Investments

Mean-Variance Analysis

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Outline

- Efficient frontier with multiple assets
- Two-fund separation
- Capital market line
- Tangency portfolio
- Sharpe Ratio
- Practical issues with trying to find the tangency portfolio

Context for Today's Lecture

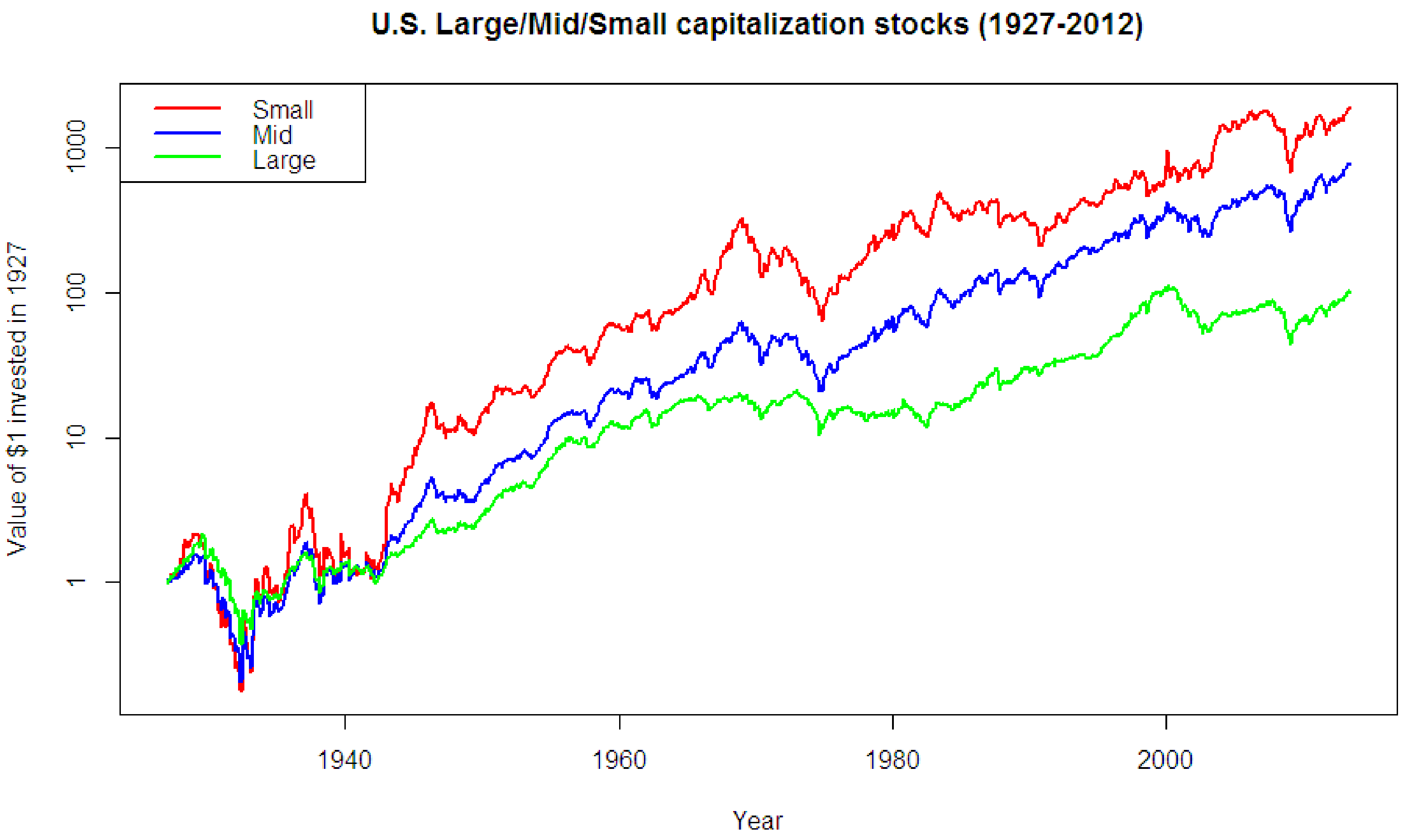
- We reviewed the history of equity/bond/bill returns
  - Large variation in returns/variability among asset classes
    - Equities have historically vastly outperformed bonds and bills
  - Motivates need for a framework to analyse these investments

- We introduced tools for summarizing a portfolio's characteristics
  - Calculated portfolio means/variances

- Introduced the concept of risk
  - Intuition
    - A risky asset has high returns when you are rich and low returns when you are poor.
    - A safe asset has high returns when you are poor and low returns when you are rich.
  - Portfolio variance as a measure of risk
  - covariance ≈ marginal variance, or how the asset contributes to the risk of a portfolio.

- Now we will study how to identify the best portfolios among those that are feasible

Why portfolio theory?



Why portfolio theory cont'd

- But what now?

	Small-cap	Mid-cap	Large-cap
Mean return	15.0%	11.2%	7.1%
Std. Deviation	35.4%	24.8%	17.7%
	(annualized monthly data 1927-2012)		

- Small-cap stocks have higher returns, but are more volatile too…

Why portfolio theory cont'd

- So small cap stocks earn 15.0% with sd of 35.4%
- Look at this portfolio

	Small-cap	Mid-cap	Large-cap
Portfolio weight	0.82	0.43	-0.24

  - It has the same sd as small-cap stocks but earns 15.4%
  - Wouldn't you prefer this portfolio?
- Portfolio theory provides a framework for evaluating (and constructing) investment portfolios.

What makes a good investment?

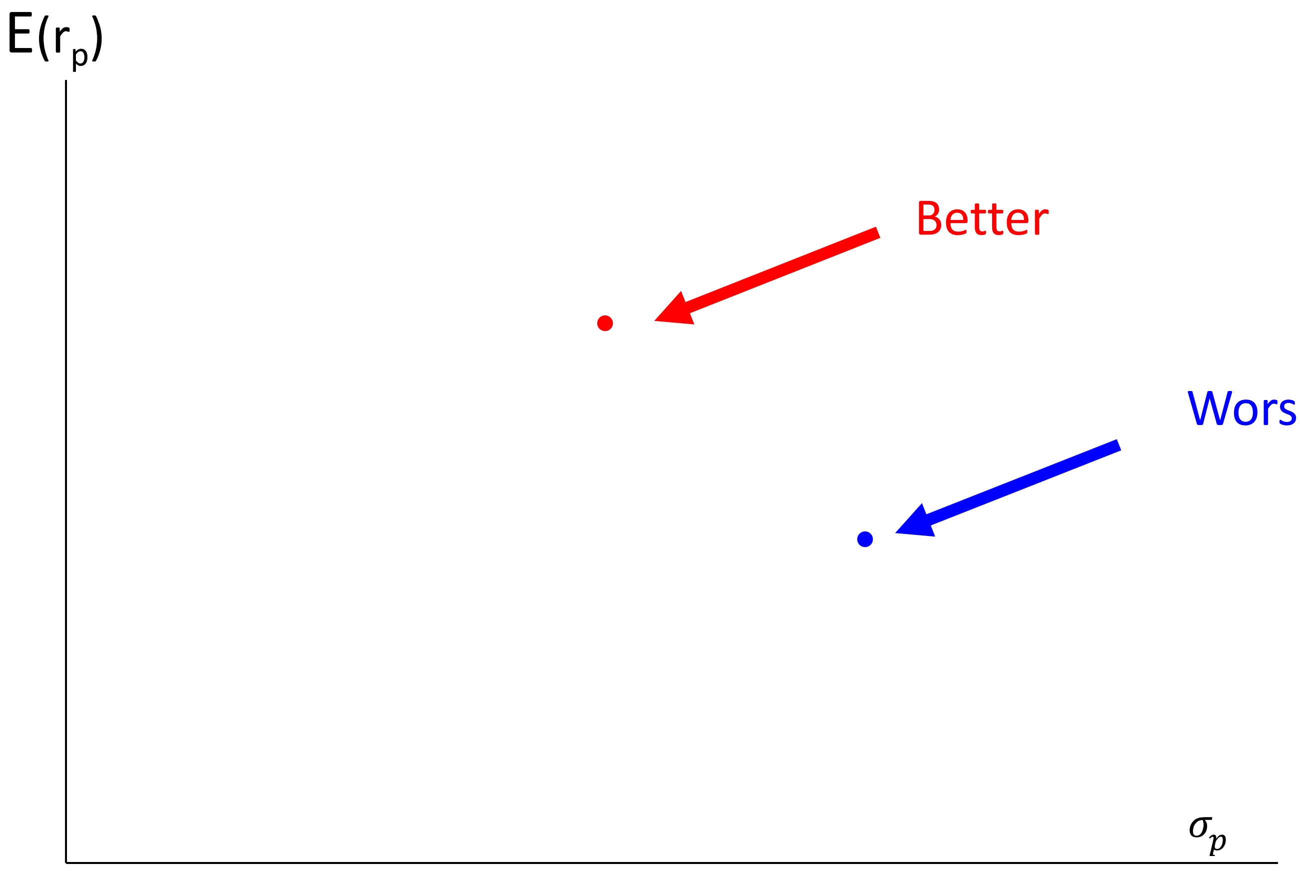
- Equivalent problem
  - What is the optimal set of portfolio weights $\{w_1, w_2, \ldots, w_j\}$?
- Optimal weights determined by preferences
  - Do you want to maximize your chance of having $1m?
  - Do you prefer to minimize the chance you aren't poor?
- Solution requires a model for preferences
  - Trade-off of risk vs. reward

Mean-Variance Preferences

- Assumptions
  - Investors care only about means (good) and variances (bad)
  - Financial markets are frictionless

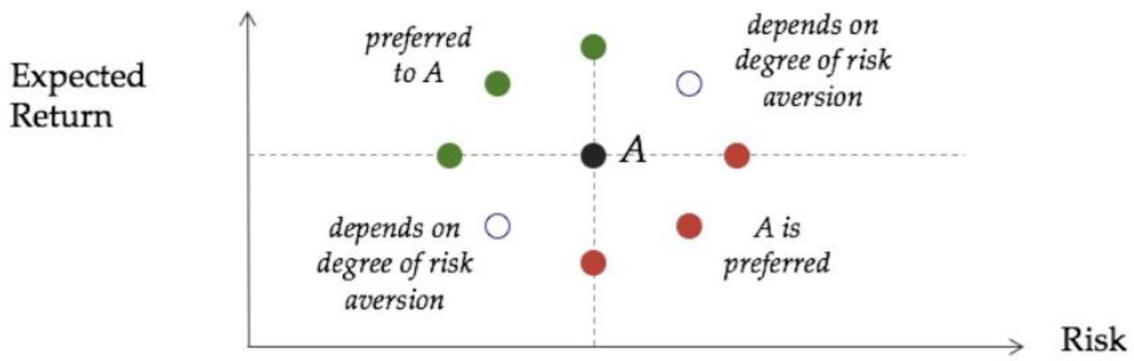
- What does this give us?
  - Can use the mean-variance diagram to interpret how desirable a portfolio is.
  - The more northwest a portfolio, the better

Red here is more northwest



Investor Preferences

- Assume that investors are risk averse:
  - for the same expected return, prefer less risk
  - for the same risk, prefer more return
- Consider a risk averse investor at point A below. Which portfolios would (s)he prefer?



Validity of assumptions

- High expected returns are good—uncontroversial
- Variance = risk
  - Intuition—Investors prefer less dispersion in wealth outcomes
  - Also holds if returns are normally distributed
  - Potential concerns
    - Investors may care about other things
      - Do they care about “higher moments” (skewness, kurtosis, etc)?
      - Do they care mostly about having high returns during recessions?
      - Do they care about staying above a baseline wealth level they are used to (habit formation)?
    - Security returns are not really normally distributed
      - E.g., equities during 2008, options at any point in time, etc.
- Frictionless markets
  - Securities may be traded at any price/quantity without transaction costs
  - Simplifies analysis, may or may not have an important effect

Feasible means and variances?

- First question: What portfolio means and variances are possible?
- Go back to our equations for $\bar{R}_p$ and σ_p^2, and vary the weights w_j

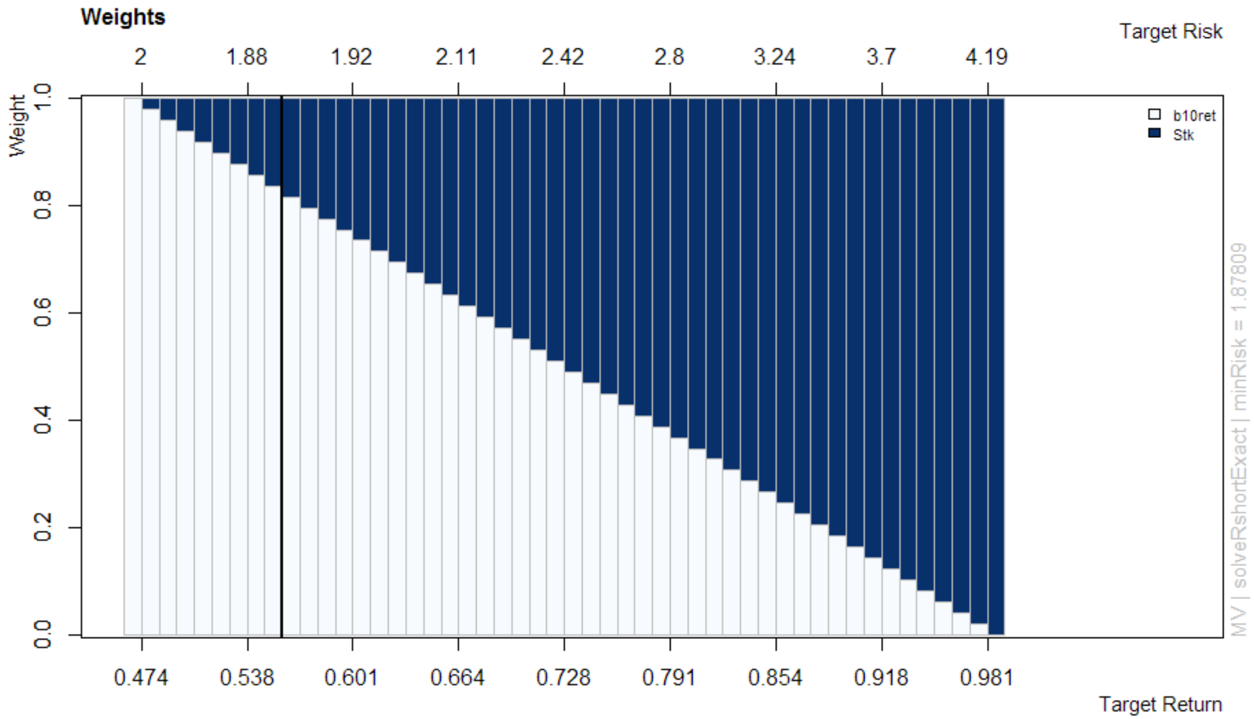
$$\bar{R}_p = \sum_{j=1}^{N} w_j \bar{r}_j$$

$$\sigma_p^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + 2 \sum_{j<i} w_i w_j \sigma_{ij}$$

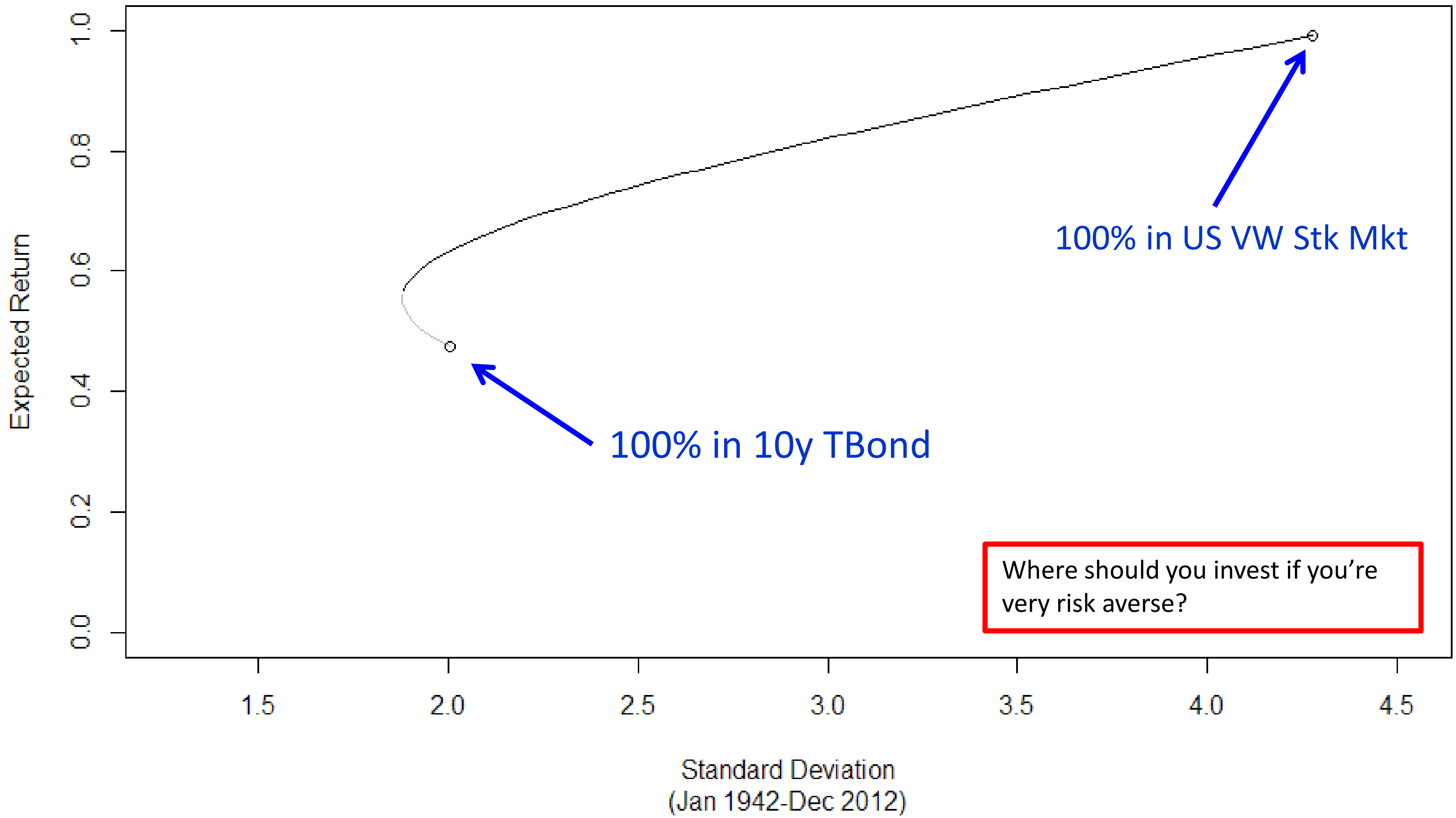
- Then figure out which is the best.

Consider a stock & bond portfolio

- Vary weights from 100% bond (white) to 100% stock (blue)



Then plot mean vs. variance.

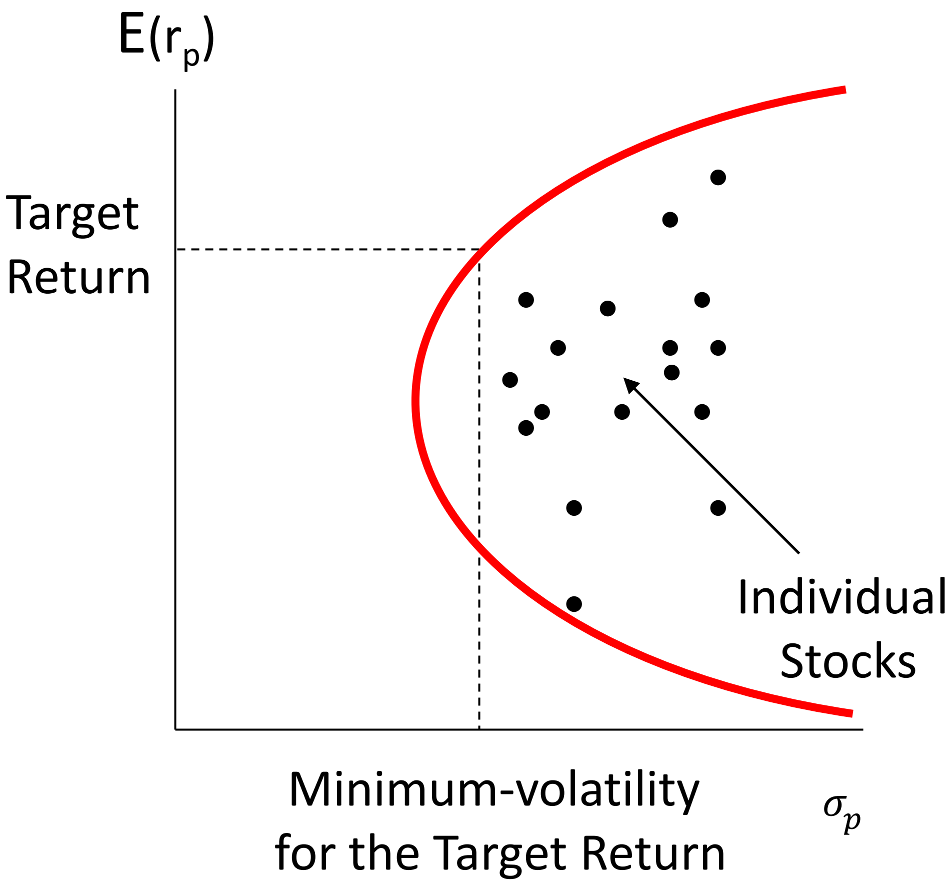


The efficient frontier

- Looks simple with two assets.
- With many assets:
  - Many more possibilities for means and variances.
  - Many portfolios will deliver the same expected return.

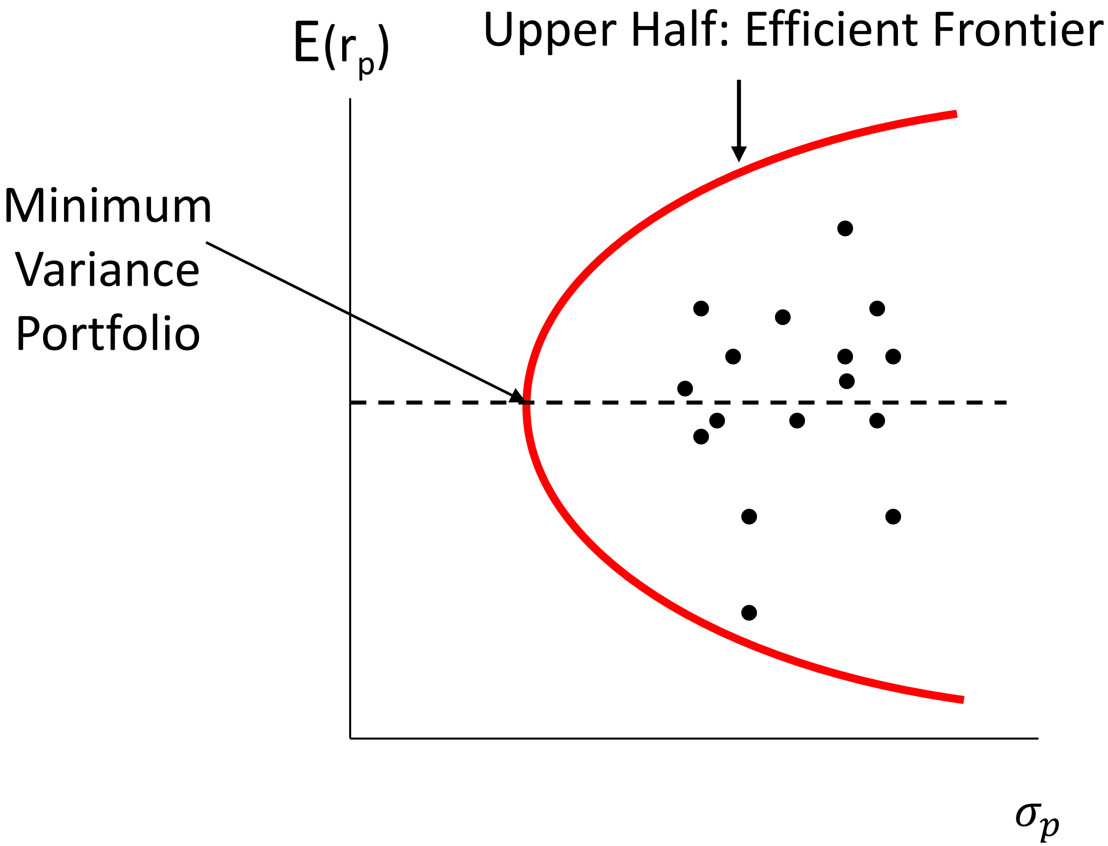
- Simplify problem by identifying **Efficient frontier**
  - Portfolios with the lowest variance for a given $E[\tilde{r}]$
  - Dominates all other portfolios with the same $E[\tilde{r}]$
  - Portfolios on the frontier are **efficient portfolios**

Possible risky asset portfolios



- Everything east of the red line is a "feasible portfolio"
- For a given $E[\tilde{r}]$, find portfolio weights, x_1, x_2,…, x_N that ***minimize portfolio volatility.***
- That gives you the red line.

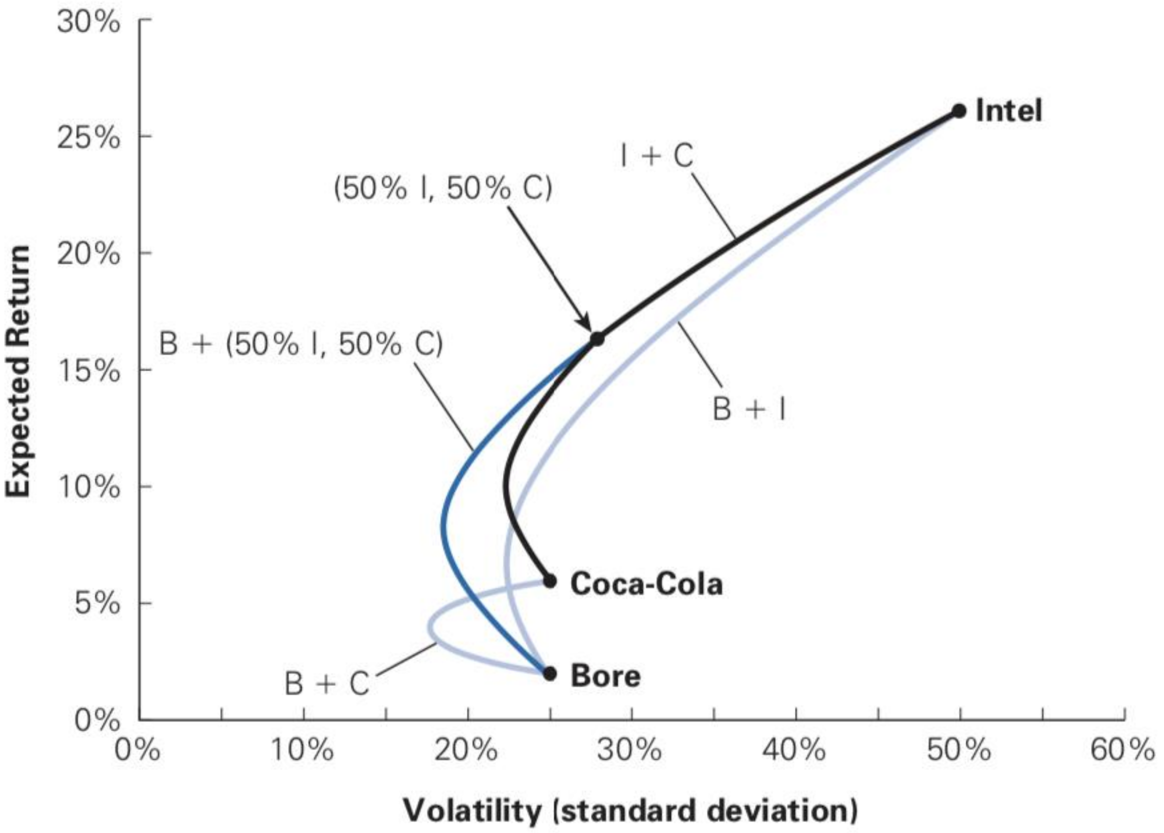
Minimum Variance Frontier



- Invest on (upper part of) the solid line: risky investment opportunity set
- Power of diversification
- Portfolios on upper half of red line are efficient portfolios.

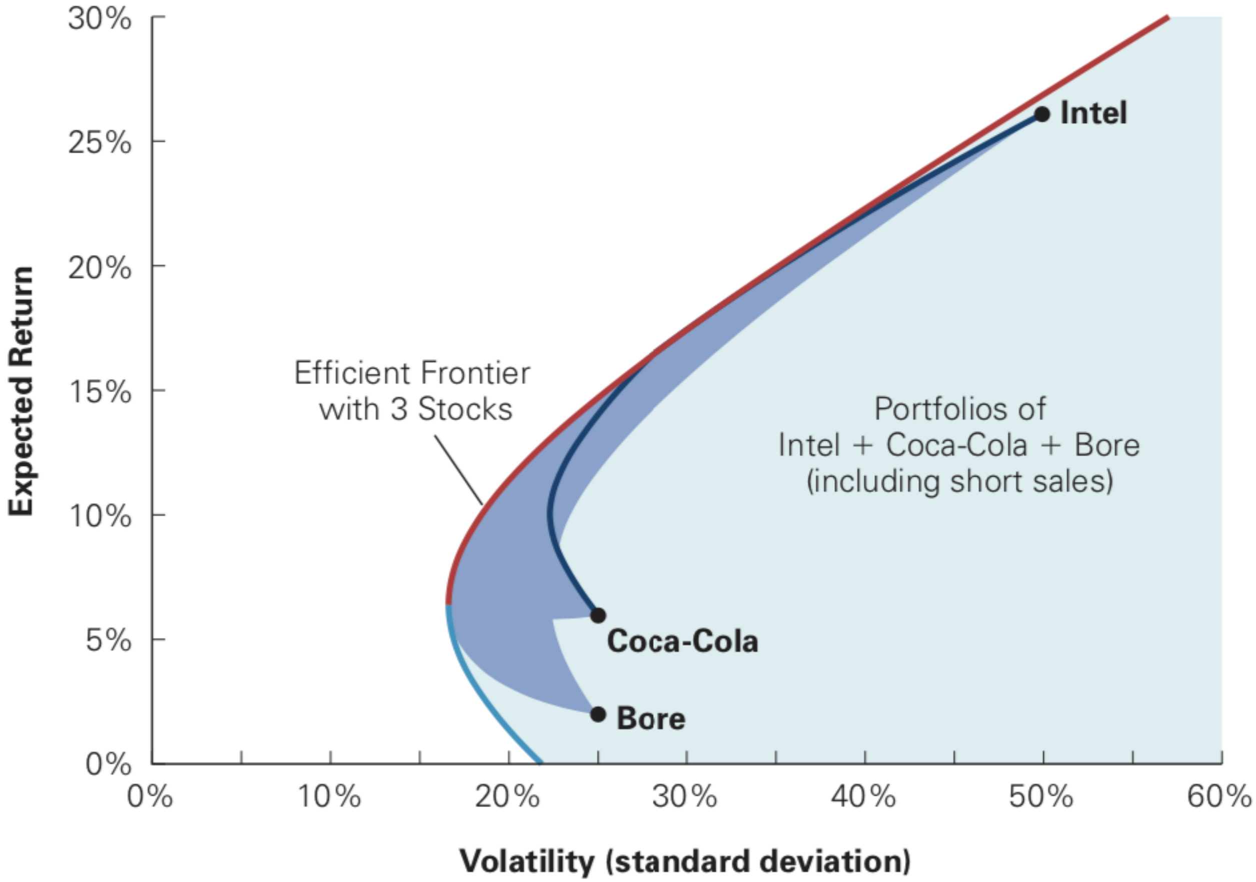
Portfolios of 3 Risky Securities

Starting from pairwise combinations of 3 stocks:



Portfolios of 3 Risky Securities

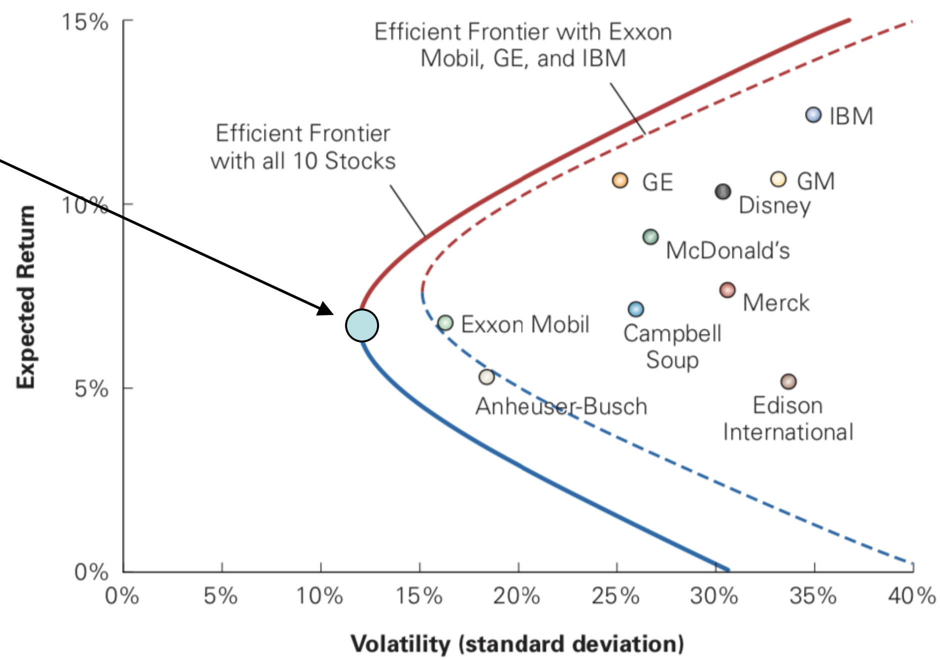
All possible combinations of 3 stocks and the **efficient frontier**



Large Portfolios Risky Securities

- If you continue to add risky securities:
  - efficient frontier improves but at a decreasing rate: decreasing marginal benefits of diversification
  - efficient frontier dominates individual securities

Minimum Variance Portfolio



Implications

- Adding securities ***shifts*** the ***frontier to the left***
- As we add more and more securities, these ***shifts become smaller*** and smaller
  - Adding all DJIA stocks has a smaller impact than adding IBM to MMM+PG
  - Adding S&P 500 has no impact at all
- ***Individual securities are dominated by the frontier!***

Limits to Diversification

- Total portfolio risk will typically not go down to zero even when you hold a very large number of securities:
  - Diversify away firm-specific, non-systematic, idiosyncratic, unique risk
  - Left with exposure to non-diversiftable, systematic, "market" risk
- Most of the diversification benefits come from adding the first 15-30 securities (chosen at random)

Limits to Diversification

- Recall for a portfolio of N stocks
  - $\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij} = \sum_{i=1}^{n} w_i^2 \sigma_i^2 + 2 \sum_{j<i} w_i w_j \sigma_{ij}$
  - If we take $w_i = w_j = 1/n$ we have*:

Variance of an Equally Weighted Portfolio of *n* Stocks

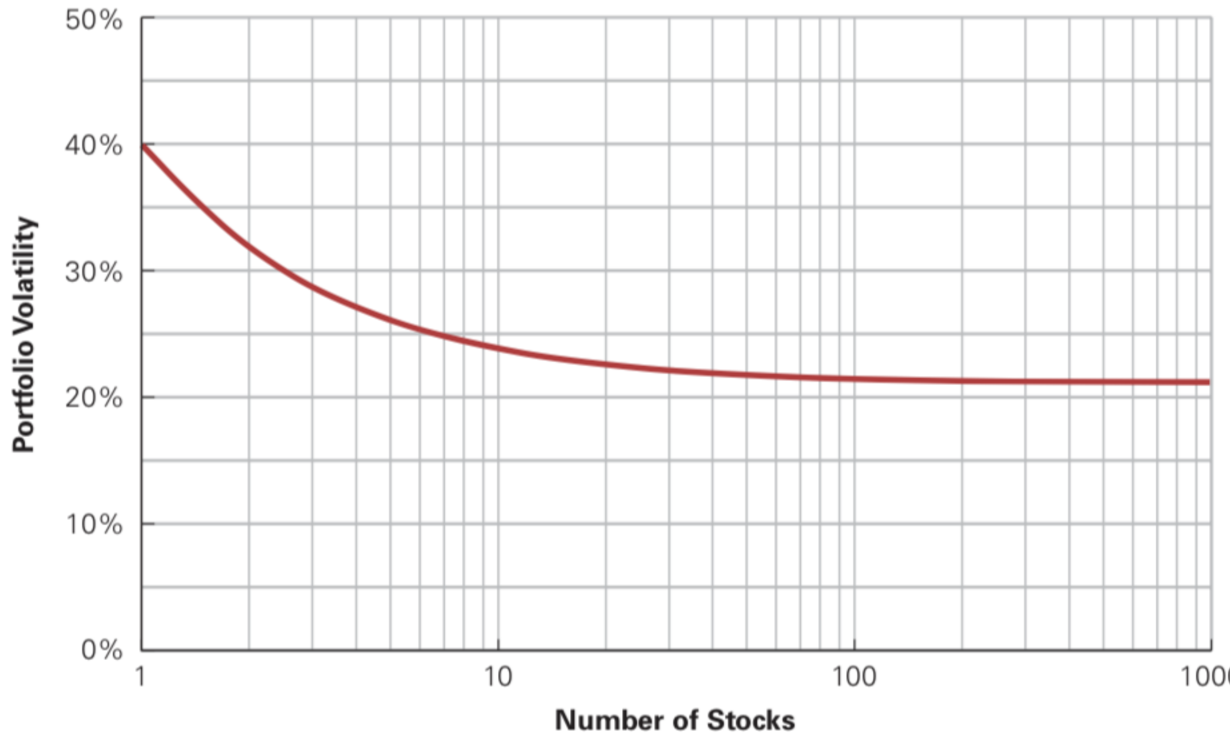
$$Var(R_P) = \frac{1}{n}(\text{Average Variance of the Individual Stocks})$$
$$+\left(1 - \frac{1}{n}\right)(\text{Average Covariance between the Stocks})$$

* Note that there are n variances each with $1/n^2$ weight . There are $n^2 - n$ covariance terms each with weight $1/n^2$

Limits to Diversification

Risk reduction from adding more stocks in portfolio is marginal after approx. 20 stocks

Assume here the individual stocks have an average standard deviation of 0.4 and average correlation of 0.28



$$SD(R_P) = \sqrt{\frac{1}{n}(0.40^2) + \left(1 - \frac{1}{n}\right)(0.28 \times 0.40 \times 0.40)}$$

Review of efficient frontier

- Gives the set of efficient portfolios
  - The portfolios with minimum variance given expected return.
- Does not tell us which efficient portfolio to choose
  - Depends on risk aversion
  - Very risk-averse investors will choose the minimum variance portfolio.
- How do we construct our ideal portfolio?
  1. Perform an optimization to find unique portfolio weights
  2. Or, use a short-cut property called two-fund separation

Two-fund Separation

- All frontier portfolios may be constructed from a weighted average of any two frontier portfolios.
- So once you have two efficient portfolios, the entire efficient frontier may then be constructed

Ex. Two-fund separation

- Two efficient portfolios

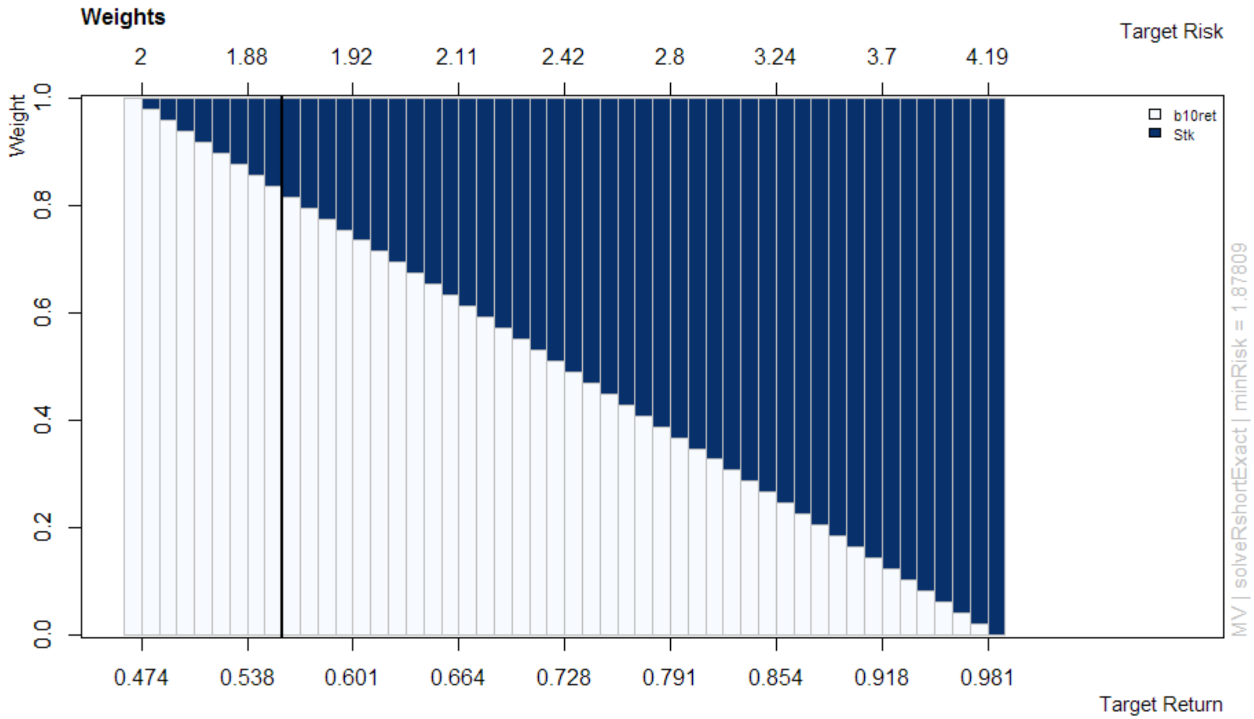
  $P_1 : \{x_1 = .2, x_2 = .3, x_3 = .1, x_4 = .4\}$

  $P_2 : \{x_1 = .25, x_2 = .25, x_3 = .25, x_4 = .25\}$

- Weights of all other efficient portfolios
  - $x_1 = .2w + .25(1 - w)$
  - $x_2 = .3w + .25(1 - w)$
  - $x_3 = .1w + .25(1 - w)$
  - $x_4 = .4\underbrace{w}_{\text{Weight on } P_1} + .25\underbrace{(1 - w)}_{\text{Weight on } P_2}$

Frontier weights with Bond/Stock Ex.

- Both bonds and stocks were on the frontier, so every frontier portfolio is a combination of the two.



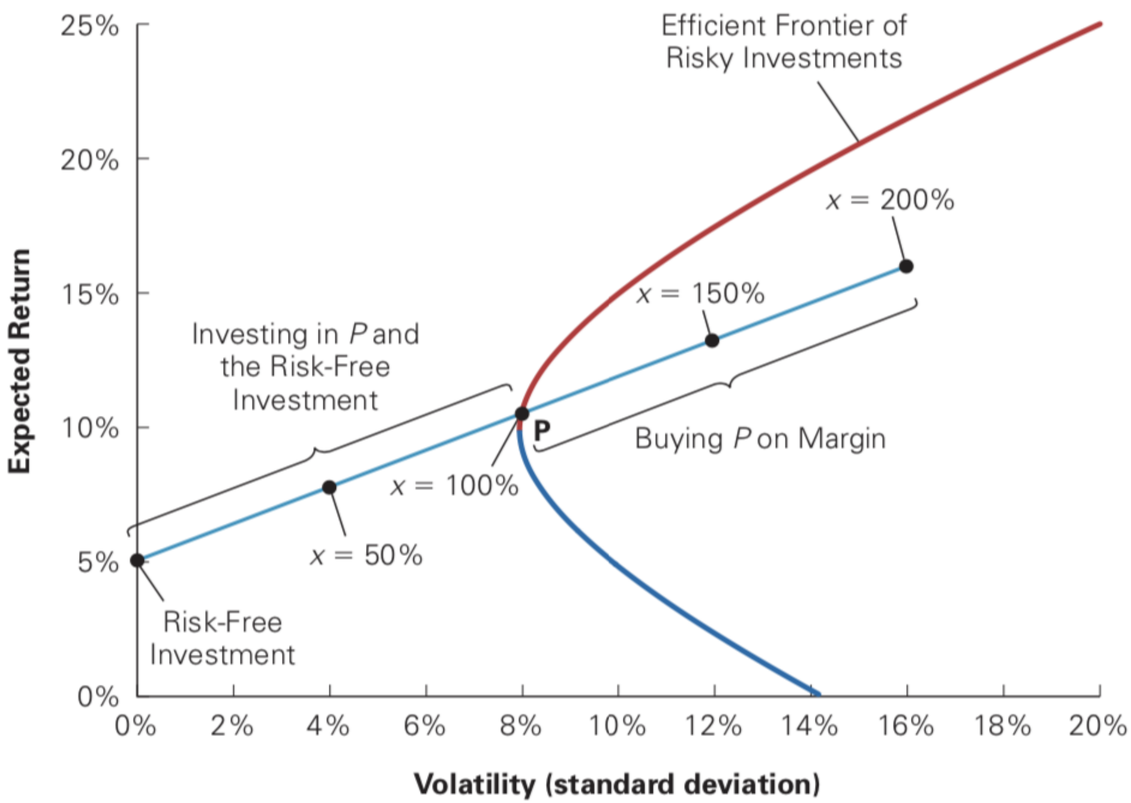
Adding the risk-free asset to the efficient frontier

- We calculated and observed properties of the frontier formed from risky assets

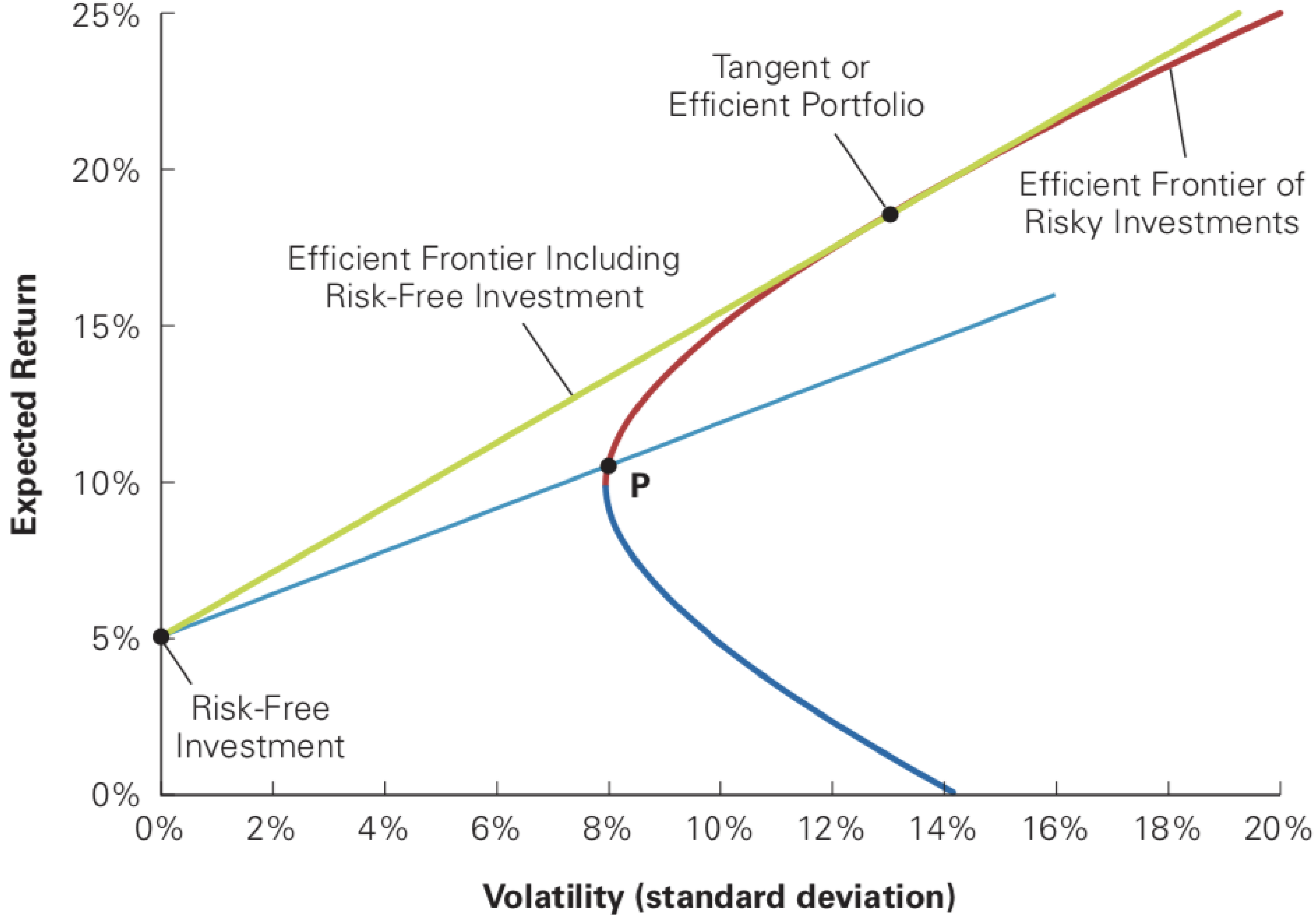
- Adding a risk-free asset has a large effect on the frontier
  - Greatly expands the set of feasible portfolios
  - Frontier changes from a curve to a straight line
- The risk-free asset is on the frontier
  - It is the minimum variance portfolio when available.
  - Implies the frontier may be constructed from it and one risky portfolio on the frontier (due to two-fund separation)

Adding a Riskfree Asset

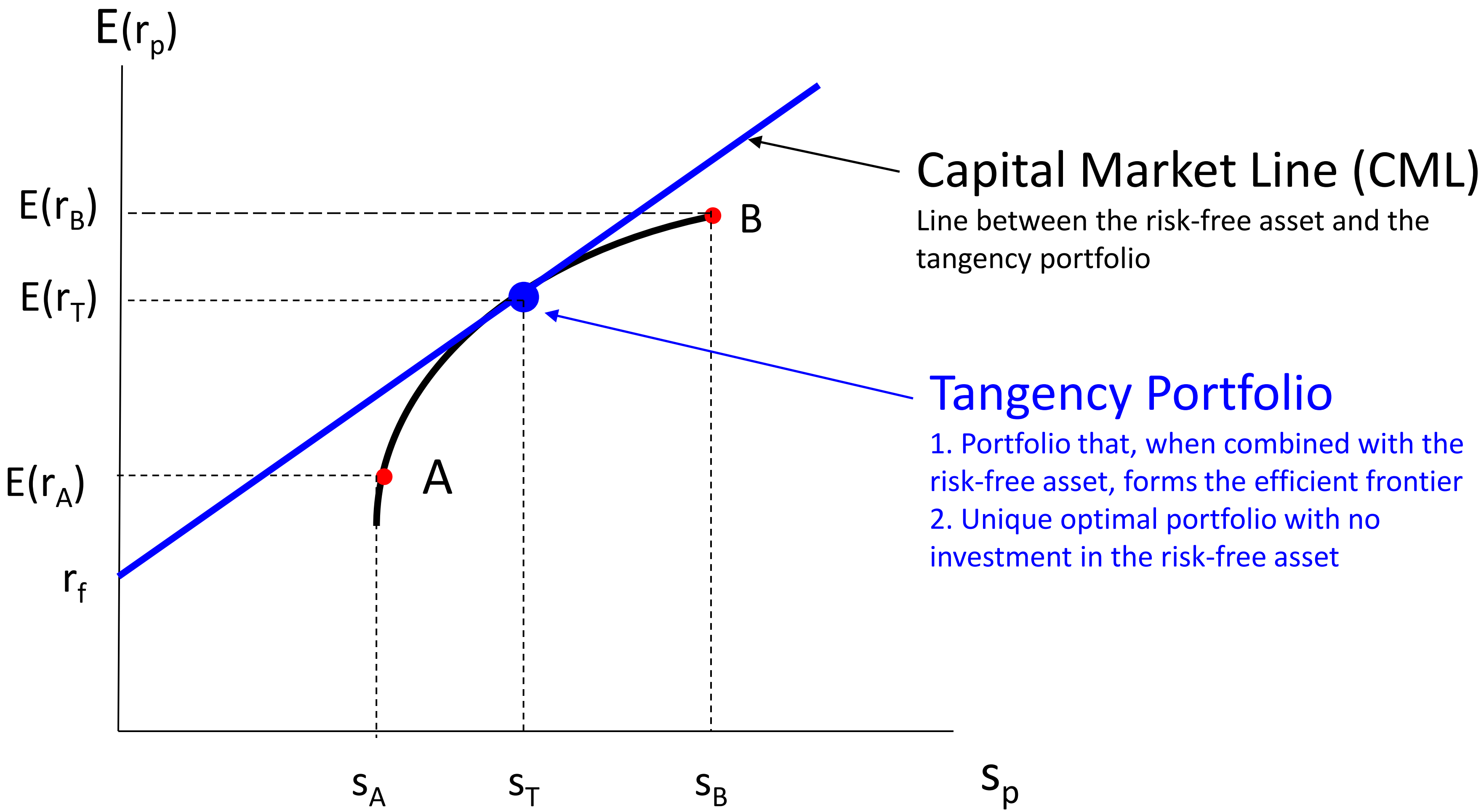
Investing in P and the risk-free asset (here $x = w_P$):



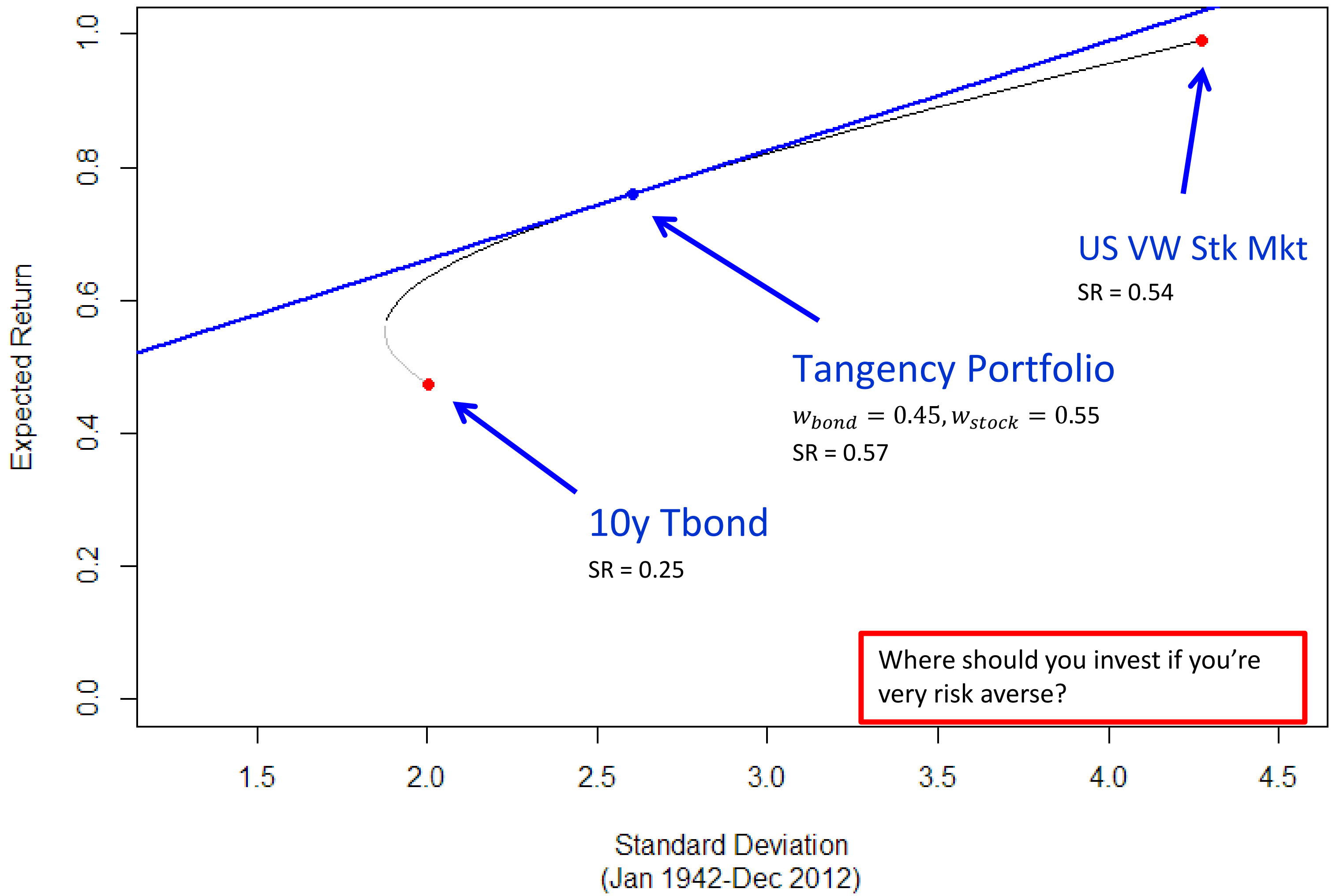
Adding a Risk-free Asset



Efficient frontier with a risk-free asset



Frontier Example (bonds & stocks)



Where should you invest if you're very risk averse?

Finding the tangency portfolio

- The ratio of each investment's risk premium to it's covariance with the tangency portfolio is constant

$$\frac{\bar{r}_s - r_f}{cov(\tilde{r}_s, \tilde{r}_t)} \text{ identical for all investments}$$

- Why?
  - Ratio of a stock's return to *marginal variance*
  - If stock B's ratio were higher than stock A's, could add stock B and remove stock A and produce a better portfolio that the tangency portfolio—which cannot be true.

- So if you found the portfolio where the above property is true, then it is the tangency portfolio

Capital Market Line equation

- Expected return and standard deviation are weighted averages of those of the constituent assets

$$\bar{R}_p = w_{rf} r_f + w_t \bar{r}_t$$
$$\sigma_p = w_t \sigma_t$$

- This implies

$$w_t = \frac{\sigma_p}{\sigma_t} \Rightarrow w_{rf} = 1 - \frac{\sigma_p}{\sigma_t}$$

- Substituting into the expected return equation

$$\bar{R}_p = \left(1 - \frac{\sigma_p}{\sigma_t}\right) r_f + \frac{\sigma_p}{\sigma_t} \bar{r}_t = r_f + \frac{\left(\bar{r}_t - r_f\right)}{\sigma_t} \sigma_p$$

Sharpe Ratio

- A Sharpe Ratio is defined as

$$\frac{E[\tilde{r}] - r_f}{\sigma}$$

  – Measures the trade-off between risk and return

- Thus the slope of the CML is the Sharpe Ratio of the tangency portfolio.
- Example ratios (annual, 1927-2011)

	Small-cap	Mid-cap	Large-cap
Mean return	.169	.148	.113
R – Rf	.132	.112	.077
SD(R-Rf)	.350	.268	.200
Sharpe Ratio	0.378	0.417	0.383

- What does this imply about the slope of the CML?

Implication of Capital Market Line

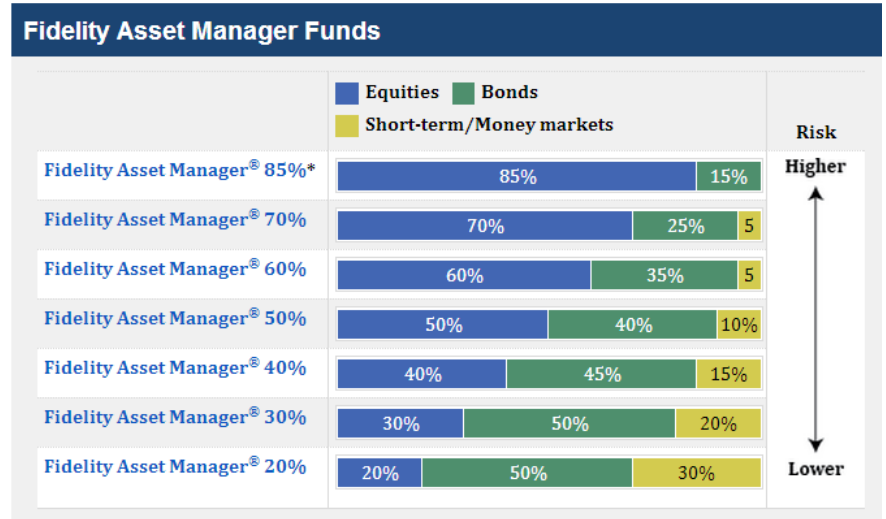
- Mean-variance optimizers will all hold portfolios on the capital market line.
  - Very risk-averse investors—put a lot of weight on the risk-free asset
  - Slightly Risk-averse investors—more weight on the tangency portfolio

- This is two-fund separation.

Two-fund separation in practice?

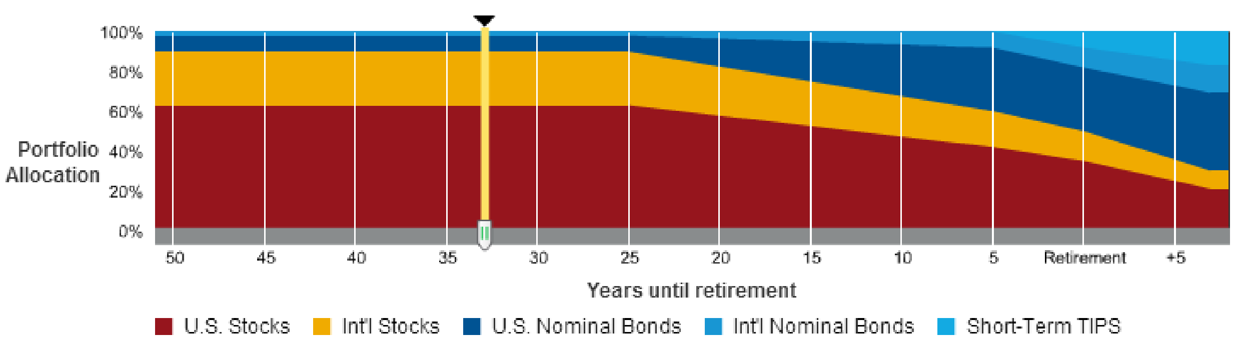
Many asset allocation recommendations do not follow two-fund separation

- Canner, Mankiw, Weil (1997 AER)



Vanguard Target Retirement Fund asset allocation tool

Move the slider to see the fund's asset allocation become more conservative as you approach retirement.



Intuition for life-cycle funds?

- The Vanguard target funds shift from stocks to bonds as the investor ages.
- What is the popular explanation for holding fewer stocks when you are older?

- A model-based explanation
  - Assume labor income is not very risky.
  - As you get older, your wealth attributable to the (safe) labor income declines. So to maintain constant risk you need to increase the weight on less risky assets.
  - See Cocco, Gomes, Maenhout (2005 RFS).

Issues when implementing Mean-Variance Optimal Portfolios

ADVANCED PORTFOLIO THEORY

How well does MVA work in practice?

- Perform an implementable test of optimization:
  - 10y-bond/Stocks 1942-2012
  - Every month, estimate based on past 10 years.
  - Form portfolio with same return as past equal-weighted average.
  - Hold for 1-month, after skipping a month between portfolio formation and holding period.
- Sharpe ratios for Mean-Variance vs. Equal-Weight
  - MV: 0.53
  - EW: 0.49

  (In-sample SR = 0.55)
- Markowitz looks pretty good!

Add an asset far from frontier

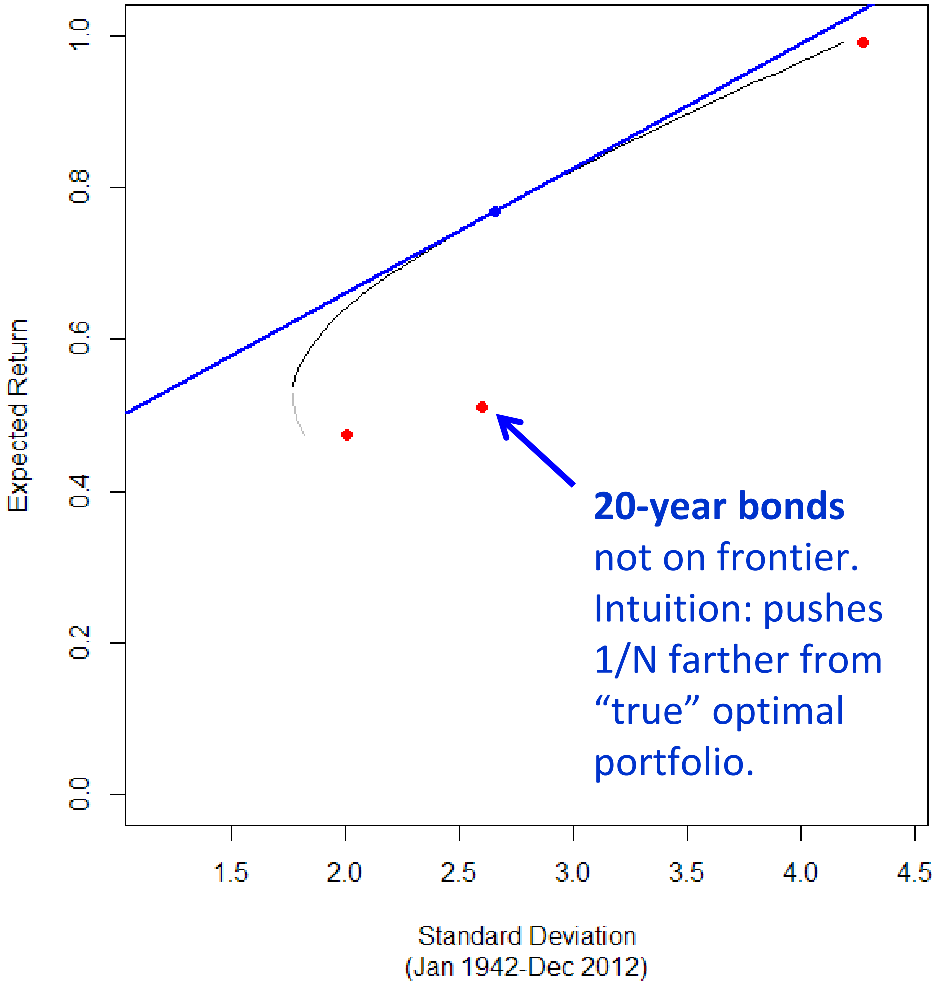
This makes optimization look better.

New 3 asset SR vs. Old 2 asset SR

MV: 0.54 vs. 0.53

EW: 0.45 vs. 0.49

(In Sample MV SR = 0.55)



Take-away from Bond + Stock example

- With only a few assets, optimization work well.
  - The Markowitz-style optimal portfolio had a higher (out of sample) Sharpe ratio than the naïve (1/N) portfolio.

- This example benefits from two things:
  - Long time series
  - Few covariances to estimate

- If the time series is shorter or we have more assets, problems arise …

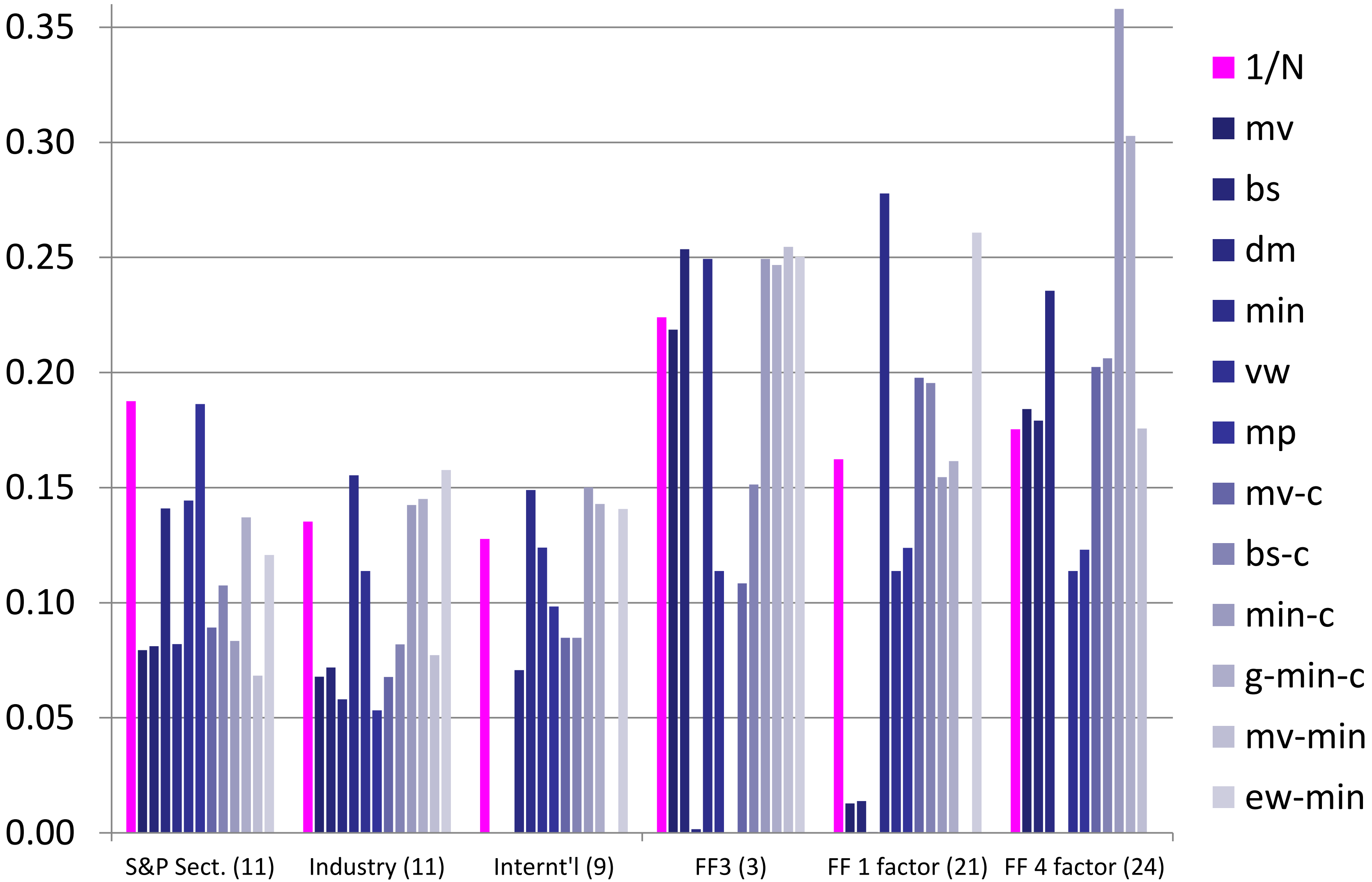
DeMiguel, Garlappi, Uppal (2009)

- What happens when you apply optimization in more complex settings?
  - More assets
  - Shorter time-series
- The authors compare 1/N to the classic mean-variance portfolio and more complicated variants.

Table 1
List of various asset-allocation models considered

#	Model	Abbreviation
Naive		
0.	$1/N$ with rebalancing *(benchmark strategy)*	ew or $1/N$
Classical approach that ignores estimation error		
1.	Sample-based mean-variance	mv
Bayesian approach to estimation error		
2.	Bayesian diffuse-prior	Not reported
3.	Bayes-Stein	bs
4.	Bayesian Data-and-Model	dm
Moment restrictions		
5.	Minimum-variance	min
6.	Value-weighted market portfolio	vw
7.	MacKinlay and Pastor's (2000) missing-factor model	mp
Portfolio constraints		
8.	Sample-based mean-variance with shortsale constraints	mv-c
9.	Bayes-Stein with shortsale constraints	bs-c
10.	Minimum-variance with shortsale constraints	min-c
11.	Minimum-variance with generalized constraints	g-min-c
Optimal combinations of portfolios		
12.	Kan and Zhou's (2007) "three-fund" model	mv-min
13.	Mixture of minimum-variance and $1/N$	ew-min
14.	Garlappi, Uppal, and Wang's (2007) multi-prior model	Not reported

Out-of-sample Sharpe Ratios



Take aways

- No method consistently better than 1/N
- How long a time-series for MV dominate 1/N?
  - Need 3000 months with 25 assets!
  - 6000 months with 50 assets!
- When will more complex methods dominate?
  - If estimation window is long
  - If efficient Sharpe ratio far from 1/N Sharpe ratio
  - When the number of assets is small

Example with random stocks:

- Test set-up
  - Estimate mean-variance optimal portfolios (MV) and compare Sharpe Ratio (SR) to 1/N
  - Time period: 1/1985-9/2013
  - Weekly returns: MMM, GE, BA, WMT, MCD, XRX, XOM, CAT, AXP, VZ, AAPL, KO
  - Estimation based on the past 156 weeks of data
- In-sample performance
  - Evaluates the model in the same data sample where the parameters were estimated.
  - This is like estimating what the best portfolio to hold in 2013 was, and then check how well you would have done if you had bought that portfolio in 2013.
- Out-of-sample performance
  - Evaluates the model in a data sample that is different from the one where the parameters were estimated.
  - This is like a trading strategy—first estimate the best portfolio for 2014 using 2013 data, next buy that portfolio and at the end of 2014 see how well you did.

- IN-SAMPLE SR:
  - MV: 0.71
  - MV long-only: 0.57
  - 1/N: 0.49
- OUT-OF-SAMPLE SR:
  - MV: 0.21
  - MV long-only: 0.48
  - 1/N: 0.49

Where are we?

- We now understand how to make capital allocation decisions ***given*** the required inputs:
  - Expected returns
  - Volatilities
  - Correlations
- A big part of the task is identifying the tangency portfolio. Difficult!
- Where do we get the inputs?
  - Historical information
  - ***Models***
- Next time: the CAPM

Key Terms

- Minimum variance frontier
- Diversification
- Idiosyncratic risk
- Systematic risk
- Tangency portfolio
- Capital Market Line
- Sharpe ratio

APPENDIX: ADVANCED PORTFOLIO SELECTION

Advanced: (μ,σ^2)-efficient Portfolio

- To compute the mean-variance efficient (MVE), or tangency portfolio, maximize the Sharpe ratio:

$$\max_{w} \frac{E\left(r_p\right)-r_f}{\sigma_p}$$

where
$$E\left(r_p\right) = wE\left(r_A\right) + (1-w)E\left(r_B\right)$$
$$\sigma_p = \left[w^2\sigma_A^2 + (1-w)^2\sigma_B^2 + 2w(1-w)\rho_{A,B}\sigma_A\sigma_B\right]^{1/2}$$

- Solution to this is ugly?

Solution of the MVE problem

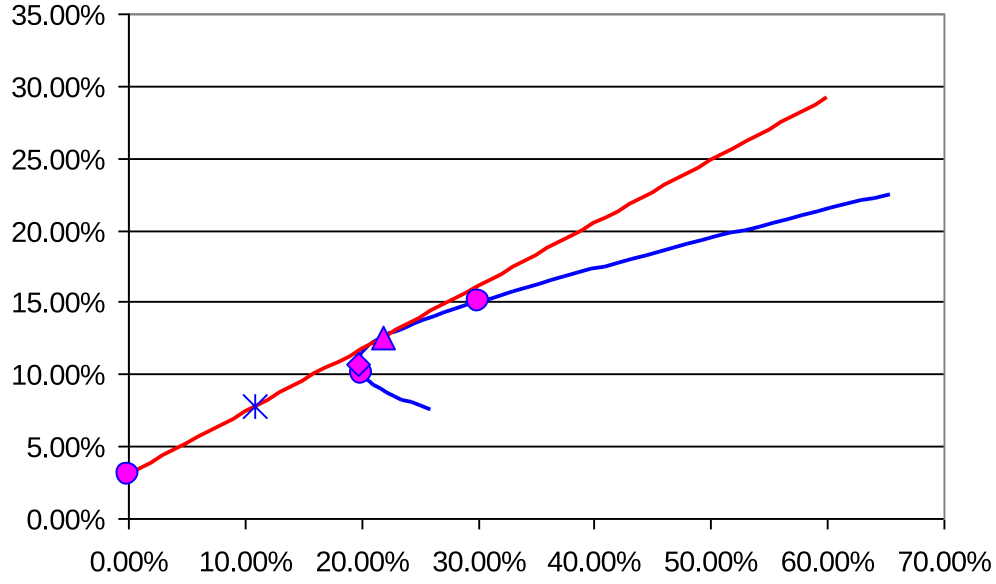
- The tangency portfolio weight for A:

$$w_T = \frac{E(r_A^e)\sigma_B^2 - E(r_B^e)\rho_{A,B}\sigma_A\sigma_B}{E(r_A^e)\sigma_B^2 + E(r_B^e)\sigma_A^2 - [E(r_A^e) + E(r_B^e)]\rho_{A,B}\sigma_A\sigma_B}$$

- Where r^e = excess returns:

$$E(r_i^e) = E(r_i) - r_f \quad i = A,\ B$$

	Expected Return	Standard Deviation
Riskless Rate	3%	0%
Risky Asset 1	10%	20%
Risky Asset 2	15%	30%
Correlation	0.5	
Risk Aversion	4	
Global		
Minimum Variance	10.75%	19.64%
Weight of Asset 1	0.85	
Tangency	12.50%	21.79%
Weight of Asset 1	0.5	
Your portfolio	7.75%	10.90%
Weight of Tangency Portfolio	0.5	



Numerical Example

- Tangency portfolio weights:

$$w_T = \frac{\underbrace{(0.07)}_{E(r_A^e)} \cdot \underbrace{0.30^2}_{\sigma_B^2} - \underbrace{(0.12)}_{E(r_B^e)} \cdot \underbrace{0.5 \cdot 0.2 \cdot 0.3}_{\rho_{A,B}\sigma_A\sigma_B}}{\underbrace{(0.07)}_{E(r_A^e)} \cdot \underbrace{0.30^2}_{\sigma_B^2} + \underbrace{(0.12)}_{E(r_B^e)} \cdot \underbrace{0.20^2}_{\sigma_A^2} - [\underbrace{(0.07)}_{E(r_A^e)} + \underbrace{(0.12)}_{E(r_B^e)}] \cdot \underbrace{0.5 \cdot 0.2 \cdot 0.3}_{\rho_{A,B}\sigma_A\sigma_B}} = .5$$

- How much you should put in B?

The tangency portfolio

- Given the weights, we can compute the risk and return of the tangency portfolio:

$$E(r_T) = w_T E(r_A) + (1 - w_T) E(r_B)$$
$$= 0.5 \cdot 0.10 + 0.5 \cdot 0.15 = 0.125 = 12.5\%$$

$$\sigma_T = \left[w^2 \sigma_A^2 + (1-w)^2 \sigma_B^2 + 2w(1-w)\rho_{A,B}\sigma_A\sigma_B \right]^{1/2} = 21.79\%$$

Maximal Sharpe Ratio?

- This portfolio should have the maximal Sharpe ratio

- Does it? Compute Sharpe Ratios

The Sharpe Ratio

- Sharpe ratios of the stocks A and B

$$SR_A = \frac{E(r_A)-r_f}{\sigma_A} = \frac{0.10-0.03}{0.20} = 0.35$$

$$SR_B = \frac{E(r_B)-r_f}{\sigma_B} = \frac{0.15-0.03}{0.30} = 0.40$$

- Sharpe ratio of the tangency portfolio

$$SR_T = \frac{E(r_T)-r_f}{\sigma_T} = \frac{0.125-0.03}{0.2179} = 0.4359$$

Capital Allocation: modelling the trade-off between risk and return

- Utility theory quantifies the subjective desirability of something
  - Investors choose the portfolio that delivers the highest utility
- A model of mean-variance utility

$$u(r_t) = -e^{-Ar}$$

  - If returns are normally distributed

$$E[u(r_t)] = -e^{-A\left(\bar{r}-\frac{A}{2}\sigma^2\right)}$$

  - Maximizing is equivalent to maximizing

$$u(r_t) = \bar{r} - \frac{A}{2}\sigma^2$$

  - *A* measures the investor's level of risk aversion:
    - $A > 0$: risk averse
    - $A = 0$: risk neutral
    - $A < 0$: risk seeking

Optimal Portfolios

- CML gives possible risk-return trade-offs
- Find the portfolio weight that maximizes utility
- ***Formally:***

$$\boldsymbol{U^* = \max_w \left(E(r_p) - 0.5A\mathrm{var}(r_p)\right)}$$

$$\boldsymbol{= \max_w \left(w(\bar{r}_T - r_f) + r_f - 0.5Aw^2\sigma_T^2\right)}$$

Optimal weight?

- Problem is well defined because utility is a quadratic function of the portfolio weight
- To find the maximum, take derivative and set equal to zero to get the optimal weight in the tangency portfolio:

$$w^* = \frac{\bar{r}_T - r_f}{A\sigma_T^2}$$

Interpretation of Portfolio Rule

- The risky asset holdings are larger when
  - Risk premium $(\bar{r}_T - r_f)$ is larger
  - Investor is less risk averse (A is smaller)
  - Volatility (σ_T^2) is lower

$$w^* = \frac{\bar{r}_T - r_f}{A\sigma_T^2}$$

Example: VW US MKT (1927-2012)

- Assume means and variances are constant over time
  - r_f = 4%,
  - E(r) = 11%,
  - σ(r) = 19%

$$w^* = \frac{\bar{r}_T - r_f}{A\sigma_T^2}$$

A	**w***	**E(r_p)**	**σ**
0.5	3.88	31.15	73.68
1.0	1.94	17.57	36.84
2.0	0.97	10.79	18.42
4.0	0.48	7.39	9.21
8.0	0.24	5.70	4.61