Continuous RVs

In most real-world settings, a continuous sample space is much more natural than discrete (e.g. time, velocity, dist, etc.).

- must define prob. in terms of sets/intervals rather

Q/ How do we def sample spaces for cont. sample spaces?

$X \sim Unif(0,1)$

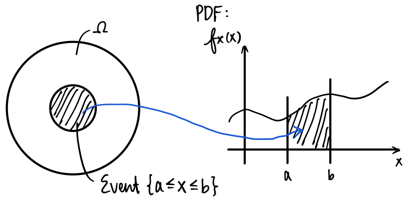
$\Rightarrow P(X=0.4) = 0$

on $\mathbb{R}$, sets that can qualify as "legal" events can be assigned prob.

↳ called Borel subsets of form:

$(-\infty, a]$ or (a,b) or $[a,b)$ or $(a,b) \cup (c,d)$, etc.

→ Any reasonable set is fine.



X is a CRV if:

① $\exists$ a non-neg $f_X(x)$ s.t.

$$P(X \in B) = \int_B f_X(x)\,dx \quad \forall B \in \mathbb{R}$$

② $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

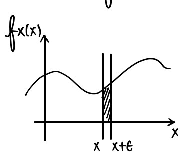
Implications:

- $P(a \le X \le b) = \int_a^b f_X(x)\,dx$
- $P(X=a) = 0$
- $P(X<a) = P(X \le a)$

Probability Density (PDF)

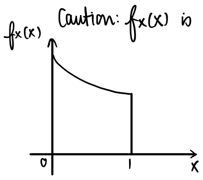
$$P(X \in [x, x+\epsilon]) = \int_x^{x+\epsilon} f_X(t)\,dt$$

$$\approx f_X(x) \cdot \epsilon \quad \text{for } \epsilon \ll 1$$



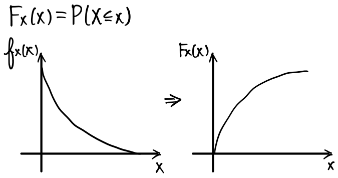
$f_X(x) \approx \dfrac{P(X \in [x, x+\epsilon])}{\epsilon}$ (prob for unit length ⇒ density)

Caution: $f_X(x)$ is NOT a probability!



$$f_X(x) = \begin{cases} \frac{1}{2\sqrt{x}}, & 0 \le x \le 1 \\ 0, & \text{else} \end{cases}$$

CDF: $F_X(x) = P(X \le x)$



Independence

$X \perp Y$ (CRVs) if $\{X \le x\}$ & $\{Y \le y\}$ $\forall x,y \in \mathbb{R}$

$\underbrace{P(X \le x, Y \le y)}_{P_{X,Y}} = F_X(x) F_Y(y)$

If $X \perp Y$, $f(x) \perp g(y)$ for any function f, g

TSF : $\mathbb{E}(X) = \int_0^\infty \underbrace{(1-F_X(x))}_{\text{survival / complementary CDF} : P(X \ge x)} dx$

CRV Facts :

$\mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x) dx$

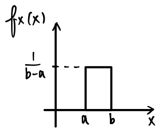
$\mathbb{E}[g(x)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$

$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \int_{-\infty}^{\infty} (X - \mathbb{E}[X])^2 f_X(x) dx$

$a Var(aX+b) = a^2 Var(X)$

Common CRVs

① Unif (a,b)

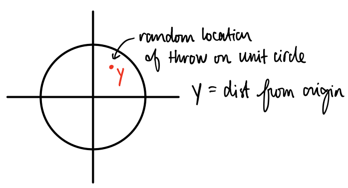


$f_X(x) = \frac{1}{b-a}$

$\mathbb{E}[X] = \int_a^b x \frac{1}{b-a} dx = \frac{a+b}{2}$

$Var(X) = \frac{(b-a)^2}{12}$

Ex: Random Dart on Circle



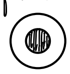
Q/ PDF of Y

↳ find CDF of Y & differentiate

$P(Y \le y) =$ Ratio

$= \frac{\pi y^2}{\pi 1^2}$

$= y^2, \ 0 \le y \le 1$

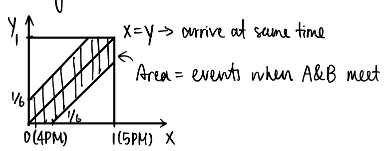


$f_Y(y) = \frac{d}{dy} F_Y(y) = 2y$

Ex: A & B arrive uniformly at random btw 4 & 5PM independently.

First to arrive waits 10 min & leaves.

Q/ P(they meet) = $\frac{11}{36}$



(X,Y) = when X, Y arrive in hours

② $Exp(\lambda)$

$f_X(x) = \lambda e^{-\lambda x}, \ x \geq 0$

$\mathbb{E}[X] = \int_0^\infty x \lambda e^{-\lambda x} \, dx = \frac{1}{\lambda}$

$Var(X) = \frac{1}{\lambda^2}$

Memoryless Property

$P(X > s+t \mid X > t) = P(X > s)$

↳ reset the system, no "breaks"

Proof: $P(X > s+t \mid X > t) = \frac{P(X > s+t, X > t)}{P(X > t)}$

$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}}$

$= e^{-\lambda s} = P(X > s)$

Theorem: Let T be some RV that satisfies memorylessness, then $f_T(t) = \lambda e^{-\lambda t}$, i.e. $T \sim Exp(\lambda)$.

Ex: Min & Max of Exp RVs

Let $T_1, \dots, T_n$ be independent Exp RVs w/ parameters $\lambda_1, \dots, \lambda_n$ respectively.

1/ $\min\{T_1, \dots, T_n\} \sim Exp(\lambda_1 + \dots + \lambda_n)$ ↳ Let T_i be independent lifetimes of lightbulbs.

Proof: Use survival for min

$P(\min\{T_1, \dots, T_n\} > t) = P(T_1 > t) \dots P(T_n > t)$

$= e^{-\lambda_1 t} \dots e^{-\lambda_n t}$

$= e^{-(\lambda_1 + \dots + \lambda_n)t}$

2/ $\max\{T_1, \dots, T_n\}$

Proof: use CDF

$P(\max\{T_1, \dots, T_n\} < t) = P(T_1 < t) \dots P(T_n < t)$

$= (1 - e^{-\lambda_1 t}) \dots (1 - e^{-\lambda_n t})$

$\mathbb{E}[\max\{T_1, \dots, T_n\}]$ = exp time for all lightbulbs to burn out

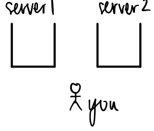
↳ use memorylessness & recursion

$$\mathbb{E}[\text{time for } n \text{ bulbs to burn out}] = \underbrace{\mathbb{E}[\text{time for first bulb to burn out}]}_{\substack{\min\{T_1, \dots, T_n\} \\ = Exp(n), \text{ for all } \lambda_i = 1 \\ \mathbb{E}[\min] = \frac{1}{n}}} + \underbrace{\mathbb{E}[\text{time for remaining } n-1 \text{ bulbs to burn out}]}_{\substack{\text{By memorylessness, same as} \\ \mathbb{E}[\text{time for } n-1 \text{ bulbs to burn out}]}}$$

Recurrence: $\mathbb{E}[S_n] = \frac{1}{n} + \mathbb{E}[S_{n-1}]$ harmonic sum

$= \sum_{k=1}^{n} \frac{1}{k} \approx \ln n$

Ex: Waiting at the P.O.



- 2 clerks, each serving a customer when you arrive
- service times of servers are independent $Exp(\lambda)$

Q/ P(you will be last customer to leave)

↳ use memorylessness ⇒ competing exponentials

Geom(p) & Exp(λ)

- both memoryless
- flip a coin N times/sec w.p. $P(\text{head}) = \frac{\lambda}{N}$ where $N \gg 1$

- Let X be the time (clock) until first head

Fact: $X \approx Exp(\lambda)$

Why? $P(X > t) = P(\text{first } Nt \text{ flips are tails})$

$$= \left(1 - \frac{\lambda}{N}\right)^{Nt} \xrightarrow[N \to \infty]{} e^{-\lambda t} \quad \left(\text{Recall } \left(1 - \frac{a}{N}\right)^N \to e^{-a}\right)$$