Linearity Behavior of RVs

We want to analyze "tail bounds"

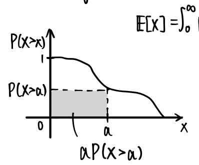
$\boxed{\text{Markov's Inequality: If } X \text{ is a non-neg RV, } P(X \geq a) \leq \frac{\mathbb{E}[X]}{a}.}$

Proof: $\mathbb{1}_{X>a} \leq \frac{X}{a}$ ← Convince yourself this is true

$\mathbb{E}[\mathbb{1}_{X>a}] \leq \mathbb{E}[\frac{X}{a}]$

$P(X>a) \leq \frac{\mathbb{E}[X]}{a}$

Alt. proof:



$\mathbb{E}[X] = \int_0^\infty P(X>x)\,dx$ (TSP)

$\mathbb{E}[X]$ = entire area under curve

$P(X>a) \cdot a$ = shaded area

$P(X>a) \cdot a < \mathbb{E}[X]$

$P(X>a) \leq \frac{\mathbb{E}[X]}{a}$

Chebyshev

$\boxed{\text{If } X \text{ is a RV w/ finite mean } \mu \text{ \& variance } \sigma^2, \; P(|X-\mathbb{E}[X]| \geq c) \leq \frac{\sigma^2}{c^2} \;\; \forall c>0}$

Special case: $c = k\sigma$ $\quad P(|X-\mathbb{E}[X]| \geq k\sigma) \leq \frac{1}{k^2}$

Proof: $P(|X-\mu| \geq c) = P(|X-\mu|^2 \geq c^2)$

$= \frac{\mathbb{E}[|X-\mu|^2]}{c^2}$

$= \frac{\sigma^2}{c^2}$

Chernoff

$P(X>a) = P(e^{sX} > e^{sa}) \leq \frac{\mathbb{E}[e^{sX}]}{e^{sa}}$ by Markov

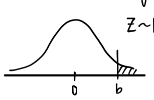
$\boxed{P(X>a) \leq \min_{s\geq 0} \frac{M_X(s)}{e^{sa}} \;\; \forall s>0}$

Since RHS is a function of s, choose s wisely to get the tightest bound.

$\Rightarrow P(X<a) \leq \min_{s>0} \frac{\mathbb{E}[e^{-sX}]}{e^{-sa}}$

$P(X<a) \leq \min_{s<0} \frac{\mathbb{E}[e^{sX}]}{e^{sa}}$

Let's see how good the bound is:



$P(Z>b) \leq \min_{s>0} \left[\frac{\mathbb{E}[e^{sZ}]}{e^{sb}}\right]$

$P(Z>b) \leq \min_{s>0} [\underbrace{e^{s^2/2} e^{-sb}}]$

$f(s) = e^{s^2/2 - sb}$

$\Rightarrow \min_{s>0} f(s) = \min_{s>0} \frac{s^2}{2} - sb$

$f'(s) = s - b = 0$

$s = b$

$\Rightarrow \boxed{P(Z>b) \leq e^{-b^2/2}}$ Chernoff Bound for Std Normal

Ex: $X = \sum_{i=1}^{n} X_i$, where $X_i = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases}$

X_i's indep $\mu = \mathbb{E}[X] = \sum_{i=1}^{\infty} p_i$

Then, $\boxed{P(|X-\mu| \geq \delta\mu) \leq 2e^{-\mu\delta^2/3} \text{ for } 0<\delta<1}$