Numerical solution of PDEs.

Prototypical problem:

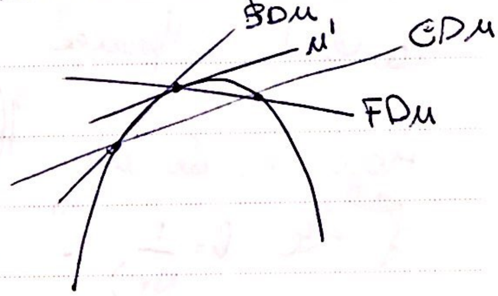
$$\text{ND} \begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Heat equation, Elasticity, Maxwell, etc.

$$\text{1D} \begin{cases} -u'' = f & \text{in } [0,1] \\ u = 0 & \text{on } \partial\Omega = 0,1 \end{cases}$$

Finite Difference: Approximate the Differential Operators.

Finite Element: " " Functional Spaces



1D, FD $\quad FDu := \dfrac{u(x+h) - u(x)}{h} \approx u'(x)$

BD $\quad \dfrac{-u(x-h) + u(x)}{h} \approx u'(x)$

CD $\quad \dfrac{u(x+h) - u(x-h)}{h} \approx u'(x)$

Error? Taylor expansion: (assume u C^∞)

$$u(x+h) = u(x) + u'(x)h + \frac{u''(\xi)}{2}h^2$$

$$\Rightarrow \text{FD}: \left| u' - FDu \right| = \left| \frac{u''(\xi)}{2} \right| h$$

BD = FD

CD: two points...

$$u(x-h) = u(x) - u'(x)h + \frac{u''(x)}{2}h^2 - \frac{u'''(x)}{6}h^3$$

$$u(x+h) - u(x-h)$$

for u'' ?

Use Taylor Directly :

$$u(x+h) = u(x) + u'(x)h + \frac{u''(x)}{2}h^2 + \frac{u'''(x)}{6}h^3 + \dots$$

$$u(x-h) = u(x) - u'(x)h + \frac{u''(x)}{2}h^2 - \frac{u'''(x)}{6}h^3 + \dots$$

$$u(x+h) + u(x-h) = 2u(x) + u''(x)h^2 + O(h^4)$$

$$\Rightarrow u''(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + O(h^2)$$

centered difference scheme approximation —

How do we use this ? split $[0,1]$ in N intervals of size $h = \frac{1}{(N+1)}$ - Sample the above in all internal points (first and last are zero) : we get

$$x^i = ih \qquad i = 1 \dots N-2 \qquad x^0 = 0, \; x^{N-1} = 0$$

$$\frac{1}{h^2}\left(u^{i+1} - 2u^i + u^{i-1}\right) = f^i \qquad i = 1 \dots N-2$$

$$\Rightarrow A_h u = f \qquad A_h := \frac{1}{h^2}\begin{pmatrix} -2 & 1 & \dots & & \\ 1 & -2 & 1 & \dots & \\ 0 & 1 & -2 & 1 & \dots \\ & \dots & & & \dots \\ & & \dots & 1 & -2 \end{pmatrix}$$

Error? → Taylor up to order 4 !!

What happens if u is <u>not</u> C^4? How about H^1? <u>NO</u>

A_h is SPD : ~~each~~ $v^T A v := -2v_1^2 + v_2 v_1$

Galerkin Method (~~or Finite Element~~)

(PV) find $u \in H_0^1([0,1])$ s.t.

$$\int_0^1 u' v' = \int_0^1 f v \qquad \forall v \in H_0^1([0,1])$$

$$H_0^1 := \left\{ u \text{ s.t. } \int_0^1 u^2 < +\infty, \int_0^1 u' < +\infty, \ u\Big|_{0,1} = 0 \right\}$$

Hilbert with scalar products

$(u,v) = \int_0^1 u' v'$ or $(u,v) \int_0^1 uv + \int_0^1 u'v'$

they are equivalent norms in H_0^1, i.e. $\exists\, c_1, c_2$ s.t.

$$|u|_1 \overset{①}{\leq} c_1 \|u\|_1 \overset{②}{\leq} c_2 |u|_1$$

$$\|u\|_1 \leq \frac{c_2}{c_1} |u|_1 \leq \frac{c_1}{c_2} \|u\|_1$$

① is obvious, for ② we use Poincaré inequality

Uniqueness: $\exists\, u_1 \neq u_2$ s.t. (PV)

$\Rightarrow \int_0^1 (u_1 - u_2)' v' = 0 \quad \forall v \Rightarrow$ i.e. also for $v = u_1 - u_2$

$\Rightarrow |u_1 - u_2|^2 = 0 \Rightarrow u_1 = u_2$ Absurd.

Existence? We prove it for an abstract problem:

find $u \in V$ s.t.

(PV) $$a(u,v) = F(v) \qquad \forall v \in V$$

(a) • V Hilbert

(b) • $a(u,v)$ bilinear, continuous, coercive form:
① ② ③

① $a(\alpha u + \beta v, w) = \alpha a(u,w) + \beta a(v,w) \quad \forall \alpha,\beta \in \mathbb{R}$
$a(u, \alpha w + \beta v) = \alpha a(u,w) + \beta a(u,v) \quad \forall u,v,w \in V$

② $a(u,v) \leq C \|u\|_V \|v\|_V \qquad \forall u,v \in V$

③ $a(u,u) \geqslant \alpha \|u\|^2 \qquad \forall u \in V$

(c) • $L(v)$ continuous and linear on V

• $|L(v)| \leq M \|v\|_V$

• $L(\alpha v + \beta w) = \alpha L(v) + \beta L(w)$

If (a)+(b)+(c) $\Rightarrow$ $\exists!$ solution to (PV) <u>and</u>

$$\|u\|_V \leqslant C_F$$

se a è simmetrica $\Rightarrow$ PV $\Leftrightarrow$ find u s.t.

$$J(u) \leq J(v) \quad \forall v \in V$$

$$J(u) := \frac{1}{2} a(u,u) - L(u)$$

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Galerkin : proof of Lax Milgram for separable spaces. (constructive) 3

Take V_N, finite dimensional of dim N, s.t. $V_N \subset V$

(V is Hilbert $\rightarrow$ ~~Separable~~ Normed, Complete, Vector space, with inner product)

PHARMACIA

write PV in V_N

find $u_N \in V_N$ s.t.

$$a(u_N, v_N) = F(v_N) \qquad \forall v_N \in V_N$$

Let $\{\varphi^i\}_{i=1}^N$ be a orthonormal basis for V_N

$$\Rightarrow \sum_j a(u^j \varphi^j, \varphi^i) = F(\varphi^i)$$

$$\Rightarrow A_N u_N = f \qquad A_{ij} := a(\varphi^j, \varphi^i)$$
$$f_i := F(\varphi^i)$$

A is a matrix, positive definite : $u_N^T A_N u_N \geq \alpha \|u_N\|^2 = \alpha \|\sum_i u^i \varphi^i_N\|^2 = \alpha(\sum (u^i)^2)$

$\rightarrow$ A is invertible $\Rightarrow \exists!\ u_N$ s.t. $Au_N = f$.

if V is separable, then $\forall \varepsilon\ \exists N$ s.t. $\|u - \tilde{u}_N\|_V < \varepsilon$

what can we say about u_N and $\tilde{u}_N$?

$$a(u - u_N, v_N) = 0 \qquad (v_N \subset V_N \subset V)$$

$$a(u,v) = F(v) \qquad \forall\, v \in V$$

$$a(u_N,v) = F(v) \qquad \forall\, v \in V_N$$

$$(*) \quad a(u-u_N, v) = 0 \qquad \forall\, v \in V_N$$

coercivity:

$$\alpha \|u-u_N\|_V^2 \leq a(u-u_N, u-u_N) + \overset{0\,(*)}{a(u-u_N, v_N)} - \overset{0\,(*)}{a(u-u_N, v_N)}$$

$$\leq a(u-u_N, u-v_N) \leq M \|u-u_N\| \, \|u-v_N\|$$

if $u = v_N \Rightarrow$ trivial

$\Rightarrow$ assume $u \neq u_N \Rightarrow \|u-u_N\|_V \leq \frac{M}{\alpha} \|u-v_N\|_V \quad \underline{\underline{\forall\, v_N}}$

$$\Rightarrow \|u-u_N\|_V \leq \frac{M}{\alpha} \inf_{v_N \in V_N} \|u-v_N\|$$

in particular, if $\|u-\tilde{u}_N\|_V \leq \varepsilon$ (if $\underline{\underline{V}}$ is separable)

then $\|u-\tilde{u}_N\|_V \leq \frac{M}{\alpha}\varepsilon \Rightarrow \underline{\underline{u_N \to u}}$

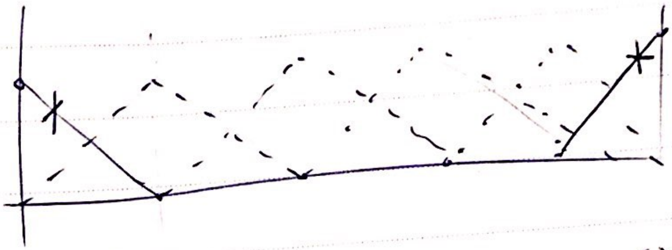
And we can quantify $\|u-u_N\|_V$

How to choose the spaces V_N? <u>Finite Elements</u>

Given N, consider $h = \frac{1}{(N+1)}$, and the space

$$V_N := \{ u \in C^0([0,1]),\ \text{s.t. } u|_{(Jh, Jh+1)} \in \mathcal{P}^k(Jh, Jh+1)\quad \forall J = 0, \dots, N-2 \}$$

Basis $\varphi_i :=$



we want the evaluation of $\varphi_i(x_J) = \delta_{iJ}$ $\quad \varphi_i(0) = \varphi_i(1) = 0$

$\varphi_i \in V_N$ s.t. $\varphi_i(x_J) = \delta_{iJ}$, $\quad \varphi_i(0) = \varphi_i(1) = 0$

$$A_{iJ} := a(\varphi_J, \varphi_i) = \int_0^1 \varphi_J' \varphi_i' \qquad \varphi_J' = \begin{cases} \frac{1}{h} & \text{if } x \in x_{i-1}, x_i \\ -\frac{1}{h} & \text{if } x \in x_i, x_{i+1} \\ 0 & \text{elsewhere} \end{cases}$$

$$\Rightarrow A_{iJ} \quad i = J: \Rightarrow \frac{1}{h^2} \cdot 2h = \frac{2}{h}$$

$$i = J-1 \Rightarrow \frac{-1}{h^2} \cdot h = -\frac{1}{h}$$

$$i = J+1 = -\frac{1}{h^2} \cdot h = -\frac{1}{h}$$

$$\Rightarrow \frac{1}{h} \begin{pmatrix} 2 & -1 & & \dots & \\ -1 & 2 & -1 & \dots & \\ \dots & & \dots & -1 & 2 \end{pmatrix}$$

<u>uguale</u>, a meno di h, a <u>FD</u>!