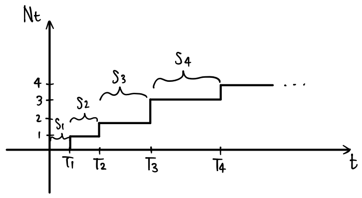
Poisson Process

Let $N = \{N(t), t \geq 0\}$ be defined as follows:

For $t \geq 0$, $N(t)$ (or N_t) be the num of arrivals on $(0,t)$. The arrival times are $\{T_n\}_{n\geq 1}$ & the interarrival times $\{S_n\}_{n\geq 1}$ are iid and exponentially distributed w/ parameter λ.
i.e. They are $Exp(\lambda)$ RVs w/ mean $\frac{1}{\lambda}$. This arrival process is called a Poisson Process. $\Rightarrow N \sim PP(\lambda)$

Motivation: PP is a good model for real world events, e.g. arrival of packets, customers, photons.



$T_1 = S_1$

$T_2 = S_1 + S_2$

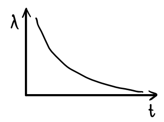
$\vdots$

$T_n = \sum_{i=1}^{n} S_i$

- S_i are iid $Exp(\lambda)$ RVs $(\lambda > 0)$
- T_i are arrival times
- N_t is the num of arrivals in $(0,t)$: $N_t = \max_{n\geq 1}\{n \mid T_n \leq t\}$
  ↳ N_t is a piecewise, constant nondecreasing counting process w/ jumps of $+1$ at the T_i's.

Recap: $Exp(\lambda)$ is a memoryless RV.

1/ $F_\tau(t) = \begin{cases} 1 - e^{-\lambda t}, & t \geq 0 \\ 0, & \text{else} \end{cases}$; $F(\tau > t) = e^{-\lambda t}$



2/ $\mathbb{E}[\tau] = \frac{1}{\lambda}$; $Var(\tau) = \frac{1}{\lambda^2}$

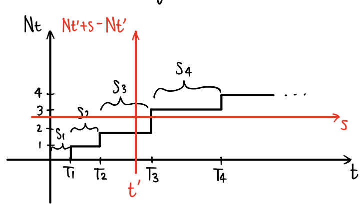
3/ $P(\tau > t+s \mid \tau > s) = P(\tau > t)$

4/ $P(\tau \leq t+\epsilon \mid \tau > t) = \lambda\epsilon + o(\epsilon)$ $\quad \lim_{\epsilon\to 0} \frac{o(\epsilon)}{\epsilon} = 0$ $\quad$ e.g. $\epsilon^2 = o(\epsilon)$

Proof of 4/:

$P(\tau > t+\epsilon \mid \tau > t) = P(\tau > \epsilon) = e^{-\lambda\epsilon} = 1 - \lambda\epsilon + o(\epsilon) = P(0 \text{ arrival in } (t, t+\epsilon))$

Theorem: $PP(\lambda)$ is a memoryless process.



Note: During each interval, there is either an arrival or not.

If $N_t \sim PP(\lambda)$, then so is $N_{t'+s} - N_{t'}$

Implications:

1/ $PP(\lambda)$ has independent & stationary increments.

$\Rightarrow$ For any $0 \leq t_1 < t_2 < \ldots$ $\{N_{t_{n+1}} - N_{t_n}\}$ are indep & distr depends only on $(t_{n+1} - t_n)$

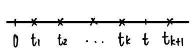
Sketch of proof:

$PP(\lambda)$ has interarrival times that are iid $Exp(\lambda)$ RVs. For $t > T_3$, it's obvious interarrival times are $Exp(\lambda)$ by construction.

Only possible issue is with the first interarr time $s < T_3 - t'$. But by the memoryless prop of the exp RV, $P(\text{arr time} > \theta + t' - T_2 \mid \text{arr time} > t' - T_2) = P(\text{arr time} > \theta)$

$\boxed{\text{Theorem 13.7: If } N_t \text{ is a } PP(\lambda)\text{, then } N_t = \#\text{arrivals in } (0,t) \text{ has } Pois(\lambda t) \text{ RV distribution.}}$

Proof:



Joint density of $T_1, T_2, \ldots, T_{k+1}$

$$P(T_1 \in \{t_1, t_1+dt_1\}, T_2 \in \{t_2, t_2+dt_2\}, \dots, T_k \in \{t_k, t_k+dt_k\}, T_{k+1} > t)$$

$$= P(S_1 \in \{t_1, t_1+dt_1\}, S_2 \in \{t_2-t_1, t_2-t_1+dt_2\}, \dots, S_k \in \{t_k - t_{k-1}, t_k - t_{k-1} + dt_k\}, S_{k+1} > t - t_k)$$

$$= (\lambda e^{-\lambda t_1} dt_1)(\lambda e^{-\lambda(t_2-t_1)} dt_2) \dots (\lambda e^{-\lambda(t_k - t_{k-1})} dt_k)(e^{-\lambda(t-t_k)})$$

$$= \lambda^k e^{-\lambda t} dt_1 dt_2 \dots dt_k$$

$$\boxed{f_{T_1T_2\dots T_k}(t_1, t_2, \dots, t_k) = \lambda^k e^{-\lambda t}}$$

Note:

1/ $f_{T_1\dots T_k}(t_1, \dots, t_k)$ is uniform over the support of (T_1, T_k)

(Given that k arrs have occurred in $(0,t)$, how are they distr?)

$$\underbrace{N_t(k)}_{P(k \text{ arrivals in } (0,t))} = \int_{t_1}\int_{t_2} \dots \int_{t_k} f_{T_1T_2\dots T_k}(t_1 \dots t_k)\, dt_1 dt_2 \dots dt_k = \lambda^k e^{-\lambda t} \int_0^{t_1} \dots \int_0^{t_k} dt_1 \dots dt_k$$

↑ ORDER matters!

- Without any constraints, $Vol(S) = t^k$.
- But this includes all permutations of $(t_1, \dots, t_k)$.
- By symmetry, all permutations have the same volume.

$$Vol(S) = \frac{t^k}{k!}$$

$$\Rightarrow N_t(k) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}, \quad k = 0, 1, \dots$$

Ex: Fishing

Bob catches fish $\sim PP(\lambda = 0.6/hr)$. If he catches at least one fish in first 2hrs, he's done; else, he continues until catches 1 fish.

a) $P(\text{Bob fishes} > 2\text{hrs}) = P(N_2 = 0) = e^{-\lambda \cdot 2} = e^{-1.2}$ ← P(no fish in first 2 hrs)

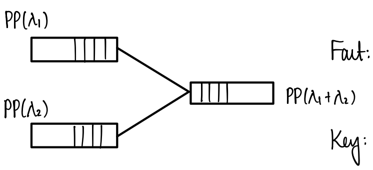
b) $P(\text{Bob catches at least 2 fish}) = 1 - P(N_2 = 0) - P(N_2 = 1) = 1 - e^{-2}(1+2)$ ← Bob must have been fishing for 2 hrs & caught ≥ 2 fish

c) $E[\text{fish caught}] = \text{Fish caught in } t \in (0,2) + \text{Fish in } t \in (2, \infty) = 1 \cdot 2 + 1 \cdot P(\text{still fishing}) = 1 \cdot 2 + 1 \cdot e^{-1.2}$

d) $E[\text{fishing time} \mid T > 4\text{hrs}] = \frac{1}{0.6} + 4$

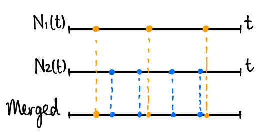
Merging

Let $N_1(t) \sim PP(\lambda_1)$ & $N_2(t) \sim PP(\lambda_2)$.

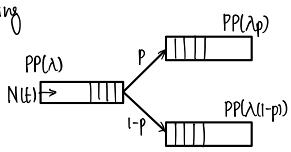


Fact: $N(t) = N_1(t) + N_2(t) \sim PP(\lambda_1 + \lambda_2)$

Key: Sum of 2 indep Pois RVs w/ param μ_1 & μ_2 = $Pois(\mu_1 + \mu_2)$



Splitting



$N(t) \sim PP(\lambda)$ ← we divide $N(t)$ into 2 processes $N_1(t)$ & $N_2(t)$.

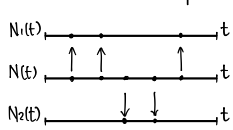
For each arrival in $N(t)$, we flip a coin w/ bias p & send it up or down.

Facts:

1/ $N_1(t) \sim PP(\lambda p)$

2/ $N_2(t) \sim PP(\lambda(1-p))$

3/ $N_1(t)$ & $N_2(t)$ are indep



Ex: 2 lightbulbs have indep & exponential lifetime T_a, T_b w/ param λ_a, λ_b. Find distr of $Z = \min\{T_a, T_b\}$.

$P(Z \geq z) = P(T_a \geq z) \cdot P(T_b \geq z) = e^{-\lambda_a z} \cdot e^{-\lambda_b z} = e^{-(\lambda_a + \lambda_b) z} \Rightarrow Z \sim Exp(\lambda_a + \lambda_b)$

Alt soln: Treat T_a & T_b as times of 1st arrivals of 2 indep PP w/ rates λ_a, λ_b.

↳ If we merge these 2 processes, the 1st arrival is $\min(T_a, T_b)$.

The first arrival of merged $PP(\lambda_a + \lambda_b)$ is $Exp(\lambda_a + \lambda_b) \Rightarrow \min\{T_a, T_b\} \sim Exp(\lambda_a + \lambda_b)$