Homework 4: Homographies

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1 Theory

2 Introduction

2.1 Planar Homographies: Theory Warm Up

Suppose we have two cameras C_1 and C_2 looking at a common plane Π in 3D space. Any 3D point P on Π generates a projected 2D point located at $p \cong (u_1, v_1, 1)^T$ on the first camera, C_1, and $q \cong (u_2, v_2, 1)^T$ on the second camera, C_2. Since P is confined to the plane Π, we expect that there is a relation ship between p and q. In particular, there exists a common 3×3 matrix H, so that for any P, the following condition holds:

$$p \cong Hq\,. \tag{1}$$

We call this relationship *planar homography*. Recall that both p and q are in homogeneous coordinates and the equality $\cong$ means p is proportional to Hq (recall homogeneous coordinates). It turns out this relationship is also true for cameras that are related by pure rotation without the planar constraint.

Question

We have a set of points $p = \{p_1, p_2, \ldots, p_N\}$ in an image taken by camera C_1 and corresponding points $q = \{q_1, q_2, \ldots, q_N\}$ in an image taken by C_2. Suppose we know there exists an unknown homography H between corresponding points for all $i \in \{1, 2, \ldots, N\}$. This formally means that $\exists H$ such that:

$$p^i \cong Hq^i\,, \tag{2}$$

where $p^i = (x_i, y_i, 1)$ and $q^i = (u_i, v_i, 1)$ are homogeneous coordinates of image points each from an image taken with C_1 and C_2, respectively.

1. Given N correspondences in p and q and using Equation 1, derive a set of $2N$ independent linear equations in the form:

   $$Ah = 0\,, \tag{3}$$

   where h is a vector of the elements of H, and A is a matrix composed of elements derived from the point coordinates. Write down an expression for A?

2. How many elements are there in h?

3. How many point pairs (correspondences) are required to solve this system? Why?

Solution

Writing Equation 2 more explicitly, we achieve that

$$\begin{bmatrix} u_i \\ v_i \\ 1 \end{bmatrix} \cong \begin{bmatrix} w \cdot u_i \\ w \cdot v_i \\ w \end{bmatrix} = \begin{bmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{bmatrix} \begin{bmatrix} x_i \\ y_i \\ 1 \end{bmatrix}\,. \tag{4}$$

Notice $(u_i, v_i, 1)^T$ is projectively similar to $(w \cdot u_i, w \cdot v_i, w)^T$. We then expand the matrix multiplication and divide out the unknown scale factor, w, and achieve a pair of equations for each point:

$$\begin{aligned} u_i &= \frac{h_1 x_i + h_2 y_i + h_3}{h_7 x_i + h_8 y_i + h_9} \\ v_i &= \frac{h_4 x_i + h_5 y_i + h_6}{h_7 x_i + h_8 y_i + h_9} . \end{aligned} \tag{5}$$

We form linear equations:

$$\begin{aligned} h_1 x_i + h_2 y_i + h_3 - u_i(h_7 x_i + h_8 y_i + h_9) = 0 \\ h_4 x_i + h_5 y_i + h_6 - v_i(h_7 x_i + h_8 y_i + h_9) = 0 \, , \end{aligned} \tag{6}$$

and re-write the equations to a matrix form to achieve an homogeneous set of equations, as follows:

$$\begin{bmatrix} x_i & y_i & 1 & 0 & 0 & 0 & -u_i x_i & -u_i y_i & -u_i \\ 0 & 0 & 0 & x_i & y_i & 1 & -v_i x_i & -v_i y_i & -v_i \end{bmatrix} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \\ h_9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} . \tag{7}$$

This equation, however, is for a single coordinate. For N coordinates, the following set of $2N$ independent linear equations is formed:

$$\begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 & -u_1 x_1 & -u_1 y_1 & -u_1 \\ 0 & 0 & 0 & x_1 & y_1 & 1 & -v_1 x_1 & -v_1 y_1 & -v_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_N & y_N & 1 & 0 & 0 & 0 & -u_N x_N & -u_N y_N & -u_N \\ 0 & 0 & 0 & x_N & y_N & 1 & -v_N x_N & -v_N y_N & -v_N \end{bmatrix}_{\mathrm{2N\times 9}} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \\ h_7 \\ h_8 \\ h_9 \end{bmatrix}_{9\times 1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}_{\mathrm{2N\times 1}} , \tag{8}$$

where the first, $2N \times 9$, matrix corresponds to the A matrix, and the second matrix, h, has 9 elements.

To solve this system, 4 points are required, one pair of equations for each point — a total of 8 equations. Even though there are 9 elements in h, there's redundancy due to the scale introduces by the homogeneous coordinates. Therefore, there are only 8 degrees of freedom.

2.2 Planar Homographies: Practice

See Python implementation.

3 Creating Your Augmented Reality Application

See Python implementation.