Brownian Motion and Poisson Process

She: What is white noise?
He: It is the best model of a totally unpredictable process.
She: Are you implying, I am white noise?
He: No, it does not exist.

Dialogue of an unknown couple.

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Brownian Motion

The American mathematician Norbert Wiener stipulated the following assumptions for a stationary random process $W(\cdot,\cdot)$ with independent increments in 1923:

Definition 1. Brownian motion
A stochastic process W(t) is called Brownian motion if

1. *Independence:* $W(t+\Delta t) - W(t)$ *is independent of* $\{W(\tau)\}$ *for all* $\tau \le t$.
2. *Stationarity: The distribution of* $W(t+\Delta t) - W(t)$ *does not depend on* t.
3. *Continuity:* $\lim\limits_{\Delta t \downarrow 0} \dfrac{P(|W(t+\Delta t) - W(t)| \ge \delta)}{\Delta t} = 0$ *for all* $\delta > 0$.

Note that the third definition is expressed in probabilities.

Brownian Motion

This definition induces the distribution of the process W_t:

Theorem 1. Normally distributed increments of Brownian motion

If $W(t)$ is a Brownian motion, then $W(t) - W(0)$ is a normal random variable with mean μt and variance $\sigma^2 t$, where μ and σ are constant real numbers.

As a result of this theorem, we have the following density function of a Brownian motion:

$$f_{W(t)}(x) = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(x-\mu t)^2}{2\sigma^2 t}}$$

An irritating property of Brownian motion is that its sample paths are not differentiable. We can easily verify that this density function fulfills all three properties stipulated by Wiener.

Brownian Motion

1. Independence: $W(t+\Delta t) - W(t) = W(t) + W(\Delta t) - W(t) = W(\Delta t)$
   This is possible because the normal distribution is self-similar. $W(\Delta t)$ is independent of $W(t)$.
2. Stationarity : $W(t+\Delta t) - W(t) = W(\Delta t)$
   The increments are always $W(\Delta t)$.
3. Continuity: $\lim\limits_{\Delta t \downarrow 0} \dfrac{P(|W(t+\Delta t) - W(t)| \geq \delta)}{\Delta t} = \lim\limits_{\Delta t \downarrow 0} \dfrac{P(|W(\Delta t)| \geq \delta)}{\Delta t}$
   $\lim\limits_{\Delta t \downarrow 0} P(|W(\Delta t)| \geq \delta) = 0$, because a normal distribution with expectation and variance equal to zero is reduce to a point mass at position 0.
   $\Rightarrow \lim_{\Delta t \downarrow 0} \frac{P(|W(\Delta t)| \geq \delta)}{\Delta t} = 0$

Brownian Motion

The Brownian motion has many more bizarre and intriguing properties:

- Autocovariance function: $E\{(W(t)-\mu t)(W(\tau)-\mu\sigma)\} = \sigma^2 \min(t,\tau)$
- $Var\left\{\frac{W(t)}{t}\right\} = \frac{\sigma^2}{t}$
- $\lim\limits_{t\to\infty} \frac{W(t)-\mu t}{t} = 0$ with probability 1
- The total variation of the Brownian motion over a finite interval $[0,T]$ is infinite!
- The "sum of squares" of a drift-free Brownian motion is deterministic:
  $$\lim_{N\to\infty} \sum_{k=1}^{N} \left(W\Big(k\frac{T}{N}\Big) - W\Big((k-1)\frac{T}{N}\Big)\right)^2 = \sigma^2 T$$

Brownian Motion

- Infinite oscillations:
  Let Y_0, Y_1, ... be mutually independent random variables with identical normal distributions $\mathcal{N}(0,1)$. The random process
  $$X(t) = \frac{Y_0}{\sqrt{\pi}} t + \sum_{k=1}^{\infty} \sqrt{\frac{2}{\pi}} \frac{Y_k}{k} \sin kt \quad \text{for } t \in [0, \pi]$$
  is a normalized Brownian motion on the interval $[0, \pi]$.
- If $W(\cdot)$ is a Wiener process on the interval $[0, \infty)$,then the following process $W^*(\cdot)$ is a Wiener process as well:
  $$W^*(t) = \begin{cases} tW(\frac{1}{t}), & \text{for } t > 0; \\ 0, & \text{for } t = 0. \end{cases}$$
- Zero crossings:
  In a finite interval $[0, T]$, every sample of a drift-free Brownian motion has infinitely many zero-crossings. The set of zero-crossings is dense in $[0, T]$, i.e., no sample path has isolated zero-crossings!

Brownian Motion

Definition 2. Standard Brownian motion
A Brownian motion is standard if

$$\begin{aligned} W(0) &= 0 \;\; a.s., \\ E[W(t)] &= 0 \;\; (\mu = 0), \\ E[W^2(t)] &= t \;\; (\sigma^2 = 1). \end{aligned}$$

Note that Brownian motion is usually assumed to be standard if not explicitly mentioned.

Definition 3. Differential of standard Brownian motion
We define, formally, the differential $dW(t)$ *to be the limit*

$$dW(t) = \lim_{\Delta t \to dt} \left(W(t + \Delta t) - W(t) \right) .$$

The probability density function of $dW(t)$ *is* $\mathcal{N}(0, dt)$.

Brownian Motion

We already stated that the "sum of squares" of a drift-free Brownian motion is deterministic and possesses the value $\sigma^2 T$. This can be formulated more generally as

Theorem 2. Quadratic variation of standard Brownian motion
The quadratic variation of standard Brownian motion over $[0, t]$ *exists and equals* t*. Formally we can state* $(dW(t))^2 = dt$*.*

We are able now to show that the derivative of standard Brownian motion has infinite variance.

$$\lim_{dt \to 0} \mathsf{Var}\left[\frac{dW(t)}{dt}\right] = \lim_{dt \to 0} \frac{\mathsf{Var}[dW(t)]}{dt^2} = \lim_{dt \to 0} \frac{dt}{dt^2} = \infty$$

For this reason mathematician regard *white noise* is a construction that is not well defined.

Brownian Motion

Nevertheless, in engineering circles, it is customary to define a random process $v(\cdot)$ called *stationary white noise* as the formal derivative of a general Brownian motion $W(\cdot)$ with the drift parameter μ and the variance parameter σ^2:

$$v(t) = \frac{dW(t)}{dt} \quad .$$

Usually, the "initial" time is shifted from $t = 0$ to $t = -\infty$. In this way, the white noise $v(\cdot)$ becomes truly stationary on the infinite time interval $(-\infty, \infty)$.
Without loss of generality, we may assume that $v(t)$ is Gaussian for all t.

Brownian Motion

This stationary white noise is characterized uniquely as follows:

- Expected value:

$$E\{v(t)\} \equiv \mu$$

- Autocovariance function:

$$\Sigma(\tau) = E\{[v(t+\tau) - \mu][v(t) - \mu]\} \equiv \sigma^2 \delta(\tau)$$

- Spectral density function:

$$S(\omega) = \mathcal{F}\{\Sigma(\tau)\} = \int_{-\infty}^{\infty} e^{-j\omega\tau} \Sigma(\tau)\, d\tau \equiv \sigma^2 \quad .$$

Of course, the characterizations by the autocovariance function *and* the spectral density function are redundant.

Brownian Motion

The Brownian motion W on the time interval $[0, \infty)$ can be retrieved from the stationary white noise v by integration:

$$W(t) = \int_0^t v(\alpha)\, d\alpha \quad .$$

Mathematician prefer to write this equation in the following way:

$$W(t) = \int_0^t v(\alpha)\, d\alpha = \int_0^t \frac{dW(\alpha)}{d\alpha}\, d\alpha = \int_0^t dW(\alpha) \quad .$$

Consequently, a Brownian motion X with the drift parameter μ, and the variance parameter σ^2, and the initial time $t = 0$ satisfies the following stochastic differential equation, where W is a standard Brownian motion:

$$dX(t) = \mu dt + \sigma dW(t)\,, \quad X(0) = 0\,.$$

Poisson Process

- Introduction of Brownian motion: random process with independent and stationary increments which is continuous .
- Many stochastic processes can be modelled by an continuous process, e.g., interest rates, macroeconomic indicators, technical measurement, etc.
- By observations of real data there exists the need to describe a process with independent and stationary increments which is not continuous.
- An example of such observation is given in Figure (1).
- The Poisson Process is suitable model for such behavior.

Poisson Process

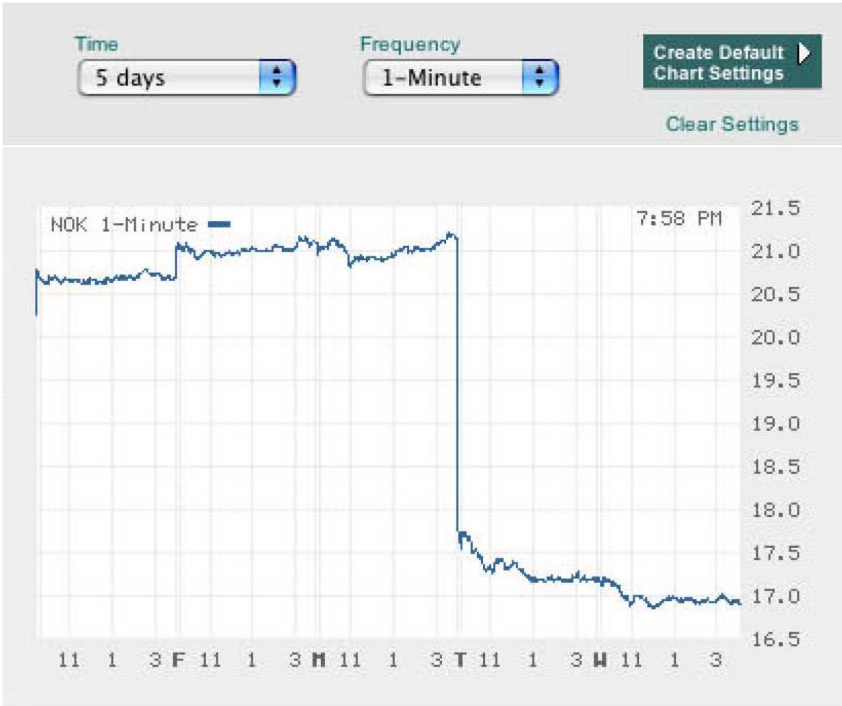

Figure 1: Discontinuity: Nokia stock price on 6th of April 2005

Poisson Process

Definition 4. Poisson process

A Poisson process with parameter λ is a collection of random variables $Q(t),\ t \in [0, \infty)$ defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ having the state space $N = \{0, 1, 2, \dots\}$ and satisfying the following properties:

1. *Q(0)=0 with probability one.*
2. *For each $0 < t_1 < t_2 < \dots < t_n$ the increments $Q(t_2) - Q(t_1),\ Q(t_3) - Q(t_2),\ \dots,\ Q(t_n) - Q(t_{n-1})$ are independent.*
3. *For $0 \leq s < t < \infty$ the increment $Q(t) - Q(s)$ has a Poisson distribution with parameter λ, i.e., the distribution of the increments is give by*

$$P([Q(t) - Q(s)] = k) = \frac{\lambda^k (t-s)^k}{k!} e^{-\lambda(t-s)}$$

for $k \in N$.

Poisson Process

The Poisson process is a continuous time process with discrete realizations, because the state space contains only discrete numbers. The realizations are always positive by definition of N. First, the probability of at least one event happening in a time period of duration Δt is given by

$$P(Q(t + \Delta t) - Q(t)) = \lambda \Delta t + o(\Delta t^2)$$

with $\lambda > 0$ and $\Delta t \to 0$. Note that the probability of an event happening during Δt is proportional to time period of duration. Secondly, the probability of two or more events happening during Δt is of order $o(\Delta t^2)$, therefore making this probability extremely small.

Poisson Process

Let $Q(t + \Delta t) - Q(t)$ be a Poisson process as defined above parameter λ.

Definition 5. Differential of a Poisson process

We define, formally, the differential $dQ(t)$ to be the limit

$$dQ(t) = \lim_{\Delta t \to dt} \left(Q(t + \Delta t) - Q(t) \right) .$$

From the definition of the Poisson process follows that $dQ(t)$ possesses the following properties:

1. $dQ(t) = 0$ with probability $1 - \lambda dt$
2. $dQ(t) = 1$ with probability λdt