Rotational Motion

Objectives

- define angular displacement, angular velocity, and angular acceleration and expresses in SI units.
- relate rpm value and angular velocity.
- relate liner displacement to angular displacement, tangential speed to angular speed and tangential acceleration to angular acceleration.
- describes rotational motion using time period and frequency.
- writes equation of rotational motion
- explain that the moment of inertia is the measure of rotational inertia.
- expresses moments of inertia of a point mass about an axis as $I = mr2$

- expresses moment of inertia of a body about an axis as $I = \sum_i m_i r_i^2$
- demonstrates that moment of inertia depends on mass, axis of rotation and mass distribution.
- relate moment of inertia and angular acceleration to the torque acting on it $\tau = I\alpha$
- predicts the motion of a rotating body by determining the torque acting on it.
- expresses angular momentum as the product of moments of inertia and angular velocity.
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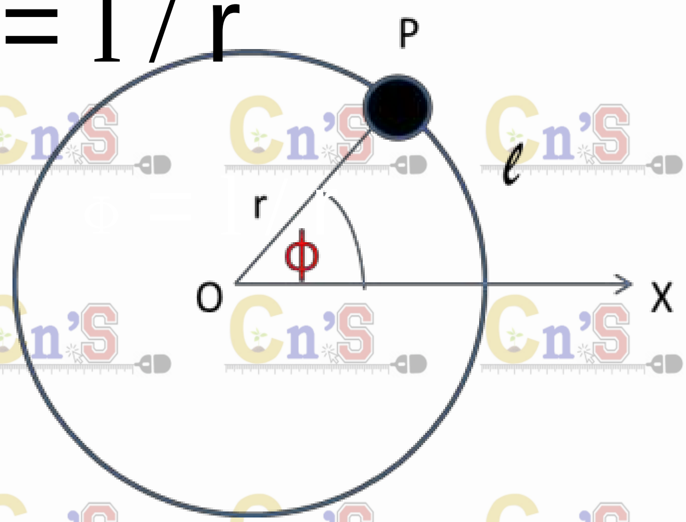
- gives examples related to principle of conservation of angular momentum.

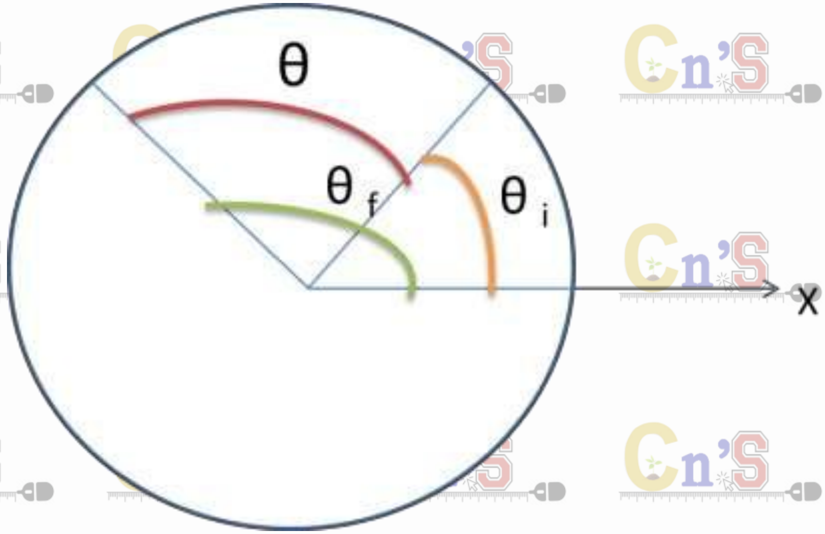
Definitions

1. Angular position (Φ) - The angular position of a particle is the angle Φ made between the line connecting the particle to the original and the positive direction of the x-axis, measured in a counterclockwise direction

$$\Phi = l / r$$

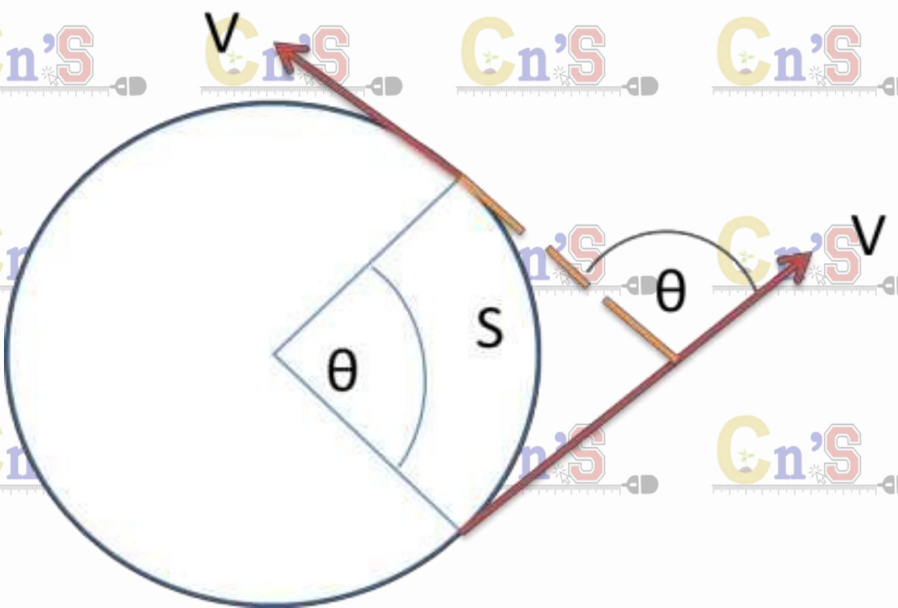


2. Angular displacement (θ) - The radian value of the angle displaced by an object on the center of its path in circular motion from the initial position to the final position is called the angular displacement.



$$\theta = \theta_f - \theta_i$$

3. Angular Velocity (ω)- Angular velocity of an object in circular motion is the rate of change of angular displacement



$$\omega = \theta / t$$

Unit - $rads^{-1}$

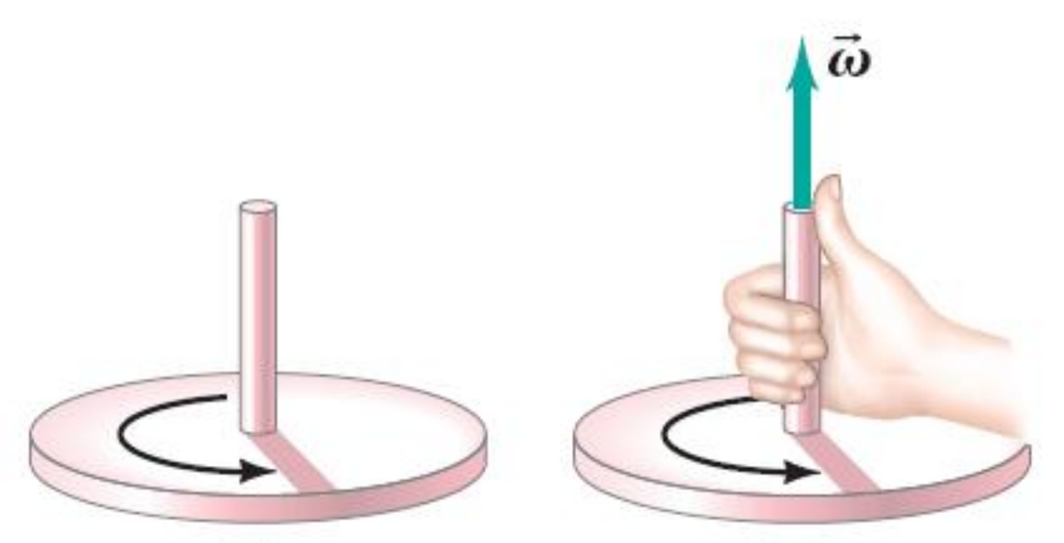
❖ Vector direction by Right hand rule

Vector characteristics

Magnitude of angular velocity is the angular speed.
<u>But angular velocity is a vector</u>.

Use right hand rule to obtain direction of angular velocity.

Curl fingers in direction of rotation, and thumb gives direction of angular velocity! (go back and check). Vector direction is perpendicular to screen.



Conceptual Quiz:

You look at a bicycle as it moves from your left to your right. The angular velocity of the rear wheel is directed

A) up

B) to the left

C) to the right

D) towards you

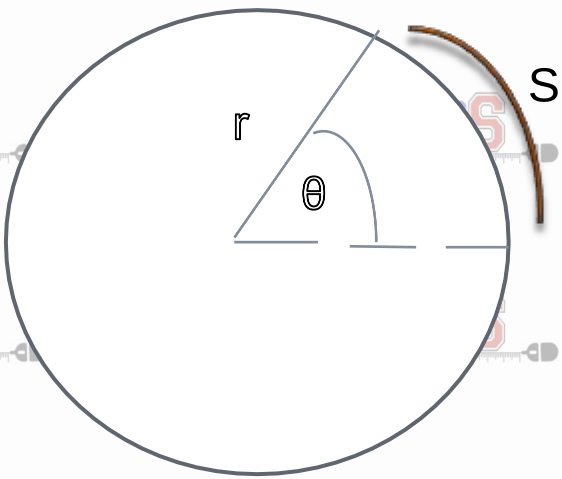
E) away from you

Answer: E – away from you

Use the right hand rule. The angular velocity is into the screen and away from you.

Relationship between physical quantities measured in angular motion and that in linear motion

1. Linear displacement- Angular displacement



$S = (2\pi r / 2\pi) \times \theta = r\theta$

$S = r\theta$

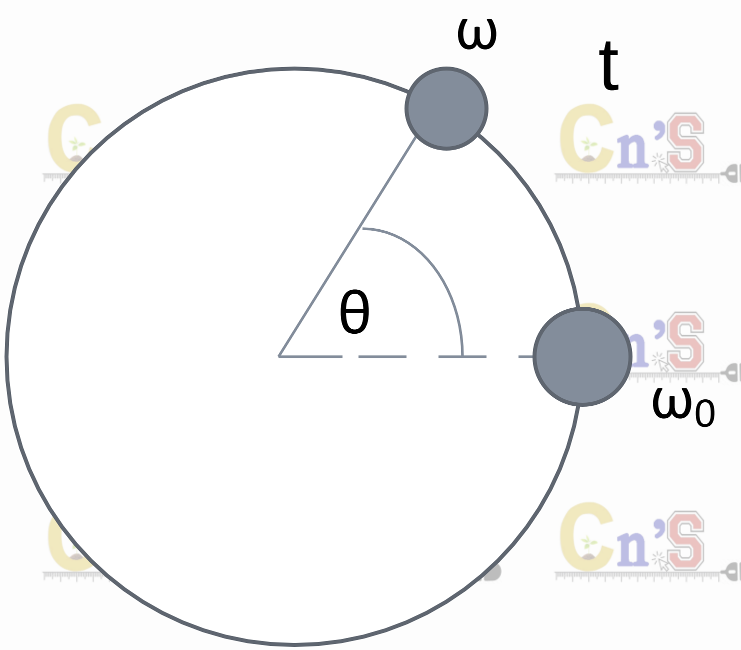
Tangential Versus Centripetal Acceleration

$a_t = r\alpha$	due to changing angular speed
$a_{cp} = r\omega^2$	due to changing direction of motion

2. Linear velocity- Angular velocity

$S / t = r\theta / t \Rightarrow V = r\omega$

4. Angular acceleration- Angular acceleration of an object in circular motion is the rate of change of angular velocity



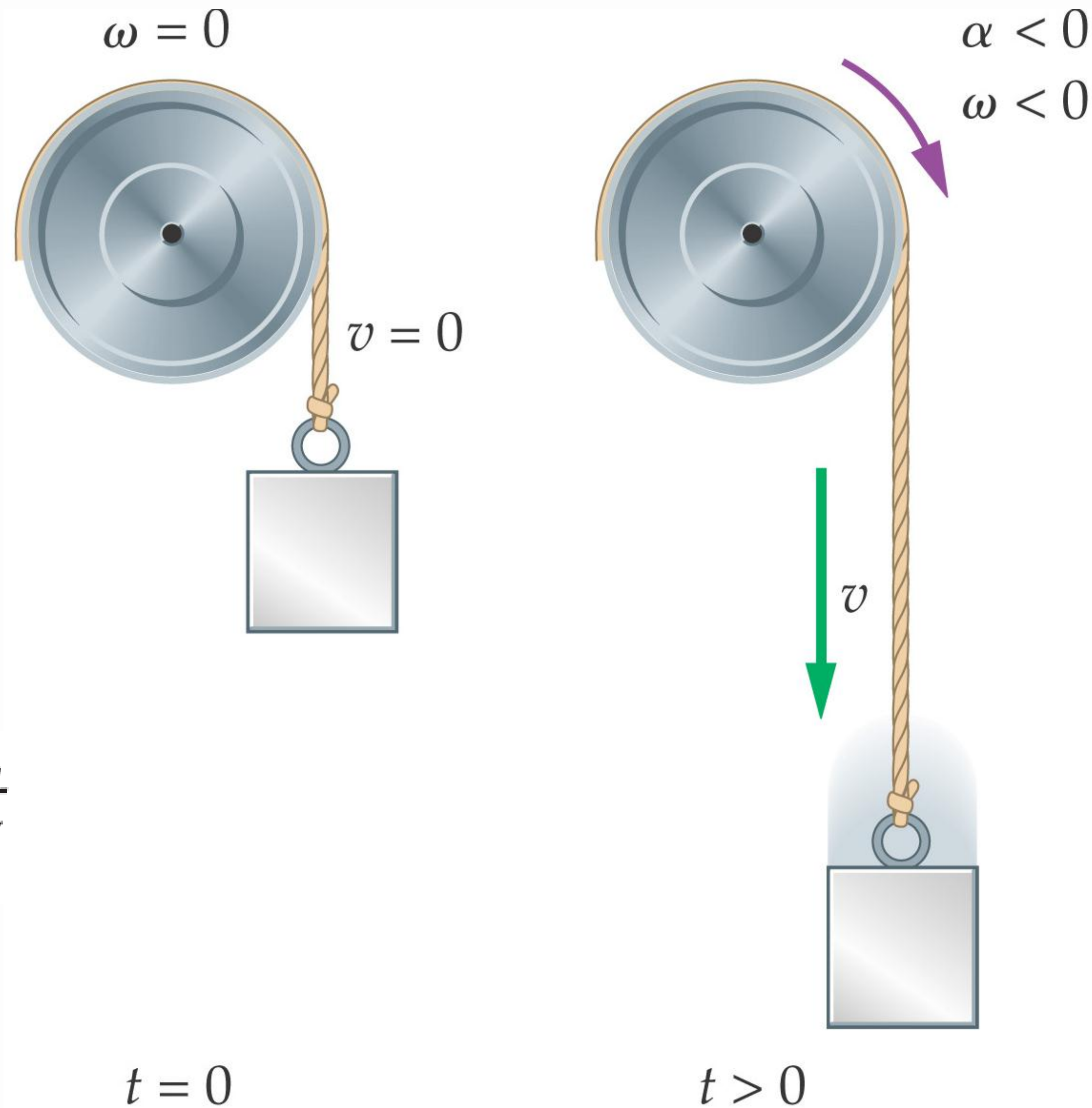
$\alpha = (\omega - \omega_0) / t$

Unit- rads^{-2}
Direction- By right hand rule

10-2 Rotational Kinematics

If the angular acceleration is constant:

$$\omega = \omega_0 + \alpha t$$



Angular equations of movement

$\alpha = (\omega - \omega_0) / t$ $\Rightarrow$ $\omega = \omega_0 + \alpha t$

$(\omega + \omega_0)/2 = \theta / t$ $\Rightarrow$ $\theta = (\omega + \omega_0)t/2$

$(\omega + \omega_0)/2 = \theta / t$, $\omega t = 2\theta - \omega_0 t$,

$(\omega_0 + \alpha t)t = 2\theta - \omega_0 t$ $\Rightarrow$ $\theta = \omega_0 t + ½ \alpha t^2$

$\theta = (\omega + \omega_0)t/2$, $\theta = \dfrac{(\omega + \omega_0)}{2}\dfrac{(\omega - \omega_0)}{\alpha}$

$\omega^2 = \omega_0^2 + 2\alpha\theta$

Therefore the four equations of angular movement are:

1. $\omega = \omega_0 + \alpha t$
2. $\theta = (\omega + \omega_0)t/2$
3. $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$
4. $\omega^2 = \omega_0^2 + 2\alpha\theta$

It should be noted that these four equations are analogous to the four linear equations of motion:

1. $V = U + at$
2. $S = (V + U)t/2$
3. $S = Ut + \frac{1}{2}at^2$
4. $V^2 = U^2 + 2as$

3. Linear acceleration- Angular acceleration

$$\alpha = (\omega - \omega_0) / t$$

$$\alpha r = (\omega - \omega_0) r / t$$

$$\alpha r = (\omega r - \omega_0 r) / t$$

$$\alpha r = (V - V_0) / t$$

$$\alpha r = a \Rightarrow a = r\alpha$$

	Displacement	Velocity	Acceleratiom
Translational motion	S	V	**a**
Rotational motion	θ	ω	**α**
Relationship	S = rθ	**V = rω**	**a = rα**

The period and the frequency of an object in rotation motion

Period (T) is the time taken by an object in rotational motion to complete one complete circle.

Frequency (f) is the no. of cycles an object rotates around its axis of rotation

Thus, $f = 1/T$ .However, $\omega = \theta / t$

Therefore, $\omega = 2\pi / T$

Therefore, $\omega = 2\pi / (1/f)$

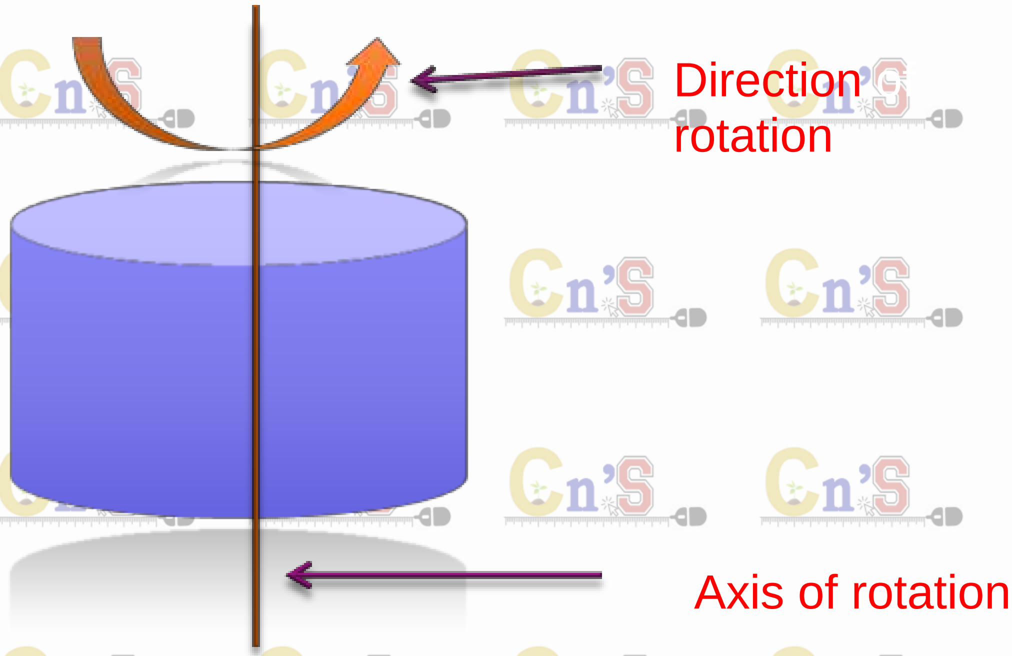
Thus, $\omega = 2\pi f$

Right hand rule

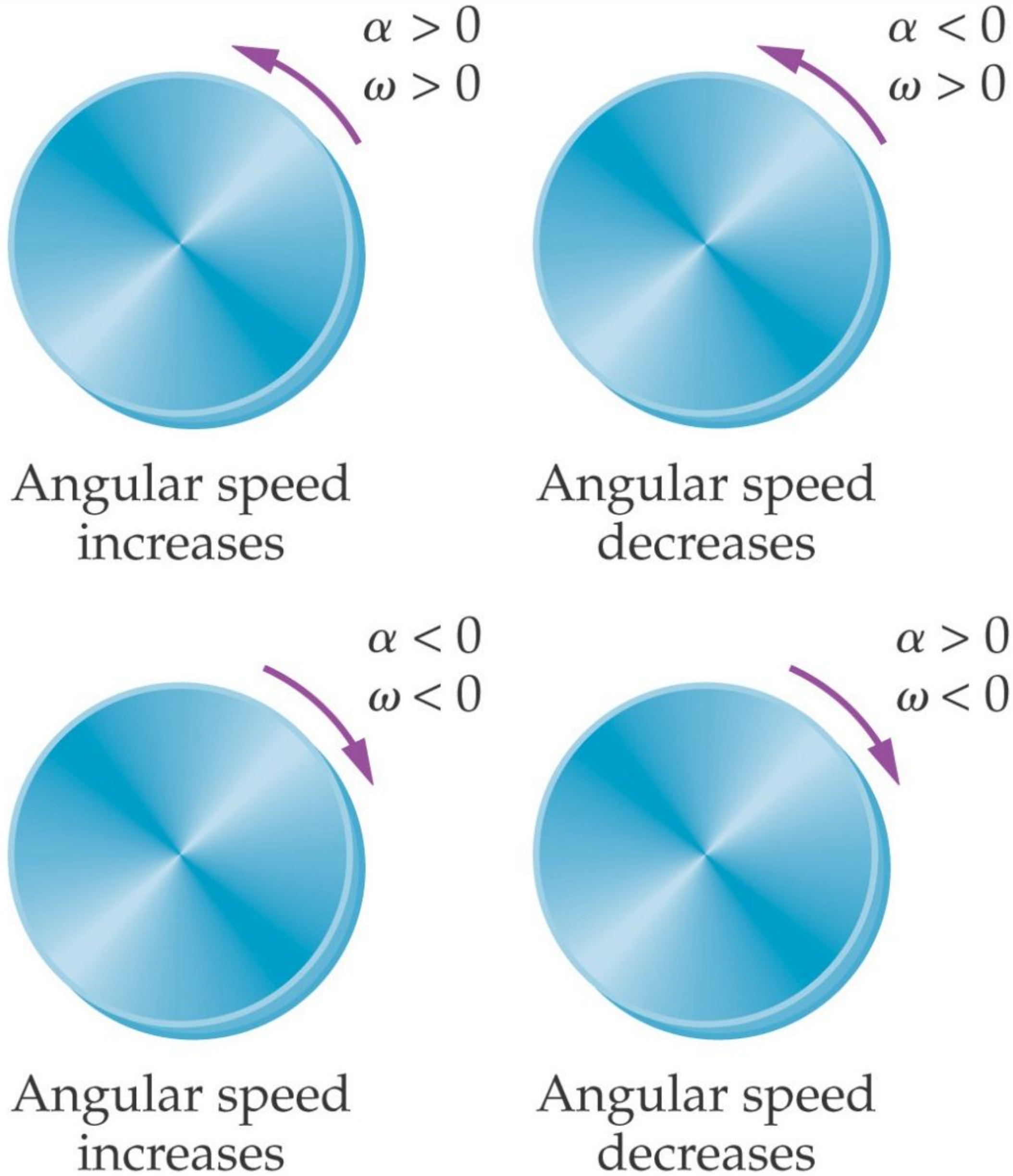
Take your right hand and curl your fingers along the direction of the rotation. Your thumb directs along the specific vector you need.

(angular velocity, angular acceleration, angular momentum etc.)

Thus, right hand rule is used whenever, in rotational motion, to measure the direction of a particular vector

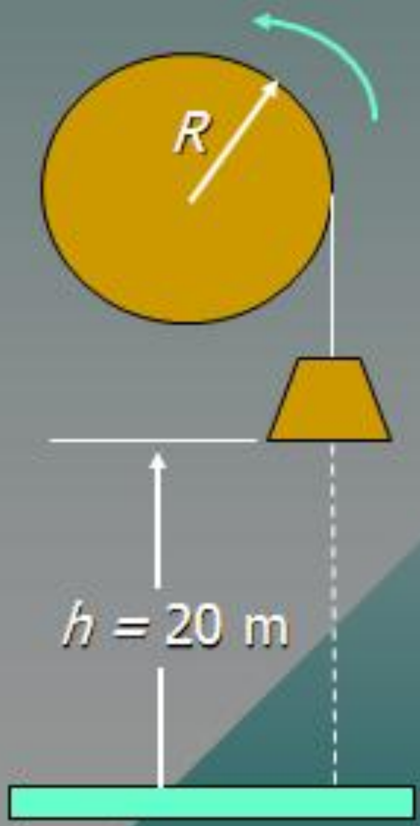


Angular Position, Velocity, and Acceleration



Example 1: A rope is wrapped many times around a drum of radius 50 cm. How many revolutions of the drum are required to raise a bucket to a height of 20 m?

$$\theta = \frac{s}{R} = \frac{20\text{ m}}{0.50\text{ m}} \qquad \theta = 40\text{ rad}$$



Now, 1 rev = 2π rad

$$\theta = (40\,\cancel{\text{rad}})\left(\frac{1\text{ rev}}{2\pi\,\cancel{\text{rad}}}\right)$$

$$\boxed{\theta = 6.37\text{ rev}}$$

Example 2: A bicycle tire has a radius of 25 cm. If the wheel makes 400 rev, how far will the bike have traveled?

$$\theta = (400 \cancel{\text{rev}})\left(\frac{2\pi \text{ rad}}{1 \cancel{\text{rev}}}\right)$$

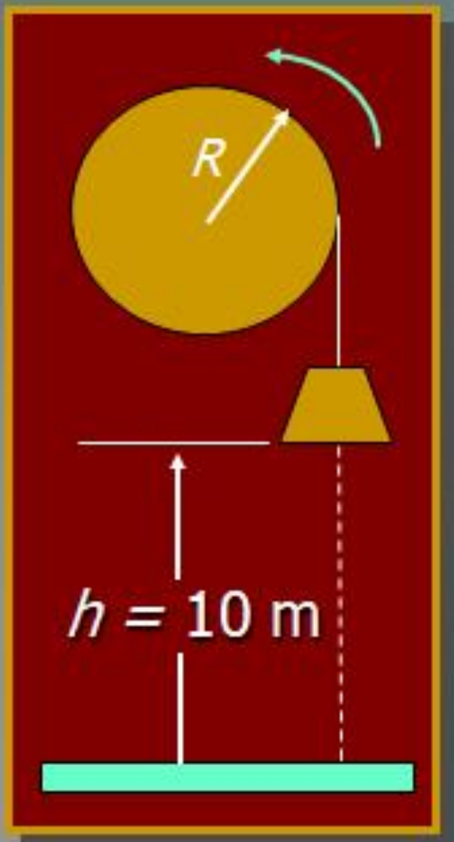
$$\theta = 2513 \text{ rad}$$

$$s = \theta R = 2513 \text{ rad } (0.25 \text{ m})$$

$$\boxed{s = 628 \text{ m}}$$

Example 3: A rope is wrapped many times around a drum of radius 20 cm. What is the angular velocity of the drum if it lifts the bucket to 10 m in 5 s?

$$\theta = \frac{s}{R} = \frac{10\text{ m}}{0.20\text{ m}} \qquad \theta = 50\text{ rad}$$



$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{50\text{ rad}}{5\text{ s}}$$

$$\boxed{\omega = 10.0\text{ rad/s}}$$

Example 4: In the previous example, what is the frequency of revolution for the drum? Recall that $\omega = 10.0$ rad/s.

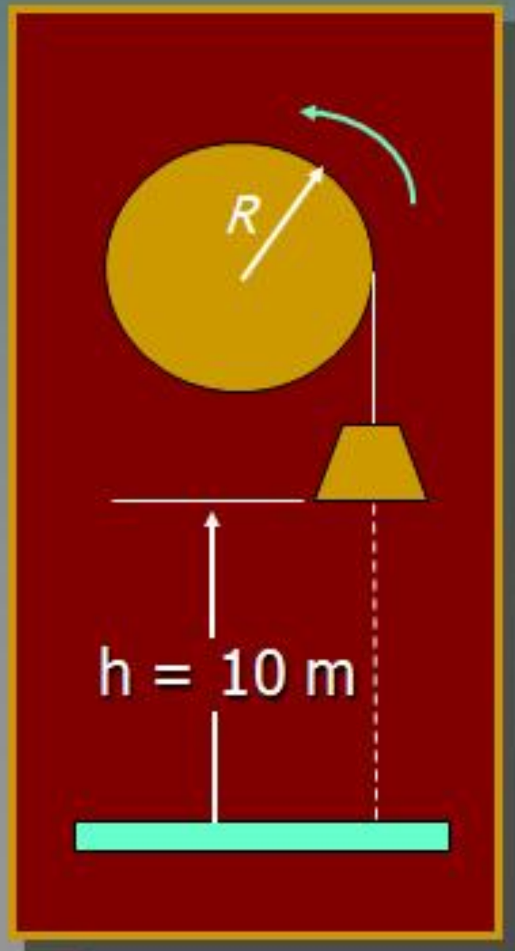
$$\omega = 2\pi f \quad \text{or} \quad f = \frac{\omega}{2\pi}$$

$$f = \frac{10.0 \text{ rad/s}}{2\pi \text{ rad/rev}} = 1.59 \text{ rev/s}$$

Or, since 60 s = 1 min:

$$f = 1.59 \frac{\text{rev}}{\cancel{\text{s}}}\left(\frac{60 \cancel{\text{s}}}{1 \text{ min}}\right) = 95.5 \frac{\text{rev}}{\text{min}}$$

$$\boxed{f = 95.5 \text{ rpm}}$$

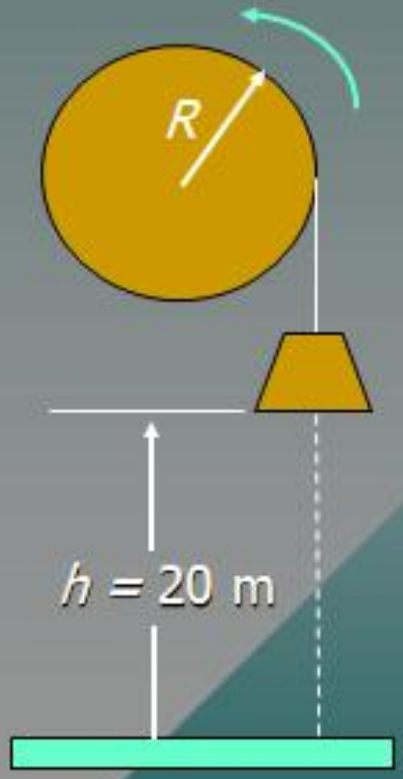


Example 5: The block is lifted from rest until the angular velocity of the drum is 16 rad/s after a time of 4 s. What is the average angular acceleration?

$$\alpha = \frac{\omega_f - \cancel{\omega_o}^{\,0}}{t} \quad or \quad \alpha = \frac{\omega_f}{t}$$

$$\alpha = \frac{16\text{ rad/s}}{4\text{ s}} = 4.00\,\frac{\text{rad}}{\text{s}^2}$$

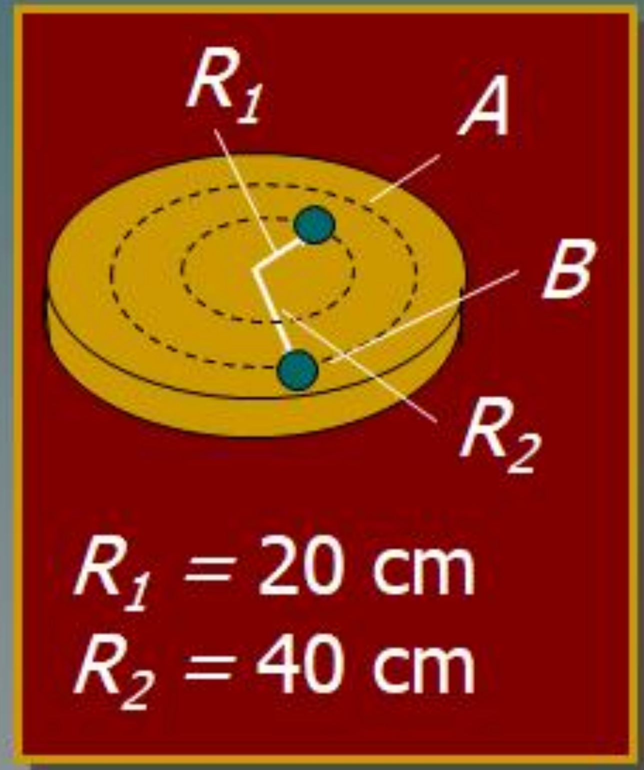
$$\boxed{\alpha = 4.00\text{ rad/s}^2}$$



Consider flat rotating disk:

$\omega_o = 0; \quad \omega_f = 20$ rad/s

$t = 4$ s



What is final linear speed at points A and B?

$v_{Af} = \omega_{Af} R_1 = (20 \text{ rad/s})(0.2 \text{ m});$ $\quad v_{Af} = 4$ m/s

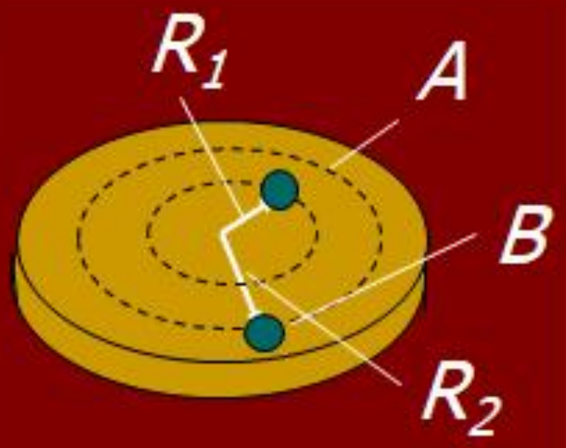
$v_{Af} = \omega_{Bf} R_1 = (20 \text{ rad/s})(0.4 \text{ m});$ $\quad v_{Bf} = 8$ m/s

Acceleration Example

Consider flat rotating disk:

$\omega_o = 0; \quad \omega_f = 20 \text{ rad/s}$

$t = 4 \text{ s}$



$R_1 = 20$ cm

$R_2 = 40$ cm

What is the average angular and linear acceleration at B?

$$\alpha = \frac{\omega_f - \omega_0}{t} = \frac{20 \text{ rad/s}}{4 \text{ s}}$$

$$\boxed{\alpha = 5.00 \text{ rad/s}^2}$$

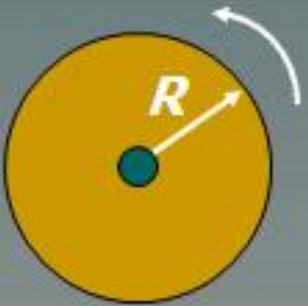
$$a = \alpha R = (5 \text{ rad/s}^2)(0.4 \text{ m})$$

$$\boxed{a = 2.00 \text{ m/s}^2}$$

Angular analogy: A disk (R = 50 cm), rotating at 600 rev/min comes to a stop after making 50 rev. What is the acceleration?

Select Equation:

$$\boxed{2\alpha\theta = \omega_f^2 - \omega_0^2}$$



$\omega_o = 600$ rpm

$\omega_f = 0$ rpm

$\theta = 50$ rev

$$600\frac{rev}{\text{min}}\left(\frac{2\pi \text{ rad}}{1 \text{ rev}}\right)\left(\frac{1 \text{ min}}{60 \text{ s}}\right) = 62.8 \text{ rad/s}$$

50 rev = 314 rad

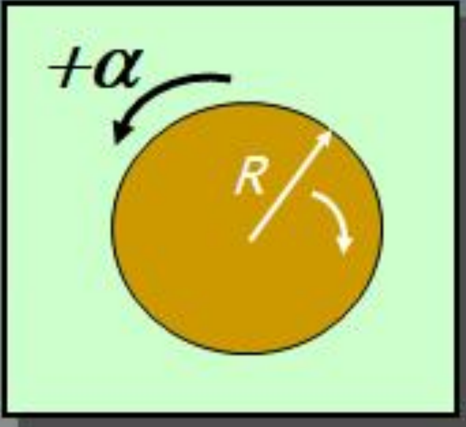
$$\alpha = \frac{0 - \omega_o^2}{2\theta} = \frac{-(62.8 \text{ rad/s})^2}{2(314 \text{ rad})}$$

$$\boxed{\alpha = -6.29 \text{ m/s}^2}$$

Example 6: A drum is rotating clockwise initially at 100 rpm and undergoes a constant counterclockwise acceleration of 3 rad/s² for 2 s. What is the angular displacement?

Given: ω_o = -100 rpm; t = 2 s

α = +2 rad/s²



$$100 \frac{\text{rev}}{\text{min}}\left(\frac{1\text{ min}}{60\text{ s}}\right)\left(\frac{2\pi\text{ rad}}{1\text{ rev}}\right) = 10.5\text{ rad/s}$$

$$\theta = \omega_o t + \tfrac{1}{2}\alpha t^2 = (-10.5)(2) + \tfrac{1}{2}(3)(2)^2$$

θ = -20.9 rad + 6 rad

$$\boxed{\theta = \text{-}14.9\text{ rad}}$$

Net displacement is clockwise (-)

Ex. To throw a curve ball, a pitcher gives the ball an initial angular speed of 36.0 rad/s. When the catcher gloves the ball 0.595 s later, its angular speed has decreased (due to air resistance) to 34.2 rad /s. (a) What is the ball's angular acceleration, assuming it to be constant? (b) How many revolutions does the ball make before being caught?

$\omega = \omega_o + \alpha t \rightarrow \alpha = (\omega - \omega o) / t$

$\alpha = (34.2 \text{ rad / s} - 36.0 \text{ rad /s}) / 0.595 \text{ s} = -3.03 \text{ rad /s}^2$

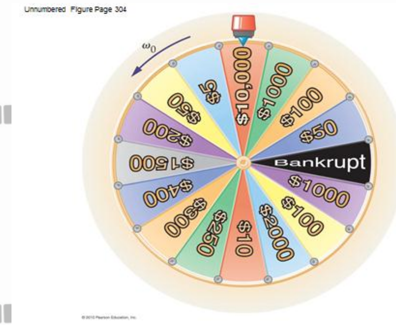
(b). Calculate the angular displacement $\theta - \theta_o = \omega_o + \frac{1}{2} \alpha t^2$

$= (36.0 \text{ rad /s})(0.595 \text{ s}) + \frac{1}{2}(-3.03 \text{ rad/ s}^2) (0.595 \text{ s})^2$

$= 20.9 \text{ rad}$

Convert the angular displacement to revolutions : 20.9 rad = 20.9 rad (1 rev / 2π rad) = 3.33 rev

On a certain show, contestants spin a wheel when it is their turn. One contestant gives the wheel an initial angular speed of 3.40 rad / s. it then rotates through one – one – quarter revolutions and comes to rest on the BANKRUPT space. (a) Find the angular acceleration of the wheel, assuming it to be constant. (b). How long does it take for the wheel to come to rest?



$\omega^2 = \omega_o^2 + 2\alpha(\theta - \theta_o) \rightarrow \alpha = (\omega^2 - \omega_o^2) / 2(\theta - \theta_o)$

$\theta - \theta_o = 1.25 \text{ rev} = 1.25 \text{ rev } (2\pi \text{ rad} / 1 \text{ rev}) = 7.85 \text{ rad}$

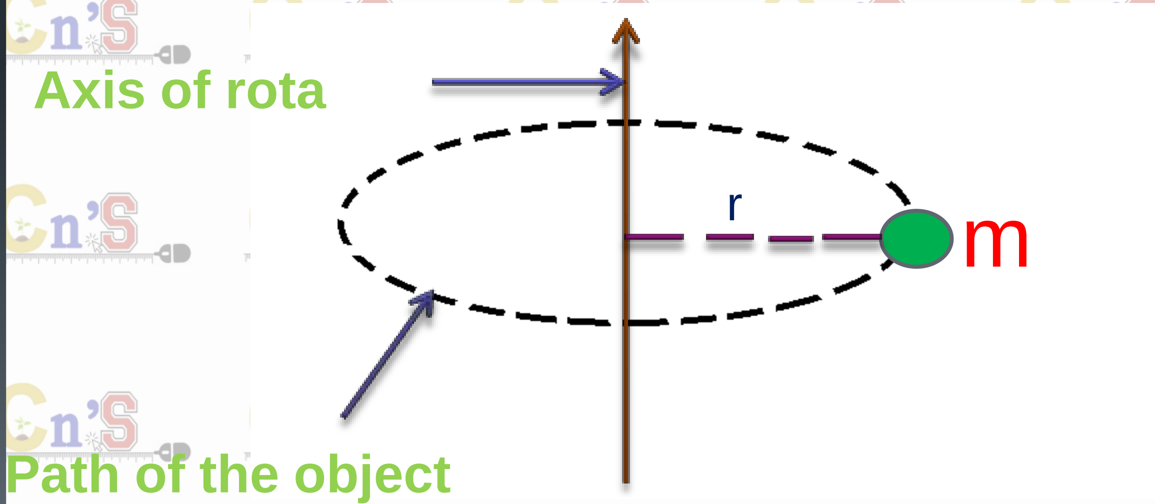
$\alpha = (\omega^2 - \omega_o^2) / 2(\theta - \theta_o) = 0 - (3.40 \text{ rad/s})^2 / 2(7.85 \text{ rad}) = -0.736 \text{ rad/s}^2$

$\omega = \omega_o + \alpha t \implies t = (\omega - \omega o) / \alpha$ ==== $0 - 3.40 \text{ rad / s}) / (-0.736 \text{ rad / s}^2) = 4.62 \text{ s}$

Moment of Inertia

Unlike in the case of linear movement's inertia (reluctance to move or stop) , inertia of circular/rotational motion depends both upon the mass of the object and the distribution of mass (how the mass is spread across the object)

Momenat of Inertia of a single object-



$$I = mr^2$$

Moment of Inertia a scalar quantity

Moment of Inertia

- *Moment of inertia,* I: rotational analog to mass

$$I = \sum_i m_i r_i^2$$

  - r defined relative to rotation axis
  - SI units are kg m^2
- Depends on mass <u>and</u> its distribution.
- If mass is distributed further from axis of rotation, moment of inertia will be larger.

Moment of Inertia of a Uniform Ring

$$I = \Sigma m_i r_i^2 = MR^2$$

- Divide ring into segments
- The radius of each segment is R

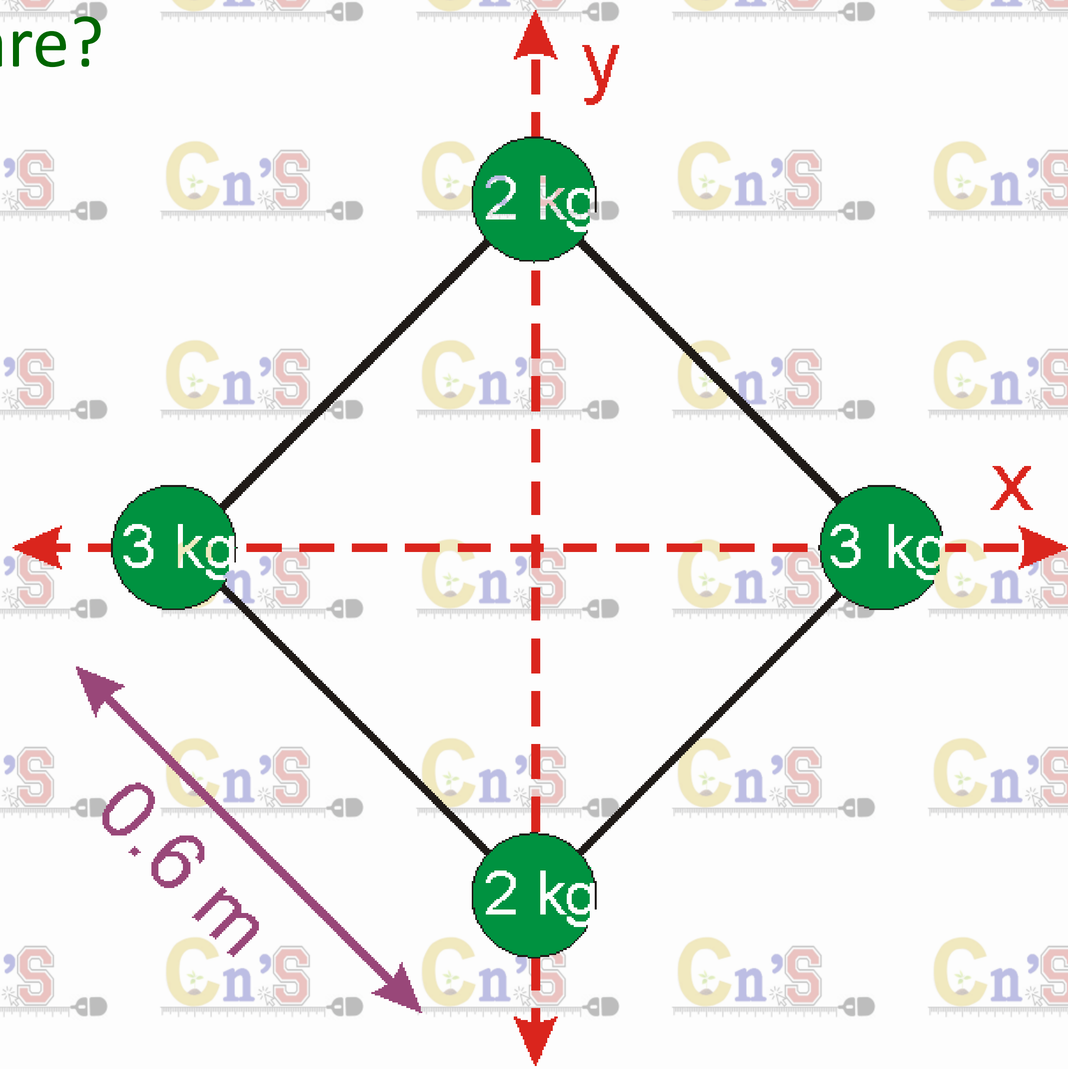
Example

What is the moment of inertia of the following point masses arranged in a square?

a) about the x-axis?

b) about the y-axis?

c) about the z-axis?



a) 0.72 kg·m^2
b) 1.08 kg·m^2
c) 1.8 kg·m^2

Moment of Inertia of some common shapes

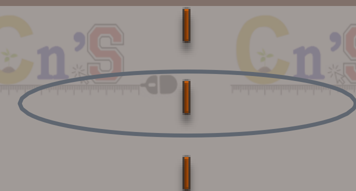
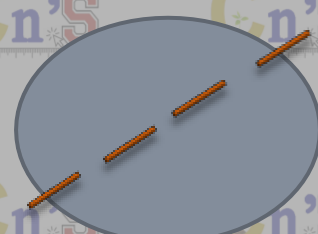
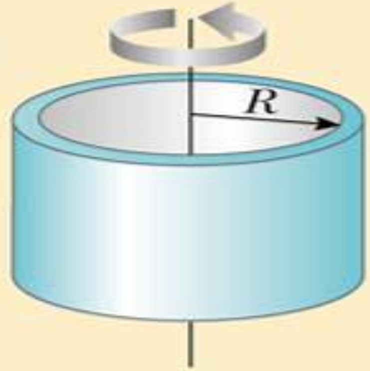
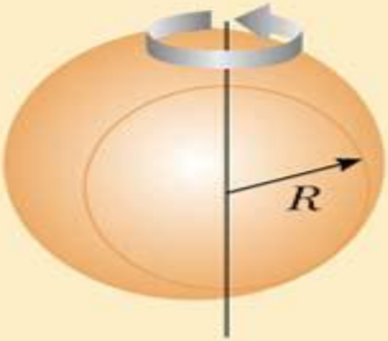
Body	Axis	Figure	
Ring (RadiusR)	Perpendicular to the plane at the center		MR^2
Disc (Radius R)	Perpendicular to the plane at the center		$\frac{1}{2} MR^2$
Solid Cylinder (Radius R)	Axis of cylinder		$\frac{1}{2} MR^2$
Solid Sphere (Radius R)	Diameter		$\frac{2}{5} MR^2$

TABLE 8.1 Moments of Inertia for Various Rigid Objects of Uniform Composition

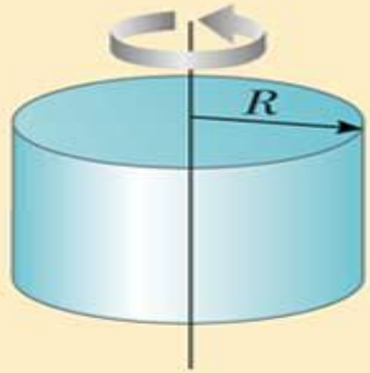
Hoop or thin cylindrical shell
$I = MR^2$



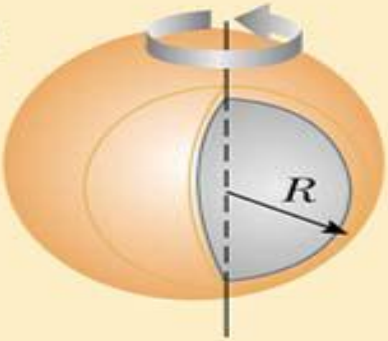
Solid sphere
$I = \frac{2}{5} MR^2$



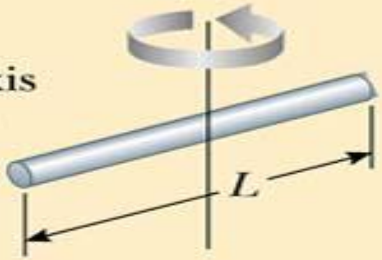
Solid cylinder or disk
$I = \frac{1}{2} MR^2$



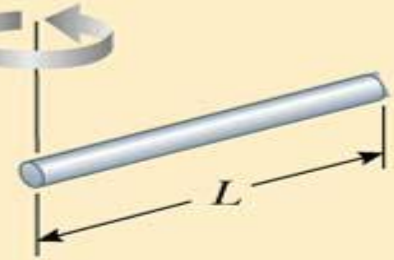
Thin spherical shell
$I = \frac{2}{3} MR^2$

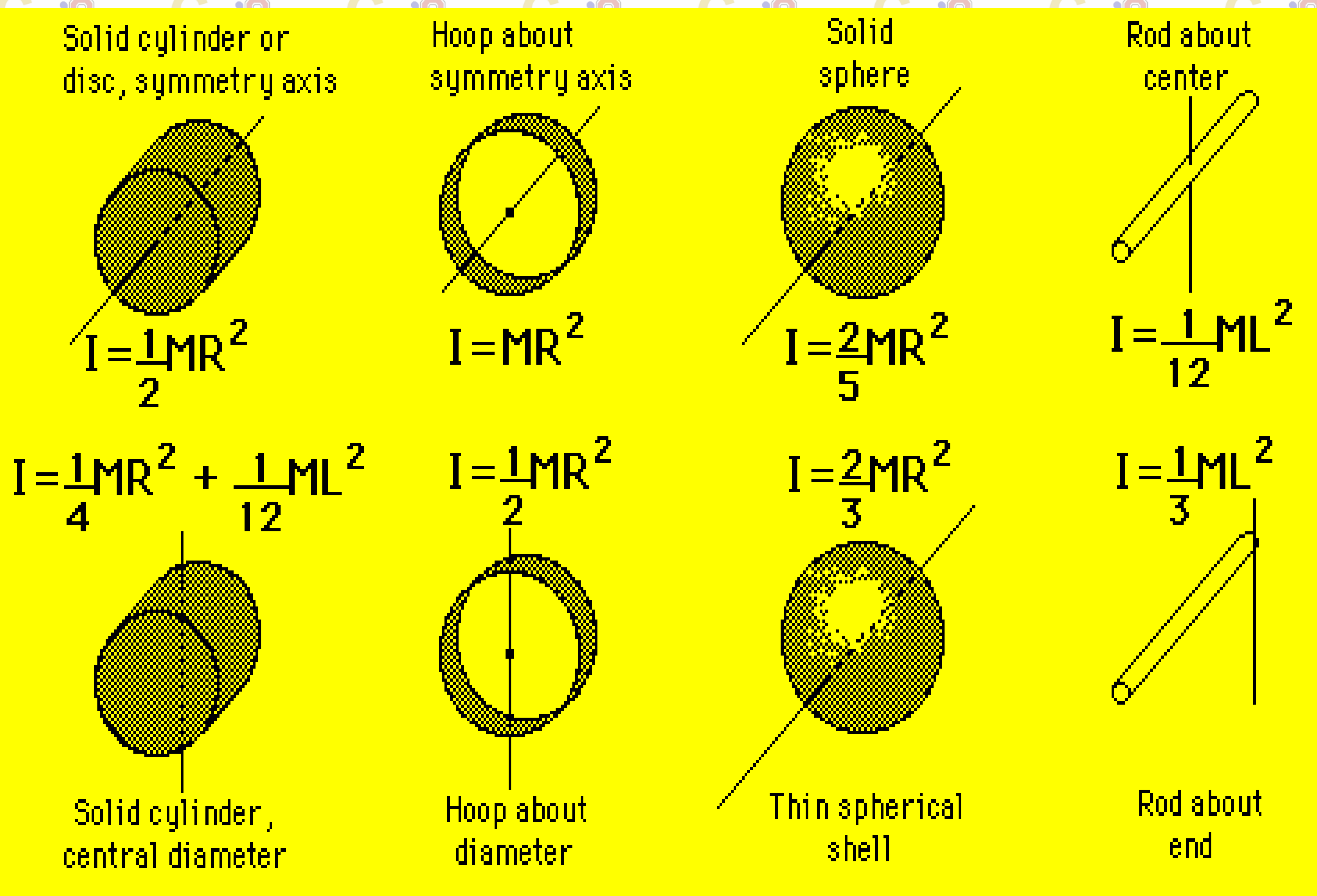


Long thin rod with rotation axis through center
$I = \frac{1}{12} ML^2$



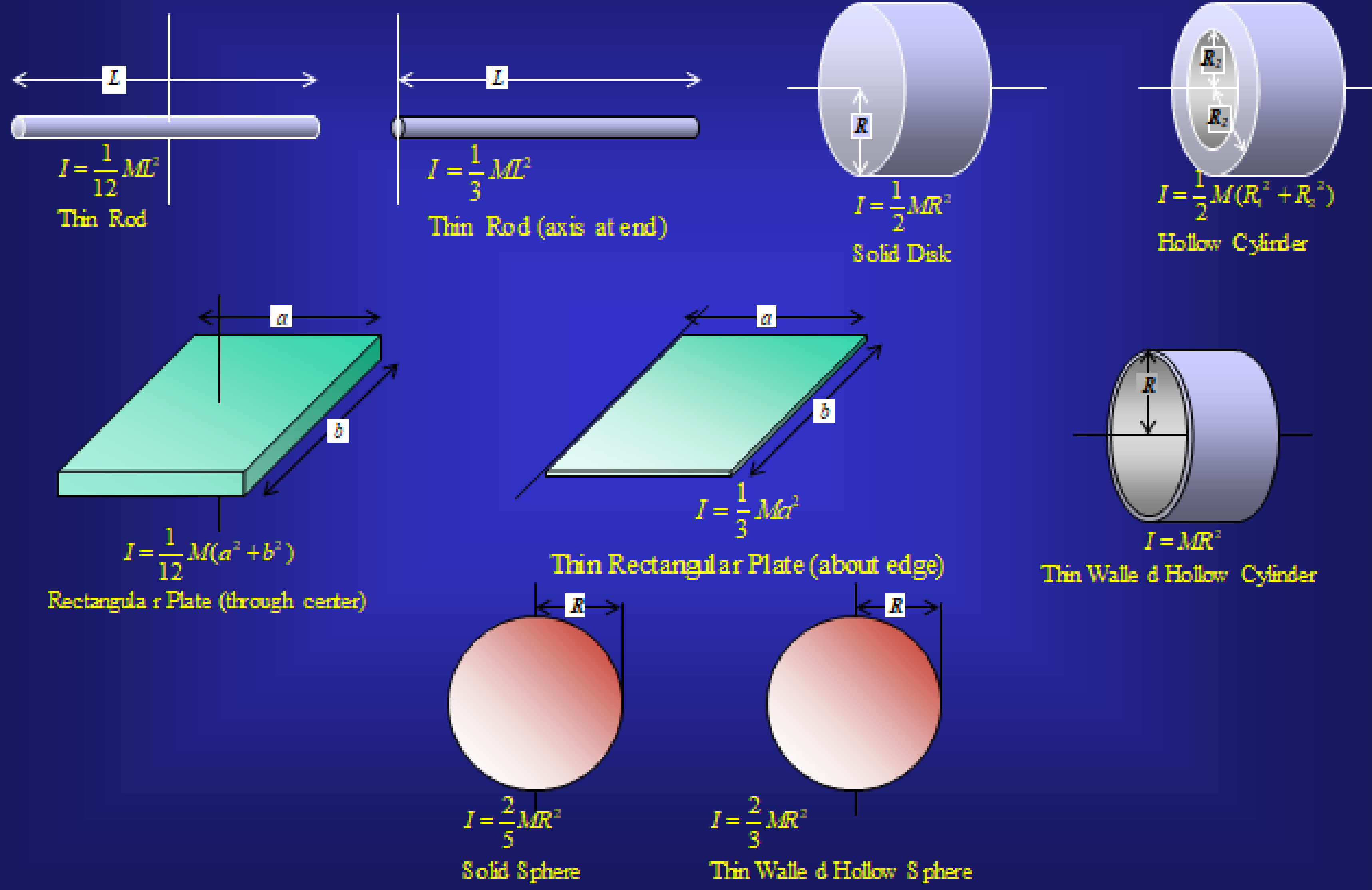
Long thin rod with rotation axis through end
$I = \frac{1}{3} ML^2$

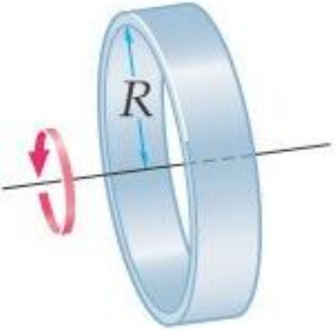




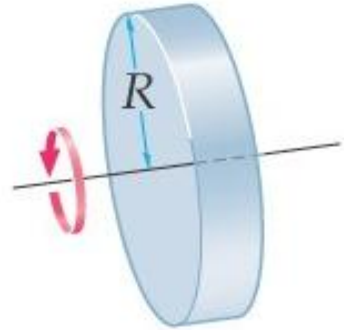
Solid cylinder or disc, symmetry axis
$I = \frac{1}{2}MR^2$
Hoop about symmetry axis
$I = MR^2$
Solid sphere
$I = \frac{2}{5}MR^2$
Rod about center
$I = \frac{1}{12}ML^2$
$I = \frac{1}{4}MR^2 + \frac{1}{12}ML^2$
Solid cylinder, central diameter
$I = \frac{1}{2}MR^2$
Hoop about diameter
$I = \frac{2}{3}MR^2$
Thin spherical shell
$I = \frac{1}{3}ML^2$
Rod about end

Moments of inertia for some common geometric solids

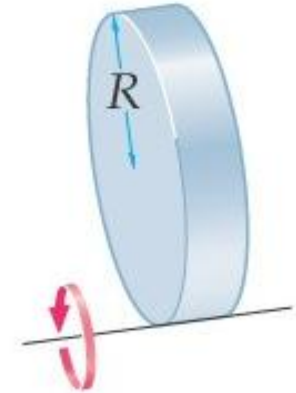




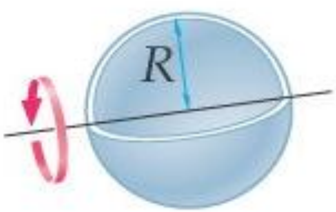
Hoop or
cylindrical shell
$I = MR^2$



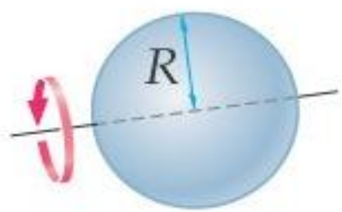
Disk or
solid cylinder
$I = \frac{1}{2} MR^2$



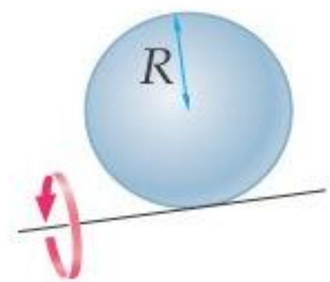
Disk or
solid cylinder
(axis at rim)
$I = \frac{3}{2} MR^2$



Hollow sphere
$I = \frac{2}{3} MR^2$



Solid sphere
$I = \frac{2}{5} MR^2$



Solid sphere
(axis at rim)
$I = \frac{7}{5} MR^2$

Other Moments of Inertia

cylindrical shell : $I = MR^2$ ← bicycle rim

solid cylinder : $I = \frac{1}{2}MR^2$ ← filled can of coke

rod about center : $I = \frac{1}{12}ML^2$ ← baton

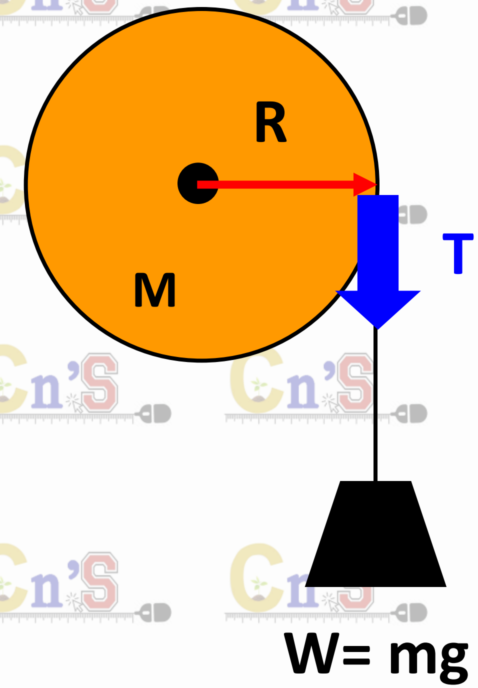
rod about end : $I = \frac{1}{3}ML^2$ ← baseball bat

spherical shell : $I = \frac{2}{3}MR^2$ ← basketball

solid sphere : $I = \frac{2}{5}MR^2$ ← boulder

Rotational inertia and torque

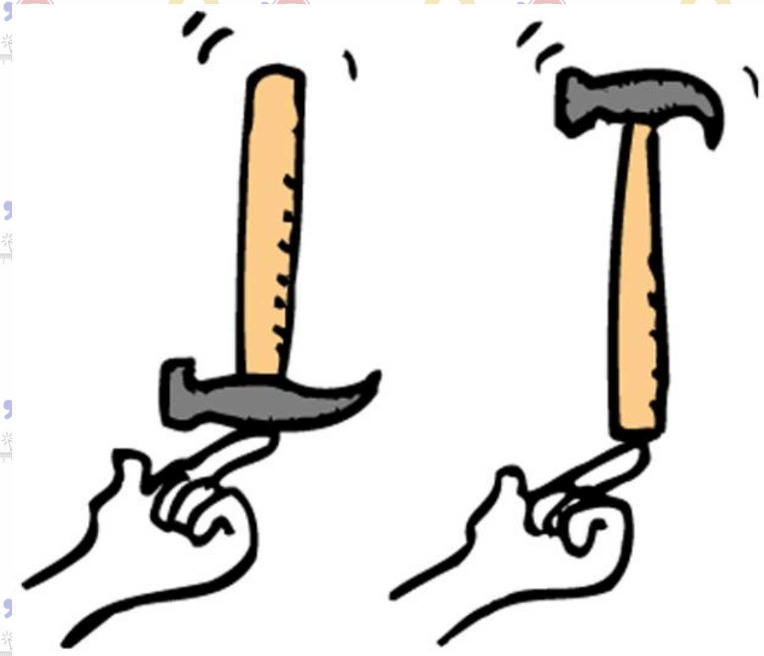
- To start an object spinning, a torque must be applied to it
- The amount of torque required depends on the rotational inertia (I) of the object
- The rotational inertia (I) depends on the mass of the object, its shape, *and on how the mass is distributed*
- Solid disk: $I = \frac{1}{2} M R^2$
- The higher the rotation inertia, the more torque that is required to make an object spin



Torque = T · R

An object rotating about an axis tends to remain rotating about the same axis, unless an external influence (*torque*, s) is acting. (c.f. 1st law)

Consider balancing a hammer upright on the tip of your finger. Would it be easier to balance in the left-hand picture or the right-hand picture, and why?



Easier on the right, because it has more rotational inertia (heavy part further away from your finger), so is more resistant to a rotational change.

Eg. Tight-rope walker carries a pole to increase his rotational inertia - if he starts to wobble, the pole starts to rotate but its inertia resists this, so the tight-rope walker has time to adjust balance and not rotate and fall.

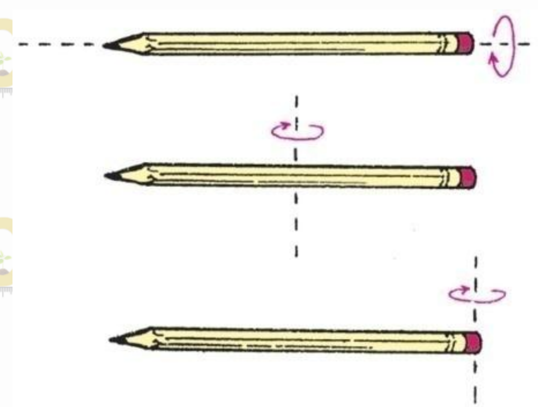
Better balance (more rotational inertia) if pole is longer and has weights at the ends.

- Rotational inertia depends on the axis around which it rotates:

Eg With a pencil: Easiest to spin here (smallest I)

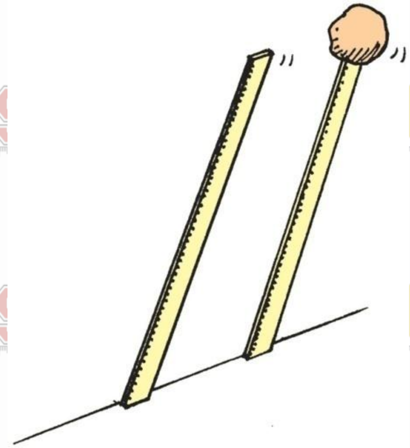
Harder here

Even harder here



Question

Consider the meter sticks shown, one with a glob of clay at the top end. If released from upright position, which reaches the ground first?



A) The one without the clay

B) The one with the clay

C) They both reach the ground together

Answer: A

The one without the clay! Because the stick-with-clay has more rotational inertia, so resists rotation to ground more.

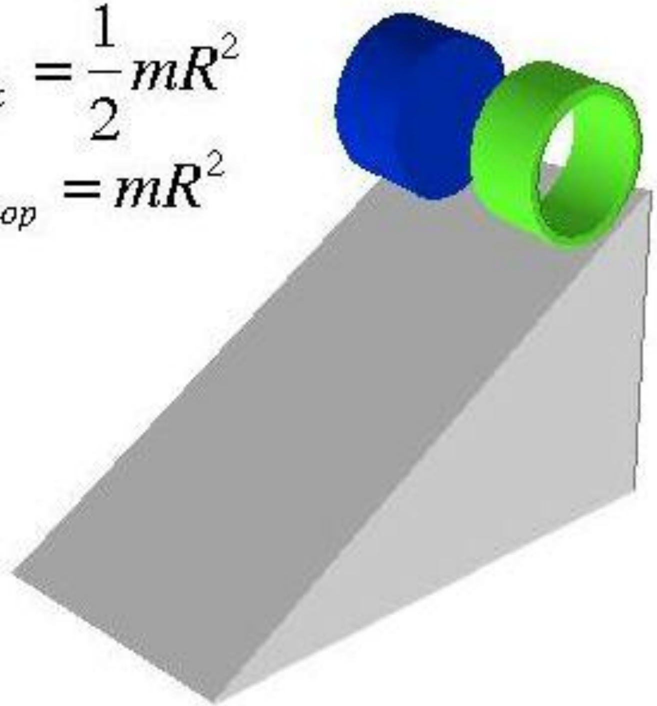
(Also, experiences more torque…see later)

Which object will reach the bottom of the incline first?

They have the same radius and are the same mass

$$I_{disk} = \frac{1}{2}mR^2$$

$$I_{hoop} = mR^2$$



The hoop has a larger moment of inertia and therefore requires more energy to get it started.

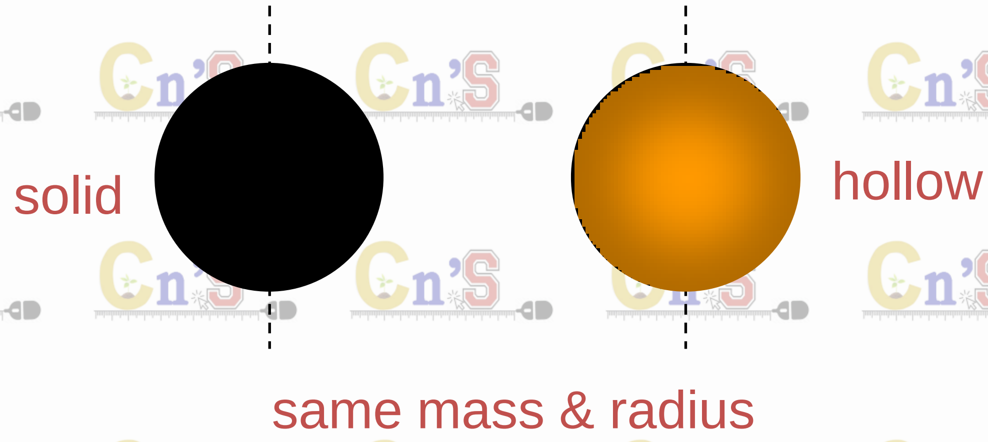
Two spheres have the same radius and equal masses. One is made of solid aluminum, and the other is made from a hollow shell of gold.

Which one has the bigger moment of inertia about an axis through its center?

a) solid aluminum

b) hollow gold

c) same



Moment of Inertia

Two spheres have the same radius and equal masses. One is made of solid aluminum, and the other is made from a hollow shell of gold.

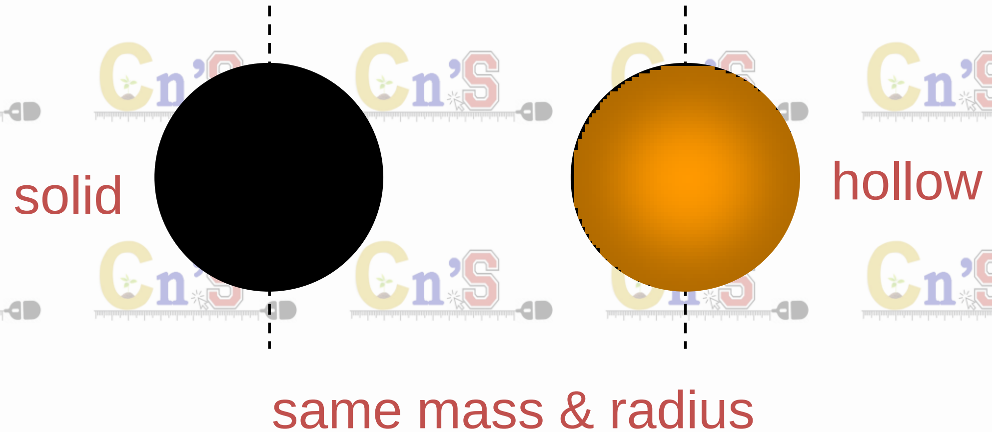
Which one has the bigger moment of inertia about an axis through its center?

a) solid aluminum

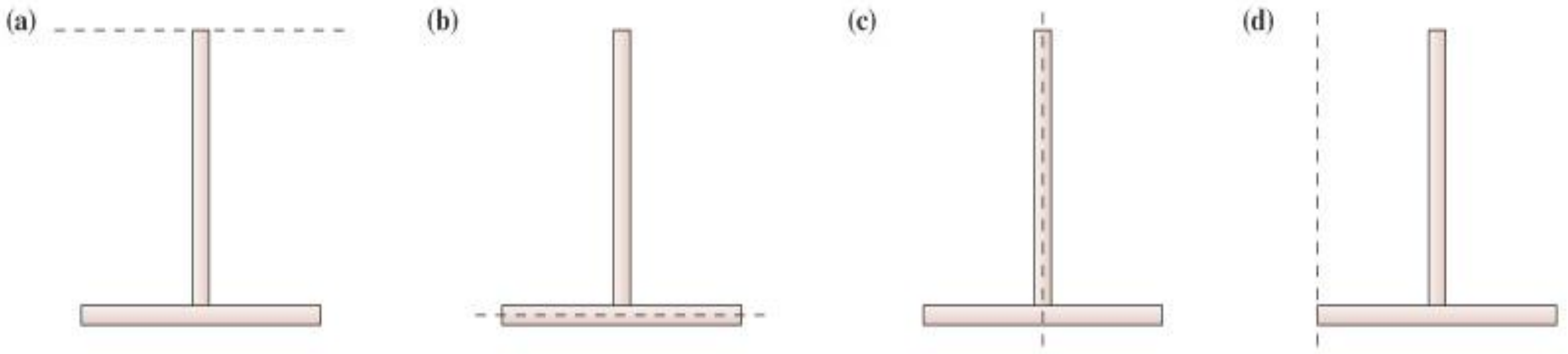
b) hollow gold

c) same

Moment of inertia depends on mass and distance from axis squared. It is bigger for the shell since its mass is located farther from the center.

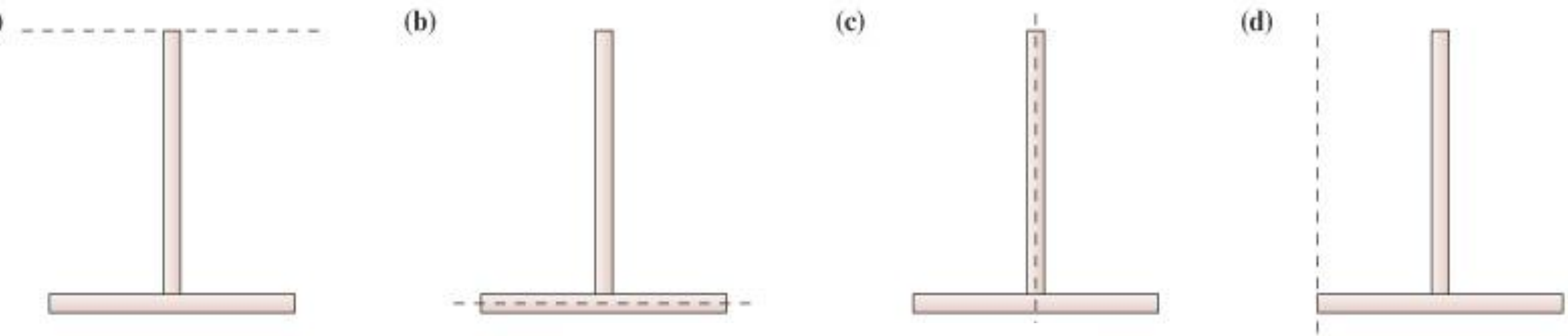


Four Ts are made from two identical rods of equal mass and length. Rank in order, from largest to smallest, the moments of inertia I_a to I_d for rotation about the dotted line.



a. $I_c > I_b > I_d > I_a$
b. $I_c = I_d > I_a = I_b$
c. $I_a = I_b > I_c = I_d$
d. $I_a > I_d > I_b > I_c$
e. $I_a > I_b > I_d > I_c$

Four Ts are made from two identical rods of equal mass and length. Rank in order, from largest to smallest, the moments of inertia I_a to I_d for rotation about the dotted line.



a. $I_c > I_b > I_d > I_a$

b. $I_c = I_d > I_a = I_b$

c. $I_a = I_b > I_c = I_d$

d. $\boldsymbol{I_a > I_d > I_b > I_c}$

e. $I_a > I_b > I_d > I_c$

Moment of inertia is

a. the rotational equivalent of mass.
b. the point at which all forces appear to act.
c. the time at which inertia occurs.
d. an alternative term for *moment arm*.

Moment of inertia is

✔ **a. the rotational equivalent of mass.**

b. the point at which all forces appear to act.

c. the time at which inertia occurs.

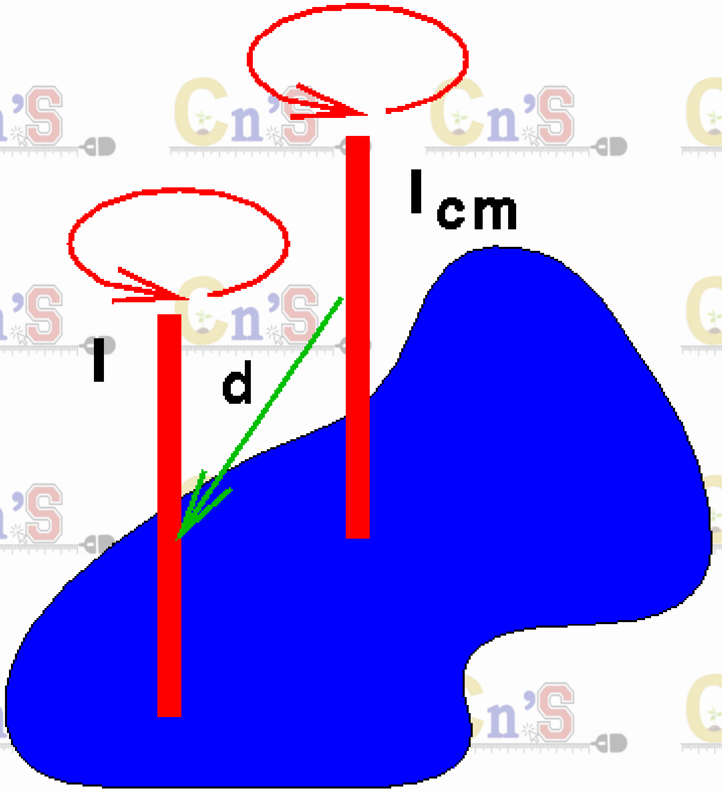
d. an alternative term for *moment arm*.

Theorem of Parallel Axes

Let I_{CM} be the moment of inertia of a body of mass M about an axis passing through the center of mass and let I be the moment of inertia about a parallel axis at a distance d from the first axis.

Then, $I = I_{CM} + Md^2$

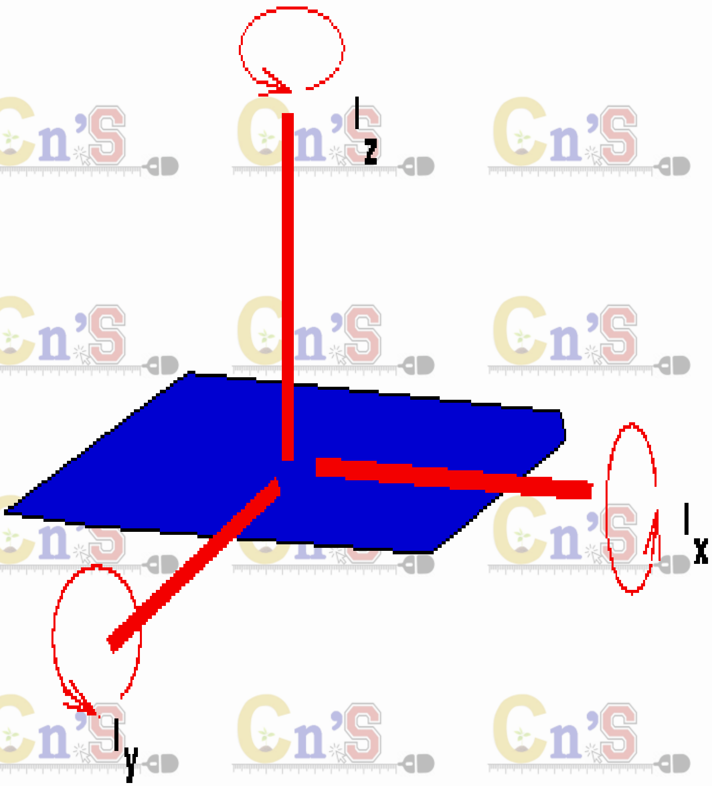
Thus the minimum moment of inertia for any object is at the center of mass, as x in the above expression is zero.

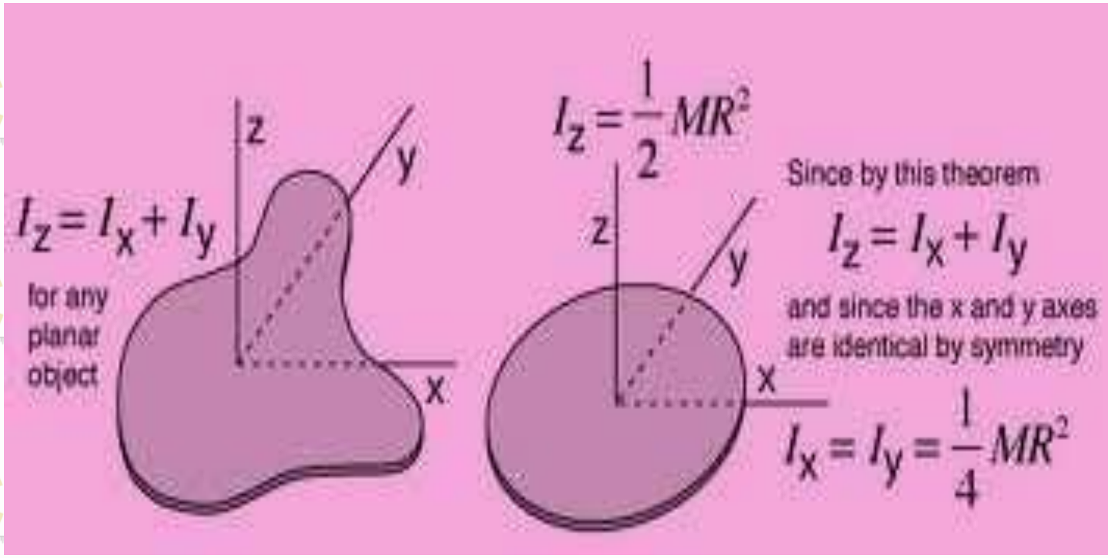


Theorem of perpendicular axis

The moment of inertia of a body about an axis perpendicular to its plane is equal to the sum of moments of inertia about two mutually perpendicular axis in its own plane and crossing through the point through which the perpendicular axis passes

$$I_z = I_x + I_y$$





Example

Treat the spindle as a solid cylinder.

a) What is the moment of Inertia of the spindle? (M=5.0 kg, R=0.6 m)

b) If the tension in the rope is 10 N, what is the angular acceleration of the wheel?

c) What is the acceleration of the bucket?

d) What is the mass of the bucket?

a) $0.9\ kg{\cdot}m^2$ b) $6.67\ rad/s^2$ c) $4\ m/s^2$ d) 1.72 kg

Example

A 600-kg solid cylinder of radius 0.6 m which can rotate freely about its axis is accelerated by hanging a 240 kg mass from the end by a string which is wrapped about the cylinder.

a) Find the linear acceleration of the mass.

b) What is the speed of the mass after it has dropped 2.5 m?

4.36 m/s^2 4.67 m/s

Example: What is the moment of inertia 'I' of the Earth?

Earth:

$r = 6400$ km

$m = 6 \times 10^{24}$ kg

For a solid sphere: $I = \frac{2}{5} m.r^2$

$I = \frac{2}{5} (6 \times 10^{24}) \times (6.4 \times 10^{6})^2$

$I = 9.8 \times 10^{37}$ kg.m^2

The **rotational inertia** of the **Earth** is therefore **enormous** and a **tremendous torque** would be needed to **slow** its rotation down (around 10^{29} N.m)

Question: Would it be more difficult to slow the Earth if it were flat?

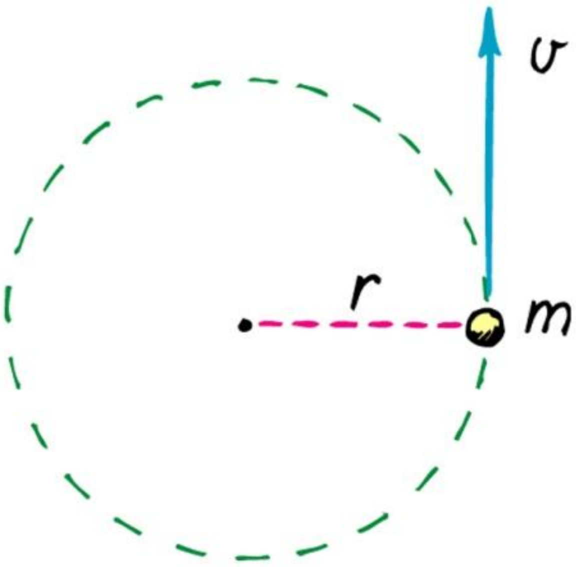
For a flat disk: $I = ½\, m.r^2$

$I = 12.3 \times 10^{37}$ kg.m^2

So it would take **even more torque** to slow a flat Earth down!

In general the larger the **mass** and its **length** or **radius** from axis of rotation the **larger** the **moment of inertia** of an object.

Angular Momentum



$L = m v r$

$L = r P$

However, $P = mv$,

(Linear moment= mass × linear velocity)

Therefore, $L = m r v$

$= m r (\omega r)$ (as $V = r\omega$)

$= mr^2 \omega$

Therefore, $L = I\omega$ (as $I = mr^2$)

Angular momentum

Angular Momentum = rotational inertia x rotational velocity

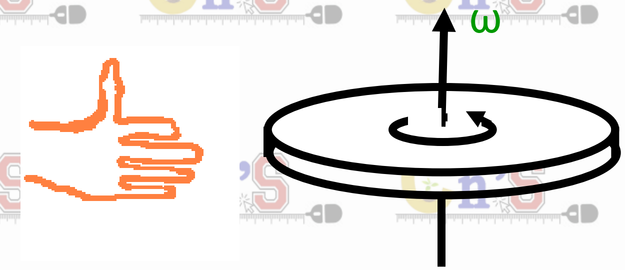
$$L = I\,\omega$$

Use right hand rule to obtain direction of angular velocity.

Angular momentum is a vector quantity,

Angular Momentum and Stability

- Angular momentum is a **vector** and both its **magnitude** and **direction** are **conserved** (…as with linear momentum).
- Recap: Linear momentum ‘P’ is in same direction as velocity.
- Angular momentum is due to angular velocity ‘ω’.



Right hand rule: The angular velocity for counter clockwise rotation is directed upwards (and vice versa).

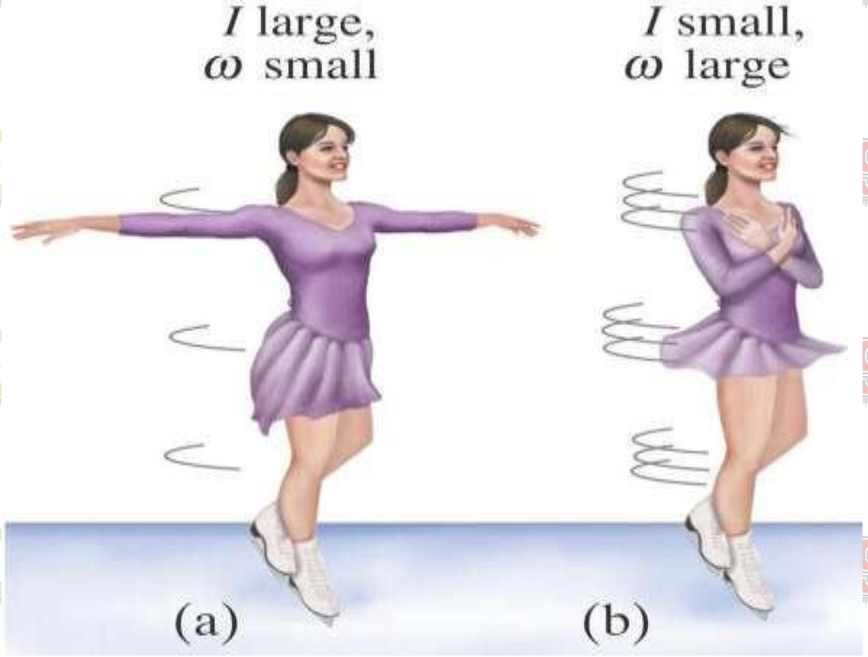
- i.e. ‘ω’ and ‘L’ in direction of extended thumb.
- Thus, the **direction of ‘L’** is important as it requires a **torque** to change it.
- **Result:** It is difficult to **change** the axis of a spinning object.

Law of Conservation of Angular Momentum

Law- If the resultant external torque on a system is zero, its total angular momentum remains constant.

That is, if $\tau = 0$, $dL / dt = 0$, which means that L is a constant

This is the rotational analogue of the law of conservation of linear momentum.



Applications Using Conserved Angular Momentum

Spinning Ice Skater:

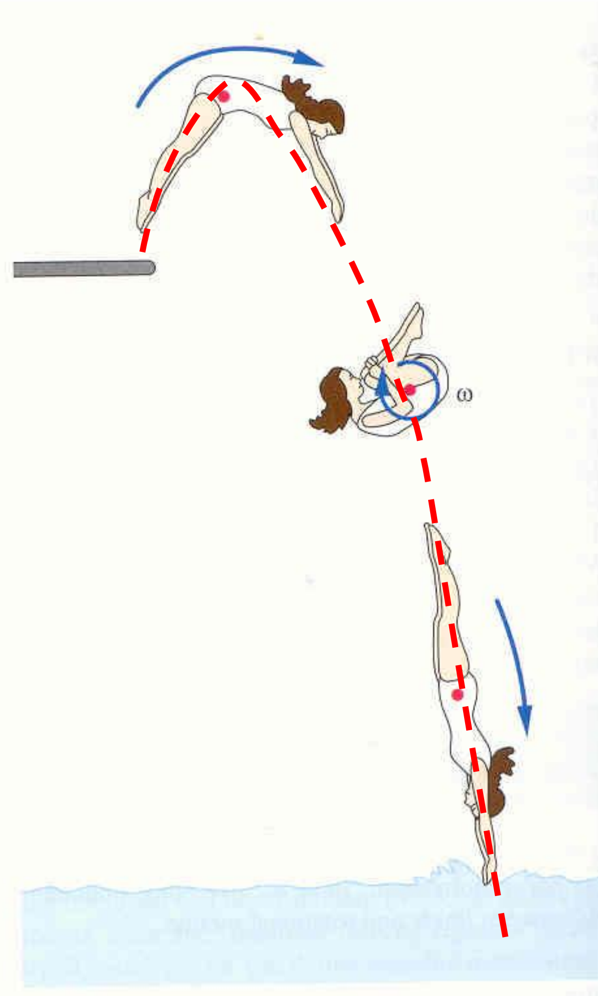
- Starts by pushing on ice - with both arms and then one leg fully extended.
- By pulling in arms and the extended leg closer to her body the skater's rotational velocity 'ω' increases rapidly.

Why?

- Her **angular momentum is conserved** as the external torque acting on the skater about the axis of rotation is very small.
- When both arms and 1 leg are extended they contribute significantly to the moment of inertia 'I'…
- This is because '**I**' depends on **mass distribution** and **distance^2** from **axis of rotation** ($I \sim m.r^2$).
- When her arms and leg are pulled in, her **moment of inertia reduces** significantly and to conserve angular momentum her **rotational velocity increases** (as $L = I.\omega$ = conserved).
- To **slow down** the skater simply **extends** her arms again…

Acrobatic Diving:

- Diver initially extends body and starts to rotate about **center of gravity.**
- Diver then goes into a "tuck" position by pulling in arms and legs to **drastically reduce** moment of inertia.
- **Rotational velocity** therefore **increases** as no external torque on diver .
- Before entering water diver extends body to reduce 'ω' again.



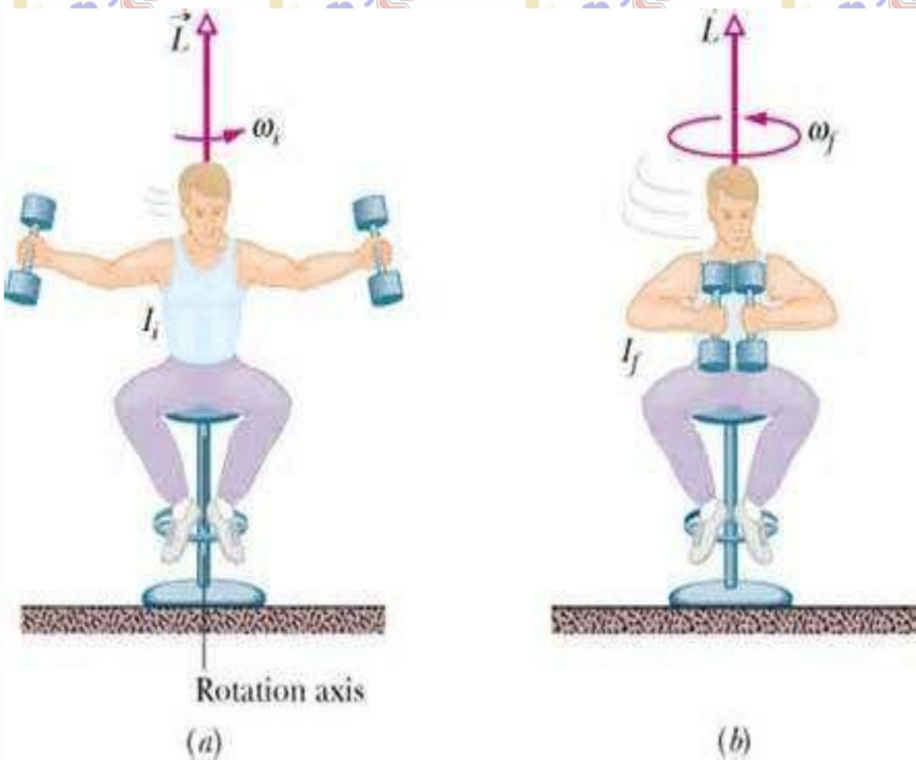
Her CG follows the same path as a projectile

Conservation of Angular Momentum

An object or system of objects will maintain its angular momentum unless acted upon by an unbalanced external torque.

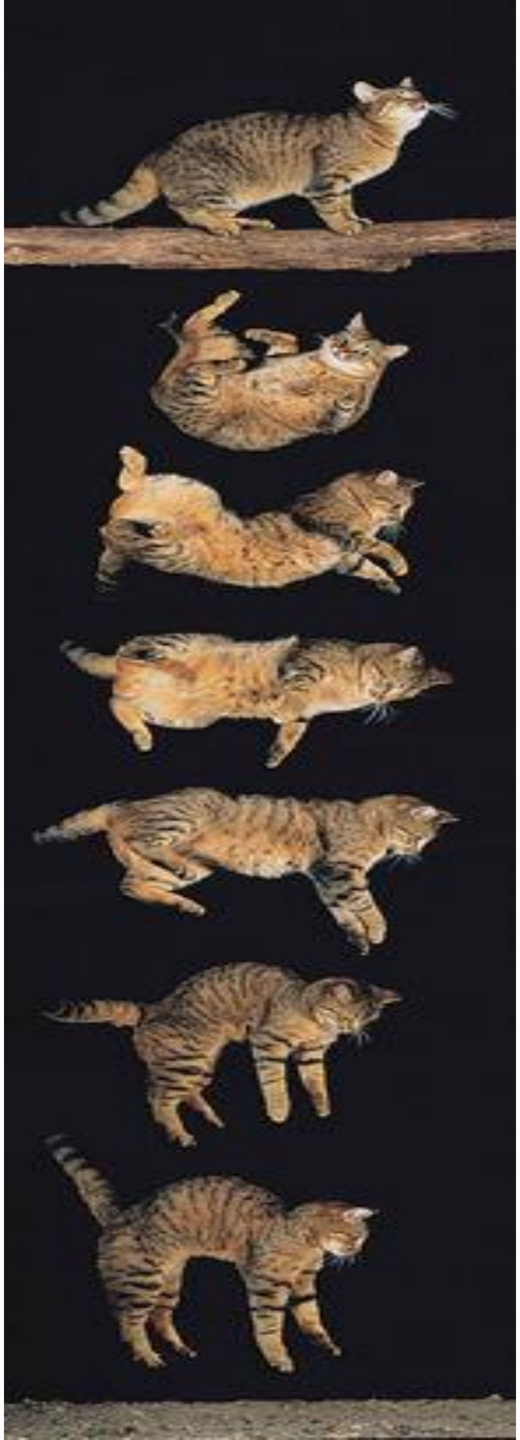
- So, if there is no net torque, ang mom is conserved.

DEMO: Sit on a rotating stool, holding weights away from you. Then pull the weights in – you go much faster! Your *I* decreases when you pull in the masses, and your ω compensates, to keep $I\ \omega$ constant.



A falling cat.

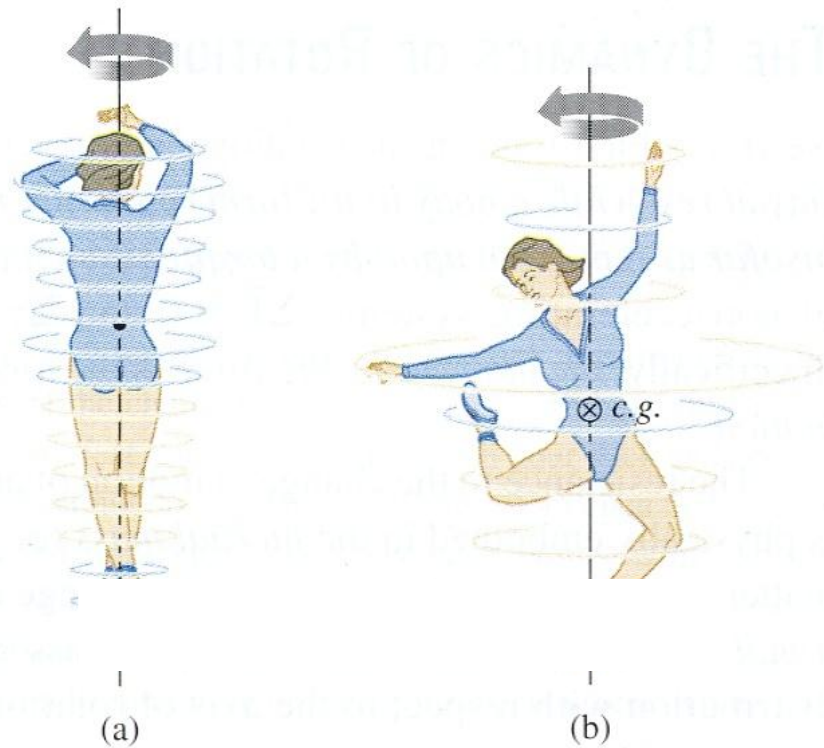
Cat begins to fall upside-down but rights itself by twisting yet conserving zero angular momentum: twist parts of its body in such a way that it rotates through 180 degrees but keeping zero ang mom!



Ice skater at Olympic games

Initial I = 3.5 kg.m^2,
Initial ω = 1.0 rev /s,

Final I = 1.0 kg.m^2,
Final ω = ?



As L is conserved:

$L_{final} = L_{initial}$

$I_f . \omega_f = I_i . \omega_i$

$$\omega_f = \frac{I_i . \omega_i}{I_f} = \frac{3.5 \times 1.0}{1.0}$$

ω_f **= 3.5 rev /s.**

Thus, for spin finish ω has increased by a factor of 3.5 times.

Relationship between Angular momentum and Angular velocity

$\tau = I\alpha = I\, d\omega/dt = d(I\omega)/dt$

But, $\tau = dL/dt$

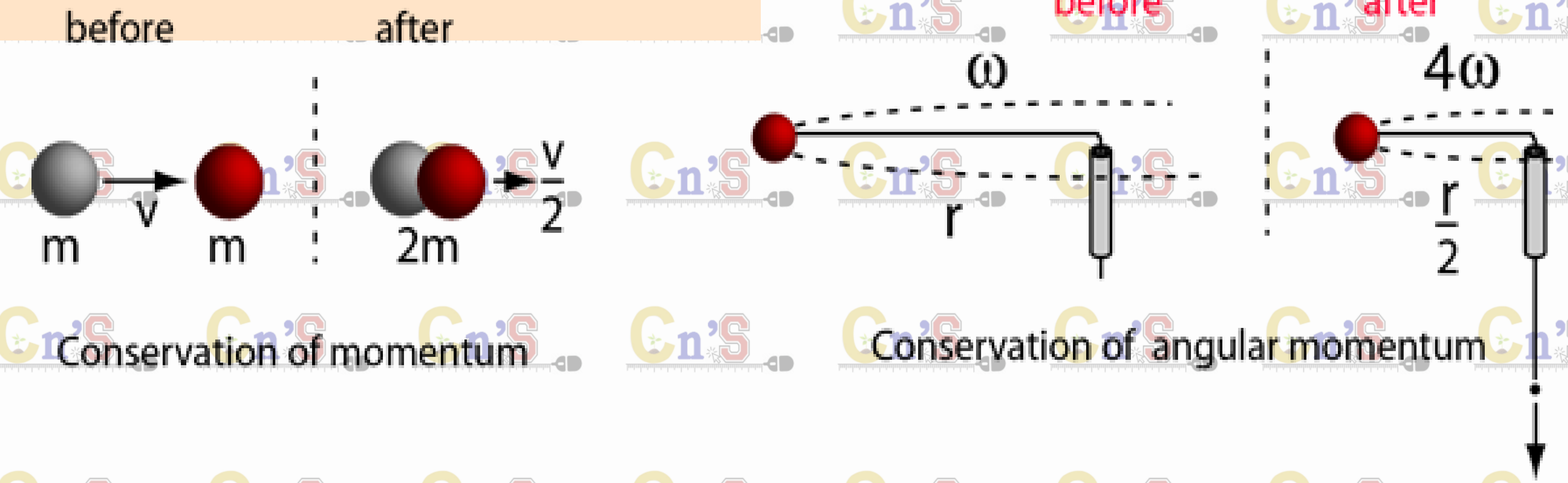
Thus, $dL/dt = d(I\omega)/dt$

By integrating both sides,

It can be shown that, $L = I\omega$

This is the rotational analogue of $P = mv$

Rotational-Linear Parallels



Conservation of linear momentum dictates that when a mass strikes an equal mass at rest and sticks to it, the combination must move at half the velocity, because the product of mass and velocity must remain constant.

Using a string through a tube, a mass is moved in a horizontal circle with angular velocity ω . If the string is pulled down so that the radius is half the original radius, then conservation of angular momentum dictates that the ball must have four times the angular velocity. This is because the product of moment of inertia and angular velocity must remain constant, and halving the radius reduces the moment of inertia by a factor of four.

Torque

The rate of change of angular momentum of an object in rotational motion is proportional to the external unbalanced torque. The direction of the torque also lies in the direction of the angular momentum.

Torque is called the moment of force and is a measure of the turning effect of the force about a given axis.

A torque is needed to rotate an object at rest or to change the rotational mode of an object.

$\tau = (I\omega - I\omega_0)/t$

$\tau = I\,[(\omega - \omega_0)/t]$

Therefore, $\tau = I\alpha$

By Newton's second law of motion

$F = ma$

$Fr = mra = mr\,(r\alpha)$ (as $a = r\alpha$)

$Fr = mr^2\,\alpha$

$Fr = I\alpha$ (as $I = mr^2$)

However, from the above derivation, $I\alpha = \tau$.

Therefore, $Fr = \tau$

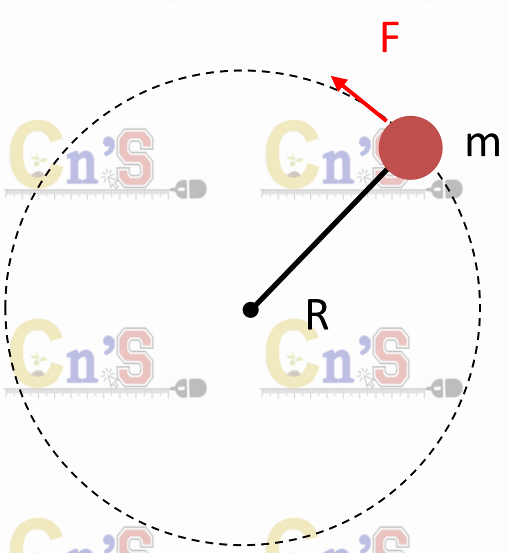
Thus, $\tau = Fr$

Torque and Angular Acceleration

Analogous to relation between F and a

$$F = ma, \qquad \tau = I\alpha$$

Moment of Inertia



$$\tau = FR = (ma_t)R = m(\alpha R)R$$

$$\tau = mR^2\alpha$$

Relationship between Torque and Angular Momentum-

$$\vec{\tau} = d\vec{L} / dt$$

$\vec{\tau}$ – Net torque

$\vec{L}$- Angular momentum

This result is the rotational analogue of Newton's second law:

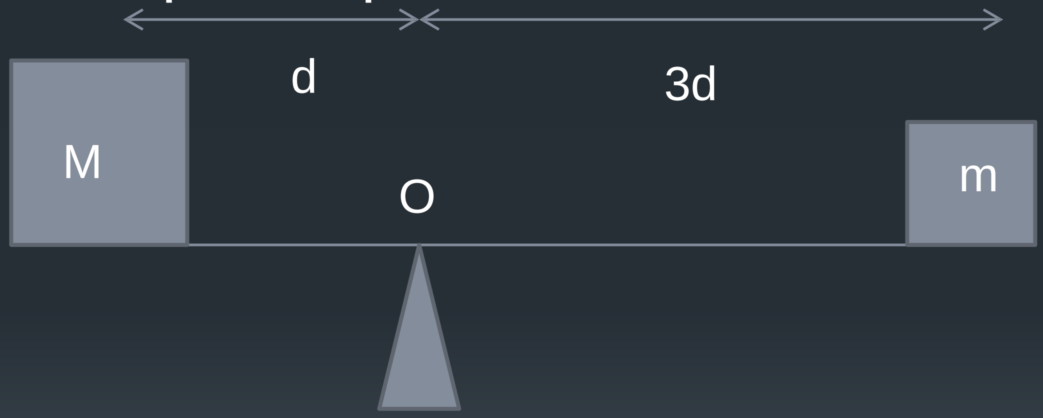
$$\vec{F} = d\vec{P} / dt$$

Applying Newton's laws to rotational motion

1. Newton's First law and equilibrium-

If the net torque acting on a rigid object is zero, it will rotate with a constant angular velocity.

Concept of equilibrium-



If the system is At equilibrium, total torque around O should be zero

Therefore, $\tau_M + \tau_m = 0$

$Mgd + [-(mg(3d))] = 0$

Therefore, $m = M/3$

Equilibrium of a rigid body

1. Transitional Equilibrium

For a body to be in transitional equilibrium, the vector sum of all the external forces on the body must be zero.

$$\vec{F}_{ext} = M\vec{a}_{cm}$$

and a_{cm} must be zero for transitional equilibrium.

2. Rotational Equilibrium

For a body to be in rotational equilibrium, the vector sum of all the external torques on the body about any axis must be zero.

$\tau_{ext} = I\alpha$ and α must be zero for rotational equilibrium.

2. Newton's second law of motion

The rate of change of angular momentum of an object in rotational motion is proportional to the external unbalanced torque. The direction of the torque also lies in the direction of the angular momentum.

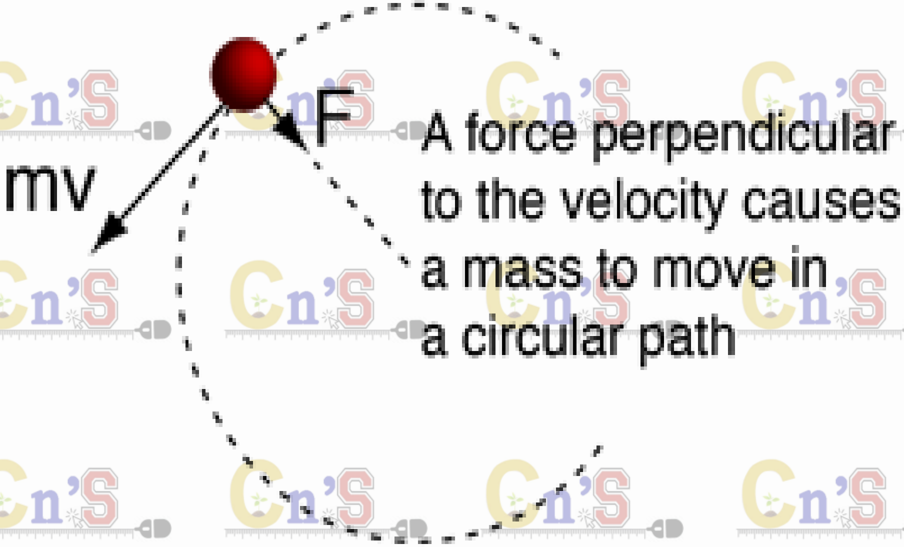
$\tau \;\alpha\; \alpha$

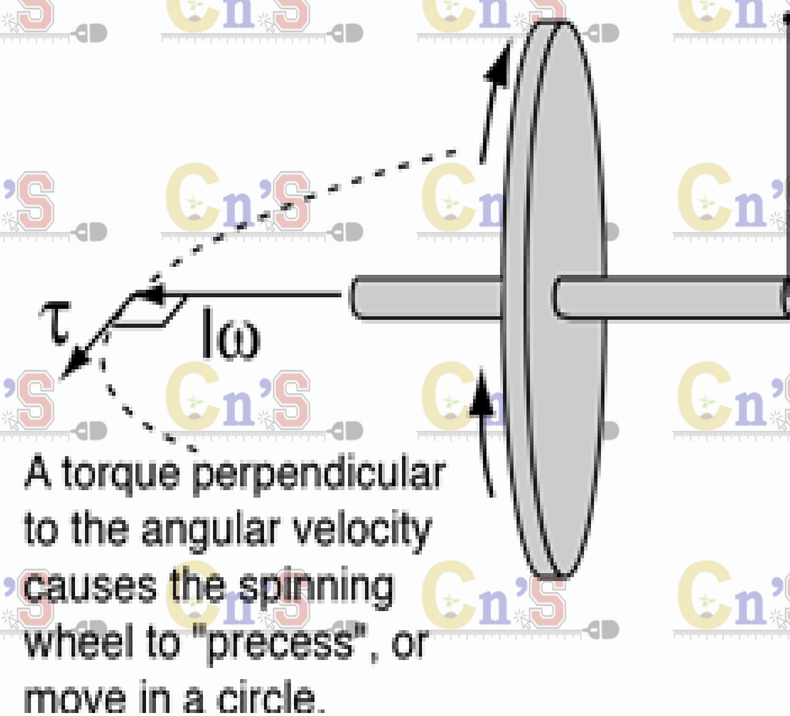
$\tau = I\alpha$

This is analogous to the linear equivalent, which states,

The rate of change of momentum is directly proportional to the external unbalanced force applied on an object and that force lies in the direction of the net momentum.

Thus, torque could be treated as the rotational analogue of the force applied on an object.


F
mv
A force perpendicular
to the velocity causes
a mass to move in
a circular path

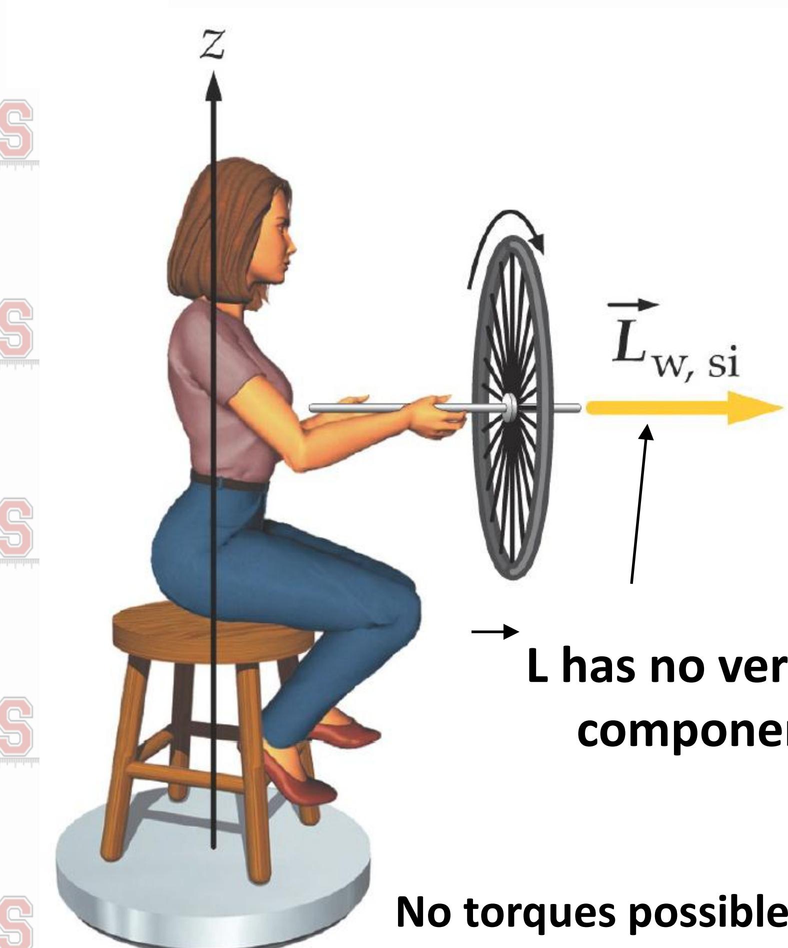

τ
Iω
A torque perpendicular
to the angular velocity
causes the spinning
wheel to "precess", or
move in a circle.

With the appropriate balance of force, a circular orbit can be produced by a force acting toward the center. Acting perpendicular to the velocity, it provides the necessary centripetal force to keep it in a circle.

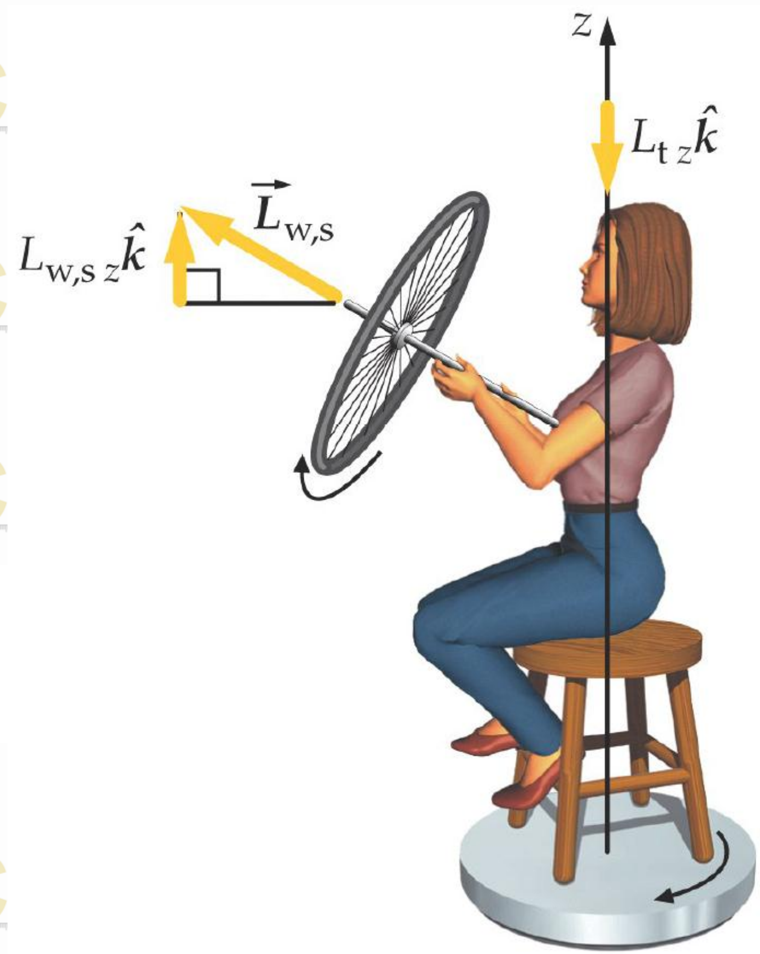
If a spinning wheel and axle is supported by one end of the axle, then the torque produced by the weight of the wheel and axle produces a torque that is perpendicular to the angular momentum of the wheel. This changes its direction but not its magnitude, causing the tip of the axle to trace out a circle. This is called precession, and is analogous to the orbit of a mass under a central force.

Conservation of angular momentum

Girl spins:
net vertical
component of L
still = 0

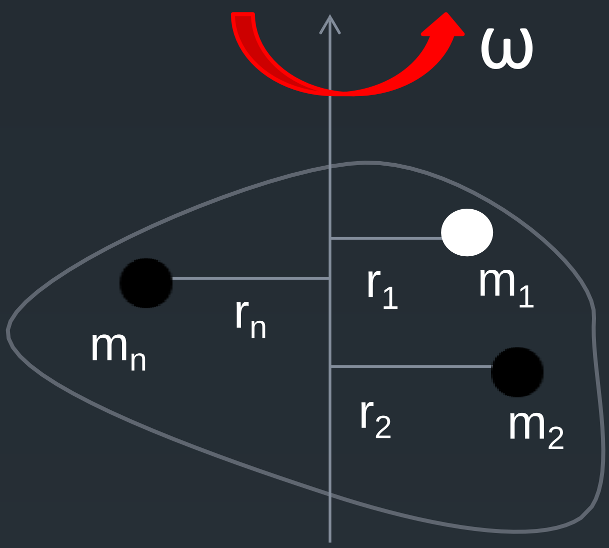


$\vec{L}$ has no vertical component



No torques possible
Around vertical axis➔
vertical component of L= const

Rotational Kinetic Energy



$E_k = \Sigma\, ½\, m_i v_i^2$

But, $v_i = r_i\omega$

So, $E_k = \Sigma\, ½\, m_i r_i^2\omega^2$

$= ½\, (\Sigma\, m_i r_i^2)\, \omega^2$

Therefore, $E_k = ½\, I\, \omega^2$

$$E_k = ½\, I\, \omega^2$$

Rotational kinetic energy is the rotational analogue of the translational kinetic energy, which is $E_K = ½\, mv^2$. In fact, rotational kinetic energy equation can be deduced by substituting $v = r\omega$ and viceversa.

Work done by a torque

$$dW = \tau\, d\theta$$

Therefore, the total work done in rotating the body from an angular displacement of θ_1 to an angle displacement θ_2 is

Therefore, $W = \tau\,[\theta_2 - \theta_1]$

Power

The rate at which work is done by a torque is called Power

$$P = dw/dt = \tau \, d\theta/dt = \tau \, \omega$$

Therefore $P = \tau \omega$

Work - Energy Principle

From $\omega^2 = \omega_0^2 + 2\alpha\theta$ and $W = \tau\theta$, it is clear that the work done by the net torque is equal to the change in rotational kinetic energy.

$\tau = I\alpha$ and $\omega^2 = \omega_0^2 + 2\alpha\theta$.

Therefore, $\omega^2 = \omega_0^2 + 2(\tau/I)\theta$.

Thus, $\omega^2 = \omega_0^2 + 2[(\tau\theta)/I]$.

Thus, $\omega^2 = \omega_0^2 + 2W/I$

OR

$W = \frac{1}{2} I (\omega^2 - \omega_0^2)$. This is called the work-energy principle.

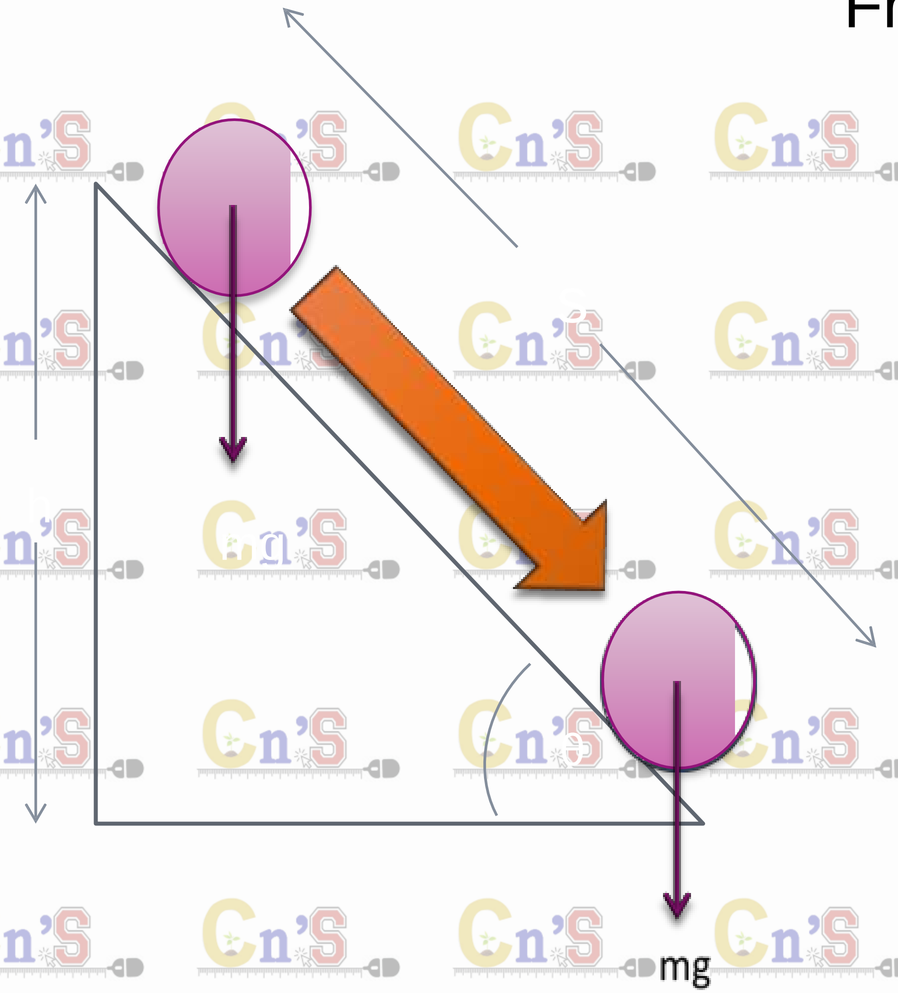
Kinetic energy of an object rolling without slipping:

$$E_{total} = E_{transitional} + E_{rotational}$$

$$E = ½\, mv^2 + ½\, I\omega^2$$

$$E = ½ m\omega^2 (R^2)$$

A body rolling down an inclined plane



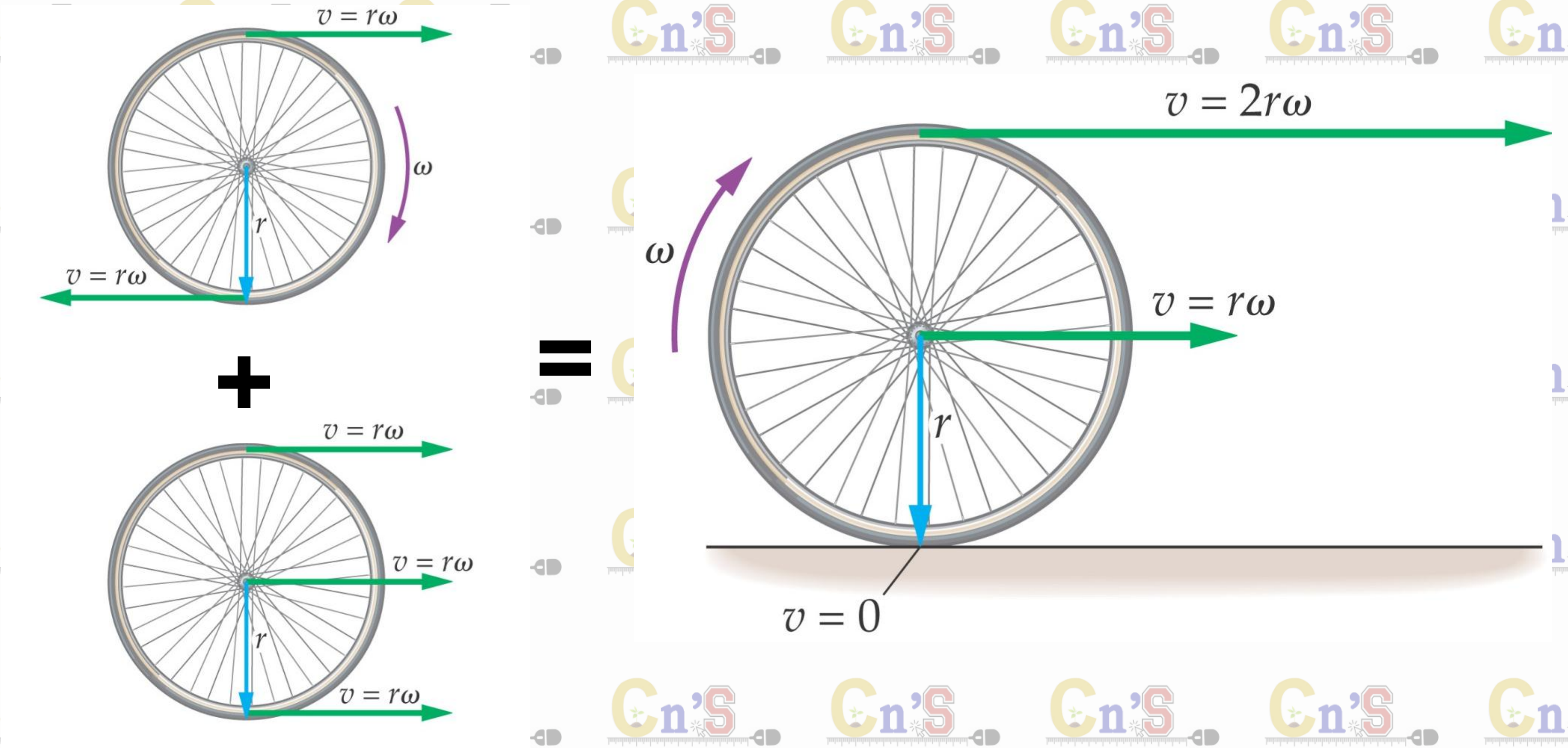
From the conservation of energy,

$$Mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

	Linear Motion	Rotational Motion	
Position	x	θ	Angular position
Velocity	v	ω	Angular velocity
Acceleration	a	α	Angular acceleration
Motion equations	$x = \bar{v}\, t$	$\theta = \bar{\omega} t$	Motion equations
	$v = v_0 + at$	$\omega = \omega_0 + \alpha t$	
	$x = v_0 t + \frac{1}{2}at^2$	$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$	
	$v^2 = v_0^2 + 2ax$	$\omega^2 = \omega_0^2 + 2\alpha\theta$	
Mass (linear inertia)	m	I	Moment of inertia
Newton's second law	$F = ma$	$\tau = I\alpha$	Newton's second law
Momentum	$p = mv$	$L = I\omega$	Angular momentum
Work	Fd	$\tau\theta$	Work
Kinetic energy	$\frac{1}{2}mv^2$	$\frac{1}{2}I\omega^2$	Kinetic energy
Power	Fv	$\tau\omega$	Power

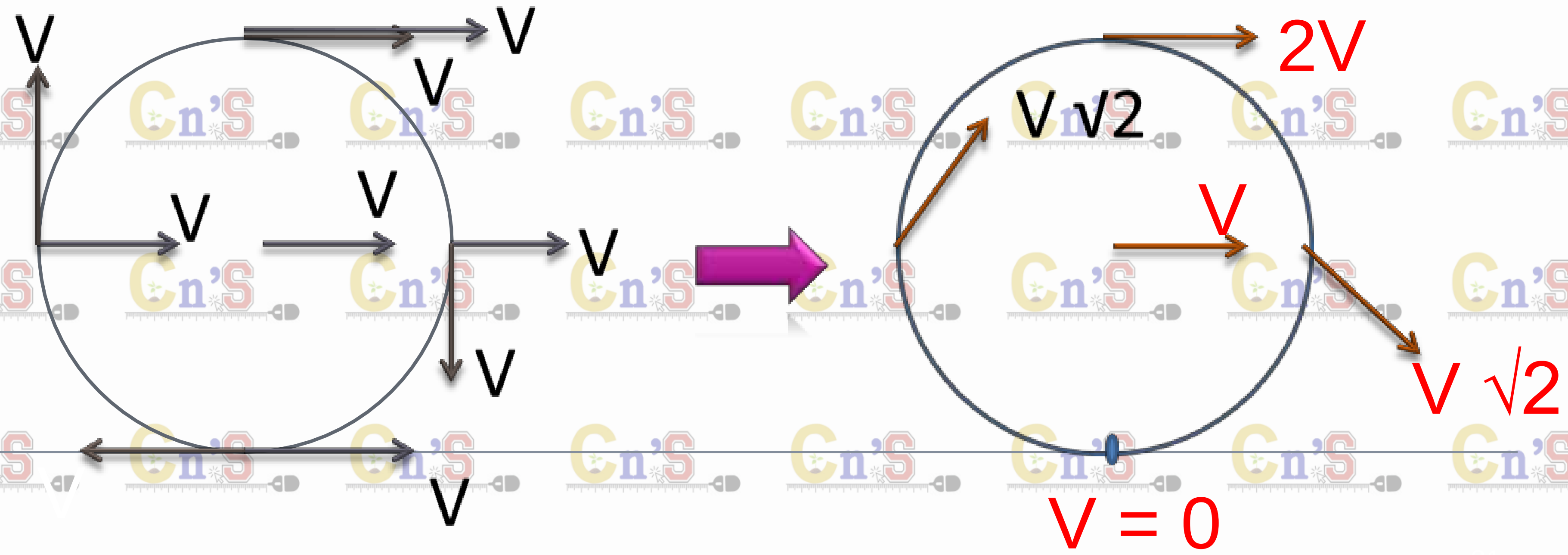
Rolling Motion

We may also consider rolling motion to be a combination of pure rotational and pure translational motion:

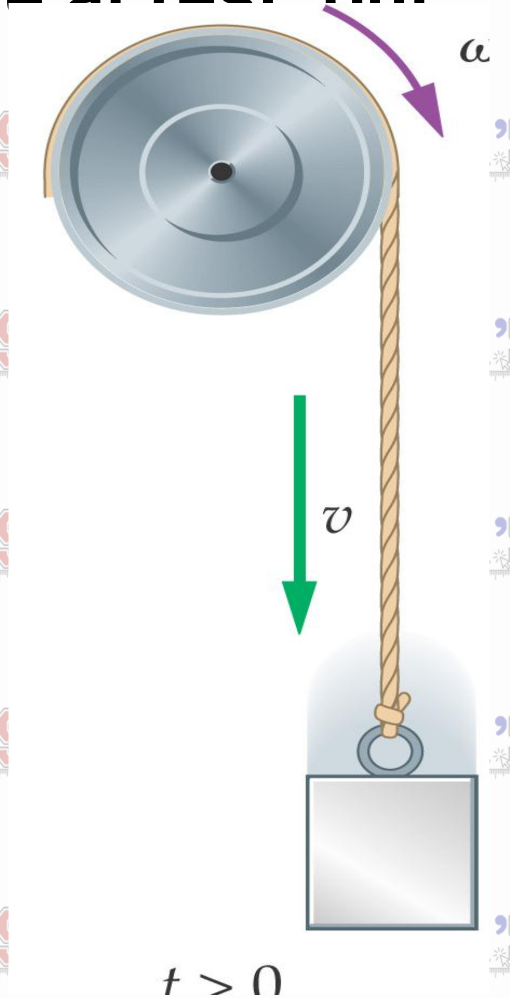


(b)

This two systems (rotational and transitional) can be combined together to understand how actually the sphere above moves in the plane. The final distribution shows clearly that in reality the ball always instantly experiences a zero velocity at the point of contact with the surface and the maximum velocity is at the top



A block of mass 1.5 kg is attached to a string that is wrapped around the circumference of a wheel of radius 30 cm and mass 5.0 kg, with uniform mass density. Initially the mass and wheel are at rest, but then the mass is allowed to fall.



What is the velocity of the mass after it falls 1 meter?

2.7m/s

Figure Skater

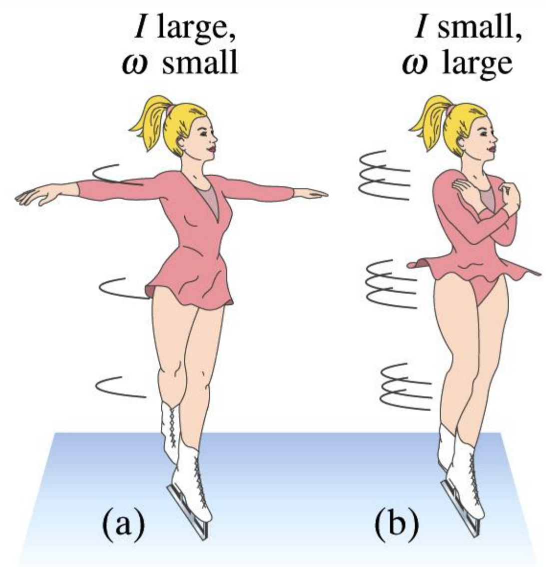
A figure skater spins with her arms extended. When she pulls in her arms, she reduces her rotational inertia and spins faster so that her angular momentum is conserved. Compared to her initial rotational kinetic energy, her rotational kinetic energy after she pulls in her arms must be

a) the same

b) larger because she's rotating faster

c) smaller because her rotational inertia is smaller

$KE_{rot} = 1/2\ I\ \omega^2 = 1/2\ L\ \omega$ (used $L = I\omega$). Since L is conserved, larger ω means larger KE_{rot}. The "extra" energy comes from the work she does on her arms.



Two Disks

Two different spinning disks have the same angular momentum, but disk 1 has more kinetic energy than disk 2.

Which one has the bigger moment of inertia?

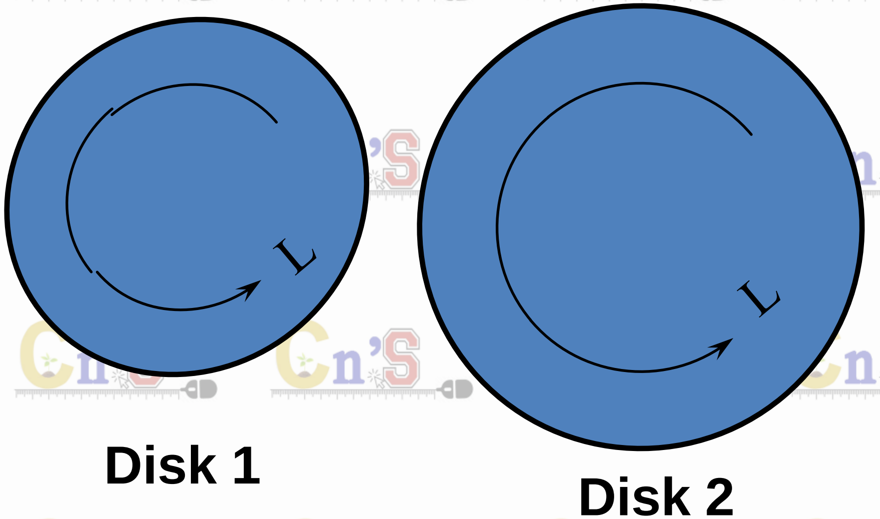
a) disk 1

b) disk 2

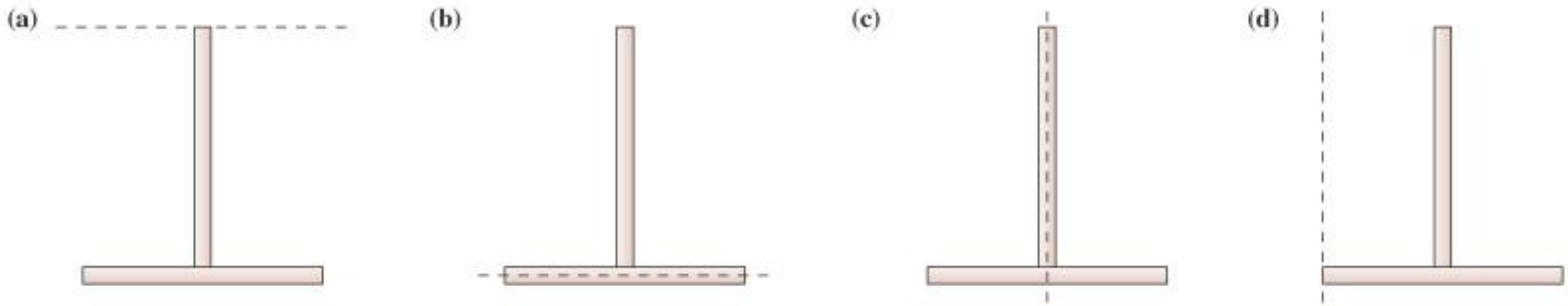
c) not enough info

$KE = 1/2\ I\ \omega^2 = L^2/(2\ I)$
(used $L = I\ \omega$).

Since L is the same, bigger I means smaller KE.

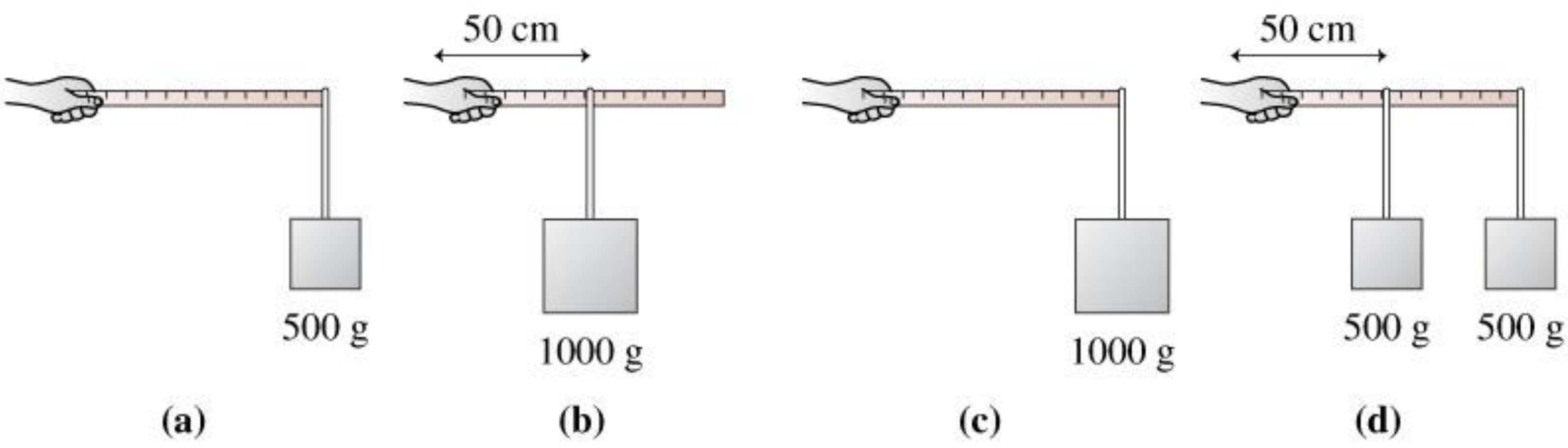


Four Ts are made from two identical rods of equal mass and length. Rank in order, from largest to smallest, the moments of inertia I_a to I_d for rotation about the dotted line.



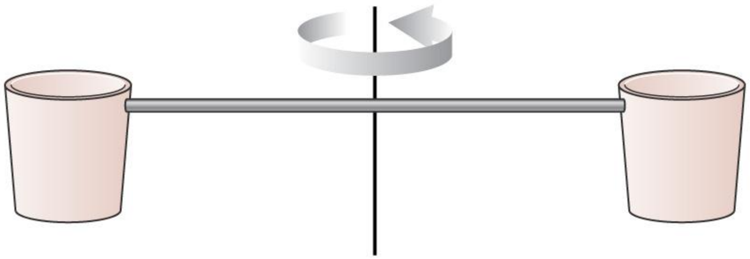
a. $I_c > I_b > I_d > I_a$
b. $I_c = I_d > I_a = I_b$
c. $I_a = I_b > I_c = I_d$
d. $I_a > I_d > I_b > I_c$
e. $I_a > I_b > I_d > I_c$

A student holds a meter stick straight out with one or more masses dangling from it. Rank in order, from most difficult to least difficult, how hard it will be for the student to keep the meter stick from rotating.



a. c > d > b > a
b. b = c = d > a
c. c > b > d > a
d. b > d > c > a
e. c > d > a = b

Two buckets spin around in a horizontal circle on frictionless bearings. Suddenly, it starts to rain. As a result,



a. The buckets slow down because the angular momentum of the bucket + rain system is conserved.

b. The buckets continue to rotate at constant angular velocity because the rain is falling vertically while the buckets move in a horizontal plane.

c. The buckets continue to rotate at constant angular velocity because the total mechanical energy of the bucket + rain system is conserved.

d. The buckets speed up because the potential energy of the rain is transformed into kinetic energy.

e. None of the above.

A rigid body is in equilibrium if

a. $\vec{F}_{\text{net}} = 0.$

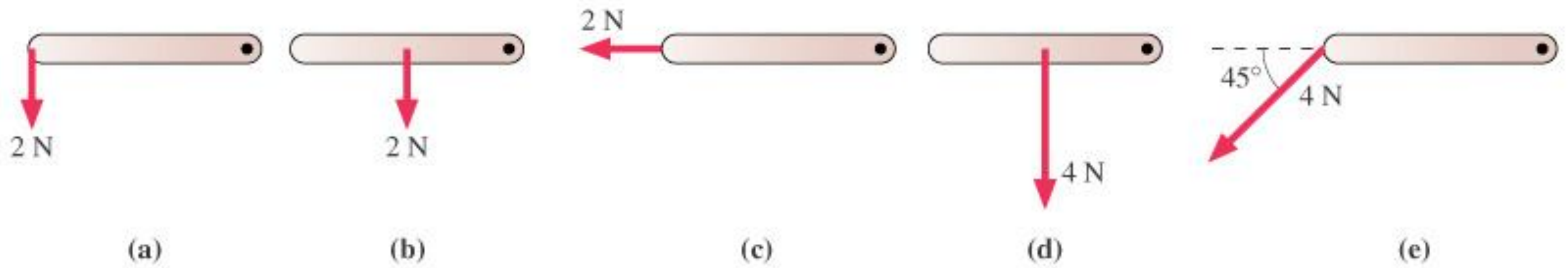
b. $\vec{\tau}_{\text{net}} = 0.$

c. neither 1 nor 2.

d. either 1 or 2.

e. both 1 and 2.

Rank in order, from largest to smallest, the five torques $\tau_a - \tau_e$. The rods all have the same length and are pivoted at the dot.



$\tau_e > \tau_a = \tau_d > \tau_b > \tau_c$

$\tau_d = \tau_e > \tau_a = \tau_b = \tau_c$

$\tau_d > \tau_e > \tau_a = \tau_b > \tau_c$

$\tau_d = \tau_e > \tau_d = \tau_b > \tau_c$

$\tau_e > \tau_a > \tau_d > \tau_b > \tau_c$