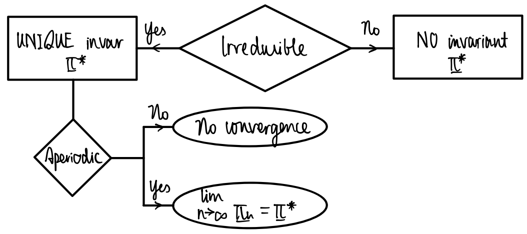
Big Theorem 1.2 (Walrand) for Finite State Markov Chain



$\lim_{N\to\infty} \frac{1}{N}\sum \{\mathbb{1}\}_{X_n=i} = \pi(i)$

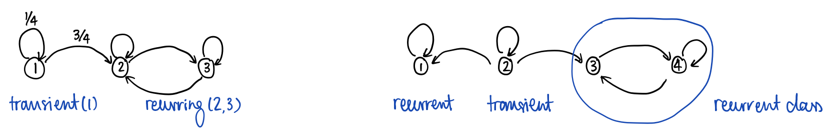
long-term fraction of time that $X_n = i$

Classification of General DTMCs

Transience: A state i is transient if given that starting in state i, there is a nonzero prob that we will never return to i.

Otherwise, the state is recurrent (i.e. zero prob we never return).

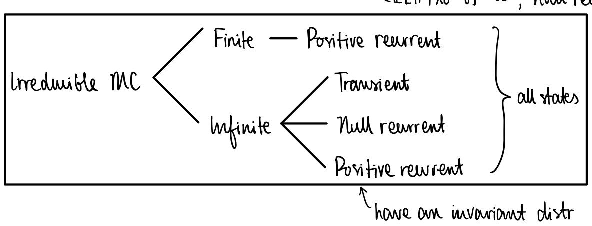
Ex:



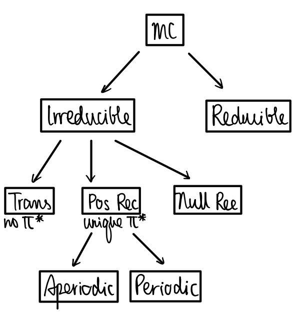
Let $T_i = \min\{n>0 \mid X_n = i\}$ capture the "first return" of i (i.e. return time).

If MC is irreducible, then $P(T_i < \infty \mid X_0 = i) = \begin{cases} 1 & \text{if recur} \\ <1 & \text{if trans} \end{cases}$

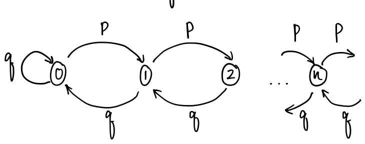
If irreducible MC is recurrent, then if $\begin{cases} \mathbb{E}[T_i \mid X_0 = i] < \infty, & \text{positive recurrent} \\ \mathbb{E}[T_i \mid X_0 = i] = \infty, & \text{null recurrent} \end{cases}$



Big Theorem for General MCs



Ex: Random walk reflected at 0

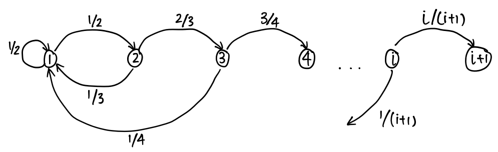


① $p < \frac{1}{2}$: pos rec

② $p > \frac{1}{2}$: transient

③ $p = \frac{1}{2}$: null rec

Ex:



$P(\text{do not return to state } ① \text{ at step } n \mid \text{start at state } ①) = P(① \to ② \to ③ \to \dots \infty)$

$= \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{3}{4}\right) \dots \left(\frac{n}{n+1}\right)$

$= \frac{1}{n+1}$

$\lim_{n\to\infty} P(\text{never returning at } ① \text{ after starting at } ①)$

$= \lim_{n\to\infty} \frac{1}{n+1} = 0 \Rightarrow$ recurrent

Given $X_0 = i$, let $T_i = \min\{n>0 \mid X_n = i\}$

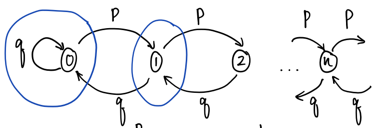
$\mathbb{E}[T_i \mid X_0 = i] = \begin{cases} <\infty : \text{pos rec} \\ =\infty : \text{null rec} \end{cases}$

$\mathbb{E}[T_1 \mid X_0 = 1] = \left(\frac{1}{2}\right)\cdot 1 + \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\cdot 2 + \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{1}{4}\right)\cdot 3 + \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)\left(\frac{1}{5}\right)\cdot 4 + \dots$

$= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ ← harmonic sum

$= \sum_{i=1}^{\infty} \frac{1}{i+1} \to \infty \Rightarrow$ $\boxed{\text{Null Recurrent}}$

Ex: Random walk reflected at 0



① $p < \frac{1}{2}$: pos rec

② $p > \frac{1}{2}$: transient

③ $p = \frac{1}{2}$: null rec

* Let $\rho = \frac{p}{q} < 1$ since $p < \frac{1}{2}$

$p<\frac{1}{2}$: $\pi_0 p = \pi_1 q \to \pi_1 = \frac{p}{q}\pi_0 = \rho\pi_0$

$\pi_1 = p\pi_0 + q\pi_2 \to \pi_1 = q\pi_1 + q\pi_2 \to \pi_2 = \frac{p}{q}\pi_1 = \rho\pi_1 = \rho^2\pi_0$

$\vdots$

$\pi_n = \rho^n \pi_0 \to \sum_{n=0}^{\infty} \pi_n = 1 \to \pi_0 = 1-\rho \to \boxed{\pi_n = (1-\rho)\rho^n}$

$p > \frac{1}{2}$: Clear intuitively that $X_n \to \infty$ as $n \to \infty$

Let $Z_n \overset{iid}{\sim} \begin{cases} P(Z_n = +1) = p \\ P(Z_n = -1) = q = 1-p \end{cases}$

$X_n = \max(X_{n-1} + Z_n, 0)$

$X_n \geq X_0 + Z_1 + Z_2 + \dots + Z_{n-1}$

$\frac{X_n}{n} = \frac{X_0 + Z_1 + Z_2 + \dots + Z_n}{n} \to \mathbb{E}[Z_n]$ by SLLN

$\mathbb{E}[Z_n] = p + (-1)(1-p) = 2p - 1 > 0$ if $p > \frac{1}{2}$

Thus $X_n \to \infty$ as $n \to \infty \Rightarrow X_n$ visits each state only finitely many times $\Rightarrow$ $\boxed{\text{Transient}}$

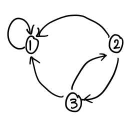
recurrent states are visited infinitely many times

Reversibility

Assume we have an irreducible & positively recurrent MC at its invariant distr π.

Suppose for every n, $(X_0, X_1, \dots, X_n)$ has the same PMF as its time-reversed version $(X_n, X_{n-1}, \dots, X_0)$, then we call the MC reversible.

Ex:



forward: 1 1 2 3 2 3 1

backward: 1 3 2 3 2 1 1 ← NOT reversible

Facts:

1/ A MC run backwards is always a MC.

2/ If it is reversible, then it is the same MC.

Proof:

1/ $P[X_k = i \mid X_{k+1} = j, X_{k+2} = i_{k+2}, \dots, X_n = i_n]$

$= \frac{P(X_k = i, X_{k+1} = j, \dots, X_n = i_n)}{P(X_{k+1} = j, \dots, X_n = i_n)}$

$= \frac{\pi(i) P_{i,j} \cancel{P_{j,i_{k+2}}} \dots \cancel{P_{i_{n-1},i_n}}}{\pi(j) \cancel{P_{j,i_{k+2}}} \dots \cancel{P_{i_{n-1},i_n}}}$ (Markov property)

$\Rightarrow$ reversed MC is also MC

$\tilde{P}_{j,i} = \frac{\pi(i)}{\pi(j)} P_{i,j}$

↳ Since this is a function of i, j only, $\tilde{P}_{ji} = P$[backwards chain goes from j to i]

2/ Condition for reversibility:

$\tilde{P}_{ji} = P_{ji} = \frac{\pi(i)}{\pi(j)} P_{ij}$

$\boxed{\pi(i) P_{ij} = \pi(j) P_{ji}}$ $\forall i, j \in \mathcal{X}$

↳ Detailed Balance Equations (local, stronger than global)

Theorem: If a MC is reversible, it has an invariant distr π.

Proof: We need to show that $\forall j$, $\pi(j) = \sum_i \pi(i) P_{i,j}$

$\sum \pi(i) P_{ij} = \sum \pi(j) P_{ji} = \pi(j) \underbrace{\sum_i P_{ji}}_{1} = \pi(j)$

Fact: Start w/ graph associated w/ MC, forget all the arrows (directions), remove multiple edges btw nodes & remove all self loops.

If resulting graph is a tree (cycle free), then stationary distr satisfies detailed balanced eqns.