Independence

Two events are independent if the occurrence of one provides no info about the occurrence of the other.

$P(A|B) = P(A)$

$P(A \cap B) = P(A) \cdot P(B)$

Remarks:

1/ Disjoint $\neq$ Independence - A & B are disjoint iff $P(A \cap B) = 0$

If 2 events are disjoint & indep, then $P(A) = 0$ or $P(B) = 0$.

2/ The base outcomes of any exp are disjoint & they have nonzero prob $\Rightarrow$ always dependent

Ex: 1/ A: 1st die 6

B: sum of 2 dice = 7

$P(A|B) = \frac{1}{6}$, $P(A) = \frac{1}{6}$ $\Rightarrow$ A,B indep

2/ A: 1st die 6

C: sum of 2 dice = 11

$P(A|C) = \frac{1}{2}$, $P(A) = \frac{1}{6}$ $\Rightarrow$ A, C dep

Conditional Independence

$P(A \cap B|C) = P(A|C) \cdot P(B|C)$

Notes: 1/ Dependent events can be cond. indep

2/ Indep events can be cond. dep

Ex: 2 indistinguishable coins - one is two-tailed, other is two-headed

You pick one of the 2 coins at random & flip it twice.

H_i: ith flip is heads

1/ Are H_1 & H_2 indep? No

2/ If not, what's $P(H_2|H_1)$? $= 1$

3/ Conditioned on picking a coin, are the 2 flips indep?

A = Pick 2-headed coin

$P(H_1 \cap H_2 | A) = P(H_1|A) \cdot P(H_2 | H_1 \cap A)$

$= P(H_1|A) \cdot P(H_2|A)$ $\leftarrow$ indep

Indep events can be conditionally dependent.

Ex: 2 tosses of a fair coin

H_1: 1st toss heads

H_2: 2nd toss heads

D : 1st $\neq$ 2nd toss

1/ Are H_1 & H_2 indep? Yes

2/ Are H_1 & H_2 indep, given D?

$P(H_1 \cap H_2 | D) = 0 \neq P(H_1|D) \cdot P(H_2|D) = \frac{1}{2} \cdot \frac{1}{2}$

Independence of collection of events:

$P(\bigcap_{i \in S} A_i) = \prod_{i \in S} P(A_i)$

Pairwise indep $\not\Rightarrow$ joint indep

Ex (prev): - H_1 & H_2 indep

- H_1 & D indep

- H_2 & D indep

$\}$ BUT H_1, H_2, D not indep

Ex: Rolling a 6 before a 7 (sum)

Roll 2 dice. What's the prob of rolling a 6 before a 7? ↙E

Use indep:

- Condition on 1st roll

① Let S be the event that 1st roll is 6

② Let T be the event that 1st roll is 7

$$P(E) = \underbrace{P(E|S)}_{1}\underbrace{P(S)}_{5/36} + \underbrace{P(E|T)}_{0}P(T) + \underbrace{P(E|(S\cup T)^c)}_{P(E)}\underbrace{P((S\cup T)^c)}_{25/36}$$

Solve: $P(E) = \frac{5}{11}$

$\{(1,5),(2,4),\ldots,(5,1)\}$ all outcomes that constitute a 6

$\{(1,6),\ldots,(6,1)\}$ all outcomes that constitute a 7 + $\{(1,5),(2,4),\ldots,(5,1)\}$ all outcomes that constitute a 6 } 11 total

Bayes' Theorem

- $A_1, \ldots, A_n$ partition Ω
- we know $P(A_i)$
- we also know $P(B|A_i)$
- we want $P(A_i|B)$

Ex: Medical Test

A_1 = malignant tumor

A_2 = benign tumor

A_3 = other

B = test is positive

We know $P(B|A_i)$ ← accuracy of test

We want $P(A_i|B)$ ← what are the odds I have a bad tumor given positive test?

$$\boxed{P(A_i|B) = \frac{P(A_i\cap B)}{P(B)} = \frac{P(B|A_i)\cdot P(A_i)}{\sum_{i=1}^{N} P(A_i)\cdot P(B|A_i)}}$$

False Positive Test

- If person has disease, test is pos w.p. 0.95
- If person does not have disease, test is neg w.p. 0.95
- Random person has disease w.p. 0.001

prior

A person tests pos. P(person has disease) = ?

A: has disease

B: tests pos

$$P(A|B) = \frac{P(B|A)P(A)}{P(A)P(B|A) + P(A^c)P(B|A^c)} = \frac{(0.001)(0.95)}{(0.001)(0.95)+(0.999)(0.05)}$$

$$= 0.0187 \quad (<2\%)$$

Note: $P(A|B)$ heavily influenced by the prior $B|A$.

Discrete RVs

From events to RVs

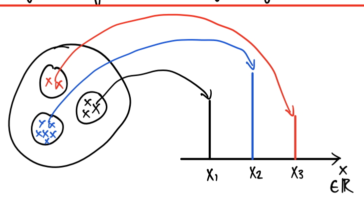
- RVs associate a real # w/ each event

Ex: ① The RV X has value i if the throw of a die is {i}.

② X^2: perfectly legit RV

③ Assigning real #s allows us to do statistics

↳ More formally, a RV is a func from $\Omega \to \mathbb{R}$



Probability Mass Function (PMF)

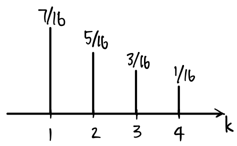
$P(X=x)$ or $P_X(x)$

Ex: 2 4-sided dice

	1	2	3	4
1	1	1	1	1
2	1	2	2	2
3	1	2	3	3
4	1	2	3	4

M_k: Event that min of the 2 is k

M: RV that is equal to min of the 2 dice



PMF / Distribution (histogram)