Imperial College London

NOTES

IMPERIAL COLLEGE LONDON

DEPARTMENT OF COMPUTING

422 Computational Finance

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1 Theory of Interest

1. Relationship between effective interest rate r_{eff} and nominal rate (m period compounding), $r^{(m)}$:

$$1 + r_{eff} = \left(1 + \frac{r^{(m)}}{m}\right)^m$$

2. **Definitions.** An ideal bank
   - applies the same interest rates for borrowing and lending
   - no transaction costs
   - has the same rate for any size of principal

3. **Definitions.** An ideal bank has an interest value that is independent of the length of time of which it applies, it is called a constant ideal bank.

4. **Theorem.** The cash flow streams $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ are equivalent for a constant ideal bank with interest rate r if and only if their PVs are equal.

5. **Definitions.** The spot rate s_t is the annualized interest rate charged for money held from the present until time t. Properties of spot rate
   - Long commitments tend to offer higher interest rates than short commitments
   - The spot rate curve undulates around in time.
   - Spot rate curve is normally curved if it is increasing; and inverted if it is decreasing.
   - Spot rate curve is smooth.

6. **Definitions.** Forward rate between times t_1 and t_2 is denoted by f_{t_1,t_2}. It is the interest rate charged for borrowing money at time t_1 which is to be repaid at t_2. f_{t_1,t_2} is agreed on today.

$$(1 + s_{t_2})^{t_2} = (1 + s_{t_1})^{t_1}(1 + f_{t_1,t_2})^{t_2 - t_1}$$

7. The forward rate $f_{1,2}$ is
   - the implied rate for money loaned for 1 year, a year from now
   - the market expectation today of what the 1-year spot rate will be next year

2 Fixed Income Securities

1. Important sequences:

   - Geometric Progression

$$S_n = \sum_{k=1}^{n} ar^{k-1} = \frac{a(1-r^n)}{1-r}$$
$$S_\infty = \sum_{k=0}^{\infty} ar^{k-1} = \frac{1}{1-r}$$

   - Arithmetic - Geometric Progression

$$S_n = \sum_{k=1}^{n} [a+(k-1)d]\, r^{k-1} = \frac{a-[a+(n-1)d]\, r^n}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2}$$

2. Price of various basic securities

   - Annuities

$$a_{\overline{n}|} = \frac{1-v^n}{r} \qquad a_{\overline{n}|}^{(m)} = \frac{1-v^n}{r^m}$$

   - Perpetuity

$$a_{\overline{\infty}|} = \frac{1}{r} \qquad a_{\overline{\infty}|} = \frac{1}{r^m}$$

   - Varying annuity

$$(Ia)_{\overline{n}|} = Pa_{\overline{n}|} + D\left[\frac{a_{\overline{n}|} - nv^n}{i}\right]$$

   - Bond

$$P = NCa_{\overline{n}|} + Nv^n$$

3. **Definition.** A bond's **yield to maturity** if the flat interest rate at which the PV of the CFs is equal to the current price. The bond's **current yield** is $\frac{NC}{P}$.

   - There's an inverse relationship between price and yield
   - The longer the time to maturity, the more sensitive is the price of the bond to the yield (think of duration!)
   - If yield to maturity is the same as the coupon rate, the bond price would be the face value

4. **Duration.** Duration measures the sensitivity of the bond with respect to the interest rate

$$
\begin{aligned}
D_{mac} &= \frac{\sum_{t=1}^{n} t \times PV_t}{\sum_{t=1}^{n} PV_t} \\
D_{mod} &= -\frac{1}{P(r_0)} \left.\frac{dP(r)}{dr}\right|_{r=r_0} = \frac{D_{mac}}{(1+r_0)}
\end{aligned}
$$

5. **Convexity.** Convexity is defined as

$$
C = \frac{1}{P(r_0)} \left.\frac{d^2P(r)}{dr^2}\right|_{r=r_0} = \frac{1}{P(r_0)} \sum_{t=1}^{n} t^2 PV_t
$$

6. Estimation of bond price change

$$
\Delta P \approx -D_{mod} P(r_0)\Delta r + \frac{1}{2} P(r_0)(\Delta r)^2
$$

7. **Immunization.** Let A be assets and L be liabilities.

   - $PV(A) = PV(L)$
   - $D(A) = D(L)$
   - $C(A) > C(L)$

3 Mean Variance Portfolio Theory

1. A portfolio's total return and rate of return are

$$R = \sum_{i}^{n} R_i$$

$$r = \sum_{i}^{n} r_i$$

2. Mean and variance of portfolio return

$$\overline{r} = \mathbb{E}(r) = \sum_{i=1}^{n} w_i \bar{(r)}_i$$

$$\sigma^2 = \mathbb{V}(r) = \sum_{i,j=1}^{n} w_i w_j \sigma_{ij}$$

3. Assumptions of the Markowitz model:

   - Investors consider each investment alternative as being represented by a probability distribution of expected returns over some holding period.
   - Investors maximize one-period expected utility and their utility curves demonstrate diminishing marginal utility of wealth.
   - Investors estimate risk on basis of variability of expected returns and base decisions solely on expected return and risk.
   - Investors prefer higher returns to lower risk and lower risk for the same level of return

4. Markowitz Model:

   - Assumed that there are n risky assets with mean return $\{\overline{r}_i\}_{i=1}^{n}$ and covariances $\{\sigma_{ij}\}_{i,j=1}^{n}$, the portfolio has mean return and variance

$$\overline{r}_P = \sum_{i=1}^{n} w_i \overline{r}_i$$

$$\sigma_P^2 = \sum_{i,j=1}^{n} w_i w_j \sigma_{ij}$$

   - The Markowitz model is then an optimization problem to minimize the variance subject to a portfolio target returns.

$$\min_{w_i, w_i} \frac{1}{2} \sum_{i,i=1}^{n} w_i \sigma_{ij} w_j = \frac{1}{2} \sigma_P^2$$

$$\text{s.t.} \sum_{i=1}^{n} w_i \overline{r}_i = \overline{r}_P$$
$$\sum_{i=1}^{n} w_i = 1$$

This problem can be solve by introducing the Lagrange multipliers.

$$L(w) = \frac{1}{2}\sum_{i,i=1}^{n} w_i \sigma_{ij} w_j - \lambda\left(\sum_{i=1}^{n} w_i \overline{r}_i - r_P\right) - \mu\left(\sum_{i=1}^{n} w_i - 1\right)$$

The mean-variance set is obtained by plotting the minimal σ_P^2 for different $\overline{r}_P$. The efficient frontier is the top half of the mean-variance set.

- Solution to the model can be obtained by solving the linear equations

$$w_i : \sum_{i=1}^{n} \sigma_{ij} w_j - \lambda \overline{r}_i - \mu = 0$$
$$\lambda : \sum_{i=1}^{n} w_i \overline{r}_i = \overline{r}_P$$
$$\mu : \sum_{i=1}^{n} w_i = 1$$

5. In vector notation:

$$\begin{aligned} \min_{\boldsymbol{w}} \ & \frac{1}{2}\boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w} \\ \text{s.t.} \ & \boldsymbol{w}^\top \overline{\boldsymbol{r}} - \overline{r}_P = 0 \\ & \boldsymbol{w}^\top \mathbf{1} - 1 = 0 \end{aligned}$$

$$L(\boldsymbol{w}, \lambda, \mu) = \frac{1}{2}\boldsymbol{w}^\top \Sigma \boldsymbol{w} - \lambda\left(\boldsymbol{w}^\top \overline{\boldsymbol{r}} - \overline{r}_P\right) - \mu(\boldsymbol{w}^\top \mathbf{1} - 1)$$

$$\begin{bmatrix} \boldsymbol{\Sigma} & -\overline{\boldsymbol{r}} & -\mathbf{1} \\ -\overline{\boldsymbol{r}}^\top & 0 & 0 \\ \mathbf{1}^\top & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{w} \\ \lambda \\ \mu \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ -\overline{r}_P \\ -1 \end{bmatrix}$$

6. The above formulation assume short selling is permitted. However, if there is no short selling, we need to add the constraints for the weights to be positive.

7. Estimation of the parameters can be done using historical data and MLE.

$$\hat{r} = \frac{1}{n}\sum_{i=1}^{n} r_i$$

$$\hat{\sigma^2} = \frac{1}{n-1}\sum_{i=1}^{n}(r_i - \hat{r})^2$$

$$\hat{\sigma}_{ab} = \frac{1}{n-1}\sum_{i=1}^{n}(r_{a,i} - \hat{r}_a)(r_{b,i} - \hat{r}_b)$$

4 Capital Asset Pricing Model

1. **The Two-Fund Theorem.** Let $(\boldsymbol{w}_1, \lambda_1, \mu_1)$ and $(\boldsymbol{w}_2, \lambda_2, \mu_2)$ be the Markowitz solutions for $\overline{r}_P^1$ and $\overline{r}_P^2$. Then , the Markowitz solution for

$$\overline{r}_P^3 = \alpha\overline{r}_P^1 + (1-\alpha)\overline{r}_P^2$$

is given by

$$(\boldsymbol{w}_3, \lambda_3, \mu_3) = \alpha(\boldsymbol{w}_1, \lambda_1, \mu_1) + (1-\alpha)(\boldsymbol{w}_2, \lambda_2, \mu_2).$$

The implication of the two fund theorem is that investors seeking efficient portfolios need to only invest in combinations of two efficient funds. There is no need for anyone to buy individual stocks.

2. **The One-Fund Theorem.** When risk-free borrowing and lending are available, there is a single fund F of risky assets such that any efficient portfolio can be constructed as a combination of the fund F and the risk free asset.

The implication is that every investor will buy a combination of the fund and the risk-free assets. Hence, F must be the market portfolio.

3. **Capital Market Line.** Given a risk-free asset, the efficient frontier is called the capital market line. Any asset on the CML satisfies

$$\overline{r} = r_f + \frac{\overline{r}_M - r_F}{\sigma_M}\sigma$$

4. **Capital Asset Pricing Model.** Assumed that

- All investors are Markowitz mean-variance investors
- Shorting is allowed
- There exists a risk-free asset
- The investors share same predictions of means, variances and covariances

For the market portfolio M is efficient, the expected return of $\overline{r}_i$ of any asset i satisfies

$$\begin{aligned} \overline{r}_i - r_F &= \beta_i(\overline{r}_M - r_f) \\ \beta_i &= \frac{\sigma_{iM}}{\sigma_M^2} \end{aligned}$$

Under equilibrium conditions assumed by CAPM, any assets should fall on the security market line.

5. Importance of CAPM

   - The correlation with the market determines the expected excess rate of return of an asset
   - The Capital Market Line relates the expected rate of return of an efficient portfolio to its standard deviation / risk.
   - The Security Market Line relates the expected rate of return of an in individual asset to its beta / systematic risk.

6. Proof of only systematic loss would be relevant

$$\begin{aligned} r_i = r_f + \beta_i(r_M - r_f) + \epsilon_i \sigma_{iM} && = \mathrm{cov}\Big(r_f + \beta_i(r_M - r_f) + \ epsilon_i, r_M\Big) \\ = \beta_i \mathrm{cov}(r_M, r_M) + \mathrm{cov}(\epsilon_i, r_M) \end{aligned}$$

7. CAPM can be used as a pricing formula

$$\begin{aligned} \frac{\overline{P}_1 - P_0}{P_0} &= r_f + \beta(\overline{r}_M - r_f) \\ P &= \frac{\overline{P}_1}{1 + r_f + \beta(\overline{r}_M - r_f)} \end{aligned}$$

8. Certainty Equivalent form for CAPM

$$\begin{aligned} P_0 &= \frac{\overline{P}_1}{1 + r_f + \beta(\overline{r}_M - r_f)} \\ &= \frac{1}{1 + r_f}\left(\overline{P}_1 - \frac{\mathrm{cov}(P, r_M)}{\sigma_M^2}(\overline{r}_M - r_f)\right) \end{aligned}$$

We can hence use the NPV to evaluate projects

$$NPV = -P + \frac{1}{1 + r_f}\left(\overline{P}_1 - \frac{\mathrm{cov}(P, r_M)}{\sigma_M^2}(\overline{r}_M - r_f)\right)$$

5 General Risk Principles

5.1 Utility Functions

1. The utility function is used to rank random wealth levels. It varies among decision makers, depending on their
   - Risk tolerance
   - Individual financial environment
2. Characteristic of utility functions:
   - Define on the real line (possible wealth levels) and gives a real value
   - Invariant to affine transformation

$$U_1(x) = aU(x) + b \equiv U(x), \forall a > 0.$$

3. Some examples of utility functions:

$$\begin{aligned} U(x) &= -e^{-ax}, \forall a > 0 \\ U(x) &= \ln x, \forall x > 0 \\ U(x) &= bx^b, \forall b \leq 1, b \neq 0 \\ U(x) &= x - bx^2, \forall b > 0 \end{aligned}$$

5.2 Risk Aversion

1. **Definition.** A function $U : [a,b] \to \mathbb{R}$ is concave if for any $\alpha \in [0,1]$ and for any x and y in $[a,b]$ there holds

$$U(\alpha x + (1-\alpha)y) \geq \alpha U(x) + (1-\alpha)U(y)$$

2. A utility is risk averse if it is concave on $[a,b]$
3. Properties of a utility function relating to its derivatives
   - $U'(x) > 0$ because people are greedy bastards, so more will be better
   - $U''(x) < 0$ because people are risk-averse
4. Arrow-Pratt absolute risk aversion

$$a(x) = -\frac{U''(x)}{U'(x)}$$

   - $a(x)$ shows how risk-aversion changes with wealth
   - Risk aversion decreases as wealth grows
   - $a(x)$ is the same for all equivalent utility funcitons

5. Certainty Equivalent: The certainty equivalent C of a random wealth variable x is the amount of certain (deterministic) wealth that has a utility level equal to the expected utility of x.

$$U(C) = \mathbb{E}[U(x)]$$

6. Measurement of utility functions

   (a) Method I

   - Select fixed wealth levels A and B.
   - Propose a lottery that has outcome A with probability p and outcome B with probability $1 - p$.
   - For $p \in [0, 1]$ the investor is asked how much certain wealth C he or she would accept in place of the lottery

   (b) Method II

   - Select a parameterized family of utility functions
   - Then determine the parameter using the lottery as per Method I

7. Connection of Utility Function to Markowitz Model

$$\begin{aligned} U(x) &= ax - \frac{b}{2}x^2, \forall a, b, > 0, x \leq \frac{a}{b} \\ \mathbb{E}[U(x)] &= a\mathbb{E}[x] - \frac{b}{2}\mathbb{E}[x^2] \\ &= a\mathbb{E}[x] - \frac{b}{2}\mathbb{V}[x] - \frac{b}{2}\mathbb{E}[x]^2 \end{aligned}$$

5.3 Arbitrage

1. Definition: A security is a random payoff variable d. The payoff is revealed and obtained at the end of the period. Associated with the security is a price P.

2. Ideal Market:

   - Securities can be arbitrarily divided
   - There are no transaction costs
   - Short sales is allowed

3. Definition of Arbitrage:

   - Type A: $P < 0$ and $d = 0$
   - Type B: $P \leq 0$, $d \geq 0$ and $P(d > 0) > 0$

4. Linearity of pricing

   - The price of the sum of two securities is the sum of their prices.

- The price of the multiple of an asset is the same multiple of the price.

5. Portfolio Problem

$$\begin{aligned} &\max_{\theta \in \mathbb{R}^n} \mathbb{E}[U(x)] \\ \text{subject to } &\sum_{i=1}^{n} \theta_i d_i = x \\ &\sum_{i=1}^{n} \theta_i P_i \leq W. \end{aligned}$$

Assume that $U(x)$ is continuous , $U(x) \to +\infty$ as $x \to +\infty$, and there is a portfolio θ^0 such that $\sum_{i=1}^{n} \theta_i^0 d_i > 0$. Then:

$$\mathcal{P} \text{ has a solution} \Leftrightarrow \text{ there is no arbitrage possibility.}$$

Consider instead a simplified problem

$$\begin{aligned} &\max_{\theta \in \mathbb{R}^n} \mathbb{E}\left[U\left(\sum_{i=1}^{n} \theta_i d_i\right)\right] \\ \text{subject to } &\sum_{i=1}^{n} \theta_i P_i = W \end{aligned}$$

We can write the Lagrangian as

$$L(\theta, \lambda) = \mathbb{E}\left[U\left(\sum_{i=1}^{n} \theta_i d_i\right)\right] - \lambda\left(\sum_{i=1}^{n} \theta_i P_i - W\right)$$

Differentiating L w.r.t θ_i

$$\begin{aligned} \mathbb{E}[U'(x^*) d_i] &= \lambda P_i, \forall i = 1, \dots, n \\ x^* &= \sum_{i=1}^{n} \theta_i^* d_i \end{aligned}$$

6. If $x^* = \sum_{i=1}^{n} \theta_i^* d_i$ solve $\mathcal{P}$ then

$$\mathbb{E}[U'(x^*) d_i] = \lambda P_i$$

If there is a risk-free asset with total return R

$$\frac{U'(x^*) d_i}{R \mathbb{E}[U'(x^*)]} = P_i$$

6 Asset Price Dynamics

1. Additive Model:

$$S_(t+1) = aS(t) + z(t), \forall t = 0, 1, \ldots, N$$
$$\mathbb{E}\left[S(t+k)\right] = a^k S(t), \forall z(t) \sim \mathcal{N}(0, \sigma^2)$$

   Deficiencies of the model:

   - For $a > 0$ the expected value of the stock increase exponentially over time.
   - The model can give negative stock prices
   - The volatility σ is not scaled to the stock price level. Price shocks are tend to be proportional to the stock price.

2. Multiplicative Model:

$$\ln \frac{S(t+1)}{S(t)} = z(t)$$
$$S(t+k) = S(t) \exp\left(\sum_{j=1}^{k} z(j)\right)$$
$$z(t) \sim \mathcal{N}(\mu, \sigma^2)$$
$$\ln S(t+k) \sim \mathcal{N}\left(S(t) + k\mu, k\sigma^2\right)$$

   Properties and Justification of the log normal price model

   - The expected value of the log returns increase linearly with time and the stock price is log-normally distributed
   - The prices cannot become negative.
   - The noise are i.i.d. and subject to finite variance.

3. Properties of Log Normal Distribution

$$X \sim \ln \mathcal{N}(\mu, \sigma^2) \Rightarrow \ln X \sim \mathcal{N}(\mu, \sigma^2)$$
$$\mathbb{E}[X] = \exp\left(\mu + \frac{\sigma^2}{2}\right)$$
$$\mathbb{V}[X] = \left(\exp(\sigma^2) - 1\right) \exp\left(2\mu + \sigma^2\right)$$

4. Random walk

$$z(t_{k+1}) - z(t_k) = \epsilon(t_k)\sqrt{\Delta t}$$
$$t_{k+1} = t_k + \Delta t, 0 \leq k < N$$
$$\epsilon(t_k) \sim \mathcal{N}(0, 1)$$

For $j < k$, we have

$$z(t_k) - z(t_{k-1}) = \sum_{i=j}^{k-1} \epsilon(t_i)\sqrt{\Delta t}$$
$$z(t_k) - z(t_{k-1}) \sim \mathcal{N}\left(0, t_k - t_j\right)$$

5. As $\Delta t \to 0$, we have Brownian motion

$$dz(t) = \epsilon(t)\sqrt{dt}$$

6. Properties of Brownian Motion
   - For all $s < t$, $z(t) - z(s) \sim \mathcal{N}(0, t-s)$
   - For all $t_1 < t_2 \le t_3 < t_e$, $z(t_2) - z(t_1)$ and $z(t_4) - z(t_3)$ are independent.
   - $z(0) = 1$ almost surely.
7. To calibrate the Binomial model to the geometric Brownian motion model, we match the expectation and the variance of the Brownian motion to that of the binomial model.

$$\begin{aligned}
S(t+\Delta t) &= S(t)e^{\Delta z(t)}, \Delta z(t) \sim \mathcal{N}(\nu\Delta t, \sigma^2\Delta t) \\
\mathbb{E}\left[\ln\left(\frac{S(t+\Delta t)}{S(t)}\right)\right] &= p\ln u + (1-p)\ln d = \nu\Delta t \\
\mathbb{V}\left[\ln\left(\frac{S(t+\Delta t)}{S(t)}\right)\right] &= p(\ln u)^2 + (1-p)(\ln d)^2 - [p\ln u + (1-p)\ln d]^2 \\
&= p(1-p)(\ln u - \ln d)^2 \\
&= \sigma^2\Delta t
\end{aligned}$$

Setting $u = 1/d$,

$$\begin{aligned}
p &\approx \frac{1}{2} + \frac{1}{2}\left(\frac{\nu}{\sigma}\right)\sqrt{\Delta t} \\
\ln u &\approx \sigma\sqrt{\Delta t} \\
\ln d &\approx -\sigma\sqrt{\Delta t}
\end{aligned}$$

7 Basic Options Theory

7.1 General Options Related Theory

1. Put Call Parity

$$C - P = S - Ke^{-T}$$

7.2 Binomial Pricing

1. Consider a single period binomial model, we have the following

$$S(t+1) = \begin{cases} uS(t) \\ dS(t) \end{cases} \quad C(t+1) = \begin{cases} \max(uS(t)-K,0) \\ \max(dS(t)-K,0) \end{cases}$$

2. Construct a portfolio, Π that buys x units of underlying and b amount in risk free bond

$$\Pi(t) = xS(t) + b$$
$$\Pi(t+1) = \begin{cases} xuS(t) + b(1+r) \\ xdS(t) + b(1+r) \end{cases}$$

We need the replicating portfolio to match the payoff of the derivative

$$\begin{aligned} C_u(t+1) &= xuS(t) + b(1+r) \\ C_d(t+1) &= xdS(t) + b(1+r) \\ x &= \frac{C_u(t+1) - C_d(t+1)}{uS(t) - dS(t)} \\ b &= \frac{uC_d - dC_u}{(1+r)(u-d)} \end{aligned}$$

$$\begin{aligned} C(t) &= \Pi(t) \\ &= \left(\frac{C_u(t+1) - C_d(t+1)}{uS(t) - dS(t)} \right) S(t) + \frac{uC_d(t+1) - dC_u(t+1)}{(1+r)(u-d)} \\ &= \frac{1}{1+r} \left[\frac{(1+r)-u}{u-d} C_u(t+1) + \frac{u-(1+r)}{u-d} C_d(t+1) \right] \\ &= \frac{1}{1+r} [\tilde{p} C_u(t+1) + (1-\tilde{p}) C_d(t+1)] \end{aligned}$$