

1. Let events A and B have nonzero probabilities, and suppose that $\mathbb{P}(A | B) > \mathbb{P}(A)$, i.e. that occurrence of B increases the likelihood of the occurrence of A . Answer each of the following questions and justify your answers rigorously.

- (a) Does occurrence of A increase the likelihood of the occurrence of B ?
- (b) Is it possible for A and B to be independent?
- (c) Is it possible for A and B to be disjoint?

2. You roll a fair six-sided die twice with faces numbered 1 through 6, where the rolls are independent of each other. Define events

- $A = \{ \text{sum of the two rolls is } 12 \}$
- $B = \{ \text{at least one roll was a } 6 \}$
- $C = \{ \text{at least one roll was a } 2 \}$

- (a) Is event A independent of event B ? Justify your answer.
- (b) Is event A independent of event C ? Justify your answer.

3. Galadriel has two six-sided dice, one with the standard faces numbered 1 through 6 and one with three faces numbered 3 and three numbered 6. Assume the dice are fair in the sense that on a single roll of either die each of the six physical faces is equally likely to come up. Galadriel selects one of the dice at random, with probability $p \in (0, 1)$ that she selects the standard die. Assume that all subsequent rolls of the selected die are mutually independent.

- (a) She rolls the die once and a 3 comes up. What is the probability that the die is the standard one?
- (b) She rolls the die again and this time a 6 comes up. Now what is the probability that the die is the standard one?
- (c) She rolls the die a third time and a 5 comes up. Now what is the probability that the die is the standard one?

4. Smaug is hiding in one of two caves with respective probabilities p_1 and p_2 , both in $(0, 1)$ and satisfying $p_1 + p_2 = 1$. Frodo sends a drone to scope things out. If Smaug is hiding in cave i and the drone visits cave i , the drone detects Smaug with probability $d_i \in (0, 1)$ and misses him with probability $1 - d_i$. If the drone visits the cave where Smaug isn't hiding, the drone doesn't spot Smaug. Suppose the drone visits cave 1 and doesn't spot Smaug. Given this information,

- (a) what is the probability that Smaug is hiding in cave 1?
- (b) what is the probability that Smaug is hiding in cave 2?
- (c) do the conditional probabilities you found in (a) and (b) sum to 1? What's an easy way to see whether this is true without calculating?

5. Suppose Ω is a sample space, \mathbb{P} is a probability law on Ω , and A_1, A_2, \dots, A_n are independent events with respect to \mathbb{P} . Show that

$$\mathbb{P}(A_1^c \cup A_2^c \cup \dots \cup A_n^c) = 1 - \mathbb{P}(A_1)\mathbb{P}(A_2) \dots \mathbb{P}(A_n).$$

6. Gandalf and Saruman play a match consisting of a series of games. The match ends when one of the players has won two more games than the other player. The results of the games are independent of each other, and in any given game Gandalf wins with probability $p \in (0, 1)$ and Saruman with probability $1 - p$. What is the probability that Gandalf wins the match? Here are some suggestions. For each integer $n > 0$, let A_n be the event that the players have tied after $2n$ games. What is the probability that Gandalf wins in exactly $2n$ games? The event $A_1 \cap A_2 \cap \dots \cap A_{n-1}$ will play a role. It might be worth figuring out $\mathbb{P}(A_1)$ and $\mathbb{P}(A_2)$ to see a pattern. Once you've found the probability that Gandalf wins in exactly $2n$ games, you sum that quantity over n to get the probability that Gandalf wins. You might find useful the geometric series

$$\sum_{n=0}^{\infty} \gamma^n = \frac{1}{1-\gamma} \quad \text{when } |\gamma| < 1 .$$