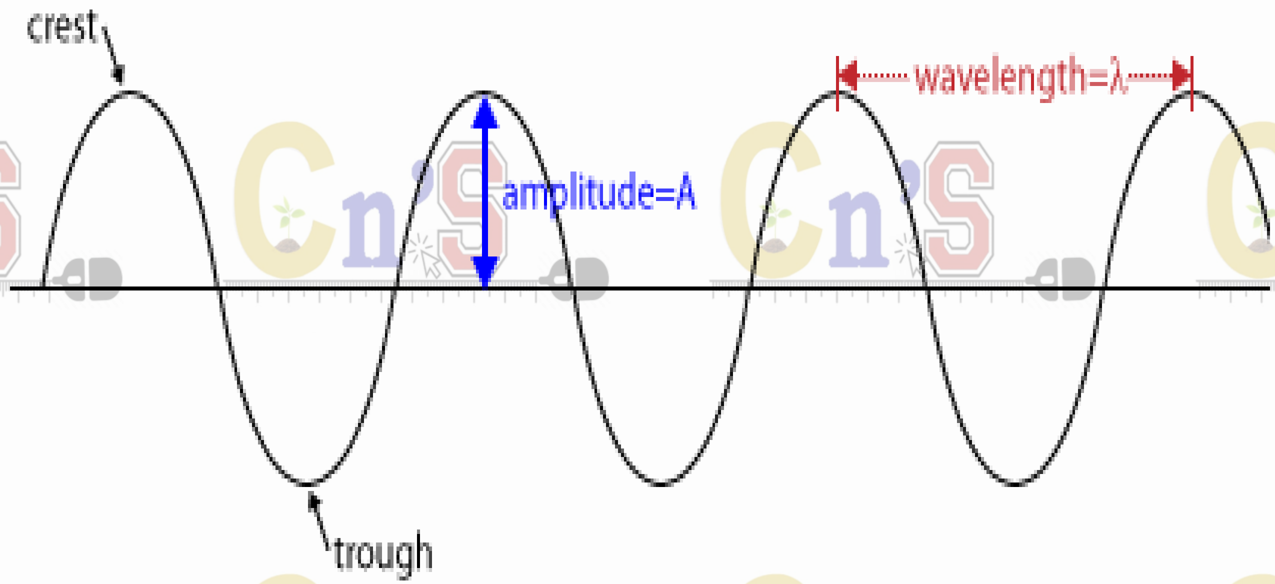
Simple Harmonic Motion



Oscillations

- **Oscillations** are the periodic varying of the position of a system about a point.
  - Examples of oscillations include:
    - A bouncing mass on a spring.
    - A pendulum.
    - The bobbing of a boat in water.
- Oscillating systems have both energy & momentum.

Restoring Force

- **Restoring Force** is any force experienced by an oscillating system that pushes it back into its original position.

- All oscillating systems have restoring forces.
- Systems oscillate because the momentum the system gains repeatedly carry it past its original position causing another restoring force.

Simple Harmonic Motion

- **Simple Harmonic Motion** is oscillatory motion of a system in which the restoring force follows Hooke's Law. "For an object attached to an elastic spring, the displacement of the object from mean position is directly proportional to the applied force".
- If an object is attached to an elastic spring of spring constant k, then the applied force F to produce the displacement x is described as: F= kx
- **Simple Harmonic Oscillators** are systems or objects that undergo simple harmonic motion.

- **“Simple Harmonic Motion”** is a type of vibratory motion in which the acceleration of the body is proportional to displacement and is directed towards its mean position. Mathematically it is described as: $a \;\alpha\; -x$ where a is the acceleration and x is the displacement of the oscillatory object from mean position. The negative sign indicates that acceleration of the object is directed towards the mean position

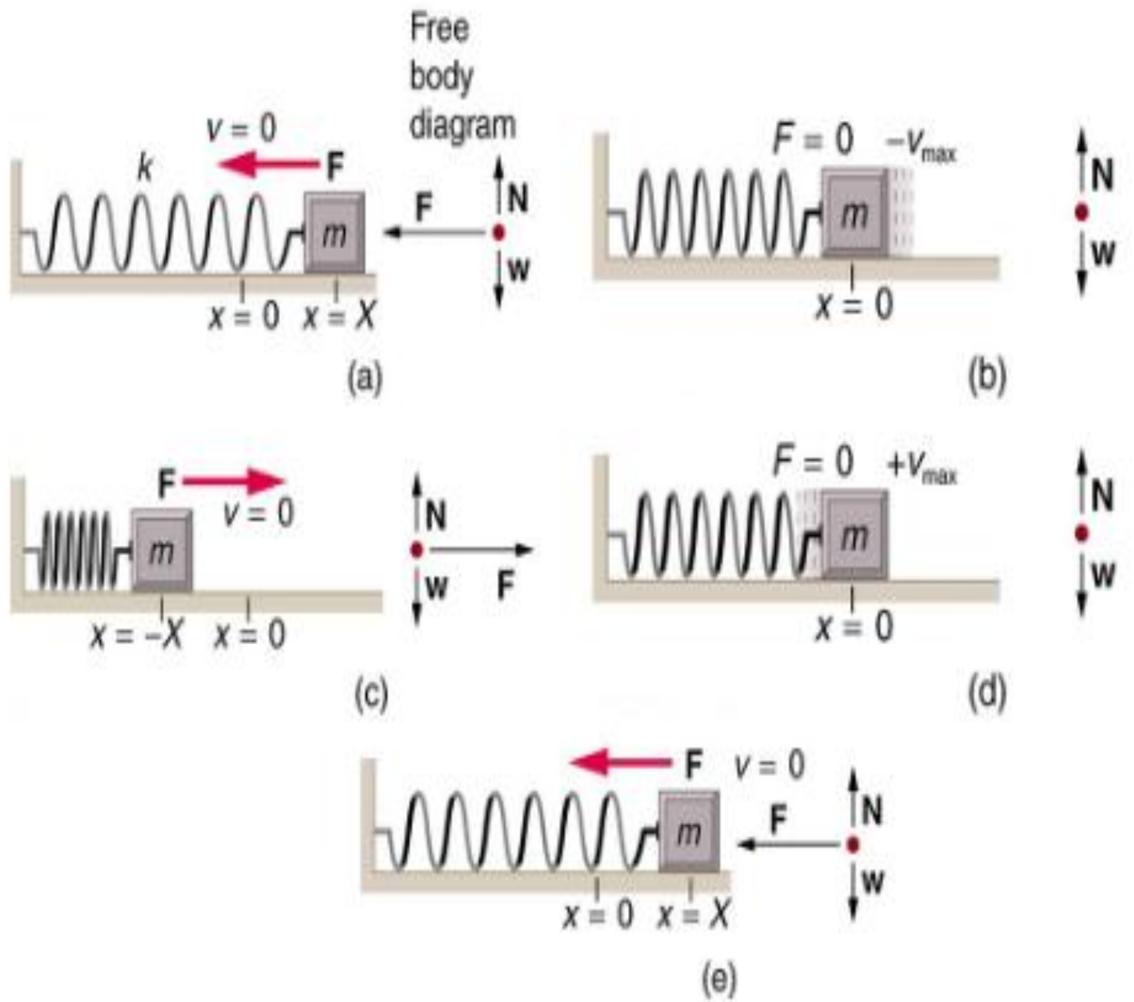
Masses on Springs

- A mass attached to a spring is an example of a simple harmonic oscillator.

- The period of oscillation for a mass on a s

$$T_s = 2\pi\sqrt{\frac{m}{k}}$$

Masses on Springs



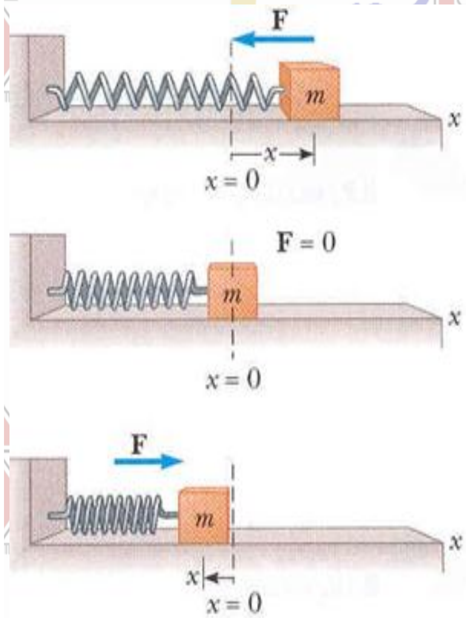
Masses on Springs

- Shock absorbers on cars keep a car from continually bouncing after it hits a bump in the road. Determine the period and frequency of a car without shock absorbers if the car has a mass of 950-kg and the springs of the car have a spring constant of 6.47 x 104 N/m.

Simple harmonic motion occurs when the net force along the direction of motion is a Hooke's law type of force. That is, when the net force is proportional to the displacement and in the opposite direction. (For example, a mass attached to a spring)

Let we define few terms:

The **amplitude, A,** is the maximum distance traveled by an object away from its equilibrium position. In the absence of friction, an object attached to a spring continues in simple harmonic motion and reaches a maximum displacement equal to the amplitude on each side of the equilibrium position during each cycle.

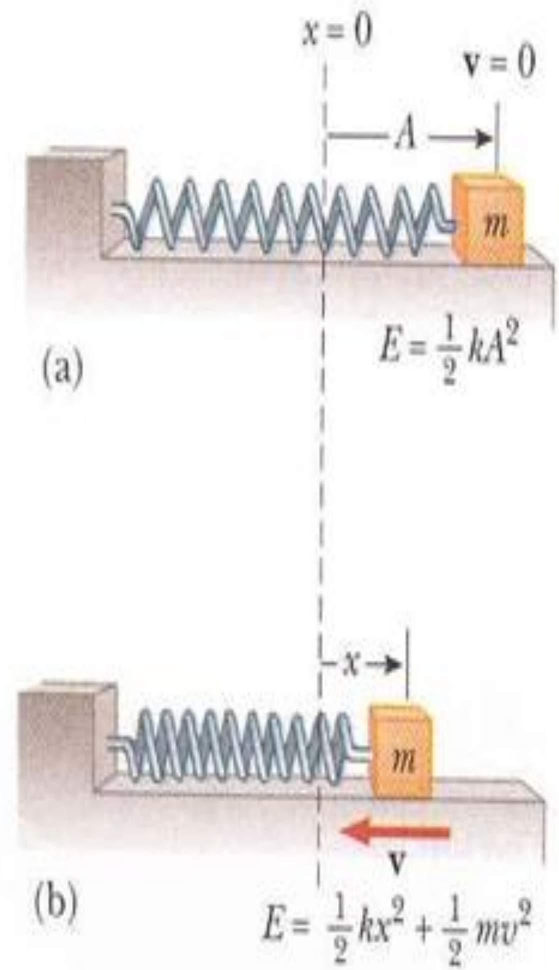


The **period, T,** is the time it takes the object to execute one complete cycle of motion.

The **frequency, f,** is the number of cycles or vibrations per unit of time.

We have worked with kinetic energy and gravitational potential energy. Here we will consider elastic potential energy.

An object has potential energy by virtue of its shape or position. As we learned an object of mass m at height h above the ground has gravitational potential energy equal to mgh. This means that the object can do work after it is released. Likewise, a compressed spring has potential energy by virtue of its shape. In this case, the compressed spring can move an object and thus do work on it.



The energy stored in a stretched or compressed spring or other elastic material is called **elastic potential energy,** $E_{p,e}$, given by

$$E_{p,e} = \frac{1}{2}kx^2$$

where k is a positive konstant, and x displacement from its unstretched position.

Note that energy is stored in an elastic material only when it is either stretched or compressed.

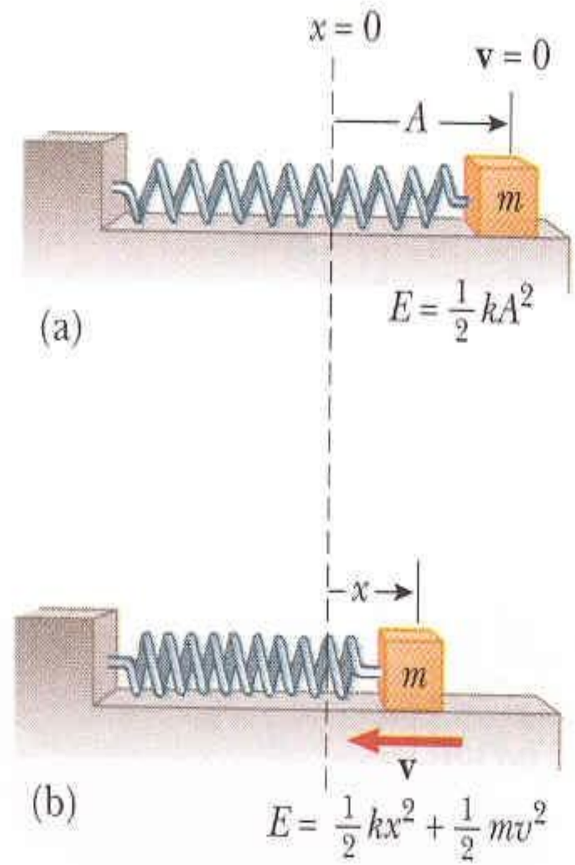
Conservation of energy provides a simple method of deriving an expression for the velocity of a mass attached to a spring undergoing periodic motion as a function of position. Let the mass be initially at its maximal extension, and then is releaced from rest. As the mass moves toward the origin to some new position x part of potential energy is transformed into kinetic energy. We can equate energies at initial and some final position:

$$\frac{1}{2}kA^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

Solving, we get speed as a function of position

$$v = \sqrt{\frac{k}{m}(A^2 - x^2)}$$

This expression shows us that the speed is a maximum at $x = 0$ and zero at the extreme positions $x = A$.



The period, T, represents the time required for one complete trip forth and back (we also say the complete cycle), and for the mass attached to the spring is

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Recall that the frequency, f, is the number of cycle per unit of time. The symmetry in the units of period and frequency should lead you to see that the period and frequency must be related inversely as

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

The units of frequency are s^{-1} or hertz (Hz).
We define angular frequency, ω, as

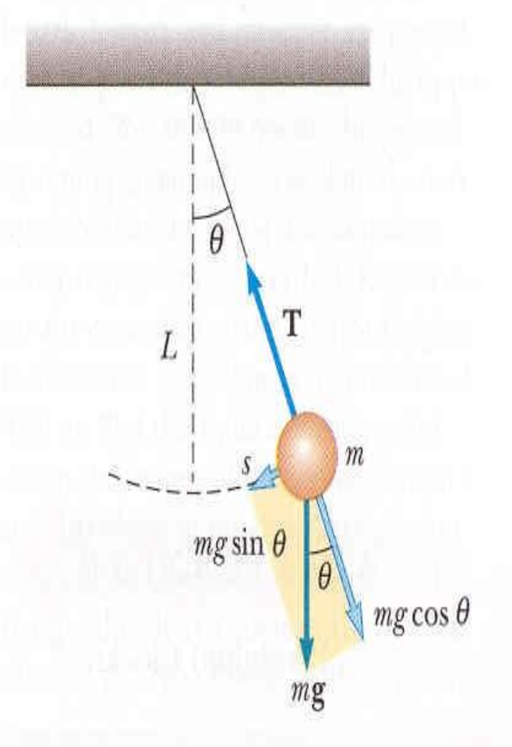
$$\omega = 2\pi f = \sqrt{\frac{k}{m}}$$

The Pendulum

- **A simple pendulum** is a mass suspended from a fixed point by a light string or wire.
  - Pendulums undergo simple harmonic motion.
- The period of a simple pendulum depends only on string length (L) and acceleration (g).

$$T_p = 2\pi\sqrt{\frac{\ell}{g}}$$

A simple pendulum is another mechanical system that exhibits periodic motion. It consists of a small bob of mass m suspended by a light string of length L fixed at its upper end.



The net force on the mass is proportional to $\sin\theta$ rather than to θ

$$F_t = -mg\sin\theta$$

Thus, in general, the motion of a pendulum is not simple harmonic. However, for small angles, less than about 15 degrees, $\sin\theta$ and θ are approximately equal. Therefore, for the motion with small angles, the net force can be written as

$$F_t = -mg\theta$$

Recalling the period of a mass-spring system, we see that the period of a simple pendulum is

$$T = 2\pi\sqrt{\frac{L}{g}}$$

(Application: pendulum clocks, prospecting for oil)

The Pendulum

- What is the length of a pendulum that has a period of 0.700-s?

- How long does it take a child on a swing to complete one swing if her center of gravity is 3.500-m below the pivot?

- What is the frequency of the swing?

Energy in SHM

- Energy in Simple Harmonic Motion is always conserved.

- For **masses on springs**, the sum of the elastic potential energy and kinetic energy is always constant.

$$PE_{spring} + KE = Constant$$

Energy in SHM

- For **pendulums**, the sum of the gravitational potential energy and kinetic energy is always constant.

$$PE_{gravitational} + KE = \text{Constant}$$

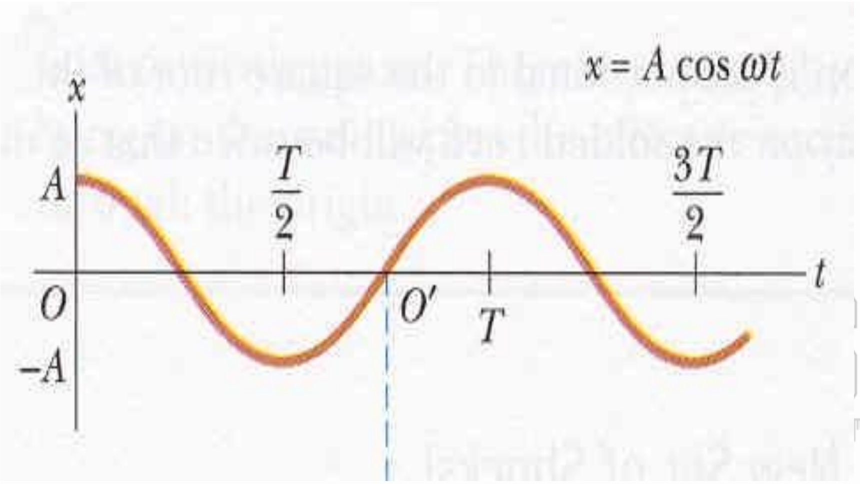
We can better understand and visualize many aspects of simple harmonic motion along s straight line by looking at their relationnships to uniform circular motion.

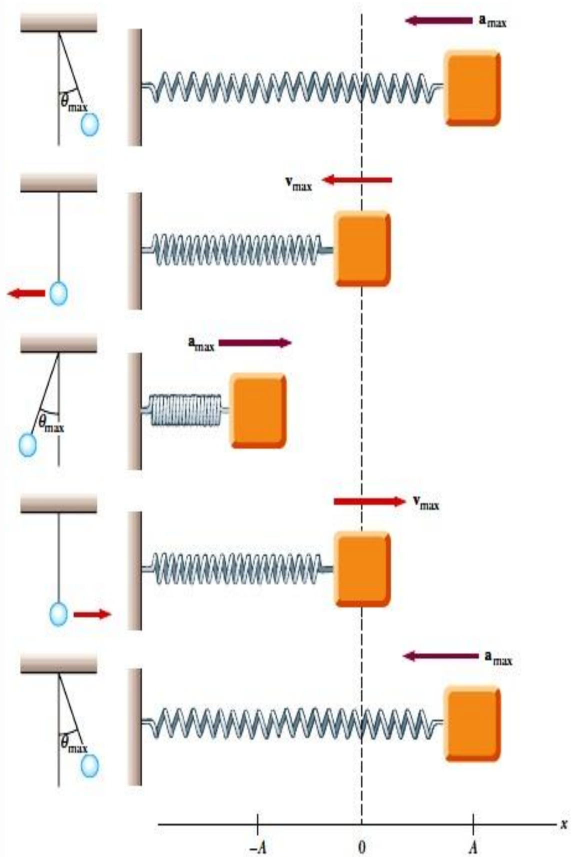
We find that as the turntable rotates with constant angular speed, the shadow of the ball moves back and forth with simple harmonic motion. The x coordinate of the ball is

$$x = A\cos(\omega t)$$

This equation represents the position of an object moving with simple harmonic motion as a function of time.

The curve should be familiar to you from trigonometry.





t	x	v	a	K	U
0	A	0	$-\omega^2 A$	0	$\frac{1}{2}kA^2$
$T/4$	0	$-\omega A$	0	$\frac{1}{2}kA^2$	0
$T/2$	$-A$	0	$\omega^2 A$	0	$\frac{1}{2}kA^2$
$3T/4$	0	ωA	0	$\frac{1}{2}kA^2$	0
T	A	0	$-\omega^2 A$	0	$\frac{1}{2}kA^2$

Damped Harmonic Motion

- **Damped Harmonic Oscillators** experience friction or other non-conservative forces that dissipate the energy of the system.

- **Dampening** is what causes objects in simple harmonic motion to decrease the amplitude of their motions and eventually come to rest.

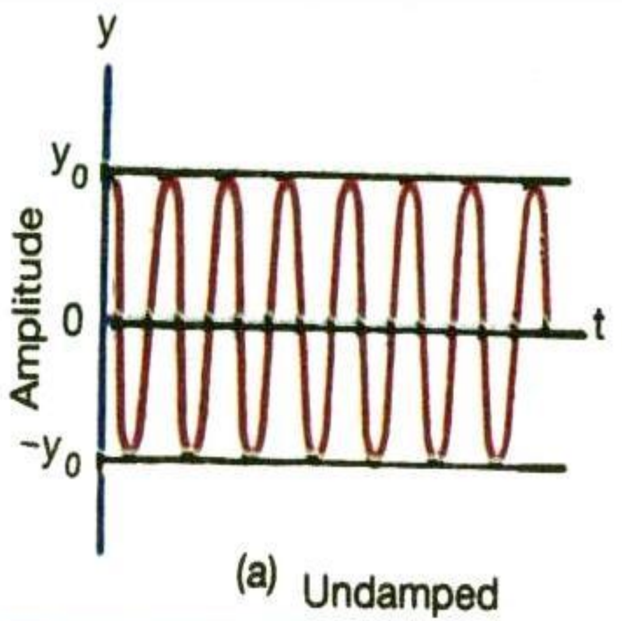
The vibrating motions we have discussed so far have taken place in ideal systems — that is, systems that oscillate indefinitely.

In many real systems, forces of friction retard the motion, which reduces the mechanical energy of the system as time passes, and the motion is said to be **damped**. Damped motion varies depending on the friction forces. The amplitude of vibration decreases in time, and we say that the system has underdamped oscillation. If the friction force is increased the mass returns to equilibrium and does not oscillate. In this case, the system is said to be critically damped. If the friction force is made grater still, the system is said to be overdamped. In this case the time required to reach equilibrium is greater than at critical dampin.

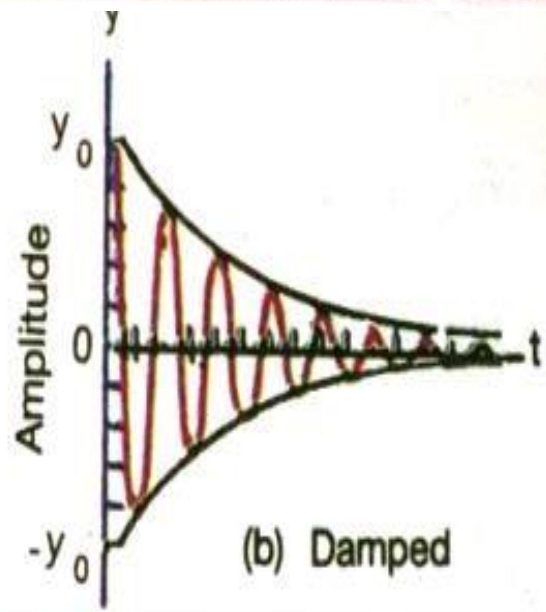
Shock absorbers in automobiles make practical application of damped motion. The shock absorbers are designed to be slightly underdamped.

DAMPING

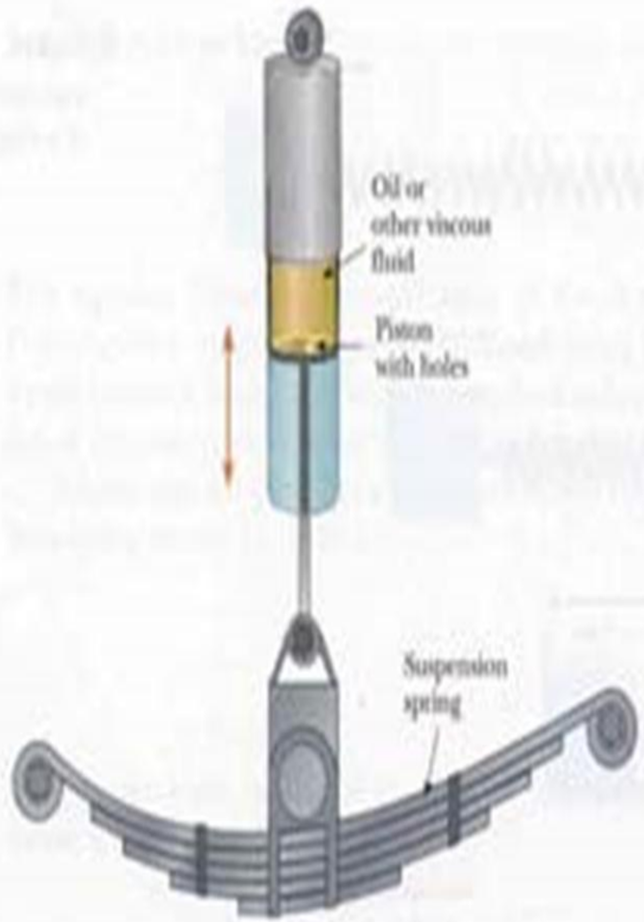
- Damping refers to the loss of energy from an oscillating system to the environment due to dissipative forces {eg, friction, viscous forces, eddy currents}
- Light Damping: The system oscillates about the equilibrium position with decreasing amplitude over a period of time.
- Critical Damping: The system does not oscillate & damping is just adequate such that the system returns to its equilibrium position in the shortest

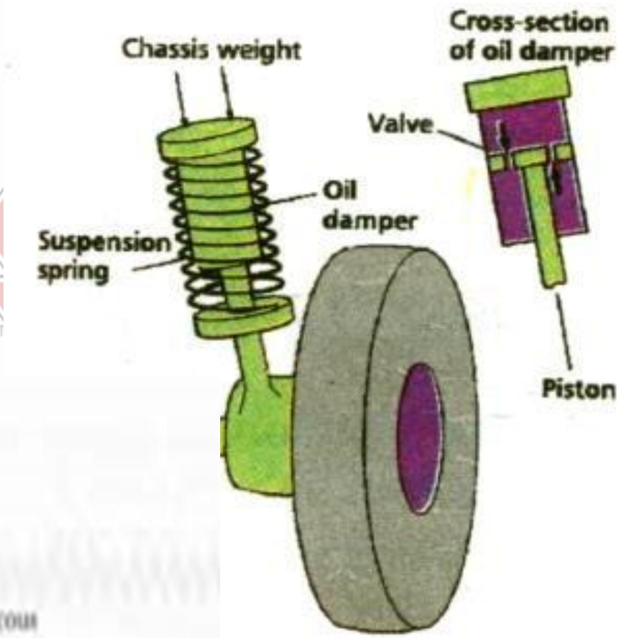


Graph between amplitude and time



Graph between amplitude and time





- Damping is the process whereby energy is dissipated from the oscillating system.

The oscillation in which the amplitude decreases steadily with time are called damped oscillations.

- In everyday life, the motion of any microscopic system is accompanied by frictional effects. For the case of SHM, the amplitude of simple harmonic oscillator gradually becomes smaller and smaller. The energy of oscillator is used up in doing work against the resistive forces An application of damped oscillations is the shock absorber of a car which provides a damping force to prevent extensive oscillations.

Damped Harmonic Motion

- Example of dampening are:
  - Shocks in a car.
  - A pendulum or mass on a spring coming to a stop.
  - A guitar string coming to a rest.

Questions with Answers

- A mass-spring system undergoes simple harmonic motion with an amplitude A. Does the total energy change if the mass is dubled but the amplitude is not changed? Are the kinetic and potential energies at a given point in its motion affected by the change in mass? Explain.

NO! Because the total energy $E = \frac{1}{2}kA^2$, changing the mass while keeping A constant has no effect on the total energy. When the mass is at displacement x from equilibrium, the potential energy is $\frac{1}{2}kx^2$, independent of mass, and the kinetic energy is $E - \frac{1}{2}kx^2$. The larger mass must move slower to have the same kinetic energy. At the particular instant in time, both kinetic and potential energy would change as the mass is increased.

- A grandfather clock depends on the period of a pendulum to keep correct time. Suppose the clock is calibrated correctly and then the temperature of the room in which it resides increases. Does the clock run slow, fast, or correctly? (A metal expands when its temperature increases.)

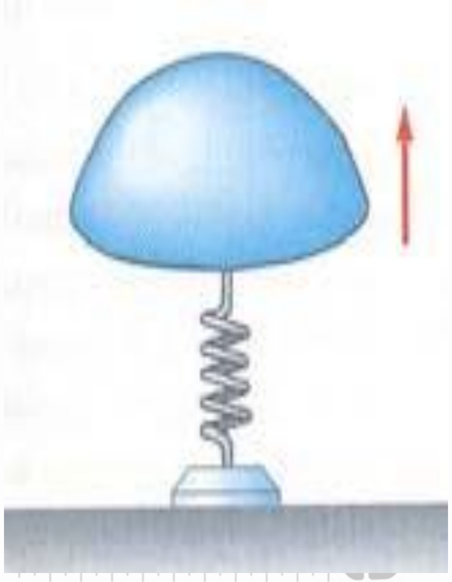
As the temperature increases, the length of the pendulum will increase. due to thermal expansion. With a longer length, the period of the pendulum will increase. Thus, it will take longer to exceute each swing, so that each second according to the clock will take longer than an actual second. Thus, the clock will run slow.

- What is the total distance traveled by a body executing simple harmonic motion in a time equal to its period if its amplitude is A?

  It traveles a distance of $4A$.

- Determine whether or not the following quantities can be in the same direction for a simple harmonic oscillator: displacement and velocity, velocity and acceleration, displacement and acceleration.

  There are times when both the displacement and the velocity are in the same direction.
  There are also times when the velocity and the acceleration are in the same direction.
  The displacement and the acceleration are always in opposite directions.



- A 0.4-kg mass is attached to a spring with a spring constant of 160 N/m so that the mass is allowed to move on a horizontal frictionless surface. The mass is released from rest when the spring is compressed 0.15 m. Find the force on the mass and its acceleration at this instant.

$$\begin{aligned} F &= -kx = -(160\ \mathrm{N/m})(0.15\ \mathrm{m}) = -24\ \mathrm{N} \\ a &= \frac{F}{m} = \frac{-24\ \mathrm{N}}{0.4\ \mathrm{kg}} = -60\ \mathrm{m/s^2} \end{aligned}$$

- A child's toy consists of a piece of plastic attached to a spring. The spring is compressed against the floor a distance of 2 cm, and the toy is released. If the toy has a mass of 100 g and rises to a maximum height of 60 cm, estimate the spring constant.

$$\begin{aligned} \frac{1}{2}kx^2 &= mgh \\ k &= \frac{2mgh}{x^2} = \frac{2(0.1\ \mathrm{m})(9.8\ \mathrm{m/s^2})(0.6\ \mathrm{m})}{(0.02\ \mathrm{m})^2} = 2.94 \cdot 10^3\ \mathrm{N/m} \end{aligned}$$

- A mass of 0.4 kg connected to a light spring with a spring constant of 19.6 N/m oscillates on a frictionless horizontal surface. If the spring is compressed 4 cm and released from rest, determine the maximum speed of the mass, the speed of the mass when the spring is compressed 1.5 cm, and the speed of the mass when the spring is streched 1.5 cm.

$$v = \sqrt{\frac{k}{m}(A^2 - x^2)}$$

$$v_{max} = \sqrt{\frac{19.6\ \mathrm{N/m}}{0.4\ \mathrm{kg}}((0.04\ \mathrm{m})^2 - 0^2)} = 0.28\ \mathrm{m/s}$$

$$v_{com.} = \sqrt{\frac{19.6\ \mathrm{N/m}}{0.4\ \mathrm{kg}}((0.04\ \mathrm{m})^2 - (-0.015\ \mathrm{m})^2)} = 0.26\ \mathrm{m/s}$$

$$v_{str.} = \sqrt{\frac{19.6\ \mathrm{N/m}}{0.4\ \mathrm{kg}}((0.04\ \mathrm{m})^2 - (+0.015\ \mathrm{m})^2)} = 0.26\ \mathrm{m/s}$$

- The motion of an object is described by the equation

$$x = (0.3\text{ m})\cos[(\frac{\pi}{3}\text{ Hz})t]$$

Find the position of the object at $t = 0$ and $t = 0.6$ s, the amplitude of the motion, the frequancy of the motion, and the period of the motion.

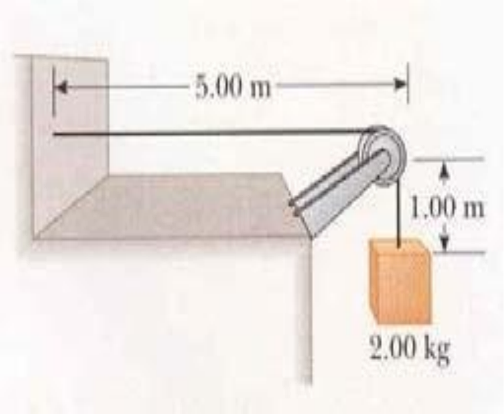
$$\begin{aligned}
x &= A\cos(\omega t) \\
x(t = 0\text{ s}) &= (0.3\text{ m})\cos[(\frac{\pi}{3}\text{ Hz})(0\text{ s})] = 0.3\text{ m} \\
x(t = 0.6\text{ s}) &= (0.3\text{ m})\cos[(\frac{\pi}{3}\text{ Hz})(0.6\text{ s})] = 0.24\text{ m} \\
A &= 0.3\text{ m} \\
\omega &= \frac{\pi}{3}\text{ Hz} \\
f &= \frac{\omega}{2\pi} = \frac{\frac{\pi}{3}\text{ Hz}}{2\pi} = \frac{1}{6}\text{ Hz} \\
T &= \frac{1}{f} = \frac{1}{\frac{1}{6}\text{ Hz}} = 6\text{ s}
\end{aligned}$$

- A simple 2-m-long pendulum oscillates in a location where g = 9.8 m/s². How many complete oscillations does it make in 5 min?

$$T = 2\pi\sqrt{\frac{L}{g}} = 2\pi\sqrt{\frac{2\text{ m}}{9.8\text{ m/s}^2}} = 2.84\text{ s}$$

$$n = \frac{t}{T} = \frac{300\text{ s}}{2.84\text{ s}} = 105.6 \qquad \text{(105 complete oscilations)}$$

- A uniform string has a mass of 0.3 kg and a length of 6 m. Tension is maintained in the spring by suspending a 2-kg block from one end. Find the speed of a pulse on this string.



$$F_T = mg = (2\text{ kg})(9.8\text{ m/s}^2) = 19.6\text{ N}$$

$$\mu = \frac{m_s}{l} = \frac{0.3\text{ kg}}{6\text{ m}} = 0.05\text{ kg/m}$$

$$v = \sqrt{\frac{F_T}{\mu}} = \sqrt{\frac{19.6\text{ N}}{0.05\text{ kg/m}}} = 19.8\text{ m/s}$$

Q # 13. Show that in SHM, the accleration is zero when the velocity is greatest and the velocity is zero when the acceleration is greatest?

Ans. The expressions of velocity and acceleration of the body executing SHM are as follow:

$$a = -\omega^2 x$$

$$v = \omega\sqrt{x_0^2 - x^2}$$

<u>At Mean Position (x = 0)</u> Acceleration of SHO: $a = -\omega^2(0) = 0$

Velocity of SHO: $v = \omega\sqrt{x_0^2 - (0)^2} = \omega x_0$

So at mean position, the acceleration of SHO is zero but velocity is greatest

<u>At Extreme Position (   )</u> Acceleration of SHO:

Velocity of SHO:

So at extreme position, the velocity of SHO is zero but acceleration is greatest **Q**

Q # 17. A block weighing 4.0 kg extends a spring by 0.16 m from its unstretched position. The block is removed and a 0.50 kg body is hung from the same spring. If the spring is now stretched