

# TMA4285 Time series models

## Exercise 8: Comparison of a state space model and a SARIMA model

*Sivert Selnes, Kristine L. Mathisen, Gina Magnussen*

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## Remember:

- Note how uncertainty should be written: 0.056(7) (See sources on web page)

## Title

## Abstract

## Introduction

## Theory

## Theory is written here (in theory.Rmd)

## \* State-space models

General:

$$\begin{aligned} Y_t &= G_t \mathbf{X}_t + W_t & \{\mathbf{W}_t\} &\sim \text{WN}(\mathbf{0}, \{R_t\}) \\ \mathbf{X}_{t+1} &= F_t \mathbf{X}_t + \mathbf{V}_t & \{\mathbf{V}_t\} &\sim \text{WN}(\mathbf{0}, \{Q_t\}) \end{aligned}$$

When  $F$ ,  $G$ ,  $R$  and  $Q$  are time independent:

$$\begin{aligned} Y_t &= G\mathbf{X}_t + W_t & \{W_t\} &\sim \text{WN}(\mathbf{0}, R) \\ \mathbf{X}_{t+1} &= F\mathbf{X}_t + \mathbf{V}_t & \{\mathbf{V}_t\} &\sim \text{WN}(\mathbf{0}, Q) \end{aligned}$$

### \* SARIMA

A SARIMA model is an extended version of the ARIMA model which can also consider a seasonal component in a time series. Its general form

$$SARIMA(p, d, q) \times (P, D, Q)_s$$

is a seasonal ARIMA process with period  $s$  and is written as

$$\phi(B)\Phi(B^S)Y_t = \theta(B)\Theta(B^S)Z_t, \quad \{Z_t\} \sim \text{WN}(0, \sigma^2)$$

if the differenced series  $Y_t = (1 - B)^d(1 - B^s)^D X_t$  is a causal ARMA process defined as above. Here,  $\phi(z) = 1 - \phi_1 z - \dots - \phi_p z^p$ ,  $\Phi(z) = 1 - \Phi_1 z - \dots - \Phi_P z^P$ ,  $\theta(z) = 1 + \theta_1 z + \dots + \theta_q z^q$  and  $\Theta(z) = 1 + \Theta_1 z + \dots + \Theta_Q z^Q$ .

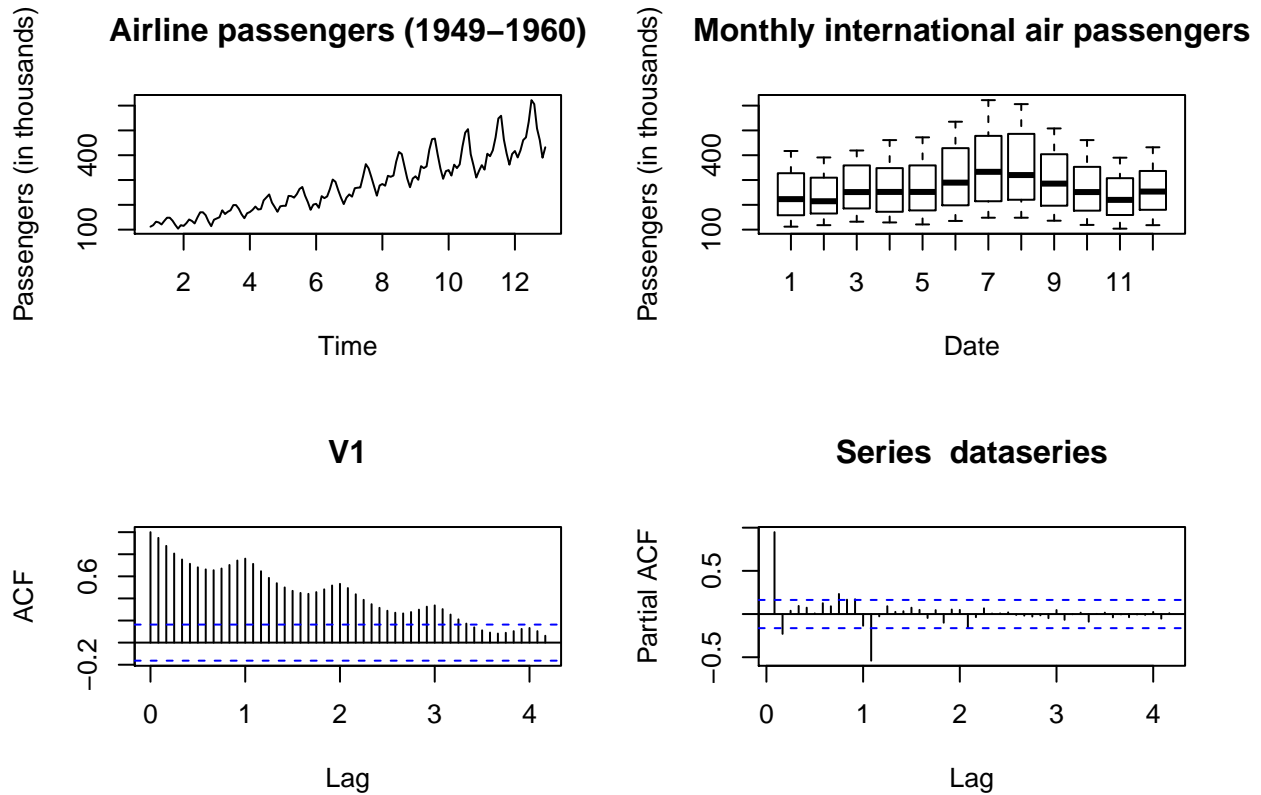
The SARIMA model has two parts: A seasonal component and a non-seasonal component.  $(P, D, Q)$  is the ARIMA model between each season and is the seasonal component of the SARIMA model. This part models behaviour for for instance the January months in a time series with monthly observations and where the season is a year long.  $(p, d, q)$  is the non-seasonal part of the model and is an ARIMA model *within* each season. The interpretation of  $p, d, q, P, D$  and  $Q$  are known from previous knowledge on the ARIMA models, and  $B$  is the backshift operator as before.

## Data analysis

1. Exploring the data with plotting of relevant statistics
2. Justification of the choice of model
3. Model parameter estimation including uncertainty
4. Model prediction at the given sample points and for the next year including uncertainty of the best linear predictions
5. Diagnostics, and model choice discussion including comparison of the two models

### $SARIMA(p, d, q) \times (P, D, Q)_s$ model

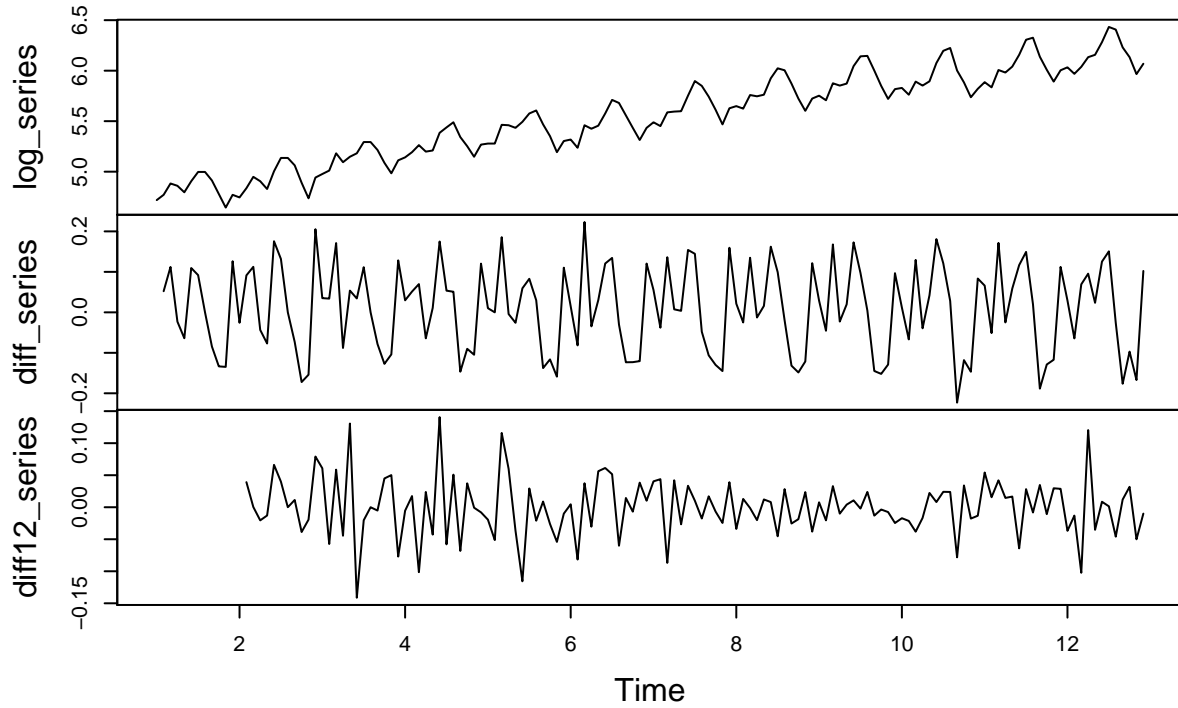
The data set consists of total number of international airline passengers, in thousands, for each month from January 1949 to December 1960 (reference to data set), giving a total number of  $N = 144$  observations.



From the time series plot, this is obviously a non-stationary series as both trend and seasonality is visible in the visualization. Trend is there because the total number of airline passengers on average increases for each month, and seasonality because of the wave-like behaviour of the series. Non-stationarity is further supported by looking at the autocorrelation function, which shows considerable correlation between observations even up to lag  $h$  of size 30, and again a periodic wave-like pattern. Furthermore, the box plot shows both higher mean number of passengers and also higher variance for months 6 to 9 in the year, i.e. June til September.

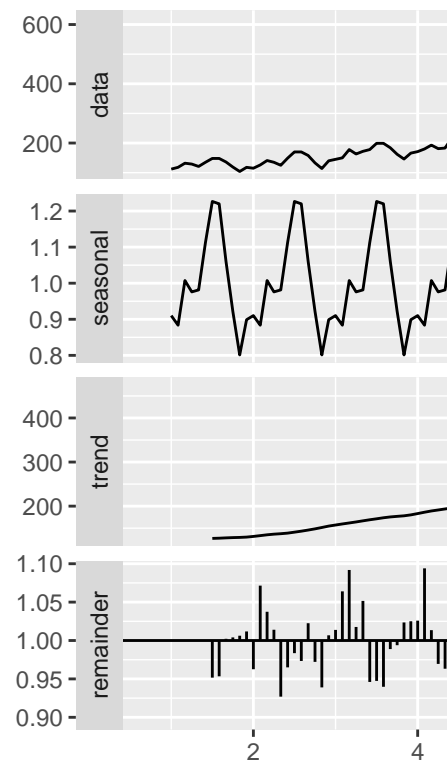
To investigate the data, we first find a transformation to the time series. A typical transformation for making the series more stationary is the  $\log()$ -transformation. This seems to stabilize the multiplicative behaviour of the variance so that the variance is not increasing with time.

## Transformed and differenced data

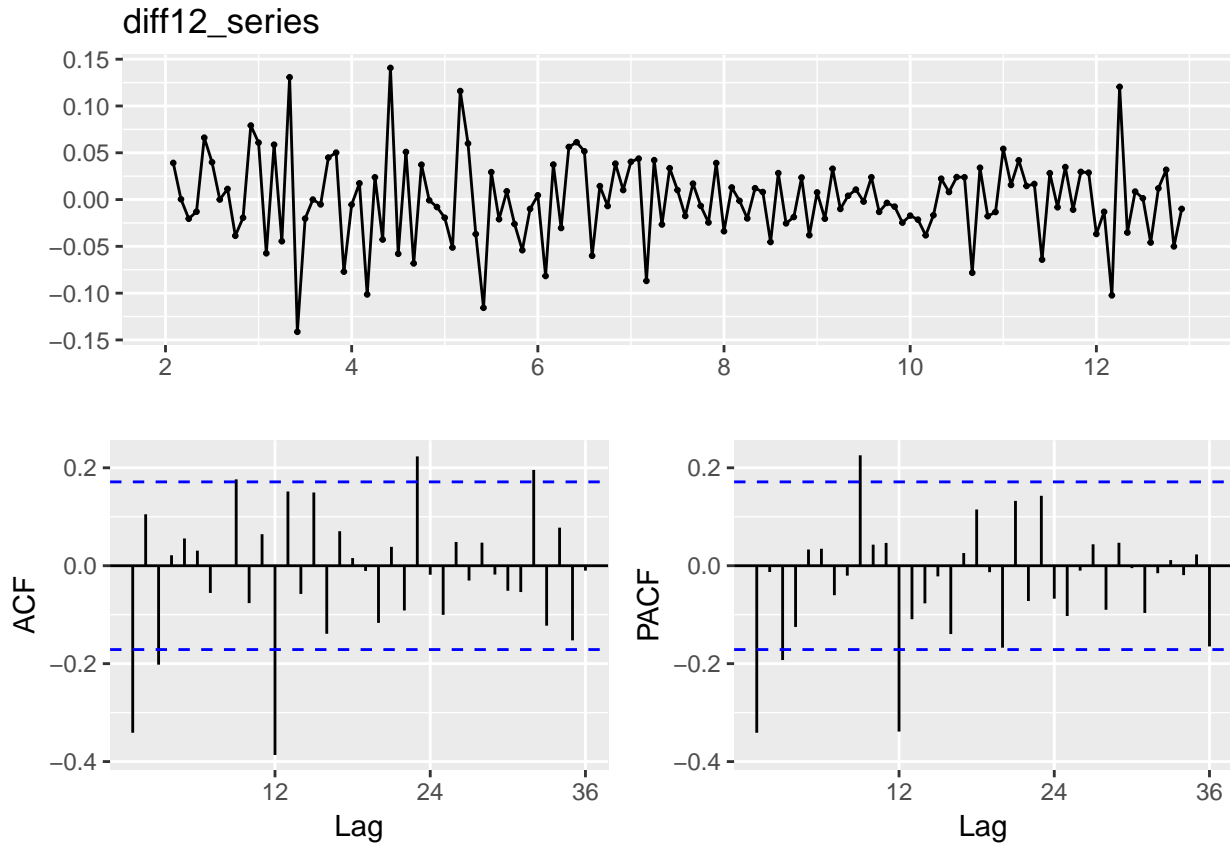


Secondly, the data is differenced to remove trend, i.e. stabilizing the mean. The parameter  $d$  is related to trend within a season, and since this trend seems to be removed by differencing once, we try  $d = 1$ . See the plot in the middle of the figure for transformed and differenced data. After differencing there still is a wave pattern present, one for each year. This indicates a seasonal trend equal to the length of a year, i.e. we suspect a  $s = 12$  in a seasonal ARIMA model. Then  $(1 - B^{12})$  is applied to the series. The parameter  $D$  is related to the trend between seasons, i.e. the trend one can see from one time of the year to the next, and the next, and so on. We therefore set  $D = 1$ . Differencing once indicates linear trend, both within and between the seasons.

## Decomposition of mu



Our model parameter choices can be checked by comparing to the decomposed time series: which clearly shows linear trend and a season equal to a year.



The obtained series after applying the twelfth-order difference AGAIN: PROBABLY NOT RIGHT seems stationary without any trend or seasonality. We can then estimate the remaining parameters by considering the ACF and the PACF. This could be done by looking at lags equal to  $1s, 2s, \dots, s = 12$  to determine  $P$  and  $Q$  in the seasonal component and by looking at smaller lags in each season to determine  $p$  and  $q$  in the non-seasonal component. In this case, however, we have used the AICC criterion and tested models with  $d = 1, D = 1, s = 12$ , which is determined before, and  $p, q, P, Q < 5$  which is based by looking at the ACF and PACF. In addition we prefer models with smaller parameters for simplicity if they are a good fit.

```
##
## ARIMA(2,1,2)(1,1,1)[12] : Inf
## ARIMA(0,1,0)(0,1,0)[12] : -434.799
## ARIMA(1,1,0)(1,1,0)[12] : -474.6299
## ARIMA(0,1,1)(0,1,1)[12] : -483.2101
## ARIMA(0,1,1)(1,1,1)[12] : -481.5957
## ARIMA(0,1,1)(0,1,0)[12] : -449.8857
## ARIMA(0,1,1)(0,1,2)[12] : -481.6451
## ARIMA(0,1,1)(1,1,2)[12] : Inf
## ARIMA(1,1,1)(0,1,1)[12] : -481.582
## ARIMA(0,1,0)(0,1,1)[12] : -467.4644
## ARIMA(0,1,2)(0,1,1)[12] : -481.2991
## ARIMA(1,1,2)(0,1,1)[12] : -481.5633
##
## Best model: ARIMA(0,1,1)(0,1,1)[12]
```

Both parameters are significant in the model, and the residual analysis plot shows that our model is a good fit. In addition, the uncertainty of the parameters are given from `fit$table`. Our chosen model is thus

$$\text{SARIMA}(0, 1, 1) \times (0, 1, 1)_{12}$$

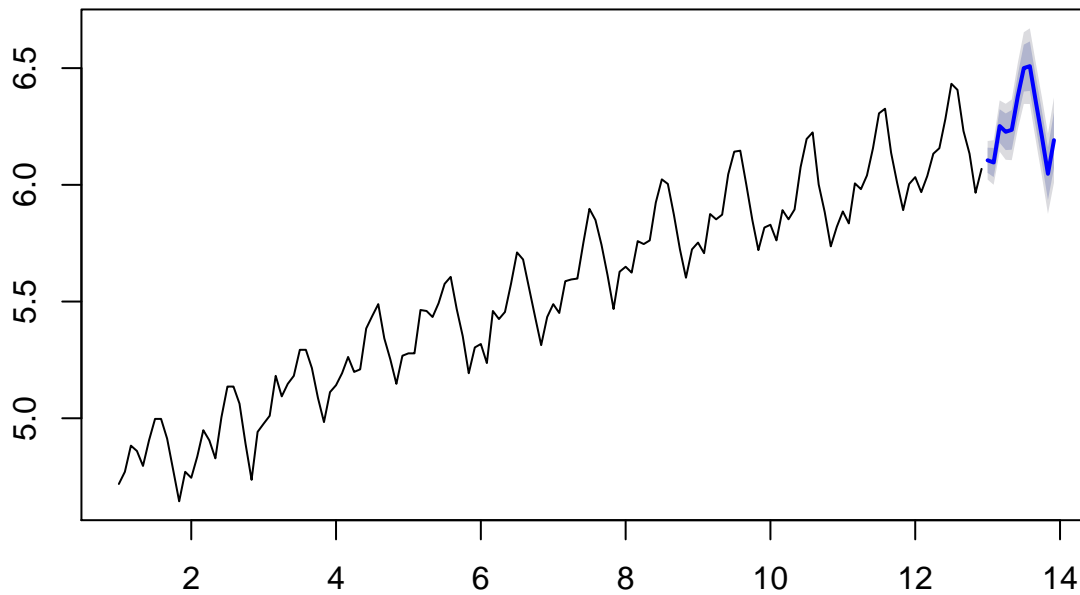
which is equal to

$$\phi(B)\Phi(B^{12})(1-B)(1-B^{12})X_t = \theta(B)\Theta(B^{12})Z_t$$

When we have found our model, we can do forecasting with the SARIMA model for the next twelve months. From the fitting functions in R, we can also extract credible intervals of the forecasts and the fitted values at our given sample points.

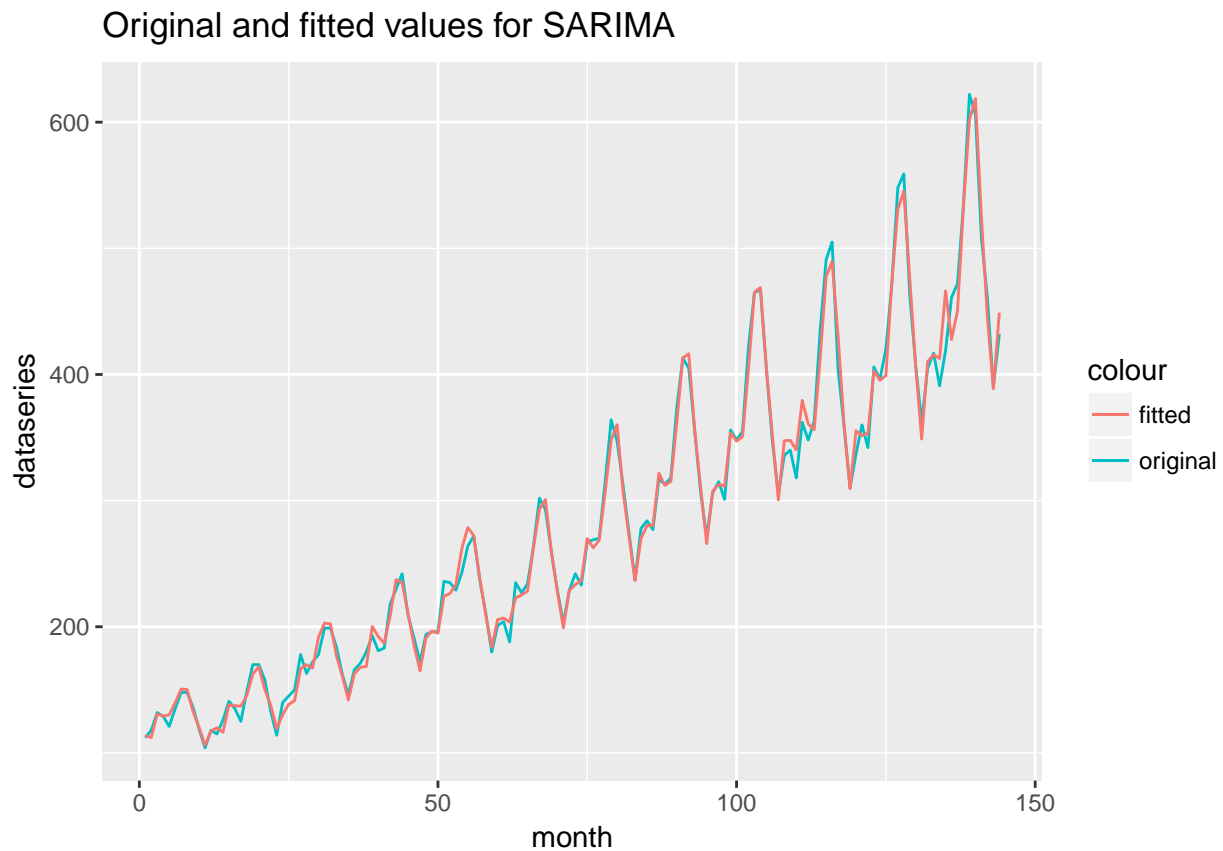
The following plot shows the time series and the forecasted values for the next twelve months:

### Forecasts from ETS(M,A,M)



The credible intervals for the forecasts are

##	X95percent_lower	X95percent_upper
## 1	412.6662	486.7393
## 2	403.4885	488.4474
## 3	465.0367	579.2250
## 4	449.1705	570.9354
## 5	448.4353	581.3636
## 6	511.4191	679.4079
## 7	570.6165	775.6796
## 8	570.3655	788.6390
## 9	487.0434	678.6601
## 10	418.5346	587.0612
## 11	356.2822	502.1102
## 12	407.1544	586.1755



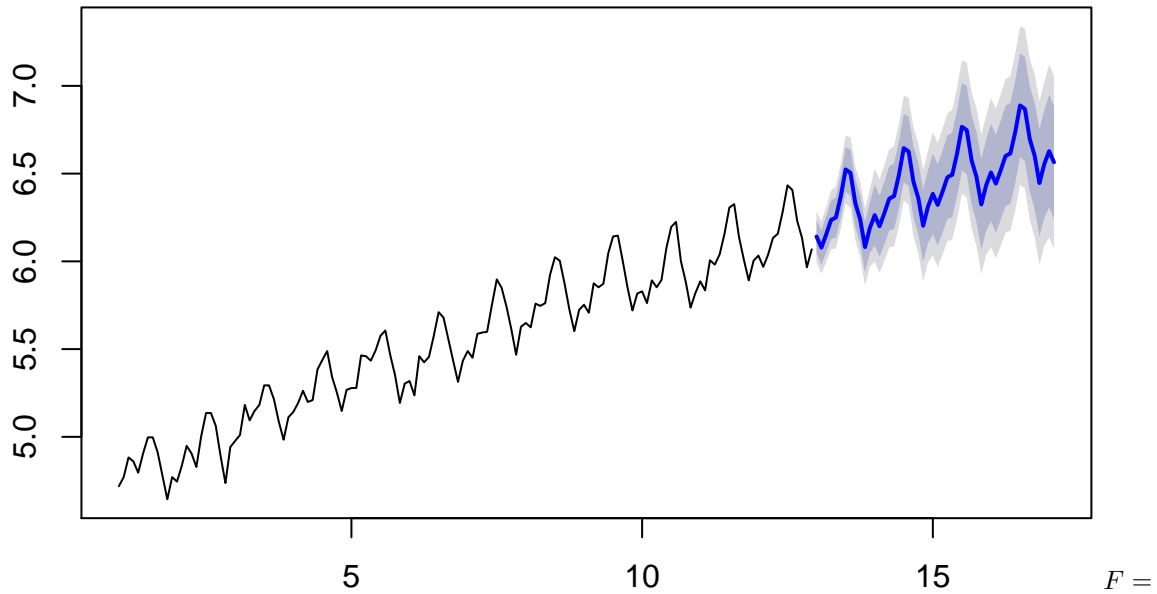
The fitted data looks good compared to the original data and the forecasting seems to continue the season and trend of the data well.

### State-space model

We build a state-space model by [ref to theory](#) or just use Rfunctions, which gives the following matrices



## Forecasts from Basic structural model



```
##      [,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10] [,11] [,12] [,13]
## [1,]    1    1    0    0    0    0    0    0    0    0    0    0    0
## [2,]    0    1    0    0    0    0    0    0    0    0    0    0    0
## [3,]    0    0   -1   -1   -1   -1   -1   -1   -1   -1   -1   -1   -1
## [4,]    0    0    1    0    0    0    0    0    0    0    0    0    0
## [5,]    0    0    0    1    0    0    0    0    0    0    0    0    0
## [6,]    0    0    0    0    1    0    0    0    0    0    0    0    0
## [7,]    0    0    0    0    0    1    0    0    0    0    0    0    0
## [8,]    0    0    0    0    0    0    1    0    0    0    0    0    0
## [9,]    0    0    0    0    0    0    0    1    0    0    0    0    0
## [10,]   0    0    0    0    0    0    0    0    1    0    0    0    0
## [11,]   0    0    0    0    0    0    0    0    0    1    0    0    0
## [12,]   0    0    0    0    0    0    0    0    0    0    1    0    0
## [13,]   0    0    0    0    0    0    0    0    0    0    0    1    0
```

$G =$

```
## [1] 1 0 1 0 0 0 0 0 0 0 0 0 0
```

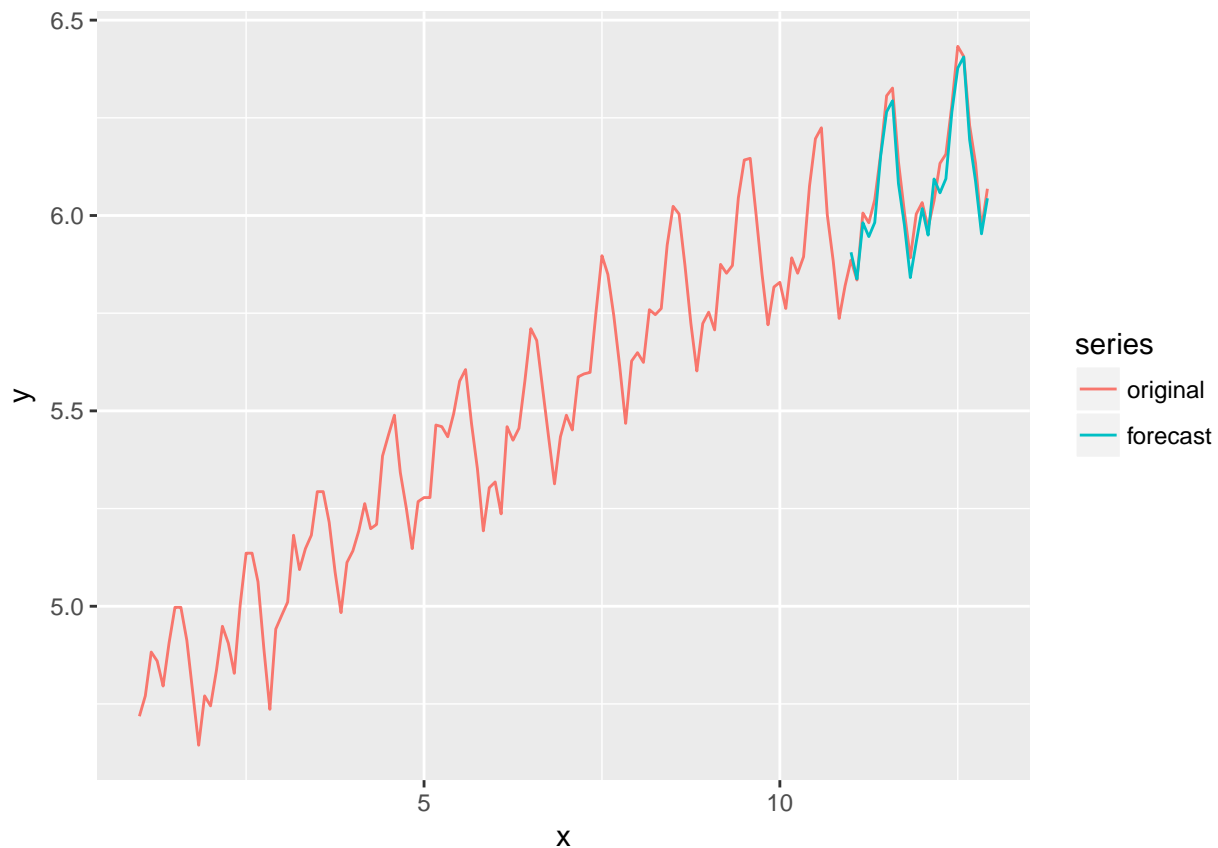
$Q =$

```
##      [,1]      [,2]      [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10] [,11]
## [1,]    0  0.0000  0.00000    0    0    0    0    0    0    0    0
## [2,]    0 160.9755  0.00000    0    0    0    0    0    0    0    0
## [3,]    0  0.0000 29.84652    0    0    0    0    0    0    0    0
## [4,]    0  0.0000  0.00000    0    0    0    0    0    0    0    0
## [5,]    0  0.0000  0.00000    0    0    0    0    0    0    0    0
## [6,]    0  0.0000  0.00000    0    0    0    0    0    0    0    0
## [7,]    0  0.0000  0.00000    0    0    0    0    0    0    0    0
## [8,]    0  0.0000  0.00000    0    0    0    0    0    0    0    0
## [9,]    0  0.0000  0.00000    0    0    0    0    0    0    0    0
## [10,]   0  0.0000  0.00000    0    0    0    0    0    0    0    0
## [11,]   0  0.0000  0.00000    0    0    0    0    0    0    0    0
## [12,]   0  0.0000  0.00000    0    0    0    0    0    0    0    0
```

```
## [13,]    0    0.0000    0.00000    0    0    0    0    0    0    0    0    0
##      [,12] [,13]
## [1,]    0    0
## [2,]    0    0
## [3,]    0    0
## [4,]    0    0
## [5,]    0    0
## [6,]    0    0
## [7,]    0    0
## [8,]    0    0
## [9,]    0    0
## [10,]   0    0
## [11,]   0    0
## [12,]   0    0
## [13,]   0    0
```

This gives  $(\hat{\sigma}_1^2, \hat{\sigma}_2^2, \hat{\sigma}_2^3) = (0.0000, 160.9755, 29.84652)$ .

```
## [1] 417.8507614 -29.6143515 14.1492386 -57.4651129 -23.6949346
## [6] -12.2523959 56.6479592 60.1013022 -9.8206840 -37.0447315
## [11] -4.5661018 -0.8086603 -7.7654562
```



?? Also need to forecast for given sample points here and find uncertainty. ??

## Bootstrap

We bootstrap the estimated parameters for the state space model.

- Model diagnostics for state-space? I don't know. \
  - Model comparison + discussion. \
  - Theory \ ??
- 

**Discussion**

**Conclusion**

**Appendix**

**References**