What is this course about?

- most real-world systems involve uncertainty: prediction, strategy/decision making, design
- "artificial" randomness is useful: prob algorithms

Probability is a principled way of reasoning about uncertainty.

- origins in gambling

Two diverse skillsets needed:

Model the real world in language of prob	$\longleftrightarrow$	Solve the problem mathematically
understand the problem as an experiment		calcus, logic, etc.

Probability Basics

Sample Space (Ω): The sample space of an experiment is the set of all possible outcomes of the experiment.

Ex: 1/ Experiment: Toss 2 fair coins $\Rightarrow$ $\Omega = \{HH, HT, TH, TT\}$
bare outcome

2/ Experiment: Toss a coin until heads $\Rightarrow$ $\Omega = \{H, TH, TTH, \ldots\}$
↳ countably infinite

3/ Experiment: Wait at bus stop for next bus for time T $\Rightarrow$ $\Omega = (0, T)$
↳ continuous sample space

Bare framework: the "random" experiment

Concept	Example
	(Toss fair coin twice)
① Bare outcomes	① $\{HH, HT, TH, TT\}$
② Outcomes are: - collectively exhaustive (CE) - mutually exclusive (ME)	② above covers: - all possibilities - are disjoint
③ Each outcome is assigned a non-neg value (prob measure P)	③ each outcome equally likely
④ Prob summed over all outcomes = 1	④ each outcome has prob $\frac{1}{4}$

Events: Allowable subsets of Ω

Ex: 1/ Get 1 head in exp 1

2/ Get even number of tossed in exp 2

3/ Bus arrives in < 5min in exp 3

A prob space $(\Omega, \mathcal{F}, P)$ is a math construct to model experiments.

Ω: set of all possible outcomes

$\mathcal{F}$: set of all events

P: assignment/mapping of prob to events ($P: \mathcal{F} \to [0,1]$)

All of probability theory rests on basically 2 axioms (Kolmogorov).

Axioms
1/ $P(\underset{\text{null}}{\emptyset}) = 0$; $P(\Omega) = 1$ (normalization)
2/ $P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots$
for disjoint events $A_1, A_2, \ldots$

Fundamental Prob Facts
1/ $P(A^c) = 1 - P(A)$
2/ $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
Union Bound: $P(A_1 \cup A_2 \cup \ldots \cup A_n) \leq \sum_{i=1}^{n} P(A_i)$

Generalized Inclusion-Exclusion
Ex: $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$
Induction on number of events:
$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{k=1}^{n} \sum_{1 \leq i_1 < \ldots < i_k \leq n} \ldots \sum (-1)^{k+1} P(A_{i_1} \cap \ldots \cap A_{i_k})$$

Discrete Prob: $P(A) = \sum_{w \in A} P\{w\}$
- Unif sample space: $P(A) = \frac{|A|}{|\Omega|}$

Ex: Birthday Paradox
P{at least 2 ppl in a group of size n shares the same birthday}
Let k = # days in a year = 365
$|\Omega| = 365^n$
! carefully define the events
Let A: event that at least 2 people have the same bday
A^c: no 2 ppl share a bday
$$P(A^c) = \frac{|A^c|}{|\Omega|} = \frac{365 \times 364 \times \ldots \times (365-n+1)}{365^n} \leftarrow \text{365 ordered } n$$
$$= 1\,\underbrace{(1-\tfrac{1}{k})}_{\approx e^{-1/k}}\,\underbrace{(1-\tfrac{2}{k})}_{e^{-2/k}} \ldots \underbrace{(1-\tfrac{n-1}{k})}_{e^{-(n-1)/k}}$$
Recall Taylor approx: $e^x \approx 1 + x$ for $x \ll 1$
$$\boxed{P(A^c) \approx e^{-(1+2+\ldots+(n-1))/k} \approx e^{-n^2/2k}}$$

When $n = 23 \Rightarrow P(A) > 50\%$
$n = 60 \Rightarrow P(A) \approx 99\%$
$n = 100 \Rightarrow P(A) \approx 99.994\%$

Conditional Prob
$P(A|B) = P(A \text{ occurred given } B \text{ occurred})$
"B" is the new "Ω".
If $P(B) \neq 0$, $P(A|B) = \frac{P(A \cap B)}{P(B)}$

Ex: 2 6-sided dice
A: 1st die is a 6
B: sum of 2 dice = 7
$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P\{6,1\}}{P\{6,1\} + P\{5,2\} + \ldots + P\{1,6\}} = \frac{1/36}{6/36} = \frac{1}{6}$$

Ex: 3 card problem : R/R, R/B, B/B

Pick one card at random & lay on table.

You see a R.

Q/ What is P(other side is B)?

[Flawed argument: can't be B/B, so ans is 50%.]

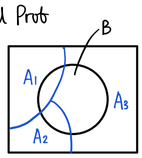
Solution:

$$\underset{\substack{\uparrow \\ \text{selected card}}}{P(RB} \mid \underset{\substack{\uparrow \\ \text{upturned card}}}{R)} = \frac{P(RB \cap R)}{P(R)} = \frac{(\frac{1}{3})(\frac{1}{2})}{1/2} = \frac{1}{3}$$

Product Rule

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1) \cdot P(A_2 \mid A_1) \cdot P(A_3 \mid A_1 \cap A_2) \cdots P(A_n \mid A_1 \cap \ldots \cap A_{n-1})$$

Total Prob



$A_1 \ldots A_n$ are ME & CE

the A_i's partition Ω

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \ldots + P(A_n \cap B)$$
$$= \sum_{i=1}^{N} P(A_i) P(B \mid A_i)$$