RECURRENCE RELATIONS

zyBooks Chapter: 8.2, 8.15

LOGISTICS

- Lab 1 – TONIGHT at 11:59pm (~5 hrs, NO extension)
- HW 7 – due next Monday, July 6 at 11:59pm

RECAP

- Recurrence relation
  - A rule that defines a term a_n as a function of previous terms in the sequence.
  - Linear
  - Homogenous
- Recursive form
- Closed form
- Recursive Form of LHRR → Closed Form

Q: Is the following recurrence relation linear and homogenous?

$s_0 = 3$

$s_1 = 10$

$s_n = 7\, s_{n-1} - 12\, s_{n-2}$ for $n \geq 2$

Q: Find the closed form

$s_0 = 3$

$s_1 = 10$

$s_n = 7\, s_{n-1} - 12\, s_{n-2}$ for $n \geq 2$

Step 0: Is it a LHRR?

Yes, LHRR, degree = 2

Step 1: Assume $s_n = c * x^n$

Step 2: Substitute $s_n = c * x^n$ to find x

Hence, $c * x^n = 7 * c * x^{n-1} - 12 * c * x^{n-2}$

$x^n = 7\, x^{n-1} - 12\, x^{n-2}$ (divide c for both sides)

$x^2 = 7x - 12$ (divide x^{n-2} for both sides)

$x^2 - 7x + 12 = 0$

$(x - 4)\,(x - 3) = 0$

Therefore, we have $x = 4$ and $x = 3$.

Hence, $s_n = c_1 * 4^n + c_2 * 3^n$

Step 3: Use the base case to find c

$s_0 = c_1 * 4^0 + c_2 * 3^0 = 3 \rightarrow c_1 + c_2 = 3$

$s_1 = c_1 * 4^1 + c_2 * 3^1 = 10 \rightarrow 4c_1 + 3c_2 = 10$

Therefore, $c_1 = 1,\ c_2 = 2$

That is, $s_n = 4^n + 2 * 3^n$

VERIFY – PROOF BY INDUCTION

Given:

$s_0 = 3$

$s_1 = 10$

$s_n = 7\, s_{n-1} - 12\, s_{n-2}$ for $n \geq 2$

Prove:

$s_n = 4^n + 2 * 3^n$

Prove that

$7s_{n-1} - 12s_{n-2} = 4^n + 2 * 3^n$ for $n \geq 2$

Base Case:

Let $n = 0$, LHS $= 3 = 4^0 + 2 * 3^0 =$ RHS

Let $n = 1$, LHS $= 10 = 4^1 + 2 * 3^1 =$ RHS

Inductive Steps:

Assume s_{k-1} is true for some integer k, $k \geq 2$.

Then we have $s_{k-1} = 4^{k-1} + 2 * 3^{k-1}$.

Assume s_{k-2} is true for some integer k, $k \geq 2$.

Then we have $s_{k-2} = 4^{k-2} + 2 * 3^{k-2}$.

Prove that $s_k = 4^k + 2 * 3^k$ is true for some integer k, $k \geq 2$.

VERIFY – PROOF BY INDUCTION

Given:

$s_0 = 3$

$s_1 = 10$

$s_n = 7\, s_{n-1} - 12\, s_{n-2}$ for $n \geq 2$

Prove:

$s_n = 4^n + 2 * 3^n$

Prove that

$7s_{n-1} - 12s_{n-2} = 4^n + 2 * 3^n$ for $n \geq 2$

Inductive Steps:

… Then we have $s_{k-1} = 4^{k-1} + 2*3^{k-1}$ and $s_{k-2} = 4^{k-2} + 2*3^{k-2}$.

Prove that $s_k = 4^k + 2 * 3^k$ is true for some integer k, $k \geq 2$.

$LHS = 7 * (4^{k-1} + 2 * 3^{k-1}) - 12 * (4^{k-2} + 2 * 3^{k-2})$

$= 7 * 4^{k-1} + 14 * 3^{k-1} - 12 * 4^{k-2} - 24 * 3^{k-2}$

$= 7 * 4^{k-1} + 14 * 3^{k-1} - 12 * \frac{1}{4} * 4^{k-1} - 24 * \frac{1}{3} * 3^{k-1}$

$= 7 * 4^{k-1} + 14 * 3^{k-1} - 3 * 4^{k-1} - 8 * 3^{k-1}$

$= (7 - 3) * 4^{k-1} + (14 - 8) * 3^{k-1}$

$= 4 * 4^{k-1} + 6 * 3^{k-1} = 4 * 4^{k-1} + 2 * 3 * 3^{k-1}$

$= 4^k + 2 * 3^k = RHS$

CLOSED FORM → RECURSIVE FORM

Q: Find the recursive form given the closed form: $g_m = (3m + 1)(m - 1)$

$$g_{m-1} = (3(m-1) + 1)((m-1) - 1)$$

$$= (3m - 2)(m - 2)$$

$$g_m - g_{m-1} = (3m + 1)(m - 1) - (3m - 2)(m - 2)$$

$$= 3m^2 - 2m - 1 - (3m^2 - 8m + 4)$$

$$= 3m^2 - 2m - 1 - 3m^2 + 8m - 4$$

$$= 6m - 5$$

$$g_m = g_{m-1} + 6m - 5 \quad \text{for } m \geq 1$$

$$g_0 = (3 * 0 + 1)(0 - 1) = -1$$

Q: Find the recursive form given the closed form: $g_m = (m + 2)(2m - 1)$

$$g_{m-1} = ((m-1) + 2)(2(m-1) - 1)$$

$$= (m + 1)(2m - 3)$$

$$g_m - g_{m-1} = (m + 2)(2m - 1) - (m + 1)(2m - 3)$$

$$= 2m^2 + 3m - 2 - (2m^2 - m - 3)$$

$$= 2m^2 + 3m - 2 - 2m^2 + m + 3$$

$$= 4m + 1$$

$$g_m = g_{m-1} + 4m + 1 \text{ for } m \geq 1$$

$$g_0 = (0 + 2)(2 * 0 - 1) = -2$$

Q: Find the recursive form given the closed form: $s_n = 4 * 3^n + 3 * 4^n$

Therefore, $x_1 = 3$, $x_2 = 4$

So we have $(x - 3)(x - 4) = 0$

$$x^2 - 7x + 12 = 0$$

$$x^2 = 7x - 12$$

Then we have $x^n = 7x^{n-1} - 12x^{n-2}$ (multiply both sides by x^{n-2})

$s_n = 7\, s_{n-1} - 12\, s_{n-2}$ for $n \geq 2$

$s_0 = 4 * 3^0 + 3 * 4^0 = 7$

$s_1 = 4 * 3^1 + 3 * 4^1 = 24$

$s_n = c_1 * x_1^{\,n} + c_2 * x_2^{\,n}$

*** LHRR ***

Q: Find the recursive form given the closed form: $s_n = 3^n + 3 * 2^n$

Therefore, $x_1 = 3$, $x_2 = 2$

So we have $(x - 3)(x - 2) = 0$

$$x^2 - 5x + 6 = 0$$

$$x^2 = 5x - 6$$

Then we have $x^n = 5x^{n-1} - 6x^{n-2}$ (multiply both sides by x^{n-2})

$s_n = 5 s_{n-1} - 6 s_{n-2}$ for $n \geq 2$

$s_0 = 3^0 + 3 * 2^0 = 4$

$s_1 = 3^1 + 3 * 2^1 = 9$

Q: Find the recursive form given the closed form: $s_n = 5 * 2^n + 2 * 4^n$

Therefore, $x_1 = 2$, $x_2 = 4$

So we have $(x - 2)(x - 4) = 0$

$$x^2 - 6x + 8 = 0$$

$$x^2 = 6x - 8$$

Then we have $x^n = 6x^{n-1} - 8x^{n-2}$ (multiply both sides by x^{n-2})

$s_n = 6\, s_{n-1} - 8\, s_{n-2}$ for $n \geq 2$

$s_0 = 5 * 2^0 + 2 * 4^0 = 7$

$s_1 = 5 * 2^1 + 2 * 4^1 = 18$