Markov Chains

A stochastic process $X \in \{X_t\}_{t \in T}$ is a collection of sequence of RVs.

X models the evolution of a seq of RVs as a function of time.

$X = \{x_1, x_2, \dots, x_T\}$

As T gets large, the joint distr of $(x_1, x_2, \dots, x_T)$ gets messy.

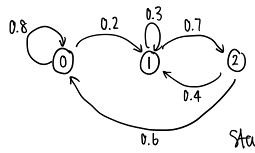
↳ Need to impose structure ⇒ MC

$$\boxed{\text{Markov Property: } P(X_n = x_n \mid X_{n-1} = x_{n-1}, \dots, X_0 = x_0) = P(X_n = x_n \mid X_{n-1} = x_{n-1}) = P(x_{n-1}, x_n)}$$

called the "state" at time n — from — to

Given the present, the past is independent of the future.

Ex:



$\mathcal{X} = \{0,1,2\}$

State-transition Diagram

from \ to	0	1	2
0	0.8	0.2	0
1	0	0.3	0.7
2	0.6	0.4	0

State-transition Matrix (P)

$\pi_0 = 0.8\pi_0 + 0.6\pi_2$

$\pi_1 = 0.2\pi_0 + 0.3\pi_1 + 0.4\pi_2$

$\pi_2 = 0.7\pi_1$

Initial Distribution: $\underline{\pi_0} = [\pi_0(0) \;\; \pi_0(1) \;\; \pi_0(2)]$

row vector

$\underline{\pi_n} = [\pi_n(0) \;\; \pi_n(1) \;\; \pi_n(2)]$

prob that at time n, you are in state 2

Q/ How to find $\underline{\pi_n}$ from $\underline{\pi_0}$?

$$\underbrace{P(X_1 = j)}_{\pi_1(j)} = \sum_{i=0}^{2} \underbrace{P(X_1 = j \mid X_0 = i)}_{P(i,j)} \underbrace{P(X_0 = i)}_{\pi_0(i)}$$

$$\pi_1(j) = \sum_{i \in \mathcal{X}} P_{i,j}\, \pi_0(i) \quad \forall j \in \mathcal{X}$$

In matrix form:

$$[\pi_1(0) \;\; \pi_1(1) \;\; \pi_1(2)] = [\pi_0(0) \;\; \pi_0(1) \;\; \pi_0(2)] \begin{bmatrix} P_{0,0} & P_{0,1} & P_{0,2} \\ P_{1,0} & P_{1,1} & P_{1,2} \\ P_{2,0} & P_{2,1} & P_{2,2} \end{bmatrix}$$

3 equations, 1 is redundant

need another eqn: $\boxed{\sum \pi = 1}$

equivalently: Flow in = Flow out

$\underline{\pi_1} = \underline{\pi_0} P$

$\underline{\pi_n} = \underline{\pi_{n-1}} P = \underline{\pi_{n-2}} P^2 = \dots = \underline{\pi_0} P^n$

$\Rightarrow \boxed{\underline{\pi_n} = \underline{\pi_0} P^n}$

If $\underline{\pi}^* = \underline{\pi}^* P$, then $\underline{\pi}^*$ is called an invariant / stationary distribution of the Markov chain.

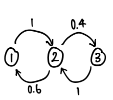
Def: Irreducible, Aperiodic

a) Irreducible if a MC can go from any state to any other state possibly after many steps.

b) $d(i) = \gcd\{n \geq 1 \mid \underbrace{P^n(i,i)}_{\text{prob of going from i back to i in n steps}} > 0\}$

Aperiodic if $d = 1$

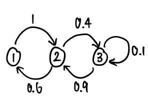
Ex:



Irreducible

$P^n(i,i) > 0 : \{n = 2, 4, \dots\}$

$d = 2 \Rightarrow$ Periodic

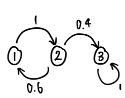


Irreducible

self loop

$P^n(i,i) > 0 : \{n = 2, 4, 5, \dots\}$

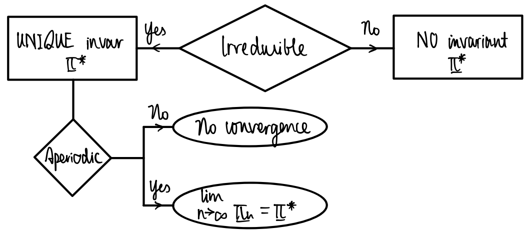
$d = 1 \Rightarrow$ Aperiodic



NOT irreducible - state 3 is absorbing

Aperiodic

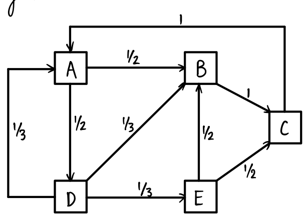
Big Theorem 1.2 (Walrand) for Finite State Markov Chain



$\lim_{N\to\infty} \frac{1}{N}\sum \{\mathbb{1}\}_{X_n=i} = \pi(i)$

long-term fraction of time that $X_n = i$

① Page Rank



After multiplying by P many times, π converges to π^*.

Hitting Time Eqns

Q/ Starting at A, how many steps does it take to reach E?

↳ this is called the hitting time / first passage time for state E, denoted as T_E.

Mean Hitting Time

$\boxed{\beta(A) = \mathbb{E}[T_E \mid X_0 = A]}$ starting at state A at timestep 0

First Step Equations

Key is to calculate $\beta(i)$ for $i = A, B, C, D, E$.

Why? Because $\beta(i)$'s are coupled.

take 1 step

$\beta(A) = 1 + \frac{1}{2}\beta(B) + \frac{1}{2}\beta(D)$

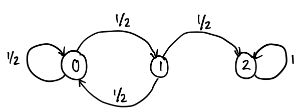
$\beta(B) = 1 + \beta(C)$

$\beta(C) = 1 + \beta(A)$

$\beta(D) = 1 + \frac{1}{3}\beta(A) + \frac{1}{3}\beta(B) + \frac{1}{3}\beta(E)$

$\beta(E) = 0$

② Coin Toss



$\mathbb{E}$[# tosses until 2 consec Hs] = ?

FSEs: Def $\beta_i = \mathbb{E}[T_2 \mid X_0 = i]$

$\beta_0 = 1 + \frac{1}{2}\beta_0 + \frac{1}{2}\beta_1$

$\beta_1 = 1 + \frac{1}{2}\beta_2 + \frac{1}{2}\beta_0$

$\beta_2 = 0$

③ Hitting C Before E

$\alpha_A = P(\text{hitting C before E} \mid X_0 = A)$

We have a set of eqns involving states A-E:

$\alpha_A = \frac{1}{2}\alpha_B + \frac{1}{2}\alpha_D$

$\alpha_B = \alpha_C$

$\alpha_C = 1$ (good state)

$\alpha_D = \frac{1}{3}\alpha_A + \frac{1}{3}\alpha_B + \frac{1}{3}\alpha_E$

$\alpha_E = 0$ (bad state)

FSEs for DTMCs

$\mathcal{X} = \{1, 2, \dots, N\}$

$A \subset \mathcal{X}$

$T_A = \min\{n \geq 0 \mid X_{(n)} = A\}$

$\beta_A(i) = \mathbb{E}[T_A \mid X_0 = i] \ \forall i \in \mathcal{X}$

$$\boxed{\text{FSE: } \beta_A(i) = \begin{cases} 1 + \sum_{j \in \mathcal{X}} P(i,j)\,\beta_A(j) \\ 0 \text{ if } i \in A \end{cases}}$$

$\alpha(i) = P(T_A < T_B \mid X_0 = i) \ \forall i \in \mathcal{X}$

$$\boxed{\text{FSE: } \alpha(i) = \begin{cases} \sum_j P(i,j)\,\alpha(j) \text{ if } i \notin A \cup B \\ 1 \text{ if } i \in A \\ 0 \text{ if } i \in B \end{cases}}$$