Bramble Hilbert

Summary of previous lecture:

LM) let $a(u,v) : V \times V \to \mathbb{R}$ ◦) continuous, bilinear

◦◦) coercive

$$a(u,v) \leq c \|u\| \|v\| \quad \forall u,v \in V$$

$$a(u,u) \geq \alpha \|u\|^2$$

Then $\exists!$ u s.t. $a(u,v) = f(v) \quad \forall f \in V^*$

$V_h \subset V$, $\dim V_h = n_h$, $V_h = \operatorname{span}\{v_i\}_{i=1}^{n_h}$

$$a(u_h, v_h) = f(v_h) \quad \Leftarrow LM$$

$$Au = F \qquad A_{ij} := a(v_j, v_i) \qquad F_i := f(v_i)$$

A is invertible.

CEA's Lemma:

$$a(u_h - u, v_h) = 0 \implies \|u - u_h\| \leq \frac{c}{\alpha} \inf_{v_h \in V_h} \|u - v_h\|$$

Today's goals: 1) Construct V_h

2) Estimate $\inf_{v_h \in V_h} \|u - v_h\|$

Finite Element Method: $\Omega = \bigcup_{i=1}^{M} \overline{K_i}$, K_i simplices or quads / hexes

◦) $\overline{K_i} \cap \overline{K_j}$ is a full edge, or a vertex

◦) create polynomial space on each K_i, s.t. $v_h|_{K_i} \in \mathcal{P}^m$

◦) glue together the elements s.t. $v_h \in C^e(\bigcup_{i=1}^{M} \overline{K_i})$

◦◦) e: degree of continuity

◦◦) m: degree of approximation $m \geq e$

Local Trial functions — (1 bis)

Finite Element : Triple $\{ \hat{K}, \hat{P}_K, \hat{N}_K \}$

$\hat{K}$: simplex or d-rectangle

$\hat{P}_k$: polynomial space on $\hat{K}$, $\dim(\hat{P}_K) = n_K$

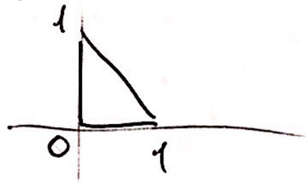
$\hat{N}_k$: set of basis functions for $(\hat{P}_K)^* := \{ e^i : P_K \to \mathbb{R} \}_{i=1}^{n_K}$ (nodal functions)

Σ_K: Local degrees of freedom $:= \{ q^i \in \mathbb{R} = e^i(q),\ q \in P_K \}_{i=1}^{n_K}$
(for any q, we associate n_k numbers $= e^i(q)$)

V_K : Local basis functions $:= \{ e_J \in P_K \mid e^i(e_J) = \delta^i_J$, $i = 1, \dots, n_K \}_{J=1}^{n_K}$

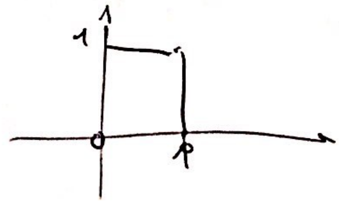
If e^i (or e_i) are linearly independent and P_K (is a complete polynomial space?), then FE is unisolvent_

We usually start by defining a "Reference Finite Element" $\{ \hat{K}, \hat{P}_{\hat{K}}, \hat{N}_{\hat{K}} \}$. $\hat{K} :=$



or

$\hat{K} :=$



Lagrangian Finite Elements $(d=2)$

$$\hat{N}_{\hat{k}} := \{ e^i := \delta(x - v_i) \;,\; v_i \text{ vertex of triangle (quad)} \}_{i=1}^{3/4}$$

$$\hat{e}^i(q) := \int_{\hat{k}} q\, e^i \, d\hat{k} = q(v_i)$$

$\Rightarrow$ ~~Local~~ Degrees of freedom are the values of the function on the vertices of the triangulation — : $\forall q \in P^k$: $q(x) = e^i(q)\, e_i(x) = q^i e_i(x)$

How about the local basis functions?

$$\hat{e}_i := q \in P^1(\hat{k}) := P_{\hat{k}} \;\Big|\; e^j(e_i) := \delta^j_i \qquad \forall\, i,j = 1,2,3 \;(\cancel{,4}) \quad \text{tria}$$

$$\hat{e}_j := q \in Q^1(\hat{k}) := P_{\hat{k}} \;\Big|\; e^i(e_j) := \delta^i_j \qquad i,j = 1,2,3,4 \quad \text{quads}$$

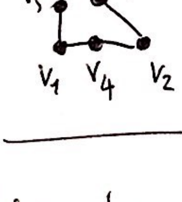
$e_1 :=$ polinomial of order 1, which is 1 in v_1, 0 in v_2, v_3

$e_2 :=$ " " , " " " v_2, " " v_1, v_3

$e_3 :=$ " " , " " v_3, " " v_1, v_2

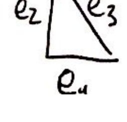
$P^2(K)$: $\operatorname{span}\{x^2, y^2, xy, x, y, 1.\}$: $\dim = 6$.

"support points" :



$$e^i := \delta(x - v_i)$$

More esoteric definitions: $\underline{e}^i := \int_{e_i} \delta(x-y)\, \underline{n}\, d\Gamma$ for $P_k := (P^m)$



$$\Rightarrow \underline{e}^i(q) = \int_{e_i} \underline{q}(y) \cdot \underline{n}\, dy \qquad \text{Raviar-Thomas}$$

In general, given $\{\hat{K}, \hat{P}_{\hat{K}}, \hat{N}_K\}$ and $\Omega = \bigcup_{i=1}^{m} K_i$ with $K_i = F_i(\hat{K})$, we define the local finite element by Affine transformations, i.e., $\{K_i, P_{K_i}, N_{K_i}\}$

$$K_i = F_i(\hat{K})$$

$$P_{K_i} := \operatorname{span}\{e_j := \hat{e}_j \circ F_i^{-1}\}_{J=1}^{n_K}$$

$$\hat{N}_{\hat{K}} := \operatorname{span}\{e^J := e^J \circ F_i^{-1}\}_{J=1}^{n_K}$$

with F_i affine, invertible, with $\det(JF_i := B) \geq 0$ a.e. in

$$\boxed{F_i(x) = \underline{\underline{B}}_i \, \underline{\hat{x}} + \underline{b}} \qquad \underline{\underline{JF_i}} := \underline{\underline{B}}_i$$

$$\rho := \sup_i \{\operatorname{diam}(\mathbb{B}) \mid \mathbb{B} \subset K_i\}$$

$$h := \sup\{\operatorname{diam}(K_i)\}$$

(Frobenious Norm)

Then $\|B_i\| \leq \dfrac{h}{\hat{\rho}}$ $\qquad \|B_i^{-1}\| \leq \dfrac{\rho}{\hat{h}}$

$$|\det(B_i)| = \frac{\operatorname{meas}(\cancel{} K_i)}{\operatorname{meas}(\hat{K})}$$

$$|\hat{v}|_{m,p,\hat{K}} \leq C \|B\|^m |\det(B)|^{-\frac{1}{p}} |v|_{m,p,K} \quad \Big| \quad |v|_{m,p,\cancel{\hat{K}}} \leq C \|B^{-1}\|^m |\det(B)|^{\frac{1}{p}} |\hat{v}|_{m,p,\hat{K}}$$

We construct a global Finite Element space by "Gluing" together the local elements, ensuring C^e continuity, i.e., ③

$v_h \in V_h$ is such that $v_h|_{K_i} \in P_{K_i}$, and

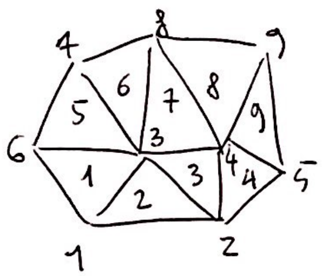
$v_h \in C^e(\overline{\Omega})$.

For C^0, Lagrangian, it is enough to "share" the same degrees of freedom between adjacent triangles / quadrilaterals

$\Rightarrow$ 1) number all vertices (support points) globally

2) associate global numbers to local numbers

9 elements, 9 vertices



For P_1: 9 degrees of freedom.

Then we transform all integrals on the reference element.

$$a(v_J, v_i) = \sum_{K_m} \int_{K_m} \nabla v_J \, \nabla v_i \, dK$$

$$= \sum_{K_m} \int_{\hat{K}_\xi} B_m^{-T} \hat{\nabla} \hat{v}_J \; B_m^{-T} \hat{\nabla} \hat{v}_i \; \det(B_m) \, d\hat{K}$$

: we need only $\underline{1}$ quadrature formula, and we need to construct only B_m, B_m^{-1}, $\det(B_m)$

Can we estimate local transformations, and errors?

Not yet — Let $\hat{P}_{\hat{k}} \subset \mathbb{P}^m(\hat{k})$

Consider $\{\hat{k}, \hat{P}_{\hat{k}}, \hat{N}_{\hat{k}}\}$ and define

$\hat{\Pi}\hat{u}$ the Interpolation of $\hat{u} \in H^{m+1}(\hat{k})$ as

$$\hat{\Pi}\hat{u} := \tilde{\hat{e}}^{i}(\hat{u})$$

$\tilde{\hat{e}}^{i}$ is a extension to $\left(H^{m+1}(\hat{k})\right)^*$ of $\hat{e}^{i}$
(Hann Banach)

$$\hat{\Pi}\hat{q} = \hat{q} \quad \forall \hat{q} \in \hat{P}_{\hat{k}}$$ (it is "polynomial preservi

We would like to estimate

$$\| \hat{\Pi}\hat{u} - \hat{u} \|_{m+1}$$

$$\leq \| (\hat{\Pi} - I)\hat{u} \|_{m+1}$$

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