Imperial College London

NOTES

IMPERIAL COLLEGE LONDON

DEPARTMENT OF COMPUTING

493 Data Analysis and Probabilistic Inference

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1 Bayes Theorem and Bayesian Inference

1. Bayes Theorem:

$$P(D \cap S) = P(D|S)P(S) = P(S|D)P(D)$$
$$P(D|S) = \alpha \times P(D) \times P(S|D)$$

2. Law of Total Probability:

$$P(F) = \sum_{i=1}^{n} P(F \cap E_i) = \sum_{i=1}^{n} P(F|E_i)P(E_i)$$

3. Conditional Independence: Two events D and S are conditionally independent given G if $P(G) \neq 0$ and one of the following holds:
   (a) $P(D|S \cap G) = P(D|G)$ and $P(D|G) \neq 0, P(S|G) \neq 0$
   (b) $P(D|G) = 0$ or $P(S|G) = 0$
   (c) $P(D \cap S|G) = P(D|G)P(S|G)$

4. Bayesian Inference: Given a set of competing hypothesis which explain a data set, for each hypothesis:
   (a) Convert the prior and likelihood information in the data into probabilities and take their product
   (b) Normalize the result to get the posterior probabilities of each hypothesis given the evidence
   (c) Select the most probably hypothesis

2 Simple Bayesian Networks

1. We have to assume the Causal Markov Condition has the following difficulties inherent in large instances

   (a) The joint probabilities are hard to estimates

   (b) Even if the joint probabilities can be obtained, there are too many number of instances

2. **Causal Markov Condition:** Suppose we have a joint probability distribution P of the random variables in some set $\mathcal{V}$ and a DAG $\mathbb{G} = (\mathcal{V}, E)$. We say that $(\mathbb{G}, P)$ satisfies the Markov condition if for each variable $X \in \mathcal{V}$, $\{X\}$ is conditional independent of **the set of all its nondescendents given the set of all its parents**. Let ND_X be the non-descendents and PA_X be the parents of X.

$$I_P(\{X\}, ND_X | PA_X)$$

3. Possible violations of the Causal Markov Condition:

   (a) Hidden cause: X and Y is said to have a common cause if there exists some variable that has causal paths into both of them. If we fail to model this common cause, in short (exists a hidden cause), the Markov condition would be violated as it assumes independence.

   (b) Selection bias: It is similar to hidden cause. The variables we observe shows independence when actually because of our sampling error.

   (c) Feedback loops: Causal relationships need to be only one way. The child node under no circumstances should influence the parent node.

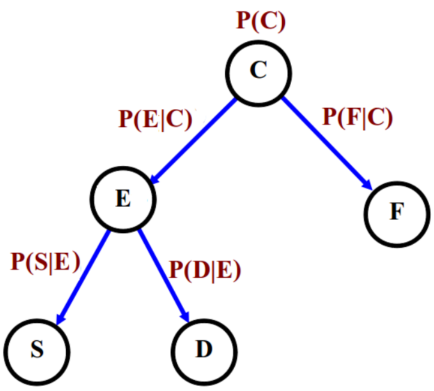


Figure 1: Example of naive bayes network. Given the parent C, the node E and F meet the causal markov condition, i.e. they are conditionally independent.

4. Each arc in a simple network is represented by a link matrix (conditional probability matrix)

$$\boldsymbol{P}(\boldsymbol{D}|\boldsymbol{C}) = \left[P(d_i|c_j)\right] = \begin{bmatrix} P(d_1|c_1) & P(d_1|c_2) \\ P(d_2|c_1) & P(d_2|c_2) \\ P(d_3|c_1) & P(d_3|c_2) \\ P(d_4|c_1) & P(d_4|c_2) \end{bmatrix}$$

5. The root nodes of a network do not have any parents. They have a vector giving the prior probabilities

$$\boldsymbol{P}(\boldsymbol{C}) = [P(c_i)] = \left[P(c_1) \quad P(c_2)\right]$$

6. **Instantiation** means setting the value of a node.

7. **Bayesian Classifiers** using the above network.

   (a) We cannot compute E as it is a latent variable that we compute and we do not have measurements. However, we can computed the likelihood of E.

   $$\begin{aligned} P(E|S \cap D) &= \alpha P(E)P(S|E)P(D|E) & &= \alpha P(E)L(E|S \cap D) \\ L(E|S \cap D) &= (S|E)P(D|E) \end{aligned}$$

   (b) Then we look at the root note C. Given $F = f_5$:

   $$P(C|E \cap F) = \alpha P(C)P(E|C)P(F|C)$$

   $$P(e|c_k) = \sum_{i=1}^{3} P(e_i|c_k)L(e_i)$$

   $$P'(c_k) = P(c_k|e \cap f_5) = \alpha P(c_k)\left(\sum_{i=1}^{3} P(e_i|c_k)L(e_i)\right)P(f_5|c_k)$$

   (c) We see the the evidence for C comes from

      i. Evidence coming from E and its sub-tree
      ii. Evidence from everywhere else.

   $$\begin{aligned} P_E(C) &= \alpha P(C)P(F|C) \\ \boldsymbol{P}(\boldsymbol{E}) &= \boldsymbol{P}(\boldsymbol{E}|\boldsymbol{C})\boldsymbol{P}_{\boldsymbol{E}}(\boldsymbol{C}) \end{aligned}$$

   (d) Suppose if we have the instantiations $S = s_3$ and $D = d_2$

   $$P'(e_i) = \alpha P(e_i)P(s_3|e_i)P(d_2|e_i)$$

3 Evidence and Message Passing: Pearl's Algorithm

1. New concepts to deal with complex networks with intermediate nodes:

   - **Evidence** is the information that we have at a node -this may be gathered through instantiation (exact value or virtual evidence), or inferred from passing messages. Evidence is unnormalized probabilities so the absolute values are meaningless, but they are useful for making comparisons.
   - **Messages** is the information (evidence) passed between nodes to provide evidence to another node.

2. **Theorem:** Let $(\mathbb{G}, P)$ be a Bayesian network whose DAG is a tree, where $\mathbb{G} = (V, E)$, and a be a set of values of a subset $A \subset V$.

   (a) λ **messages**: For each child Y of X, $\forall x \in X$

   $$\lambda_Y(x) = \sum_y P(y|x)\lambda(y)$$

   (b) λ **values**:

   i. If $X \in A$ and X is instantiated to $\hat{x}$

   $$\lambda(\hat{x}) = 1, \text{ for } x = \hat{x}$$
   $$\lambda(x) = 0, \text{ for } x \neq \hat{x}$$

   ii. if $X \notin A$ and X is a leaf, $\forall x \in X$,

   $$\lambda(x) = 1$$

   iii. If $X \notin A$ and X is not a leaf, $\forall x \in X$

   $$\lambda(x) = \prod_{U \in CH_X} \lambda_U(x),$$

   where CH_X denotes the set of the children of X.

   (c) π **messages**: If Z is the parent of X, then $\forall z \in Z$

   $$\pi_X(z) = \pi(z) \prod_{U \in CH_Z - \{X\}} \lambda_U(z)$$

   (d) π **values**:

   i. If $X \in A$ and X is instantiated to $\hat{x}$:

   $$\pi(\hat{x}) = 1, \text{ for } x = \hat{x}$$
   $$\pi(x) = 0, \text{ for } x \neq \hat{x}$$

   ii. If $X \notin A$ and X is the root, $\forall x \in X$

   $$\pi(X) = P(x)$$

iii. If $X \notin A$, X is not the root and Z is the parent of X, $\forall x\ inX$

$$\pi(x) = \sum_z P(x|z)\pi_X(z)$$

(e) Given the definitions, for each variable X, we have for all values of x,

$$P(x|a) = \alpha\lambda(x)\pi(x)$$

3. The π values are basically evidence from the parents and it generalises the concept of prior. The λ values are basically evidence from the descendents and it generalises the concept of likelihood probability.

4. Mnemonic: **p**i (π), **p**rior, and "**p**arent" all start with letter "p"; and **l**ambda (λ), **l**ikelihood, and "**l**ad" all start with "l". Further, prior comes *before* so from parents.

5. If we use virtual evidence at the leaf nodes, we can use the conditioning equation (above b(iii))

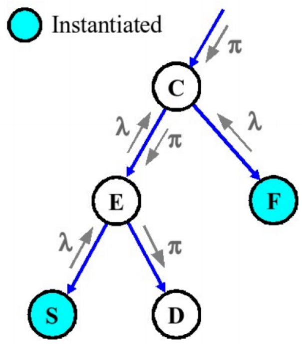


Figure 2: Upon instantiation of a node, we can propagate the λ and π messages

6. Important equations:

$$\boldsymbol{\lambda_S(E) = \lambda(S)P(S|E)} \qquad \boldsymbol{\pi(E) = P(E|C)\pi_E(C)}$$

$$\lambda(e_i) = \prod_{CH_E}\left[\sum_j \lambda(s_j)P(s_j|e_i)\right] \qquad \pi(e_i) = \sum_j \left[P(e_i|c_j)\pi(c_j)\prod_{k\backslash E}\lambda_k(c_j)\right]$$

7. Notation

(a) $\boldsymbol{\lambda_F(c)}$ means λ evidence from node F to node C.

(b) $\boldsymbol{\pi_F(c)}$ means π from all nodes besides F for node C. (This can alternatively viewed as the information from C to F)

4 Probability Propagation: Single Connected Networks

1. A DAG is singly connected if there is at most one path between any two nodes. In a singly-connected network, each node can have more than 1 parents.

2. The main properties of these networks are:

   (a) The parents of a node are always independent given their common child (i.e. they don't have a common parent). This lets us calculate their joint probability as the product of their marginals.

   (b) When updating evidence of a node, the belief propagated through the net to update all nodes is guaranteed to reach a steady state.

3. An example of a link matrix for a node with multiple parents looks as follows:

$$\boldsymbol{P(e|w,c)} = \begin{bmatrix} P(e_1|w_1,c_1) & P(e_1|w_1,c_2) & P(e_1|w_2,c_1) & P(e_1|w_2,c_2) \\ (e_2|w_1,c_1) & P(e_2|w_1,c_2) & P(e_2|w_2,c_1) & P(e_2|w_2,c_2) \\ P(e_3|w_1,c_1) & P(e_3|w_1,c_2) & P(e_3|w_2,c_1) & P(e_3|w_2,c_2) \end{bmatrix}$$

4. To calculate the π evidence of a node with 2 parents (assuming independence between the parents):

$$\pi(\boldsymbol{E}) = \boldsymbol{P(e|w,c)\pi_e(w,c)} = \boldsymbol{P(e|w,c)\pi_e(w)\pi_e(c)}$$

5. To calculate the λ evidence of a node with 2 parents, we have to calculate one λ message for each of the parents. If c has parents a and b, then the evidence from c to a is as follows:

$$\lambda_c(a_i) = \sum_{j=1}^{n} \pi_c(b_j) \sum_{k=1}^{m} P(c_k|a_i,b)\lambda(c_k)$$

   where n is the number of values that b takes, and m is the number of values c takes.

6. **Theorem:** Let $(\mathbb{G},P)$ be a Bayesian network that is singly-connected, where $\mathbb{G} = (V,E)$, and a be a set of values of a subset $A \subset V$.

   (a) λ **messages**: For each child Y of X, $\forall x \in X$

$$\lambda_Y(x) = \sum_{y} \left[\sum_{w_1,\dots,w_k} \left(P(y|x,w_i,\dots,w_k) \prod_{i=1}^{k} \pi_Y(w_i) \right) \right] \lambda(y)$$

   where $\{W_i\}_{i=1}^{k}$ are the other parents of Y.

   (b) λ **values**:

      i. If $X \in A$ and X is instantiated to $\hat{x}$

$$\lambda(\hat{x}) = 1, \text{ for } x = \hat{x}$$
$$\lambda(x) = 0, \text{ for } x \neq \hat{x}$$

ii. if $X \notin A$ and X is a leaf, $\forall x \in X$,

$$\lambda(x) = 1$$

iii. If $X \notin A$ and X is not a leaf, $\forall x \in X$

$$\lambda(x) = \prod_{U \in CH_X} \lambda_U(x),$$

where CH_X denotes the set of the children of X.

(c) π **messages**: If Z is the parent of X, then $\forall z \in Z$

$$\pi_X(z) = \pi(z) \prod_{U \in CH_Z - \{X\}} \lambda_U(z)$$

(d) π **values**:

i. If $X \in A$ and X is instantiated to $\hat{x}$:

$$\pi(\hat{x}) = 1, \text{ for } x = \hat{x}$$
$$\pi(x) = 0, \text{ for } x \neq \hat{x}$$

ii. If $X \notin A$ and X is the root, $\forall x \in X$

$$\pi(X) = P(x)$$

iii. If $X \notin A$, X is not the root and $\{Z_i\}_{i=1}^{j}$ are the parents of X, $\forall x\ inX$

$$\pi(x) = \sum_{z_1,\ldots,z_j} \left(P(x|z_1,\ldots,z_j) \prod_{i=1}^{j} \pi_X(z_i) \right)$$

(e) Given the definitions, for each variable X, we have for all values of x,

$$P(x|a) = \alpha \lambda(x) \pi(x)$$

7. **The Operating Equations for Probability Propagation**

(a) The λ Message

$$\lambda_C(a_i) = \sum_{j=1}^{m} \pi_C(b_j) \sum_{k=1}^{n} P(c_k|a_i \cap b_j) \lambda(c_k)$$

$$\boldsymbol{\lambda_C(A) = \lambda(C) P(C|A)}$$
$$\lambda_C(a_i) = \sum_{j=1}^{m} \pi_C(b_j) \lambda_C(a_i \cap b_j)$$

(b) The π Message: If C is a child of A, the π message from A to C is:

$$\pi_C(a_i) = \begin{cases} 1 & \text{if } A \text{ is instantiated for } a_i \\ 0 & \text{if } A \text{ is instantiated but not for } a_i \\ P'(a_i)/\lambda_C(a_i) & \text{if } A \text{ is not instantiated} \end{cases}$$

(c) The λ Evidence: If C is a node with n children $D_1,\ldots,D_n$, then the λ evidence for C is

$$\lambda(c_k) = \begin{cases} 1 & \text{if } C \text{ is instantiated for } c_k \\ 0 & \text{if } C \text{ is instantiated but not for } c_k \\ \prod_i \lambda_{D_i}(c_k) & \text{if } C \text{ is not instantiated} \end{cases}$$

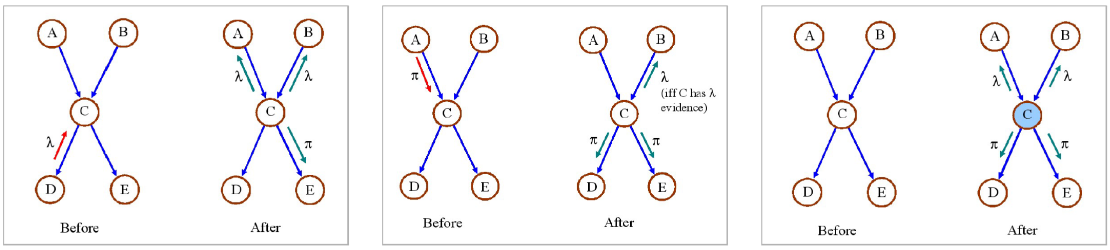
(d) The π Evidence: If C is a child of two parents A and B, the pi evidence for C is:

$$\pi(c_k) = \sum_{i=1}^{l}\sum_{j=1}^{m} P(c_k|a_i \cap b_j)\pi_C(a_i)\pi_C(b_i)$$

$$\boldsymbol{\pi(C) = P(C|A)\pi_C(A)}$$

(e) The Posterior Probabilities: If C is a variable, the posterior probability of C based on the evidence received is

$$P'(c_k) = \alpha\lambda(c_k)\pi(c_k)$$



(a) Upward Propagation (b) Downward propagation (c) Instantiation

Figure 3: Probability propagation

8. **Message Passing**

   (a) **Initialization:** For all the root nodes, the π values are set to the prior probabilities and propagate the π messages using downward propagation

   (b) **Upward Propagation (only for uninstantiated nodes):** A node C receives λ from a child.

      - Compute the new $\lambda(C)$ and $P'(C)$. Then, post a λ message to all its parents and post a π message to all C's other children.

(c) **Downward Propagation:** If a variable C receives a π message from one parent.
   - If C is not instantiated: Compute $\pi(C)$ and $P'(C)$ and post π message to each child.
   - If there is evidence in C: Post λ message to the other parents

(d) **Instantiation:** If C is instantiated for state c_k,
   - $P'(c_k) = 1$ and $P'(c_j) = 0 \forall j \neq k$. Compute $\lambda(C)$
   - Post λ and π message to each parent and each child of C respectively.

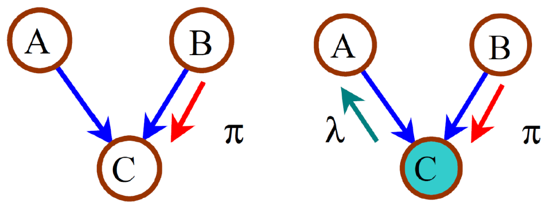


Figure 4: Converging Connections

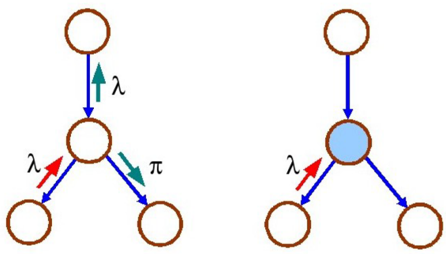


Figure 5: Diverging Connection

9. **Blocked Path (very important to understand this):**
   - For a diverging path: when the node is instantiated, it will block the passing message between other nodes.
   - For a converging path: it is blocked when there is no λ evidence on the child node, but unblocked when there is λ evidence or when the node is instantiated.

$$\lambda_C(a_i) = \sum_{j=1}^{m} \pi_C(b_j) \sum_{k=1}^{n} P(c_k | a_i \cap b_j) \lambda(c_k)$$

$$\lambda(c_k) = 1, \forall c_k \Rightarrow \lambda_C(a_i) = \sum_{j=1}^{m} \pi_C(b_j)$$

5 Building Networks from Data

1. **Measures of dependencies**

   - $L1$ **dependency measure**:

   $$Dep(A,B) = \sum_{A\times B} \left|P(a_i \cap b_j) - P(a_i)P(b_j)\right|$$
   $$Dep(A,B) = \sum_{A\times B} P(a_i \cap b_j) \times \left|P(a_i \cap b_j) - P(a_i)P(b_j)\right|$$

   - $L2$ **dependency measure**:

   $$Dep(A,B) = \sum_{A\times B} \left(P(a_i \cap b_j) - P(a_i)P(b_j)\right)^2$$
   $$Dep(A,B) = \sum_{A\times B} P(a_i \cap b_j) \times \left(P(a_i \cap b_j) - P(a_i)P(b_j)\right)^2$$

     As the probabilities becomes small they contribute less to the dependency, and this effect is acceptable since we have little information on rare events. The weighted version of the $L1$ and $L2$ further reduces the dependency for low probability values.

   - **Kullback-Leibler Measure (mutual entropy)**
     - It is zero when two variables are completely independent
     - It is positive and increasing with dependency when applied to probability distributions
     - It is independent of the actual value of probability
     - Can be computed as follows

   $$Dep(A,B) = \sum_{A\times B} P(a_i \cap b_j) \log_2 \left[\frac{P(a_i \cap b_j)}{P(a_i)P(b_j)}\right]$$

   - **Correlation**
     - Measures only linear dependency whereas mutual entropy can characterise higher order dependencies more accurately.
     - Can be computed as follows:

   $$C(A,B) = \frac{\Sigma_{AB}}{\sqrt{\sigma_A \sigma_B}}$$
   $$\Sigma_{AB} = \frac{1}{N-1} \sum_{i=1}^{N} (a_i - \bar{a}_i)(b_i - \bar{b}_i)$$
   $$\sigma_A = \frac{1}{N-1} \sum_{i=1}^{N} (a_i - \bar{a}_i)^2$$

6 Cause and Independence

(d-separation) Let $\mathbb{G} = (V, E)$ be a DAG, $A \subseteq V$ and X and Y be distinct nodes in $V - A$. X and Y are d-separated by A in $\mathbb{G}$ if every chain between X and Y is blocked by A.

7 Model Accuracy

8 Approximate Inference

9 Exact Inference

10 Probability Propagation: Join Trees