Project 3: The Building Blocks of Planet Formation

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***Abstract*—Using the "streaming instability" model, this project investigates the fascinating process of planet formation within protoplanetary disks. Numerical simulations study dust and gas interactions, turbulence and particle growth to determine planetesimal formation conditions. Turbulent diffusion obstructs small-particle aggregation, but it is found that the growth of particle sizes allows the dust to overcome the turbulence and form planetsimals. Numerical methods ensured the stability of the simulation and minimized the numerical diffusion, confirming the robustness of the findings. Astrophysical observations confirm the model's consistency with Solar System planetary formation patterns.**

I. Introduction

PLANETS form in a dynamic and complex environment known as a protoplanetary disc, a flat structure of gas and dust surrounding a newborn star. Observations of our Solar System provide clues about the role of these discs in the formation of celestial objects: they seem to nearly lie in a plane aligned with the Sun's spin axis, suggesting an emergence from an organized and coherent structure. Referred to as accretion discs, they are composed primarily of hydrogen and helium, with a crucial small percentage of dust particles derived from the remnants of earlier generations of stars, allowing the formation of protoplanetary objects. The protoplanetary disc structure can be naturally described in cylindrical coordinates.

The formation model I used in this project is the "streaming instability", where local enhancements in the dust-to-gas density ratio allow the gravitational clumping of particles. While the gas can maintain verticality against gravity due to pressure, the dust particles, lacking the support, settle towards the disc's midplane, naturally augmenting their density.

Gas density follows a vertical hydrostatic equilibrium, governed by a Gaussian profile, while dust evolution is dictated by advection-diffusion dynamics under gravity.

Theoretical model adopted

In this work, I started by modeling the vertical settling of dust particles using a flux-conservative advection equation:

$$\frac{\partial}{\partial z}\left(u_z \rho_d\right) = -\frac{\partial \rho_d}{\partial t} \tag{I.1}$$

where:

$$u_z = -\Omega^2 z t_s$$

where $\Omega = \sqrt{GM/R^3}$ is the angular frequency of a circular Keplerian orbit at a radius R with M as the Sun's mass and $t_s = \sqrt{\frac{\pi}{8}}\frac{\rho_{in} a}{\rho_g c_s}$ (settling timescale, depending on the dust particle's size a_d, its internal density $\rho_{in} = 1000$ kg/m^{-3}, $c_s = \sqrt{k_b T(R)/(2m_H)}$ is the gas disk's sound-speed with $T(R)$ the temperature, k_b the Boltzmann constant and m_H the mass of the hydrogen atom). The temperature depends on the radius by the following:

$$T(R) = 264 \text{ K} \left(\frac{R}{1\text{AU}}\right)^{-1/2}$$

As stated previously, the gas follows a vertical gaussian profile that is modelized as:

$$\rho_g = \rho_g(R, z = 0) \cdot \exp\left(-\frac{z^2}{2H(R)^2}\right)^{1}$$

while its radial function is:

$$\rho_g(R, z = 0) = 5 \times 10^{-7} \text{ kg m}^{-3} \left(\frac{R}{1\text{AU}}\right)^{-3}$$

II. Aims

[1] $H(R) = \frac{c_s}{\Omega}$ is the scale height of the disk

THE aims of this multi-faceted project can be summarized as:

- Enhance the understanding on the physics of planet formation and investigate how planetesimals form in protoplanetary discs around young stars.
- Develop and implement a computational model, constructing a simulation using numerical methods to solve the problems and in doing so ensuring proper handling of the selection of timesteps, boundary conditions, domain size, and grid resolution to maintain a satisfying level of accuracy.
- Investigate on the role of gas and dust interactions in the discs, particularly how these can be modeled as diffusion. Then, examine how turbulence affects the ability of dust to settle and form planetesimals.
- Study how dust particles grow via collisions by sticking to each other, thus enabling them to overcome the turbulences and settle into the mid-plane. Consequently, determine the sizes the dust particles must achieve to make planetesimal formation possible at various radial distances from the center of the disc.
- Ensure accuracy in measuring numerical diffusion introduced by the aforementioned computational scheme, thus demonstrating that the simulation results are robust and not significantly affected by numerical artifacts, confirming the reliability and reproducibility of the physical conclusions.
- Finally, analyze how the findings align with observed properties of our solar system, such as locations and compositions of planets.

III. Problem setup and considerations

I first chose the natural units. I appropriately used AU (astronomic units) for the radius R, years for time and $H(R)$ for the height of the disk at radius R, with upper and lower boundaries $\pm 2H(R)$ as a compromise between having a small enough density near the boundary and limiting the size of u_{max}, in fact, as I will discuss later, the time step was chosen taking in account the value of u_{max} which, if too small, would slow down the computation time significantly. For the dust and gas density I chose the unit $\rho_0 = 5 \cdot 10^{-7}\ \mathrm{kg \cdot m^{-3}}$ in order to express the gas density as

$$\rho_{\text{gas}} = \frac{1}{R^3} \cdot e^{-\frac{z^2}{2}} \text{ , where } R \text{ and } z$$

are expressed in the aforementioned natural units. Consequentially I converted u_z in $(H(R)/yrs)$.

Analysis of the advection equation

Then to make a prediction in the evolution of the system I further analyzed the speed[2] u_z:

$$u_z = -\Omega^2 z t_s = -\Omega^2 z \cdot \sqrt{\frac{\pi}{8}} \frac{\rho_{in} a_d}{\rho_g c_s} =$$

$$= -\frac{GM_S}{R^3} z \cdot \sqrt{\frac{\pi}{8}} \frac{\rho_{in} a_d}{\left(\frac{1}{R^3} \cdot e^{\frac{-z^2}{2}}\right) \cdot \left(\sqrt{k_b T(R)/2m_H}\right)} =$$

$$= -GM_S \cdot \sqrt{\frac{\pi}{8}} \rho_{in} a_d \cdot \sqrt{\frac{2m_H}{k_b}} z e^{\frac{z^2}{2}} \cdot \frac{1}{\sqrt{T_0 \cdot \frac{1}{\sqrt{R}}}}$$

$$= -6 GM_S \rho_{\text{in}}\, a_d \cdot \sqrt{\frac{\pi m_H}{4 k_b T_0}} \cdot z e^{\frac{z^2}{2}} \cdot \sqrt{R}$$

Notice that u_z depends exponentially on z and is always directed towards the center (since the dependance on $-z$). We can expect by consequence that the system evolves by moving the dust particles in a centripetal motion, where eventually the local dust density will surpass the one of the gas.

Numerical methods for advection

To verify this, I set up the first-order upwind scheme[3] to solve the advection equation I.1, where the update function is:

$$\rho_i^{n+1} = \rho_i^n + \Delta t \cdot \text{adv\_term}_i^n$$

and the adv_term is calculated using the backwards finite difference if $u_i > 0$ (which means $z < 0$)

$$\text{adv\_term}_i^n = -\frac{\rho_i^n u_i - \rho_{i-1}^n u_{i-1}}{\Delta z}$$

[2] where $T_0 = 264$ K

[3] described in the lecture notes in section 11.3.1

if instead $u_i < 0$ ($z > 0$) I used the forward difference:

$$\text{adv\_term}_i^n = -\frac{\rho_{i+1}^n u_{i+1} - \rho_i^n, u_i}{\Delta z}$$

I also had to make sure to choose an appropriate Δt to respect the CFL condition $\left(\left(u_i \frac{\Delta t}{\Delta z} = a_i\right) < 1\right)$. So, if $|a_{\max}| = 0.8$ then I choose $\Delta t = |a_{max}| \cdot \frac{\Delta z}{|u_{max}|}$, where u_{max} is calculated at the boundary ($z = 2$). This way I was sure that every time I solved the equation for a different R the stability conditions would be respected. As the Dirichlet boundary conditions, I imposed the dust density at the border to be zero. The validity of this choice will be discussed in the next section.

IV. Dust settling

To simulate the dust settling I initialized the density uniformly as $\rho_d^i(t = 0) = 0.01\rho_g^i$, since if the initial gas distribution is Gaussian in nature, the initial dust distribution is Gaussian as well. Since the z domain has ± 2 as boundaries, imposing a Dirichlet condition does not lead to a discontinuity (as $0.01 \cdot e^{-\frac{(z=2)^2}{2}} \approx 0$) at any given instant, providing the distribution remains Gaussian and simply thins out over time. The time domain has been chosen at $5Myrs$, as that is a higher value than even the longest settling time (that happens at $R = 1AU$).

If we observe in figure 2 the amount of time in which the local dust density surpasses the local gas density ($\rho_d > \rho_g$) we can notice that the formation time is in the order of magnitude of $1Myrs$ and decreases with increasing radial distance. This is in accord with the equation III since u_z is proportional to $\sqrt{R}$ (given its dependance on the temperature) so the dust moves faster at larger radial distances.

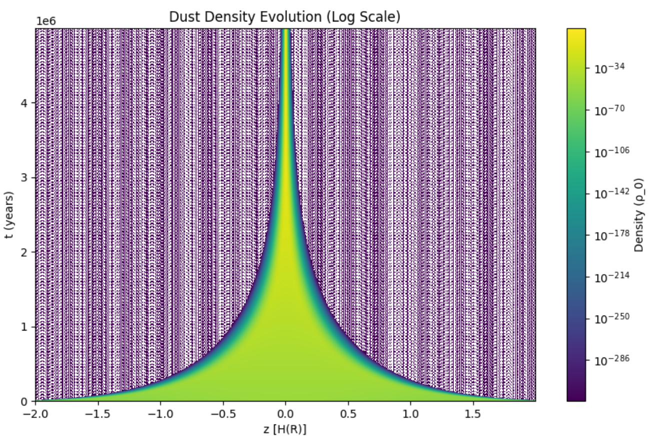

Figure 1: Dust evolution for $R = 1, H(R = 1) = 5.25 \cdot 10^9\text{m} \approx 0.03\text{AU}$. Notice how dust converges towards the middle as expected

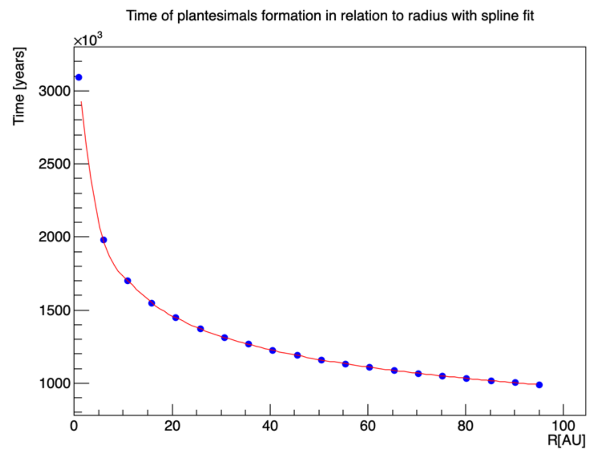

Figure 2: The fit allows us to qualitatively infer that the formation time is $\sim \frac{1}{\sqrt{R}}$, as predicted by the theory

V. Turbulent diffusion of the dust

My code predicts that at $R = 100AU$, after 1 Myr, the dust should have settled into a thin layer around the center, as you can see in figure 3.

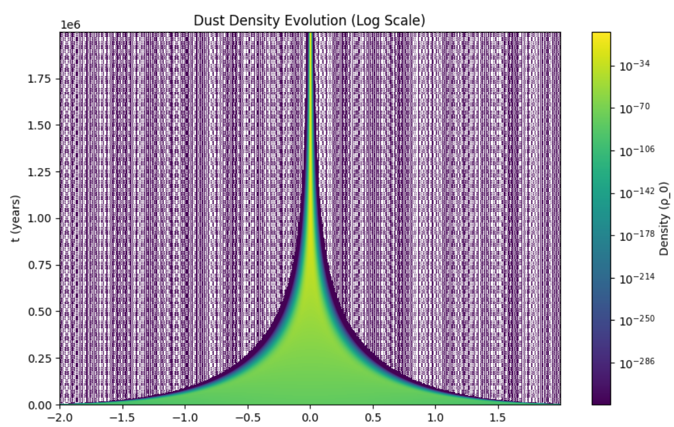

Figure 3: Dust evolution with no turbulence, time domain = $2Myrs$, $R = 100, H(R = 100) = 1.66 \cdot 10^{12}\text{m} \approx 12\text{AU}$

However, this is contradicted by our observations. It is possible to extimate the disk's height

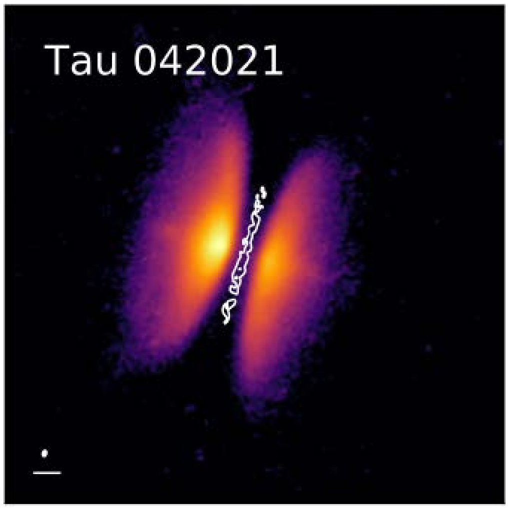

Figure 4: An image of a planet-forming disc in the Taurus star forming cluster, taken at a wavelength of $\sim 1\mu m$. The dark region corresponds to the location of dust particles, as they are blocking the light from the young star. The bar in the left corner represents a distance of 33 AU.

(at ~ 100AU) from the picture 4, which I placed at 10AU. These discrepancies between my computational predictions and the experimental observations can be explained by introducing turbulences in the theoretical modellization. In particular, we can insert a diffusion term into the advection equation

$$\frac{\partial \rho_d}{\partial t} = -\frac{\partial}{\partial z}\left(u_z \rho_d\right) + D\frac{\partial}{\partial t}\left[\rho_g \frac{\partial}{\partial z}\left(\frac{\rho_d}{\rho_g}\right)\right]$$

Where D is the diffusion constant equal to $\alpha\frac{c_s^2}{\Omega}$. We cannot extrapolate α from theory alone, it can only be determined by experimental observations.

Numerical methods for diffusion

The diffusive part of the equation was implemented in the code using the Central Difference Scheme[4] for both derivatives (internal and external) in the following way[5]:
Internal

$$\frac{\partial}{\partial z}\left(\frac{\rho_d}{\rho_g}\right)\Bigg|_i \approx \frac{\rho_d^{i+1}/\rho_g^{i+1} - \rho_a^{i-1}/\rho_g^{i-1}}{2\Delta z} = \mathrm{gr}^i$$

External

$$\frac{\partial}{\partial z}\left[\rho_g \frac{\partial}{\partial z}\left(\frac{\rho_d}{\rho_g}\right)\right]\Bigg|_i \approx \frac{\rho_g^{i+1}\mathrm{gr}^{i+1} - \rho_g^{i-1}\mathrm{gr}^{i-1}}{2\Delta z}$$

The diffusive term is:

$$\text{diff\_term}\ {}_i^n = D \cdot \frac{\rho_g^{i+1}\mathrm{gr}^{i+1} - \rho_g^{i-1}\mathrm{gr}^{i-1}}{2\Delta z}$$

Thus the new update function will be:

$$\rho_i^{n+1} = \rho_i^n + \Delta t \cdot \left(\text{adv\_term}_i^n + \text{diff\_term}_i^n\right)$$

It's important to note that the i-domain has boundaries 2 and $Nz-2$ since the second derivative takes terms from $i-2$ to $i+2$. It is thus necessary to add ulterior Dirichlet conditions. Thinking about the physics involved near the borders we notice that u_z is big, so even by factoring in the diffusion there shouldn't be any dust. We can infer that the internal derivative gr^i is zero near the border (where $i = 0, 1, Nz-2, Nz-1$) and so is the diffusive term. We then have to check the CFL condition $\left(d = \frac{D\Delta t}{\Delta z^2}\right) < \frac{1}{2}$. Thus, every time the temporal step is too big, it's instead changed to $\Delta t = \frac{\Delta z^2}{3D}\left(d = \frac{1}{3}\right)$, to respect the advection and diffusion CFLs. Once this code was implemented it was possible to extimate α via observing the height[6] of the dust cloud after 1 Myr and finding a parameter that fit the previous observation. I extimated α to be $\approx 10^{-4}$

VI. Dust particle growth

We notice that the diffusion prevents the dust from collecting towards the center and form planetsimals so, once more, the model does not seem to include the possibility of planet formation. Observations inform us, though, that the dust particle

[4]a reference can be found in lecture the notes at section 11.2

[5]$z_i = \left(i - \frac{Nz}{2}\right)\Delta z$, where Nz is the number of points in the grid in z.

[6]For finding how high the dust cloud was I used the criteria of considering the density non zero if $\rho_d > 10^{-3}\cdot rho_d^{center}$ since my Dirichelet boundary conditions already assume that $0.01\cdot e^{\frac{(z=2)^2}{2}} \approx 10^{-3} \approx 0$ and furthermore if I counted the boundary as 10^{-4} it made almost no difference while with 10^{-2} I was finding inconsistent results.

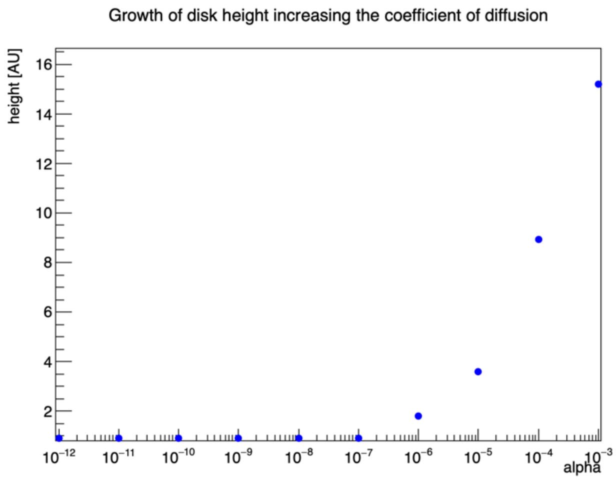

Figure 5: Height of the disk after $1Myrs$ at $R = 100AU$ in relation to α coefficient

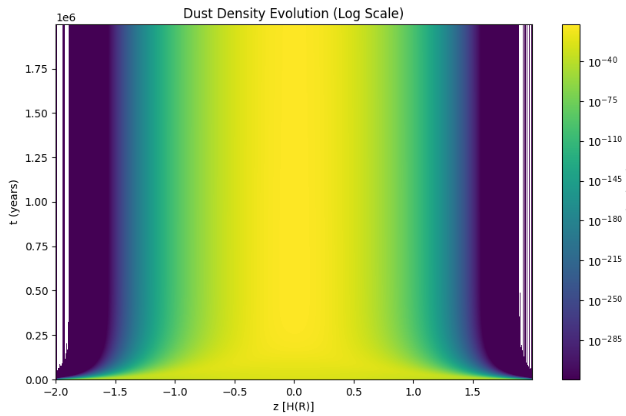

Figure 6: Added turbulence, R = 100, $\alpha = 10^{-4}$. We assume that the yellow part ($\rho_d > 10^{-3} \cdot rho_d^{center}$) is the part which physically contains the dust. So the height of the disk at around $1Myrs$ is $\approx 0.8 \cdot H(R = 100) \approx 9AU$.

dimension keeps rising with time because of the multiple inelastic collisions between them, thus allowing them to meld together. The formation of planetesimals becomes possible after the particles reach a certain size. This is in accordance with the modellization because, as you can see in III, $u_z \sim a_d$[7], so adv_term $\sim a_d$, thus adv_term $>>$ diff_term for a certain value of a_d and the dust will return to accumulate in the center. I calculated the necessary dust particle size in order to see planetesimal formations at the gas giants' (and the outer Kuiper Belt's) distance from the sun R. Thus I verified that a_d is the minimum particle size necessary for planetesimals formation for each radius.

R [AU]	a_d $[m]$	Location
5.2	0.5	Jupiter
9.6	0.3	Saturn
19	0.06	Uranus
30	0.03	Neptune
100	0.004	outer Kuiper Belt

It is important to note that, since u_z grows really big (it becomes roughly 10^5 the scale we used in the past sections), we need to adapt the time-domain as well, otherwise our code will run for a long time pointlessly. Thus, for this part, I used a time-domain of $100yrs$.

VII. NUMERICAL DIFFUSION

Resolving a PDE using an explicit method always brings along some amount of numerical diffusion which can, if too elevated, produce significant errors in the solution. For this reason it is required to evaluate the strength of the diffusion that comes from our choices of resolution.

I began by extimating the diffusion present in the z-grid I used for the past calculations, which is composed by Nz=1000 points, by analyzing the discrepancy for $\alpha = 0$. It is possible to see that

α for Nz=1000	disk height [1Myrs]
0	0.31 AU
10^{-10}	0.31 AU
10^{-9}	0.35 AU
10^{-7}	0.57 AU
10^{-6}	1.4 AU
10^{-5}	3.7 AU
10^{-4}	9.1 AU

for values of $\alpha < 10^{-9}$ there is no difference between the height of the disk after 1 Myr in the modellizations with and without turbulence. We can thus extimate that the numerical diffusion (for $Nz = 1000$) is lower than the physical one with $\alpha = 10^{-9}$.

To further validate this result, I lowered Nz to 100 and verified that the numerical diffusion increases as foreseen by the theory on numerical methods. It is easy to extimate the numerical diffusion as certainly lower than a physical one with $\alpha = 10^{-6}$.

[7]where a_d is the size of the dust particles

α for Nz = 100	disk height [1Myrs]
0	0.9 AU
10^{-7}	0.9 AU
10^{-6}	1.8 AU
10^{-5}	3.6 AU
10^{-4}	8.9 AU

This shows that, while it is certainly higher, it still remains negligible at the level in which our interest lies ($\alpha = 10^{-4}$).
As further confirmation of the negligibility of the numerical diffusion in the calculations for the size of the dust particles, I executed the code with $Nz = 5000$ arriving at the same exact results, thus proving that the numerical diffusion does not affect the estimation of the critical values a_d with a resolution of 1 significant figure.

VIII. Conclusions

THE results I arrived at provide some significant insights into the conditions necessary for planet growth.

Physical Implications

- *Dust Settling*: I found that the time required for planetsimals formation decreases with increasing radial distance. This result aligns with the theoretical observation that dust particles settle more rapidly in the outer regions of the disk, where vertical velocity is higher.
- *Turbulent Diffusion*: By introducing turbulent diffusion, I showed that it prevents complete dust settling for particle sizes small enough.
- *Particle Growth*: It is indicated by the results that a sufficient number of dust particles need to attain critical size to overcome turbulence and settle effectively. The observed composition of giant planets[8] confirms this finding.
- *Comparison with the Solar System*: Estimated lifetimes of active protoplanetary disks are generally between 1 to 10 Myr. These lifetimes are roughly matched by our timescales for planetesimal formation. Current planetary formation theories[9] support the finding that outer planets formed before inner ones because colder regions experienced faster material accretion (for Jupiter and Saturn $\approx 1-3Myrs$ and from 10 to $50Myrs$ for Earth and Mars).

Numerical Considerations

- *Simulation Robustness*: The analysis of numerical diffusion confirmed that the physical results are unaffected by numerical artifacts, ensuring the validity of the conclusions. I proved that, for my number of grid points, numerical diffusion is negligible compared to physical diffusion.
- *Accuracy and Stability*:BY respecting the CFL conditions[10] for my explicit schemes of choice (upwind method for advection and centered differences for diffusion) I ensured the stability of the simulation.

General Conclusion

The ”streaming instability” model was successfully implemented to to predict the behavior of protoplanetary disks, leading to results that are consistent with astrophysical observations.

[8] See this article of The Planetary Society
[9] See this Solar system timeline
[10] See lecture notes at 11.3.2 and 11.2.2 for stability analysis reference