SDE and its financial applications

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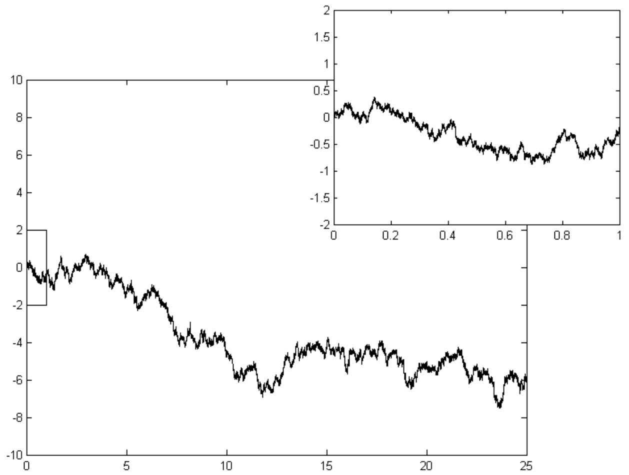
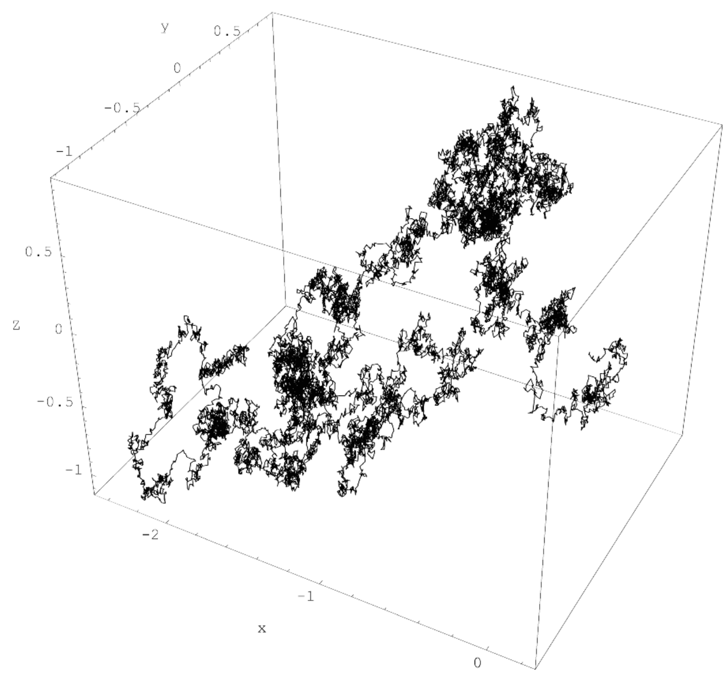


A single realization of a one-dimensional Wiener process



A single realization of a three-dimensional Wiener process

In mathematics, the **Wiener process** is a continuous-time stochastic process named in honor of Norbert Wiener. It is often called standard **Brownian motion**, after Robert Brown. It is one of the best known Lévy processes (càdlàg stochastic processes with stationary independent increments) and occurs frequently in pure and applied mathematics, economics, quantitative finance and physics.

The Wiener process plays an important role both in pure and applied mathematics. In pure mathematics, the Wiener process gave rise to the study of continuous time martingales. It is a key process in terms of which more complicated stochastic processes can be described. As such, it plays a vital role in stochastic calculus, diffusion processes and even potential theory. It is the driving process of Schramm–Loewner evolution. In applied mathematics, the Wiener process is used to represent the integral of a Gaussian white noise process, and so is useful as a model of noise in electronics engineering, instrument errors in filtering theory and unknown forces in control theory.

The Wiener process has applications throughout the mathematical sciences. In physics it is used to study Brownian motion, the diffusion of minute particles suspended in fluid, and other types of diffusion via the Fokker–Planck and Langevin equations. It also forms the basis for the rigorous path integral formulation of quantum mechanics (by the Feynman–Kac formula, a solution to the Schrödinger equation can be represented in terms of the Wiener process) and the study of eternal inflation in physical cosmology. It is also prominent in the mathematical theory of finance, in particular the Black–Scholes option pricing model.

Characterizations of the Wiener process

The Wiener process W_t is characterized by three properties:[1]

1. $W_0 = 0$
2. The function $t \to W_t$ is almost surely everywhere continuous
3. W_t has independent increments with $W_t - W_s \sim N(0, t-s)$ (for $0 \le s < t$), where $N(\mu, \sigma^2)$ denotes the normal distribution with expected value μ and variance σ^2.

The last condition means that if $0 \le s_1 < t_1 \le s_2 < t_2$ then $W_t1 - W_s1$ and $W_t2 - W_s2$ are independent random variables, and the similar condition holds for n increments.

An alternative characterization of the Wiener process is the so-called *Lévy characterization* that says that the Wiener process is an almost surely continuous martingale with $W_0 = 0$ and quadratic variation $[W_t, W_t] = t$ (which means that $W_t^2 - t$ is also a martingale).

A third characterization is that the Wiener process has a spectral representation as a sine series whose coefficients are independent $N(0, 1)$ random variables. This representation can be obtained using the Karhunen–Loève theorem.

The Wiener process can be constructed as the scaling limit of a random walk, or other discrete-time stochastic processes with stationary independent increments. This is known as Donsker's theorem. Like the random walk, the Wiener process is recurrent in one or two dimensions (meaning that it returns almost surely to any fixed neighborhood of the origin infinitely often) whereas it is not recurrent in dimensions three and higher. Unlike the random walk, it is scale invariant, meaning that

$$\alpha^{-1} W_{\alpha^2 t}$$

is a Wiener process for any nonzero constant α. The **Wiener measure** is the probability law on the space of continuous functions g, with $g(0) = 0$, induced by the Wiener process. An integral based on Wiener measure may be called a **Wiener integral**.

Properties of a one-dimensional Wiener process

Basic properties

The unconditional probability density function at a fixed time t:

$$f_{W_t}(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}.$$

The expectation is zero:

$$E[W_t] = 0.$$

The variance, using the computational formula, is t:

$$\mathrm{Var}(W_t) = E\left[W_t^2\right] - E^2[W_t] = E\left[W_t^2\right] - 0 = E\left[W_t^2\right] = t.$$

The covariance and correlation:

$$\mathrm{cov}(W_s, W_t) = \min(s, t),$$

$$\mathrm{corr}(W_s, W_t) = \frac{\mathrm{cov}(W_s, W_t)}{\sigma_{W_s}\sigma_{W_t}} = \frac{\min(s, t)}{\sqrt{st}} = \sqrt{\frac{\min(s, t)}{\max(s, t)}}.$$

The results for the expectation and variance follow immediately from the definition that increments have a normal distribution, centered at zero. Thus

$$W_t = W_t - W_0 \sim N(0, t).$$

The results for the covariance and correlation follow from the definition that non-overlapping increments are independent, of which only the property that they are uncorrelated is used. Suppose that $t_1 < t_2$.

$$\operatorname{cov}(W_{t_1}, W_{t_2}) = E\left[(W_{t_1} - E[W_{t_1}]) \cdot (W_{t_2} - E[W_{t_2}])\right] = E\left[W_{t_1} \cdot W_{t_2}\right].$$

Substituting

$$W_{t_2} = (W_{t_2} - W_{t_1}) + W_{t_1}$$

we arrive at:

$$E[W_{t_1} \cdot W_{t_2}] = E\left[W_{t_1} \cdot ((W_{t_2} - W_{t_1}) + W_{t_1})\right] = E\left[W_{t_1} \cdot (W_{t_2} - W_{t_1})\right] + E\left[W_{t_1}^2\right].$$

Since $W(t_1) = W(t_1) - W(t_0)$ and $W(t_2) - W(t_1)$, are independent,

$$E\left[W_{t_1} \cdot (W_{t_2} - W_{t_1})\right] = E[W_{t_1}] \cdot E[W_{t_2} - W_{t_1}] = 0.$$

Thus

$$\operatorname{cov}(W_{t_1}, W_{t_2}) = E\left[W_{t_1}^2\right] = t_1.$$

Running maximum

The joint distribution of the running maximum

$$M_t = \max_{0 \le s \le t} W_s$$

and W_t is

$$f_{M_t, W_t}(m, w) = \frac{2(2m - w)}{t\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}}, \qquad m \ge 0, w \le m.$$

To get the unconditional distribution of f_{M_t}, integrate over $-\infty < w \le m$

$$f_{M_t}(m) = \int_{-\infty}^{m} f_{M_t, W_t}(m, w)\, dw = \int_{-\infty}^{m} \frac{2(2m - w)}{t\sqrt{2\pi t}} e^{-\frac{(2m-w)^2}{2t}}\, dw = \sqrt{\frac{2}{\pi t}} e^{-\frac{m^2}{2t}}$$

And the expectation[2]

$$E[M_t] = \int_0^\infty m f_{M_t}(m)\, dm = \int_0^\infty m \sqrt{\frac{2}{\pi t}} e^{-\frac{m^2}{2t}}\, dm = \sqrt{\frac{2t}{\pi}}$$

Self-similarity

Brownian scaling

For every $c > 0$ the process $V_t = (1/\sqrt{c}) W_{ct}$ is another Wiener process.

Time reversal

The process $V_t = W_1 - W_{1-t}$ for $0 \le t \le 1$ is distributed like W_t for $0 \le t \le 1$.

Time inversion

The process $V_t = t W_{1/t}$ is another Wiener process.

A class of Brownian martingales

If a polynomial $p(x, t)$ satisfies the PDE

$$\left(\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right) p(x, t) = 0$$

then the stochastic process

$$M_t = p(W_t, t)$$

is a martingale.

Example: $W_t^2 - t$ is a martingale, which shows that the quadratic variation of *W* on [0, *t*] is equal to *t*. It follows that the expected time of first exit of *W* from (−*c*, *c*) is equal to c^2.
More generally, for every polynomial *p*(*x*, *t*) the following stochastic process is a martingale:

$$M_t = p(W_t, t) - \int_0^t a(W_s, s)\,\mathrm{d}s,$$

where *a* is the polynomial

$$a(x,t) = \left(\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right) p(x,t).$$

Example: $p(x,t) = (x^2 - t)^2,\ a(x,t) = 4x^2;$ the process

$$(W_t^2 - t)^2 - 4\int_0^t W_s^2\,\mathrm{d}s$$

is a martingale, which shows that the quadratic variation of the martingale $W_t^2 - t$ on [0, *t*] is equal to

$$4\int_0^t W_s^2\,\mathrm{d}s.$$

About functions *p*(*xa*, *t*) more general than polynomials, see local martingales.

Some properties of sample paths

The set of all functions *w* with these properties is of full Wiener measure. That is, a path (sample function) of the Wiener process has all these properties almost surely.

Qualitative properties

- For every ε > 0, the function *w* takes both (strictly) positive and (strictly) negative values on (0, ε).
- The function *w* is continuous everywhere but differentiable nowhere (like the Weierstrass function).
- Points of local maximum of the function *w* are a dense countable set; the maximum values are pairwise different; each local maximum is sharp in the following sense: if *w* has a local maximum at *t* then

$$\lim_{s\to t} \frac{|w(s) - w(t)|}{|s-t|} \to \infty.$$

  The same holds for local minima.

- The function *w* has no points of local increase, that is, no *t* > 0 satisfies the following for some ε in (0, *t*): first, *w*(*s*) ≤ *w*(*t*) for all *s* in (*t* − ε, *t*), and second, *w*(*s*) ≥ *w*(*t*) for all *s* in (*t*, *t* + ε). (Local increase is a weaker condition than that *w* is increasing on (*t* − ε, *t* + ε).) The same holds for local decrease.
- The function *w* is of unbounded variation on every interval.
- Zeros of the function *w* are a nowhere dense perfect set of Lebesgue measure 0 and Hausdorff dimension 1/2 (therefore, uncountable).

Quantitative properties

Law of the iterated logarithm

$$\limsup_{t\to+\infty} \frac{|w(t)|}{\sqrt{2t\log\log t}} = 1, \quad \text{almost surely.}$$

Modulus of continuity

Local modulus of continuity:

$$\limsup_{\varepsilon\to 0+} \frac{|w(\varepsilon)|}{\sqrt{2\varepsilon\log\log(1/\varepsilon)}} = 1, \qquad \text{almost surely.}$$

Global modulus of continuity (Lévy):

$$\limsup_{\varepsilon\to 0+} \sup_{0\le s<t\le 1, t-s\le\varepsilon} \frac{|w(s)-w(t)|}{\sqrt{2\varepsilon\log(1/\varepsilon)}} = 1, \qquad \text{almost surely.}$$

Local time

The image of the Lebesgue measure on [0, t] under the map w (the pushforward measure) has a density $L_t(\cdot)$. Thus,

$$\int_0^t f(w(s))\,\mathrm{d}s = \int_{-\infty}^{+\infty} f(x)L_t(x)\,\mathrm{d}x$$

for a wide class of functions f (namely: all continuous functions; all locally integrable functions; all non-negative measurable functions). The density L_t is (more exactly, can and will be chosen to be) continuous. The number $L_t(x)$ is called the local time at x of w on [0, t]. It is strictly positive for all x of the interval (a, b) where a and b are the least and the greatest value of w on [0, t], respectively. (For x outside this interval the local time evidently vanishes.) Treated as a function of two variables x and t, the local time is still continuous. Treated as a function of t (while x is fixed), the local time is a singular function corresponding to a nonatomic measure on the set of zeros of w.

These continuity properties are fairly non-trivial. Consider that the local time can also be defined (as the density of the pushforward measure) for a smooth function. Then, however, the density is discontinuous, unless the given function is monotone. In other words, there is a conflict between good behavior of a function and good behavior of its local time. In this sense, the continuity of the local time of the Wiener process is another manifestation of non-smoothness of the trajectory.

Related processes

The stochastic process defined by

$$X_t = \mu t + \sigma W_t$$

is called a **Wiener process with drift μ** and infinitesimal variance σ^2. These processes exhaust continuous Lévy processes.

The generator of a Brownian motion is ½ times the Laplace–Beltrami operator. The image above is of the Brownian motion on a special manifold: the surface of a sphere.

Two random processes on the time interval [0, 1] appear, roughly speaking, when conditioning the Wiener process to vanish on both ends of [0,1]. With no further conditioning, the process takes both positive and negative values on [0, 1] and is called Brownian bridge. Conditioned also to stay positive on (0, 1), the process is called Brownian excursion.[3] In both cases a rigorous treatment involves a limiting procedure, since the formula $P(A|B) = P(A \cap B)/P(B)$ does not apply when $P(B) = 0$.

A geometric Brownian motion can be written

$$e^{\mu t - \frac{\sigma^2 t}{2} + \sigma W_t}.$$

It is a stochastic process which is used to model processes that can never take on negative values, such as the value of stocks.

The stochastic process

$$X_t = e^{-t} W_{e^{2t}}$$

is distributed like the Ornstein–Uhlenbeck process.

The time of hitting a single point $x > 0$ by the Wiener process is a random variable with the Lévy distribution. The family of these random variables (indexed by all positive numbers x) is a left-continuous modification of a Lévy process. The right-continuous modification of this process is given by times of first exit from closed intervals $[0, x]$.

The local time $L = (L^x_t)_{x \in \mathbf{R},\, t \geq 0}$ of a Brownian motion describes the time that the process spends at the point x. Formally

$$L^x(t) = \int_0^t \delta(x - B_t)\, ds$$

where δ is the Dirac delta function. The behaviour of the local time is characterised by Ray–Knight theorems.

Brownian martingales

Let A be an event related to the Wiener process (more formally: a set, measurable with respect to the Wiener measure, in the space of functions), and X_t the conditional probability of A given the Wiener process on the time interval $[0, t]$ (more formally: the Wiener measure of the set of trajectories whose concatenation with the given partial trajectory on $[0, t]$ belongs to A). Then the process X_t is a continuous martingale. Its martingale property follows immediately from the definitions, but its continuity is a very special fact – a special case of a general theorem stating that all Brownian martingales are continuous. A Brownian martingale is, by definition, a martingale adapted to the Brownian filtration; and the Brownian filtration is, by definition, the filtration generated by the Wiener process.

Integrated Brownian motion

The time-integral of the Wiener process

$$W^{(-1)}(t) := \int_0^t W(s) ds$$

is called **integrated Brownian motion** or **integrated Wiener process**. It arises in many applications and can be shown to have the distribution $N(0, t^3/3)$, calculus lead using the fact that the covariation of the Wiener process is $t \wedge s$.[4]

Time change

Every continuous martingale (starting at the origin) is a time changed Wiener process.

Example: $2W_t = V(4t)$ where V is another Wiener process (different from W but distributed like W).

Example. $W_t^2 - t = V_{A(t)}$ where $A(t) = 4 \int_0^t W_s^2 \, \mathrm{d}s$ and V is another Wiener process.

In general, if M is a continuous martingale then $M_t - M_0 = V_{A(t)}$ where $A(t)$ is the quadratic variation of M on $[0, t]$, and V is a Wiener process.

Corollary. (See also Doob's martingale convergence theorems) Let M_t be a continuous martingale, and

$$M^-_\infty = \liminf_{t\to\infty} M_t,$$
$$M^+_\infty = \limsup_{t\to\infty} M_t.$$

Then only the following two cases are possible:

$$-\infty < M_\infty^- = M_\infty^+ < +\infty,$$
$$-\infty = M_\infty^- < M_\infty^+ = +\infty;$$

other cases (such as $M_\infty^- = M_\infty^+ = +\infty, \quad M_\infty^- < M_\infty^+ < +\infty$ etc.) are of probability 0.

Especially, a nonnegative continuous martingale has a finite limit (as $t \to \infty$) almost surely.

All stated (in this subsection) for martingales holds also for local martingales.

Change of measure

A wide class of continuous semimartingales (especially, of diffusion processes) is related to the Wiener process via a combination of time change and change of measure.

Using this fact, the qualitative properties stated above for the Wiener process can be generalized to a wide class of continuous semimartingales.

Complex-valued Wiener process

The complex-valued Wiener process may be defined as a complex-valued random process of the form $Z_t = X_t + iY_t$ where X_t, Y_t are independent Wiener processes (real-valued).[5]

Self-similarity

Brownian scaling, time reversal, time inversion: the same as in the real-valued case.

Rotation invariance: for every complex number c such that $|c| = 1$ the process cZ_t is another complex-valued Wiener process.

Time change

If f is an entire function then the process $f(Z_t) - f(0)$ is a time-changed complex-valued Wiener process.

Example: $Z_t^2 = (X_t^2 - Y_t^2) + 2X_tY_ti = U_{A(t)}$ where

$$A(t) = 4\int_0^t |Z_s|^2 \, \mathrm{d}s$$

and U is another complex-valued Wiener process.

In contrast to the real-valued case, a complex-valued martingale is generally not a time-changed complex-valued Wiener process. For example, the martingale $2X_t + iY_t$ is not (here X_t, Y_t are independent Wiener processes, as before).

Notes

[1] Durrett 1996, Sect. 7.1
[2] Shreve, Steven E (2008). *Stochastic Calculus for Finance II: Continuous Time Models*. Springer. pp. 114. ISBN 978-0-387-40101-0.
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[5] Navarro-moreno, J.; Estudillo-martinez, M.D; Fernandez-alcala, R.M.; Ruiz-molina, J.C., "Estimation of Improper Complex-Valued Random Signals in Colored Noise by Using the Hilbert Space Theory" (http://ieeexplore.ieee.org/Xplore/login.jsp?url=http://ieeexplore.ieee.org/iel5/18/4957623/04957648.pdf?arnumber=4957648&authDecision=-203), *IEEE Transactions on Information Theory* **55** (6): 2859–2867, doi:10.1109/TIT.2009.2018329, , retrieved 2010-03-30

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External links

- Brownian motion java simulation (http://galileo.phys.virginia.edu/classes/109N/more_stuff/Applets/brownian/applet.html)
- Article for the school-going child (http://xxx.imsc.res.in/abs/physics/0412132)
- Einstein on Brownian Motion (http://www.bun.kyoto-u.ac.jp/~suchii/einsteinBM.html)
- Brownian Motion, "Diverse and Undulating" (http://arxiv.org/abs/0705.1951)
- Short Movie on Brownian Motion (http://www.composite-agency.com/brownian_movement.htm)
- Discusses history, botany and physics of Brown's original observations, with videos (http://physerver.hamilton.edu/Research/Brownian/index.html)
- "Einstein's prediction finally witnessed one century later" (http://www.gizmag.com/einsteins-prediction-finally-witnessed/16212/) : a test to observe the velocity of Brownian motion

Stochastic differential equation

A **stochastic differential equation** (**SDE**) is a differential equation in which one or more of the terms is a stochastic process, resulting in a solution which is itself a stochastic process. SDE are used to model diverse phenomena such as fluctuating stock prices or physical system subject to thermal fluctuations. Typically, SDEs incorporate white noise which can be thought of as the derivative of Brownian motion (or the Wiener process); however, it should be mentioned that other types of random fluctuations are possible, such as jump processes.

Background

The earliest work on SDEs was done to describe Brownian motion in Einstein's famous paper, and at the same time by Smoluchowski. However, one of the earlier works related to Brownian motion is credited to Bachelier (1900) in his thesis 'Theory of Speculation'. This work was followed upon by Langevin. Later Itō and Stratonovich put SDEs on more solid mathematical footing.

Terminology

In physical science, SDEs are usually written as Langevin equations. These are sometimes confusingly called "the Langevin equation" even though there are many possible forms. These consist of an ordinary differential equation containing a deterministic part and an additional random white noise term. A second form is the Smoluchowski equation and, more generally, the Fokker-Planck equation. These are partial differential equations that describe the time evolution of probability distribution functions. The third form is the stochastic differential equation that is used most frequently in mathematics and quantitative finance (see below). This is similar to the Langevin form, but it is usually written in differential form. SDEs come in two varieties, corresponding to two versions of stochastic calculus.

Stochastic Calculus

Brownian motion or the Wiener process was discovered to be exceptionally complex mathematically. The Wiener process is nowhere differentiable; thus, it requires its own rules of calculus. There are two dominating versions of stochastic calculus, the Ito stochastic calculus and the Stratonovich stochastic calculus. Each of the two has advantages and disadvantages, and newcomers are often confused whether the one is more appropriate than the other in a given situation. Guidelines exist (e.g. Øksendal, 2003) and conveniently, one can readily convert an Ito SDE to an equivalent Stratonovich SDE and back again. Still, one must be careful which calculus to use when the SDE is initially written down.

Numerical Solutions

Numerical solution of stochastic differential equations and especially stochastic partial differential equations is a young field relatively speaking. Almost all algorithms that are used for the solution of ordinary differential equations will work very poorly for SDEs, having very poor numerical convergence. A textbook describing many different algorithms is Kloeden & Platen (1995).

Methods include the Euler–Maruyama method, Milstein method and Runge–Kutta method (SDE).

Use in Physics

In physics, SDEs are typically written in the Langevin form and referred to as "the Langevin equation." For example, a general coupled set of first-order SDEs is often written in the form:

$$\dot{x}_i = \frac{dx_i}{dt} = f_i(\mathbf{x}) + \sum_{m=1}^{n} g_i^m(\mathbf{x})\eta_m(t),$$

where $\mathbf{x} = \{x_i | 1 \le i \le k\}$ is the set of unknowns, the f_i and g_i are arbitrary functions and the η_m are random functions of time, often referred to as "noise terms". This form is usually usable because there are standard techniques for transforming higher-order equations into several coupled first-order equations by introducing new unknowns. If the g_i are constants, the system is said to be subject to additive noise, otherwise it is said to be subject to multiplicative noise. This term is somewhat misleading as it has come to mean the general case even though it appears to imply the limited case where : $g(x) \propto x$. Additive noise is the simpler of the two cases; in that situation the correct solution can often be found using ordinary calculus and in particular the ordinary chain rule of calculus. However, in the case of multiplicative noise, the Langevin equation is not a well-defined entity on its own, and it must be specified whether the Langevin equation should be interpreted as an Ito SDE or a Stratonovich SDE.

In physics, the main method of solution is to find the probability distribution function as a function of time using the equivalent Fokker-Planck equation (FPE). The Fokker-Planck equation is a deterministic partial differential equation. It tells how the probability distribution function evolves in time similarly to how the Schrödinger equation gives the time evolution of the quantum wave function or the diffusion equation gives the time evolution of chemical concentration. Alternatively numerical solutions can be obtained by Monte Carlo simulation. Other techniques include the path integration that draws on the analogy between statistical physics and quantum mechanics (for example, the Fokker-Planck equation can be transformed into the Schrödinger equation by rescaling a few variables) or by writing down ordinary differential equations for the statistical moments of the probability distribution function.

Use in probability and mathematical finance

The notation used in probability theory (and in many applications of probability theory, for instance mathematical finance) is slightly different. This notation makes the exotic nature of the random function of time η_m in the physics formulation more explicit. It is also the notation used in publications on numerical methods for solving stochastic differential equations. In strict mathematical terms, η_m can not be chosen as a usual function, but only as a generalized function. The mathematical formulation treats this complication with less ambiguity than the physics formulation.

A typical equation is of the form

$$\mathrm{d}X_t = \mu(X_t, t)\,\mathrm{d}t + \sigma(X_t, t)\,\mathrm{d}B_t,$$

where B denotes a Wiener process (Standard Brownian motion). This equation should be interpreted as an informal way of expressing the corresponding integral equation

$$X_{t+s} - X_t = \int_t^{t+s} \mu(X_u, u)\mathrm{d}u + \int_t^{t+s} \sigma(X_u, u)\,\mathrm{d}B_u.$$

The equation above characterizes the behavior of the continuous time stochastic process X_t as the sum of an ordinary Lebesgue integral and an Itō integral. A heuristic (but very helpful) interpretation of the stochastic differential equation is that in a small time interval of length δ the stochastic process X_t changes its value by an amount that is normally distributed with expectation $\mu(X_t, t)\,\delta$ and variance $\sigma(X_t, t)^2\,\delta$ and is independent of the past behavior of the process. This is so because the increments of a Wiener process are independent and normally distributed. The function μ is referred to as the drift coefficient, while σ is called the diffusion coefficient. The stochastic process X_t is called a diffusion process, and is usually a Markov process.

The formal interpretation of an SDE is given in terms of what constitutes a solution to the SDE. There are two main definitions of a solution to an SDE, a strong solution and a weak solution. Both require the existence of a process X_t that solves the integral equation version of the SDE. The difference between the two lies in the underlying probability space (Ω, F, Pr). A weak solution consists of a probability space and a process that satisfies the integral equation, while a strong solution is a process that satisfies the equation and is defined on a given probability space.

An important example is the equation for geometric Brownian motion

$$\mathrm{d}X_t = \mu X_t\,\mathrm{d}t + \sigma X_t\,\mathrm{d}B_t.$$

which is the equation for the dynamics of the price of a stock in the Black Scholes options pricing model of financial mathematics.

There are also more general stochastic differential equations where the coefficients μ and σ depend not only on the present value of the process X_t, but also on previous values of the process and possibly on present or previous values of other processes too. In that case the solution process, X, is not a Markov process, and it is called an Itō process and not a diffusion process. When the coefficients depends only on present and past values of X, the defining equation is called a stochastic delay differential equation.

Existence and uniqueness of solutions

As with deterministic ordinary and partial differential equations, it is important to know whether a given SDE has a solution, and whether or not it is unique. The following is a typical existence and uniqueness theorem for Itō SDEs taking values in n-dimensional Euclidean space $\mathbf{R}^n$ and driven by an m-dimensional Brownian motion B; the proof may be found in Øksendal (2003, §5.2).

Let $T > 0$, and let

$$\mu : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n;$$
$$\sigma : \mathbb{R}^n \times [0,T] \to \mathbb{R}^{n\times m};$$

be measurable functions for which there exist constants C and D such that

$$|\mu(x,t)| + |\sigma(x,t)| \leq C(1+|x|);$$
$$|\mu(x,t) - \mu(y,t)| + |\sigma(x,t) - \sigma(y,t)| \leq D|x-y|;$$

for all $t \in [0, T]$ and all x and $y \in \mathbf{R}^n$, where

$$|\sigma|^2 = \sum_{i,j=1}^{n} |\sigma_{ij}|^2.$$

Let Z be a random variable that is independent of the σ-algebra generated by B_s, $s \geq 0$, and with finite second moment:

$$\mathbb{E}\big[|Z|^2\big] < +\infty.$$

Then the stochastic differential equation/initial value problem

$$\mathrm{d}X_t = \mu(X_t,t)\,\mathrm{d}t + \sigma(X_t,t)\,\mathrm{d}B_t \text{ for } t \in [0,T];$$
$$X_0 = Z;$$

has a Pr-almost surely unique t-continuous solution $(t, \omega) \mapsto X_t(\omega)$ such that X is adapted to the filtration F_t^Z generated by Z and B_s, $s \leq t$, and

$$\mathbb{E}\left[\int_0^T |X_t|^2\,\mathrm{d}t\right] < +\infty.$$

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Diffusion process

In probability theory, a branch of mathematics, a **diffusion process** is a solution to a stochastic differential equation. It is a continuous-time Markov process with almost surely continuous sample paths. Brownian motion, reflected Brownian motion and Ornstein–Uhlenbeck processes are examples of diffusion processes.

A sample path of a diffusion process models the trajectory of a particle embedded in a flowing fluid and subjected to random displacements due to collisions with molecules, which is called Brownian motion. The position of the particle is then random; its probability density function as a function of space and time is governed by an advection-diffusion equation.

Mathematical definition

A *diffusion process* is a Markov process with continuous sample paths for which the Kolmogorov forward equation is the Fokker-Planck equation.[1]

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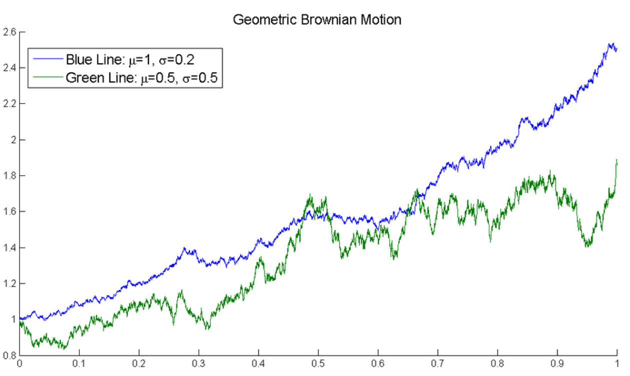
Geometric Brownian motion

A **geometric Brownian motion (GBM)** (also known as **exponential Brownian motion**) is a continuous-time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion (also called a Wiener process) with drift.[1] It is an important example of stochastic processes satisfying a stochastic differential equation (SDE); in particular, it is used in mathematical finance to model stock prices in the Black–Scholes model.



Two sample paths of Geometric Brownian motion, with different parameters. The blue line has larger drift, the green line has larger variance.

Technical definition: the SDE

A stochastic process S_t is said to follow a GBM if it satisfies the following stochastic differential equation (SDE):

$$dS_t = \mu S_t\,dt + \sigma S_t\,dW_t$$

where W_t is a Wiener process or Brownian motion and μ ('the percentage drift') and σ ('the percentage volatility') are constants. The latter term is often used to model a set of unpredictable events occurring during this motion, while the former is used to model deterministic trends.

Solving the SDE

For an arbitrary initial value S_0 the above SDE has the analytic solution (under Itō's interpretation):

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right).$$

To arrive at this formula, let us divide the SDE by S_t, and write it in Itō integral form:

$$\int_0^t \frac{dS_t}{S_t} = \mu\, t + \sigma\, W_t\,, \qquad \text{assuming } W_0 = 0\,.$$

Of course, $\frac{dS_t}{S_t}$ looks related to the derivative of $\ln S_t$; however, S_t being an Itō process, we need to use Itō calculus: by Itō's formula, we have

$$d(\ln S_t) = \frac{dS_t}{S_t} - \frac{1}{2}\,\sigma^2\,dt\,.$$

Plugging back to the equation we got from the SDE, we obtain

$$\ln\frac{S_t}{S_0} = \left(\mu - \frac{1}{2}\,\sigma^2\right)t + \sigma W_t\,.$$

Exponentiating gives the solution claimed above.

Properties of GBM

The above solution S_t (for any value of t) is a log-normally distributed random variable with expected value and variance given by[2]

$$\mathbb{E}(S_t) = S_0 e^{\mu t},$$
$$\operatorname{Var}(S_t) = S_0^2 e^{2\mu t}\left(e^{\sigma^2 t} - 1\right),$$

that is the probability density function of a S_t is:

$$f_{S_t}(s;\mu,\sigma,t) = \frac{1}{\sqrt{2\pi}}\,\frac{1}{s\sigma\sqrt{t}}\,\exp\left(-\frac{\left(\ln s - \ln S_0 - \left(\mu - \frac{1}{2}\sigma^2\right)t\right)^2}{2\sigma^2 t}\right).$$

When deriving further properties of GBM, use can be made of the SDE of which GBM is the solution, or the explicit solution given above can be used. For example, consider the stochastic process $\log(S_t)$. This is an interesting process, because in the Black–Scholes model it is related to the log return of the stock price. Using Itō's lemma with $f(S) = \log(S)$ gives

$$\begin{aligned}
d\log(S) &= f'(S)\,dS + \frac{1}{2}f''(S)S^2\sigma^2\,dt \\
&= \frac{1}{S}\left(\sigma S\,dW_t + \mu S\,dt\right) - \frac{1}{2}\sigma^2\,dt \\
&= \sigma\,dW_t + (\mu - \sigma^2/2)\,dt.
\end{aligned}$$

It follows that $\mathbb{E}\log(S_t) = \log(S_0) + (\mu - \sigma^2/2)t$.

This result can also be derived by applying the logarithm to the explicit solution of GBM:

$$\begin{aligned}
\log(S_t) &= \log\left(S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)\right) \\
&= \log(S_0) + \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t.
\end{aligned}$$

Taking the expectation yields the same result as above: $\mathbb{E}\log(S_t) = \log(S_0) + (\mu - \sigma^2/2)t$.

Multivariate Geometric Brownian motion

GBM can be extended to the cast where there are multiple correlated price paths.

Each price path follows the underlying process

$$dS_t^i = \mu_i S_t^i \, dt + \sigma_i S_t^i \, dW_t^i,$$

where the Wiener processes are correlated such that $\mathbb{E}(dW_t^i dW_t^j) = \rho_{i,j} dt$ where $\rho_{i,i} = 1$.

For the multivariate case, this implies that

$$\mathrm{Cov}(S_t^i, S_t^j) = S_0^i S_0^j e^{(\mu_i + \mu_j)t} \left(e^{\rho_{i,j}\sigma_i\sigma_j t} - 1 \right).$$

Use of GBM in finance

Geometric Brownian Motion is used to model stock prices in the Black–Scholes model and is the most widely used model of stock price behavior.[3]

Some of the arguments for using GBM to model stock prices are:

- The expected returns of GBM are independent of the value of the process (stock price), which agrees with what we would expect in reality.[3]
- A GBM process only assumes positive values, just like real stock prices.
- A GBM process shows the same kind of 'roughness' in its paths as we see in real stock prices.
- Calculations with GBM processes are relatively easy.

However, GBM is not a completely realistic model, in particular it falls short of reality in the following points:

- In real stock prices, volatility changes over time (possibly stochastically), but in GBM, volatility is assumed constant.
- In real stock prices, returns are usually not normally distributed (real stock returns have higher kurtosis ('fatter tails'), which means there is a higher chance of large price changes).[4]

Extensions of GBM

In an attempt to make GBM more realistic as a model for stock prices, one can drop the assumption that the volatility (σ) is constant. If we assume that the volatility is a deterministic function of the stock price and time, this is called a local volatility model. If instead we assume that the volatility has a randomness of its own—often described by a different equation driven by a different Brownian Motion—the model is called a stochastic volatility model.

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[1] Ross, Sheldon M. (2007). "10.3.2". *Introduction to Probability Models*.
[2] Oksendal, Bernt K. (2002), *Stochastic Differential Equations: An Introduction with Applications*, Springer, pp. 326, ISBN 3-540-63720-6
[3] Hull, John (2009). "12.3". *Options, Futures, and other Derivatives* (7 ed.).
[4] Wilmott, Paul (2006). "16.4". *Paul Wilmott on Quantitative Finance* (2 ed.).

External links

- Geometric Brownian motion models for stock movement except in rare events. (http://www.math.nyu.edu/financial_mathematics/content/02_financial/02.html)
- R and C# Simulation of a Geometric Brownian Motion (http://forevermar.com/Brownian.php)

Black–Scholes

The **Black–Scholes** model /ˌblækˈʃoʊlz/[1] or **Black–Scholes–Merton** is a mathematical model of a financial market containing certain derivative investment instruments. From the model, one can deduce the **Black–Scholes formula**, which gives the price of European-style options. The formula led to a boom in options trading and legitimised scientifically the activities of the Chicago Board Options Exchange and other options markets around the world.[2] It is widely used by options market participants.[3]:751 Many empirical tests have shown the Black–Scholes price is "fairly close" to the observed prices, although there are well-known discrepancies such as the “option smile”.[3]:770–771

The model was first articulated by Fischer Black and Myron Scholes in their 1973 paper, "The Pricing of Options and Corporate Liabilities", published in the *Journal of Political Economy*. They derived a partial differential equation, now called the **Black–Scholes equation**, which governs the price of the option over time. The key idea behind the derivation was to hedge perfectly the option by buying and selling the underlying asset in just the right way and consequently "eliminate risk". This hedge is called delta hedging and is the basis of more complicated hedging strategies such as those engaged in by Wall Street investment banks. The hedge implies there is only one right price for the option and it is given by the Black–Scholes formula.

Robert C. Merton was the first to publish a paper expanding the mathematical understanding of the options pricing model and coined the term Black–Scholes options pricing model. Merton and Scholes received the 1997 Nobel Prize in Economics (*The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel*) for their work. Though ineligible for the prize because of his death in 1995, Black was mentioned as a contributor by the Swedish academy.[4]

Assumptions

The Black–Scholes model of the market for a particular stock makes the following explicit assumptions:

- There is no arbitrage opportunity (i.e., there is no way to make a riskless profit).
- It is possible to borrow and lend cash at a known constant risk-free interest rate.
- It is possible to buy and sell any amount, even fractional, of stock (this includes short selling).
- The above transactions do not incur any fees or costs (i.e., frictionless market).
- The stock price follows a geometric Brownian motion with constant drift and volatility.
- The underlying security does not pay a dividend.[5]

From these assumptions, Black and Scholes showed that “it is possible to create a hedged position, consisting of a long position in the stock and a short position in the option, whose value will not depend on the price of the stock.”[6]

Several of these assumptions of the original model have been removed in subsequent extensions of the model. Modern versions account for changing interest rates (Merton, 1976), transaction costs and taxes (Ingersoll, 1976), and dividend payout.[7]

Notation

Let

S, be the price of the stock (please note inconsistencies as below).

$V(S,t)$, the price of a derivative as a function of time and stock price.

$C(S,t)$ the price of a European call option and $P(S,t)$ the price of a European put option.

K, the strike of the option.

r, the annualized risk-free interest rate, continuously compounded (the force of interest).

μ, the drift rate of S, annualized.

σ, the volatility of the stock's returns; this is the square root of the quadratic variation of the stock's log price process.

t, a time in years; we generally use: now=0, expiry=T.

Π, the value of a portfolio.

Finally we will use $N(x)$ which denotes the standard normal cumulative distribution function,

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{z^2}{2}} \, dz \, .$$

$N'(x)$ which denotes the standard normal probability density function,

$$N'(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}.$$

Inconsistencies

The reader is warned of the inconsistent notation that appears in this article. Thus the letter S is used as:

1. a constant denoting the current price of the stock
2. a real variable denoting the price at an arbitrary time
3. a random variable denoting the price at maturity
4. a stochastic process denoting the price at an arbitrary time

It is also used in the meaning of (4) with a subscript denoting time, but here the subscript is merely a mnemonic.

In the partial derivatives, the letters in the numerators and denominators are, of course, real variables, and the partial derivatives themselves are, initially, real functions of real variables. But after the substitution of a stochastic process for one of the arguments they become stochastic processes.

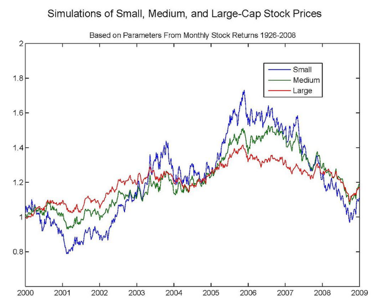
The Black–Scholes PDE is, initially, a statement about the stochastic process S, but when S is reinterpreted as a real variable, it becomes an ordinary PDE. It is only then that we can ask about its solution.

The parameter u that appears in the discrete-dividend model and the elementary derivation is not the same as the parameter μ that appears elsewhere in the article. For the relationship between them see Geometric Brownian motion.

The Black–Scholes equation

As above, the **Black–Scholes equation** is a partial differential equation, which describes the price of the option over time. The key idea behind the equation is that one can perfectly hedge the option by buying and selling the underlying asset in just the right way and consequently “eliminate risk". This hedge, in turn, implies that there is only one right price for the option, as returned by the Black–Scholes formula given in the next section. The Equation:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0.$$



Simulated geometric Brownian motions with parameters from market data

Derivation

The following derivation is given in Hull's *Options, Futures, and Other Derivatives*.[8]:287–288 That, in turn, is based on the classic argument in the original Black–Scholes paper.

Per the model assumptions above, the price of the underlying asset (typically a stock) follows a geometric Brownian motion. That is

$$\frac{dS}{S}=\mu\,dt+\sigma\,dW$$

where *W* is Brownian motion. Note that *W*, and consequently its infinitesimal increment *dW*, represents the only source of uncertainty in the price history of the stock. Intuitively, *W(t)* is a process that "wiggles up and down" in such a random way that its expected change over any time interval is 0. (In addition, its variance over time *T* is equal to *T*; see Wiener process: Basic properties); a good discrete analogue for *W* is a simple random walk. Thus the above equation states that the infinitesimal rate of return on the stock has an expected value of *μ dt* and a variance of $\sigma^2 dt$.

The payoff of an option $V(S,T)$ at maturity is known. To find its value at an earlier time we need to know how V evolves as a function of S and t. By Itō's lemma for two variables we have

$$dV=\left(\mu S\frac{\partial V}{\partial S}+\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt+\sigma S\frac{\partial V}{\partial S}\,dW.$$

Now consider a certain portfolio, called the delta-hedge portfolio, consisting of being short one option and long $\frac{\partial V}{\partial S}$ shares at time t. The value of these holdings is

$$\Pi=-V+\frac{\partial V}{\partial S}S.$$

Over the time period $[t,t+\Delta t]$, the total profit or loss from changes in the values of the holdings is:

$$\Delta\Pi=-\Delta V+\frac{\partial V}{\partial S}\,\Delta S.$$

Now discretize the equations for *dS/S* and *dV* by replacing differentials with deltas:

$$\Delta S=\mu S\,\Delta t+\sigma S\,\Delta W$$

$$\Delta V=\left(\mu S\frac{\partial V}{\partial S}+\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)\Delta t+\sigma S\frac{\partial V}{\partial S}\,\Delta W.$$

and appropriately substitute them into the expression for $\Delta\Pi$:

$$\Delta\Pi = \left(-\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)\Delta t.$$

Notice that the ΔW term has vanished. Thus uncertainty has been eliminated and the portfolio is effectively riskless. The rate of return on this portfolio must be equal to the rate of return on any other riskless instrument; otherwise, there would be opportunities for arbitrage. Now assuming the risk-free rate of return is r we must have over the time period $[t, t + \Delta t]$

$$r\Pi\,\Delta t = \Delta\Pi.$$

If we now equate our two formulas for $\Delta\Pi$ we obtain:

$$\left(-\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)\Delta t = r\left(-V + S\frac{\partial V}{\partial S}\right)\Delta t.$$

Simplifying, we arrive at the celebrated Black–Scholes partial differential equation:

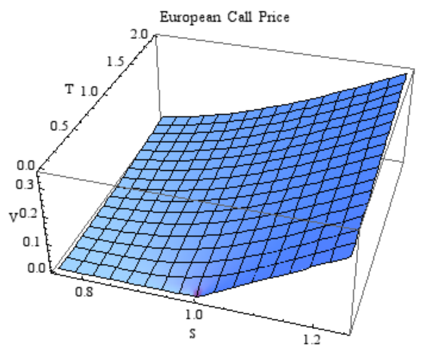
$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

With the assumptions of the Black–Scholes model, this second order partial differential equation holds for any type of option as long as its price function V is twice differentiable with respect to S and once with respect to t. Different pricing formulae for various options will arise from the choice of payoff function at expiry and appropriate boundary conditions.

Black–Scholes formula

The Black–Scholes formula calculates the price of European put and call options. This price is consistent with the Black–Scholes equation as above; this follows since the formula can be obtained by solving the equation for the corresponding terminal and boundary conditions.

The value of a call option for a non-dividend paying underlying stock in terms of the Black–Scholes parameters is:



A European call valued using the Black-Scholes pricing equation for varying asset price S and time-to-expiry T. In this particular example, the strike price is set to unity.

$$C(S,t) = N(d_1)\; S - N(d_2)\; Ke^{-r(T-t)},$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln\left(\frac{S}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}.$$

The price of a corresponding put option based on put-call parity is:

$$\begin{aligned} P(S,t) &= Ke^{-r(T-t)} - S + C(S,t) \\ &= N(-d_2)\; Ke^{-r(T-t)} - N(-d_1)\; S. \end{aligned}$$

For both, as above:

- $N(\cdot)$ is the cumulative distribution function of the standard normal distribution
- $T-t$ is the time to maturity
- S is the spot price of the underlying asset
- K is the strike price
- r is the risk free rate (annual rate, expressed in terms of continuous compounding)
- σ is the volatility of returns of the underlying asset

Alternative formulation

Introducing some auxiliary variables allows the formula to be simplified and reformulated in a more intuitive form:

$$\begin{aligned} C(F,\tau) &= D\big(N(d_+)\ F - N(d_-)\ K\big), \\ d_\pm &= \frac{\ln(F/K) \pm (\sigma^2/2)\tau}{\sigma\sqrt{\tau}}, \\ d_\mp &= d_\pm \mp \sigma\sqrt{\tau} \qquad \big(d_- = d_+ - \sigma\sqrt{\tau} \quad \text{and} \quad d_+ = d_- + \sigma\sqrt{\tau}\big). \end{aligned}$$

The auxiliary variables are:

- $\tau = T - t$ is the time to expiry (remaining time, backwards time)
- $D = e^{-r\tau}$ is the discount factor
- $F = e^{r\tau}S = S/D$ is the forward price of the underlying asset, and $S = D\ F$

with $d_+ = d_1$ and $d_- = d_2$ to clarify notation.

Given put-call parity, which is expressed in these terms as:

$$C - P = D(F-K) = S - D\ K$$

the price of a put option is:

$$P(F,\tau) = D\big(N(-d_-)\ K - N(-d_+)\ F\big).$$

Interpretation

The Black–Scholes formula can be interpreted fairly easily, with the main subtlety the interpretation of the $N(d_\pm)$ (and a fortiori $d_\pm$) terms, particularly d_+ and why there are two different terms.[9]

The formula can be interpreted by first decomposing a call option into the difference of two binary options: an asset-or-nothing call minus a cash-or-nothing call (long an asset-or-nothing call, short a cash-or-nothing call). A call option exchanges cash for an asset at expiry, while an asset-or-nothing call just yields the asset (with no cash in exchange) and a cash-or-nothing call just yields cash (with no asset in exchange). The Black–Scholes formula is a difference of two terms, and these two terms equal the value of the binary call options. These binary options are much less frequently traded than vanilla call options, but are easier to analyze.

Thus the formula:

$$C = D\big(N(d_+)\ F - N(d_-)\ K\big)$$

breaks up as:

$$C = D\ N(d_+)\ F - D\ N(d_-)\ K,$$

where $D\ N(d_+)\ F$ is the present value of an asset-or-nothing call and $D\ N(d_-)\ K$ is the present value of a cash-or-nothing call. The *D* factor is for discounting, because the expiration date is in future, and removing it changes *present* value to *future* value (value at expiry). Thus $N(d_+)\ F$ is the future value of an asset-or-nothing call and $N(d_-)\ K$ is the future value of a cash-or-nothing call. In risk-neutral terms, these are the expected value of the asset and the expected value of the cash in the risk-neutral measure.

The naive – and not quite correct – interpretation of these terms is that $N(d_+)\ F$ is the probability of the option expiring in the money $N(d_+)$, times the value of the underlying at expiry *F,* while $N(d_-)\ K$ is the probability of the option expiring in the money $N(d_-)$, times the value of the cash at expiry *K*. This is obviously incorrect, as either both binaries expire in the money or both expire out of the money (either cash is exchanged for asset or it is not), but the probabilities $N(d_+)$ and $N(d_-)$ are not equal. In fact, $d_\pm$ can be interpreted as measures of moneyness (in standard deviations) a probabilities of expiring ITM (*percent moneyness*), in the respective numéraire, as discussed below. Simply put, the interpretation of the cash option, $N(d_-)\ K$, is correct, as the value of the cash is independent of movements of the underlying, and thus can be interpreted as a simple product of "probability times value", while the $N(d_+)\ F$ is more complicated, as the probability of expiring in the money and the value of the asset at expiry are not independent.[9] More precisely, the value of the asset at expiry is variable in terms of cash, but is constant in terms of the asset itself (a fixed quantity of the asset), and thus these quantities are independent if one changes numéraire to the asset rather than cash.

If one uses spot *S* instead of forward *F*, in $d_\pm$ instead of the $\sigma^2/2$ term there is $(r \pm \sigma^2/2)\tau$, which can be interpreted as a drift factor (in the risk-neutral measure for appropriate numéraire). The use of d_- for moneyness rather than the standardized moneyness $m = \ln(F/K) \big/ \sigma\sqrt{\tau}$ – in other words, the reason for the $\sigma^2/2$ factor – is due to the difference between the median and mean of the log-normal distribution; it is the same factor as in Itō's lemma applied to geometric Brownian motion. In addition, another way to see that the naive interpretation is incorrect is that replacing $N(d_+)$ by $N(d_-)$ in the formula yields a negative value for out-of-the-money call options.[9]:6

In detail, the terms $N(d_1), N(d_2)$ are the *probabilities of the option expiring in-the-money* under the equivalent exponential martingale probability measure (numéraire=stock) and the equivalent martingale probability measure (numéraire=risk free asset), respectively.[9] The risk neutral probability density for the stock price $S_T \in (0, \infty)$ is

$$p(S,T) = \frac{N'(d_2(S_T))}{S_T \sigma \sqrt{T}}$$

where $d_2 = d_2(K)$ is defined as above.

Specifically, $N(d_2)$ is the probability that the call will be exercised provided one assumes that the asset drift is the risk-free rate. $N(d_1)$, however, does not lend itself to a simple probability interpretation. $SN(d_1)$ is correctly interpreted as the present value, using the risk-free interest rate, of the expected asset price at expiration, given that the asset price at expiration is above the exercise price.[10] For related discussion – and graphical representation – see section "Interpretation" under Datar–Mathews method for real option valuation.

The equivalent martingale probability measure is also called the risk-neutral probability measure. Note that both of these are *probabilities* in a measure theoretic sense, and neither of these is the true probability of expiring in-the-money under the real probability measure. To calculate the probability under the real (“physical”) probability measure, additional information is required—the drift term in the physical measure, or equivalently, the market price of risk.

Derivation

We now show how to get from the general Black–Scholes PDE to a specific valuation for an option. Consider as an example the Black–Scholes price of a call option, for which the PDE above has boundary conditions

$$C(0,t) = 0 \text{ for all } t$$
$$C(S,t) \to S \text{ as } S \to \infty$$
$$C(S,T) = \max\{S - K, 0\}.$$

The last condition gives the value of the option at the time that the option matures. The solution of the PDE gives the value of the option at any earlier time, $\mathbb{E}\left[\max\{S-K,0\}\right]$. To solve the PDE we recognize that it is a Cauchy–Euler equation which can be transformed into a diffusion equation by introducing the change-of-variable transformation

$$\tau = T - t$$

$$u = Ce^{r\tau}$$

$$x = \ln(S/K) + \left(r - \frac{\sigma^2}{2}\right)\tau.$$

Then the Black–Scholes PDE becomes a diffusion equation

$$\frac{\partial u}{\partial \tau} = \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial x^2}.$$

The terminal condition $C(S,T) = \max\{S-K,0\}$ now becomes an initial condition

$$u(x,0) = u_0(x) \equiv K(e^{\max\{x,0\}} - 1).$$

Using the standard method for solving a diffusion equation we have

$$u(x,\tau) = \frac{1}{\sigma\sqrt{2\pi\tau}}\int_{-\infty}^{\infty} u_0(y)\exp\left(-\frac{(x-y)^2}{2\sigma^2\tau}\right)dy.$$

which, after some manipulations, yields

$$u(x,\tau) = Ke^{x+\sigma^2\tau/2}N(d_1) - KN(d_2)$$

where

$$d_1 = \frac{(x+\frac{1}{2}\sigma^2\tau) + \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}$$

$$d_2 = \frac{(x+\frac{1}{2}\sigma^2\tau) - \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}.$$

Reverting u, x, τ to the original set of variables yields the above stated solution to the Black–Scholes equation.

Other derivations

Above we used the method of arbitrage-free pricing ("delta-hedging") to derive the Black–Scholes PDE, and then solved the PDE to get the valuation formula. It is also possible to derive the latter directly using a Risk neutrality argument.[9] This method gives the price as the expectation of the option payoff under a particular probability measure, called the risk-neutral measure, which differs from the real world measure. For the underlying logic see section "risk neutral valuation" under Rational pricing as well as section "Derivatives pricing: the Q world" under Mathematical finance; for detail, once again, see Hull.[8]:307–309

The Greeks

"The Greeks" measure the sensitivity to change of the option price under a slight change of a single parameter while holding the other parameters fixed. Formally, they are partial derivatives of the option price with respect to the independent variables (technically, one Greek, gamma, is a partial derivative of another Greek, called delta).

The Greeks are not only important for the mathematical theory of finance, but for those actively involved in trading. Financial institutions will typically set limits for the Greeks that their trader cannot exceed. Delta is the most important Greek and traders will zero their delta at the end of the day. Gamma and vega are also important but not as closely monitored.

The Greeks for Black–Scholes are given in closed form below. They can be obtained by straightforward differentiation of the Black–Scholes formula.[11]

	What	Calls	Puts
delta	$\frac{\partial C}{\partial S}$	$N(d_1)$	$-N(-d_1) = N(d_1) - 1$
gamma	$\frac{\partial^2 C}{\partial S^2}$	$\frac{N'(d_1)}{S\sigma\sqrt{T-t}}$	
vega	$\frac{\partial C}{\partial \sigma}$	$SN'(d_1)\sqrt{T-t}$	
theta	$\frac{\partial C}{\partial t}$	$-\frac{SN'(d_1)\sigma}{2\sqrt{T-t}} - rKe^{-r(T-t)}N(d_2)$	$-\frac{SN'(d_1)\sigma}{2\sqrt{T-t}} + rKe^{-r(T-t)}N(-d_2)$
rho	$\frac{\partial C}{\partial r}$	$K(T-t)e^{-r(T-t)}N(d_2)$	$-K(T-t)e^{-r(T-t)}N(-d_2)$

Note that the gamma and vega formulas are the same for calls and puts. This can be seen directly from put–call parity, since the difference of a put and a call is a forward, which is linear in S and independent of σ (so the gamma and vega of a forward vanish).

In practice, some sensitivities are usually quoted in scaled-down terms, to match the scale of likely changes in the parameters. For example, rho is often reported multiplied by 10,000 (1bp rate change), vega by 100 (1 vol point change), and theta by 365 or 252 (1 day decay based on either calendar days or trading days per year).

Extensions of the model

The above model can be extended for variable (but deterministic) rates and volatilities. The model may also be used to value European options on instruments paying dividends. In this case, closed-form solutions are available if the dividend is a known proportion of the stock price. American options and options on stocks paying a known cash dividend (in the short term, more realistic than a proportional dividend) are more difficult to value, and a choice of solution techniques is available (for example lattices and grids).

Instruments paying continuous yield dividends

For options on indexes, it is reasonable to make the simplifying assumption that dividends are paid continuously, and that the dividend amount is proportional to the level of the index.

The dividend payment paid over the time period $[t, t + dt)$ is then modelled as

$$qS_t\,dt$$

for some constant q (the dividend yield).

Under this formulation the arbitrage-free price implied by the Black–Scholes model can be shown to be

$$C(S_0, t) = e^{-r(T-t)}(FN(d_1) - KN(d_2))$$

and

$$P(S_0, t) = e^{-r(T-t)}(KN(-d_2) - FN(-d_1))$$

where now

$$F = S_0 e^{(r-q)(T-t)}$$

is the modified forward price that occurs in the terms d_1, d_2:

$$d_1 = \frac{\ln(F/K) + (\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

Exactly the same formula is used to price options on foreign exchange rates, except that now q plays the role of the foreign risk-free interest rate and S is the spot exchange rate. This is the **Garman–Kohlhagen model** (1983).

Instruments paying discrete proportional dividends

It is also possible to extend the Black–Scholes framework to options on instruments paying discrete proportional dividends. This is useful when the option is struck on a single stock.

A typical model is to assume that a proportion δ of the stock price is paid out at pre-determined times $t_1, t_2, \ldots$. The price of the stock is then modelled as

$$S_t = S_0(1-\delta)^{n(t)} e^{ut+\sigma W_t}$$

where $n(t)$ is the number of dividends that have been paid by time t .

The price of a call option on such a stock is again

$$C(S_0, T) = e^{-rT}(FN(d_1) - KN(d_2))$$

where now

$$F = S_0(1-\delta)^{n(T)} e^{rT}$$

is the forward price for the dividend paying stock.

American options

The problem of finding the price of an American option is related to the optimal stopping problem of finding the time to execute the option. Since the American option can be exercised at any time before the expiration date, the Black-Scholes equation becomes an inequality of the form

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \leq 0.$$ [12]

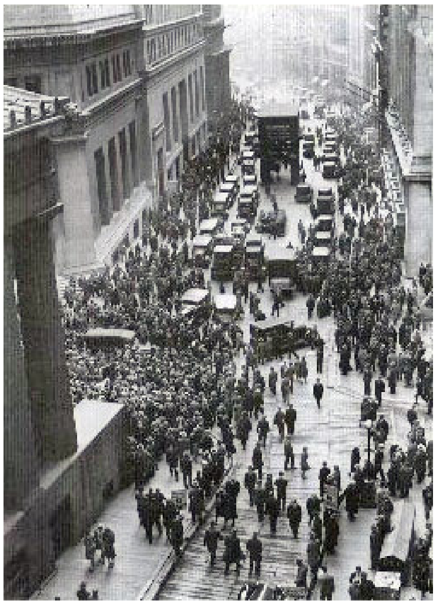
With the terminal and (free) boundary conditions: $V(S,T) = H(S)$ and $V(S,t) \geq H(S)$ where $H(S)$ denotes the payoff at stock price S

In general this inequality does not have a closed form solution, though an American call with no dividends is equal to a European call and the Roll-Geske-Whaley method provides a solution for an American call with one dividend.[13][14]

Barone-Adesi and Whaley[15] is a further approximation formula. Here, the stochastic differential equation (which is valid for the value of any derivative) is split into two components: the European option value and the early exercise premium. With some assumptions, a quadratic equation that approximates the solution for the latter is then obtained. This solution involves finding the critical value, $s*$, such that one is indifferent between early exercise and holding to maturity.[16][17]

Bjerksund and Stensland[18] provide an approximation based on an exercise strategy corresponding to a trigger price. Here, if the underlying asset price is greater than or equal to the trigger price it is optimal to exercise, and the value must equal $S - X$, otherwise the option "boils down to: (i) a European up-and-out call option… and (ii) a rebate that is received at the knock-out date if the option is knocked out prior to the maturity date." The formula is readily modified for the valuation of a put option, using put call parity. This approximation is computationally inexpensive and the method is fast, with evidence indicating that the approximation may be more accurate in pricing long dated options than Barone-Adesi and Whaley. [19]

Black–Scholes in practice



The normality assumption of the Black–Scholes model does not capture extreme movements such as stock market crashes.

The Black–Scholes model disagrees with reality in a number of ways, some significant. It is widely employed as a useful approximation, but proper application requires understanding its limitations – blindly following the model exposes the user to unexpected risk.

Among the most significant limitations are:

- the underestimation of extreme moves, yielding tail risk, which can be hedged with out-of-the-money options;
- the assumption of instant, cost-less trading, yielding liquidity risk, which is difficult to hedge;
- the assumption of a stationary process, yielding volatility risk, which can be hedged with volatility hedging;
- the assumption of continuous time and continuous trading, yielding gap risk, which can be hedged with Gamma hedging.

In short, while in the Black–Scholes model one can perfectly hedge options by simply Delta hedging, in practice there are many other sources of risk.

Results using the Black–Scholes model differ from real world prices because of simplifying assumptions of the model. One significant limitation is that in reality security prices do not follow a strict stationary log-normal process, nor is the risk-free interest actually known (and is not constant over time). The variance has been observed to be non-constant leading to models such as GARCH to model volatility changes. Pricing discrepancies between empirical and the Black–Scholes model have long been observed in options that are far out-of-the-money, corresponding to extreme price changes; such events would be very rare if returns were lognormally distributed, but are observed much more often in practice.

Nevertheless, Black–Scholes pricing is widely used in practice,[20][3]:751 for it is:

- easy to calculate
- a useful approximation, particularly when analyzing the direction in which prices move when crossing critical points
- a robust basis for more refined models
- reversible, as the model's original output -- price -- can be used as an input and one of the other variables solved for; the implied volatility calculated in this way is often used to quote option prices (that is, as a *quoting convention*)

The first point is self-evidently useful. The others can be further discussed:

Useful approximation: although volatility is not constant, results from the model are often helpful in setting up hedges in the correct proportions to minimize risk. Even when the results are not completely accurate, they serve as a first approximation to which adjustments can be made.

Basis for more refined models: The Black–Scholes model is *robust* in that it can be adjusted to deal with some of its failures. Rather than considering some parameters (such as volatility or interest rates) as *constant,* one considers them as *variables,* and thus added sources of risk. This is reflected in the Greeks (the change in option value for a change in these parameters, or equivalently the partial derivatives with respect to these variables), and hedging these Greeks mitigates the risk caused by the non-constant nature of these parameters. Other defects cannot be mitigated by modifying the model, however, notably tail risk and liquidity risk, and these are instead managed outside the model, chiefly by minimizing these risks and by stress testing.

Explicit modeling: this feature mean that, rather than *assuming* a volatility *a priori* and computing prices from it, one can use the model to solve for volatility, which gives the implied volatility of an option at given prices, durations and exercise prices. Solving for volatility over a given set of durations and strike prices one can construct an implied volatility surface. In this application of the Black–Scholes model, a coordinate transformation from the *price domain* to the *volatility domain* is obtained. Rather than quoting option prices in terms of dollars per unit (which are hard to compare across strikes and tenors), option prices can thus be quoted in terms of implied volatility, which leads to trading of volatility in option markets.

The volatility smile

One of the attractive features of the Black–Scholes model is that the parameters in the model (other than the volatility) — the time to maturity, the strike, the risk-free interest rate, and the current underlying price – are unequivocally observable. All other things being equal, an option's theoretical value is a monotonic increasing function of implied volatility.

By computing the implied volatility for traded options with different strikes and maturities, the Black–Scholes model can be tested. If the Black–Scholes model held, then the implied volatility for a particular stock would be the same for all strikes and maturities. In practice, the volatility surface (the 3D graph of implied volatility against strike and maturity) is not flat.

The typical shape of the implied volatility curve for a given maturity depends on the underlying instrument. Equities tend to have skewed curves: compared to at-the-money, implied volatility is substantially higher for low strikes, and slightly lower for high strikes. Currencies tend to have more symmetrical curves, with implied volatility lowest at-the-money, and higher volatilities in both wings. Commodities often have the reverse behavior to equities, with higher implied volatility for higher strikes.

Despite the existence of the volatility smile (and the violation of all the other assumptions of the Black–Scholes model), the Black–Scholes PDE and Black–Scholes formula are still used extensively in practice. A typical approach is to regard the volatility surface as a fact about the market, and use an implied volatility from it in a Black–Scholes valuation model. This has been described as using "the wrong number in the wrong formula to get the right price."[21] This approach also gives usable values for the hedge ratios (the Greeks).

Even when more advanced models are used, traders prefer to think in terms of volatility as it allows them to evaluate and compare options of different maturities, strikes, and so on.

Valuing bond options

Black–Scholes cannot be applied directly to bond securities because of pull-to-par. As the bond reaches its maturity date, all of the prices involved with the bond become known, thereby decreasing its volatility, and the simple Black–Scholes model does not reflect this process. A large number of extensions to Black–Scholes, beginning with the Black model, have been used to deal with this phenomenon.[22] See Bond option: Valuation.

Interest-rate curve

In practice, interest rates are not constant – they vary by tenor, giving an interest rate curve which may be interpolated to pick an appropriate rate to use in the Black–Scholes formula. Another consideration is that interest rates vary over time. This volatility may make a significant contribution to the price, especially of long-dated options.This is simply like the interest rate and bond price relationship which is inversely related.

Short stock rate

It is not free to take a short stock position. Similarly, it may be possible to lend out a long stock position for a small fee. In either case, this can be treated as a continuous dividend for the purposes of a Black–Scholes valuation, provided that there is no glaring asymmetry between the short stock borrowing cost and the long stock lending income.

Criticism

Espen Gaarder Haug and Nassim Nicholas Taleb argue that the Black–Scholes model merely recast existing widely used models in terms of practically impossible "dynamic hedging" rather than "risk," to make them more compatible with mainstream neoclassical economic theory.[23] Similar arguments were made in an earlier paper by Emanuel Derman and Nassim Taleb.[24] In response, Paul Wilmott has defended the model.[20][25]

Notes

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[2] MacKenzie, Donald (2006). *An Engine, Not a Camera: How Financial Models Shape Markets*. Cambridge, MA: MIT Press. ISBN 0-262-13460-8.

[3] Bodie, Zvi; Alex Kane, Alan J. Marcus (2008). *Investments* (7th ed.). New York: McGraw-Hill/Irwin. ISBN 978-0-07-326967-2.

[4] "Nobel prize foundation, 1997 Press release" (http://nobelprize.org/nobel_prizes/economics/laureates/1997/press.html). October 14, 1997. . Retrieved March 26, 2012.

[5] Although the original model assumed no dividends, trivial extensions to the model can accommodate a continuous dividend yield factor.

[6] Black, Fischer; Scholes, Myron. "The Pricing of Options and Corporate Liabilities". *Journal of Political Economy* **81** (3): 637–654.

[7] Merton, Robert. "Theory of Rational Option Pricing". *Bell Journal of Economics and Management Science* **4** (1): 141–183. doi:10.2307/3003143..

[8] Hull, John C. (2008). *Options, Futures and Other Derivatives* (7 ed.). Prentice Hall. ISBN 0-13-505283-1.

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[10] Don Chance (June 3, 2011). "Derivation and Interpretation of the Black–Scholes Model" (http://www.bus.lsu.edu/academics/finance/faculty/dchance/Instructional/TN99-02.pdf) (PDF). . Retrieved March 27, 2012.

[11] Although with significant algebra; see, for example, Hong-Yi Chen, Cheng-Few Lee and Weikang Shih (2010). Derivations and Applications of Greek Letters: Review and Integration (https://docs.google.com/viewer?a=v&q=cache:ai5xEtbLxCIJ:centerforpbbefr.rutgers.edu/TaipeiPBFR&D/01-16-09%20papers/5-4%20Greek%20letters.doc+Derivations+and+Applications+of+Greek+Letters+ â+Review+and+Integration&hl=en&pid=bl& srcid=ADGEEShU4q28apOYjO-BmqXOJTOHj2BG0BgnxtLn-ccCfh27FYlCDla0nspYCidFFFWiPfYjM2PTT0_109Lth79rFwKsenMFpawjU9BtpBSQO81hUj0 sig=AHIEtbREe6Jg8SlzylhuYC9xEoG0eG3dGg), *Handbook of Quantitative Finance and Risk Management*, III:491–503.

[12] André Jaun. "The Black-Scholes equation for American options" (http://www.lifelong-learners.com/opt/com/SYL/s6node6.php). . Retrieved May 5, 2012.

[13] Bernt Ødegaard (2003). "Extending the Black Scholes formula" (http://finance.bi.no/~bernt/gcc_prog/recipes/recipes/node9.html#SECTION00920000000000000000). . Retrieved May 5, 2012.

[14] Don Chance (2008). "Closed-Form American Call Option Pricing: Roll-Geske-Whaley" (http://www.bus.lsu.edu/academics/finance/faculty/dchance/Instructional/TN98-01.pdf). . Retrieved May 16, 2012.

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[16] Bernt Ødegaard (2003). "A quadratic approximation to American prices due to Barone-Adesi and Whaley" (http://finance.bi.no/~bernt/gcc_prog/recipes/recipes/node13.html). . Retrieved June 25, 2012.

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[20] Paul Wilmott (2008): In defence of Black Scholes and Merton (http://www.wilmott.com/blogs/paul/index.cfm/2008/4/29/Science-in-Finance-IX-In-defence-of-Black-Scholes-and-Merton), Dynamic hedging and further defence of Black-Scholes (http://www.wilmott.com/blogs/paul/index.cfm/2008/7/23/Science-in-Finance-X-Dynamic-hedging-and-further-defence-of-BlackScholes)

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[25] See also: Doriana Ruffinno and Jonathan Treussard (2006). *Derman and Taleb's The Illusions of Dynamic Replication: A Comment* (http://wayback.archive.org/web/*/http://www.bu.edu/econ/workingpapers/papers/RuffinoTreussardDT.pdf), WP2006-019, Boston University - Department of Economics.

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Historical and sociological aspects

- Bernstein, Peter (1992). *Capital Ideas: The Improbable Origins of Modern Wall Street*. The Free Press. ISBN 0-02-903012-9.
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- MacKenzie, Donald (2006). *An Engine, not a Camera: How Financial Models Shape Markets*. MIT Press. ISBN 0-262-13460-8.

Further reading

- Haug, E. G (2007). "Option Pricing and Hedging from Theory to Practice". *Derivatives: Models on Models*. Wiley. ISBN 978-0-470-01322-9. The book gives a series of historical references supporting the theory that option traders use much more robust hedging and pricing principles than the Black, Scholes and Merton model.
- Triana, Pablo (2009). *Lecturing Birds on Flying: Can Mathematical Theories Destroy the Financial Markets?*. Wiley. ISBN 978-0-470-40675-5. The book takes a critical look at the Black, Scholes and Merton model.

External links

Discussion of the model

- Ajay Shah. Black, Merton and Scholes: Their work and its consequences. Economic and Political Weekly, XXXII(52):3337–3342, December 1997 link (http://www.mayin.org/ajayshah/PDFDOCS/Shah1997_bms.pdf)
- Inside Wall Street's Black Hole (http://www.portfolio.com/news-markets/national-news/portfolio/2008/02/19/Black-Scholes-Pricing-Model?print=true) by Michael Lewis, March 2008 Issue of portfolio.com
- Whither Black–Scholes? (http://www.forbes.com/opinions/2008/04/07/black-scholes-options-oped-cx_ptp_{0}408black.html) by Pablo Triana, April 2008 Issue of Forbes.com
- Black Scholes model lecture (http://wikilecture.org/Black_Scholes) by Professor Robert Shiller from Yale
- The mathematical equation that caused the banks to crash (http://www.guardian.co.uk/science/2012/feb/12/black-scholes-equation-credit-crunch) by Ian Stewart in The Observer, February 12, 2012

Derivation and solution

- Derivation of the Black–Scholes Equation for Option Value (http://www.sjsu.edu/faculty/watkins/blacksch.htm), Prof. Thayer Watkins
- Solution of the Black–Scholes Equation Using the Green's Function (http://www.physics.uci.edu/~silverma/bseqn/bs/bs.html), Prof. Dennis Silverman
- Solution via risk neutral pricing or via the PDE approach using Fourier transforms (http://homepages.nyu.edu/~sl1544/KnownClosedForms.pdf) (includes discussion of other option types), Simon Leger
- Step-by-step solution of the Black–Scholes PDE (http://planetmath.org/encyclopedia/AnalyticSolutionOfBlackScholesPDE.html), planetmath.org.
- On the Black–Scholes Equation: Various Derivations (http://www.stanford.edu/~japrimbs/Publications/OnBlackScholesEq.pdf), Manabu Kishimoto
- The Black–Scholes Equation (http://terrytao.wordpress.com/2008/07/01/the-black-scholes-equation/) Expository article by mathematician Terence Tao.

Revisiting the model

- When You Cannot Hedge Continuously: The Corrections to Black–Scholes (http://www.ederman.com/new/docs/risk-non_continuous_hedge.pdf), Emanuel Derman
- Arbitrage and Stock Option Pricing: A Fresh Look At The Binomial Model (http://www.soa.org/library/newsletters/risks-and-rewards/2011/august/rar-2011-iss58-joss.pdf)

Computer implementations

- Black–Scholes in Multiple Languages (http://www.espenhaug.com/black_scholes.html), espenhaug.com
- Chicago Option Pricing Model (Graphing Version) (http://sourceforge.net/projects/chipricingmodel/), sourceforge.net
- Black-Scholes-Merton Implied Volatility Surface Model (Java) (https://github.com/OpenGamma/OG-Platform/blob/master/projects/OG-Analytics/src/main/java/com/opengamma/analytics/financial/model/volatility/surface/BlackScholesMertonImpliedVolatilitySurfaceModel.java), github.com

Historical

- Trillion Dollar Bet (http://www.pbs.org/wgbh/nova/stockmarket/)—Companion Web site to a Nova episode originally broadcast on February 8, 2000. *"The film tells the fascinating story of the invention of the Black–Scholes Formula, a mathematical Holy Grail that forever altered the world of finance and earned its creators the 1997 Nobel Prize in Economics."*
- BBC Horizon (http://www.bbc.co.uk/science/horizon/1999/midas.shtml) A TV-programme on the so-called Midas formula and the bankruptcy of Long-Term Capital Management (LTCM)
- BBC News Magazine (http://www.bbc.co.uk/news/magazine-17866646) Black-Scholes: The maths formula linked to the financial crash (April 27, 2012 article)

Brownian model of financial markets

The Brownian motion models for financial markets are based on the work of Robert C. Merton and Paul A. Samuelson, as extensions to the one-period market models of Harold Markowitz and William Sharpe, and are concerned with defining the concepts of financial assets and markets, portfolios, gains and wealth in terms of continuous-time stochastic processes.

Under this model, these assets have continuous prices evolving continuously in time and are driven by Brownian motion processes.[1] This model requires an assumption of perfectly divisible assets and a frictionless market (i.e. that no transaction costs occur either for buying or selling). Another assumption is that asset prices have no jumps, that is there are no surprises in the market. This last assumption is removed in jump diffusion models.

Financial market processes

Consider a financial market consisting of $N+1$ financial assets, where one of these assets, called a *bond* or *money-market*, is risk free while the remaining N assets, called *stocks*, are risky.

Definition

A *financial market* is defined as $\mathcal{M} = (r, \mathbf{b}, \delta, \sigma, A, \mathbf{S}(0))$:

1. A probability space $(\Omega, \mathcal{F}, P)$
2. A time interval $[0, T]$
3. A D -dimensional Brownian process $\mathbf{W}(t) = (W_1(t) \ldots W_D(t))',\ 0 \leq t \leq T$ adapted to the augmented filtration $\{\mathcal{F}(t);\ 0 \leq t \leq T\}$
4. A measurable risk-free money market rate process $r(t) \in L_1[0, T]$
5. A measurable mean rate of return process $\mathbf{b} : [0, T] \times \mathbb{R}^N \to \mathbb{R} \in L_2[0, T]$.
6. A measurable dividend rate of return process $\delta : [0, T] \times \mathbb{R}^N \to \mathbb{R} \in L_2[0, T]$.
7. A measurable volatility process $\sigma : [0, T] \times \mathbb{R}^{N \times D} \to \mathbb{R}$ such that $\sum_{n=1}^{N} \sum_{d=1}^{D} \int_0^T \sigma_{n,d}^2(s) ds < \infty$.
8. A measurable, finite variation, singularly continuous stochastic $A(t)$
9. The initial conditions given by $\mathbf{S}(0) = (S_0(0), \ldots S_N(0))'$

The augmented filtration

Let $(\Omega, \mathcal{F}, p)$ be a probability space, and a $\mathbf{W}(t) = (W_1(t) \ldots W_D(t))',\ 0 \le t \le T$ be D-dimensional Brownian motion stochastic process, with the natural filtration:

$$\mathcal{F}^{\mathbf{W}}(t) \triangleq \sigma\left(\{\mathbf{W}(s);\ 0 \le s \le t\}\right), \quad \forall t \in [0, T].$$

If $\mathcal{N}$ are the measure 0 (i.e. null under measure P) subsets of $\mathcal{F}^{\mathbf{W}}(t)$, then define the *augmented* filtration:

$$\mathcal{F}(t) \triangleq \sigma\left(\mathcal{F}^{\mathbf{W}}(t) \cup \mathcal{N}\right), \quad \forall t \in [0, T]$$

The difference between $\{\mathcal{F}^{\mathbf{W}}(t);\ 0 \le t \le T\}$ and $\{\mathcal{F}(t);\ 0 \le t \le T\}$ is that the latter is both left-continuous, in the sense that:

$$\mathcal{F}(t) = \sigma\left(\bigcup_{0 \le s < t} \mathcal{F}(s)\right),$$

and right-continuous, such that:

$$\mathcal{F}(t) = \bigcap_{t < s \le T} \mathcal{F}(s),$$

while the former is only left-continuous.[2]

Bond

A share of a bond (money market) has price $S_0(t) > 0$ at time t with $S_0(0) = 1$, is continuous, $\{\mathcal{F}(t);\ 0 \le t \le T\}$ adapted, and has finite variation. Because it has finite variation, it can be decomposed into an absolutely continuous part $S_0^a(t)$ and a singularly continuous part $S_0^s(t)$, by Lebesgue's decomposition theorem. Define:

$$r(t) \triangleq \frac{1}{S_0(t)} \frac{d}{dt} S_0^a(t), \text{ and}$$

$$A(t) \triangleq \int_0^t \frac{1}{S_0^s(s)} dS_0(s),$$

resulting in the SDE:

$$dS_0(t) = S_0(t)[r(t)dt + dA(t)], \quad \forall 0 \le t \le T,$$

which gives:

$$S_0(t) = \exp\left(\int_0^t r(s)ds + A(t)\right), \quad \forall 0 \le t \le T.$$

Thus, it can be easily seen that if $S_0(t)$ is absolutely continuous (i.e. $A(\cdot) = 0$), then the price of the bond evolves like the value of a risk-free savings account with instantaneous interest rate $r(t)$, which is random, time-dependent and $\mathcal{F}(t)$ measurable.

Stocks

Stock prices are modeled as being similar to that of bonds, except with a randomly fluctuating component (called its volatility). As a premium for the risk originating from these random fluctuations, the mean rate of return of a stock is higher than that of a bond.

Let $S_1(t) \ldots S_N(t)$ be the strictly positive prices per share of the N stocks, which are continuous stochastic processes satisfying:

$$dS_n(t) = S_n(t)\left[b_n(t)dt + dA(t) + \sum_{d=1}^{D} \sigma_{n,d}(t) dW_d(t)\right], \quad \forall 0 \le t \le T, \quad n = 1 \ldots N.$$

Here, $\sigma_{n,d}(t),\ d = 1 \ldots D$ gives the volatility of the n-th stock, while $b_n(t)$ is its mean rate of return.

In order for an arbitrage-free pricing scenario, $A(t)$ must be as defined above. The solution to this is:

$$S_n(t) = S_n(0)\exp\left(\int_0^t \sum_{d=1}^D \sigma_{n,d}(s)dW_d(s) + \int_0^t \left[b_n(s) - \frac{1}{2}\sum_{d=1}^D \sigma_{n,d}^2(s)\right]ds + A(t)\right), \quad \forall 0 \le t \le T, \quad n = 1 \dots N,$$

and the discounted stock prices are:

$$\frac{S_n(t)}{S_0(t)} = S_n(0)\exp\left(\int_0^t \sum_{d=1}^D \sigma_{n,d}(s)dW_d(s) + \int_0^t \left[b_n(s) - \frac{1}{2}\sum_{d=1}^D \sigma_{n,d}^2(s)\right]ds\right), \quad \forall 0 \le t \le T, \quad n = 1 \dots N.$$

Note that the contribution due to the discontinuites in the bond price $A(t)$ does not appear in this equation.

Dividend rate

Each stock may have an associated dividend rate process $\delta_n(t)$ giving the rate of dividend payment per unit price of the stock at time t . Accounting for this in the model, gives the *yield* process $Y_n(t)$:

$$dY_n(t) = S_n(t)\left[b_n(t)dt + dA(t) + \sum_{d=1}^D \sigma_{n,d}(t)dW_d(t) + \delta_n(t)\right], \quad \forall 0 \le t \le T, \quad n = 1 \dots N.$$

Portfolio and gain processes

Definition

Consider a financial market $\mathcal{M} = (r, \mathbf{b}, \delta, \sigma, A, \mathbf{S}(0))$.

A *portfolio process* $(\pi_0, \pi_1, \dots \pi_N)$ for this market is an $\mathcal{F}(t)$ measurable, $\mathbb{R}^{N+1}$ valued process such that:

$$\int_0^T |\sum_{n=0}^N \pi_n(t)|\,[|r(t)|dt + dA(t)] < \infty \text{, almost surely,}$$

$$\int_0^T |\sum_{n=1}^N \pi_n(t)[b_n(t) + \delta_n(t) - r(t)]|dt < \infty \text{, almost surely, and}$$

$$\int_0^T \sum_{d=1}^D |\sum_{n=1}^N \sigma_{n,d}(t)\pi_n(t)|^2 dt < \infty \text{, almost surely.}$$

The *gains process* for this porfolio is:

$$G(t) \triangleq \int_0^t \left[\sum_{n=0}^N \pi_n(t)\right](r(s)ds + dA(s)) + \int_0^t \left[\sum_{n=1}^N \pi_n(t)\left(b_n(t) + \delta_n(t) - r(t)\right)\right]dt + \int_0^t \sum_{d=1}^D \sum_{n=1}^N \sigma_{n,d}(t)\pi_n(t)dW_d(s) \quad 0 \le t \le T$$

We say that the porfolio is *self-financed* if:

$$G(t) = \sum_{n=0}^N \pi_n(t).$$

It turns out that for a self-financed portfolio, the appropriate value of π_0 is determined from $\pi = (\pi_1, \dots \pi_N)$ and therefore sometimes π is referred to as the portfolio process. Also, $\pi_0 < 0$ implies borrowing money from the money-market, while $\pi_n < 0$ implies taking a short position on the stock.

The term $b_n(t) + \delta_n(t) - r(t)$ in the SDE of $G(t)$ is the *risk premium* process, and it is the compensation received in return for investing in the n -th stock.

Motivation

Consider time intervals $0 = t_0 < t_1 < \ldots < t_M = T$, and let $\nu_n(t_m)$be the number of shares of asset $n = 0 \ldots N$, held in a portfolio during time interval at time $[t_m, t_{m+1}$ $m = 0 \ldots M-1$. To avoid the case of insider trading (i.e. foreknowledge of the future), it is required that $\nu_n(t_m)$is $\mathcal{F}(t_m)$ measurable.
Therefore, the incremental gains at each trading interval from such a portfolio is:

$$G(0) = 0,$$

$$G(tm+1) - G(t_m) = \sum_{n=0}^{N} \nu_n(t_m)[Y_n(t_{m+1}) - Y_n(t_m)], \quad m = 0 \ldots M-1,$$

and $G(m)$is the total gain over time $[0, t_m]$, while the total value of the portfolio is $\sum_{n=0}^{N} \nu_n(t_m) S_n(t_m)$.

Define $\pi_n(t) \triangleq \nu_n(t)$, let the time partition go to zero, and substitute for $Y(t)$as defined earlier, to get the corresponding SDE for the gains process. Here $\pi_n(t)$denotes the dollar amount invested in asset n at time t, not the number of shares held.

Income and wealth processes

Definition

Given a financial market $\mathcal{M}$, then a *cumulative income process* $\Gamma(t)$ $0 \le t \le T$is a semimartingale and represents the income accumulated over time $[0, t]$, due to sources other than the investments in the $N+1$assets of the financial market.
A *wealth process* $X(t)$is then defined as:

$$X(t) \triangleq G(t) + \Gamma(t)$$

and represents the total wealth of an investor at time $0 \le t \le T$. The portfolio is said to be $\Gamma(t)$*-financed* if:

$$X(t) = \sum_{n=0}^{N} \pi_n(t).$$

The corresponding SDE for the wealth process, through appropriate substitutions, becomes:

$$dX(t) = d\Gamma(t) + X(t)\left[r(t)dt + dA(t)\right] + \sum_{n=1}^{N} \left[\pi_n(t)\left(b_n(t) + \delta_n(t) - r(t)\right)\right] + \sum_{d=1}^{D} \left[\sum_{n=1}^{N} \pi_n(t)\sigma_{n,d}(t)\right] dW_d(t).$$

Note, that again in this case, the value of $\pi_0$can be determined from $\pi_n, \ n = 1 \ldots N$.

Viable markets

The standard theory of mathematical finance is restricted to viable financial markets, i.e. those in which there are no opportunities for arbitrage. If such opportunities exists, it implies the possibility of making an arbitrarily large risk-free profit.

Definition

In a financial market $\mathcal{M}$, a self-financed portfolio process $\pi(t)$is said to be an *arbitrage opportunity* if the associated gains process $G(T) \ge 0$, almost surely and $P[G(T) > 0] > 0$strictly. A market $\mathcal{M}$ in which no such portfolio exists is said to be *viable*.

Implications

In a viable market $\mathcal{M}$, there exists a $\mathcal{F}(t)$ adapted process $\theta : [0,T] \times \mathbb{R}^D \longrightarrow \mathbb{R}$ such that for almost every $t \in [0,T]$:

$$b_n(t) + \delta_n(t) - r(t) = \sum_{d=1}^{D} \sigma_{n,d}(t)\theta_d(t).$$

This θ is called the *market price of risk* and relates the premium for the n -the stock with its volatility $\sigma_{n,\cdot}$.

Conversely, if there exists a D-dimensional process $\theta(t)$ such that it satifies the above requirement, and:

$$\int_0^T \sum_{d=1}^{D} |\theta_d(t)|^2 dt < \infty$$

$$\mathbb{E}\left[\exp\left\{-\int_0^T \sum_{d=1}^{D} \theta_d(t) dW_d(t) - \frac{1}{2}\int_0^T \sum_{d=1}^{D} |\theta_d(t)|^2 dt\right\}\right] = 1,$$

then the market is viable.

Also, a viable market $\mathcal{M}$ can have only one money-market (bond) and hence only one risk-free rate. Therefore, if the n -th stock entails no risk (i.e. $\sigma_{n,d} = 0,\ d = 1 \ldots D$) and pays no dividend (i.e. $\delta_n(t) = 0$), then its rate of return is equal to the money market rate (i.e. $b_n(t) = r(t)$) and its price tracks that of the bond (i.e. $S_n(t) = S_n(0)S_0(t)$).

Standard financial market

Definition

A financial market $\mathcal{M}$ is said to be *standard* if:

(i) It is viable.

(ii) The number of stocks N is not greater than the dimension D of the underlying Brownian motion process $\mathbf{W}(t)$.

(iii) The market price of risk process θ satisfies:

$$\int_0^T \sum_{d=1}^{D} |\theta_d(t)|^2 dt < \infty$$, almost surely.

(iv) The positive process $Z_0(t) = \exp\left\{-\int_0^t \sum_{d=1}^{D} \theta_d(t) dW_d(t) - \frac{1}{2}\int_0^t \sum_{d=1}^{D} |\theta_d(t)|^2 dt\right\}$ is a martingale.

Comments

In case the number of stocks N is greater than the dimension D, in violation of point (ii), from linear algebra, it can be seen that there are $N-D$ stocks whose volatilies (given by the vector $(\sigma_{n,1} \ldots \sigma_{n,D})$) are linear combination of the volatilities of D other stocks (because the rank of σ is D). Therefore, the N stocks can be replaced by D equivalent mutual funds.

The *standard martingale measure* P_0 on $\mathcal{F}(T)$ for the standard market, is defined as:

$$P_0(A) \triangleq \mathbb{E}[Z_0(T)\mathbf{1}_A], \quad \forall A \in \mathcal{F}(T).$$

Note that P and P_0 are absolutely continuous with respect to each other, i.e. they are equivalent. Also, according to Girsanov's theorem,

$$\mathbf{W}_0(t) \triangleq \mathbf{W}(t) + \int_0^t \theta(s) ds,$$

is a D -dimensional Brownian motion process on the filtration $\{\mathcal{F}(t);\ 0 \le t \le T\}$ with respect to P_0.

Complete financial markets

A complete financial market is one that allows effective hedging of the risk inherent in any investment strategy.

Definition

Let $\mathcal{M}$ be a standard financial market, and B be an $\mathcal{F}(T)$-measurable random variable, such that:

$$P_0\left[\frac{B}{S_0(T)} > -\infty\right] = 1.$$

$$x \triangleq \mathbb{E}_0\left[\frac{B}{S_0(T)}\right] < \infty\,,$$

The market $\mathcal{M}$ is said to be *complete* if every such B is *financeable*, i.e. if there is an x -financed portfolio process $(\pi_n(t);\ n = 1 \ldots N)$, such that its associated wealth process $X(t)$ satisfies

$X(t) = B$, almost surely.

Motivation

If a particular investment strategy calls for a payment B at time T , the amount of which is unknown at time $t = 0$, then a conservative strategy would be to set aside an amount $x = \sup_\omega B(\omega)$ in order to cover the payment. However, in a complete market it is possible to set aside less capital (viz. x) and invest it so that at time T it has grown to match the size of B .

Corollary

A standard financial market $\mathcal{M}$ is complete if and only if $N = D$, and the $N \times D$ volalatity process $\sigma(t)$ is non-singular for almost every $t \in [0, T]$, with respect to the Lebesgue measure.

Notes

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Stochastic volatility

Stochastic volatility models are used in the field of mathematical finance to evaluate derivative securities, such as options. The name derives from the models' treatment of the underlying security's volatility as a random process, governed by state variables such as the price level of the underlying security, the tendency of volatility to revert to some long-run mean value, and the variance of the volatility process itself, among others.

Stochastic volatility models are one approach to resolve a shortcoming of the Black–Scholes model. In particular, these models assume that the underlying volatility is constant over the life of the derivative, and unaffected by the changes in the price level of the underlying security. However, these models cannot explain long-observed features of the implied volatility surface such as volatility smile and skew, which indicate that implied volatility does tend to vary with respect to strike price and expiry. By assuming that the volatility of the underlying price is a stochastic process rather than a constant, it becomes possible to model derivatives more accurately.

Basic model

Starting from a constant volatility approach, assume that the derivative's underlying price follows a standard model for geometric brownian motion:

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$$

where μ is the constant drift (i.e. expected return) of the security price S_t, σ is the constant volatility, and dW_t is a standard Wiener process with zero mean and unit rate of variance. The explicit solution of this stochastic differential equation is

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}.$$

The Maximum likelihood estimator to estimate the constant volatility σ for given stock prices S_t at different times t_i is

$$\begin{aligned}\hat{\sigma}^2 &= \left(\frac{1}{n}\sum_{i=1}^{n}\frac{(\ln S_{t_i} - \ln S_{t_{i-1}})^2}{t_i - t_{i-1}}\right) - \frac{1}{n}\frac{(\ln S_{t_n} - \ln S_{t_0})^2}{t_n - t_0} \\ &= \frac{1}{n}\sum_{i=1}^{n}(t_i - t_{i-1})\left(\frac{\ln\frac{S_{t_i}}{S_{t_{i-1}}}}{t_i - t_{i-1}} - \frac{\ln\frac{S_{t_n}}{S_{t_0}}}{t_n - t_0}\right)^2 ;\end{aligned}$$

its expectation value is $E\left[\hat{\sigma}^2\right] = \frac{n-1}{n}\sigma^2.$

This basic model with constant volatility σ is the starting point for non-stochastic volatility models such as Black–Scholes and Cox–Ross–Rubinstein.

For a stochastic volatility model, replace the constant volatility σ with a function ν_t, that models the variance of S_t. This variance function is also modeled as brownian motion, and the form of ν_t depends on the particular SV model under study.

$$\begin{aligned}dS_t &= \mu S_t \, dt + \sqrt{\nu_t} S_t \, dW_t \\ d\nu_t &= \alpha_{S,t} \, dt + \beta_{S,t} \, dB_t\end{aligned}$$

where $\alpha_{S,t}$ and $\beta_{S,t}$ are some functions of ν and dB_t is another standard gaussian that is correlated with dW_t with constant correlation factor ρ.

Heston model

The popular Heston model is a commonly used SV model, in which the randomness of the variance process varies as the square root of variance. In this case, the differential equation for variance takes the form:

$$d\nu_t = \theta(\omega - \nu_t)dt + \xi\sqrt{\nu_t}\,dB_t$$

where ω is the mean long-term volatility, θ is the rate at which the volatility reverts toward its long-term mean, ξ is the volatility of the volatility process, and dB_t is, like dW_t, a gaussian with zero mean and unit standard deviation. However, dW_t and dB_t are correlated with the constant correlation value ρ.
In other words, the Heston SV model assumes that the variance is a random process that

1. exhibits a tendency to revert towards a long-term mean ω at a rate θ ,
2. exhibits a volatility proportional to the square root of its level
3. and whose source of randomness is correlated (with correlation ρ) with the randomness of the underlying's price processes.

CEV Model

The **CEV** model describes the relationship between volatility and price, introducing stochastic volatility:

$$dS_t = \mu S_t dt + \sigma S_t^\gamma dW_t$$

Conceptually, in some markets volatility rises when prices rise (e.g. commodities), so $\gamma > 1$. In other markets, volatility tends to rise as prices fall, modelled with $\gamma < 1$.
Some argue that because the CEV model does not incorporate its own stochastic process for volatility, it is not truly a stochastic volatility model. Instead, they call it a local volatility model.

SABR volatility model

The **SABR** model (Stochastic Alpha, Beta, Rho) describes a single forward F (related to any asset e.g. an index, interest rate, bond, currency or equity) under stochastic volatility σ :

$$dF_t = \sigma_t F_t^\beta\, dW_t,$$
$$d\sigma_t = \alpha\sigma_t\, dZ_t,$$

The initial values F_0 and σ_0 are the current forward price and volatility, whereas W_t and Z_t are two correlated Wiener processes (i.e. Brownian motions) with correlation coefficient $-1 < \rho < 1$. The constant parameters $\beta,\ \alpha$ are such that $0 \leq \beta \leq 1,\ \alpha \geq 0$.
The main feature of the SABR model is to be able to reproduce the smile effect of the volatility smile.

GARCH model

The Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model is another popular model for estimating stochastic volatility. It assumes that the randomness of the variance process varies with the variance, as opposed to the square root of the variance as in the Heston model. The standard GARCH(1,1) model has the following form for the variance differential:

$$d\nu_t = \theta(\omega - \nu_t)dt + \xi\nu_t\, dB_t$$

The GARCH model has been extended via numerous variants, including the NGARCH, TGARCH, IGARCH, LGARCH, EGARCH, GJR-GARCH, etc.

3/2 model

The 3/2 model is similar to the Heston model, but assumes that the randomness of the variance process varies with $\nu_t^{3/2}$. The form of the variance differential is:

$$d\nu_t = \theta\nu_t(\omega - \nu_t)dt + \xi\nu_t^{\frac{3}{2}}\,dB_t.$$

However the meaning of the parameters is different from Heston model. In this model both, mean reverting and volatility of variance parameters, are stochastic quantities given by $\theta\nu_t$ and $\xi\nu_t$ respectively.

Chen model

In interest rate modelings, Lin Chen in 1994 developed the first stochastic mean and stochastic volatility model, Chen model. Specifically, the dynamics of the instantaneous interest rate are given by following the stochastic differential equations:

$$dr_t = (\theta_t - \alpha_t)\,dt + \sqrt{r_t}\,\sigma_t\,dW_t,$$
$$d\alpha_t = (\zeta_t - \alpha_t)\,dt + \sqrt{\alpha_t}\,\sigma_t\,dW_t,$$
$$d\sigma_t = (\beta_t - \sigma_t)\,dt + \sqrt{\sigma_t}\,\eta_t\,dW_t.$$

Calibration

Once a particular SV model is chosen, it must be calibrated against existing market data. Calibration is the process of identifying the set of model parameters that are most likely given the observed data. One popular technique is to use Maximum Likelihood Estimation (MLE). For instance, in the Heston model, the set of model parameters $\Psi_0 = \{\omega, \theta, \xi, \rho\}$ can be estimated applying an MLE algorithm such as the Powell Directed Set method [1] to observations of historic underlying security prices.

In this case, you start with an estimate for Ψ_0, compute the residual errors when applying the historic price data to the resulting model, and then adjust Ψ to try to minimize these errors. Once the calibration has been performed, it is standard practice to re-calibrate the model periodically.

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