## Kalman Filter

The KF algorithm updates the estimate of the state of a system as shown:

$$\boxed{X(n)} \xrightarrow[\text{output}]{Y(n)} \boxed{KF} \longrightarrow \hat{X}(n)$$

The system has a state $X(n)$ and an observation/output.

$Y(n)$ at time $n = 0, 1, \ldots$ according to:

$$\left.\begin{aligned} X(n+1) &= AX(n) + V(n) \quad ; \ n \geq 0 \\ Y(n) &= CX(n) + W(n) \quad ; \ n \geq 0 \end{aligned}\right\} \text{state-space eqns}$$

The RVs $\{X(0), \{V(n), W(n)\}_{n \geq 0}\}$ are all orthogonal & zero-mean.

$Cov(V_n) = \Sigma_V$ ; $Cov(W_n) = \Sigma_W$

Objective of the KF: Estimate $\hat{X}(n) = L[X(n) \mid Y(0), \ldots, Y(n)]$ ; $n \geq 0$

**Theorem: Kalman Filter**

(1) $\hat{X}_{n|n} = A\hat{X}_{n-1|n-1} + K_n(Y_n - CA\hat{X}_{n-1|n-1})$

(2) $K_n = \Sigma_{n|n-1} C^T [C \Sigma_{n|n-1} C^T + \Sigma_W]^{-1}$

(3) $\Sigma_{n|n-1} = A \Sigma_{n-1|n-1} + \Sigma_V$

(4) $\Sigma_{n|n} = (I - K_n C) \Sigma_{n|n-1}$

Note: $\Sigma_W = Cov(W_n)$

$\Sigma_V = Cov(V_n)$

$\Sigma_{n|n-1} = Cov(\overbrace{X_n - A\hat{X}_{n-1|n-1}}^{\Delta_{n|n-1}})$

$\Sigma_{n|n} = Cov(\overbrace{X_n - \hat{X}_{n|n}}^{\Delta_{n|n}})$

Notation:

$\hat{X}_{n|n} = L[X_n \mid Y_1, \ldots, Y_n] =$ LLSE of $X_n$ at time $n$

estimate of sample at time $n$

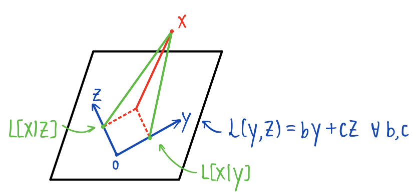
based on all observations up to time $n$

$\Delta_{n|n} = X_n - \hat{X}_{n|n} =$ estimated error of $X_n$ at time $n$

## Orthogonal Updates

Warm-up: Updating LLSE

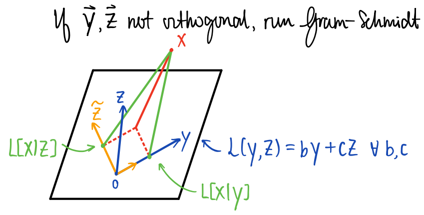
Theorem: If $X, Y, Z$ are zero-mean and $E[YZ] = 0$ $(Y \perp\!\!\!\perp Z)$, then $L[X \mid Y, Z] = L[X \mid Y] + L[X \mid Z]$.




$L[X \mid Y, Z] = L[X \mid Y] + L[X \mid Z]$ is equivalent to

$P_{\vec{Y},\vec{Z}}\vec{X} = P_{\vec{Y}}\vec{X} + P_{\vec{Z}}\vec{X}$ if $\vec{Y} \perp\!\!\!\perp \vec{Z}$

If $\vec{Y}, \vec{Z}$ not orthogonal, run Gram-Schmidt:




$L[X \mid Y, Z] = L[X \mid Y] + L[X \mid \tilde{Z}]$ where $\tilde{Z} = Z - P_Y Z = Z - L[Z \mid Y]$

KF: recursive way of updating estimates based on prediction & update (after observing new sample point) one at a time;

useful for real-time tracking scenarios

↳ estimate $X_n$ given $Y^n$ by first finding/predicting $L[X_n \mid Y_1, \ldots, Y_n]$ at time $n-1$, and then updating the estimate based on new observation $Y_n$ at time $n$

$L[X_n \mid Y_1, \ldots, Y_n] = L[X_n \mid Y_1, \ldots, Y_{n-1}] + L[X_n \mid \tilde{Y}_n]$

where $\tilde{Y}_n = Y_n - L[Y_n \mid Y_1, \ldots, Y_{n-1}]$

## Scalar Kalman Filter

State space eqn: $x_n = a x_{n-1} + v_n$ (state dynamics)

$y_n = c x_n + w_n$ (observation)

Goal / Estimate $x_n$ given $y^n = \{y_1, \dots, y_n\}$ in an online fashion (real-time).

Scalar Kalman Equations:

(1) $\hat{x}_{n|n} = \hat{x}_{n|n-1} + K_n \tilde{y}_n$ Kalman gain

(a) $\hat{x}_{n|n-1} = a \hat{x}_{n-1|n-1}$

(b) $\tilde{y}_n = y_n - \hat{x}_{n|n-1}$

(2) $K_n = \dfrac{\sigma^2_{n|n-1}}{\sigma^2_{n|n-1} + \sigma_w^2}$

(3) $\sigma^2_{n|n-1} = a^2 \sigma^2_{n-1|n-1} + \sigma_v^2$

(4) $\sigma^2_{n|n} = (1 - K_n)\, \sigma^2_{n|n-1}$

Remarks:

1/ At iteration $n$, the alg inputs $\hat{x}_{n-1|n-1}$, $\sigma^2_{n-1|n-1}$ (and new observation $y_n$) & outputs $\hat{x}_{n|n}$, $\sigma^2_{n|n}$.

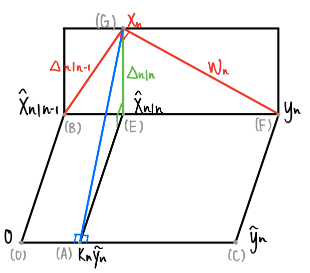
2/ The Kalman gain $K_n$ and the errors $\sigma^2_{n|n-1}$ & $\sigma^2_{n|n}$ can be precomputed.
(Only $\hat{x}_{n|n}$ needs to be computed online.)

3/ Easy to implement.

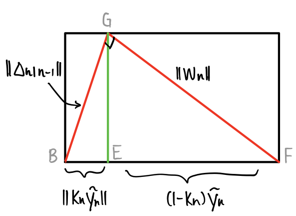
4/ If $v_n$ & $w_n$ are $\mathcal{N}(0, *)$, then we have MMSE estimate.

Proof of (1a):

$$\hat{x}_{n|n-1} = L[x_n \mid y_1 \dots y_{n-1}]$$
$$= L[a x_{n-1} + v_n \mid y_1 \dots y_{n-1}]$$
$$= a \underbrace{L[x_{n-1} \mid y_1 \dots y_{n-1}]}_{\hat{x}_{n-1|n-1}} + \underbrace{L[v_n \mid y_1 \dots y_{n-1}]}_{0}$$




1. Place 0, $x_n$
2. Place $\hat{x}_{n|n-1}$ s.t. $\hat{x}_{n|n-1} \perp x_n - \hat{x}_{n|n-1}$
3. Place $\tilde{y}_n$ s.t. $\tilde{y}_n \perp \hat{x}_{n|n-1}$
4. Place $K_n \tilde{y}_n$ as the $\perp$ proj of $x_n$ on $\tilde{y}_n$




Proof of (2), (4):

$\triangle$ BEG & BGF are similar:

$$\frac{\|K_n \tilde{y}_n\|}{\|\Delta_{n|n-1}\|} = \frac{\|\Delta_{n|n-1}\|}{\|\tilde{y}_n\|}$$

$$K_n \|y_n\|^2 = \|\Delta_{n|n-1}\|^2$$

(2) $$K_n = \frac{\|\Delta_{n|n-1}\|^2}{\|\Delta_{n|n-1}\|^2 + \|w_n\|^2} = \frac{\sigma^2_{n|n-1}}{\sigma^2_{n|n-1} + \sigma_w^2}$$

$\triangle$ BGE, GEF, BGF are similar

$$\underbrace{\frac{\|\Delta_{n|n}\|}{\|\Delta_{n|n-1}\|}}_{A} = \underbrace{\frac{(1-K_n)\|\tilde{y}_n\|}{\|w_n\|}}_{B} = \underbrace{\frac{\|w_n\|}{\|\tilde{y}_n\|}}_{C} = \sin\theta$$

$$A = B = C \Rightarrow A^2 = BC$$

(4) $$\frac{\|\Delta_{n|n}\|^2}{\|\Delta_{n|n-1}\|^2} = (1 - K_n) = \frac{\sigma^2_{n|n}}{\sigma^2_{n|n-1}}$$

Proof of (3):

$$\Delta_{n|n-1} = x_n - \hat{x}_{n|n-1}$$

$$= (a x_{n-1} + v_n) - a\hat{x}_{n-1|n-1}$$

$$= a\underbrace{(x_{n-1} - \hat{x}_{n-1|n-1})}_{\Delta_{n-1|n-1}} + v_n$$

(3) $\sigma^2_{n|n-1} = a^2 \sigma^2_{n-1|n-1} + \sigma_v^2$