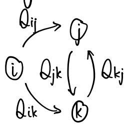
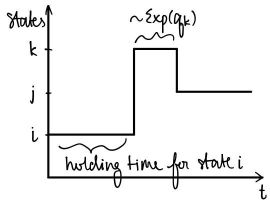
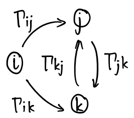
Recap of CTMCs





$$Q = \begin{bmatrix} Q_{11} & Q_{12} & & \\ Q_{21} & \cdot & & \\ \vdots & & \cdot & \\ Q_{n1} & & & Q_{nn} \end{bmatrix}$$

Jump Chain



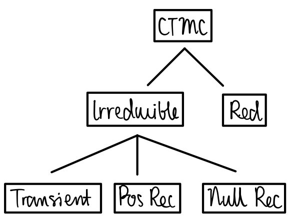
DTMC

Row sums of $Q = 0$

$\Rightarrow \sum_{j=1}^{n} Q_{ij} = -Q_{ii} = q_i \ , i \neq j$

$Q_{ij} \geq 0 \, , i \neq j$

CTMC Big Theorem

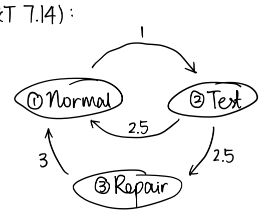


↳ unique stationary distr π

(no "periodicity" issues)

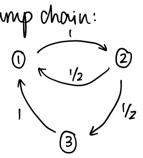
$\lim_{t\to\infty} \pi_t = \pi$

Ex (B&T 7.14):



"ignores" holding time

Jump chain:



Q/ What is the stationary distr π for the CTMC?

$\pi Q = 0$

$$Q = \begin{bmatrix} -1 & 1 & 0 \\ 2.5 & -5 & 2.5 \\ 3 & 0 & -3 \end{bmatrix}$$

Rate in = Rate out

① $2.5\pi_2 + 3\pi_3 = 1\pi_1$

② $1\pi_1 = 5\pi_2$

③ $\pi_1 + \pi_2 + \pi_3 = 1$

Solve: $\pi = \frac{1}{41}[30, 6, 5]$

Jump chain:

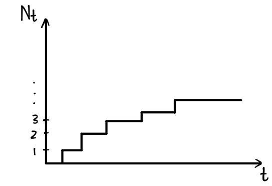
$$[p_1 \ p_2 \ p_3] = [p_1 \ p_2 \ p_3] \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \end{bmatrix}$$

$p = pP$

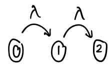
$p = \frac{1}{5}[2, 2, 1]$

* Note: $p \neq \pi$ since we ignore the holding times of the states.

Ex:



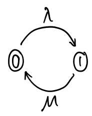
PP(λ) → CTMC



transient

$$Q = \begin{bmatrix} \boxed{-\lambda \ \lambda} & 0 & 0 & 0 & \\ 0 & \boxed{-\lambda \ \lambda} & 0 & 0 & \\ 0 & 0 & \boxed{-\lambda \ \lambda} & & \\ & & & \ddots & \\ & & & & \end{bmatrix}$$

Ex: 2-state CTMC



$$Q = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

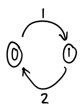
$\pi Q = 0$

$$[\pi_0 \ \pi_1] \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix} = [0 \ 0]$$

$$\pi_0 = \frac{\mu}{\mu+\lambda}, \quad \pi_1 = \frac{\lambda}{\mu+\lambda}$$

e.g. $\lambda = 1$, $\mu = 2$

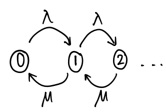
$\pi = [\frac{2}{3} \ \frac{1}{3}]$



you're "parked" in state 0 more (twice as much as in state 1)
because you come back at a faster rate from 1 to 0

Intuition: Chain spends twice as much time in state 0 as in state 1 because its transition rate from 0 to 1 is half the transition rate from 1 to 0.

Ex: Birth-death chain



$$Q = \begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 & \\ \mu & -(\mu+\lambda) & \lambda & 0 & 0 & \\ 0 & \mu & -(\mu+\lambda) & \lambda & 0 & \\ & & & & \ddots & \\ & & & & & \end{bmatrix}$$

$\lambda < \mu \Rightarrow$ Pos rec

$\lambda = \mu \Rightarrow$ Null rec

$\lambda > \mu \Rightarrow$ Trans

$\rho = \frac{\lambda}{\mu} < 1$

Use detail balanced eqns to find π

$\pi_0 \lambda = \pi_1 \mu \Rightarrow \pi_1 = \rho \pi_0$

$\pi_1 \lambda = \pi_2 \mu \Rightarrow \pi_2 = \rho \pi_1 = \rho^2 \pi_0$

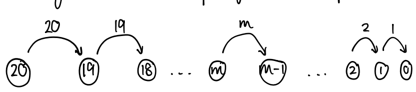
$\vdots$

$\pi_n = \rho^n \mu_0$

$\sum_{i=0}^{\infty} \pi_i = 1 \Rightarrow \pi_0 = 1-\rho \Rightarrow \pi_n = \rho^n (1-\rho), \ n \geq 0$

Hitting Times for CTMCs

Consider 20 light bulbs w/ indep lifetimes ~ Exp(1) month. On avg, how long before all bulbs are burnt out?



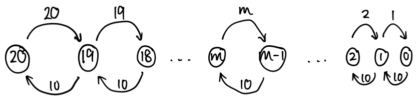
FSE: $\beta_0(m)$ = Exp time to hit 0 given that you start at m $(0 \leq m \leq 20)$

$\beta_0(m) = \frac{1}{m} + \beta_0(m-1)$

$\beta(20) = \frac{1}{20} + \frac{1}{19} + \ldots + 1 \approx 3.6$ months

Ex: Assume that burnt out bulbs are replaced after an indep Exp(10) rate (i.e. mean of 0.1 month).

What's the exp time now for all bulbs to burn out?



holding time ($\sim Exp(q_{ii}$ = sum of outgoing rates$)$)

FSE: $\beta(20) = \frac{1}{20} + \beta(19)$

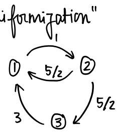
$\beta(19) = \frac{1}{29} + \frac{19}{29}\beta(18) + \frac{10}{29}\beta(20)$

$\beta(m) = \frac{1}{m+10} + \frac{m}{m+10}\beta(m-1) + \frac{10}{m+10}\beta(m+1)$

$\Rightarrow$ Solve recursively: $\beta_0(20) \approx 2488$ months

Simulating a CTMC of DTMC

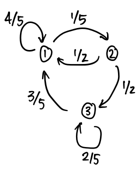
"uniformization"



$q_1 = 1,\ q_2 = 5,\ q_3 = 3$

$q = \max_i q_i = 5$ ← Normalization

$\Downarrow$ DTMC



$$P = \begin{bmatrix} 4/5 & 1/5 & 0 \\ 1/2 & 0 & 1/2 \\ 3/5 & 0 & 2/5 \end{bmatrix}$$

$\pi P = \pi$ ← Solve: $\pi = \{\frac{30}{41}, \frac{6}{41}, \frac{5}{41}\}$

same as $\pi Q = 0$

jump chain

$T_{ij} = \frac{Q_{ij}}{q_i}$, $q_i \le q$ (recall $q \ge \max_i q_i$)

$\Rightarrow P_{ij} = \frac{Q_{ij}}{q} \Rightarrow P_{ii} = 1 - \frac{q_i}{q}$ (recall $P_{ii} = 1 - \frac{\Sigma Q}{q}$)

DTMC

self-loop prob

↳ If $q_i = q$, $T_{ij} = P_{ij}$ & $P_{ii} = 0$.

Matrix form:

$\boxed{P = I + \frac{1}{q}Q}$

↑ normalization

$$\begin{cases} P_{ij} = \frac{Q_{ij}}{q}, & i \ne j \\ P_{ii} = 1 - \frac{q_i}{q} \\ \quad\ = 1 + \frac{Q_{ii}}{q} \end{cases}$$

$P - I = \frac{1}{q}Q$

$Q = q(P - I)$

$\pi Q = 0$

$\cancel{q}\pi(P - I) = 0$

$\pi P = \pi$