Building a Computational Model of a Double Pendulum

Pretented to: Computational Physics (PHY 365) Fall 2019

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Abstract

The double pendulum is a scenario often presented in Classical Mechanics courses to illustrate conservation of energy in the familiar context of an oscillating mass on a string. The double pendulum is also a great example of a simple physical system which can quickly become chaotic. We solved for the equations of motion of both masses with the given Lagrangian of the system using Equations 2, 3, and 4. We built a computational model using Spyder (Python 3.6) to plot the motion of each pendulums mass over time. We used our model to show energy was conserved in the system for small angles, and the system becomes chaotic due to changes in initial conditions and constants (mass and length of rod). This model is faster and more accurate than solving for the position or energy of either pendulums masses over time.

Background

A double pendulum is a string connected (l_1) to another mass (m_1) connected to another string (l_2) connected to another mass (m_2). A picture of this is shown in Figure 1. Each pendulum mass may move freely, and both are subject to gravity pointing downward in the negative y-direction. m_1 pivots a fixed point connected by l_1, while m_2 is suspended by the base of m_1. Traditionally, the double pendulum is explored for the case where l_1 and l_2 are rigid rods of negligible mass.[1]

An oscillating pendulum is a familiar concept for physics students as it is often used to show simple harmonic motion. In upper-level physics electives, the double pendulum is used to build on this familiar set up in the new context of the Lagrangian. The Lagrangian is defined as:

$$\mathcal{L} = T - U \tag{1}$$

where T represents the kinetic energy of the system and U represents the potential energy of the system. The Lagrangian for a double pendulum includes two potential and two kinetic energies as shown in Equation 2. This can be used to derive equations for angular acceleration for both pendulums, as shown in Equations 29 and 30.

As one can see, these equations are very long and complex to solve by hand. Additionally, it is not immediately obvious by examining these equations how the pendulums move together over time. A computational model can solve these equations for velocity and position quickly and use those equations to plot the motion of each pendulum over time. This can be used to see how the motion of one pendulum affects the motion of the other and how total energy in the system is conserved.

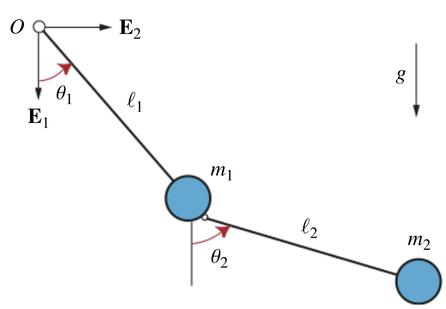

Figure 1: Pictured is a Double Pendulum with one mass connected to a fixed point and the other mass conntected to the first mass. Both masses are connected by rigid rods of negligible masses and subject to gravity, which points downward.[2]

General Equations of Motion

In the case where the two rods have the same length l, the Lagrangian for this system is given by:

$$\mathcal{L} = \frac{1}{2}(m_1+m_2)l^2\dot{\theta}_1^2 + \frac{1}{2}m_2l^2\dot{\theta}_2^2 + m_2l^2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2) + (m_1+m_2)gl\cos\theta_1 + m_2gl\cos\theta_2 \tag{2}$$

Using the Euler-Lagrange equation we can find the equations of motion for this system:

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1} - \frac{\partial\mathcal{L}}{\partial\theta_1} = 0 \tag{3}$$

and

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}_2} - \frac{\partial\mathcal{L}}{\partial\theta_2} = 0 \tag{4}$$

Where

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1} = (m_1+m_2)l^2\dot{\theta}_1 + m_2l^2\dot{\theta}_2\cos(\theta_1-\theta_2) \tag{5}$$

$$\frac{\partial\mathcal{L}}{\partial\theta_1} = -m_2l^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) - (m_1+m_2)gl\sin\theta_1 \tag{6}$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}_2} = m_2l^2\dot{\theta}_2 + m_2l^2\dot{\theta}_1\cos(\theta_1-\theta_2) \tag{7}$$

$$\frac{\partial\mathcal{L}}{\partial\theta_2} = m_2l^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) - m_2gl\sin\theta_2 \tag{8}$$

Plugging these equations in the Euler-Lagrange equation we get:

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1} - \frac{\partial\mathcal{L}}{\partial\theta_1} = (m_1+m_2)l^2\ddot{\theta}_1 + m_2l^2\ddot{\theta}_2\cos(\theta_1-\theta_2) - m_2l^2\dot{\theta}_2\sin(\theta_1-\theta_2)(\dot{\theta}_1-\dot{\theta}_2) + m_2l^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) + (m_1+m_2)gl\sin\theta_1 \tag{9}$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}_2} - \frac{\partial\mathcal{L}}{\partial\theta_2} = m_2l^2\ddot{\theta}_2 + m_2l^2\ddot{\theta}_1\cos(\theta_1-\theta_2) - m_2l^2\dot{\theta}_1\sin(\theta_1-\theta_2)(\dot{\theta}-\dot{\theta}_2) - m_2l^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) + m_2gl\sin\theta_2 \tag{10}$$

After some reducing and cancellations, the nonlinear system of two Lagrange differential equations can be written as:

$$(m_1+m_2)l\ddot{\theta}_1 + m_2l\ddot{\theta}_2\cos(\theta_1-\theta_2) + m_2l\dot{\theta}_2^2\sin(\theta_1-\theta_2) + (m_1+m_2)g\sin\theta_1 = 0 \tag{11}$$

$$l\ddot{\theta}_2 + l\ddot{\theta}_1\cos(\theta_1-\theta_2) - l\dot{\theta}_1^2\sin(\theta_1-\theta_2) + g\sin\theta_2 = 0 \tag{12}$$

Solving for $\ddot{\theta}_1$ and $\ddot{\theta}_2$ we get:

$$\ddot{\theta}_1 = m_2g\sin\theta_2\cos(\theta_1-\theta_2) - m_2\sin(\theta_1-\theta_2)[l\dot{\theta}_1^2\cos(\theta_1-\theta_2) + l\dot{\theta}_2^2] - (m_1+m_2)g\sin\theta_1(l[m_1+m_2\sin^2(\theta_1-\theta_2))^{-1} \tag{13}$$

$$\ddot{\theta}_2 = (m_1+m_2)[l\dot{\theta}_1^2\sin(\theta_1-\theta_2) - g\sin\theta_2 + g\sin\theta_1\cos(\theta_1-\theta_2)] + m_2l\dot{\theta}_2^2\sin(\theta_1-\theta_2)\cos(\theta_1-\theta_2)(l[m_1+m_2\sin^2(\theta_1-\theta_2)])^{-1} \tag{14}$$

Small Angles

For small angles, $\sin\theta \approx \theta$ and $\cos\theta \approx 1$. This comes from the first term in the Maclaurin expansion for $\sin\theta$ and $\cos\theta$. For these cases , the Euler-Lagrange equations reduce to:

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1} - \frac{\partial\mathcal{L}}{\partial\theta_1} = (m_1+m_2)l^2\ddot{\theta}_1 + m_2l^2\ddot{\theta}_2 + (m_1+m_2)gl\theta_1 \tag{15}$$

and

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}_2} - \frac{\partial\mathcal{L}}{\partial\theta_2} = m_2 l^2\ddot{\theta}_1 + m_2 l^2\ddot{\theta}_2 + m_2 g l^2\theta_2 \tag{16}$$

Solving for $\ddot{\theta}_1$ and $\ddot{\theta}_2$ we get:

$$\ddot{\theta}_1 = \frac{-(m_1+m_2)g\theta_1 + m_2 g l\theta_2}{(m_1+m_2)l - m_2 g l} \tag{17}$$

$$\ddot{\theta}_2 = \frac{-m_2 l^2\left(\frac{-(m_1+m_2)g\theta_1+m_2 g l\theta_2}{(m_1+m_2)l-m_2 g l}\right) - m_2 g l^2\theta_2}{m_2 l^2} \tag{18}$$

General Equations of Motion When $l_1 \neq l_2$

$$\begin{aligned} x_1 &= l_1\sin\theta_1 & \dot{x}_1 &= l_1\dot{\theta}_1\cos\theta_1 \\ x_2 &= l_1\sin\theta_1 + l_2\sin\theta_2 & \dot{x}_2 &= l_1\dot{\theta}_1\cos\theta_1 + l_2\dot{\theta}_2\cos\theta_2 \\ y_1 &= -l_1\cos\theta_1 & \dot{y}_1 &= l_1\dot{\theta}_1\sin\theta_1 \\ y_2 &= -l_1\cos\theta_1 - l_2\cos\theta_2 & \dot{y}_2 &= l_1\dot{\theta}_1\sin\theta_1 + l_2\dot{\theta}_2\sin\theta_2 \end{aligned} \tag{19}$$

The kinetic and potential energies are:

$$U = m_1 g y_1 + m_2 g y_2 = -(m_1+m_2)l_1 g\cos\theta_1 - m_2 l_2 g\cos\theta_2 \tag{20}$$

$$T = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_1(\dot{x}_1^2+\dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2+\dot{y}_2^2) = \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2[l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)] \tag{21}$$

We can substitute these expressions into the Lagrangian, $\mathcal{L} = T - U$:

$$\mathcal{L} = \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2[l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1 l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)] + (m_1+m_2)l_1 g\cos\theta_1 + m_2 l_2 g\cos\theta_2 \tag{22}$$

Using the Euler-Lagrange Equation:

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1} = (m_1+m_2)l_1^2\dot{\theta}_1 + m_2 l_1 l_2\dot{\theta}_2\cos(\theta_1-\theta_2) \tag{23}$$

$$\frac{\partial\mathcal{L}}{\partial\theta_1} = -m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) - (m_1+m_2)g l_1\sin\theta_1 \tag{24}$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}_2} = m_2 l_2^2\dot{\theta}_2 + m_2 l_1 l_2\dot{\theta}_1\cos(\theta_1-\theta_2) \tag{25}$$

$$\frac{\partial\mathcal{L}}{\partial\theta_2} = m_2 l_1 l_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2) - m_2 g l_2\sin\theta_2 \tag{26}$$

Plugging these equations in the Euler-Lagrange equation we get:

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1} - \frac{\partial\mathcal{L}}{\partial\theta_1} = (m_1+m_2)l_1^2\ddot{\theta}_1 + m_2 l_1 l_2\ddot{\theta}_2\cos(\theta_1-\theta_2) + m_2 l_1 l_2\dot{\theta}_2^2\sin(\theta_1-\theta_2) + (m_1+m_2)l_1 g\sin\theta_1 \tag{27}$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}_2} - \frac{\partial\mathcal{L}}{\partial\theta_2} = m_2 l_2^2\ddot{\theta}_2 + m_2 l_1 l_2\ddot{\theta}_1\cos(\theta_1-\theta_2) + m_2 l_2 g\sin\theta_2 \tag{28}$$

Solving for $\ddot{\theta}_1$ and $\ddot{\theta}_2$ we get:

$$\ddot{\theta}_1 = m_2 g\sin\theta_2\cos(\theta_1-\theta_2) - m_2\sin(\theta_1-\theta_2)[l_1\dot{\theta}_1^2\cos(\theta_1-\theta_2) + l_2\dot{\theta}_2^2] - (m_1+m_2)g\sin\theta_1(l_1[m_1+m_2\sin^2(\theta_1-\theta_2))^{-1} \tag{29}$$

$$\ddot{\theta}_2 = (m_1+m_2)[l_1\dot{\theta}_1^2\sin(\theta_1-\theta_2) - g\sin\theta_2 + g\sin\theta_1\cos(\theta_1-\theta_2)] + m_2 l_2\dot{\theta}_2^2\sin(\theta_1-\theta_2)\cos(\theta_1-\theta_2)(l_2[m_1+m_2\sin^2(\theta_1-\theta_2)])^{-1} \tag{30}$$

The Numerical Methods

Sometimes an approximation can be used to solve when the exact solution would be too difficult or long. A numerical method is an approximate, but faster solution commonly used in programming.

For example, we used the small angle approximation to reduce the number of terms in our Euler-Lagrange equations. Equations 9 and 10 show the Euler-Lagrange equations before the small angle approximation, and Equations 15 and 16 show the Euler-Lagrange equations after the small angle approximations. This reduction of terms allowed us to approximate Equations 13 and 14 to Equations 17 and 18 to solve for the angular acceleration of mass one and mass two.

We made several assumptions about the rods connecting the two masses in order to make it easier to solve. First, we assumed the rods were massless. This is a fair approximation because the pendulums we were considering had masses much more massive than the strings connecting them, such that the strings are essentially weightless. Assuming a massless rod reduced the number of terms in our equations of motion because we did not have to take the potential energy or kinetic energy of the rods.

In addition to assuming the rods were massless, we also assumed the pendulum masses were connected by solid rods instead of strings. This simplified our code because we did not have to take any stretching of the string into account. This is a fair approximation because, for most pendulums, the stretching of the string has negligible effects on the string's motion. Lastly, we also assumed the pendulum system is operating in a ideal environment such that there is no frictional force opposing the motion of the pendulums (i.e. air resistance).

Copy of the Code

This is our entire code that produces the model of our chaotic double pendulum. Please note that when $l_1 = l_2$, our equations simplify to our original equations of motion for the given Lagrangian. However, we made our code more generic to be able to vary the pendulums masses and rod lengths. We altered code from Matplotlib to produce the animations. [3]

```
"""
Final Project: The Double Pendulum
By: Gina Pantano, Miliani Hernandez, Kelli Shar

The double pendulum is a system where two masses are connected by rigid massless
rods. The double pendulum is a great example of a simple physical system which
can become quickly chaotic.
"""
#Imported Packages
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import odeint
import matplotlib.animation as animation

def Pendulum(Y,t,m1,m2,l1,l2,g):
    '''
    The function Pendulum solves for the change in theta1, theta2, omega1, and
    omega 2 at every time step starting with the intiial conditions at t = 0
    for each pendulum mass m1 and m2.
    '''
```

```
    theta1, dot_theta1, theta2, dot_theta2 = Y

    cos = np.cos(theta1-theta2)
    sin = np.sin(theta1-theta2)

    dtheta1 = dot_theta1
    ddot_theta1 = ((m2*g*np.sin(theta2)*cos) - (m2*sin*(l1*dot_theta1**2*cos + l2*
    dot_theta2**2))
    -((m1+m2)*g*np.sin(theta1))) / (l1*(m1 + m2*sin**2))
    dtheta2 = dot_theta2
    ddot_theta2 = (((m1+m2)*(l1*dot_theta1**2*sin - g*np.sin(theta2) + g*np.sin(
    theta1)*cos))
    + (m2*l2*dot_theta2**2*sin*cos)) / (l2*(m1 + m2*sin**2))

    return [dtheta1, ddot_theta1, dtheta2, ddot_theta2]

def Energy_Function(th1, dot_th1, th2, dot_th2):
    '''
    The function Energy_Function takes in theta1, theta2, omega1, and omega2 at
    every time step starting with our initial conditions at t = 0. This function
    calculates the kinetic and potential energy, and returns the total energy
    E = T + U.
    '''

    cos = np.cos(th1-th2)

    T = (.5*(m1+m2)*l1**2*dot_th1**2) + (.5*m2*l2**2*dot_th2**2) + (m2*l1*l2*
    dot_th1*dot_th2*cos)
    U = -((m1+m2)*g*l1*np.cos(th1)) - (m2*g*l2*np.cos(th2))

    return abs(T + U)

#Constants
g = 9.81
m1 = 5      #m1 and m2 in kilograms
m2 = 5
l1 = 2      #l1 and l2 in meters
l2 = 2

#Time and Initial Conditions
time,dt = np.linspace(0,20,1000, retstep = 'True')   #Time range and time steps
init = [.3, 4, 0, 1]     #[theta1, dtheta1, theta2, dtheta2]

#Solution Array
soln = odeint(Pendulum,init,time,args=(m1,m2,l1,l2,g,))

#Variable Solution
theta1, dot_theta1, theta2, dot_theta2 = soln[:,0], soln[:,1], soln[:,2], soln
    [:,3]

#Energy Calculations - Conservation of Energy
Total_Energy = Energy_Function(theta1, dot_theta1, theta2, dot_theta2)
```

```python
Initial_Energy = Energy_Function(init[0],init[1],init[2],init[3])

#Checks to see if the Total_Energy - Initial_Energy never exceeds 1e-5
#In other words, checks to see if the energy is conserved over all time.
for i in range(len(Total_Energy)):
    '''
    Will print "Energy is not conserved" at a given time step t if the
    change in energy exceeds 1e-5
    '''
    if abs(Total_Energy[i]-Initial_Energy)>1e-5:
        print("Energy is not conserved at time step {}".format(i))
        break
    else:
        pass

#Conversion to Cartesian coordinates of the two pendulum masses.
x1 = l1 * np.sin(theta1)
y1 = -l1 * np.cos(theta1)
x2 = x1 + l2 * np.sin(theta2)
y2 = y1 - l2 * np.cos(theta2)

delta_theta = abs(theta1-theta2)

#Plotting Commands
fig = plt.figure(figsize = (10,10))

#-----------------------------------------------------------------------------
"""
Source: https://matplotlib.org/3.1.1/gallery/animation/double_pendulum_sgskip.html

    This source helped us visualize our data through an animation of our
    two pendulum masses. We wanted to be able to present our data in the
    clearest way possible. This animation tracks the value of our variables at
    every time step from 0 to t.
"""
ax = fig.add_subplot(311, autoscale_on= False, xlim=(-6, 6), ylim=(-6, 0.2))
ax.set_aspect('equal')
ax.grid()

line, = ax.plot([], [], 'co-', lw=1)
time_template = 'time = %.1fs'
time_text = ax.text(0.5, 0.9, '', transform=ax.transAxes)

def init():
    line.set_data([], [])
    time_text.set_text('')
    return line, time_text


def animate(i):
    thisx = [0, x1[i], x2[i]]
    thisy = [0, y1[i], y2[i]]
```

```
    line.set_data(thisx, thisy)
    time_text.set_text(time_template % (i*dt))
    return line, time_text

ani = animation.FuncAnimation(fig, animate, range(1, len(soln)),
                              interval=dt*1000, blit=True, init_func=init)

#------------------------------------------------------------------------------
#Our Plotting Commands

#Cartesian Coordinates
plt.subplot(3,1,2)
plt.plot(x1,y1,'c', label = 'Top Mass')
plt.plot(x2,y2, label = 'Bottom Mass')
plt.ylabel('y (m)')
plt.xlabel('x (m)')
plt.title('Motion of the Double Pendulum')
plt.legend()

#Polar Coordinates
plt.subplot(3,1,3)
plt.plot(theta1,'c', label = 'Top Angle')
plt.plot(theta2, label = 'Bottom Angle')
plt.ylabel('$\\theta_1$ AND $\\theta_2$(radians)')
plt.xlabel('$\\theta_2$ (radians)')
plt.legend()

#Change in Theta over Time
'''
plt.plot(time,delta_theta)
plt.ylabel('$\Delta \\theta$')
plt.xlabel('Time (s)')
plt.title('Angular Frequency')

Note: We quoted this graph out since the most interesting result is when the
system is chaotic. We have included our images within the LaTeX file portion.
'''
plt.show()
```

Results

After successfully completing our code to run the model, we were able to start our investigation. We placed a for loop inside the code that checks the Total Energy of the system at every time step. If the Total Energy of the system is not within $1x10^{-5}$ standard deviation of our initial energy, the code will tell us our system is no longer conserved and at which time step this occurred. We decided to split up our investigation into four parts that each analyze different components of the system. The first task is to show, for small displacements, that the motion of the pendulum has two major modes: both θ_1 and θ_2 oscillate in the same direction, or they oscillate opposite of each other. In order to do this, we need to keep $m_1 = m_2 = 5$ kg and $l_1 = l_2 = 2$ m throughout these two cases. We only need to vary the angles θ_1 and θ_2 and keep the angular velocities 0. Given these initial conditions, we found the following for when the angles are the same shown in Figure 2.

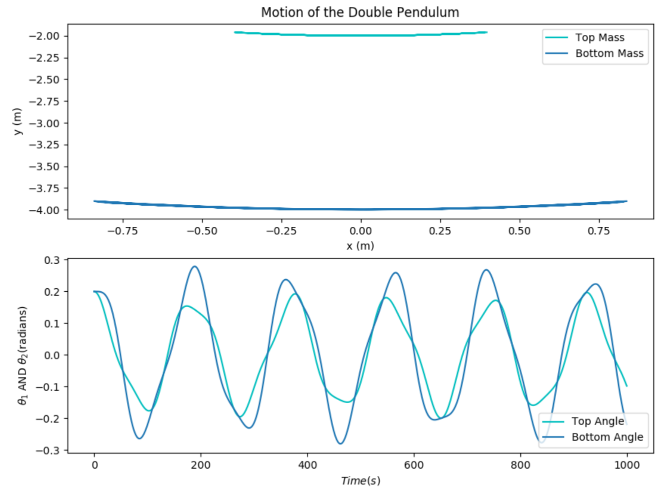

Figure 2: This graph shows when the angles are the same for both masses. θ_1 and θ_2 are both 0.2 radians.

For this case, the bottom and top mass oscillate together in a sinusoidal manner, acting similarly to a single long pendulum. As expected, the bottom mass oscillates over a larger distance that the smaller mass. This is due to the fact that the second mass is suspended by two lengths of l instead of one like the first mass. For this system, energy is entirely conserved, the lack of chaos provided by the initial conditions makes this so. Now, we changed the angle θ_1 to .4 radians and θ_2 to -.4 radians leaving everything else untouched shown in Figure 3.

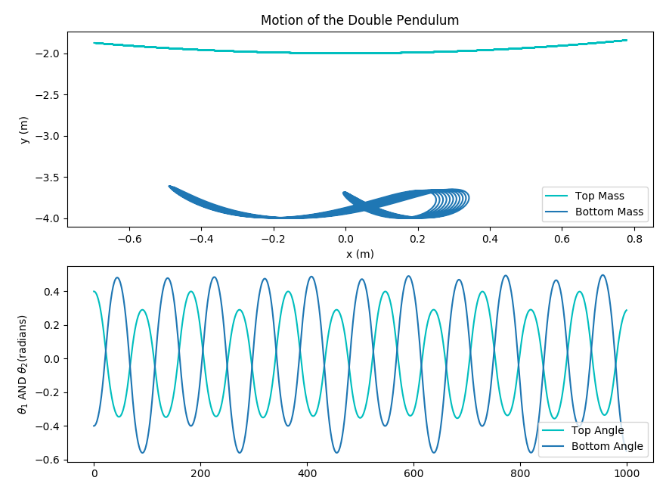

Figure 3: This graph shows when the angles are the opposite for both masses. θ_1 and θ_2 are both .4 radians, but θ_2 is negative.

For this system, the top mass's movement is similar to the case where both masses had the same initial angle however it has a much farther range. Because the top mass has a further range in the x-y plane, the bottom mass's motion has a bounce to it. Essentially the masses propel each other's motion causing sinusoidal motion in both masses that differ with a phase factor. Energy within this system is mostly conserved, however it does take on non-conservative value for times steps larger than 698. This is primarily due to the fact that the derivatives of the Lagrangian in respects to $\dot{\theta_1}$ and $\dot{\theta_2}$ are not zero and this results in the canonical momenta not being conserved quantities. The second task is to demonstrate the chaotic motion of the double pendulum by making small changes to our initial conditions. We plotted the change in theta vs time to show that the system is chaotic based on the positive Lyapunov exponent of the exponentially increasing function for each of the two cases. For our first chaotic case, we wanted to drastically change our initial values to compare them to our results when we only slightly change the initial angular velocities. We kept $m_1 = m_2 = 5$kg and $l_1 = l_2 = 2$m, and made θ_1 = 0.7 radians, $\dot{\theta_1}$= 3 rads/s, θ_2 = 0 radians, and $\dot{\theta_2}$ = 2 rads/s.

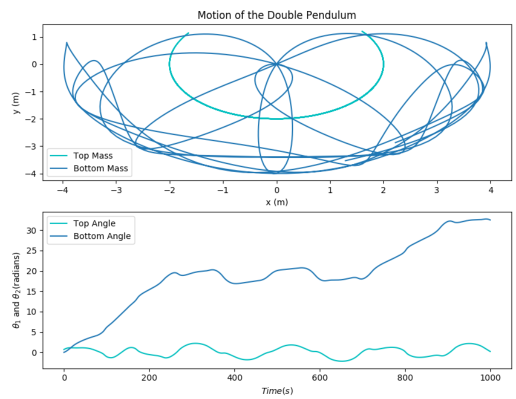

Figure 4: This graph shows when the system is chaotic for a large θ_1 angle with an initial angular velocity.

Figure 4 illustrates chaotic motion of the double pendulum. The bottom angle θ_2 is exponentially increasing over time, which shows us the system is becoming more chaotic with an increasing angle. However, notice the motion of θ_1. The motion of the top angle remains unchanged as it swings normally, which leads us to believe the first mass is dominating the motion for this system. Figure 5 is the change in theta graphed over time. We can see how the function is exponentially increasing due to the fact the system is chaotic.

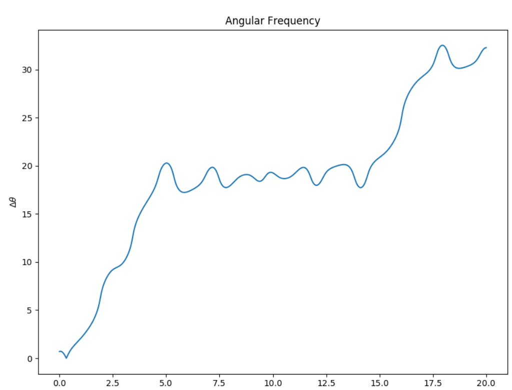

Figure 5: This graph shows the change in theta exponentially increasing over time. This represents a chaotic system.

Next, we compared our ideas of chaos to Figure 6 which only slightly varied the initial angles and angular velocities. We kept our initial angular velocities the same as before, but dropped the angle θ_1 back down to .2 radians.

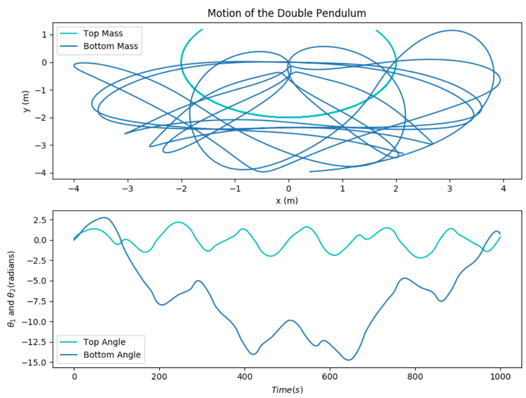

Figure 6: This figure shows the double penedulum being chaotic after making small changes to the systems initial conditions.

θ_1 and θ_2 vs time produced an interesting result. The first mass still oscillates relatively unbothered, but the first mass angle has an interesting shift. In the beginning due to the initial angular velocities, the bottom pendulum starts spinning with negative angles. Once mass 2 swings up to the top of the left side, the angles become positive again as it oscillates back in the opposite direction. We graphed the change in these angles over time in Figure 7, and still saw an exponentially increasing function (positive λ), which means the system is still clearly chaotic.

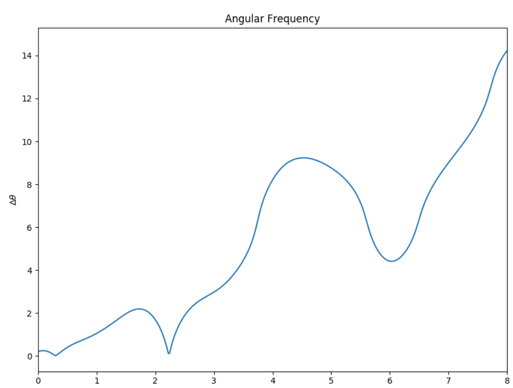

Figure 7: This graph represents the systems change in theta over time. The Lyapunov exponent is positive, which means the system is chaotic.

The third task is to show how differing the lengths of the rods and masses for small angles can affect the double pendulum. We kept the initial conditions of our angles and angular velocities the same throughout task three. We made θ_1 = 0.3 radians, $\dot{\theta}_1$= 2 rads/s, θ_2 = 0 radians, and $\dot{\theta}_2$ = 0 rads/s. We made the bigger mass 5 kg, the smaller mass 2 kg, the larger rod 4 m, and the smaller rod 2 m. We kept the mass equal to each other while exploring different lengths of the rods, and vice versa. We first looked at the effects of varying masses. Our first case is when mass 1 (5 kg) is greater than mass 2 (2 kg) shown in Figure 8.

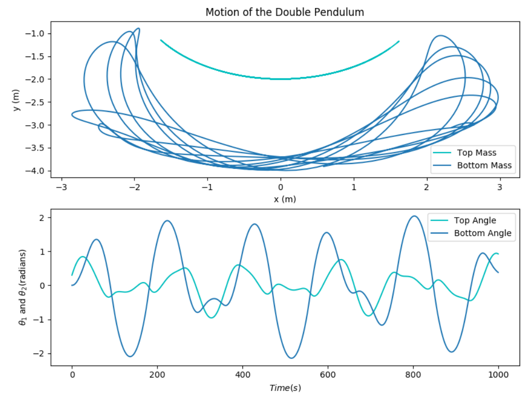

Figure 8: This graph shows the effects of when the first mass is greater than the second mass.

When mass 1 is greater than mass 2, we noticed a whipping motion occurring. Because mass 2 is less than mass 1, the bottom pendulum is subject to more motion. This is because, by decreasing the mass, the gravitational force on mass 2 is lessened which prevents the bob from staying in sinusoidal motion. The next case we switched the values of the masses around. In this scenario, the bottom mass is greater driving the entire motion of the pendulum. This led us to the conclusion that mass 1 has a greater influence on the system than mass 2. When mass 1 is less than mass 2, the pendulum swings as one single pendulum with a slight wobbly motion shown in Figure 9.

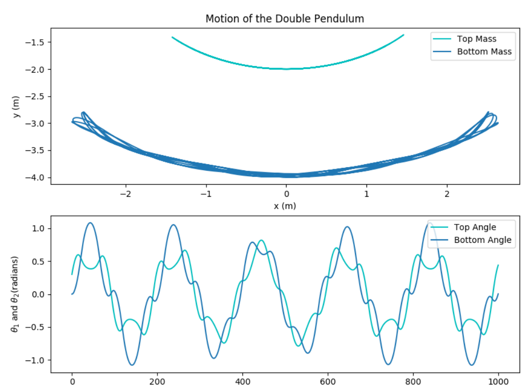

Figure 9: This graph shows the effects on the system when mass 2 is greater than mass 1

We made the masses equal to each other again, and changed the values for the lengths of rods to investigate the next case. We made the length of the first rod greater than the length of the second for our first case shown in Figure 10.

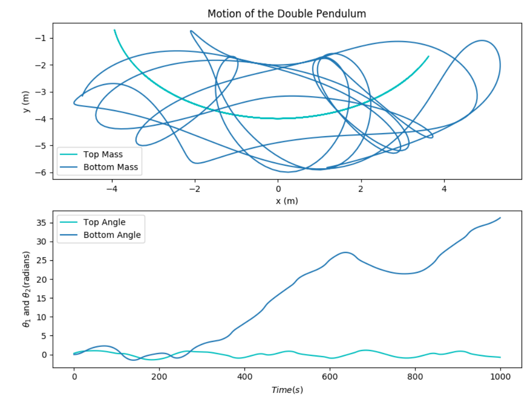

Figure 10: This graph shows the effects of when the length of the first rod is greater than the length of the second rod by 2 m.

The length of the first rod directly affects the motion of the second mass. Due to the length of the first rod being greater by 2 m, the second mass experiences chaotic motion by being whipped around a larger range. Next, we switched the lengths of the rods such that the length of the second rod is greater. The motion of the system becomes periodic again, which leaves us to conclude the length of the second rod does not affect the system nearly as much shown in Figure 11.

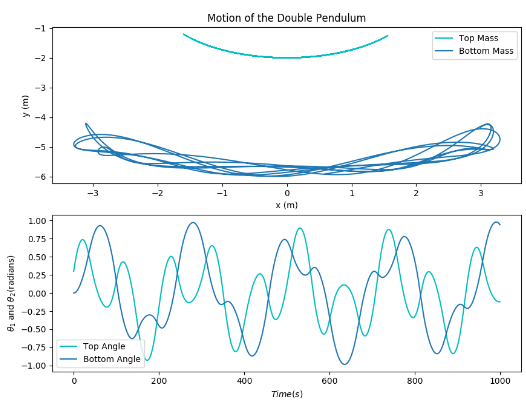

Figure 11: This graph shows when the length of the second rod is greater than the length of the first rod by 2 m.

For the fourth and final task, we experimented with three different initial conditions, including the masses and length of the rods, to produce interesting solutions to better understand the motion of the double pendulum. The initial conditions of the first interesting case in this system consist of θ_1 and θ_2 having initial angles of zero while having their angular velocities be 3 radians per second and 1 radians per second shown in Figure 12. We also made the first rod 3 m long instead of 2 m long and mass 1 (5 kg) greater than mass 2 (2 kg).

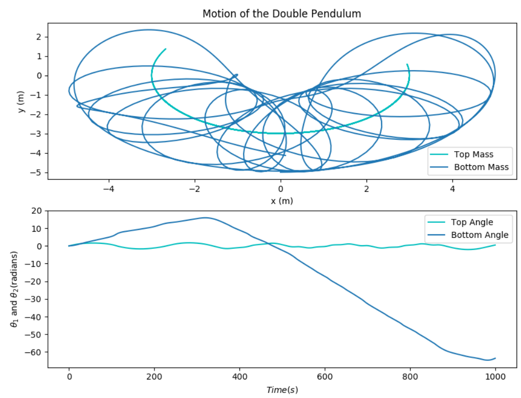

Figure 12: This graph shows the interesting chaotic motion of the double pendulum as mass 2 rapidly spins in circles.

Unlike the previous systems, this one experiments with differing values of length in the two connecting strings, mass, and initial conditions. From this graph, we are able to show that the length of rod 1 and the initial angular velocities of the masses directly contribute to the systems chaotic motion. As a result, we get a chaotic solution which demonstrates that systems that are deterministic are not always predictable. Due to the chaotic nature of this system, energy is only conserved for the first 100 time steps. The next two interesting cases we kept $m_1 = m_2$ = 5 kg and $l_1 = l_2$ = 2 m. We changed the initial conditions to be θ_1 = .3, $\dot{\theta_1}$ = -2, θ_2 = .1, and $\dot{\theta_2}$ = 3 for the next case shown in Figure 13.

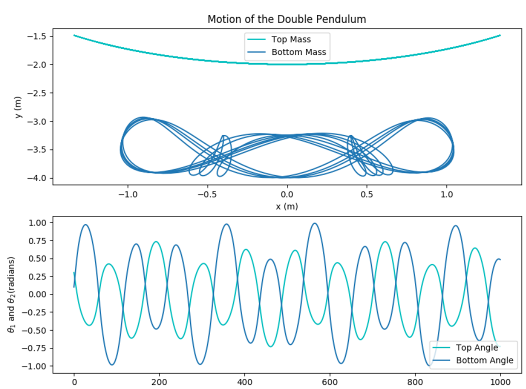

Figure 13: This graph shows the interesting motion for the initial conditions θ_1 = .3, $\dot{\theta_1}$ = -2, θ_2 = .1, and $\dot{\theta_2}$ = 3

This system behaves similarly to the system depicted in Figure 3 due to the initial angles in both cases being in opposite directions. The motion of the bottom mass mimicking an infinity loop with three sections , this is due to the motion of the top mass that causes the bottom mass to have deviations of motion in the y-direction due to a bouncing sensation. Energy is conserved for the first 30 time steps which is considerably less than what is shown in Figure 3. This movement results from the bottom mass being so much larger than the top mass. The final cases initial conditions is θ_1 = .1, $\dot{\theta_1}$ = 4, θ_2 = .1, and $\dot{\theta_2}$ = 4.

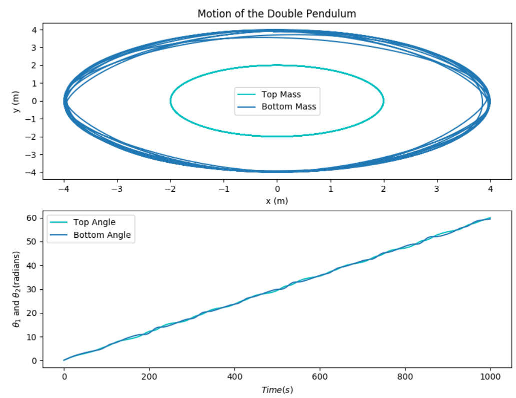

Figure 14: This graph shows the interesting motion for the initial conditions θ_1 = .1, $\dot{\theta_1}$ = 4, θ_2 = .1, and $\dot{\theta_2}$ = 4

The motion of this system results in periodic motion in both x, y and thetas directions shown in Figure 14. The graph of theta's versus time does not look periodic but that's due to the program counts each period as a new interval of 2π. In this system the double pendulum moves similarly to a single pendulum swinging around in a circle.

Discussion and Conclusions

Our computational model shows that, for small displacements, the pendulum masses either oscillate in the same direction or opposite direction. We showed this by having the two angles initially equal each other, and then change θ_2 to be negative for the second case. Our results are consistent with theory as described in our project description.

Our model also shows that energy is conserved at each time step for non-chaotic systems. Our results are consistent with Conservation of Energy, which states that total energy in a system is always conserved.[1]

Lastly, our model shows that the motion of a double pendulum can become chaotic and enters the regime with a positive Lyapunov exponent. We showed this by exploring different chaotic cases by varying our initial conditions and graphing the change in theta over time. Our results are consistent with theory as described in our project description.

Overall, our results are consistent with theory and show that our computational model is an accurate model of the motion of a double pendulum. We were able to conclude that the first mass and rod length cause the system to become more chaotic when changed than mass 2 and length of rod 2. When we increased mass 2 and the length of rod 2, we saw the system become periodic again. Additionally, our model is more efficient and accurate than graphing the motion of a double pendulum by hand. Our computational model shows students the motion of the double pendulum in an illuminating way for students, which can help students visualize the double pendulum equations without requiring physics instructors to set up the physical demonstration.

Future research will model the motion of a double pendulum connected by springs instead of rods. This will better represent the motion of a pendulum connected by strings, which stretch as they oscillate. We will also explore how damping affects the motion of the double pendulum and account for this in our models equations. This will make our model more representative of a physical system.

Member Contributions

First, Milani and Kelli derived the equations of motion for the double pendulum using the Langrangian. Next, Gina wrote the code of the equations of motion and to check if the total mechanical energy remained conserved. Together, we plugged in different initial conditions to complete the four required tasks for our analysis. Kelli wrote most of the report, Gina designed most of the presentation, and Milani typed up the long derivation in LaTeX. Overall, we worked well as a group by tailoring the project to each of our strengths.

References

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[2] Prithvi Akella. Rotations - the double pendulum. Retrieved from rotations.berkeley.edu/the-double-pendulum/.

[3] matplotlib. The double pendulum problem. Retrieved from https://matplotlib.org/3.1.1/gallery/animation/double$_{p}endulum_{s}gskip.html$.