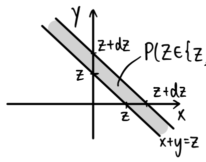
Convolution

Sum of Indep RVs

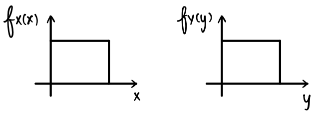
Let $Z = X + Y$ and assume X, Y are CRVs & indep.



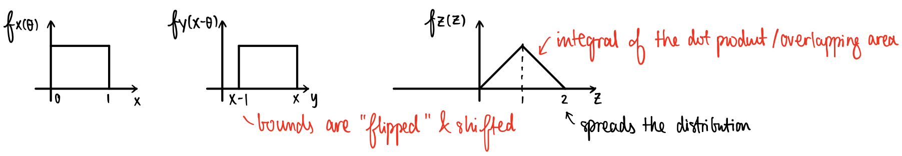
$$P(Z \in \{z, z+dz\}) \cong f_Z(z)\,dz$$
$$= \int_{x=-\infty}^{\infty} \int_{y=z-x}^{z+dz-x} f_{XY}(x,y)\,dy\,dx$$
$$= \int_{-\infty}^{\infty} \int_{z-x}^{z+dz-x} f_X(x) f_Y(y)\,dy\,dx$$
$$= \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx$$
$$= \underset{\substack{\uparrow \\ \text{convolution}}}{f_X * f_Y}(z)$$

Ex:



$$f_Z(x) = \int_0^1 f_X(\theta) f_Y(x-\theta)\,d\theta$$



Ex: $X \sim N(\mu_x, \sigma_x^2)$, $Y \sim N(\mu_y, \sigma_y^2)$, $X \perp Y$

$Z = X + Y \Rightarrow Z \sim N(\mu_x + \mu_y, \sigma_x^2 + \sigma_y^2)$

Special case: $\mu_x = \mu_y = 0$, $\sigma_x^2 = \sigma_y^2 = 1$

Convolution Theorem: $f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx$ symmetric/commutative

$$= \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi}} e^{-x^2/2}\right)\left(\frac{1}{\sqrt{2\pi}} e^{-(z-x)^2/2}\right) dx$$
$$= \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-z^2/4}$$

Order Statistics

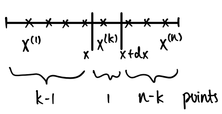
Suppose $X_1, \ldots, X_n$ are iid CRVs w/ common PDF $f_X(x)$ and CDF $F_X(x)$.

Reorder the X_i's s.t. $X^{(1)} < X^{(2)} < \ldots < X^{(n)}$.

Then $\{X^{(k)}\}_{k=1}^{n}$ is called the kth ordered statistic.

$P(X^{(k)} \in (x, x+dx)) \approx f_{X^{(k)}}(x) \cdot dx$

In order for the kth smallest point to lie in $(x, x+dx)$, we must have:



- $(k-1)$ pts $\in (-\infty, x)$
- 1 pt $\in (x, x+dx)$
- $(n-k)$ pts $\in (x+dx, \infty)$

Also, ways to choose each set of pts:

- n ways to choose which of the n pts should lie in $(x, x+dx)$
- out of remaining $n-1$ pts, $\binom{n-1}{k-1}$ ways to choose the pts in $(-\infty, x)$

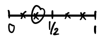
$\Rightarrow f_{X^{(k)}}(x)\cancel{dx} = P(X^{(k)} \in \{x, x+dx\}) = n\binom{n-1}{k-1} f_X(x)\cancel{dx}\,(F_X(x))^{k-1}(1-F_X(x))^{n-k}$

$$\boxed{f_{X^{(k)}}(x) = \underbrace{n\binom{n-1}{k-1}}_{\text{multinomial: } \binom{n}{k-1,\,1,\,n-k}} f_X(x)\,(F_X(x))^{k-1}(1-F_X(x))^{n-k}}$$

Ex: If the X_i's $\overset{iid}{\sim}$ Unif(0,1) CRVs, where $f_X(x)=1$ for $0\le x\le 1$, $F_X(x)=x$ for $0\le x\le 1$,

$$\boxed{f_{X^{(k)}}(x) = \frac{n!}{(k-1)!\,(n-k)!}\, x^{k-1}(1-x)^{n-k}}$$

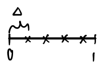
$P[\text{2nd smallest} < \frac{1}{2}]$



$= \int_0^{1/2} \frac{4!}{1!\,2!}\, x(1-x)^2 dx$

≈ 0.7

$\mathbb{E}[X^{(2)}] =$



$5\Delta = 1 \Rightarrow \Delta = \frac{1}{5}$

Moment Generating Functions

Use the result that

$e^{sx} = 1 + sx + \frac{s^2x^2}{2!} + \frac{s^3x^3}{3!} + \dots$ Taylor expansion

Let X be an RV $\Rightarrow$ use linearity of expectation:

$\boxed{\underbrace{\mathbb{E}[e^{sX}]} = 1 + s\mathbb{E}[X] + \frac{s^2}{2!}\mathbb{E}[X^2] + \dots}$

$M_X(s) = \mathbb{E}[e^{sX}]$

$= \sum_x e^{sx} P(X=x)$ (discrete ~ z-transform)

$= \int_{-\infty}^{\infty} e^{sx} f_X(x)\,dx$ (continuous ~ Laplace transform)

$M_X'(s) = \mathbb{E}[X] + s\mathbb{E}[X^2] + \frac{s^2}{2!}\mathbb{E}[X^3] + \dots$

$\frac{d^n}{ds^n} M_X(s) = \mathbb{E}[X^n] + s\mathbb{E}[X^{n+1}] + \dots$

* Notes:

1/ $M_X(0) = 1$

2/ $M_X'(0) = \mathbb{E}[X]$

3/ $M_X''(0) = \mathbb{E}[X^2]$

4/ $\frac{d^n M_X}{ds^n}\Big|_{s=0} = \mathbb{E}[X^n]$

MGF: It's often more immediate to work w/ the MGF of X $\mathbb{E}[e^{sX}]$ rather than the density $f_X(x)$.

1/ Much easier to find the moments of X: easier to differentiate than integrate.

2/ Much easier to do multiplication than convolution.

3/ Great analytical tool of proving things e.g. CLT.

If $Y = aX + b$, $M_Y(s) = \mathbb{E}[e^{s(aX+b)}] = e^{sb}\mathbb{E}[e^{saX}]$

$\Rightarrow \boxed{M_Y(s) = e^{sb} M_X(as)}$

Ex 1: $X \sim Exp(\lambda) \Rightarrow f_X(x) = \lambda e^{-\lambda x}$, $x \ge 0$

$\mathbb{E}[e^{sX}] = \lambda \int_0^\infty e^{-\lambda x} e^{sx} dx$

$= \lambda \frac{e^{-(\lambda-s)x}}{-(\lambda-s)}\Big|_0^\infty$

$= \boxed{\frac{\lambda}{\lambda - s}}$ for $s < \lambda$

$\mathbb{E}[X] = M_X'(0) = \frac{\lambda}{(\lambda-s)^2}\Big|_{s=0} = \frac{1}{\lambda}$

$\mathbb{E}[X^2] = M_X''(0) = \frac{2}{\lambda^2}$

Ex 2: $X \sim Pois(\lambda)$

$M_X(s) = \sum_{k=0}^{\infty} e^{sk} \frac{e^{-\lambda}\lambda^k}{k!} = \boxed{e^{-\lambda + \lambda e^s}}$

$\mathbb{E}[X] = M_X'(0) = e^{-\lambda} e^{\lambda e^s} \lambda e^s\Big|_{s=0} = \lambda$

$\mathbb{E}[X^2] = M_X''(0) = \lambda^2 + \lambda$

Ex 3: $X \sim N(0,1) \Rightarrow f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, $x \in \mathbb{R}$

$$M_X(s) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} e^{sx} dx$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\{-\tfrac{1}{2}(x^2 - 2xs + s^2)\} e^{s^2/2} dx$$
$$= e^{s^2/2} \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(x-s)^2/2}}_{1} dx$$
$$= \boxed{e^{s^2/2}}$$

$$Y = \mu X + \sigma \Rightarrow Y \sim N(\mu, \sigma^2) \Rightarrow M_Y(s) = e^{s\mu} M_X(\sigma s)$$
$$= \boxed{e^{s\mu} e^{\sigma^2 s^2/2}}$$

A given transform worrespond to a unique PDF since $M_X(s)$ contains all the info in $f_X(x)$.

- Inversion: done mostly w/ pattern matching

Ex: $M_X(s) = \frac{1}{2} e^{-3s} + \frac{1}{4} e^{200s} + \frac{1}{4} e^{s}$

$$P_X(X=k) = \begin{cases} 1/2, & k=-3 \\ 1/4, & k=200 \\ 1/4, & \text{else} \end{cases}$$

Sum of independent RVs

$Z = X + Y$, X, Y independent

$$M_Z(s) = \mathbb{E}[e^{s(X+Y)}] = \mathbb{E}[e^{sX} \cdot e^{sY}]$$
$$= \mathbb{E}[e^{sX}]\,\mathbb{E}[e^{sY}]$$
$$= M_X(s)\, M_Y(s)$$

```tikzcd
f_X \arrow[r, "T"] & M_X \arrow[dr] & & \\
 & & \cdot \arrow[r, "M_Z(s)\ T^{-1}"] & f_Z \\
f_Y \arrow[r, "T"] & M_Y \arrow[ur] & &
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Ex: $Z = X + Y$ $\quad X, Y \sim N(0,1)$, $X \perp Y$

Transform in MGF domain:

$$M_Z(s) = M_X(s) M_Y(s) = e^{s^2/2} e^{s^2/2} = e^{s^2} \quad \sim N(0,2)$$