Variance

$Var(X) = \mathbb{E}\{[X-\mathbb{E}(X)]^2\} = \sigma^2$

We call $\sigma = \sqrt{Var(X)}$

SD of X

Computational Formula

$Var(X) = \sum_x (X-\mathbb{E}(X))^2 P_X(x)$

$\boxed{Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2}$

$\mathbb{E}[X^n] \triangleq$ nth moment of X

Properties of Variance

$Var(\underbrace{aX+c}_{Y}) = a^2 Var(X)$

$Y = aX + c$

A nonzero mean distr & a zero-mean distr have the same variance.

$\boxed{\sigma_{aX+c} = |a|\sigma_X}$

Theorem: Variance of the sum of RVs

Let X, Y be RVs,

$$Var(X+Y) = Var(X) + Var(Y) + 2(\underbrace{\mathbb{E}[XY] - \mathbb{E}[X]\cdot\mathbb{E}[Y]}_{\text{Covariance}})$$

Collolary: Var of indep RVs

Let X, Y be indep,

$$Var(X+Y) = Var(X) + Var(Y)$$

If X, Y indep,

$\mathbb{E}[XY] = \mathbb{E}[X]\cdot\mathbb{E}[Y] \Rightarrow Cov(X,Y) = 0$

If $X_1, \ldots, X_n$ are pairwise indep,

$Var(X_1 + \ldots + X_n) = \sum_{i=1}^{n} var(X_i)$

① Bernoulli

$$X = \begin{cases} 0 & \text{w.p. } 1-p \\ 1 & \text{w.p. } p \end{cases}$$

$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

$= \mathbb{E}[X] - \mathbb{E}[X]^2$

$\boxed{Var(X) = p(1-p)}$

② Binomial

$X \sim Binom(n,p)$ is sum of iid $B(p)$

$X = X_1 + \ldots + X_n$

$Var(X) = Var(X_1 + \ldots + X_n)$

$= Var(X_1) + \ldots + Var(X_n)$

$\boxed{Var(X) = np(1-p)}$

Conditional Expectation

Note: $P(X=k \mid Y=m) = \dfrac{P(X=k, Y=m)}{P(Y=m)}$

$= P_{X|Y}(k|m)$

$P_{X|Y}(k|m)$ called the conditional PMF of X given Y.

↳ $X|Y$ is just another RV.

① $\boxed{\mathbb{E}[X \mid Y=y] = \sum_x x P(X=x \mid Y=y)}$

↳ real number like any other expectation

Note: eqn ① is defined for all value of y.

$\boxed{\mathbb{E}[X|Y] \text{ is a fuction of } Y.}$

Note: $\mathbb{E}[X|Y]$ is a RV.

Ex: Roll a die N times where N is a RV.

If $N=1$, $\mathbb{E}[X|N=1] = \frac{7}{2}$

$N=2$, $\mathbb{E}[X|N=2] = 7$

$N=n$, $\mathbb{E}[X|N=n] = \frac{7n}{2}$

$\mathbb{E}[X|N] = \frac{7}{2}N$

Law of Iterated Expectation (LIE)

$\boxed{\text{Theorem: } \mathbb{E}[\underbrace{\mathbb{E}[X|Y]}_{f(Y) \leftarrow RV}] = \mathbb{E}[X]}$

Proof: $\underset{\text{wrt } Y}{\mathbb{E}}[\underset{\text{wrt } X}{\mathbb{E}}[X|Y]]$

$= \sum_y \underbrace{\mathbb{E}[X|Y=y]}_{\sum_x x P(X=x|Y=y)} P(Y=y)$

$= \sum_y \sum_x x \underbrace{P(X=x|Y=y) P(Y=y)}_{P(X=x, Y=y)}$

$= \sum_x x \underbrace{\sum_y P(X=x, Y=y)}_{P(X=x)}$ ← switch x, y bound order

$= \sum_x x P(X=x)$

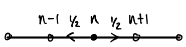
$= \mathbb{E}[X]$

Suppose in dice Ex, $N \sim Geom(p)$

We saw that $\mathbb{E}[X|N] = \frac{7}{2}N$

$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|N]] = \frac{7}{2}\mathbb{E}[N] = \frac{7}{2} \cdot \frac{1}{p}$

Ex: Random Walk



A drunk walks along real integer line. He starts at origin. W.p. $\frac{1}{2}$ he takes a positive step $(X_i = 1)$; w.p. $\frac{1}{2}$ he takes negative step $(X_i = -1)$.

$\mathbb{E}[X_n]$ = ? location of man at $t=n$ (time)

$X_{n+1} = X_n + \underbrace{\mathbb{1}_+ + \mathbb{1}_-}_{\text{indicator RVs}}$

$\mathbb{E}[X_{n+1}] = \mathbb{E}[X_n] + \frac{1}{2} - \frac{1}{2}$

Since $X_0 = 0 \Rightarrow \mathbb{E}[X_0] = 0$

$\Rightarrow \boxed{\mathbb{E}[X_n] = 0 \ \forall n}$

2/ $Var(X_n) = ?$

Let's calculate $\mathbb{E}[X_n^2]$

$P[X_{n+1}^2 = (k+1)^2 \mid X_n = k] = P[X_{n+1}^2 = (k-1)^2 \mid X_n = k] = \frac{1}{2}$

$\mathbb{E}[X_{n+1}^2 \mid X_n = k] = \frac{1}{2}(k-1)^2 + \frac{1}{2}(k+1)^2$

$= k^2 + 1$

$\mathbb{E}[X_{n+1}^2 \mid X_n] = X_n^2 + 1$

$\mathbb{E}[X_{n+1}^2] = \mathbb{E}[\underbrace{\mathbb{E}[X_{n+1}^2 \mid X_n]}_{X_n^2+1}]$

$\Rightarrow \mathbb{E}[X_{n+1}^2] = \mathbb{E}[X_n^2] + 1$

Since $\mathbb{E}[X_0^2] = 0$

$\Rightarrow \boxed{\begin{array}{l} \mathbb{E}[X_n^2] = n \\ SD = \sqrt{n} \end{array}}$

Conditional Variance

$Var(X \mid Y = y) = \mathbb{E}[X^2 \mid Y = y] - (\mathbb{E}[X \mid Y = y])^2$

$\boxed{\begin{array}{l} \text{Law of Total Variance:} \\ \quad Var(X) = \mathbb{E}[Var(X \mid Y)] + var(\mathbb{E}[X \mid Y]) \end{array}}$