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Asymptotics of Daubechies Filters, Scaling Functions, and Wavelets

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We study the asymptotic form as $p \to \infty$ of the Daubechies orthogonal minimum phase filter $h_p[n]$, scaling function $\phi_p(t)$, and wavelet $w_p(t)$. Kateb and Lemarié calculated the leading term in the phase of the frequency response $H_p(\omega)$. The infinite product $\hat{\phi}_p(\omega) = \prod H_p(\omega/2^k)$ leads us to a problem in stationary phase, for an oscillatory integral with parameter t. The leading terms change form with $\tau = t/p$ and we find three regions for $\phi_p(\tau)$:

(1) An Airy function up to near τ_0: $\sqrt[3]{42\pi/p}\ \mathrm{Ai}(-\sqrt[3]{42\pi p^2}(\tau - \tau_0)) + o(p^{-1/3})$

(2) An oscillating region $\sqrt{2/\pi p G'(\omega_\tau)} \cos[p(G^{(-1)}(\omega_\tau) - G(\omega_\tau)\omega_\tau) + \frac{\pi}{4}] + o(p^{-1/2})$

(3) A rapid decay after τ_1: $(1/p\pi)(1/(\tau - \tau_1)) \sin[p(G^{(-1)}(\pi) - \tau\pi)] + o(p^{-1})$

The numbers $\tau_0 \simeq 0.1817$ and $\tau_1 \simeq 0.3515$ are known constants. The function G and its integral $G^{(-1)}$ are independent of p. Regions 1 and 2 are matched over the interval $p^{-2/3} \ll \tau - \tau_0 \ll 1$.

The wavelets have a simpler asymptotic expression because the Airy wavefront is removed by the highpass filter. We also find the asymptotics of the impulse response $h_p[n]$—a different function $g(\omega)$ controls the three regions.

The difficulty throughout is to estimate the phase. © 1998 Academic Press

1. INTRODUCTION

Orthogonal wavelets with compact support were announced by Ingrid Daubechies in 1988. For each $p = 1, 2, \ldots,$ she created a wavelet supported on $[0, 2p - 1]$ with p vanishing moments. Our goal is to understand the asymptotic behavior of the filter coefficients and the scaling functions and the wavelets as $p \to \infty$. The construction begins with

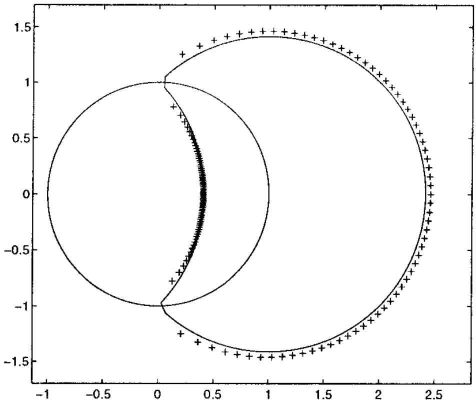

FIG. 1. H_{70} has 70 zeros at $z = -1$ (not shown in the graph) and 69 zeros inside the unit circle. Those outside the unit circle in the graph are their reciprocals.

the "maxflat minphase lowpass filter" of length $2p$. From its coefficients $h_p[k]$ we form the transfer function $H_p(z)$, and there are four main steps to analyze as $p \to \infty$:

(1) The $2p - 1$ zeros of $H_p(z) = \sum_k h_p[k]z^{-k}$.

(2) The phase of $H_p(z)$ on the unit circle $z = e^{i\omega}$.

(3) The scaling function $\phi_p(t)$ with Fourier transform $\prod_{k=1}^{\infty} H_p(\omega/2^k)$.

(4) The wavelet $w_p(t) = \sum_k (-1)^k h_p[2p - 1 - k]\phi_p(2t - k)$.

This paper brings steps 3 and 4 near to completion, building on the Kateb–Lemarié analysis of step 2. The phase is of crucial importance because orthogonal filters cannot be symmetric (beyond the Haar case $p = 1$). We show that the filter coefficients and the scaling functions have similar asymptotic behavior (but not identical! See Section 6).

The zeros of $H_{70}(z)$ are shown in Fig. 1. There are 70 zeros at $z = -1$, or $\omega = \pi$, which makes the function "maxflat." The other 69 zeros are inside the unit circle, which makes it "minphase." The graph of $|H_{70}(\omega)|$ shows that the filter is "lowpass"; the magnitude is near zero for high frequencies. This graph approaches the ideal one-zero function as $p \to \infty$. Then the magnitude of the infinite product $\hat{\phi}_p(\omega) = \prod_{k=1}^{\infty} H_p(\omega/2^k)$ approaches the characteristic function of $[-\pi, \pi]$.

The zeros inside the circle were analyzed by Kateb and Lemarié [3] and (independently but later) by Shen and Strang [8; 9, pp. 166–170]. The zeros approach the circular arc $|z + 1| = \sqrt{2}$ from inside. Their distribution on the $w = (z - z^{-1})^2/4$ plane is asymptotically uniform along the unit circle. These are half of the zeros (the other half are the reciprocals z^{-1} outside the unit circle) of a simple polynomial with integer coefficients.

From the two asymptotic formulas for the zeros (along the circular arc and near the end points), Kateb and Lemarié found the leading term in the phase $\arg(H_p(\omega))$. They multiplied the $2p - 1$ linear factors and added phases. The result is naturally expressed in terms of the *group delay,*

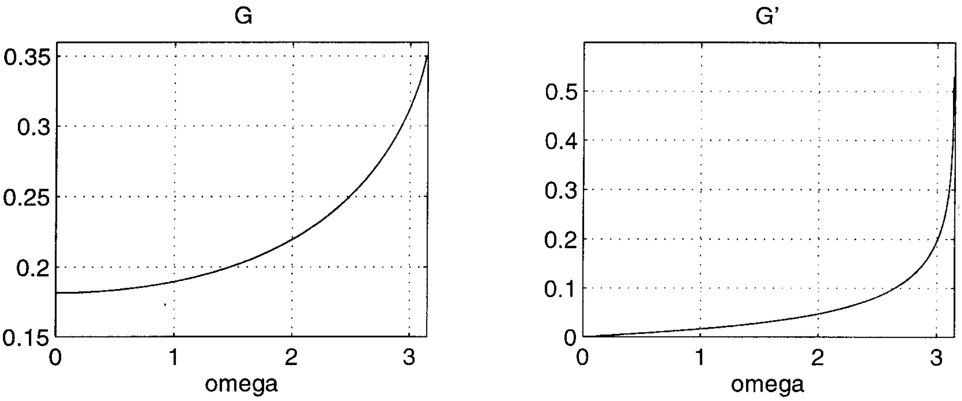

FIG. 2. $G(\omega)$ and $G'(\omega)$.

$$\mathrm{grd}(H_p(\omega)) = -\frac{d}{d\omega}(\text{phase of } H_p(\omega)) = pg(\omega) + O(p^{1/2}), \tag{1}$$

with

$$g(\omega) = \frac{1}{2} + \frac{1}{2\pi}\frac{\cos\omega}{\sin\omega}\ln\frac{1-\sin\omega}{1+\sin\omega}. \tag{2}$$

(The $\frac{1}{2}$ term was not in Kateb and Lemarié's paper [3] and appears here because we shifted the highpass filter to make it causal.) This even function $g(\omega)$ is analytic and convex on $(-\pi, \pi)$. Its Taylor expansion around $\omega = 0$ is $(1/2 - 1/\pi) + \omega^2/6\pi + O(\omega^4)$. Its derivative is infinite at $\omega = \pm\pi$.

Our step 3 in the analysis works with the infinite product $\hat{\phi}_p(\omega) = \prod_{k=1}^{\infty} H_p(\omega/2^k)$. The phases add, and the derivative for the group delay contributes a factor $1/2^k$. This makes the infinite sum converge,

$$\mathrm{grd}(\hat{\phi}_p(\omega)) = pG(\omega) + O(p^{1/2}), \tag{3}$$

with

$$G(\omega) = \sum_{k=1}^{\infty}\frac{1}{2^k}g\left(\frac{\omega}{2^k}\right). \tag{4}$$

The function $G(\omega)$ and its derivative are shown in Fig. 2. The series for $G(\omega)$ gives $G(0) = g(0) = 1/2 - 1/\pi$ and $G''(0) = g''(0)/7 = 1/21\pi$. The numbers $\tau_0 = G(0) \simeq 0.1817$ and $\tau_1 = G(\pi) \simeq 0.3515$ will be called the transition time in the eventual asymptotic formula for $\phi_p(\tau)$, with $\tau = t/p$.

We note an important difference in the time scale t/p, compared to the asymptotics of B-splines (see Unser *et al.* [4]). The splines are symmetric. They approach Gaussians with scaling $t/\sqrt{p}$. The spline wavelets approach cosine-modulated Gaussians on that scale, too. The central limit theorem is at work. Our problem requires a further step, and the technical tool will be the method of *stationary phase.* This enters when we invert the Fourier transform:

$$\phi_p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\phi}_p(\omega)e^{it\omega}d\omega \simeq \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ipG^{(-1)}(\omega)}e^{it\omega}d\omega = \Phi_p(t). \tag{5}$$

The scaled phase is approximately $G^{(-1)}(\omega) = \int_0^{\omega} G(\theta)d\theta$. Our main task is to justify this approximation $\phi_p(t) \simeq \Phi_p(t)$ based on magnitude and phase. Then we analyze the asymptotics of $\Phi_p(t)$ as $p \to \infty$. The results are summarized in our abstract.

Throughout this paper, ‘‘$A \simeq B$’’ means that A and B share the same leading term (when expanded in terms of a certain asymptotic parameter). The symbol ‘‘$a \ll 1$’’ means that a is small enough (this usually can be characterized by some asymptotic parameter). We refer to [1, 5, 7, 11] for a full theory of the asymptotic analysis of integrals.

2. ACCURACY OF APPROXIMATIONS

We define the following approximations to $\phi_p(t)$:

$$\phi_p^c(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{\phi}_p(\omega)e^{it\omega}d\omega \qquad \text{(frequency limited to } |\omega| \le \pi) \tag{6}$$

$$\psi_p(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\arg(\hat{\phi}_p)}e^{it\omega}d\omega \qquad \text{(magnitude taken as 1)} \tag{7}$$

$$\Phi_p(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ipG^{(-1)}(\omega)}e^{it\omega}d\omega \qquad \text{(leading term of phase).} \tag{8}$$

The integral $G^{(-1)}(\omega) = \int_0^{\omega} G(\theta)d\theta$ approximates the phase. Our main goal in this section is to justify these approximations. And in the next two sections, we will give the asymptotic analysis of $\Phi_p(t)$.

2.1. Spectrum of the Scaling Function $\phi_p(t)$

We take the Fourier transform and its inverse to be

$$\hat{\phi}(\omega) = \int_{-\infty}^{\infty} \phi(t)e^{-it\omega}dt \qquad \text{and} \qquad \phi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\phi}(\omega)e^{it\omega}d\omega.$$

Therefore, for any two square integrable functions $f(t)$ and $g(t)$,

$$\langle f(t), g(t)\rangle = \frac{1}{2\pi}\langle \hat{f}(\omega), \hat{g}(\omega)\rangle. \tag{9}$$

For each p, let $E_p(\omega) = |\hat{\phi}_p(\omega)|^2$ and $P_p(\omega) = |H_p(e^{i\omega})|^2$. In order to use $P_p(\omega)$ to estimate the L_q-norm of $\hat{\phi}_p(\omega)$, we need a detailed description of its behavior outside a finite spectral interval. It is known that asymptotically $\phi_p \in C^{-\mu p}$ with μ

$\simeq 0.2$, so that $\hat{\phi}_p$ decays like $\omega^{-\mu p}$ at infinity (see Daubechies [2, Chapter 7]). But this is not sufficient to justify (in the sense of L_q-norm) our attempt to drop the spectrum of $\phi_p(t)$ outside $[-\pi, \pi]$ without significant loss of energy. Meyer provides the right bound, even though it is rougher in terms of the estimation of regularity compared to the results in Daubechies [2, Chapter 7].

LEMMA 1 (See Meyer [6, p. 103]). *There exists a positive number* α, *such that* $E_p(\omega) \leq 2^{-2\alpha pj}$ *for any* p *and any* $\omega \in [\frac{\pi}{2} 2^j, \pi 2^j]$, $j = 0, 1, \ldots$. *Therefore* $E_p(\omega) \leq (\pi/\omega)^{\alpha p}$ *for any positive* ω.

COROLLARY 1. *For any positive* q *and* $0 < \epsilon \ll 1$,

$$\|\hat{\phi}_p\|_{L_q(\mathbb{R})} = \|\hat{\phi}_p\|_{L_q[-\pi-\epsilon,\pi+\epsilon]} + \textit{exponentially small term} \qquad \textit{as } p \to \infty \tag{10}$$

LEMMA 2. $0 < \epsilon \ll 1$. *There exists* $\delta_\epsilon \in (0, 1)$ *such that*

$$E_p(\omega) \begin{cases} \leq P_p\left(\dfrac{\omega}{2}\right) = O(\delta_\epsilon^p) & \omega \in [\pi + \epsilon, 2\pi] \\ \\ = P_p\left(\dfrac{\omega}{2}\right)(1 + O(\delta_\epsilon^p)) & \omega \in [0, \pi + \epsilon] \end{cases}. \tag{11}$$

Proof. (i) $P_p(\omega)$ has the following nice integral form (See Meyer [6] or Strang and Nguyen [9, p. 168]),

$$P_p(\omega) = 1 - R_p(\omega), \qquad \text{with } R_p(\omega) = c_p^{-1} \int_0^\omega \sin^{2p-1}\theta d\theta,$$

where c_p is the constant that makes P_p vanish at π. Stirling's formula shows that c_p has the leading term $\sqrt{\pi/p}$. Note $P_p(\omega) = R_p(\pi - \omega)$ is the mirror image of $R_p(\omega)$ with respect to $\omega = \pi/2$. Since $\frac{2}{\pi}\theta \leq \sin\theta \leq \theta$ on $[0, \pi/2]$, we have

$$R_p(\omega) = c_p^{-1} \int_0^\omega \sin^{2p-1}\theta d\theta \leq c_p^{-1}\omega \sin^{2p-1}\omega \leq c_p^{-1} \frac{\pi}{2} \sin^{2p}\omega. \tag{12}$$

(12) also gives $R_p(\omega) \leq c_p^{-1}\omega^{2p}$.

(ii) Noticing that $E_p(\omega) = \prod_{k=1}^{\infty} P_p(\omega/2^k)$ and $\prod_{k=1}^{K} (1 - x_k) \geq 1 - \sum_{k=1}^{K} x_k$, for any $x_k \in [0, 1]$, one has, for any $\omega \in [0, \pi + \epsilon]$,

$$E_p(\omega) = P_p\left(\frac{\omega}{2}\right) \prod_{k=2}^{\infty} \left(1 - R_p\left(\frac{\omega}{2^k}\right)\right)$$

$$\geq P_p\left(\frac{\omega}{2}\right)\left(1 - c_p^{-1} \sum_{k=2}^{\infty} \frac{\omega^{2p}}{4^{pk}}\right) = P_p\left(\frac{\omega}{2}\right)\left(1 - \frac{c_p^{-1}}{1 - 4^{-p}} \frac{\omega^{2p}}{4^{2p}}\right). \tag{13}$$

For $\omega \in [\pi + \epsilon, 2\pi]$,

$$E_p(\omega) \leq P_p\left(\frac{\omega}{2}\right) = R_p\left(\pi - \frac{\omega}{2}\right) \leq c_p^{-1} \frac{\pi}{2} \cos^{2p} \frac{\epsilon}{2}. \tag{14}$$

By (13) and (14), any $\delta_\epsilon \in (\max[(\pi + \epsilon/4)^2, \cos^2 \frac{\epsilon}{2}], 1)$ makes (11) true. ∎

LEMMA 3. *For any positive q,*

$$\int_0^{\pi/2} R_p^q(\theta) d\theta = O(p^{-1/2}) \qquad \textit{as } p \to \infty. \tag{15}$$

Proof. For $0 \leq \omega \ll 1$,

$$R_p\left(\frac{\pi}{2} - \omega\right) = \frac{1}{2} - c_p^{-1} \int_0^\omega \cos^{2p-1}\theta d\theta \simeq \frac{1}{2} - \sqrt{\frac{p - 1/2}{\pi}} \int_0^\omega e^{-(p-1/2)\theta^2} d\theta$$

$$= \frac{1}{2} - \frac{1}{\sqrt{\pi}} \int_0^{\sqrt{p-1/2}\,\omega} e^{-\theta^2} d\theta = \frac{1}{2} \operatorname{erfc}(\sqrt{p - 1/2}\,\omega),$$

where the complementary error function is defined by

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-t^2} dt.$$

Since $R_q(\omega)$ has a boundary layer near $\omega = \pi$, we have (for any $q > 0$)

$$\int_0^{\pi/2} [R_p(\omega)]^q d\omega = \int_0^{\pi/2} \left[R_p\left(\frac{\pi}{2} - \omega\right) \right]^q d\omega \simeq \int_0^\epsilon \left[R_p\left(\frac{\pi}{2} - \omega\right) \right]^q d\omega$$

$$= \frac{1}{2^q} \int_0^\epsilon [\operatorname{erfc}(\sqrt{p - 1/2}\,\omega)]^q d\omega$$

$$= \frac{1}{2^q \sqrt{p - 1/2}} \int_0^{\sqrt{p-1/2}\,\epsilon} [\operatorname{erfc}(\theta)]^q d\theta = O(p^{-1/2}). \quad ∎$$

Now we can analyze the accuracy of our approximations.

2.2. *Approximating $\phi_p(t)$ with $\phi_p^c(t)$*

THEOREM 1. *Let $r_p(t) = \phi_p(t) - \phi_p^c(t)$. Then,*

$$\|r_p(t)\|_{L_\infty(\mathbb{R})} = O(p^{-1/2}) \tag{16}$$

$$\|r_p(t)\|_{L_2(\mathbb{R})} = O(p^{-1/4}). \tag{17}$$

Proof. By definition, $\hat{r}_p(\omega)$ is just the truncated spectrum of $\hat{\phi}_p(\omega)$ on $\mathbb{R}\backslash[-\pi, \pi]$. By the corollary of Lemma 1, the L_q norm of $\hat{r}_p$ is determined by its restriction on $[-2\pi, -\pi] \cup [\pi, 2\pi]$ up to a p-exponentially small error. Then,

$$\begin{aligned}
\|\hat{r}_p\|_{L_q(\mathbb{R})} &\simeq 2^{1/q}\|\hat{r}_p\|_{L_q[\pi,2\pi]} = 2^{1/q}\|\sqrt{E_p(\omega)}\,\|_{L_q[\pi,2\pi]} \\
&\simeq 2^{1/q}\left\|\sqrt{P_p\left(\frac{\omega}{2}\right)}\,\right\|_{L_q[\pi,2\pi]} && \text{(by Lemma 2)} \\
&= 4^{1/q}\|\sqrt{R_p(\omega)}\,\|_{L_q[0,\pi/2]} && \text{(by mirror relation)} \\
&= O(p^{-1/2q}) && \text{(by Lemma 3).} \qquad (18)
\end{aligned}$$

Then (16) and (17) follow immediately from

$$\|r_p(t)\|_{L_2(\mathbb{R})} = \frac{1}{\sqrt{2\pi}}\,\|\hat{r}_p\|_{L_2(\mathbb{R})} \qquad \text{and} \qquad \|r_p(t)\|_{L_\infty(\mathbb{R})} \leq \frac{1}{2\pi}\,\|\hat{r}_p\|_{L_1(\mathbb{R})}. \blacksquare$$

2.3. *Approximating $\phi_p^c(t)$ with $\psi_p(t)$*

THEOREM 2. *Let $s_p(t) = \phi_p^c(t) - \psi_p(t)$. Then*

$$\|s_p(t)\|_{L_\infty(\mathbb{R})} = O(p^{-1/2}) \qquad (19)$$

$$\|s_p(t)\|_{L_2(\mathbb{R})} = O(p^{-1/4}). \qquad (20)$$

Proof. (i) For (19),

$$\begin{aligned}
|s_p(t)| &= \frac{1}{2\pi}\left|\int_{-\pi}^{\pi}(\hat{\phi}_p - \hat{\phi}_p/|\hat{\phi}_p|)e^{it\omega}d\omega\right| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}|1 - |\hat{\phi}_p|\;|d\omega \\
&\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}(1 - |\hat{\phi}_p|)(1 + |\hat{\phi}_p|)d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}(1 - E_p(\omega))d\omega \\
&\simeq \frac{1}{\pi}\int_0^{\pi}\left(1 - P_p\left(\frac{\omega}{2}\right)\right)d\omega = \frac{2}{\pi}\int_0^{\pi/2}R_p(\theta)d\theta = O(p^{-1/2}),
\end{aligned}$$

where the approximation has an exponentially small error (by Lemma 2). This argument is valid for any real t. Therefore (19) is true.

(ii) For (20),

$$\begin{aligned}
\|s_p\|^2_{L_2(\mathbb{R})} &= \frac{1}{2\pi}\,\|\hat{\phi}_p - \hat{\phi}_p/|\hat{\phi}_p|\;\|^2_{L_2[-\pi,\pi]} = \frac{1}{2\pi}\,\|1 - |\hat{\phi}_p|\;\|^2_{L_2[-\pi,\pi]} \\
&\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}(1 - |\hat{\phi}_p|)^2(1 + |\hat{\phi}_p|)^2d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}[1 - E_p(\omega)]^2d\omega \\
&\simeq \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[1 - P_p\left(\frac{\omega}{2}\right)\right]^2d\omega = \frac{2}{\pi}\int_0^{\pi/2}[R_p(\theta)]^2d\theta = O(p^{-1/2}). \blacksquare
\end{aligned}$$

2.4. Approximating $\psi_p(t)$ with $\Phi_p(t)$ (I)

Both $\psi_p(t)$ and $\Phi_p(t)$ are entirely determined by their phases. By (3), the phase difference has order $O(p^{1/2})$, which is very large in the usual sense. This prevents any attempt to explain their similarity by estimating the L_p norms of their spectra. A new mechanism has to be introduced to explain their close relation. That is the stationary phase method we will discuss in the next section. Before finishing this section, we interpret the results obtained so far.

By Theorems 1 and 2, one has

$$\|\phi_p(t) - \psi_p(t)\|_{L_\infty(\mathbb{R})} = O(p^{-1/2}) \tag{21}$$

$$\|\phi_p(t) - \psi_p(t)\|_{L_2(\mathbb{R})} = O(p^{-1/4}). \tag{22}$$

(21) is not so satisfactory since we will see later that $O(p^{-1/2})$ itself is the characteristic magnitude of the scaling function $\phi_p(t)$ (see Section 4.1). However, with the help of (22), one can show that the set on which $|\phi_p(t) - \psi_p(t)|$ reaches $O(p^{-1/2})$ is small. The exact statement is Theorem 3.

THEOREM 3. *For any positive α and C, we define*

$$A_{\alpha,C} = \{t \in \mathbb{R}\colon |\phi_p(t) - \psi_p(t)| \geq Cp^{-\alpha}\}.$$

Then $\mu(A_{\alpha,C}/p) \leq C'p^{2(\alpha-3/4)}$, where μ is Lebesgue measure on the real line.

The proof uses the Chebyshev inequality to estimate the measure of $A_{\alpha,C}$ by the L_2 norm of $\phi_p(t) - \psi_p(t)$.

COROLLARY 2. *Set $\tau = t/p$, and denote any $f(t) = f(p\tau)$ still by $f(\tau)$ for simplicity. Then for any $\alpha < 3/4$ and $C > 0$, $\lim_{p\to\infty} \mu\{\tau \in \mathbb{R}\colon |\phi_p(\tau) - \psi_p(\tau)| \geq Cp^{-\alpha}\} = 0$.*

This result tells what one can hope for from the approximation to $\phi_p(t)$ by $\Phi_p(t)$ (or $\phi_p(\tau)$ by $\Phi_p(\tau)$, $\tau = t/p$). The approximations defined by (6) and (7) have already introduced a non-negligible error of at least order $O(p^{-3/4})$. Therefore for the further approximation by (8), it is only meaningful to talk about an accuracy of order $O(p^{-\alpha})$ with $\alpha < 3/4$.

3. FOURIER INTEGRALS WITH LARGE PARAMETERS

Before we investigate the asymptotic form of the approximative scaling function $\Phi_p(t)$, it is helpful to review and extend some results in asymptotic analysis.

Fourier integrals with one large parameter have the form

$$I_\lambda = \int_a^b f(\omega)e^{-i\lambda F(\omega)}d\omega \tag{23}$$

with real λ, f, F. The asymptotic analysis usually deals with $\lambda \gg 1$. The interval $[a,$

$b]$ can be finite or infinite. Our problem is the finite case. The basic results can be stated as follows:

STATEMENT 1 (End Point Contribution). *Suppose that F is a C^1 function and has no critical point inside the closed interval (or equivalently, no stationary phase), and f is a continuous function. Then the leading asymptotic magnitude is proportional to $1/\lambda$:*

$$I_\lambda = \frac{1}{i\lambda}\left[f(a)\frac{e^{-i\lambda F(a)}}{F'(a)} - f(b)\frac{e^{-i\lambda F(b)}}{F'(b)} \right] + o(\lambda^{-1}). \tag{24}$$

STATEMENT 2 (Stationary Phase Contribution). *Suppose*

(*i*) *F and f are C^1 and C^0 functions on $[a, b]$, respectively,*
(*ii*) *$c \in (a, b)$ is the only critical point of F on $[a, b]$ and $f(c) \neq 0$,*
(*iii*) *F is C^2 around this critical point and $F''(c) \neq 0$.*

Then the leading asymptotic magnitude is proportional to $1/\sqrt{\lambda}$:

$$I_\lambda = f(c)\sqrt{\frac{2\pi}{|\lambda F''(c)|}}\, e^{-i[\lambda F(c) + sign(F''(c))\pi/4]} + o(\lambda^{-1/2}). \tag{25}$$

The proofs of these two statements can be found in many asymptotic analysis textbooks (for instance [1, 5, 7, 11]) with a little modification on the regularity of F.

Next, let's consider the *doubly parameterized Fourier integral* (DPFI):

$$I_\lambda(\tau) = \int_a^b e^{-i\lambda F(\omega,\tau)} d\omega, \qquad \text{for real } \tau. \tag{26}$$

At any fixed time τ, Statements 1 and 2 can be applied to DPFI. As long as the regularity conditions for ω are satisfied uniformly during a certain period of time, the approximations hold uniformly with respect to τ. Attention must be paid to the so-called *transition period* of τ. It could happen that one period of time belongs to the case of Statement 1 uniformly, and some other period to that of Statement 2, while the rest is a transition period between these two cases.

The transition phenomenon is structurally stable and therefore universal. It occurs near "turning points." To begin, we consider an idealized DPFI with $a = -1$, $b = 1$, and $F(\omega, \tau) = \omega^3/3 - \tau\omega$. We plot the critical points of $F(\omega, \tau)$ as a function of ω with parameter τ on the $\tau - \omega$ plane. F has two critical points for $\tau \in (0, 1)$. Outside $[0, 1]$, there are no critical points on the interval $[a, b]$. Two classes of transitions with different origins occur here (see Fig. 3):

(i) Near $\tau = 0$, the two critical points coming from the right side collide and cancel each other (and actually go to the imaginary axis).

(ii) Near $\tau = 1$, the pair of critical points coming from the left go out of the integral domain.

In this example, the leading magnitude of $I_\lambda(\tau)$ is $1/\lambda$ uniformly on any compact set

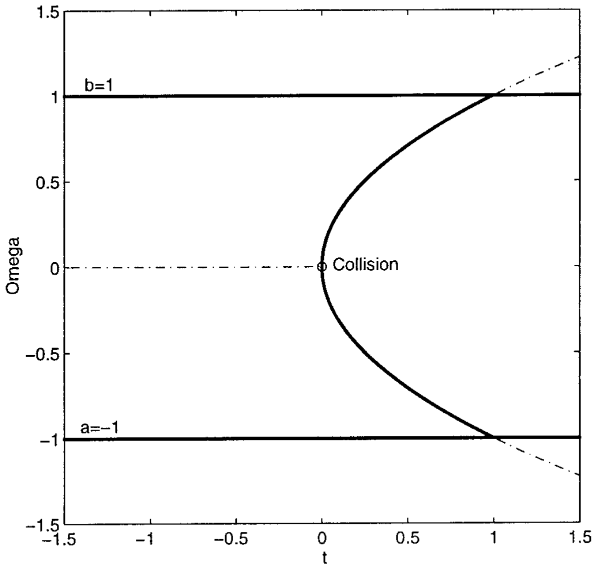

FIG. 3. Transition phenomenon.

of negative time (by Statement 1) and $1/\sqrt{\lambda}$ uniformly on any compact set of positive time (by Statement 2). Then, how is this jump in magnitude realized around time zero? The answer is given by Theorem 4. We only sketch the proof, since a strict proof may take unsuitably long. Similar work on uniform approximations of integrals can be found in Wong [11, Chapter 7, Section 4].

THEOREM 4. *Let $L(\omega)$ be a function on $[a, b]$ $(a < 0 < b)$ that satisfies*

(*i*) *L is a C^1 function;*
(*ii*) *$\omega = 0$ is the unique critical point of L;*
(*iii*) *L is C^3 around 0, and $L(0) = L'(0) = L''(0) = 0$, $L'''(0) = \alpha > 0$.*

Suppose $F(\omega, \tau) = L(\omega) - \tau\omega$. *Then for $|\tau| \ll 1$ (or precisely, $\tau = O(\lambda^{-2/3})$)*

$$I_\lambda(\tau) = 2\pi\sqrt[3]{2/\alpha\lambda}\,\mathrm{Ai}\left(-\sqrt[3]{2\lambda^2/\alpha}\,\tau\right) + o(\lambda^{-1/3}), \tag{27}$$

where Ai *is the Airy function.*

Proof. Since $|\tau| \ll 1$, the leading magnitude of $I_\lambda(\tau)$ is completely determined by the local property of L near $\omega = 0$. Therefore we have

$$\begin{aligned} I_\lambda(\tau) &= \int_a^b e^{-i\lambda(L(\omega)-\tau\omega)}d\omega \simeq \int_{-\delta}^{\delta} e^{-i\lambda(L(\omega)-\tau\omega)}d\omega \\ &\simeq \int_{-\delta}^{\delta} e^{-i\lambda(\alpha\omega^3/6-\tau\omega)}d\omega \simeq \int_{-\infty}^{\infty} e^{-i\lambda(\alpha\omega^3/6-\tau\omega)}d\omega. \end{aligned} \tag{28}$$

Scaling the variables in (28) by setting $\tau = -\sqrt[3]{\alpha/2\lambda^2}\,t$ and $\omega = \sqrt[3]{2/\alpha\lambda}\,\theta$ yields

$$
\begin{aligned}
(28) &= \sqrt[3]{2/\alpha\lambda}\int_{-\infty}^{\infty} e^{-i(\theta^3/3+t\theta)}d\theta + o(\lambda^{-1/3})\\
&= 2\pi\sqrt[3]{2/\alpha\lambda}\ \mathrm{Ai}(t) + o(\lambda^{-1/3})\\
&= 2\pi\sqrt[3]{2/\alpha\lambda}\ \mathrm{Ai}(-\sqrt[3]{2\lambda^2/\alpha}\,\tau) + o(\lambda^{-1/3}).\ \blacksquare
\end{aligned}
$$

Now let us consider the second type of transition. A generic case is given by the following theorem.

THEOREM 5. *Suppose $F(\omega, \tau) = L(\omega) - \tau\omega$, where $L(\omega) \in C^1[-a, a]$ satisfies*

(*i*) $L(\omega) = L(-\omega)$;
(*ii*) $\omega = 0$ *is the only critical point for L;*
(*iii*) *L is C^2 around $\omega = a$ and $L''(a) \neq 0, \infty$.*

Then for $|\tau - A_1| \ll 1$ (*precisely,* $|\tau - A_1| = O(\lambda^{-1/2})$), *we have*

$$
I_\lambda(\tau) = \sqrt{\frac{8\pi}{\lambda|A_2|}}\left[\cos\left(\frac{\lambda}{2}T\right)\operatorname{coserf}(\sqrt{\lambda}S) + \operatorname{sign}(A_2)\sin\left(\frac{\lambda}{2}T\right)\operatorname{sinerf}(\sqrt{\lambda}S)\right] + o(\lambda^{-1/2}) \quad (29)
$$

$$
S = \operatorname{sign}(A_2)\frac{A_1 - \tau}{\sqrt{|A_2|}}, \qquad T = 2(\tau a - A_0) + \frac{(A_1 - \tau)^2}{A_2}
$$

with $A_0 = L(a)$, $A_1 = L'(a)$, $A_2 = L''(a)$, and two Fresnel integrals,

$$
\operatorname{coserf}(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{s}\cos\frac{\theta^2}{2}\,d\theta, \qquad \operatorname{sinerf}(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{s}\sin\frac{\theta^2}{2}\,d\theta. \quad (30)
$$

Proof. Again we only sketch the proof and refer to Wong [11, Chapter 7].

[1] One only needs to prove the case when $A_2 > 0$. For negative A_2, simply replace L and τ by $-L$ and $-\tau$.

[2] As $|\tau - A_1| \ll 1$,

$$
\begin{aligned}
I_\lambda(\tau) &= \int_{-a}^{a} e^{-i\lambda(L(\omega)-\tau\omega)}d\omega \simeq \int_{a-\delta}^{a} + \int_{-a}^{-a+\delta} e^{-i\lambda(L(\omega)-\tau\omega)}d\omega\\
&= 2\ \mathrm{Re}\int_{a-\delta}^{a} e^{-i\lambda(L(\omega)-\tau\omega)}d\omega \simeq 2\ \mathrm{Re}\int_{a-\delta}^{a} e^{-i\lambda[A_0+A_1(\omega-a)+(A_2/2)(\omega-a)^2-\tau\omega]}d\omega\\
&\simeq 2\ \mathrm{Re}\ e^{-i\lambda(A_0-\tau a)}\int_{-\delta}^{0} e^{-i\lambda[(A_1-\tau)\theta+(A_2/2)\theta^2]}d\theta \qquad (\theta = \omega - a)\\
&\simeq 2\ \mathrm{Re}\ e^{-i\lambda(A_0-\tau a)}\int_{-\infty}^{0} e^{-i\lambda[(A_1-\tau)\theta+(A_2/2)\theta^2]}d\theta \qquad (\text{since } |A_1 - \tau| \ll 1)
\end{aligned}
$$

$$= \frac{2}{\sqrt{\lambda A_2}} \operatorname{Re} e^{-i\lambda(A_0-\tau a)} e^{i(\lambda/2)(A_1-\tau)^2/A_2} \int_{-\infty}^{\sqrt{\lambda}S} e^{-i(u^2/2)} du$$

$$= \sqrt{\frac{8\pi}{\lambda A_2}} \left[\cos\left(\frac{\lambda}{2} T\right) \operatorname{coserf}(\sqrt{\lambda}S) + \sin\left(\frac{\lambda}{2} T\right) \operatorname{sinerf}(\sqrt{\lambda}S) \right]. \quad \blacksquare$$

Unfortunately, in our case, $L(\omega) = G^{(-1)}(\omega)$, hence $A_2 = \infty$. The generic result of Theorem 5 does not apply to this special case. In fact for $\Phi_p(\tau)$, one has to deal with the following type of local integral with large parameter λ:

$$I_\lambda(\tau) = \int_0^c e^{-i\lambda(\tau\theta - a\theta^2 \log\theta + b\theta^2)} d\theta \qquad \tau \simeq 0,\ a,\ b,\ c > 0. \tag{31}$$

The asymptotic analysis of (31) may be very involved. However, this transition is of less importance to us for the reason indicated in the last paragraph of Section 2.

4. ASYMPTOTIC STRUCTURE OF $\Phi_p(t)$ AND $\psi_p(t)$

4.1. Asymptotic Form of $\Phi_p(t)$

We are ready now to establish the asymptotic form of $\Phi_p(t)$. Recall that we introduced the lower transition time $\tau_0 = G(0) \simeq 0.1817$ and upper transition time $\tau_1 = G(\pi) \simeq 0.3515$. The definition of $\Phi_p(t)$ in (8) and DPFI in (26) implies a scaling $t = \tau p$. For simplicity, we continue to use $\Phi_p(\tau)$ to denote $\Phi_p(p\tau)$.

RESULT 1 (Stationary Phase). *Uniformly on any compact subset K of* (τ_0, τ_1),

$$\Phi_p(\tau) = \sqrt{\frac{2}{\pi p G'(\omega_\tau)}} \cos\left[p(G^{(-1)}(\omega_\tau) - G(\omega_\tau)\omega_\tau) + \frac{\pi}{4} \right] + o(p^{-1/2}). \tag{32}$$

ω_τ *is the unique* $\omega \in (0, \pi)$ *such that* $G(\omega) = \tau$.

Let $F = G^{(-1)}(\omega) - \tau\omega$ and apply the result of Statement 2 to DPFI.

RESULT 2 (Airy Transition). *For* $|\tau - \tau_0| \ll 1$,

$$\Phi_p(\tau) = \sqrt[3]{\frac{42\pi}{p}} \operatorname{Ai}(-\sqrt[3]{42\pi p^2}(\tau - \tau_0)) + o(p^{-1/3}). \tag{33}$$

Use Theorem 4 with $L(\omega) = G^{(-1)}(\omega)$, and use $\tau - \tau_0$ instead of τ. Note $\alpha = 1/21\pi$ in this case.

RESULT 3 (End Points). *Uniformly on any compact subset K of* $[\tau_0, \tau_1]^c$,

$$\Phi_p(\tau) = \frac{1}{p\pi(\tau - \tau_1)} \sin[p(G^{(-1)}(\pi) - \tau\pi)] + o(p^{-1}). \tag{34}$$

Apply Statement 1 to DPFI in the case $F = G^{(-1)} - \tau\omega$, $a = -\pi$, $b = \pi$.

RESULT 4 (Front Matching). (32) *and* (33) *match for* $p^{-2/3} \ll \tau - \tau_0 \ll 1$.

Proof. Let $x = \sqrt[3]{42\pi p^2}(\tau - \tau_0)$ and $K(\omega) = G^{(-1)}(\omega) - G(\omega)\omega$.

(i) $\tau - \tau_0 \gg p^{-2/3}$ implies $x \gg 1$. Since

$$\mathrm{Ai}(-x) = \frac{1}{\sqrt{\pi}} x^{-1/4} \sin\left[\frac{2}{3}x^{3/2} + \frac{\pi}{4}\right] + O(x^{-7/4}) \qquad \text{as } x \gg 1,$$

the leading term of the expression in (33) is given by

$$\left(\frac{42}{\pi}\right)^{1/4} p^{-1/2}(\tau - \tau_0)^{-1/4} \sin\left[\frac{2}{3}(42\pi)^{1/2} p(\tau - \tau_0)^{3/2} + \frac{\pi}{4}\right]. \tag{35}$$

(ii) $0 \ll \tau - \tau_0 \ll 1$ implies $\omega_\tau \ll 1$ and $G(\omega) \simeq G(0) + (G''(0)/2)\omega^2$ as $|\omega| \ll 1$. Therefore $G''(0) = 1/21\pi$ gives $\omega_\tau \simeq [42\pi(\tau - \tau_0)]^{1/2}$ and

$$K(\omega_\tau) \simeq \frac{K'''(0)}{6}\omega_\tau^3 = -\frac{G''(0)}{3}\omega_\tau^3 = -\frac{1}{63\pi}\omega_\tau^3 \simeq -\frac{2}{3}(42\pi)^{1/2}(\tau - \tau_0)^{3/2} \tag{36}$$

and

$$G'(\omega_\tau) \simeq G''(0)\omega_\tau \simeq \left(\frac{2}{21\pi}\right)^{1/2}(\tau - \tau_0)^{1/2}. \tag{37}$$

Substituting (36) and (37) into the expression in (32), the leading term of (32) is given exactly by (35) for $0 \ll \tau - \tau_0 \ll 1$. ■

The end point contribution is of magnitude $O(p^{-1})$. This order is more delicate than our approximation precision (see the last paragraph of Section 2). Therefore this modulated sine wave of order p^{-1} is only the behavior of $\Phi_p(t)$, not that of the scaling function. These ripples are introduced by the ideal lowpass filtering (truncation) in (6). However, as one can see from Fig. 4, part of the earliest ripples given by (34) can match $\phi_p(\tau)$ quite well.

The stationary phase contribution lasts asymptotically $\tau_1 - \tau_0 = 0.1698$ in the scaled time τ. During this period, the magnitude has an order of $O(p^{-1/2})$ as noted below Eq. (22). A better way to interpret (32) is that the phase inside the cosine is basically amplified (by p) and shifted (by $\pi/4$) from the Legendre transform of $G^{(-1)}$. $G(\omega)$ provides a natural parameterization for the scaled time period (τ_0, τ_1). Therefore, $\tau = G(\omega_\tau)$ together with (32) is the ω_τ-parameterized version of $\Phi_p(\tau)$ on (τ_0, τ_1). This is quite useful for computer plotting (the explicit inverse of G is unnecessary). The total wave number k during this period is entirely determined by the area bounded by $G = G(\omega)$, $\omega = 0$, and $G = \tau_1$ in the left subplot of Fig. 2, which is approximately 0.4152. Therefore $k \simeq 0.4152p/2\pi \simeq 0.0661p$. Every increment of

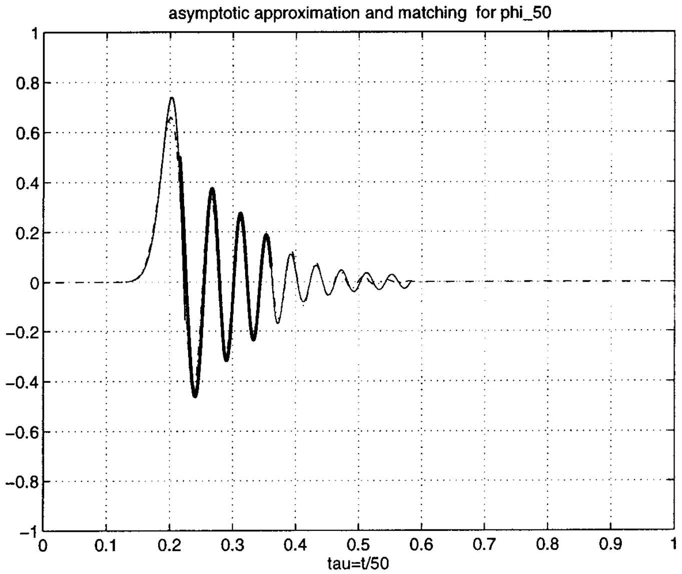

FIG. 4. Asymptotic approximation and matching for $\phi_{50}(\tau)$ (dashed line). The left and right solid lines are Airy transition and end points contribution. The dark dotted line in the middle is the stationary phase. Each approximation has been uniformly shifted by approximately 0.01 ($=O(p^{-1})$) (see next subsection).

30 in p adds two complete waves during this period, asymptotically. This is confirmed by numerical results.

The Airy transition reveals the rich structure of the main lobe (or the wavefront). The main lobe lasts about $p^{-2/3}$ in the scaled time τ and has a magnitude $O(p^{-1/3})$. This makes it the real leader of all waves that follow it: it is much wider than other waves ($p^{-2/3}$ to p^{-1}) and also much higher ($p^{-1/3}$ to $p^{-1/2}$). However, from the viewpoint of energy, Airy transition is insignificant since its total energy is of order $O(p^{-1/3})$.

We have plotted our approximation in Fig. 4.

4.2. *Approximating $\psi_p(t)$ with $\Phi_p(t)$ (Revisited)*

Now we show that $\psi_p(\tau)$ and $\Phi_p(\tau)$ will share the same envelope for the stationary phase period and their graphs have separation $O(p^{-1})$ during this period.

For convenience, let $Q(\omega) = \mathrm{grd}(\hat{\phi}_p)/p$ and $Q^{(-1)} = \int_0^\omega Q(\omega)d\omega = -\arg(\hat{\phi}_p)/p$ for any fixed p. We state the following facts:

(i) $\psi_p(\tau) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ip(Q^{(-1)}(\omega)-\tau\omega)}d\omega$.

(ii) $Q(-\omega) = Q(\omega)$, and $Q(\omega)$ is strictly increasing on $[0, \pi]$.

(iii) By (3), $|Q(\omega) - G(\omega)| \leq cp^{-1/2}$ for any $\omega \in [-\pi, \pi]$. On any compact subset K of $(-\pi, \pi)$ this estimation is differentiable:

$$|Q^{(n)}(\omega) - G^{(n)}(\omega)| \leq c_n p^{-1/2}. \tag{38}$$

Therefore we can apply stationary phase to $\psi_p(\tau)$ uniformly on any compact set K of (τ_0, τ_1) to find that

$$\psi_p(\tau) = \sqrt{\frac{2}{\pi p Q'(\omega'_\tau)}} \cos\left[p(Q^{(-1)}(\omega'_\tau) - Q(\omega'_\tau)\omega'_\tau) + \frac{\pi}{4}\right] + o(p^{-1/2}). \quad (39)$$

ω'_τ is the unique $\omega \in (0, \pi)$ such that $Q(\omega) = \tau$.

By (38), on the compact set K, one can replace $Q^{(n)}$ and ω'_τ with $G^{(n)}$ and ω_τ (as defined in (32)) at the price of an order $O(p^{-1/2})$ phase perturbation, i.e.,

$$\psi_p(\tau) = \sqrt{\frac{2}{\pi p G'(\omega_\tau)}} \cos\left[p(G^{(-1)}(\omega_\tau) - G(\omega_\tau)\omega_\tau + O(p^{-1/2})) + \frac{\pi}{4}\right] + o(p^{-1/2}). \quad (40)$$

Comparing (40) with (32), one sees that during the stationary period, $\psi_p(\tau)$ and $\Phi_p(\tau)$ share the same envelope $\sqrt{2/\pi p G'(\omega_\tau)}$. And at each time τ, the phase of the leading term can be made the same (mod 2π) up to an order $O(p^{-1})$ shifting of the scaled time τ. This means that the graphs of $\psi_p(\tau)$ and $\Phi_p(\tau)$ during this period have separation $O(p^{-3/2})$. Since the residue terms in (40) and (32) actually have magnitude $O(p^{-1})$ (due to the good regularity of both G and Q on K), the graphs of $\psi_p(\tau)$ and $\Phi_p(\tau)$ have distance $O(p^{-1})$.

Similar work can be done for the Airy transition; the phase perturbation still exists.

5. ASYMPTOTIC STRUCTURE OF WAVELETS

5.1. *Goodness of Approximation*

The orthogonal highpass filter F_p is the alternating flip of the lowpass filter H_p. In the z-domain, $F_p(z) = -z^{-N}H_p(-z^{-1})$ with $N = 2p - 1$. The delay factor z^{-N} makes F_p causal. Its group delay is $\mathrm{grd}(F_p) = N - \mathrm{grd}(H_p)(\omega + \pi)$.

The transform of the wavelet $w_p(t)$ is $\hat{w}_p(\omega) = F_p(\omega/2)\hat{\phi}_p(\omega/2)$, so that $\mathrm{grd}(\hat{w}_p) = \frac{1}{2}[\mathrm{grd}(F_p)(\omega/2) + \mathrm{grd}(\hat{\phi}_p)(\omega/2)]$. Using the same notation as (3) and (4), the group delay for wavelets has leading term

$$\left| \mathrm{grd}(\hat{w}_p) - \left[p\left(\frac{1}{2} + G(\omega)\right) - \frac{1}{2}\right]\right| \le C\sqrt{p}. \quad (41)$$

We already know that the spectrum of $\hat{\phi}_p$ is mainly concentrated on $[-\pi, \pi]$ and the magnitude $|F_p(\omega)|$ converges to the ideal highpass filter. Therefore it is natural to introduce $W_p(t)$ to approximate the wavelet $w_p(t)$:

$$W_p(t) = -\frac{1}{2\pi}\left(\int_{-2\pi}^{-\pi} + \int_{\pi}^{2\pi}\right) e^{-ip(G^{(-1)}(\omega)+\omega/2)+i\omega/2} e^{it\omega} d\omega. \tag{42}$$

The minus sign before the integral is because the phase of $\hat{w}_p$ near $\omega = 0$ is near π by our definition of the highpass filter $F_p(z) = -z^{-N}H_p(-z^{-1})$, though its magnitude is zero at $\omega = 0$. One can repeat the work already done for the scaling functions and obtain the analogues of (21) and (22). The discussion at the end of Section 2.4 applies similarly.

5.2. *Asymptotic Form of $W_p(t)$*

It is always good to use the scaled time $\tau = t/p$. First we have

$$W_p(\tau) = -\frac{1}{2\pi}\left(\int_{-2\pi}^{-\pi} + \int_{\pi}^{2\pi}\right) e^{-ip(G^{(-1)}(\omega)-(\tau-0.5)\omega)} e^{i\omega/2} d\omega. \tag{43}$$

Let $\tau_0^w = 0.5 + G(\pi) = 0.5 + \tau_1 \approx 0.8515$ and $\tau_1^w = 0.5 + G(2\pi) \approx 1.0849$ be the lower and upper transition times for the scaled wavelets. Notice that

(i) G is smooth inside $(\pi, 2\pi)$ and continuous on $[\pi, 2\pi]$.
(ii) G is monotonically increasing on $[\pi, 2\pi]$.

RESULT 5 (Stationary Phase). Uniformly on any compact subset K of (τ_0^w, τ_1^w),

$$W_p(\tau) = -\sqrt{\frac{2}{\pi p G'(\omega_\tau)}} \cos\left[p(G^{(-1)}(\omega_\tau) - G(\omega_\tau)\omega_\tau) + \frac{\pi}{4} + \frac{\omega_\tau}{2}\right] + o(p^{-1/2}). \tag{44}$$

Here ω_τ is the unique $\omega \in (\pi, 2\pi)$ such that $G(\omega) = \tau - 0.5$.

And similarly, one can write the end point contribution:

RESULT 6 (End Points). *Uniformly on any compact subset K of $[\tau_0^w, \tau_1^w]^c$,*

$$\begin{aligned} W_p(\tau) = \frac{1}{p\pi}\Bigg(&\frac{1}{\tau_1^w - \tau}\sin[p(G^{(-1)}(2\pi) - (\tau - 0.5)2\pi) + \pi] \\ &+ \frac{1}{\tau - \tau_0^w}\sin[p(G^{(-1)}(\pi) - (\tau - 0.5)\pi) + \pi/2]\Bigg) + o(p^{-1}). \end{aligned} \tag{45}$$

There is no Airy transition for wavelets. The group delay of $\hat{w}_p$ on $[-2\pi, -\pi] \cup [\pi, 2\pi]$ has no critical point. Another interpretation is that the Airy wavefront has been removed by the highpass filter F_p. Therefore the characteristic scale for wavelets is as follows: magnitude $= O(p^{-1/2})$; lasting time (scaled) $= \tau_1^w - \tau_0^w \approx 0.2334$.

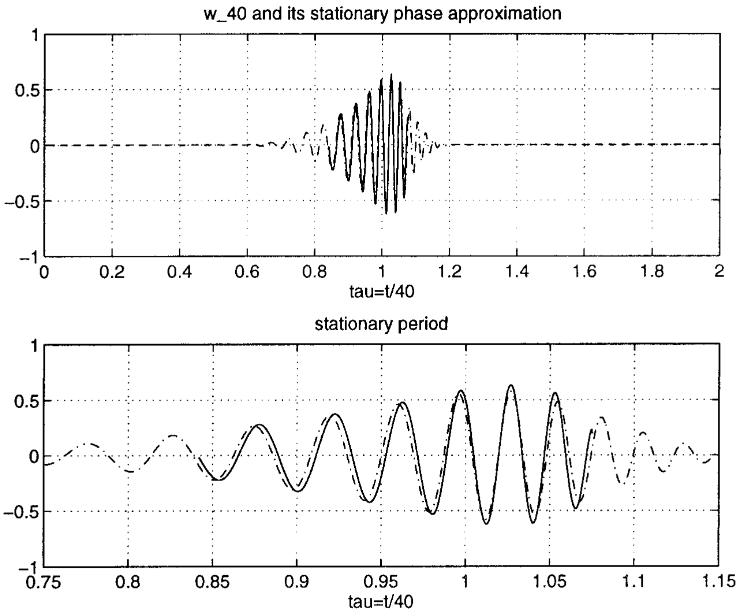

FIG. 5. Accuracy of the stationary phase approximation for the wavelet $w_p(t)$. The dashed line represents w_{40} and the solid line is given by (44) with $p = 40$ and τ shifted by $0.01 = O(p^{-1})$.

We have plotted our stationary phase approximation in Fig. 5. Beyond this stationary phase period, the magnitude of $w_p(t)$ is at most $O(p^{-1})$, which is energy insignificant.

6. ASYMPTOTIC STRUCTURE OF THE FILTER COEFFICIENTS

6.1. Three Asymptotic Regions

The asymptotic analysis method for the scaling function ϕ_p also applies to the filter coefficients $h_p[n]$. Comparing (1) to (3), one only needs to replace G and $G^{(-1)}$ by g and $g^{(-1)} = \int_0^\omega g(\theta)\,d\theta$. A natural approximation to the impulse response $h_p[n]$ is $h_p^*[n]$:

$$h_p^*[n] = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{-ipg^{(-1)}(\omega)} e^{in\omega} d\omega. \tag{46}$$

We can extend the definition of h_p^* to allow non-integer index t by

$$h_p^*(t) = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} e^{-ipg^{(-1)}(\omega)} e^{it\omega} d\omega. \tag{47}$$

Scaling t by a factor p, $\tau = t/p$ yields

$$h_p^*(\tau) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{-ip[g^{(-1)}(\omega) - \tau\omega]} d\omega. \tag{48}$$

Define two filter transition times τ_0^h and τ_1^h by $\tau_0^h = g(0) = G(0) \simeq 0.1817$ and $\tau_1^h = g(\pi/2) = 0.5$. Then the asymptotic form of $h_p^*(\tau)$ is described by the following three results:

RESULT 7 (Stationary Phase). *Uniformly on any compact subset K of* (τ_0^h, τ_1^h),

$$h_p^*(\tau) = \sqrt{\frac{2}{\pi p g'(\omega_\tau)}} \cos\left[p(g^{(-1)}(\omega_\tau) - g(\omega_\tau)\omega_\tau) + \frac{\pi}{4} \right] + o(p^{-1/2}). \tag{49}$$

Here ω_τ is the unique $\omega \in (0, \pi/2)$ such that $g(\omega) = \tau$.

RESULT 8 (Airy Transition). *For* $|\tau - \tau_0^h| \ll 1$,

$$h_p^*(\tau) = \sqrt[3]{\frac{6\pi}{p}} \operatorname{Ai}(-\sqrt[3]{6\pi p^2}(\tau - \tau_0^h)) + o(p^{-1/3}). \tag{50}$$

RESULT 9 (End Points). *Uniformly on any compact subset K of* $[\tau_0^h, \tau_1^h]^c$,

$$h_p^*(\tau) = \frac{1}{p\pi(\tau - \tau_1^h)} \sin\left[p\left(g^{(-1)}\left(\frac{\pi}{2}\right) - \tau \frac{\pi}{2} \right) \right] + o(p^{-1}). \tag{51}$$

Of course, the Airy transition and the stationary phase are matched over the interval $p^{-2/3} \ll \tau - \tau_0 \ll 1$.

These results have the following interpretations:

(i) Stationary phase. The impulse response $h_p[n]$ in this period is energy significant. Its characteristic magnitude is $p^{-1/2}$. This period lasts approximately 0.3183 p from $n \simeq \tau_0^h p$ to $\tau_1^h p = p/2$. Its total energy is $O[(p^{-1/2})^2(\tau_1^h - \tau_0^h)p] =$ which is O (1).

(ii) Airy transition. Though highest, it is energy insignificant. Its leading order is $O[p^{-1/3}]$ but its duration is only $O[p^{-2/3}] \cdot O[p] = O[p^{-1/3}]$. Therefore its total energy is of order $O[(p^{-1/3})^2 p^{1/3}] = O[p^{-1/3}]$.

(iii) End points. The magnitude is no more than $O[p^{-1}]$. Therefore it is energy insignificant.

In one word, the energy of the impulse response $h_p[n]$ is asymptotically concentrated on the interval $n \in [\tau_0^h p, \tau_1^h p]$. Our analysis of these coefficients began with Nico

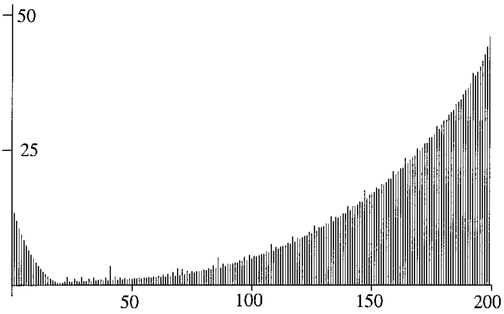

FIG. 6. Plot of $-\log_{10}|h_p[n]|$ with $p = 100$, $0 \leq n \leq 2p - 1 = 199$. The sharp Airy transition ends near $n = \tau_0^h p \simeq 18$, and the stationary phase extends to $n = 50$ with $-\log_{10}|h_p[n]| \simeq 1$. The long tail has small coefficients that are energy insignificant.

Temme's plot of $h_{100}[n]$ in [10], which is reproduced in Fig. 6. In this case $p = 100$, and the leading order is $-\log_{10} 1/\sqrt{100} = 1$. Readers can observe the stationary period up to $n = p/2 = 50$.

6.2. *The Similarity and Difference between $\phi_p(t)$ and $h_p[n]$*

A frequent conjecture is that *as p goes to ∞, $h_p[n]$ should look like $\phi_p(t)$ at integer times $t = n$, $0 \leq n \leq 2p - 1$.* This is partly correct and partly wrong. We can summarize three essential points of similarity:

- (i) Both $h_p[n]$ and $\phi_p(t)$ have the same support interval $[0, 2p - 1]$.
- (ii) For large p, as n and t increase from 0 to $2p - 1$, both $h_p[n]$ and $\phi_p(t)$ undergo the following three stages:
  - (a) Airy transition (wavefront) with leading magnitude $O(p^{-1/3})$ and lasting time $O(p^{1/3})$.
  - (b) Stationary phase (steady oscillation) with leading magnitude $O(p^{-1/2})$ and lasting time $O(p)$.
  - (c) End points with leading magnitude $O(p^{-1})$.
- (iii) Both wavefronts start near time $p(1/2 - 1/\pi) \simeq .1817p$.

However, this structural similarity *does not* imply that the conjecture is entirely true. A significant difference also exists. *The stationary phase period of the filter impulse response $h_p[n]$ is much longer than that of the scaling function.* The scaling function stops near $t \simeq 0.3515p$, much earlier than the impulse response does (near $n \simeq 0.5p$). Therefore the impulse response cannot be the sampling of the scaling function.

Numerically the dilation equation is solved by the cascade algorithm, which iterates the lowpass filter (with time rescaling). A natural choice of the initial data is the impulse $\delta[n]$. Then the first iteration gives exactly the filter impulse response $h_p[n]$. Usually within seven or eight steps, one can obtain the scaling function with satisfactory accuracy.

From our results, we can see what is happening during this algorithm. After the first step, the values corresponding to the time interval $(0.3515p, 0.5p)$ will be attenuated again and again until the leading magnitude falls from $O(p^{-1/2})$ to $O(p^{-1})$.

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