Time series models.

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Time series.

A time series is an ordered sequence of values

$$\{X_\tau\}_{\tau=1}^{\infty} \equiv X_1, X_2, \ldots, X_t, \ldots$$

- Example:
  - Gaussian white noise: Sequence of independent random variables (irv's) sampled from the normal distribution $\mathcal{N}(0, \sigma)$

$$\epsilon_1, \epsilon_2, \ldots \epsilon_t, \ldots$$

  - Brownian motion: Sequence of random variables

$$0, \epsilon_1, \sum_{\tau=1}^{2} \epsilon_\tau, \sum_{\tau=1}^{3} \epsilon_\tau, \ldots \sum_{\tau=1}^{t} \epsilon_\tau, \ldots$$

    in which l $\{\epsilon_\tau\}_{\tau=1}^{\infty}$ is Gaussian white noise.

We assume that the time series $X_1, X_2, \ldots, X_n$ is sampled from the probability distribution density

$$P\left(x_1, x_2, \ldots, x_T\right)$$

- Example: A deterministic trajectory is unique. Therefore, the pdf is a product of $delta$ distributions

$$P\left(x_1, x_2, \ldots, x_T\right) = \prod_{\tau=1}^{T} \delta\left(x_\tau - G(x_{\tau-1}, \tau)\right)$$

- Example: The pdf for Gaussian white noise is factorized

$$P\left(\{\epsilon_\tau\}_{\tau=1}^{T}\right) = \prod_{\tau=1}^{T}\left[\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\epsilon_\tau^2\right\}\right]$$

The marginal ditribution of the random variable X_t can be obtained by integrating the multivariate pdf over the remaining variables

$$\begin{aligned} P\left(x_t\right) &= \int dx_1 \int dx_2 \ldots \int dx_{t-1} \int dx_{t+1} \ldots \\ &\quad \ldots \int dx_T \; P\left(x_1, x_2, \ldots, x_T\right). \end{aligned}$$

Expected values.

The expected value of a given funtion of the values of a time series $F(\{X_\tau\}_{\tau=1}^{T})$ is

$$\mathbf{E}\left[F\left(\{X_\tau\}_{\tau=1}^{T}\right)\right] = \int dX_1 \int dX_2 \ldots \int dX_T \; F(\{X_\tau\}_{\tau=1}^{T}) P\left(\{X_\tau\}_{\tau=1}^{T}\right)$$

In practice, the actual form of the actual multivariate pdf is not known. If one has access to different realizations of the time series,

$$\left\{X_\tau^{(i)}\right\}_{\tau=1}^{T} \equiv X_1^{(i)}, X_2^{(i)}, \ldots, X_T^{(i)}, \quad i = 1, 2, \ldots I$$

it is possible to obtain sample estimates of these expected values

$$\langle F \rangle = \frac{1}{I} \sum_{i=1}^{I} F\left(\left\{X_\tau^{(i)}\right\}_{\tau=1}^{T}\right),$$

By the law of large numbers, in the limit $I \to \infty$, this estimate converges to the exact value of the expected value

$$\langle F \rangle \to \mathbf{E}\left[F\right], \quad \text{cuando } I \to \infty.$$

- Average:

$$\mathbf{E}\left[X_t\right] = \mu_t.$$

- Variance: Defining

$$\hat{X}_t = X_t - \mu_t,$$

the variance is

$$\mathbf{E}\left[\hat{X}_t^2\right] = \sigma_t^2.$$

- Autocovariance:

$$\mathbf{E}\left[\hat{X}_{t+\tau}\hat{X}_t\right] = \gamma(t;\tau).$$

- Autocorrelation:

$$\rho(t;\tau) = \frac{\gamma(t;\tau)}{\sigma_t^2}.$$

- Example: Gaussian white noise.
  - $\mathbf{E}\left[\epsilon_t\right] = 0$
  - $\mathbf{E}\left[\epsilon_{t+\tau}\epsilon_t\right] = \sigma^2\delta_{\tau,0}$
  - $\epsilon_t \approx \mathcal{N}(0,\sigma)$

Stationary process.

- The process $X_0, X_1, \ldots, X_t, \ldots$ is **stationary in a strong sense** if its pdf fulfills

$$P(X_{t_1}, X_{t_2}, \ldots, X_{t_r}) = P(X_{t_1+\tau}, X_{t_2+\tau}, \ldots, X_{t_r+\tau}).$$

- The process $X_0, X_1, \ldots, X_t, \ldots$ is **weakly stationary**, covariance-stationary wrt the covariance iff it fulfills the conditions

$$\begin{aligned} \mathbf{E}\left[X_t\right] &= \mu \\ \mathbf{E}\left[\hat{X}_{t+\tau}\hat{X}_t\right] &= \gamma_\tau \end{aligned}$$

  Strong stationarity implies weak stationarity if the first two moments of the distributions $P(x_{t-\tau}, x_t)$ exist.

  - A stationary process is **ergodic** wrt the mean if

$$\langle X \rangle = \frac{1}{T}\sum_{\tau=1}^{T} X_\tau \ \rightarrow \mu, \quad T \rightarrow \infty$$

  - A stationary process is **ergodic** wrt the variance if

$$\frac{1}{T-\tau}\sum_{t=1}^{T-\tau} \hat{X}_{t+\tau}\hat{X}_t \ \rightarrow \gamma_\tau, \quad T \rightarrow \infty$$

Autocovariance / autocorrelation function.

The covariance function of a weakly stationary process is

$$\gamma_\tau = \mathbf{E}\left[X_{t+\tau} X_t\right].$$

The autocorrelation coefficient is

$$\rho_\tau = \frac{\gamma_\tau}{\gamma_0}.$$

The value

$$-1 \;\le\; \rho_\tau \;\le 1.$$

Defining lag operator

$$L X_t = X_{t-1};$$

with the properties

$$\begin{aligned} L^0 X_t &= X_t; \\ L^{-1} X_t &= X_{t+1}; \\ L^\tau X_t &= X_{t-\tau}; \end{aligned}$$

First order difference equations.

Consider the first order difference equation

$$X_t = \phi X_{t-1} + \epsilon_t.$$

The solution of this equation is

$$X_t = \phi^t X_0 + \sum_{\tau=0}^{t-1} \phi^\tau \epsilon_{t-\tau}$$

Possible asymptotic behavior of the solution

- With $\phi > 1$ the solution explodes.
- With $\phi < -1$ the solution is oscillatory and explodes.
- With $0 \leq \phi < 1$ the solution exhibits exponential decay.
- With $-1 < \phi \leq 0$ the solution has exponentially damped oscillations.

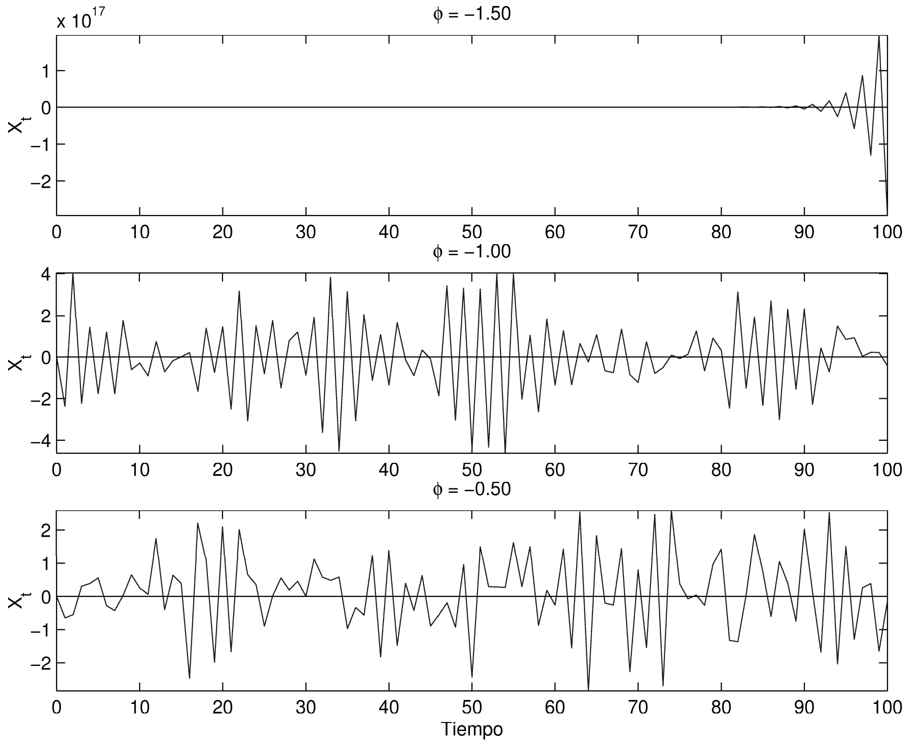

Figure 1: Simulations ($\sigma = 1$).

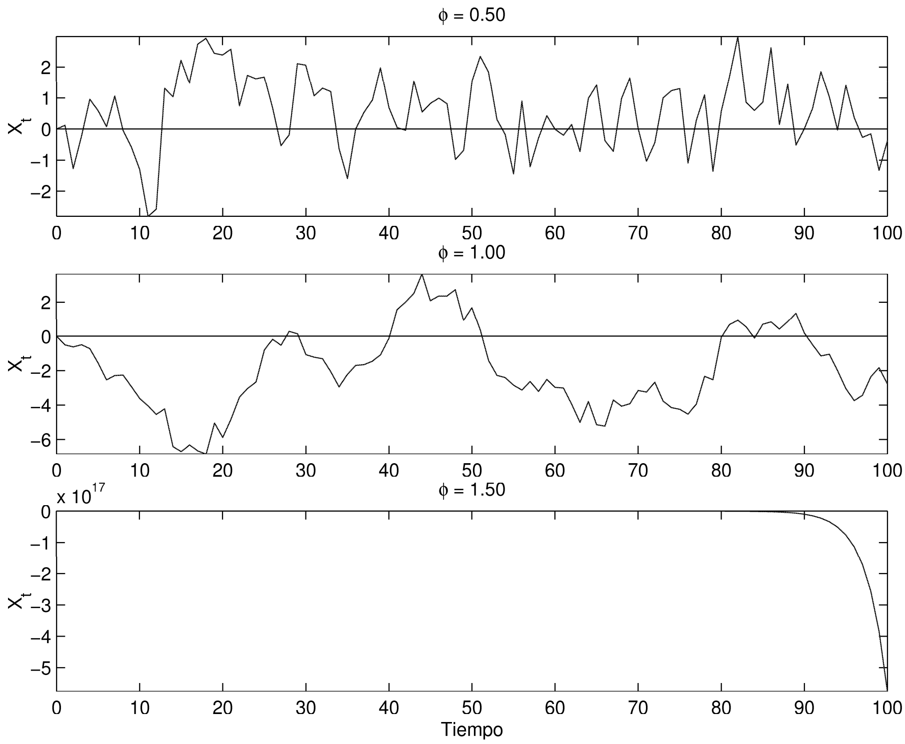

Figure 2: Simulations ($\sigma = 1$).

Difference equations of order p.

Consider the difference equation of order p

$$X_t \; = \; \sum_{i=1}^{p} \phi_i \; X_{t-i} + \epsilon_t.$$

Defining the state vector

$$\boldsymbol{\xi}_t^\dagger = \left(X_t \; X_{t-1} \; \ldots \; X_{t-p+1}\right),$$

the evolution can be expressed as

$$\boldsymbol{\xi}_t = \mathbf{F} \cdot \boldsymbol{\xi}_{t-1} \; + \; \boldsymbol{\epsilon}_t,$$

with the definitions

$$\begin{aligned} \boldsymbol{\epsilon}_t^\dagger &= (\epsilon_t \; 0 \; 0 \; \ldots \; 0 \; 0) \\ \mathbf{F} &= \left(\begin{array}{cccccc} \phi_1 & \phi_2 & \phi_3 & \ldots & \phi_{p-1} & \phi_p \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ & & & \ldots & & \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{array} \right). \end{aligned}$$

The asymptotic evolution of the process depends on the eigenvalues of $\mathbf{F}$.

Example:

Consider the difference equation of order 2

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \epsilon_t.$$

The equation can be written equivalently as

$$\begin{pmatrix} X_t \\ X_{t-1} \end{pmatrix} = \begin{pmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} X_{t-1} \\ X_{t-2} \end{pmatrix} + \begin{pmatrix} \epsilon_t \\ 0 \end{pmatrix}.$$

The asymptotoc behavior of the solution depends on $\mathbf{F}$

$$\begin{aligned} \lambda_1 &= \frac{1}{2}\left(\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}\right), \\ \lambda_2 &= \frac{1}{2}\left(\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}\right). \end{aligned}$$