Imperial College London

NOTES

IMPERIAL COLLEGE LONDON

DEPARTMENT OF COMPUTING

422 Computational Finance

Author:
Thomas Teh (CID: 0124 3008)

Date: February 12, 2017

1 Theory of Interest

1. Relationship between effective interest rate r_{eff} and nominal rate (m period compounding), $r^{(m)}$:

$$1 + r_{eff} = \left(1 + \frac{r^{(m)}}{m}\right)^m$$

2. **Definitions.** An ideal bank
   - applies the same interest rates for borrowing and lending
   - no transaction costs
   - has the same rate for any size of principal

3. **Definitions.** An ideal bank has an interest value that is independent of the length of time of which it applies, it is called a constant ideal bank.

4. **Theorem.** The cash flow streams $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ are equivalent for a constant ideal bank with interest rate r if and only if their PVs are equal.

5. **Definitions.** The spot rate s_t is the annualized interest rate charged for money held from the present until time t. Properties of spot rate
   - Long commitments tend to offer higher interest rates than short commitments
   - The spot rate curve undulates around in time.
   - Spot rate curve is normally curved if it is increasing; and inverted if it is decreasing.
   - Spot rate curve is smooth.

6. **Definitions.** Forward rate between times t_1 and t_2 is denoted by f_{t_1,t_2}. It is the interest rate charged for borrowing money at time t_1 which is to be repaid at t_2. f_{t_1,t_2} is agreed on today.

$$(1 + s_{t_2})^{t_2} = (1 + s_{t_1})^{t_1}(1 + f_{t_1,t_2})^{t_2 - t_1}$$

7. The forward rate $f_{1,2}$ is
   - the implied rate for money loaned for 1 year, a year from now
   - the market expectation today of what the 1-year spot rate will be next year

2 Fixed Income Securities

1. Important sequences:

   - Geometric Progression

$$S_n = \sum_{k=1}^{n} ar^{k-1} = \frac{a(1-r^n)}{1-r}$$
$$S_\infty = \sum_{k=0}^{\infty} ar^{k-1} = \frac{1}{1-r}$$

   - Arithmetic - Geometric Progression

$$S_n = \sum_{k=1}^{n} [a+(k-1)d]\, r^{k-1} = \frac{a-[a+(n-1)d]\, r^n}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2}$$

2. Price of various basic securities

   - Annuities

$$a_{\overline{n}|} = \frac{1-v^n}{r} \qquad a_{\overline{n}|}^{(m)} = \frac{1-v^n}{r^m}$$

   - Perpetuity

$$a_{\overline{\infty}|} = \frac{1}{r} \qquad a_{\overline{\infty}|} = \frac{1}{r^m}$$

   - Varying annuity

$$(Ia)_{\overline{n}|} = Pa_{\overline{n}|} + D\left[\frac{a_{\overline{n}|} - nv^n}{i}\right]$$

   - Bond

$$P = NCa_{\overline{n}|} + Nv^n$$

3. **Definition.** A bond's **yield to maturity** if the flat interest rate at which the PV of the CFs is equal to the current price. The bond's **current yield** is $\frac{NC}{P}$.

   - There's an inverse relationship between price and yield
   - The longer the time to maturity, the more sensitive is the price of the bond to the yield (think of duration!)
   - If yield to maturity is the same as the coupon rate, the bond price would be the face value

4. **Duration.** Duration measures the sensitivity of the bond with respect to the interest rate

$$D_{mac} = \frac{\sum_{t=1}^{n} t \times PV_t}{\sum_{t=1}^{n} PV_t}$$
$$D_{mod} = -\frac{1}{P(r_0)} \frac{dP(r)}{dr}\bigg|_{r=r_0} = \frac{D_{mac}}{(1+r_0)}$$

5. **Convexity.** Convexity is defined as

$$C = \frac{1}{P(r_0)} \frac{d^2P(r)}{dr^2}\bigg|_{r=r_0} = \frac{1}{P(r_0)} \sum_{t=1}^{n} t^2 PV_t$$

6. Estimation of bond price change

$$\Delta P \approx -D_{mod} P(r_0)\Delta r + \frac{1}{2} P(r_0)(\Delta r)^2$$

7. **Immunization.** Let A be assets and L be liabilities.

    - $PV(A) = PV(L)$
    - $D(A) = D(L)$
    - $C(A) > C(L)$

3 Mean Variance Portfolio Theory

1. A portfolio's total return and rate of return are

$$R = \sum_{i}^{n} R_i$$
$$r = \sum_{i}^{n} r_i$$

2. Mean and variance of portfolio return

$$\overline{r} = \mathbb{E}(r) = \sum_{i=1}^{n} w_i \bar{(r)}_i$$
$$\sigma^2 = \mathbb{V}(r) = \sum_{i,j=1}^{n} w_i w_j \sigma_{ij}$$

3. Assumptions of the Markowitz model:

   - Investors consider each investment alternative as being represented by a probability distribution of expected returns over some holding period.
   - Investors maximize one-period expected utility and their utility curves demonstrate diminishing marginal utility of wealth.
   - Investors estimate risk on basis of variability of expected returns and base decisions solely on expected return and risk.
   - Investors prefer higher returns to lower risk and lower risk for the same level of return

4. Markowitz Model:

   - Assumed that there are n risky assets with mean return $\{\overline{r}_i\}_{i=1}^{n}$ and covariances $\{\sigma_{ij}\}_{i,j=1}^{n}$, the portfolio has mean return and variance

$$\overline{r}_P = \sum_{i=1}^{n} w_i \overline{r}_i$$
$$\sigma_P^2 = \sum_{i,j=1}^{n} w_i w_j \sigma_{ij}$$

   - The Markowitz model is then an optimization problem to minimize the variance subject to a portfolio target returns.

$$\min_{w_i, w_i} \frac{1}{2} \sum_{i,i=1}^{n} w_i \sigma_{ij} w_j = \frac{1}{2} \sigma_P^2$$

$$\text{s.t.} \sum_{i=1}^{n} w_i \overline{r}_i = \overline{r}_P$$
$$\sum_{i=1}^{n} w_i = 1$$

This problem can be solve by introducing the Lagrange multipliers.

$$L(w) = \frac{1}{2} \sum_{i,i=1}^{n} w_i \sigma_{ij} w_j - \lambda \left(\sum_{i=1}^{n} w_i \overline{r}_i - r_P \right) - \mu \left(\sum_{i=1}^{n} w_i - 1 \right)$$

The mean-variance set is obtained by plotting the minimal σ_P^2 for different $\overline{r}_P$. The efficient frontier is the top half of the mean-variance set.

- Solution to the model can be obtained by solving the linear equations

$$w_i : \sum_{i=1}^{n} \sigma_{ij} w_j - \lambda \overline{r}_i - \mu = 0$$
$$\lambda : \sum_{i=1}^{n} w_i \overline{r}_i = \overline{r}_P$$
$$\mu : \sum_{i=1}^{n} w_i = 1$$

5. In vector notation:

$$\min_{\boldsymbol{w}} \ \frac{1}{2} \boldsymbol{w}^\top \boldsymbol{\Sigma} \boldsymbol{w}$$
$$\text{s.t. } \boldsymbol{w}^\top \overline{\boldsymbol{r}} - \overline{r}_P = 0$$
$$\boldsymbol{w}^\top \mathbf{1} - 1 = 0$$

$$L(\boldsymbol{w}, \lambda, \mu) = \frac{1}{2} \boldsymbol{w}^\top \Sigma \boldsymbol{w} - \lambda \left(\boldsymbol{w}^\top \overline{\boldsymbol{r}} - \overline{r}_P \right) - \mu (\boldsymbol{w}^\top \mathbf{1} - 1)$$

$$\begin{bmatrix} \boldsymbol{\Sigma} & -\overline{\boldsymbol{r}} & -\mathbf{1} \\ -\overline{\boldsymbol{r}}^\top & 0 & 0 \\ \mathbf{1}^\top & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{w} \\ \lambda \\ \mu \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ -\overline{r}_P \\ -1 \end{bmatrix}$$

6. The above formulation assume short selling is permitted. However, if there is no short selling, we need to add the constraints for the weights to be positive.

7. Estimation of the parameters can be done using historical data and MLE.

$$\hat{r} = \frac{1}{n}\sum_{i=1}^{n} r_i$$
$$\hat{\sigma^2} = \frac{1}{n-1}\sum_{i=1}^{n}(r_i - \hat{r})^2$$
$$\hat{\sigma}_{ab} = \frac{1}{n-1}\sum_{i=1}^{n}(r_{a,i} - \hat{r}_a)(r_{b,i} - \hat{r}_b)$$

4 CAPM