PREDICATES & QUANTIFIERS

zyBooks Chapter: 4.6 ~ 4.13

LOGISTICS

- Midterm 1
  - Submit regrade request by11:00 PM Wednesday, June 10.
    - Send regrade request to TA and copy me
- HW5 – due next Monday, June 15 at Midnight

RECAP – PREDICATES & QUANTIFIERS

- Predicates
  - A function that evaluates to a proposition
  - Domain of a a variable: a set of ALL possible values
- Quantifiers – Size of the domain
  - Universal Quantifier – $\forall x P(x)$
  - Existential Quantifier – $\exists x P(x)$
- Bound and Free variables
  - A proposition does NOT contain free variable
- De Morgan’s Law

PREDICATES AND QUANTIFIERS

D(x): x likes dogs.

English	Existential	Universal
Someone does not like dogs	$\exists x\ \neg D(x)$	$\neg\forall x\ D(x)$
No one likes dogs	$\neg\exists x\ D(x)$	$\forall x\ \neg D(x)$
Someone likes dogs	$\exists x\ D(x)$	$\neg\forall x\ \neg D(x)$
Everyone likes dogs	$\neg\exists x\ \neg D(x)$	$\forall x\ D(x)$

De Morgan's Law

$$\neg\forall x\ P(x) \equiv \exists x\ \neg P(x)$$

$$\neg\exists x\ P(x) \equiv \forall x\ \neg P(x)$$

NESTED QUANTIFIERS

If a predicate has more than one variable, each variable must be bound by a separate quantifier. For example, $\forall x \forall y\ L(x, y)$

A logical expression with more than one quantifier that bind different variables in the same predicate is said to have nested quantifiers.

NESTED QUANTIFIERS

F(x, y): x flies to y

- Every bird flies to everywhere

  $\forall x \forall y\ F(x, y)$

- Some birds fly to somewhere

  $\exists x \exists y\ F(x, y)$

- Every bird flies to somewhere

  $\forall x \exists y\ F(x, y)$

- There is a place where every bird flies to

  $\forall x \exists y\ F(x, y)$

NESTED QUANTIFIERS

Q: Determine the truth value of each expression below. The domain is the set of all real numbers (Exercise 4.9.3 in zyBooks).

- $\forall x \exists y\ (xy > 0)$
- $\exists x \forall y\ (xy = 0)$
- $\forall x \forall y \exists z\ (z = (x - y)/3)$
- $\forall x \exists y \forall z\ (z = (x - y)/3)$
- $\forall x \forall y\ (xy = yx)$
- $\exists x \exists y \exists z\ (x^2 + y^2 = z^2)$
- $\forall x \exists y\ (y^2 = x)$
- $\forall x \exists y\ (x < 0 \lor y^2 = x)$
- $\exists x \exists y\ (x^2 = y^2 \land x \neq y)$
- $\exists x \exists y\ (x^2 = y^2 \land |x| \neq |y|)$
- $\forall x \forall y\ (x^2 \neq y^2 \lor |x| = |y|)$

Q: Determine the truth value of each expression below. The domain is the set of all real numbers (Exercise 4.9.3 in zyBooks).

- $\forall x \exists y \, (xy > 0)$
  - False. If $x = 0$, there is no y such that $xy > 0$.
- $\exists x \forall y \, (xy = 0)$
  - True. If $x = 0$, then for all y, $xy = 0$.
- $\forall x \forall y \exists z \, (z = (x - y)/3)$
  - True. For any x and y, set the value of z to be $(x - y)/3$.
- $\forall x \exists y \forall z \, (z = (x - y)/3)$
  - False. Once x and y are determined, then if z is any value besides $(x - y)/3$, the equality is false.
- $\forall x \forall y \, (xy = yx)$
  - True.

- $\exists x \exists y \exists z\ (x^2 + y^2 = z^2)$
  - True. One example is $x = 3$, $y = 4$, $z = 5$.
- $\forall x \exists y\ (y^2 = x)$
  - False. If $x < 0$, then there is no y such that $y^2 = x$.
- $\forall x \exists y\ (x < 0 \vee y^2 = x)$
  - True. If $x \geq 0$, then there is a y such that $y^2 = x$.
- $\exists x \exists y\ (x^2 = y^2 \wedge x \neq y)$
  - True. For example, $x = 2$ and $y = -2$.
- $\exists x \exists y\ (x^2 = y^2 \wedge |x| \neq |y|)$
  - False.
- $\forall x \forall y\ (x^2 \neq y^2 \vee |x| = |y|)$
  - True. If the first part is false for some x and y (i.e., $x^2 = y^2$), then $|x| = |y|$ is true.

DE MORGAN'S ON NESTED QUANTIFIERS

$$\neg\forall x\exists y\ P(x, y) \equiv \exists x\forall y\ \neg P(x, y)$$

$$\neg\exists x\forall y\ P(x, y) \equiv \forall x\exists y\ \neg P(x, y)$$

DE MORGAN'S LAW

$$\neg\forall x\ P(x) \equiv \exists x\ \neg P(x)$$

$$\neg\exists x\ P(x) \equiv \forall x\ \neg P(x)$$

$$\neg\forall x\exists y\ P(x, y) \equiv \exists x\forall y\ \neg P(x, y)$$

$$\neg\exists x\forall y\ P(x, y) \equiv \forall x\exists y\ \neg P(x, y)$$

Q: Simplify to remove all $\neg$ and $\rightarrow$

$$\neg\forall x\exists y\exists z\left((x > z) \rightarrow (x > y)\right)$$

Q: Simplify to remove all $\neg$ and $\rightarrow$

$$\neg\forall x \exists y \exists z \left((x > z) \rightarrow (x > y) \right)$$

$\equiv \exists x \forall y \forall z \, \neg\left((x > z) \rightarrow (x > y) \right)$	De Morgan's
$\equiv \exists x \forall y \forall z \, \neg\left(\neg(x > z) \vee (x > y) \right)$	Implication
$\equiv \exists x \forall y \forall z \, \neg\neg(x > z) \wedge \neg(x > y)$	De Morgan's
$\equiv \exists x \forall y \forall z \, (x > z) \wedge (x \leq y)$	Double Negation

Q: The domain for variables x and y is a group of people. The predicate $F(x, y)$ is true if and only if x is a friend of y. For the purposes of this problem, assume that for any person x and person y, either x is a friend of y or x is an enemy of y. Therefore, $\neg F(x, y)$ means that x is an enemy of y.

Steps:

i. Translate each statement into a logical expression.

ii. Negate the expression by adding a negation operation to the beginning of the expression.

iii. Apply De Morgan's law until the negation operation applies directly to the predicate

iv. Translate the logical expression back into English.

a) Everyone is a friend of everyone.
b) Someone is a friend of someone.
c) Someone is a friend of everyone.
d) Everyone is a friend of someone.

- Everyone is a friend of everyone.
  - Logical expression: $\forall x \forall y\, F(x, y)$
  - Negation: $\neg \forall x \forall y\, F(x, y)$
  - Apply De Morgan's: $\exists x \exists y\, \neg F(x, y)$
  - English: Someone is an enemy of someone.
- Someone is a friend of someone.
  - Logical expression: $\exists x \exists y\, F(x, y)$
  - Negation: $\neg \exists x \exists y\, F(x, y)$
  - Apply De Morgan's: $\forall x \forall y\, \neg F(x, y)$
  - English: Everyone is an enemy of everyone.

- Someone is a friend of everyone.
  - Logical expression: $\exists x \forall y\ F(x, y)$
  - Negation: $\neg\ \exists x \forall y\ F(x, y)$
  - Apply De Morgan's: $\forall x \exists y\ \neg F(x, y)$
  - English: Everyone is an enemy of someone.

- Everyone is a friend of someone.
  - Logical expression: $\forall x \exists y\ F(x, y)$
  - Negation: $\neg\ \forall x \exists y\ F(x, y)$
  - Apply De Morgan's: $\exists x \forall y\ \neg F(x, y)$
  - English: Someone is an enemy of everyone.