[thm]Note

Direct Fidelity Estimation

Ahmed Aman

August 2025

What is DFE and Why?

- DFE is a new technique to estimate if an arbitrary pure quantum state is close to a desired ideal state.
- It's crucial for verifying the quality of state preparation in quantum computation.
- This estimation uses a few "constant" numbers of Pauli measurements, selected randomly based on an importance-weighted rule.
- DFE is faster and requires fewer resources than traditional Quantum State Tomography (QST).
- It allows for the validation of a quantum state's closeness to a desired state without needing to reconstruct the full state itself.
- It can be applied to a wider range of states than the Entanglement Witness method.

Setting of the Protocol

- **Hilbert Space** $\mathcal{H}$ with $dim(\mathcal{H}) = d = 2^n$.

Assumptions

1. The ideal state ρ is pure.
2. We can measure n-qubit Pauli observables, which are tensor products of single-qubit Pauli operations.
3. There are no assumptions of additional structure or symmetry on ρ.

Fidelity Estimation

The fidelity between the ideal state ρ and the actual state σ is given by:

$$F(\rho, \sigma) = \mathrm{tr}([(\rho^{1/2}\sigma\rho^{1/2})^{1/2}]) = \mathrm{tr}(\rho\sigma) \tag{1}$$

This can be expressed in terms of Pauli expectation values.

- Let W_k $(k = 1, \ldots, d^2)$ be all possible Pauli operators.
- We define the characteristic function:

$$\mathcal{X}_\rho(k) = \mathrm{tr}(\rho W_k / d^{1/2}) \tag{2}$$

- The fidelity can then be written as:

$$F(\rho, \sigma) = \sum_k \mathcal{X}_\rho(k)\mathcal{X}_\sigma(k) \tag{3}$$

Estimation is performed by randomly selecting Pauli operators according to an importance-weighted rule, with the probability of selecting W_k given by:

$$p(k) = (\mathcal{X}_\rho(k))^2 \tag{4}$$

DFE Protocol Summarized

1) Generate ℓ independent and identically distributed random samples $k_1, ..., k_\ell$ from the importance sampling distribution $p(k_i)$, where $\ell = \lceil 1/(\epsilon^2 \delta) \rceil$.
2) For each k_i, measure the observable W_{k_i} a total of m_i times, where m_i is defined as:
$$m_i = \left\lceil \frac{2}{(\mathcal{X}_\rho(k_i))^2 d\ell\epsilon^2} \ln \frac{2}{\delta} \right\rceil$$
3) For each measurement i, use the outcomes $A_{ij} \in \{-1, 1\}$ to calculate the estimator $\tilde{X}_i$:
$$\tilde{X}_i = \frac{1}{m_i \sqrt{d} \mathcal{X}_\rho(k_i)} \sum_{j=1}^{m_i} A_{ij}$$

DFE Protocol Summarized

4) Calculate the final fidelity estimate $\tilde{Y}$:

$$\tilde{Y} = \frac{1}{\ell} \sum_{i=1}^{\ell} \tilde{X}_i$$

5) Return $\tilde{Y}$ as the fidelity estimator. This provides an unbiased estimate of $F(\rho, \sigma)$ with an accuracy of 2ϵ and a confidence level of $1 - 2\delta$.

Calculation of Required Sample Counts

This section details the derivation for the number of measurement settings, ℓ, and the number of measurements per setting, m_i.

- The calculations rely on two fundamental concentration inequalities: Chebyshev's inequality for ℓ and Hoeffding's inequality for m_i.
- The total number of copies of the state σ required is $m = \sum_{i=1}^{\ell} m_i$.
- While this depends on the random choices of k_i, we can bound its expectation value.

The expected number of total copies can be shown to be:

$$(m) \leq 1 + \frac{2d}{\epsilon^2 \delta} + \frac{2d}{\epsilon^2} \log(2/\delta) \tag{5}$$

Numerics

- The goal was to recreate the graph from the DFE paper.
- Parameters used: $\epsilon = 0.05$ and $\delta = 0.05$, which results in $\ell = 8000$.
- The number of qubits n is 4.

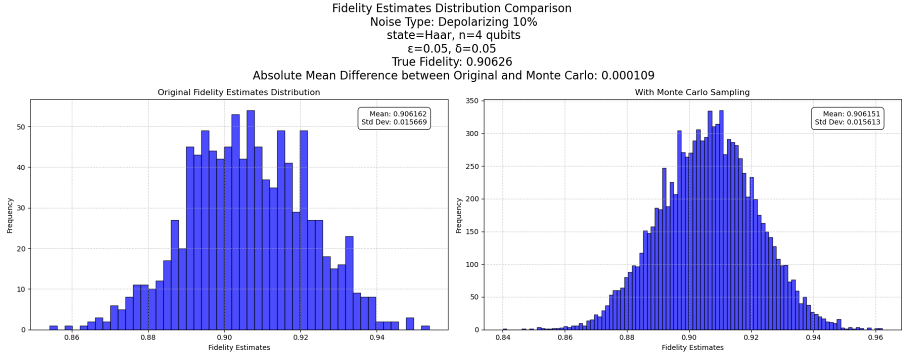
States Investigated

- Haar Random States
- W States
- GHZ States

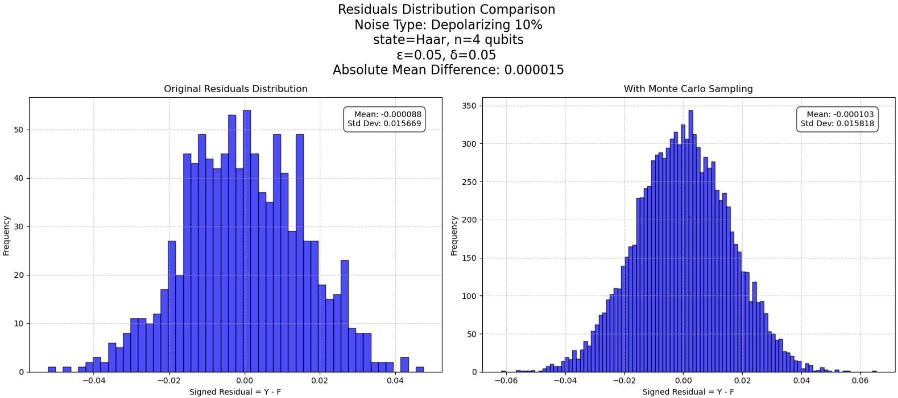
Errors Applied (10%)

- Depolarizing Noise
- Phase Damping Noise
- Amplitude Damping Noise
- General Amplitude Damping

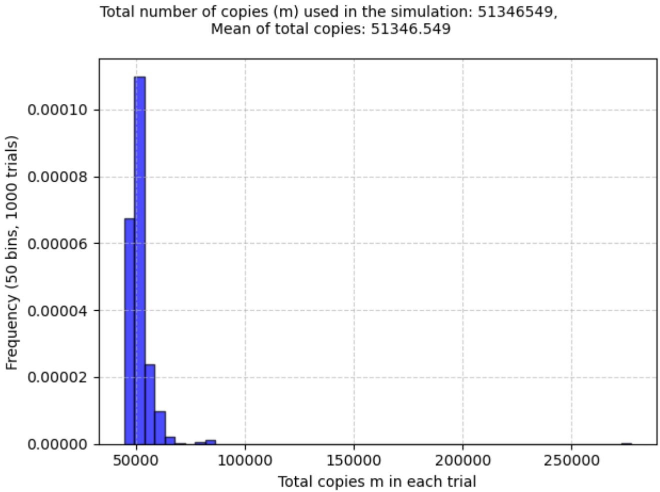
Plots


Fidelity Estimates Distribution Comparison
Noise Type: Depolarizing 10%
state=Haar, n=4 qubits
ε=0.05, δ=0.05
True Fidelity: 0.90626
Absolute Mean Difference between Original and Monte Carlo: 0.000109
Original Fidelity Estimates Distribution
Mean: 0.906162
Std Dev: 0.015669
Frequency
Fidelity Estimates
With Monte Carlo Sampling
Mean: 0.906151
Std Dev: 0.015613
Frequency
Fidelity Estimates

Plots

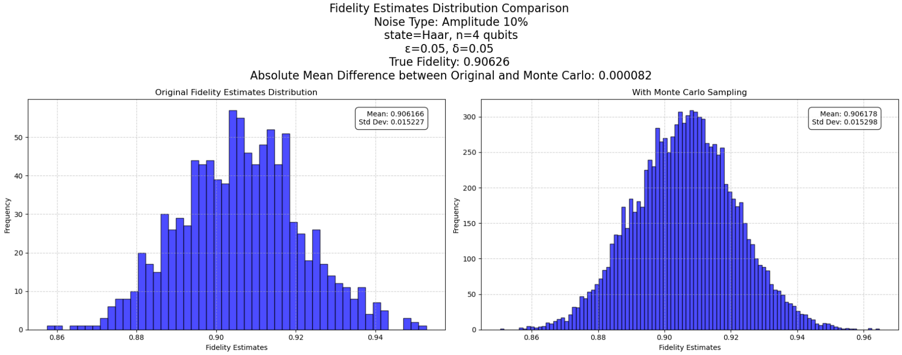

Residuals Distribution Comparison
Noise Type: Depolarizing 10%
state=Haar, n=4 qubits
ε=0.05, δ=0.05
Absolute Mean Difference: 0.000015
Original Residuals Distribution
Mean: -0.000088
Std Dev: 0.015669
Frequency
Signed Residual = Y - F
With Monte Carlo Sampling
Mean: -0.000103
Std Dev: 0.015818
Frequency
Signed Residual = Y - F

Plots

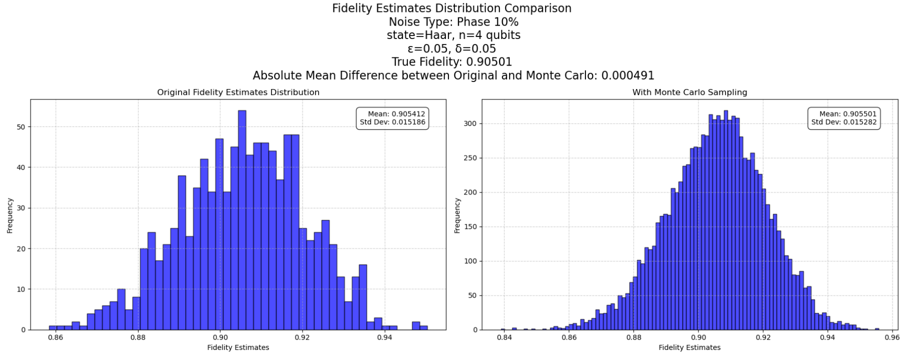


Total number of copies (m) used in the simulation: 51346549,
Mean of total copies: 51346.549
0.00010
0.00008
0.00006
0.00004
0.00002
0.00000
Frequency (50 bins, 1000 trials)
50000
100000
150000
200000
250000
Total copies m in each trial

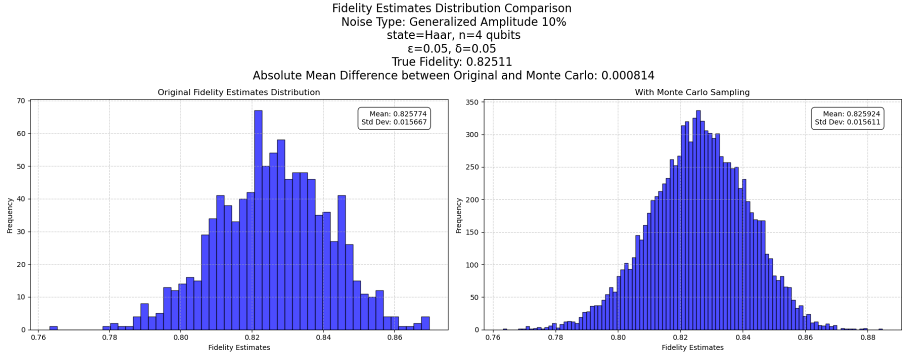
Plots


Fidelity Estimates Distribution Comparison
Noise Type: Amplitude 10%
state=Haar, n=4 qubits
ε=0.05, δ=0.05
True Fidelity: 0.90626
Absolute Mean Difference between Original and Monte Carlo: 0.000082
Original Fidelity Estimates Distribution
Mean: 0.906166
Std Dev: 0.015227
Frequency
Fidelity Estimates
0
10
20
30
40
50
0.86
0.88
0.90
0.92
0.94
With Monte Carlo Sampling
Mean: 0.906178
Std Dev: 0.015298
Frequency
Fidelity Estimates
0
50
100
150
200
250
300
0.86
0.88
0.90
0.92
0.94
0.96

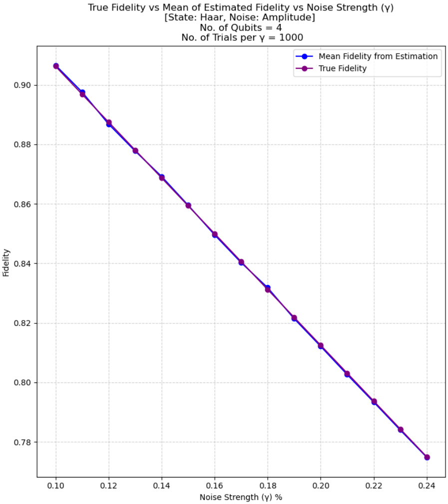
Plots


Fidelity Estimates Distribution Comparison
Noise Type: Phase 10%
state=Haar, n=4 qubits
ε=0.05, δ=0.05
True Fidelity: 0.90501
Absolute Mean Difference between Original and Monte Carlo: 0.000491
Original Fidelity Estimates Distribution
Mean: 0.905412
Std Dev: 0.015186
Frequency
Fidelity Estimates
0.86
0.88
0.90
0.92
0.94
0
10
20
30
40
50
With Monte Carlo Sampling
Mean: 0.905501
Std Dev: 0.015282
Frequency
Fidelity Estimates
0.84
0.86
0.88
0.90
0.92
0.94
0.96
0
50
100
150
200
250
300

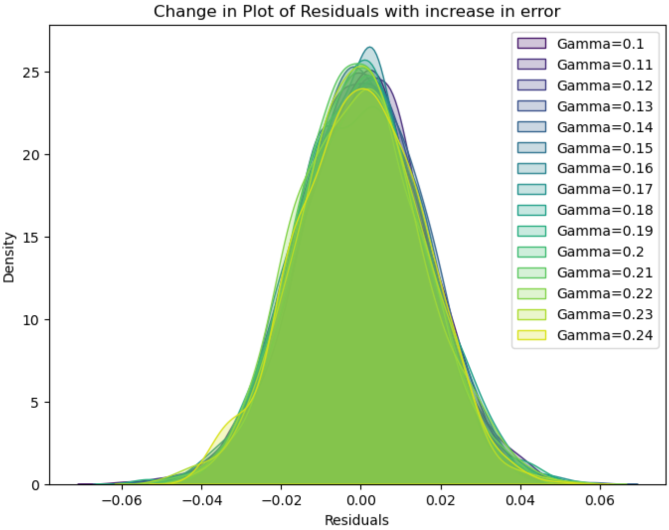
Plots


Fidelity Estimates Distribution Comparison
Noise Type: Generalized Amplitude 10%
state=Haar, n=4 qubits
ε=0.05, δ=0.05
True Fidelity: 0.82511
Absolute Mean Difference between Original and Monte Carlo: 0.000814
Original Fidelity Estimates Distribution
Mean: 0.825774
Std Dev: 0.015667
Frequency
Fidelity Estimates
0.76
0.78
0.80
0.82
0.84
0.86
0
10
20
30
40
50
60
70
With Monte Carlo Sampling
Mean: 0.825924
Std Dev: 0.015611
Frequency
Fidelity Estimates
0.76
0.78
0.80
0.82
0.84
0.86
0.88
0
50
100
150
200
250
300
350

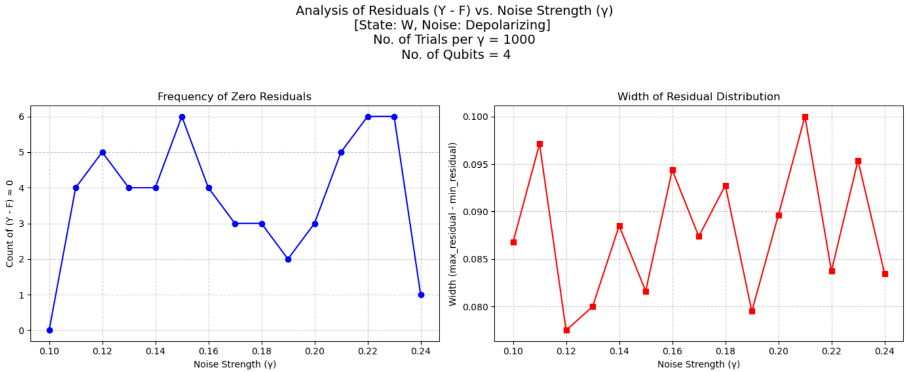
Plots


True Fidelity vs Mean of Estimated Fidelity vs Noise Strength (γ)
[State: Haar, Noise: Amplitude]
No. of Qubits = 4
No. of Trials per γ = 1000
Mean Fidelity from Estimation
True Fidelity
Fidelity
0.90
0.88
0.86
0.84
0.82
0.80
0.78
0.10
0.12
0.14
0.16
0.18
0.20
0.22
0.24
Noise Strength (γ) %

Plots

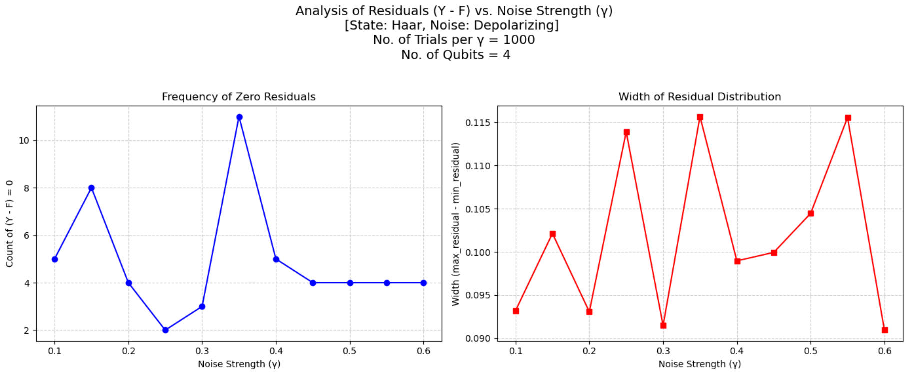

Change in Plot of Residuals with increase in error
Density
Residuals
0
5
10
15
20
25
−0.06
−0.04
−0.02
0.00
0.02
0.04
0.06
Gamma=0.1
Gamma=0.11
Gamma=0.12
Gamma=0.13
Gamma=0.14
Gamma=0.15
Gamma=0.16
Gamma=0.17
Gamma=0.18
Gamma=0.19
Gamma=0.2
Gamma=0.21
Gamma=0.22
Gamma=0.23
Gamma=0.24

Plots

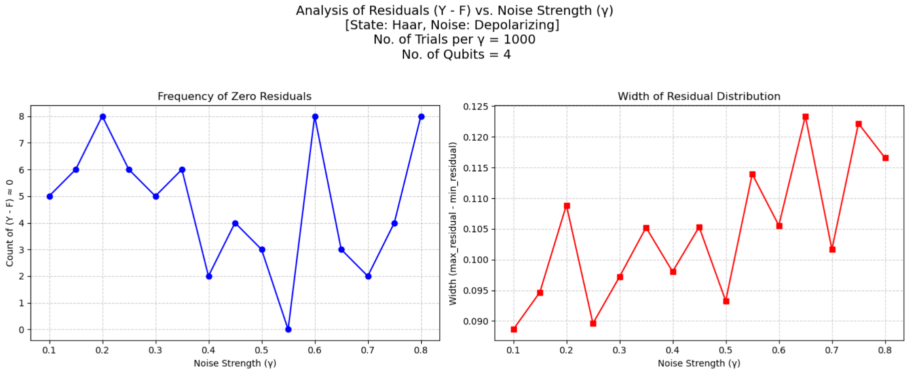


Analysis of Residuals (Y - F) vs. Noise Strength (γ)
[State: W, Noise: Depolarizing]
No. of Trials per γ = 1000
No. of Qubits = 4
Frequency of Zero Residuals
Count of (Y - F) ≈ 0
Noise Strength (γ)
0.10
0.12
0.14
0.16
0.18
0.20
0.22
0.24
0
1
2
3
4
5
6
Width of Residual Distribution
Width (max_residual - min_residual)
Noise Strength (γ)
0.080
0.085
0.090
0.095
0.100

Plots


Analysis of Residuals (Y - F) vs. Noise Strength (γ)
[State: Haar, Noise: Depolarizing]
No. of Trials per γ = 1000
No. of Qubits = 4
Frequency of Zero Residuals
Count of (Y - F) ≈ 0
Noise Strength (γ)
Width of Residual Distribution
Width (max_residual - min_residual)
Noise Strength (γ)

Plots


Analysis of Residuals (Y - F) vs. Noise Strength (γ)
[State: Haar, Noise: Depolarizing]
No. of Trials per γ = 1000
No. of Qubits = 4
Frequency of Zero Residuals
Count of (Y - F) ≈ 0
Noise Strength (γ)
Width of Residual Distribution
Width (max_residual - min_residual)
Noise Strength (γ)

Other Plots and Code

- **Click here** to check out the other plots, as well as the code for the simulation.