Joint Gaussian RVs

If $f_{X_1,X_2}(x_1,x_2)$ is s.t. $\alpha_1 X_1+\alpha_2 X_2 \sim$ Normal for every $\alpha_1,\alpha_2 \in \mathbb{R}$, then X_1, X_2 are JG RVs if **every** linear combination $\alpha_1 X_1+\alpha_2 X_2$ is a normal pdf.

Alt. def:

$Y_1, \ldots, Y_n$ are JG RVs

$\equiv \; Y = \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix}$ has a multivariate normal pdf

$Y \sim N(\mu_Y, \Sigma_Y)$ is JG w/ mean μ_Y & covariance Σ_Y if $\boxed{Y = AX + \mu_Y}$ where $X = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$, $X_i \overset{iid}{\sim} N(0,\sigma^2)$, $X \sim N(0,1)$

Note:
$$\begin{aligned}\Sigma_Y &= \mathbb{E}[(Y-\mu_Y)(Y-\mu_Y)^T] \\ &= \mathbb{E}[(AX)(AX)^T] \\ &= \mathbb{E}[AXX^TA^T] \\ &= A\underbrace{\mathbb{E}[XX^T]}_{I}A^T\end{aligned}$$
$\Rightarrow \Sigma_Y = AA^T$

> Theorem 6.3 (Walrand):
> Joint density of $Y \sim N(\mu_Y, \Sigma_Y)$ is given by
> $$f_Y(y) = \frac{1}{(2\pi)^{n/2}\sqrt{|\Sigma_Y|}} \exp\left\{-\frac{1}{2}(y-\mu_Y)^T \Sigma_Y^{-1} (y-\mu_Y)\right\}$$

Recall: Univariate Normal
$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_Y^2}} \exp\{-(y-\mu_Y)^2/2\sigma_Y^2\}$$

Note: $|\Sigma_Y|$ means $\det(\Sigma_Y)$

Ex: $n=2$

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix}}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \qquad X_i \overset{iid}{\sim} N(0,1)$$

$Y \;=\; A \;\; X$

with $\Sigma_Y = AA^T = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$

$$\Rightarrow f_Y(y_1,y_2) = \frac{1}{2\pi\sigma_1\sigma_2} \exp\left\{\underbrace{-\frac{1}{2}[y_1 \; y_2]^T \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{-\frac{1}{2}\left[\frac{y_1^2}{2\sigma_1^2} + \frac{y_2^2}{2\sigma_2^2}\right]}\right\}$$

Recall: $\Sigma_Y = \begin{bmatrix} Var(Y_1) & cov(Y_1,Y_2) \\ cov(Y_2,Y_1) & Var(Y_2) \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$

$\rho = \dfrac{cov(Y_1,Y_2)}{\sqrt{Var(Y_1)}\sqrt{Var(Y_2)}}$

$$\begin{aligned} f_{Y_1,Y_2}(y_1,y_2) &= \left(\frac{1}{\sqrt{2\pi}\sigma_1} e^{-y_1^2/2}\right)\left(\frac{1}{\sqrt{2\pi}\sigma_2} e^{-y_2^2/2}\right) \\ &= f_{Y_1}(y_1)\, f_{Y_2}(y_2) \end{aligned}$$
, i.e. Y_1, Y_2 are indep

> Note: If X_1, X_2 are JG & uncorrelated, then X_1, X_2 are also independent RVs.

Theorem 6.4 (Walrand):

For jointly Gaussian RVs $\{X_1, X_2\}$, if X_1 & X_2 are uncorrelated (i.e. $\mathbb{E}[(X_1-\mu_1)(X_2-\mu_2)] = 0$), then X_1 and X_2 are also independent.

Theorem 6.5 (Walrand):

Linear combinations of JG RVs are JG.

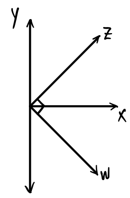
Ex: $X, Y \sim N(0,1)$ iid RVs

$Z = X+Y$, $W = X-Y$

Q/ Are Z, W indep?

$$\begin{aligned}\text{cov}(Z,W) &= \mathbb{E}[ZW] - \cancelto{0}{\mathbb{E}[Z]}\mathbb{E}[W] \\ &= \mathbb{E}[(X+Y)(X-Y)] \\ &= \mathbb{E}[X^2] - \mathbb{E}[Y^2] \\ &= 0\end{aligned}$$

Since Z, W are JG & uncorrelated, they are indep.



Z, W orthogonal, therefore indep

Ex: $W = \begin{cases} 1 & \text{w.p. } 1/2 \\ -1 & \text{w.p. } 1/2 \end{cases}$ $\mathbb{E}[W] = 0$

$X \sim N(0,1)$; X, W indep

$Y = X \cdot W$

1/ Are X, Y correlated?

$\mathbb{E}[XY] = \mathbb{E}[X^2 W] = \mathbb{E}[X^2] \cdot \mathbb{E}[W] = 0$ ← uncorrelated

2/ What is the distribution of Y?

$Y = \begin{cases} X & \text{w.p. } 1/2 \\ -X & \text{w.p. } 1/2 \end{cases} \rightarrow Y \sim N(0,1)$

3/ Are X, Y indep?

Clearly $Y = W \cdot X$ is NOT indep of X.

Note: X, Y are marginally $N(0,1)$ RVs but not JG!

e.g. $X+Y = \begin{cases} 2X & \text{w.p. } 1/2 \\ 0 & \text{w.p. } 1/2 \end{cases} \not\Rightarrow N(\mu, \sigma^2)$

Not all lin combo of X, Y is normal → not JG

$\boxed{\text{* Theorem: If } X, Y \text{ are JG, } \mathbb{E}[X|Y] = L[X|Y]}$

a) Recall: If X, Y are JG, then all linear combos are JG.

b) If X, Y are JG & uncorrelated, they are indep.

Proof:

1/ $X - L[X|Y] \perp Y$ (projection property of $L[X|Y]$)

2/ If X, Y are JG, so are lin combos of X, Y, namely $X - L[X|Y]$ and Y.

3/ Therefore $X - L[X|Y]$ and Y are indep.

4/ Thus $X - L[X|Y]$ and $\phi(Y)$ are indep.

function of Y, since funcs of indep RVs are indep

5/ Thus $X - L[X|Y]$ and $\phi(Y)$ for any ϕ are uncorrelated, since independence implies uncorrelated.

6/ $L[X|Y]$ must be the same as $\mathbb{E}[X|Y]$, since $X - \mathbb{E}[X|Y]$ is unique and $\perp$ to every func of Y.

$\Rightarrow L[X|Y] = \mathbb{E}[X|Y]$