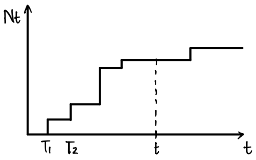
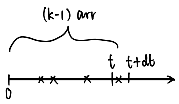
Erlang Distribution



$P(N_t = k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}$, $k = 0, 1, 2, \dots$

Erlang distr of kth order: $T_k = S_1 + \dots + S_k$

↳ sum of k iid Exp(λ) RVs, i.e. time until kth arrival in Poisson process



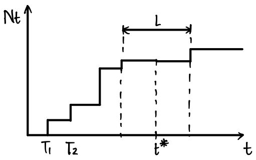
$P(\text{kth arrival in } \{t, t+dt\}) = f_{T_k}(t)\cdot dt = P(k\text{-1 arrivals in } (0,t))\cdot P(\text{1 arrival in } (t, t+dt))$

$= e^{-\lambda t}\frac{(\lambda t)^{k-1}}{(k-1)!}\lambda dt$

$\boxed{\text{Erlang}(k;\lambda) = \frac{\lambda^k t^{k-1}}{(k-1)!}e^{-\lambda t}}$, $k = 1, 2, \dots$

↳ kth order Erlang

Random Incidence Paradox

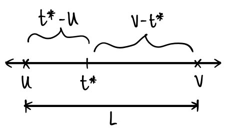


Fix a time t^* and consider the length L of the interarrival time that contains t^*.

Q/ How is L distributed?

L is distributed like a typical interarrival interval, which follows a distr ~ Exp(λ) $\Rightarrow \mathbb{E}[L] = \frac{1}{\lambda}$

↳ this is WRONG!



L contains t^*; U, V are neighboring arrivals

$L = \underbrace{(t^* - U)}_{\text{Exp}(\lambda)} + \underbrace{(V - t^*)}_{\text{Exp}(\lambda) \text{ by memorylessness}}$

$P(t^* - U > x) = P(\text{no arrivals in } [t^* - x, t^*])$

$= P(\text{no arrivals in } [0, x])$

$= e^{-\lambda x}$ ← A Poisson process backwards is still a Poisson process

$\mathbb{E}[\text{Erlang}(k;\lambda)] = \frac{k}{\lambda}$, $\text{Var}(\text{Erlang}(k;\lambda)) = \frac{k}{\lambda^2}$

$\boxed{\mathbb{E}[L] = \frac{2}{\lambda}}$ Intuition: more likely to fall into a larger interval