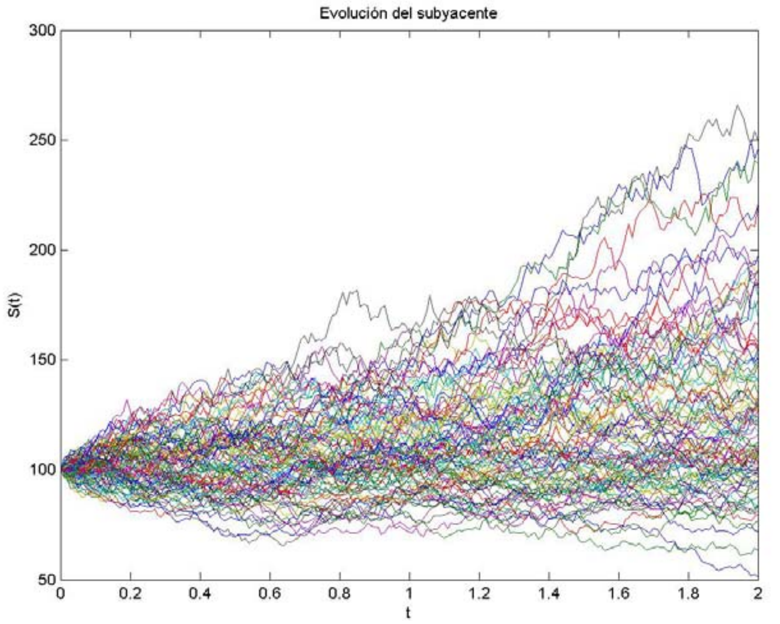
Brownian motion



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Wiener process

$$W(0) = 0; \qquad W(t + \Delta t) = W(t) + X\sqrt{\Delta t}$$

$$W(t) \equiv W(t + \Delta t) - W(t) = X\sqrt{\Delta t};$$

$$X \sim N(0,1) \Rightarrow \Delta W(t) \sim N(0, \sqrt{\Delta t})$$

$$W(t) \sim N(0, \sqrt{t})$$

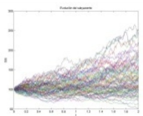
Simulate M trajectories: $W^{(m)}(t_0) = 0; \qquad t_0, t_1, t_2, \ldots, t_N$

$$W_n^{(m)} \equiv W^{(m)}(t_n); \quad \Delta t_n = t_n - t_{n-1};$$

$$W_n^{(m)} = W_{n-1}^{(m)} + \sqrt{\Delta t_n}\, X_n^{(m)}; \qquad X_n^{(m)} \sim N(0,1); \quad n = 1,2,\ldots,N;$$

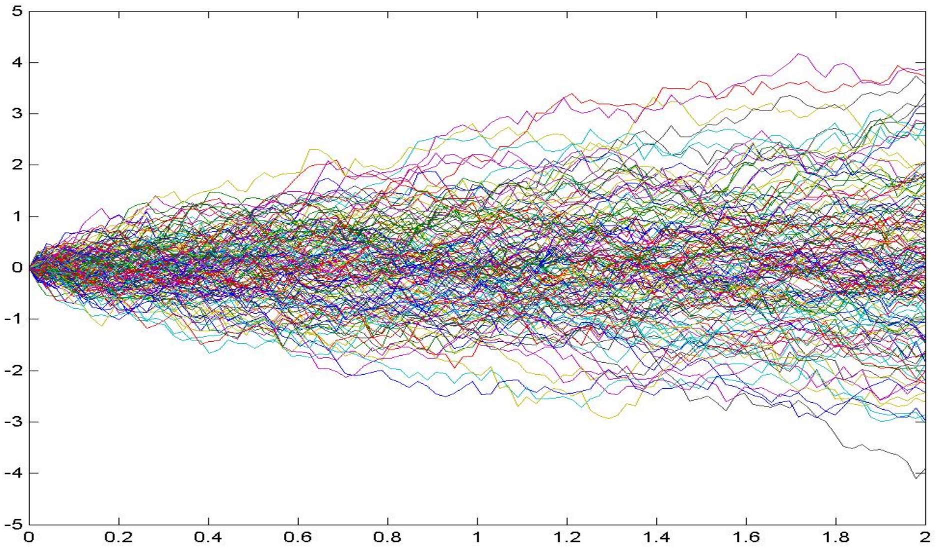
$$m = 1,2,\ldots,M;$$

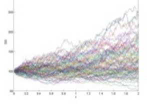
Δt_n is arbitrary



Wiener process: simulation

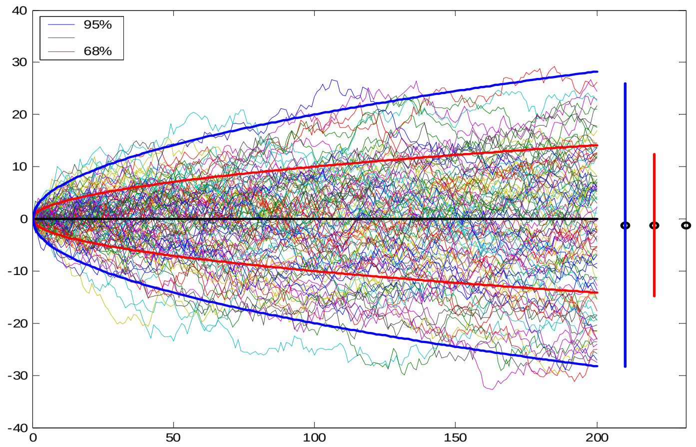
```
function W = simulateWiener(M,times)
%simulateWiener: Simulate Wiener Process
%
% EXAMPLE 1:
%       T = 2; N = 100; times = linspace(0,T,N);
%       M = 150;
%       W = simulateWiener(M,times);
%       figure(1); plot(times,W'); xlabel('t'); ylabel('W(t)')
```





Wiener process: properties

- W(t) is normally distributed
  - Average = 0
  - Variance = t

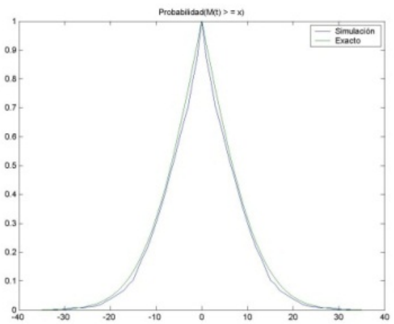


- Autocovariance

$$Cov[W(t)W(s)] = \min(t, s)$$

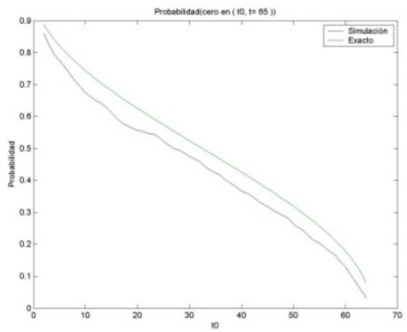
- Distribution of maxima

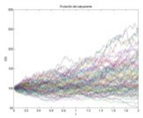
$$\mathrm{P}\left[Max_{[0,T]}[W(t)] > x\right] = \sqrt{\frac{2}{\pi T}} \int_x^{\infty} du\, exp\left\{-\frac{u^2}{2T}\right\}$$



- Probability of return to zero

$$P[zero\ in\ (t_0, t_1)] = \frac{2}{\pi} \cos^{-1} \sqrt{\frac{t_0}{t_1}}$$





Generalized Wiener Process

$$\Delta B(t) = \mu \Delta t + \sigma \Delta W(t);$$

$$\Delta B(t) \equiv B(t + \Delta t) - B(t)$$

$$\Delta W(t) \sim N(0, \Delta t) \Rightarrow \Delta B(t) \sim N(\mu \Delta t, \sigma \sqrt{\Delta t})$$

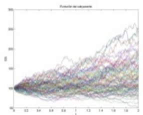
$$B(t) \sim N(B_0 + \mu t, \sigma \sqrt{t})$$

Simulate M trajectories: $B^{(m)}(t_0) = B_0; \quad t_0, t_1, t_2, \ldots, t_N$

$$B_n^{(m)} \equiv B^{(m)}(t_n); \quad \Delta t_n = t_n - t_{n-1};$$

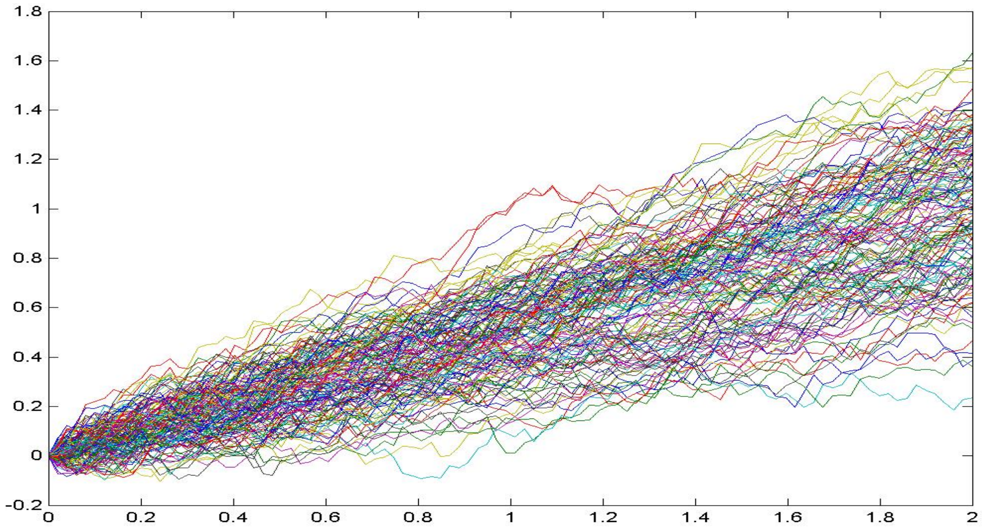
$$B_n^{(m)} = B_{n-1}^{(m)} + \mu \Delta t_n + \sigma \sqrt{\Delta t_n} Z_n^{(m)}; \quad X_n^{(m)} \sim N(0,1); \quad n = 1,2,\ldots,N;$$

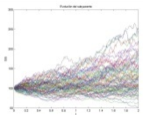
$$m = 1,2,\ldots,M;$$



Generalizec Wiener process: simulation

```
function B = simulateGeneralizedWiener(M,B0,mu,sigma,times)
%simulateGeneralizedWiener: Simulate generalized Wiener Process
% (arithmetic Brownian motion)
%
% EXAMPLE 1:
%       B0 = 1.0;  mu = 0.5; sigma = 0.2;
%       T = 2; N = 100; times = linspace(0,T,N);
%       M = 150;
%       B = simulateGeneralizedWiener(M,B0,mu,sigma,times);
%       figure(1); plot(times,B'); xlabel('t'); ylabel('B(t)')
```





Geometric Brownian Motion

$$\Delta S(t) = \mu\, S(t)\Delta t + \sigma\, S(t)\Delta W_t$$

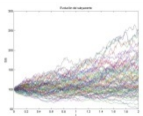
$$S(t) = S_0 \exp\left((\mu - \frac{\sigma^2}{2})\; t + \sigma\; X \sqrt{t} \right)$$

Simulation of M trajectories: $S^{(m)}(t_0) = S_0; \qquad t_0, t_1, t_2, \ldots, t_N$

$$S_n^{(m)} \equiv S^{(m)}(t_n); \quad \Delta t_n = t_n - t_{n-1};$$

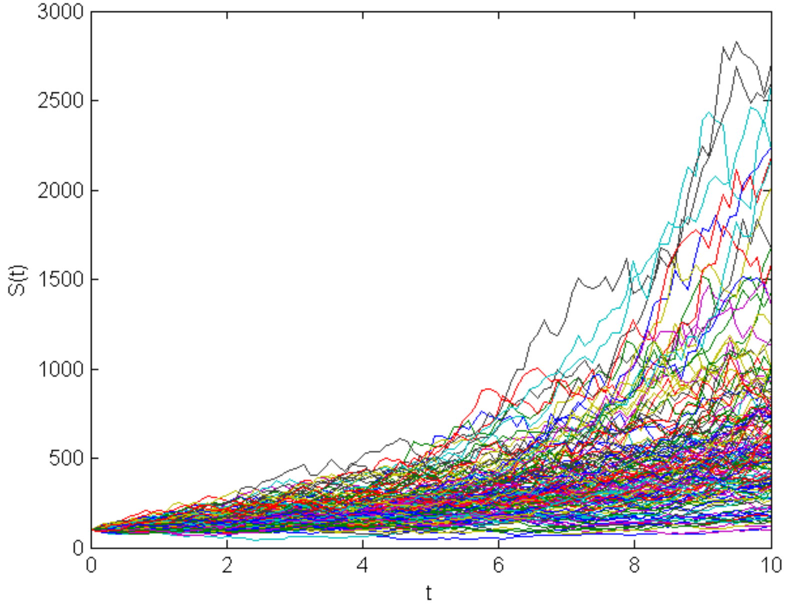
$$S_n^{(m)} = S_{n-1}^{(m)} \exp\left\{ \left(\mu - \frac{\sigma^2}{2} \right) \Delta t_n + \sigma \sqrt{\Delta t_n}\, X_n^{(m)} \right\};$$

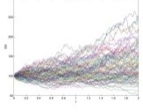
$$X_n^{(m)} \sim N(0,1); \quad n = 1,2,\ldots,N; \qquad m = 1,2,\ldots,M;$$



Geometric Brownian motion: Simulation

```
function S = simulateGeometricBrownianMotion(M,S0,mu,sigma,times)
%simulateGeometricBrownianMotion: Simulate Brownian motion
%
% EXAMPLE 1:
%       S0 = 100.0;  mu = 0.2; sigma = 0.2;
%       T = 10; N = 100; times = linspace(0,T,N);
%       M = 150;
%       S = simulateGeometricBrownianMotion(M,S0,mu,sigma,times);
%       figure(1); plot(times,S'); xlabel('t'); ylabel('S(t)')
```





Brownian bridge

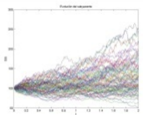
- Goal: Given a series of values for the Brownian process $B(t_0)=B_0$, $B(t_1)=B_1$, $B(t_2)=B_2$, …, $B(t_N)=B_N$
  simulate a generalized Wiener process that takes these vales

$$B(t_n) = B_n, \qquad B(t_{n+1}) = B_{n+1},$$

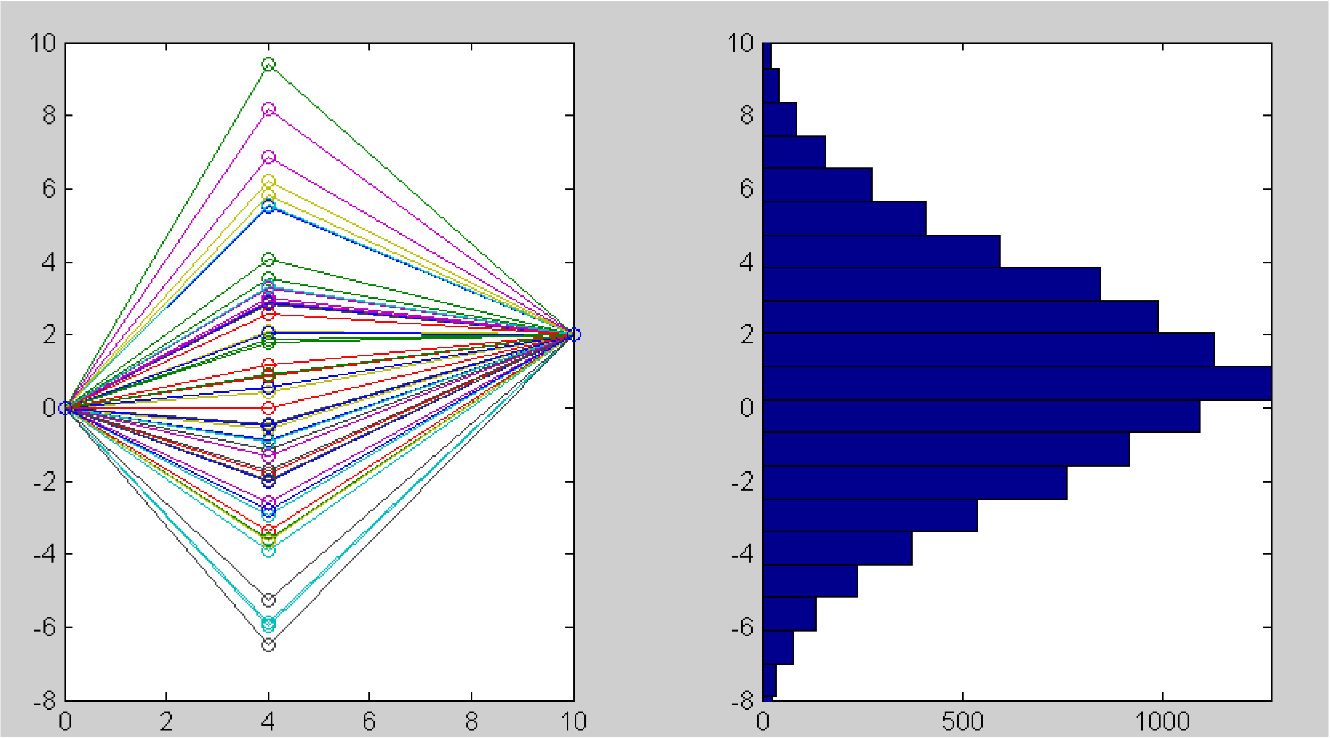
$$B^{(m)}(t) = \left[B_n + \frac{t - t_n}{t_{n+1} - t_n}\left(B_{n+1} - B_n\right) \right] + \sigma \sqrt{\frac{(t - t_n)(t_{n+1} - t)}{t_{n+1} - t_n}}\, X^{(m)};$$

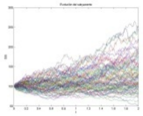
$$t_n \leq t \leq t_{n+1}; \qquad n = 0,1,\ldots,(N\text{-}1);$$

$$X^{(m)} \sim N(0,1); \qquad m = 1,2,\ldots,M;$$

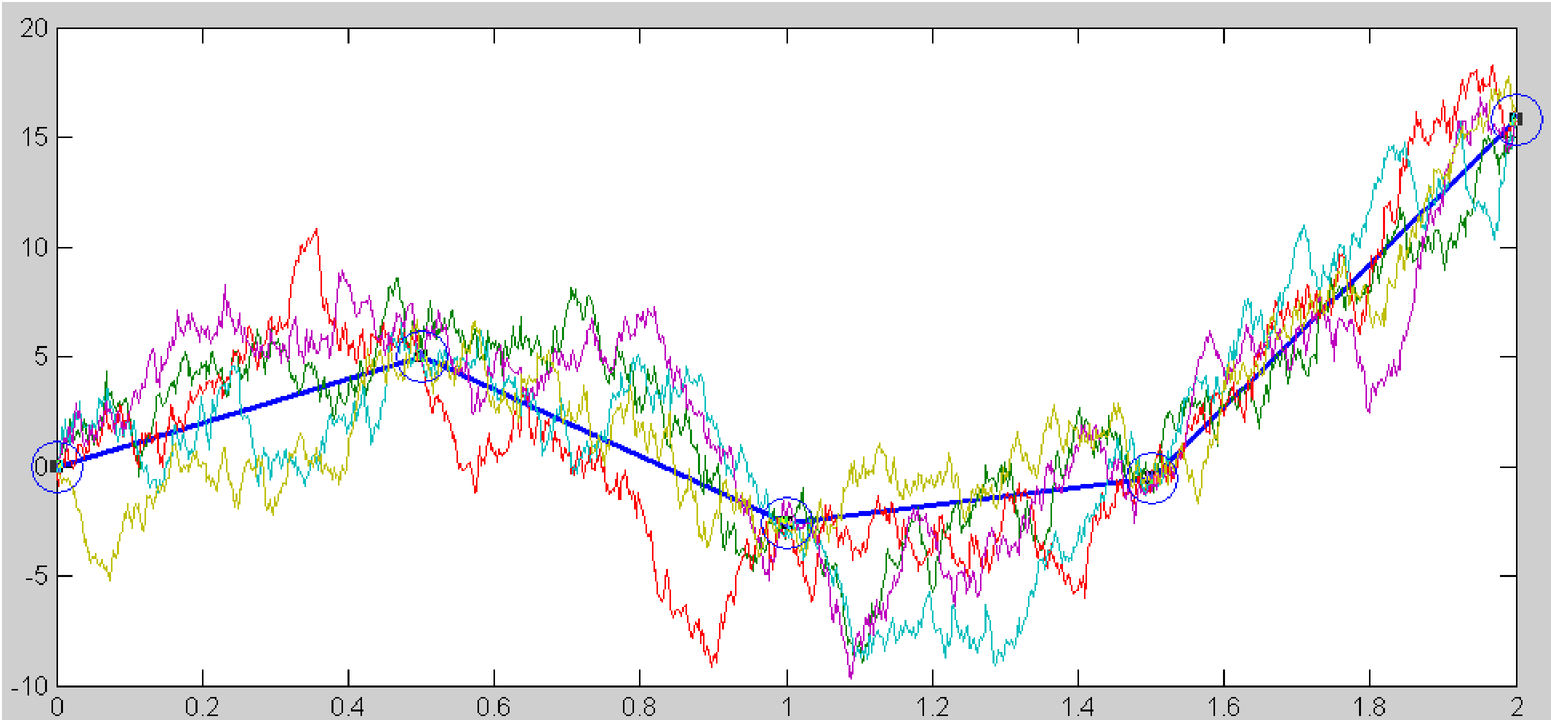


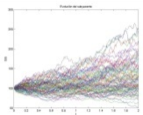
Brownian bridge: Simulation I





Brownian bridge: Simulation II





Brownian bridge: Simulation III