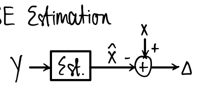
MMSE Estimation



Goal/ $\min_{\hat{X}} \mathbb{E}[(X-\hat{X}(Y))^2]$, where $\hat{X}(Y)$ is any estimator of X given Y.

Intuition:

a) If Y is not observed, what is the MMSE estimate of X (given nothing)?

$\text{MMSE}[X \mid \text{no obs}] = \mathbb{E}[X]$

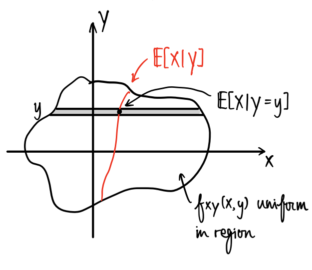
can show that $\arg\min_{c} \mathbb{E}[(X-c)^2] = \mathbb{E}[X]$

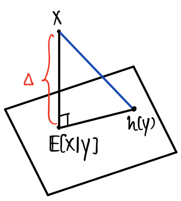
b) What if Y is given?

$\boxed{\text{MMSE}[X|Y] = \mathbb{E}[X|Y]}$

Recall:

$\mathbb{E}[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,dx$





$\mathcal{G}(Y) = \{g(Y) \mid g(\cdot) \text{ is a function}\}$

Theorem 7.4 (Walrand):

The MMSE of X given Y is given by $g(Y) = \mathbb{E}[X|Y]$.

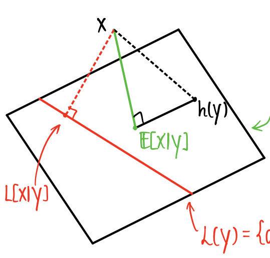
Lemma 7.6 (Walrand):

a) $\forall$ function $\phi(\cdot)$, $\mathbb{E}[(\underbrace{X-\mathbb{E}[X|Y]}_{\Delta})\phi(Y)] = 0$.

The MMSE opt solution is orthogonal to any function of Y.

b) If $\exists$ a function $g(Y)$ s.t. $\mathbb{E}[(X-g(Y))\phi(Y)] = 0 \ \forall \phi(\cdot)$, then $g(Y) = \mathbb{E}[X|Y]$.

The MMSE opt solution is unique.



$\mathcal{G}(Y) = \{g(Y) \mid g(\cdot) \text{ is a func}\}$

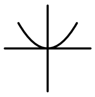
$\mathcal{L}(Y) = \{a+bY \mid a,b \in \mathbb{R}\}$

$L[X|Y] = \mathbb{E}[X] + \frac{\text{cov}(X,Y)}{\text{Var}(Y)}(Y-\mathbb{E}[Y])$

$\mathbb{E}[X|Y] = \arg\min_{f(\cdot)} \mathbb{E}[(X-f(Y))^2]$

$\begin{cases} \text{In general, } L[X|Y] \neq \mathbb{E}[X|Y]. \\ \boxed{\text{If } X, Y \text{ are jointly Gaussian, then } \mathbb{E}[X|Y] = L[X|Y].} \end{cases}$

To verify $y \in U[-1,1]$



$X = Y^2$

$\mathbb{E}[X|Y] = \mathbb{E}[Y^2|Y] = Y^2$

$L[X|Y] = \mathbb{E}[X] + \frac{\mathbb{E}[XY]}{\mathbb{E}[Y^2]} Y$

$\mathbb{E}[XY] = \mathbb{E}[Y^3] = 0$

$L[X|Y] = \mathbb{E}[X]$