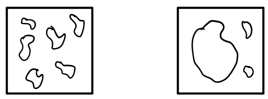
Random Graphs

Erdos-Renyi (ER) Random Graphs

↳ Model: Given a positive integer n & prob value p, G(n,p) is a random graph which is an undirected graph on n vertices s.t. each of the $\binom{n}{2}$ edges exist w.p. p.

Results: Certain thresholds mark emergence of various structural properties.

- $p=\frac{1}{n^2}$ ⇒ first edge appears
- $p=\frac{1}{n^{3/2}}$ ⇒ first "3-node-trees" emerge
- $p=\frac{1}{n}$ ⇒ first cycles emerge
- $p=\frac{1}{n}$ ⇒ "Giant component" emerges



$p=\frac{1-\epsilon}{n}$ $\qquad$ $p=\frac{1+\epsilon}{n}$

Formally, G(n,p) describes a distribution on the set of undirected graphs on n vertices.

Ex: 1. $\mathbb{E}[\#\text{ of edges in } G] = \binom{n}{2}p$

2. Pick an arbitrary node, let D be its degree. What is the distribution of D?

   $D \sim Bin(n-1, p)$

   $\mathbb{E}[D] = (n-1)p$

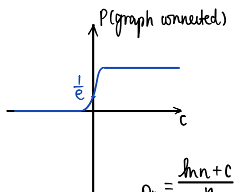
3. What's the prob q that a node is isolate?

   $q = (1-p)^{n-1}$

Threshold for Connectivity

- graph NOT connected: $p=\frac{(1-\epsilon)\log n}{n}$
- graph connected: $p=\frac{(1+\epsilon)\log n}{n}$

> Theorem: Let $p_n = \frac{\lambda \log n}{n}$, then:
>
> a) If $\lambda < 1$, $P(G(n,p) \text{ is connected}) \to 0$ as $n \to \infty$
>
> b) If $\lambda > 1$, $P(G(n,p) \text{ is connected}) \to 1$ as $n \to \infty$



$P(\text{graph connected}) \xrightarrow{n\to\infty} e^{-e^{-c}}$

$p_n = \frac{\ln n + c}{n}$ for a constant $c \in \mathbb{R}$

Proof of a): It is sufficient to show $P(\text{no isolated nodes}) \xrightarrow{n\to\infty} 0$

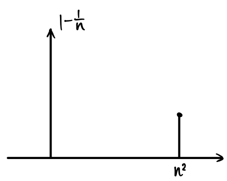
Let X be the num of isolated nodes in G(n,p). Find $\mathbb{E}[X]$.

Let I_i = indicator RV be the event node i is isolated.

$\mathbb{E}[X] = \mathbb{E}[\sum_{i=1}^{n} I_i] = \sum_{i=1}^{n}\mathbb{E}[I_i] = \sum_{i=1}^{n} P(\text{node } i \text{ is isolated}) = nq = n(1-p)^{n-1}$

$\mathbb{E}[X] = n(1-p)^{n-1} \sim ne^{-p(n-1)}$ (Taylor approx: $e^{-x} = 1-x+o(x)$)

$\sim ne^{-\lambda \log n} = n^{1-\lambda} \xrightarrow{n\to\infty} \infty$



$\mathbb{E}[W] \xrightarrow{n\to\infty} \infty$

$P(W=0) \xrightarrow{n\to\infty} 1$

We need to have a handle on the variance of X.

Lemma: If X is a nonneg integer-valued RV, then $\boxed{P(X=0) \le \frac{Var(X)}{\mathbb{E}[X]^2}}$

Proof: $Var(X) = \mathbb{E}[(X-\mathbb{E}[X])^2]$

$$= P(X=0)\,\mathbb{E}[X]^2 + P(X=1)\,\mathbb{E}[(1-\mathbb{E}[X])^2] + P(X=2)\,\mathbb{E}[(1-\mathbb{E}[X])^2] + \dots$$

$$\ge P(X=0)\,\mathbb{E}[X]^2 \quad \square$$

$$Var(X) = Var\left(\sum_{i=1}^{n} I_i\right) = \sum_{i=1}^{n} Var(I_i) + \sum_{j=1}^{n} \sum_{\substack{k=1 \\ j\neq k}}^{n} cov(I_j, I_k)$$

$$\boxed{Var(X) = n\,Var(I_1) + n(n-1)\,cov(I_1, I_2)}$$

$$Var(I_1) = q(1-q)$$

$$cov(I_1, I_2) = \mathbb{E}[\overbrace{I_1 I_2}^{\text{both nodes isolated}}] - \mathbb{E}[I_1]\mathbb{E}[I_2]$$

$$= (1-p)^{n-1}(1-p)^{n-2} - (1-p)^{n-1}(1-p)^{n-1}$$

$$= (1-p)^{2n-3} - q^2$$

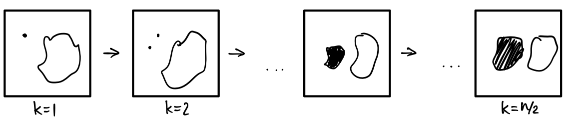
$$= \frac{q^2}{1-p} - q^2 = \frac{pq^2}{1-p}$$

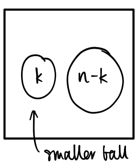
$$\Rightarrow P(X=0) \le \frac{Var(X)}{\mathbb{E}[X]^2} = \frac{nq(1-q) + n(n-1)\frac{pq^2}{1-p}}{n^2q^2} = \frac{1-q}{nq} + \left(\frac{n-1}{n}\right)\left(\frac{p}{1-p}\right) \xrightarrow{n\to\infty} 0$$

as $n\to\infty$, $p\to 0$

Proof of b): If $\lambda<1$, show that P(not connected) $\xrightarrow{n\to\infty} 0$

Key idea - Graph disconnected = $\exists$ a set of size k ($1 \le k \le \frac{n}{2}$) s.t. there is no edge btw this set & its complement





Apply Union Bound twice:

$$P(G(n,p) \text{ is not connected})$$

$$= P\left(\bigcup_{k=1}^{n/2} (\exists \text{ a smaller set of size } k \text{ that is disconnected from its complement set})\right)$$

$$\le \sum_{k=1}^{n/2} P(\exists \text{ a smaller set} \dots)$$

$$\le \sum_{k=1}^{n/2} \binom{n}{k} P(\text{a specific set of size } k \text{ is disconnected})$$

$$= \sum_{k=1}^{n/2} \binom{n}{k} (1-p)^{k(n-k)} \xrightarrow{n\to}$$