Normal Distribution

$Z \sim N(0,1)$

$\boxed{f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}}$

$f_X(x) \propto e^{-x^2/2}$ ← must be normalized

↖ no known elementary variable for this func

Integrate the normalized distr,

$c\int_{-\infty}^{\infty} e^{-x^2/2}dx = 1$ no closed form solution

We'll solve a slightly harder problem $\Rightarrow e^{-\alpha x^2/2}$, & set $\alpha=1$ at the end

Trick: $I^2 = \int_{-\infty}^{\infty} e^{-\alpha x^2/2}dx \int_{-\infty}^{\infty} e^{-\alpha y^2/2}dy$ convert to polar coordinates

$= \int_0^{2\pi}\int_0^{\infty} e^{-\alpha r^2/2} r\,dr\,d\theta$

$= \frac{2\pi}{\alpha}$

$\Rightarrow I = \sqrt{\frac{2\pi}{\alpha}}$

Set $\alpha = 1$, $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, $x \in \mathbb{R}$

CDF of $Z \sim N(0,1)$

$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}dx$

Mean: $\mathbb{E}[X] = 0$ by symmetry

Variance: $\mathbb{E}[X^2] = \int x^2 e^{-x^2/2}dx$

$= \frac{1}{\sqrt{2\pi}} \int \underbrace{x^2 e^{-\alpha x^2/2}}_{\text{works like } (-2)\frac{\partial}{\partial\alpha} e^{-\alpha x^2/2}} dx$

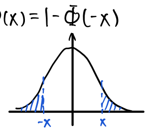
$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (-2)\frac{\partial}{\partial\alpha}\left[e^{-\alpha x^2/2}\right]dx$

$= -\frac{2}{\sqrt{2\pi}} \frac{\partial}{\partial\alpha}\left[\underbrace{\int_{-\infty}^{\infty} e^{-\alpha x^2/2}dx}_{\sqrt{2\pi/2}}\right]$

$= -\frac{2}{\sqrt{2\pi}} \frac{\partial}{\partial\alpha}\left[\alpha^{-1/2}\right]$

$= \alpha^{-3/2}$

$\Phi(x) = 1 - \Phi(-x)$ symmetric



Change of Variables

Let X be a RV and $g: \mathbb{R} \to \mathbb{R}$ is one-to-one. $Y = g(X)$

Let h be inverse of g, i.e. $h = g^{-1}$.

Then, $\boxed{f_Y(y) = f_X(h(y))\,|h'(y)|}$

Let's show it for g strictly increasing $\Rightarrow$ h also strictly increasing $\Rightarrow h'(y) > 0 \ \forall y$.

$F_Y(y) = P(Y \le y) = P(g(X) \le y) = P(X \le h(y)) = F_X(h(y))$

$\Rightarrow f_Y(y) = \frac{d}{dy} F_X(h(y)) = f_X(h(y))\,h'(y)$

If g NOT one-to-one,

$\boxed{f_Y(y) = \sum_{x: g(x)=y} f_X(x)\,|h'(y)|}$

Nonstandard Gaussian

$Y = \mu + \sigma Z$, where $Z \sim N(0,1)$

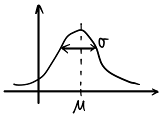
$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$, $z \in \mathbb{R}$

$F_Y(y) = P(Y \le y) = P(\mu + \sigma Z \le y) = P(Z \le \frac{y-\mu}{\sigma}) = F_Z(\frac{y-\mu}{\sigma})$

$\Rightarrow \boxed{f_Y(y) = f_Z(\frac{y-\mu}{\sigma}) \frac{1}{\sigma}}$

$\boxed{f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(y-\mu)^2/2\sigma^2}}$

$\left.\begin{array}{l} \mathbb{E}[Y] = \mu \\ Var(Y) = \sigma^2 \end{array}\right\} Y \sim N(\mu, \sigma^2)$



Ex: $X \sim$ (annual snowfall) $N(\mu, \sigma^2)$ $\mu = 60''$, $\sigma = 20''$

Q/ $P(X \ge 80'')$

Let $Z = \frac{X-\mu}{\sigma}$ (standardize)

$P(X \ge 80'') = P(\frac{X-\mu}{\sigma} \ge \frac{80''-\mu}{\sigma})$

$= P(Z \ge 1)$

$= 1 - \Phi(1)$

$= 1 - 0.8413$

$= 0.1587$