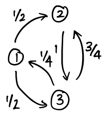
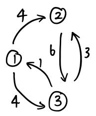
An Intuitive Intro to CTMCs



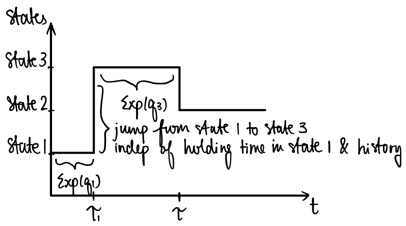
DTMC



CTMC

1. Time is no longer discrete
2. Flow is not a probability, but a rate of transition

Time spent in CTMCs (holding time) no longer one step but a RV.
Given that the chain jumped to state i at some time, the holding time in state i should not depend on when it got to state i or the past trajectory of MC.
↳ holding time in state i should be $Exp(q_i)$



2 Interpretations of Jumps in CTMCs:

1/ From state i, we have n indep jumps to states $1, 2, \dots, n$ at rates $\lambda_{i,1}, \lambda_{i,2}, \dots, \lambda_{i,n}$ rep.
e.g. from state 1, time to jump to state 2 is $Exp(\lambda_{1,2}=4)$,
time to jump to state 3 is indep & $\sim Exp(\lambda_{1,3}=4)$.
⇒ need to keep n indep $Exp(\lambda_{i,j})$ timer

2/ From state i, we have a single Exp timer at rate $(\lambda_{i,1}+\lambda_{i,2}+\dots+\lambda_{i,n})$ and one timer expires, you jump to state i w.p. $\left.\frac{\lambda_{i,j}}{\sum_{j=1}^{n}\lambda_{i,j}}\right\}$ Def Π_{ij}
e.g. from state 1, run an $Exp(\lambda_{1,2}+\lambda_{1,3})$ timer; when timer rings, jump to state 2 w.p. $\Pi_{1,2}=\frac{\lambda_{1,2}}{\lambda_{1,2}+\lambda_{1,3}}=\frac{1}{2}$,
state 3 w.p. $\Pi_{1,3}=\frac{1}{2}$

Why are these 2 interpretations equivalent? Idea: competing Exps
Recall facts:
1. If $X_1, \dots, X_n$ are indep Exp RVs w/ rates $\lambda_1, \dots, \lambda_n$, then $Y=\min\{X_i\} \sim Exp(\sum_{i=1}^{n}\lambda_i)$
2. $P(Y=X_j) = \frac{\lambda_i}{\sum\lambda_i}$

Rate Matrix

$$Q=\begin{bmatrix} -\sum_{j\neq 1}^{n}\lambda_{1,j} & \lambda_{1,2} & \cdots & \lambda_{1,n} \\ \lambda_{2,1} & -\sum_{j\neq 2}^{n}\lambda_{2,j} & & \\ \vdots & & \ddots & \\ \lambda_{n,1} & & & -\sum_{j\neq n}^{n}\lambda_{n,j} \end{bmatrix}$$

- row sums are 0
- the entries of Q are $Q_{i,i}$ for diagonal entries, $Q_{i,j}$ for off-diagonal entries

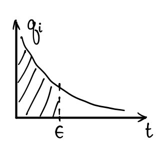
$$\boxed{q(i) \text{ or } q_i = \sum_{j\neq i}\lambda_{i,j}}$$

$$Q=\begin{bmatrix} -8 & 4 & 4 \\ 0 & -6 & 6 \\ 1 & 3 & -4 \end{bmatrix}$$

CTMC: $P(X_{t+\epsilon}=j \mid X_t=i, \overbrace{X_u, u<t}^{\text{history, MC should not depend on this part}}) = \begin{cases} \epsilon Q(i,j) + o(\epsilon) &, j\neq i \\ \underbrace{1-\sum_{j\neq i}\epsilon Q(i,j)}_{1+\epsilon Q_{i,i}} + o(\epsilon) &, j=i \end{cases}$

For small ϵ, $\lim_{\epsilon\to 0}\frac{o(\epsilon)}{\epsilon}=0$

Assume Markov property holds for small time scale & holding times are Exp.
Recall that if $Y\sim Exp(\lambda)$, $P(Y>\epsilon)=e^{-\lambda\epsilon}=1-\lambda\epsilon+o(\epsilon)$ Taylor expansion



$P(\text{no jump in interval of size } \epsilon) = 1 - q\epsilon + o(\epsilon)$

$P(1 \text{ transition in interval of size } \epsilon) = q_i\epsilon + o(\epsilon)$

$P(2 \text{ or more trans in interval of size } \epsilon) = o(\epsilon)$

Go to our 3-state ex:

CTMCs - $P(X_{t+2} = 2 \mid X_t = 1, X_u \ \forall \ u < t) = P(X_{t+\epsilon} = 2 \mid X_t = 1) = \underbrace{(8\epsilon)}_{P(\text{jump})}\overbrace{\left(\tfrac{4}{8}\right)}^{P(\text{jump to } 2)} = 4\epsilon$

$P(X_{t+\epsilon} = 3 \mid X_t = 1, \ldots) = (8\epsilon)\left(\tfrac{4}{8}\right) = 4\epsilon$

$P(X_{t+\epsilon} = 1 \mid X_t = 1, \ldots) = 1 - 8\epsilon$

Formal Def of CTMC

Def: $\mathcal{X}$, countable set: The stochastic process $\{X_t, t \geq 0\}$ is

- π is a prob distr on $\mathcal{X}$
- rate matrix $Q = \{Q_{ij}\} \ \forall \ i,j \in \mathcal{X}$

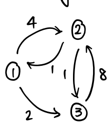
$Q(i,j) \geq 0 \ \forall \ i \neq j$ and $\sum_{j \in \mathcal{X}} Q(i,j) = 0 \ \forall \ i \in \mathcal{X}$ hold

A CTMC w/ distr π & rate matrix Q is a continuous time stoch process $\{X_t, t > 0\}$ s.t. $P(X_0 = i) = \pi(i)$ or π_i,

$$P(X_{t+\epsilon} = j \mid X_t = i, X_u, u < t) = \begin{cases} \epsilon Q(i,j) + o(\epsilon) & \text{if } i \neq j \\ 1 + \epsilon Q(i,j) + o(\epsilon) & \text{if } i \neq j \end{cases}$$

$$Q = \begin{bmatrix} Q_{11} & Q_{12} & & \\ Q_{21} & Q_{22} & & \\ \vdots & & \ddots & \\ & & & Q_{nn} \end{bmatrix}, \quad \Pi_{i,j} = \frac{Q(i,j)}{q(i)}, \quad \Pi_{i,i} = 0 \quad (\text{no self loops!})$$

Stationary Distribution



Rate out = Rate in

① $\pi_1 \cdot 6 = \pi_2 \cdot 1$

② $\pi_2 \cdot 2 = \pi_1 \cdot 4 + \pi_3 \cdot 8$

③ $\pi_3 \cdot 8 = \pi_1 \cdot 2 + \pi_2 \cdot 1$

$$[\pi_1 \ \pi_2 \ \pi_3]\begin{bmatrix} -6 & 4 & 2 \\ 1 & -2 & 1 \\ 0 & 8 & 8 \end{bmatrix} = [0 \ 0 \ 0]$$