Undergraduate Mathematics Research: The Road To Graduate School

Gina Pantano*
University of Tampa, Tampa, FL, 33606

The inverse-scattering transform is a method for solving nonlinear partial differential equations and is an active area of research in pure and applied mathematics. In this study, I attempt to learn this method to improve my overall mathematical proficiency as an undergraduate student, and to identify the essential skills sought out for in physics graduate student applicants. The purpose of the study is to help bridge the gap between undergraduate and graduate education and promote involvement in undergraduate mathematics research. My research advisor, Dr. Morgan McAnally, conducted a series of two assessments and five evaluations throughout the summer. The initial and final assessments scored my mathematical proficiency in the areas of linear algebra, real and complex analysis, and linear and nonlinear partial differential equations to show an evident change in my overall knowledge. The evaluations and assessments determine and explain how I gained non-cognitive and mathematical skills from proving a set of problems. I analyzed the data collected as either an increase in my mathematical proficiency, non-cognitive skills, or both. I found a significant increase in my mathematical proficiency and three non-cognitive skills: self-learning, resilience, and self-motivation. Based on our study, undergraduate mathematics research has numerous benefits to students including close mentorship with a professor, capacity to think critically and abstractly, and the ability to lessen the educational gap between undergraduate and graduate school.

I. Introduction

In summer 2020, I received the Summer Undergraduate Research Fellowship (SURF) from the University of Tampa to support my research with Dr. Morgan McAnally. During the ten-week program, we studied a method in mathematics known as the inverse-scattering transform. The goal of our research was to examine how learning this method aided in the development of my mathematical proficiency as an undergraduate student, specifically in the areas of linear algebra, real and complex analysis, and linear and nonlinear partial differential equations. We wanted to identify the skills I obtained from participating in undergraduate mathematics research in hopes to bridge the gap between undergraduate and graduate education. This paper seeks to answer these questions: what are the qualities that distinguish undergraduate students from graduate students, and why should students and faculty be participating in undergraduate pure and applied mathematics research?

In a report [1] by the National Research Council, mathematical proficiency is defined by five components known as the five strands of mathematical proficiency.

- conceptual understanding — The comprehension of mathematical concepts, operations, and relations.
- procedural fluency — The skill in carrying out procedures flexibly, accurately, efficiently, and appropriately.
- strategic competence — The ability to formulate, represent, and solve mathematical problems.
- adaptive reasoning — The capacity for logical thought, reflection, explanation, and justification.
- productive disposition — The habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy.

Educators can use the five strands of mathematical proficiency to evaluate a student's success in an area of mathematics. In this paper, we will use this definition of mathematical proficiency to develop assessments of my skills in the areas of linear algebra, real and complex analysis, and linear and nonlinear partial differential equations before and after my fellowship. The first objective was to establish the differences between undergraduate mathematics research and undergraduate research in the natural and social sciences. In 2006, the Mathematical Association of America (MAA)

*Department of Physics and Mathematics, University of Tampa.

published a report [2] on the costs and benefits of participating in undergraduate mathematics research for faculty and institutions. Their goal was to provide a framework for universities wanting to participate in undergraduate mathematics research to follow. The MAA defined undergraduate mathematics research by the following characteristics:

- The student is engaged in original work in pure or applied mathematics.
- The student understands and works on a problem of current research interest.
- The activity simulates publishable mathematical work even if the outcome is not publishable.
- The topic addressed is significantly beyond the standard undergraduate curriculum.

The MAA report [2] also highlighted the key differences between undergraduate research in mathematics versus research in the natural and social sciences. Researchers in the natural and social sciences commonly perform physical or social experiments in which the students collect and analyze data to produce results. This process usually requires several research and staff assistants and significantly more funding compared to research in mathematics. Student researchers in mathematics usually do not perform experiments or collect any data. Mathematics research is introspective in nature and does not have immediate implications unlike most research in the natural and social sciences. The method of solving for particular solutions is more important than the actual results themselves. Mathematics is distinct from the natural and social sciences because mathematicians seek to use critical thinking and logic to prove and understand relationships between variables and equations. Scientists use experimental and observational data as a way to provide an understanding of natural phenomena. Mathematics research can be categorized as pure or applied mathematics. The goal of research in pure mathematics is to develop novel theorems. Applied mathematics applies these new theorems and methods to problems of practical interest. Mathematics teaches students to be persistent, to question deeper, and think critically and creatively to prove problems. To fully comprehend a mathematical paper, mathematicians have to use deductive reasoning to understand how the author arrives at each equation. Thus, mathematics researchers become independent learners early on as undergraduate students which helps them prepare for graduate school.

Additionally, research in the natural and social sciences do not require the same amount of hands-on mentorship needed for research in mathematics. Students usually assist faculty in completing one of their current projects and are capable of conducting research independently. Undergraduate research projects in mathematics are tailored specifically to the student. The faculty mentor spends a significant portion of their time creating the project, guiding the student, and teaching the mathematics along the way. Although students have the benefit of working under the close surveillance of a professor, the faculty member will have less time to work on their own research projects. In the mathematics journal *Involve* [3], one of the articles published discusses the challenges of promoting undergraduate mathematics research and ways to combat these issues. Faculty, similarly to the natural and social sciences, should produce a project related to their own research interests for the student and have the intention of the project being published. This will increase the chance of the research project providing something of real value to the faculty advisor and the student as well as the chance to publish original work. Faculty will also receive the benefit of gaining skills as an educator working closely with an individual student. Professors will learn to understand students' individual strengths and weaknesses and be able to produce problems suitable to their level of understanding.

Undergraduate research in mathematics is cost-effective since most projects do not require expensive lab equipment. Universities have enough access to computer labs and programs for students to be able to construct models and run simulations if needed. As an example, the University of Tampa has a license for Mathematica for all of the main computer labs on campus. Students are able to run the program through their school accounts.

II. Literature Review

The second objective in this project was to identify the skills and qualities graduate admission committees look for in applicants. I am particularly interested in Ph.D. granting programs for physics because of my personal research interests, the broad range of career opportunities available post graduation, and it being a common path for undergraduate mathematics students. A study presented at the 2019 Physics Education Research Conference [4] sought to identify these desired qualities and the sources faculty use to determine if the applicant demonstrates them. The researchers interviewed seventeen faculty members from four institutions who serve or have served on their programs graduate admission committees. The results were that the faculty members looked for a theme of independence in candidates. This consisted of mainly three non-cognitive skills: self-learning, resilience, and self-motivation. Faculty members also look for research specific qualities including programming and research skills. Based on the study and the definitions of the three skills, Dr. McAnally and I planned to identify an increase in these soft skills through her evaluations after I proved each given set of problems. An increase in these non-cognitive skills will show that pure and applied mathematics research helps to bridge the gap between undergraduate and graduate education.

The last objective for this project before conducting research was to find previous studies that surveyed undergraduate students to identify the benefits they received from participating in similar programs. The goal was to use these surveys to defend my findings. However, the studies were either out of date or lacked representation of undergraduate mathematics researchers. Although the studies I found were promising and showed similar benefits I received from SURF, the surveys sample sizes were too small and out of date to include with my findings. Thus, it is vital to add to the literature since undergraduate mathematics research has dramatically changed over the years and provides substantial benefits to students. Students have more opportunities to access quality information and utilize computer programs to conduct research beyond the undergraduate curriculum.

III. Methods

A. Methodology

Our research consisted of an in-depth study of the peer-reviewed article "What is the Inverse-Scattering Transform" and chapter two of the book *Nonlinear Waves in Integrable and Non-integrable Systems* by Peter Miller and Jianke Yang. The author of the article, Peter Miller, does extensive research on nonlinear equations at the University of Michigan. Miller provides a great introduction to using the inverse-scattering transform on the defocusing Nonlinear Schrödinger Equation. The article focuses on the formulation of the Riemann-Hilbert problem, which is the final step in solving for the wave solution of these nonlinear equations and is often referenced as the inverse-scattering problem. The second source we read was chapter two in the book by Yang, “Integrable Theory for the Nonlinear Schrödinger Equation”, which focuses on the execution of the Riemann-Hilbert problem and shows the wave solutions for the focusing Nonlinear Schrödinger Equation.

Dr. McAnally first conducted an initial assessment of my mathematical proficiency skills in the areas of linear algebra, real and complex analysis, and linear and nonlinear partial differential equations at the beginning of my fellowship. This served as a baseline to demonstrate how my mathematical proficiency skills increased in these areas over the course of my fellowship. The first few weeks were spent understanding deductive reasoning and reading the Miller paper together. Dr. McAnally provided notes on the assumptions and reasons implied by each line in the Miller article, which helped me to comprehend chapter two in the book by Yang. After finishing reading the Miller article, Dr. McAnally assigned sets of problems each week from the book for me to prove. I collected the data through my self-reflections and evaluations done by Dr. McAnally on my performance after proving each set of problems. From the results, I constructed a list of what I learned from the process of working on problems. Dr. McAnally performed a total of five evaluations over the course of the summer.

I analyzed the data collected by using the definitions of the five strands of mathematical proficiency and the three non-cognitive skills identified in the study on the ideal physics graduate student. I categorized each point as either an increase in my mathematical proficiency, soft skills, or both. At the end of my fellowship, Dr. McAnally took a final assessment of my mathematical proficiency in the same areas to compare with my initial assessment. I summarized the ways my non-cognitive and mathematical skills improved based on our results.

B. The Inverse-Scattering Transform

Dr. McAnally and I researched the inverse-scattering transform due to its rich history in mathematics, broad range of applications, and cross-disciplinary aspects. To this day, there still does not exist a general analytical method for solving nonlinear partial differential equations (NPDEs), so each equation must be treated individually. However, the inverse-scattering transform was one of the first methods in solving an entire class of NPDEs known as integrable systems. The method was discovered in 1967 and led to a new wave of mathematics studying the solutions to these particular evolution equations [5]. NPDEs have a broad range of applications since nature is inherently nonlinear. A simple case would be the nonlinear quadratic drag equation in classical mechanics that accounts for the drag force on a moving mass. Due to the importance of nonlinear equations in science and engineering, the inverse-scattering transform remains an active area of research in pure and applied mathematics. The inverse-scattering transform crosses several disciplines in mathematics including linear algebra, real and complex analysis, and linear and nonlinear partial differential equations. Exposure to numerous fields in mathematics leads to a greater conceptual understanding and ability to think critically and logically. Thus, the inverse-scattering transform is an excellent method for undergraduate students to learn due to the non-cognitive and attitudinal gains.

The inverse-scattering transform is a method of solving NPDEs similar to the Fourier transform that solves linear differential equations. Linear partial and ordinary differential equations (LPDEs and LODEs) obey the superposition principle that states any linear combination of its solutions is a solution to the equation unlike NPDEs. The inverse-scattering transform is a three step process. The process starts with an initial condition and boundary conditions of an integrable nonlinear evolution equation which reduces to a linear problem at the boundary. An evolution equation has at least one time dependent term of the unknown function. The equation we studied was the focusing Nonlinear Schrödinger Equation (NLS) with the initial condition $u(x, 0) = u_0$ and the boundary conditions $u \to 0$ as $x \to \pm\infty$. The NLS equation represents the propagation of waves through a nonlinear medium [6],

$$iu_t + u_{xx} + 2|u|^2 u = 0. \tag{1}$$

For example, telecommunication companies that use fiber optic services, such as Frontier FiOS, carry information by infrared light inside nonlinear glass optical fibers whose motion is described by the solutions to equation (1). The nonlinear effects come from the refraction index of the glass in which the total internal reflection of the light occurs where no light is able to escape the optical fiber [6]. Unique solutions such as solitons, waves that can pass through each other without changing shape or speed, can double the amount of information carried. Equation (1) can be written equivalently as the following spectral problem:

$$Y_x = \begin{bmatrix} -i\lambda & u \\ -u^* & i\lambda \end{bmatrix} Y = (-i\lambda\Lambda + Q)Y, \tag{2}$$

$$Y_t = \begin{bmatrix} -2i\lambda^2 + i|u|^2 & iu_x + 2\lambda u \\ iu_x^* - 2\lambda u^* & 2i\lambda^2 - i|u|^2 \end{bmatrix} Y = (-2i\lambda^2\Lambda + V)Y, \tag{3}$$

where Λ =diag$(-1, 1)$, λ is the spectral parameter, and u^* is the complex conjugate of u. The matrix Q is dependent on the solution u shown by

$$Q = \begin{bmatrix} 0 & u \\ -u^* & 0 \end{bmatrix}. \tag{4}$$

Similarly, V is dependent on the solution u and the spectral parameter λ given by

$$V = \begin{bmatrix} i|u|^2 & 2\lambda u + iu_x \\ -2\lambda u^* + iu_x^* & -i|u|^2 \end{bmatrix}. \tag{5}$$

The equations (2) and (3) are LODEs, which are easier to analyze and solve. Taking the time derivative of equation (2) and the spatial derivative of equation (3), we can equate them to get equation (1) which is known as the compatibility condition. We want to analyze the solution to equation (2) under the given boundary condition $|u| \to 0$ as $x \to \pm\infty$. The first step in solving equation (1) is known as the direct scattering problem. Analyzing the far right representation of equation (2), we get the following spatial LPDE:

$$J_x = -i\lambda[\Lambda, J] + QJ. \tag{6}$$

We can construct two eigenfunction solutions known as Jost solutions J_+ and J_- to the associated spatial LPDE, equation (6), using the boundary conditions as $x \to \pm\infty$. We can show the $\det(J_+) = \det(J_-) = \mathbb{I}$ through their asymptotics at large distances ($J\pm \to \mathbb{I}$ as $x \to \pm$) and applying Abel's identity [7] and boundary conditions to equation (2). This means the column vectors of J_+ are linearly independent similar to the column vectors of J_-. However, obeying the boundary and initial conditions, the uniqueness of solutions implies J_+ and J_- must be linearly dependent since they are both solutions to equation (2). The two solutions depend on each other by the scattering matrix $S(\lambda, 0)$:

$$J_- = J_+ESE^{-1}, \tag{7}$$

where $E(x, \lambda) = e^{-i\lambda\Lambda x}$. Using the analytical properties of the Jost solutions, we can find the analytical properties of the scattering matrix retrieving our initial scattering data. The second step is finding the time evolution of the scattering data, $S(\lambda, t)$. This is done by recognizing the equation $J_{-}E = J_{+}ES$ is a solution to the associated temporal LODE, equation (8). The temporal equation (8) is retrieved similarly to equation (6),

$$J_t = -2i\lambda^2[\Lambda, J] + VJ. \tag{8}$$

By inserting $J_{-}E = J_{+}ES$ into equation (8), we find the following differential equations where s_{11}, s_{12}, s_{21}, and s_{22} are the matrix elements of $S(\lambda, t)$:

$$\frac{\partial \hat{s}_{22}}{\partial t} = \frac{\partial s_{22}}{\partial t} = 0, \tag{9}$$

and

$$\frac{\partial \hat{s}_{12}}{\partial t} = -4i\lambda^2 \hat{s}_{12}, \quad \frac{\partial s_{12}}{\partial t} = -4i\lambda^2 s_{21}. \tag{10}$$

These equations give us the time dependence of the scattering data. Next, we introduce the final step which is solving the Reimann-Hilbert problem or the inverse-scattering problem using the scattering data. We can reconstruct $u(x, t)$ from the matrix Q using the asymptotic expansion of its solution at large λ [7]:

$$P^{\pm}(x, \lambda) = \mathbb{I} + \lambda^{-1} P_1^{\pm}(x) + O(\lambda^{-2}), \tag{11}$$

where $O(\lambda^{-2})$ is the sum of terms with λ to the second order. P^+ represents the elements of the Jost solutions that are analytic in the upper half complex plane, and P^- represents the elements of the Jost solutions that are analytic in the lower half complex plane in the following way,

$$P^+ = J_{-}H_1 + J_{+}H_2, \tag{12}$$

$$P^- = H_1 J_{-}^{-1} + H_2 J_{+}^{-1}, \tag{13}$$

where $H_1 = \mathrm{diag}(1, 0)$ and $H_2 = \mathrm{diag}(0, 1)$. The Riemann-Hilbert problem is a problem of complex analysis in which an unknown piecewise-analytic function is to be reconstructed by discontinuity data given along the contour in the complex plane [7]. The discontinuities we discuss are a collection of point singularities or "zeros" that occur when the $\det(P^{\pm}) = 0$ [7]. The regular Reimann-Hilbert problem is presented as the following equation.

$$P^-(x, \lambda) P^+(x, \lambda) = G(x, \lambda). \tag{14}$$

The Reimann-Hilbert problem is said to be regular when $\det(P^{\pm}) \neq 0$ and is nonregular if $\det(P^{\pm}) = 0$. $P^{\pm}$ are related to the Jost solutions J^+ and J^-. The matrix G is related to the scattering matrix by the following form:

$$G = E \begin{bmatrix} 1 & \hat{s}_{12} \\ s_{21} & 1 \end{bmatrix} E^{-1}.$$

The nonregular Reimann-Hilbert problem can be constructed as a regular Reimann-Hilbert problem using the following equations [7]:

$$P^+(\lambda) = \hat{P}^+(\lambda)\Gamma(\lambda), \tag{15}$$

$$P^-(\lambda) = \Gamma^{-1}(\lambda)\hat{P}^-(\lambda), \tag{16}$$

where,

$$\Gamma(\lambda) = \mathbb{I} + \sum_{j,k=1}^{N} \frac{v_j (M^{-1})_{jk} \bar{v}_k}{\lambda - \bar{\lambda}_k}, \tag{17}$$

$$\Gamma^{-1}(\lambda) = \mathbb{I} - \sum_{j,k=1}^{N} \frac{v_j (M^{-1})_{jk} \bar{v}_k}{\lambda - \lambda_j}, \tag{18}$$

and M is an NxN matrix with its (j,k)th element given by

$$M_{jk} = \frac{\overline{v}_j v_k}{\overline{\lambda}_j - \lambda_k}, \quad 1 \le j, k \le N, \tag{19}$$

$$det\Gamma(\lambda) = \prod_{k=1}^{N} \frac{\lambda - \lambda_k}{\lambda - \overline{\lambda}_k}, \tag{20}$$

and $\hat{P}^{\pm}(\lambda)$ is the unique solution to the following regular Riemann-Hilbert problem:

$$\hat{P}^{-}(\lambda)\hat{P}^{+} = \Gamma(\lambda)G(\lambda)\Gamma^{-1}(\lambda), \quad \lambda \in \mathbb{R}, \tag{21}$$

where $\hat{P}^{\pm}(\lambda)$ are analytic in $\mathbb{C}_{\pm}$ and $\hat{P}^{\pm}(\lambda) \longrightarrow \mathbb{I}$ as $\lambda \longrightarrow \infty$. Thus, equation (14) when $\det(P^{\pm}) = 0$ can be written as the following regular Riemann-Hilbert problem:

$$(\hat{P}^{+})^{-1} - P^{-} = \hat{G}(\hat{P}^{+})^{-1} \tag{22}$$

where,

$$\hat{G} = \mathbb{I} - G = E \begin{bmatrix} 0 & \hat{s}_{12} \\ s_{21} & 0 \end{bmatrix} E^{-1}.$$

Using the Plemelj formula [7], we can get the solution to the nonregular Riemann-Hilbert problem, equation (14) when the $\det(P^{\pm}) = 0$, where

$$(\hat{P}^{+})^{-1}(\lambda) = \mathbb{I} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\Gamma(\xi)\hat{G}(\xi)\Gamma^{-1}(\xi)(\hat{P}^{+})^{-1}(\xi)}{\xi - \lambda} d\xi. \tag{23}$$

By expanding the solution of P^{+} with equation (11), we are able to show that Q is related to P^{+} by the following formula:

$$Q = i[\Lambda, P_1^{+}] = -i[\Lambda, P_1^{-}]. \tag{24}$$

Using equation (24), we see that the solution u is related to P^{+} by the following equation,

$$u = 2i(P_1^{+})_{12} = -2i(P_1^{-})_{12}. \tag{25}$$

From the solution of the Reimann-Hilbert problem, equation (23), we know that P_1^{+} is equal to

$$P_1^{+} \longrightarrow \sum_{j,k=1}^{N} v_j \left(M^{-1}\right)_{jk} \bar{v}_k, \quad as \;\; t \longrightarrow \infty. \tag{26}$$

The single row vector $\bar{v}_k$ is the kernel of P^{-} at certain discrete locations where the $\det(P^{-}) = 0$. The row vector is related to the column vector v_k, which is the kernel of P^{+}, by $\bar{v}_k = v_k^{\dagger}$. Using equation (26), we arrive at the solution

$$u(x,t) \longrightarrow 2i\left(\sum_{j,k=1}^{N} v_j \left(M^{-1}\right)_{jk} \bar{v}_k\right) \quad as \;\; t \longrightarrow \infty. \tag{27}$$

The process of the inverse-scattering transform can be represented by the following diagram.

$$
\begin{tikzcd}
u(x,0) \arrow[r, "\text{Direct Scattering Problem}"] \arrow[d, "\text{Solution to NPDE}"'] & S(\lambda,0) \arrow[d, "\text{Time Evolution of Scattering Data}"] \\
u(x,t) & S(\lambda,t) \arrow[l, "\text{Inverse-Scattering Problem}"]
\end{tikzcd}
$$

To summarize, the process starts with an initial condition and boundary conditions of an integrable nonlinear evolution equation to reduce it to a linear problem by analyzing the solution at the boundary. Subsequently, the direct scattering problem is solved yielding the initial scattering data in the complex plane that is linearly related to the solutions of equation (2). We can find the time evolution of the scattering data through equation (3) which gives the differential equations for each matrix element in the scattering matrix, S. The last step is to solve the Riemann-Hilbert problem or inverse-scattering problem using the Plemelj formula and manipulating the solution matrix Q.

C. Evaluations and Self-Reflections

The problems I mainly had to prove dealt with relations and the properties of matrices. The time of proving each set of problems varied between a day to a couple of weeks. An example of a set of problems I was tasked with analyzing includes the following: show that the Lax pair becomes the form (28) and (29), and that Q and V are anti-Hermitian and traceless, equation (30).

$$J_x = -i\lambda[\Lambda, J] + QJ \tag{28}$$

$$J_t = -2i\lambda^2[\Lambda, J] + VJ \tag{29}$$

$$Q^\dagger = -Q, \quad V^\dagger = -V \tag{30}$$

This particular set of problems took approximately a day to complete. After proving the set of problems, I conducted a self-reflection shown in Table 1 by asking two questions: what did I learn in a mathematical sense from proving these problems, and what skills did I learn as a student researcher? I created a brief bulleted list summarizing my experience, and reviewed the reflection with Dr. McAnally. Proceedingly, Dr. McAnally evaluated my performance proving the set of problems which is also recorded in Table 1. We both noted the related non-cognitive and mathematical skills I improved on.

Mentor Evaluation I	Self-Reflection I
Asked questions before the problems were due.	Go beyond the given literature and do not be afraid to ask questions.
Showed an understanding of exponential matrix.	Think abstractly and test to make sure assumptions are true. Learn to use deductive reasoning.
Eager to understand the material and timid to ask questions.	Look at the form of the answer beforehand to see if there is any useful information for the proof.
Looked up the Hermitian matrix and did not quite grasp the exponential matrix derivative.	Make connections to previous papers to understand mathematical notation better.
Familiarizing with new mathematical concepts: exponential matrix, analytical continuation, basis, and ODE theory.	The "when it finally clicks" feeling. Rest of the problems became easier to solve.

Table 1 This table represents the first evaluation by Dr. McAnally and my first self-reflection. I proved four problems which took approximately a days worth of work to complete.

The first set of problems was challenging due to my inexperience of reading mathematical papers and working with matrices. At the time, I had minimal knowledge of linear algebra which required additional research to be done to complete the set of problems. Through this particular set, I grew as a self-learner by going beyond the given literature, improved my conceptual understanding of matrices, understood the procedures of proving the properties of matrices, and experienced self-motivation due to the excitement of proving the problems correctly. We produced four similar tables from the other sets of problems, and used the tables to draw conclusions about my experience and the skills I obtained from learning the inverse-scattering transform. My second set of problems dealt with the real and complex analysis of the Jost solutions and initial scattering matrix. This problem set took me over three days to complete.

Mentor Evaluation II	Self-Evaluation II
The assignment was much more complicated.	You can only use the information given to formulate logical proofs. Do not assume relationships to be true.
Showed understanding of limiting behavior and using variation of parameters.	The supremum norm $\sup(\|x\|)$ is the maximum norm of the element x of a normed vector space.
Long matrix multiplication and bookkeeping were challenging, and learned to better treat matrices with care.	Stay organized, try not to be a perfectionist, and rewrite problems.
Looked up the properties of an adjoint matrix.	Most proofs rely on related theorems and definitions.
Learned about analyticity of integrals and discovered how integrals can converge.	Apply the extreme value theorem to prove integral equations converge.
Learned the comparing coefficents trick.	Utilize the properties of Hermitian adjoint and diagonal matrices.

Table 2 This table represents the second evaluation by Dr. McAnally and my second self-reflection. I proved eight problems which took over three days.

Through this set of problems, I improved my strategic competence by developing complicated proofs, conceptual understanding of matrices and converging integrals, and adaptive reasoning by understanding the entire process. I also grew as a self-learner for recognizing the importance of organization with mathematical work. My third set of problems dealt with formulating the Riemann-Hilbert problem and proving involution properties of the Jost solutions and scattering matrix. This problem set took approximately five days to complete.

Mentor Evaluation III	**Self-Reflection III**
Learned to work backwards and how to expand analytic functions.	Easier to work backwards from the equation you are trying to prove.
Rediscovered terminology.	Involution properties of functions can relate the upper and lower half complex plane.
Learned the existence and uniqueness theorem in ODE.	Apply the existence and uniqueness theorem to prove certain equations are equivalent.
Took criticism of proof well and fixed it.	Showing one equation is true can prove another.

Table 3 This table represents the third evaluation by Dr. McAnally and my third self-reflection. I proved seven problems which took approximately five days.

Through this problem set, I learned new methods and theorems for proving properties of analytic functions, useful information on the procedures of proofs, and resilience by responding to criticism positively. Dr. McAnally gave constructive criticism for each of my proofs, and thoroughly explained the solutions after completing each problem set. My fourth set of problems dealt with proving the theorem that reduces a nonregular Riemann-Hilbert problem to a regular Riemann-Hilbert problem making it easier to solve. This particular problem set took me two weeks to solve. I noticed a major increase in my self-confidence after completing this set.

Mentor Evaluation IV	**Self-Reflection IV**
Learned to keep bra and ket notation for easier simplification.	Stay in bra-ket notation to cancel fractions to be one.
Was frustrated but remained persistent.	Stay persistent. Take breaks from problems and revisit them each day.
This problem set was never completed on her own. Learned the importance of writing out all of the details and carefully simplifying.	Write out problems completely.
Proved results after a quick 15 minute help session for questions.	Ask for help from advisor if stuck on a problem.
Employed a previously learned trick of adding one.	Add one in the form of a fraction to change the form of the answer.

Table 4 This table represents the fourth evaluation by Dr. McAnally and my fourth self-reflection. I proved three problems which took two weeks to complete.

Through this problem set, I improved my strategic competence by learning to use bra-ket notation for simplification, productive disposition for appreciating the required work needed to prove properties of complicated matrices, and resilience by staying persistent and working on the problem set consistently over the two weeks. I also became more self-motivated after gaining confidence in my mathematical abilities. My final set of problems dealt with forming the solution to equation (1). This problem set took me two days to finish, which is evidence towards improvement given the set was made up of ten problems.

Mentor Evaluation V	Self-Reflection V
Proved a problem differently than I showed, which demonstrates creativity and growth.	Confidence in myself and math abilities have greatly increased.
Showed improvement in taking initiative and looked up physics definitions in an owned textboo.	Take the limit before taking the derivative.
Learned the solution is not always explicit.	The ability to explain the process of proving problems on my own has increased
Many improvements: Understand exponential matrices and product derivative rule for matrices.	Knowledge gained from previous problems has helped prove this set of problems.

Table 5 This table represents the fifth evaluation by Dr. McAnally and my fifth self-reflection. I proved ten problems which took two days to complete.

After completing this problem set, I gained confidence in myself and my mathematical abilities. I grew as a self-learner by exploring different methods outside of the material discussed in the book and proving the majority of the problems on my own. Overall, I acquired numerous skills in mathematics and enhanced my non-cognitive skills sought out by graduate admission committees specifically in physics.

IV. Results

After collecting and analyzing my self-reflections and the evaluations by Dr. McAnally, I found an increase in my overall mathematical proficiency in the areas of linear algebra, real and complex analysis, and linear and nonlinear partial differential equations. I also saw improvements with the three desired non-cognitive skills self-learning, resilience, and self-motivation. Linear algebra refers to the understanding of operations between matrices, definitions of exponential matrix, adjoint, inverse, derivative of a matrix, limit of a matrix, integration on matrices, rewriting systems of differential equations as a single matrix differential equation, non-communativity of matrices, Lax pairs, basis of a space, null space, and rank. Real and complex analysis refers to the understanding of limits, iterative integrals, Fourier transforms, analytic continuation, Taylor expansion, conjugate of vectors and matrices, and the Riemann-Hilbert problem. Linear and nonlinear partial differential equations refers to the understanding of the focusing and defocusing Schrodinger equation, uniqueness and existence theory for ordinary differential equations, derivative of exponential matrices, integral of exponential matrices, Volterra integral, Jost solutions, boundary conditions, and soliton solutions. Dr. McAnally constructed a rubric to evaluate my mathematical proficieny in these areas before and after my fellowship shown in Table 6. Dr. McAnally's initial assessment of my mathematical proficiency is represented by Table 7.

Five Strands of Math Proficiency	**1**	**2**	**3**	**4**	**5**
Conceptual Understanding	Limited understanding of ideas.	Understands some ideas.	Moderate understanding of ideas.	Mostly understands ideas.	Understands ideas fully.
Procedural Fluency	No ability to carry problems out independently, flexibly, accurately, efficiently, and appropriately	Some ability to carry problems out independently, flexibly, accurately, efficiently, and appropriately	Carries half of the problems out independently, flexibly, accurately, efficiently, and appropriately.	Mostly carries problems out independently with limited help, flexibly, accurately, efficiently, and appropriately.	Carries all problems out independently, flexibly, accurately, efficiently, and appropriately.
Strategic Competence	Limited understanding of where to start.	Somewhat knew where to start problems.	Half of the time knows where to start.	Most times know where to start.	Always knows where to start problems.
Adaptive Reasoning	Limited ability to prove results and understand their implications.	Some ability to prove results and understand their implications.	Moderate ability to prove results and understand their implications.	Advanced ability to prove results and understand their implications.	Full ability to prove results and understand their implications.
Productive Disposition	Limited productive disposition.	Some productive disposition.	Moderate productive disposition.	Above average productive disposition.	Needs no improvement on productive disposition.

Table 6 The table represents Dr. McAnally's assessment rubric used to evaluate my mathematical proficiency in the areas of linear algebra, complex and real analysis, and linear and nonlinear partial differential equations.

Five Strands of Math Proficiency	**1**	**2**	**3**	**4**	**5**	**Total Score**
Conceptual Understanding	X					1
Procedural Fluency		X				2
Strategic Competence		X				2
Adaptive Reasoning		X				2
Productive Disposition			X			3

Overall Mathematical Proficiency Score: 10

Table 7 The table represents the initial mathematics assessment conducted by Dr. McAnally at the beginning of my fellowship.

Dr. McAnally ranked conceptual understanding the lowest since my relevant coursework as an undergraduate did not cover all of the necessary definitions and theorems to, at first glance, understand the article and book. Dr. McAnally ranked me a two in three of the proceeding catagories: procedural fluency, strategic competence, and adaptive reasoning. Initially, I needed direction on all the proofs with the Miller paper and guidance on how to formulate the proofs properly. I was ranked highest in productive disposition because I exhibited excitement towards learning the inverse-scattering transform and was able to connect the methods application to my quantum mechanics class. Dr. McAnally's final mathematics assessment is represented by Table 8.

Five Strands of Math Proficiency	1	2	3	4	5	Total Score
Conceptual Understanding				X		4
Procedural Fluency				X		4
Strategic Competence				X		4
Adaptive Reasoning			X			3
Productive Disposition				X		4

Overall Math Proficiency Score: 19

Table 8 The table represents my final mathematics assessment conducted by Dr. McAnally at the end of my fellowship.

Dr. McAnally ranked my conceptual understanding three points higher than my initial assessment due to understanding the concepts covered and using the terminology correctly through my SURF presentation. Procedural fluency and strategic competence increased by two points due to working more independently, gaining a general understanding of the inverse-scattering transform, and being able to prove problems on my own. I showed the least improvement in adaptive reasoning and productive disposition. Although I made advances in formulating and proving problems related to the transform, I still made mistakes on the last problem set. Productive disposition only increased by one because I gained motivation and confidence in my mathematical abilities, but already saw mathematics as useful and worthwhile.

Through Dr. McAnally's evaluations and my self-reflections, I was able to identify an increase in the three non-cognitive skills presented in the study on the ideal physics graduate student. I gained qualities of being a self-learner, resilient, and self-motivated. I grew as a self-learner by solving problem sets on my own. Initially, Dr. McAnally provided assistance on proving the problems, but eventually I was tasked with proving them on my own and with minimal hints. I learned how to read a mathematical paper, use deductive reasoning, and understand new mathematical notation. This also included the ability to draw connections to previous papers I have read. I learned how to look up related definitions, theorems, and reliable information to help prove equations. I discovered that student researchers have to go beyond the given literature to have a full understanding of a paper. Students need to be proactive and take the initiative to comprehend mathematical concepts used and discussed in academic papers. The simple explanations given by the author is not enough to fully grasp how he or she arrived at certain equations. I learned to prioritize certain aspects of my work to be able to accomplish solving problems quickly. I became comfortable with failure by constantly proving problems wrong the first few attempts, which helped me develop resilience.

I found myself better able to adapt to uncomfortable situations, such as being in the dark on hard problems, and demonstrated resilient qualities. I learned persistence is essential in mathematics to continuously attempt problems until you finally arrive at the correct solution. Being persistent meant being patient and thinking creatively and critically using different methods revisiting the problem each time. The more problems I was able to prove the more optimistic and confident I felt analyzing the next problem.

I gained qualities of being self-motivated due to the challenges of learning mathematics beyond the normal undergraduate curriculum, and developing confidence and a sense of pride for my work. The problems were highly engaging and interesting making it easier to stay motivated and complete each problem set. My confidence grew exponentially over the course of my fellowship as we progressed through the book by Yang. Feeling successful and having confidence in my abilities increased my overall motivation. In part of my increase in productive disposition, I had a newfound sense of pride for my work and belief that my research is useful with practical interest. Oftentimes I took the initiative to rewrite my solutions to make my work more neat and readable for Dr. McAnally. I also learned the environment I conduct research in is an essential factor in staying motivated. Mathematics is introspective in nature and requires deep thought and reflection to successfully comprehend and explain problems. An environment that allows students to reflect and think creatively and critically is essential for conducting pure or applied mathematics research. Overall, I discovered an increase in my mathematical proficiency in the areas of linear algebra, real and complex analysis, and linear and nonlinear partial differential equations as well as an increase in the desired non-cognitive skills demonstrated in the ideal physics graduate student.

V. Discussion

The main objective of our research was to examine how learning the inverse-scattering transform aided in the development of my mathematical proficiency specifically in the areas of linear algebra, real and complex analysis, and linear and nonlinear partial differential equations. We wanted to identify the skills I obtained from participating in undergraduate mathematics research to help bridge the gap between undergraduate and graduate education. After completing my fellowship, I found an increase in the five areas of mathematical proficiency and improvements on the three essential non-cognitive skills demonstrated in the ideal physics graduate student: self-learning, resilience, and self-motivation. This indicates pure and applied mathematics research leads to an increase in non-cognitive skills essential in accepted graduate students. Undergraduate mathematics research has the capability to bridge the educational gap admitted graduate students struggle with during the first couple of years. Undergraduate research in mathematics also improves students' mathematical proficiency in a broad range of areas beyond the undergraduate curriculum. Mathematics is interwoven in several disciplines and can often lead to transformative research when applied to another field, which is beneficial to faculty and administrators especially at primarily undergraduate institutions [8]. Improving our understanding of mathematics resultingly will improve current technology, develop new fields, and produce more engineers and scientists. Students will benefit greatly from close mentorship under a professor, and will be trained to become skeptical and creative through undergraduate mathematics research. The benefits of learning to think logically, abstractly, and critically will help students achieve success and grow to become independent learners.

The limitations of our study includes having a lack of supporting evidence and only researching physics Ph.D. granting programs due to time restraints. The assessments and evaluations were based on Dr. McAnally's expertise and rubric created to assess my mathematical proficiency throughout the fellowship. Although Dr. McAnally has years of research experience in these fields, the assessments and evaluations are still opinionated from her perspective which inevitably has self-bias. A numerical tool, such as an official exam, would have been more accurate to evaluate the changes of my mathematical abilities. Furthermore, my self-reflections were also used to identify improvements in the three non-cognitive skills. Since the reflections were performed by me, there is possible self-bias. We attempted to limit self-bias in both cases by using frameworks from other studies conducted and published reports. Additionally, I found a lack of supporting evidence since surveys questioning undergraduate researchers were out of date or lacked the representation of undergraduate mathematics researchers. Thus, we can not draw definitive conclusions all students will receive the same benefits participating in mathematics research. However, of the studies we reviewed [9], our results coincide with their findings. Furthermore, I only researched physics Ph.D. granting programs due to time restraints. I should have compared the skills I obtained with the desired qualities faculty search for in non-STEM related graduate programs. This would build stronger evidence that undergraduate mathematics research can assist all students in lessening the educational gap between undergraduate and graduate school, not just STEM students.

For further research, I would like to survey a sufficient sample size of undergraduate mathematics researchers from a variety of institutions to help defend my findings. By interviewing other students, I will gain supporting evidence consistent with the benefits I have received from the SURF program. The survey would also include different research programs such as research experiences for undergraduates (REUs) to successfully capture all experiences involving undergraduate mathematics research. Additionally, I want to report on my experience back in the classroom after participating in undergraduate mathematics research. We are interested if the qualities and improvements I have gained from the fellowship will enhance my performance in my undergraduate level courses.

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