RECURRENCE RELATIONS

zyBooks Chapter: 8.2, 8.15

LOGISTICS

- Midterm 2 grades out
  - Submit regrade request by11:00 PM Sunday, July 5.

- Lab 1 – Wednesday, July 1st at 11:59pm (~5 hrs)
- HW 7 is out – due next Monday, July 6 at 11:59pm

COURSE SCHEDULE

- Week 8 (June 30 – July 2)
  - Recursion 1
  - Recursion 2
  - Big – O
- Week 9 (July 7 – 9)
  - Relation 1
  - Relation 2
  - Midterm 3 Review
- Week 10 (July 14 –16)
  - Midterm 3
  - Graph 1
  - Graph 2
- Week 11 (July 21 – 23)
  - NO CLASS
  - Q&A Sessions OR Appointment (vote on piazza)
- Week 12 (July 27, 6 – 9 PM)
  - FINAL (6% on graphs)
  - + Optional Retake

RECURRENCE RELATION

A rule that defines a term a_n as a function of previous terms in the sequence is called a recurrence relation.

a_0	a_1	a_2	a_3	a_4	a_5	…	a_n
3	6	12	24	48	96	…	?

a_0	a_1	a_2	a_3	a_4	a_5	…	a_n
3	6	12	24	48	96	…	?

$a_0 = 3$

$a_1 = 2 * a_0 = 6$

$a_2 = 2 * a_1 = 12$

$a_3 = 2 * a_2 = 48$

$a_4 = 2 * a_3 = 96$

…

$a_n = 2 * a_{n-1}$ for $n \geq 1$

Recursive Form

$a_0 = 3$

$a_1 = 3 * 2 = 6$

$a_2 = 3 * 2 * 2 = 12$

$a_3 = 3 * 2 * 2 * 2 = 48$

$a_4 = 3 * 2 * 2 * 2 * 2 = 96$

…

Closed Form

$a_n = 3 * 2^n$ for $n \geq 0$

a_0	a_1	a_2	a_3	a_4	a_5	…	a_n
0	2	6	12	20	30	…	?

Q: Find the RECURSIVE form

$a_1 - a_0 = 2 = 2 \times 1$

$a_2 - a_1 = 4 = 2 \times 2$

$a_3 - a_2 = 6 = 2 \times 3$

$a_4 - a_3 = 8 = 2 \times 4$

$a_5 - a_4 = 10 = 2 \times 5$

…

$a_n - a_{n-1} = 2n$ for $n \geq 1$ & $a_0 = 0$

$a_n = a_{n-1} + 2n$ for $n \geq 1$ & $a_0 = 0$

a_0	a_1	a_2	a_3	a_4	a_5	…	a_n
0	2	6	12	20	30	…	?

Q: Find the CLOSED form

$a_1 - a_0 = 2$

$a_2 - a_1 = 4$

$a_3 - a_2 = 6$

$a_4 - a_3 = 8$

$a_5 - a_4 = 10$

…

$a_n - a_{n-1} = 2n$

Q: Find the CLOSED form

$$
\begin{aligned}
&(\cancel{a_1} - a_0 = 2) \\
+\ &(\cancel{a_2} - \cancel{a_1} = 4) \\
+\ &(\cancel{a_3} - \cancel{a_2} = 6) \\
+\ &(\cancel{a_4} - \cancel{a_3} = 8) \\
+\ &(\cancel{a_5} - \cancel{a_4} = 10) \\
+\ &(\cancel{\ldots}) \\
+\ &(a_n - \cancel{a_{n-1}} = 2n) \\
\hline
a_n - a_0 &= 2 + 4 + 6 + \ldots + 2n \\
&= 2 \times (1 + 2 + 3 + \ldots + n) \\
&= 2 \times \frac{n\,(n+1)}{2} \\
&= n\,(n+1)
\end{aligned}
$$

Since $a_0 = 0$,

$$a_n = n\,(n+1)$$

LINEAR HOMOGENOUS RECURSIVE RELATION (LHRR)

- Linear
  - There are no product of recursive terms
- Homogenous
  - There are no additional terms besides previous terms in the sequence

Q: Are they linear and homogenous?

$a_0 = 3$ L & H

$a_n = 2 * a_{n-1}$ for $n \geq 1$

$a_0 = 0$ L & non-H

$a_n = a_{n-1} + 2n$ for $n \geq 1$

LINEAR HOMOGENOUS RECURSIVE RELATION (LHRR)

	Degree	Linear	Homogeneous
$a_n = 3a_{n-1}$	1	Y	Y
$a_n = n * a_{n-1} + 5a_{n-2}$	2	N	Y
$a_n = \sqrt{a_{n-2}}$	2	N	Y
$a_n = a_{n-1} + 2a_{n-2} + 3a_{n-3}$	3	Y	Y
$a_n = a_{n-4}$	4	Y	Y
$a_n = a_{n-1} + 2 + a_{n-2}$	2	Y	N
$a_n = 4n + a_{n-1}$	1	Y	N
$a_n = 5a_{n-1} - 6a_{n-2}$	2	Y	Y

- Linear – no product of recursive terms
- Homogenous – no additional terms besides previous terms in the sequence

LINEAR HOMOGENOUS RECURSIVE RELATION (LHRR)

A linear homogeneous recurrence relation of degree k has the following form:

$$f_n = c_1 f_{n-1} + c_2 f_{n-2} + \cdots + c_k f_{n-k}$$

where the c_i's are constants that do not depend on n, and $c_k \neq 0$.

LHRR → CLOSED FORM

Q: Find the closed form

$s_0 = 2$

$s_n = 3\ s_{n-1}$ for $n \geq 1$

Step 0: Is it a LHRR?

Yes, LHRR, degree = 1

Step 1: Assume $s_n = c * x^n$

Step 2: Substitute $s_n = c * x^n$ to find x

Hence, $c * x^n = 3 * c * x^{n-1}$

$x^n = 3\ x^{n-1}$ (divide c for both side)

$x = 3$ (divide x^{n-1} for both side)

Therefore, $s_n = c * 3^n$

Step 3: Use the base case to find c

$s_0 = c * 3^0 = 2$

Therefore, $c = 2$

That is, $s_n = 2 * 3^n$

Q: Find the closed form

$s_0 = 2$

$s_1 = 5$

$s_n = 5\ s_{n-1} - 6\ s_{n-2}$ for $n \geq 2$

And then verify via proof by induction

Step 0: Is it a LHRR?

Yes, LHRR, degree = 2

Step 1: Assume $s_n = c * x^n$

Step 2: Substitute $s_n = c * x^n$ to find x

Hence, $c * x^n = 5 * c * x^{n-1} - 6 * c * x^{n-2}$

$x^n = 5\ x^{n-1} - 6\ x^{n-2}$ (divide c for both side)

$x^2 = 5x - 6$ (divide x^{n-2} for both side)

$x^2 - 5x + 6 = 0$

$(x - 2)\ (x - 3) = 0$

Therefore, we have $x = 2$ and $x = 3$.

Hence, $s_n = c_1 * 2^n + c_2 * 3^n$

Step 3: Use the base cases to find c

$s_0 = c_1 * 2^0 + c_2 * 3^0 = 2 \rightarrow c_1 + c_2 = 2$

$s_1 = c_1 * 2^1 + c_2 * 3^1 = 5 \rightarrow 2c_1 + 3c_2 = 5$

Therefore, $c_1 = 1,\ c_2 = 1$

That is, $s_n = 2^n + 3^n$

VERIFY – PROOF BY INDUCTION

LHS:

$s_0 = 2$

$s_1 = 5$

$s_n = 5\ s_{n-1} - 6\ s_{n-2}$ for $n \geq 2$

RHS:

$s_n = 2^n + 3^n$

Base Case:

Let $n = 0$, LHS $= 2 = 2^0 + 3^0 =$ RHS

Let $n = 1$, LHS $= 5 = 2^1 + 3^1 =$ RHS

Inductive Steps:

Assume s_{k-1} is true for some integer k, $k \geq 2$.

Then we have $s_{k-1} = 2^{k-1} + 3^{k-1}$.

Assume s_{k-2} is true for some integer k, $k \geq 2$.

Then we have $s_{k-2} = 2^{k-2} + 3^{k-2}$.

Prove that $s_k = 2^k + 3^k$ is true for some integer k, $k \geq 2$.

VERIFY – PROOF BY INDUCTION

LHS:

$s_0 = 2$

$s_1 = 5$

$s_n = 5\ s_{n-1} - 6\ s_{n-2}$ for $n \geq 2$

RHS:

$s_n = 2^n + 3^n$

Inductive Steps:

… Then we have $s_{k-1} = 2^{k-1} + 3^{k-1}$ and $s_{k-2} = 2^{k-2} + 3^{k-2}$.

Prove that $s_k = 2^k + 3^k$ is true for some integer k, $k \geq 2$.

$$\begin{aligned}
\text{LHS} &= 5 * (2^{k-1} + 3^{k-1}) - 6 * (2^{k-2} + 3^{k-2}) \\
&= 5 * 2^{k-1} + 5 * 3^{k-1} - 6 * 2^{k-2} - 6 * 3^{k-2} \\
&= 5 * 2^{k-1} + 5 * 3^{k-1} - 6 * \frac{1}{2} * 2^{k-1} - 6 * \frac{1}{3} * 3^{k-1} \\
&= 5 * 2^{k-1} + 5 * 3^{k-1} - 3 * 2^{k-1} - 2 * 3^{k-1} \\
&= (5 - 3) * 2^{k-1} + (5 - 2) * 3^{k-1} \\
&= 2 * 2^{k-1} + 3 * 3^{k-1} \\
&= 2^k + 3^k = \text{RHS}
\end{aligned}$$

Q: Find the closed form

$s_0 = 4$

$s_1 = 1$

$s_n = 3\,s_{n-1} + 4\,s_{n-2}$ for $n \geq 2$

Step 0: Is it a LHRR?

Yes, LHRR, degree = 2

Step 1: Assume $s_n = c * x^n$

Step 2: Substitute $s_n = c * x^n$ to find x

Hence, $c * x^n = 3 * c * x^{n-1} + 4 * c * x^{n-2}$

$x^n = 3\,x^{n-1} + 4\,x^{n-2}$ (divide c for both side)

$x^2 = 3x + 4$ (divide x^{n-2} for both side)

$x^2 - 3x - 4 = 0$

$(x + 1)(x - 4) = 0$

Therefore, we have $x = -1$ and $x = 4$.

Hence, $s_n = c_1 * (-1)^n + c_2 * 4^n$

Step 3: Use the base case to find c

$s_0 = c_1 * (-1)^0 + c_2 * 4^0 = 4 \rightarrow c_1 + c_2 = 4$

$s_1 = c_1 * (-1)^1 + c_2 * 4^1 = 1 \rightarrow -c_1 + 4c_2 = 1$

Therefore, $c_1 = 3$, $c_2 = 1$

That is, $s_n = 3 * (-1)^n + 4^n$

SUMMARY

- Linear Homogeneous Recurrence Relation
- From LHRR to Closed Form
- Proof of correctness by induction

Next Class

- From Closed Form to LHRR