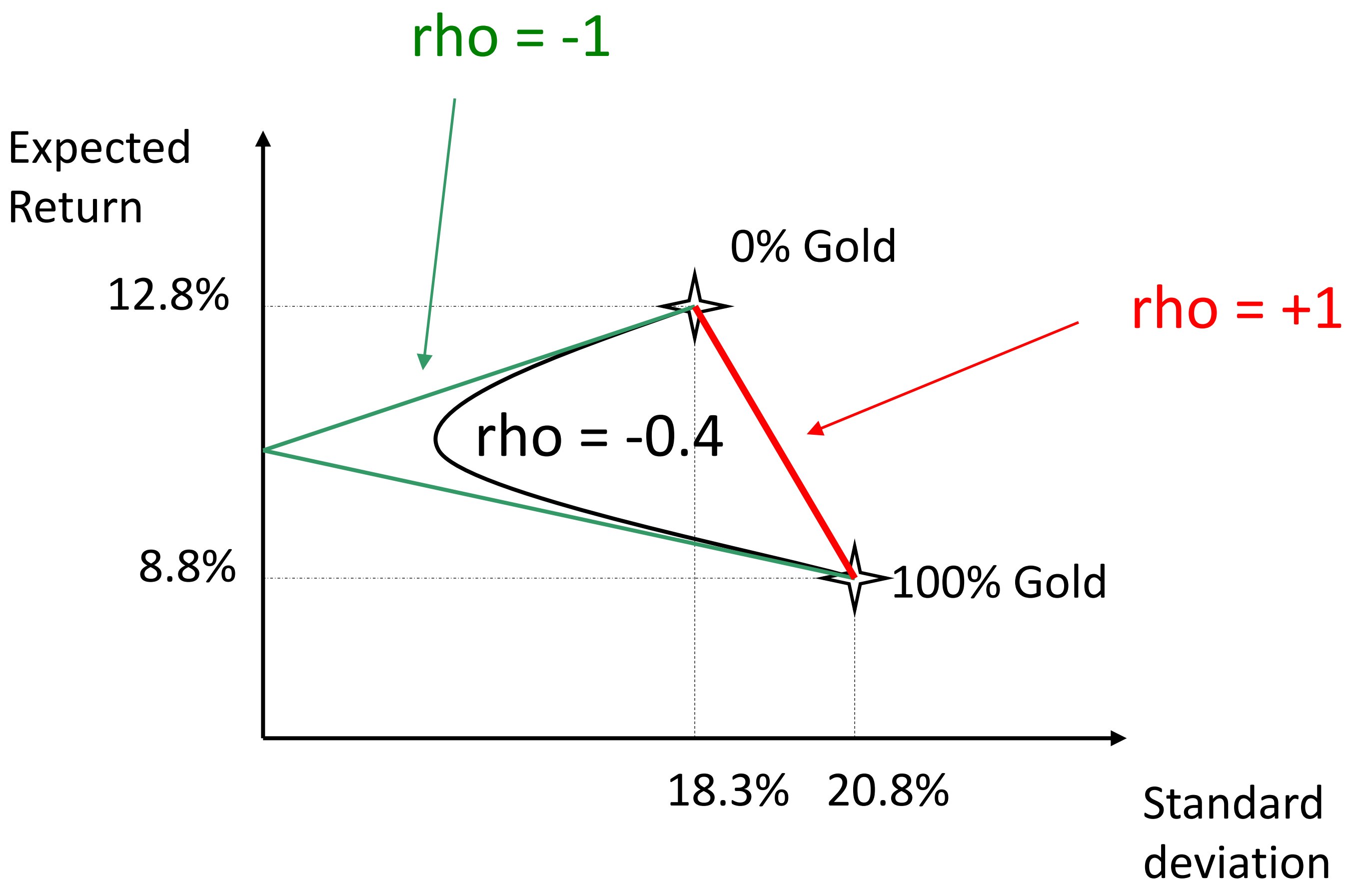


Case of perfect negative correlation



rho = -1: Maximum benefits from diversification (remember the house fire insurance example).

Correlation between +1 and -1 (e.g., Gold & S&P)



Main take away

- When stocks are less then perfectly correlated…
  - The expected return is a portfolio-weighted average
  - The standard deviation is less than the portfolio-weighted average of that of constituent stocks.
    - When combining stocks in a portfolio, the stock returns do not all move together and some variation cancels out
    - The amount of variation you can eliminate depends on the degree to which stock returns move together. The more "common risks" that affect both the stocks, the lower the risk reduction.
- If the correlation between assets is sufficiently low

  $\Rightarrow$ Diversification gives you more $\bar{r}$ per unit σ

  Benefits of diversification come without having to give up return,
  the one "free lunch" in finance!

- Similarly, if you hold portfolio weights constant

  $\Rightarrow$ higher correlations lead to higher portfolio variance

Covariance ≈ Marginal Variance

- The increase in a portfolio's variance following the addition of a small stock position is proportional to the covariance of the stock with the portfolio.
- Proof
  - Say you add x dollars to a portfolio

$$\tilde{r}_{new} = \tilde{r}_{old} + x(\tilde{r}_s - r_f)$$

$$\Rightarrow \sigma^2_{new} = var\left(\tilde{r}_{old} + x\left(\tilde{r}_s - r_f\right)\right)$$

$$= \sigma^2_{old} + x^2\sigma^2_s + 2x\sigma_{s,old}$$

  - Look at the derivative w.r.t. x

$$\frac{d\sigma^2_{new}}{dx} = 2\left(x\sigma^2_s + \sigma_{s,old}\right)$$

  - What happens as $x \to 0$?
- Implication for portfolio choice: Investors evaluate the risk of investment as it contributes to the risk of their portfolio, not the total risk (σ^2) of the investment.

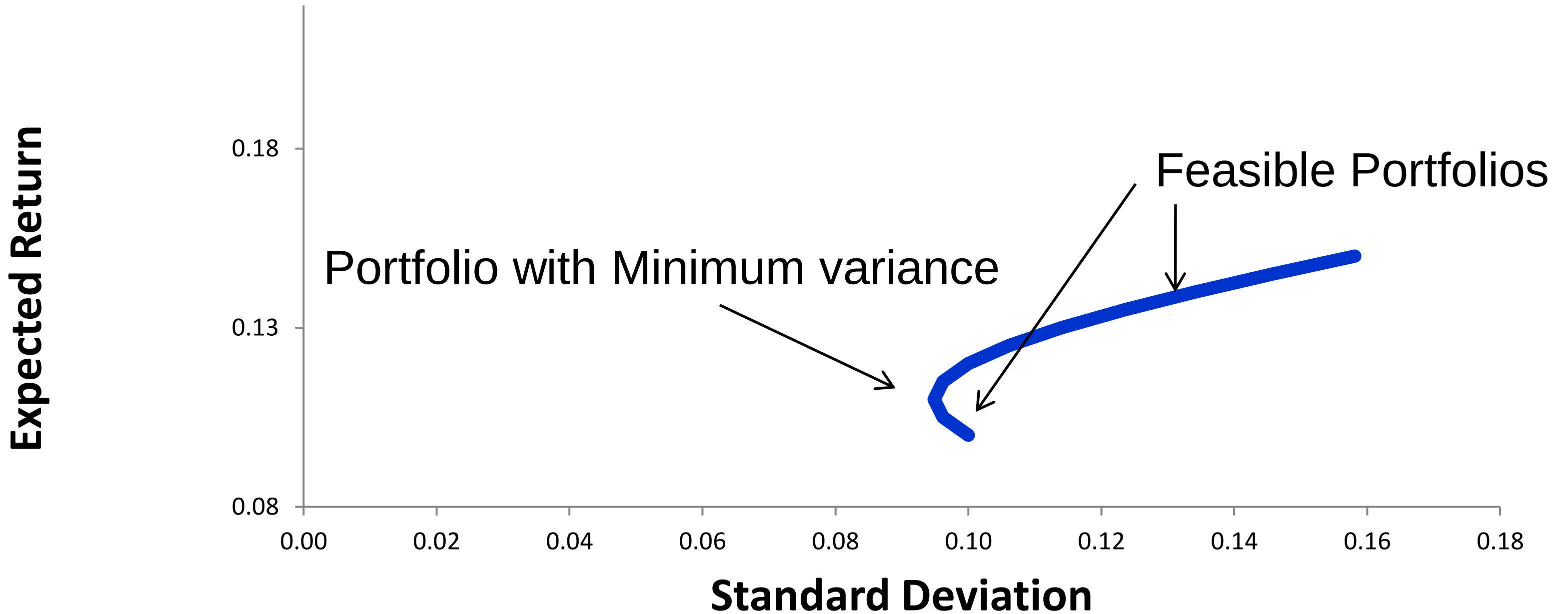
An implication of covariance as marginal variance

- Take 2 stocks in a portfolio, P
  - Stock A—covariance with P is σ_{PA}
  - Stock B—covariance with P is σ_{PB}

- If $\sigma_{PA} > \sigma_{PB} \Rightarrow$
  - Sell a small amount of Stock A and buy an equivalent amount of Stock B and you reduce portfolio variance.

The minimum variance portfolio

- The feasible portfolio with the minimum variance
  - Very risk-averse investors will hold this if there is no risk-free asset.
- How does one find it?
  - The portfolio will have equal covariance with all constituent stocks.



Recap of intro to mean-variance analysis

- Mean-variance analysis is a fundamental building block of portfolio theory
  - Assumes portfolio desirability determined solely by the mean and variance of returns.
- Portfolio variance largely determined by covariances of portfolio assets.
  - Low covariance/correlation leads to low portfolio variance
- Has implications for construction of "optimal" portfolios.
  - A security's risk is determined by how it contributes to the overall riskiness of your wealth portfolio.