Lump and line rogue wave solutions to a (2+1)-dimensional Hietarinta-type equation

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Abstract

Lump solutions are analytic rational function solutions that are localized in all spatial directions. They appear frequently in many areas of mathematical physics, particularly, fluid dynamics and usually arise as solutions to higher dimensional nonlinear equations often possessing Hirota bilinear forms. These solutions can sometimes lead to a special class of rogue wave solutions under some parameter constraint. In this article, we propose a novel nonlinear non-evolutionary equation in 2+1 dimensions and study its lump and rogue wave solutions with the aid of a computer algebra system. We give necessary and sufficient conditions that guarantee analyticity and rational localization of these solutions, and additionally, present

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illustrative examples together with 3D and contour plots.

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1. Introduction

Nonlinear partial differential equations play an important role in the study of many problems in mathematical physics. Exact solutions, such as lumps and solitons, derived from these equations describe various nonlinear phenomena in many areas of nonlinear science such as fluid dynamics [1, 2], gas dynamics [3], plasma physics [4] and nonlinear optics [5]. Lump solutions are analytic rational function solutions that are localized in all directions in space, and have been extensively studied over the last few years (see e.g., [6, 7, 8, 9, 10, 11]). They can be obtained by various methods such as Hirota's method [12, 13], the singular manifold method [14, 15, 16] or by taking long wave limits of soliton solutions [8, 17, 18].

Recently, there has been a growing interest in rogue wave solutions, which are a kind of rational function solutions localized in both space and time [19, 20]. It is known that many integrable equations such as the Kadomtsev-Petviashvili equation [21, 22], the Hirota-Satsuma-Ito equation [9, 23] and the B-type KP equation [24, 25, 26], possess lump and rogue wave solutions. Lump solutions can give rise to a special class of rogue wave solutions under some parameter constraint [27]. These rogue waves, referred to as *line rogue waves* [27], arise from a constant background with a line profile and decay back into the constant background after some time. In this article, we study lump and line rogue wave solutions of a novel (2+1)-dimensional extension of the so-called Hietarinta equation [13] via the Hirota bilinear method. The Hirota bilinear method is perhaps the most effective tool for finding exact solutions to nonlinear equations that possess Hirota bilinear forms. To use this method to find lump solutions, one constructs positive quadratic function solutions to a bilinear equation and uses logarithmic transformations to obtain the solutions (see e.g., [22, 28, 29]).

In what follows, we introduce the novel (2+1)-dimensional equation whose exact solutions we intend to study in this article. We further formulate its Hirota bilinear form and construct positive quadratic solutions to this bilinear equation. Consequently, we construct lump and line rogue waves to the newly introduced nonlinear equation. The paper concludes with many illustrative examples with 3D and contour plots and some additional concluding remarks.

2. The Hietarinta-type equation

The bilinear Hietarinta equation [13] is the equation,

$$(D_x^4 - D_x D_t^3 + \alpha D_x^2 + \beta D_x D_t + \gamma D_t^2) f \cdot f = 0. \tag{1}$$

It is an equation in (1+1)-dimensions and integrable in the sense that it has at least four soliton solutions and also passes the Painlevé test [13, 30], but interestingly, not much is known about this equation. It has not been identified within the Jimbo-Miwa classification [13, 31], and its Lax pair or Bäcklund transformation remains to be discovered [13].

A (2+1)-dimensional extension of the Hietarinta equation (hereafter referred to as a Hietarinta-type equation) is proposed as

$$P(u) = 6u_x u_{xx} + u_{xxxx} + 3u_t u_{tt} + 3u_{tx} v_{tt} + u_{xttt} + \alpha u_{xx} + \beta u_{tx} + \gamma u_{tt} - u_{xy} = 0, \tag{2}$$

where $v_x = u$, and α, β, and γ are arbitrary constants. Under the tranformation,

$$u = 2(\ln f)_x, \qquad v = 2\ln f, \tag{3}$$

the Hietarinta-type equation has the Hirota bilinear form,

$$B(f) = (D_x^4 + D_x D_t^3 + \alpha D_x^2 + \beta D_x D_t + \gamma D_t^2 - D_x D_y) f \cdot f = 0, \tag{4}$$

where D_x, D_y, and D_t are Hirota bilinear derivatives. The direct connection between (2) and (4) is given by the equation

$$P(u) = \left(\frac{B(u)}{f}\right)_x. \tag{5}$$

3. Lump solutions

We now construct lump solutions to the Hietarinta-type equation. We begin by assuming that f is a positive quadratic function of the form

$$\begin{cases} f &= f_1^2 + f_2^2 + a_9, \\ f_1 &= a_1 x + a_2 y + a_3 t + a_4, \\ f_2 &= a_5 x + a_6 y + a_7 t + a_8. \end{cases} \tag{6}$$

Now, substituting f into (4), a direct computation determines the following set of solutions:

$$\begin{cases} a_2 = \dfrac{\alpha a_1^3 + \beta a_1^2 a_3 + (\alpha a_5^2 + \gamma a_3^2 - \gamma a_7^2) a_1 + a_3 a_5 (\beta a_5 + 2\gamma a_7)}{a_1^2 + a_5^2}, \\ a_6 = \dfrac{\alpha a_5^3 + \beta a_5^2 a_7 + (\alpha a_1^2 - \gamma a_3^2 + \gamma a_7^2) a_5 + a_1 a_7 (\beta a_1 + 2\gamma a_3)}{a_1^2 + a_5^2}, \\ a_9 = -\dfrac{3(a_1^4 + 2a_1^2 a_5^2 + a_3^3 a_1 + a_3 a_7^2 a_1 + a_5 a_3^2 a_7 + a_5^4 + a_5 a_7^3)(a_1^2 + a_5^2)}{\gamma (a_1 a_7 - a_3 a_5)^2}, \end{cases} \tag{7}$$

with a_i for $i = \{1, 3, 4, 5, 7, 8\}$ as free parameters. To ensure that the functions in (3) are analytic, we require

$$a_9 > 0, \quad \gamma \neq 0, \quad a_1 a_7 - a_3 a_5 \neq 0. \tag{8}$$

Moreover, we require f_1 and f_2 to be linearly independent in order to generate lump solutions. This is equivalent to the determinant condition

$$\Delta := a_1 a_6 - a_2 a_5 = \begin{vmatrix} a_1 & a_2 \\ a_5 & a_6 \end{vmatrix} \neq 0. \tag{9}$$

Note that condition (9) actually gives rise to the last condition in (8) since

$$a_1 a_6 - a_2 a_5 = \frac{(a_1 a_7 - a_3 a_5)(\beta a_1^2 + \beta a_5^2 + 2\gamma a_1 a_3 + 2\gamma a_5 a_7)}{a_1^2 + a_5^2}. \tag{10}$$

It follows that the bilinear Hietarinta-type equation, (4), has the following positive quadratic function solutions:

$$
\begin{aligned}
f =&(a_1x + \frac{\alpha a_1^3 + \beta a_1^2 a_3 + (\alpha a_5^2 + \gamma a_3^2 - \gamma a_7^2)a_1 + a_3 a_5(\beta a_5 + 2\gamma a_7)}{a_1^2 + a_5^2} y + a_3 t + a_4)^2 \\
&+ (a_5 x + \frac{\alpha a_5^3 + \beta a_5^2 a_7 + (\alpha a_1^2 - \gamma a_3^2 + \gamma a_7^2)a_5 + a_1 a_7(\beta a_1 + 2\gamma a_3)}{a_1^2 + a_5^2} y + a_7 t + a_8)^2 \\
&- \frac{3(a_1^4 + 2a_1^2 a_5^2 + a_3^3 a_1 + a_3 a_7^2 a_1 + a_5 a_3^2 a_7 + a_5^4 + a_5 a_7^3)(a_1^2 + a_5^2)}{\gamma(a_1 a_7 - a_3 a_5)^2}.
\end{aligned}
\tag{11}
$$

Consequently, through the transformation (3), we obtain the following class of solutions to the Hietarinta-type equation (2);

$$
u = \frac{4a_5 f_2 + 4a_1 f_1}{f}, \tag{12}
$$

where f_1 and f_2 are are given by,

$$
\begin{cases}
f_1 &= a_1 x + \frac{\alpha a_1^3 + \beta a_1^2 a_3 + (\alpha a_5^2 + \gamma a_3^2 - \gamma a_7^2)a_1 + a_3 a_5(\beta a_5 + 2\gamma a_7)}{a_1^2 + a_5^2} y + a_3 t + a_4, \\
f_2 &= a_5 x + \frac{\alpha a_5^3 + \beta a_5^2 a_7 + (\alpha a_1^2 - \gamma a_3^2 + \gamma a_7^2)a_5 + a_1 a_7(\beta a_1 + 2\gamma a_3)}{a_1^2 + a_5^2} y + a_7 t + a_8.
\end{cases}
\tag{13}
$$

As a result of condition (9), the above solutions $u \to 0$ at any given time t, if and only if $x^2 + y^2 \to 0$. Thus, the condition $a_9 > 0$ and condition (9) guarantee the analyticity and rational localization of the solutions in (12). Under the above conditions, the solutions in (12) are a class of lump solutions to the Hietarinta-type equation.

4. Rogue waves

We now find line rogue wave solutions to the Hietarinta-type equation. Suppose the determinant condition (9) is not satisfied, i.e.

$$
a_1 a_6 - a_2 a_5 = \frac{(a_1 a_7 - a_3 a_5)(\beta a_1^2 + \beta a_5^2 + 2\gamma a_1 a_3 + 2\gamma a_5 a_7)}{a_1^2 + a_5^2} = 0.
$$

Then, we have

$$\beta a_1^2 + \beta a_5^2 + 2\gamma a_1 a_3 + 2\gamma a_5 a_7 = 0. \tag{14}$$

Note that we still require condition (8) to be satisfied in order for the solutions in (12) to be analytic. According to (8), at least one of the constants, $a_1,\ a_5$ is nonzero. Suppose $a_5 \neq 0$. It follows from (14) that

$$a_7 = -\frac{\beta a_1^2 + \beta a_5^2 + 2\gamma a_1 a_3}{2\gamma a_5}, \tag{15}$$

for $\gamma \neq 0$. Under this condition (15) and condition (8), the solutions in (12) yield a class of solutions that satisfy

$$\lim_{|t|\to\infty} u(x,y,t) = 0 \tag{16}$$

for $(x,y) \in \mathbb{R}^2$. Thus, these solutions are localized in time and are called line rogue wave solutions. We give illustrative examples in the next section.

5. Illustrative examples

5.1. Lump solutions

Choosing the parameters,

$$\alpha = 1, \beta = 1, \gamma = -1, a_1 = 1, a_3 = 3, a_4 = -3, a_5 = -1, a_7 = -2, a_8 = 2,$$

we obtain

$$f = \left(3t + x - \frac{9}{2}y - 3\right)^2 + \left(-2t - x + \frac{1}{2}y + 2\right)^2 + 414, \tag{17}$$

and

$$u = \frac{8(5t + 2x - 5y - 5)}{26t^2 + 20tx - 58ty + 4x^2 - 20xy + 41y^2 - 52t - 20x + 58y + 854}. \tag{18}$$

It could be easily verified that u decays in all spacial directions, ie., for any fixed t,

$$\lim_{x^2+y^2\to\infty} u(x,y,t) = 0. \tag{19}$$

For $t = -20,\ 0$ and 20, we obtain the solutions

$$u = \frac{8(-105 + 2x - 5y)}{4x^2 - 20xy + 41y^2 - 420x + 1218y + 12294}, \tag{20}$$

$$u = \frac{8(2x - 5y - 5)}{4x^2 - 20xy + 41y^2 - 20x + 58y + 854}, \tag{21}$$

and

$$u = \frac{8(95 + 2x - 5y)}{4x^2 - 20xy + 41y^2 + 380x - 1102y + 10214}, \tag{22}$$

respectively, which are depicted by 3D and contour plots below.

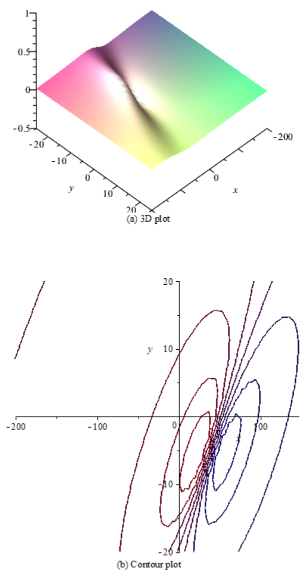

Fig. 1: Wave profile of solution 20

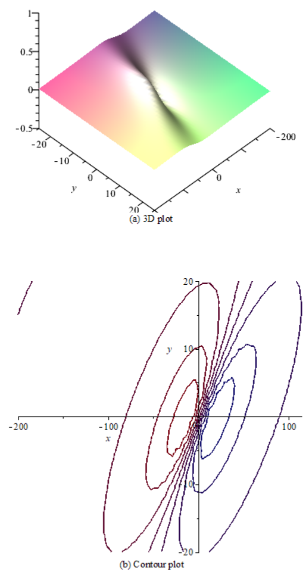

Fig. 2: Wave profile of solution 21

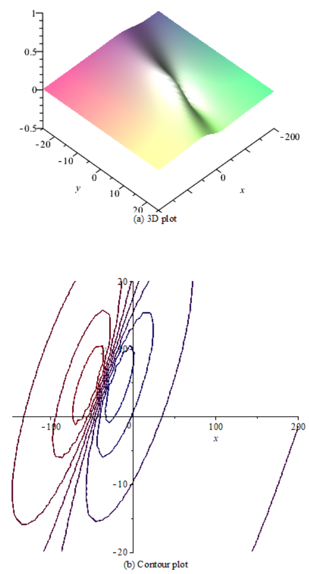

Fig. 3: Wave profile of solution 22

If we choose the parameters,

$$\alpha = 1, \beta = -1, \gamma = 1, a_1 = -1, a_3 = 2, a_4 = -2, a_5 = 1, a_7 = -3, a_8 = 3,$$

we obtain

$$f = \left(2t - x - \frac{13}{2}y - 2\right)^2 + \left(-3t + x + \frac{25}{2}y + 3\right)^2 + 366, \tag{23}$$

and

$$u = -\frac{8(5t - 2x - 19y - 5)}{26t^2 - 20tx - 202ty + 4x^2 + 76xy + 397y^2 - 52t + 20x + 202y + 758}. \quad (24)$$

Similarly, for any fixed t,

$$\lim_{x^2+y^2 \to \infty} u(x, y, t) = 0. \quad (25)$$

For $t = -20,\ 0$ and 20, we obtain the solution

$$u = \frac{8(105 + 2x + 19y)}{4x^2 + 76xy + 397y^2 + 420x + 4242y + 12198}, \quad (26)$$

$$u = \frac{8(5 + 2x + 19y)}{4x^2 + 76xy + 397y^2 + 20x + 202y + 758}, \quad (27)$$

and

$$u = \frac{8(-95 + 2x + 19y)}{4x^2 + 76xy + 397y^2 - 380x - 3838y + 10118}, \quad (28)$$

respectively, which are also depicted by 3D and contour plots below.

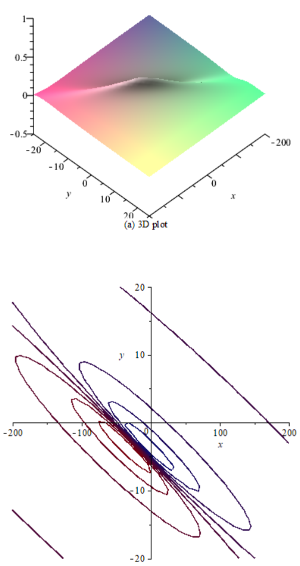

Fig. 4: Wave profile of solution 26

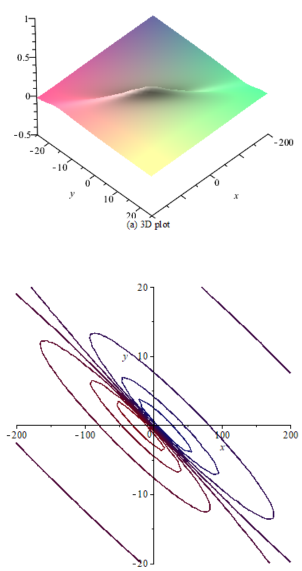

Fig. 5: Wave profile of solution 27

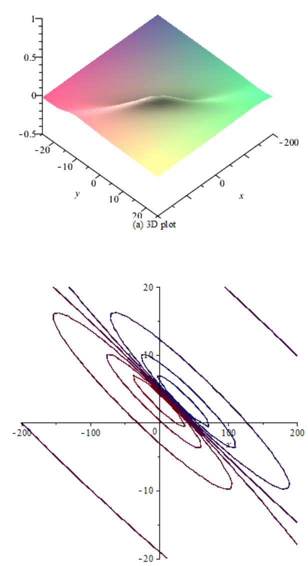

Fig. 6: Wave profile of solution 28

5.2. Line rogue waves

If we choose parameters

$$\alpha = 1, \beta = 1, \gamma = -1, a_1 = 1, a_3 = 2, a_4 = 0, a_5 = -1, a_7 = 1, a_8 = 3 = 0,$$

we obtain

$$f = \left(2t + x + \frac{7}{2}y\right)^2 + \left(t - x - \frac{7}{2}y\right)^2 + 6, \tag{29}$$

and

$$u = \frac{8(t + 2x + 7y)}{10t^2 + 4tx + 14ty + 4x^2 + 28xy + 49y^2 + 12}. \tag{30}$$

Note that the above parameters satisfy conditions (8) and (15) and u satisfies (16). For $t = -10, -4, -2, 0, 2, 4$ and 10, we obtain the solutions,

$$u = \frac{8(-10 + 2x + 7y)}{4x^2 + 28xy + 49y^2 - 40x - 140y + 1012}, \tag{31}$$

$$u = \frac{8(-4 + 2x + 7y)}{4x^2 + 28xy + 49y^2 - 16x - 56y + 172}, \tag{32}$$

$$u = \frac{8(-2 + 2x + 7y)}{4x^2 + 28xy + 49y^2 - 8x - 28y + 52}, \tag{33}$$

$$u = \frac{8(2x + 7y)}{4x^2 + 28xy + 49y^2 + 12}, \tag{34}$$

$$u = \frac{8(2 + 2x + 7y)}{4x^2 + 28xy + 49y^2 + 8x + 28y + 52}, \tag{35}$$

$$u = \frac{8(4 + 2x + 7y)}{4x^2 + 28xy + 49y^2 + 16x + 56y + 172}, \tag{36}$$

and

$$u = \frac{8(10 + 2x + 7y)}{4x^2 + 28xy + 49y^2 + 40x + 140y + 1012}, \tag{37}$$

respectively. The 3D plot and contour plots for these solutions are shown below.

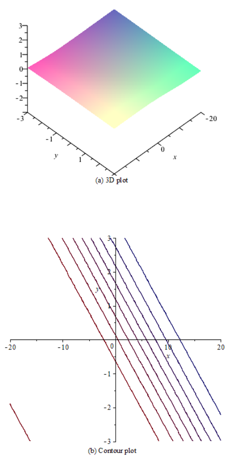

Fig. 7: Wave profile of solution 31

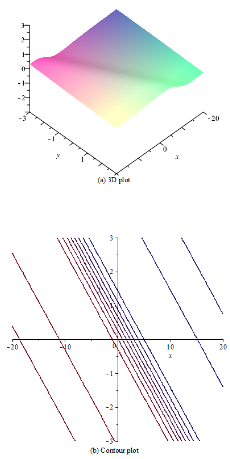

Fig. 8: Wave profile of solution 32

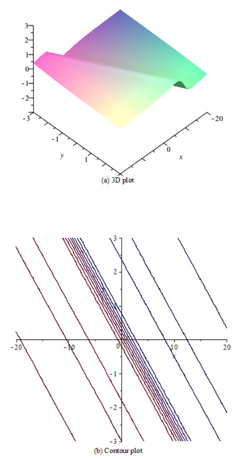

Fig. 9: Wave profile of solution 33

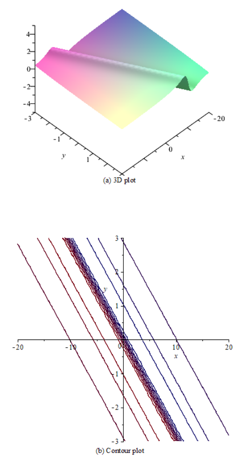

Fig. 10: Wave profile of solution 34

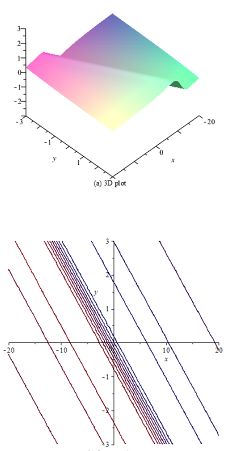

Fig. 11: Wave profile of solution 35

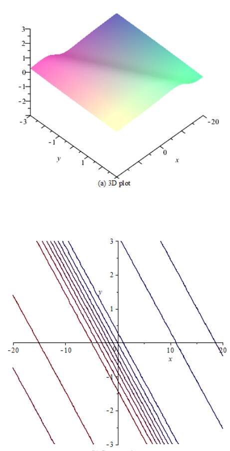

Fig. 12: Wave profile of solution 36

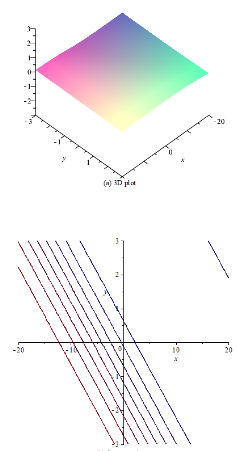

Fig. 13: Wave profile of solution 37

6. Concluding Remarks

By means of the Hirota bilinear method, we have constructed rationally localized solutions to a so-called Hietarinta-type equation. The solutions have been depicted for some selected parameters to illustrate their space-time evolution. An analysis of the lump solutions indicate that the wave profile and the amplitude of the lumps remain unchanged during propagation. Thus, these solutions do not decay in time. In contrast, the line rogue waves exhibit a different kind of dynamic behavior. As depicted by Fig.7-Fig.13, the line rogue waves appear from a constant background and decay back into the same background after peaking at around $t = 0$. This shows that the solutions actually decay in time, i.e. $\lim\limits_{|t|\to\infty} u = 0$. It is important to remark that these solutions are not rogue waves in the usual sense since they do not satisfy the condition $\lim\limits_{(x,y,t)\to\infty} u(x, y, t) = 0$.

As remarked earlier, the Hietarinta-type equation studied in this paper is a new nonlinear equation which is a (2+1)-dimensional extension of the Hietarinta equation. Like the Hietarinta equation, the new Hietartinta-type equation is also non-evolutionary. Very recently, another Hietarinta-type equation was introduced in [32] by Batwa and Ma. However, this equation does not contain the term, D_t^2, and is therefore not equivalent to the present equation. Furthermore, the Batwa-Ma equation does not possess line rogue waves which may be due to the the absence of the term D_t^2.

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