Societies

Joshua Epstein came up with “Artificial Societies”

Thomas Schelling’s mode - “Segregation”

a person is unsatisfied if immediate neighbor < p% of similar category

Hoshen Kopelman algorithm

$$Q = \frac{1}{2m} \sum_{i,j} \left(A_{ij} - \frac{k_i k_j}{2m} \right) \delta_{g_i g_j}$$

Where A is the adjacency matrix of size $N^2(1-\rho) \times N^2(1-\rho)$

$$\frac{k_i k_j}{2m}$$

expected edges i & j where k_i and k_j are known

this represents a null model

Q = 0 ⇒ the connection is mixed as in random placement

Q > 0 ⇒ segregation is present

Q < 0 ⇒ more mixed than random placements

We can think of this in the sense of Covariance

We know $\mu = \frac{1}{2m} \sum x_i k_i$

$$
\begin{aligned}
&Cov(x_i, x_j) \\
&\quad = \frac{\sum_{ij} A_{ij}(x_i - \mu)(x_j - \mu)}{\sum_{ij} A_{ij}} \\
&\quad = \frac{1}{2m}\left[\sum_{ij} A_{ij}(x_i x_j - \mu x_i - \mu x_j + \mu^2)\right] \\
&\quad = \frac{1}{2m}\left[\sum_{ij} A_{ij}(x_i x_j) - 2m\mu^2 - 2m\mu^2 + 2m\mu^2\right] \\
&\quad = \frac{1}{2m}\sum_{ij} A_{ij}(x_i x_j) - \mu^2 \\
&\quad = \frac{1}{2m}\sum_{ij}\left[A_{ij} - \frac{k_i k_j}{2m}\right] x_i x_j
\end{aligned}
$$

$$
\tilde{C} = \frac{C}{Cs(x_i = x_j)} = -1 \text{ to } 1
$$

Path based centrality

Closeness

$$
\begin{gathered}
\langle l_i \rangle = \frac{1}{n}\sum d_{ij} \\
C_i = \frac{1}{l_i} = \frac{n}{\sum d_{ij}} \\
C_i = \frac{1}{n-1}\sum_{j \neq 1} \frac{1}{d_{ij}}
\end{gathered}
$$

d_{ij} = shortest distance i → j

Tells us about how compact the network is

Betweenness

Captures how many shortest path goes through a particular node (gives an idea of how important a node is)

$$
x_i = \sum_{s,t} \frac{n^i_{s,t}}{g_{s,t}}
$$

$n_{s,t}$ = number of nodes in shortest path from s to g

$g_{s,t}$ = number of paths with $n_{s,t}$ number of nodes

Centrality

1. Degree Centrality

$$x_i = \sum_j A_{ij} = A\vec{1}$$

2. Eigen Vector Centrality

$$x_i = K^{-1} \sum_j A_{ij} x_j$$
$$X = K^{-1} A X$$
$$AX = KX$$
$$(A - KI)X = 0$$
$$\det(A - KI) = 0$$

   We look for the eigen vector for which all the values are greater than 0

   coincidentally the eigen vector corresponding to the highest eigen value is a vector which is all positive since $A \geq 0$ perron frobenius theorem

3. Katz Centrality

$$x_i = K^{-1} \sum_j A_{ij} x_j + \beta$$
$$X = K^{-1} A X + \beta$$
$$X = (I - K^{-1} A)^{-1} \beta$$

   β can be set to 1 as it is multiplying the entire thing (it is essentially scaling the result)

   If $K \to \infty$, there is no equation. If $K = 0$, only the influence of AX is present in the equation.

   There exists an inverse such that $|I - K^{-1} A| = 0$ is a bound for K

   In the same way when 0 in a node is cascaded into nodes connected to it. When a node with high centrality is connected to a normal node, its centrality also increases. essentially distributing out the information. This is highly undesirable

4. PG rank centrality

$$x_i = K^{-1} \sum_j A_{ij} \frac{x_j}{k_j} + \beta$$
$$X = K^{-1} A D^{-1} X + \beta$$
$$X = (I - K^{-1} A D^{-1})^{-1} \beta$$

   There may be an issue where $k_j = 0$. We can resolve that by either setting a small value to it or 1.

   For undirected graph, $\beta = 0$

$$x_i = K^{-1}\sum_j A_{ij}\frac{x_j}{k_j}$$
$$(I - K^{-1}AD^{-1})X = 0$$
$$(D - K^{-1}A)D^{-1}X = 0$$
$$(D - K^{-1}A)Y = 0$$

Non trivial solution for Y when $rank(D - K^{-1}A) < n$

only possible when $K = 1$

$D - A = L$

$$LY = 0$$
$$Y = 1$$
$$X = DY = \vec{k}$$

5. hubs authority

   x ⇒ authority

   y ⇒ hubs

$$x_i = \alpha\sum_j A_{ij}y_j$$
$$y_i = \beta\sum_j A_{ji}x_j$$

$$X = \alpha AY$$
$$Y = \beta A^T X$$

$$X = \alpha\beta AA^T X$$
$$Y = \alpha\beta A^T AY$$

$$AA^T X = \lambda X$$
$$A^T AY = \lambda Y$$

   where $\lambda = (\alpha\beta)^{-1}$

Local Connectivity

emergence can have effect on topology and topology can have effect on emergence

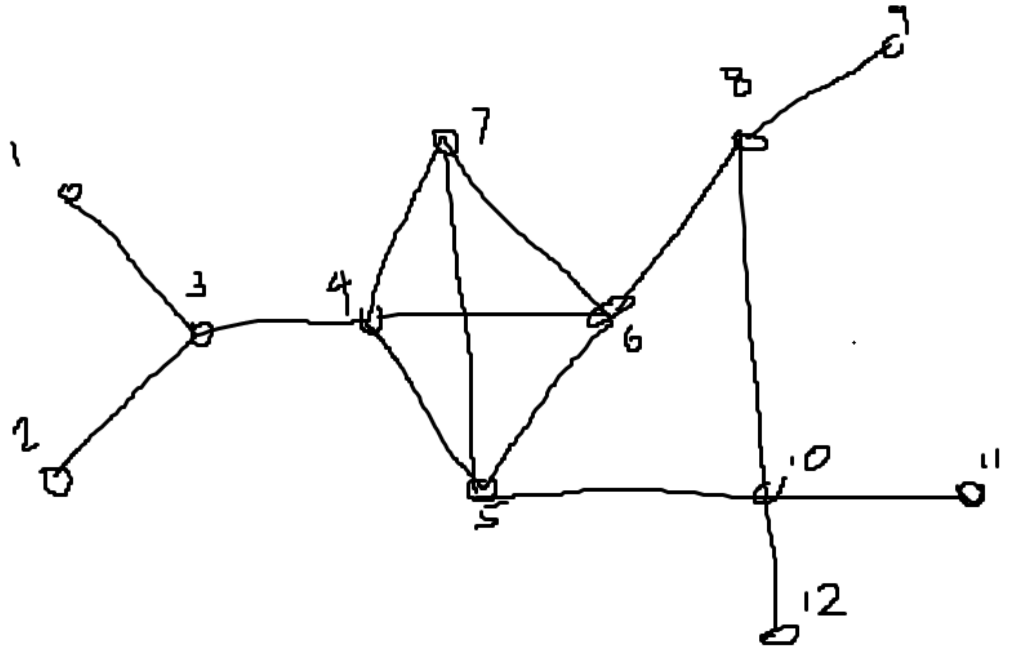
Cliques

subset of nodes where every node is connected to every other node

$\frac{n(n-1)}{2}$connections

k-Core

subset of nodes where each node in the subset has at least k connections within that subset



1-core is the entire thing

2-core is 4-5-6-7-8-10

3-core is 4-5-6-7 (which also appears to be a clique)

larger-k cores are subsets of smaller-k cores (that’s where the name comes from)

k-components

at least k paths between two nodes in the subset of nodes

2-comp

    at least 2 paths between each pair of nodes

▼ edge-independent and node-independent

For edge-independent paths:

- k-edge-component: a maximal subset of nodes where there are k edge-independent paths between any pair of nodes in the subset
- Paths that don't share any edges but can share nodes

For node-independent paths:

- k-node-component: a maximal subset of nodes where there are k node-independent paths between any pair of nodes in the subset
- Paths that don't share any nodes except the start and end nodes

Node-independent paths are more restrictive than edge-independent paths, as node-independent paths are also edge-independent, but not vice versa.

Reciprocity

in directed network. when $A_{ij} = 1 = A_{ji}$

$$r = \frac{1}{m} \sum_{ij} A_{ij} A_{ji}$$
$$r = \frac{1}{m} tr(A^2)$$

in undirected: clusters

Local Clustering

$$C_i = \frac{\text{number of pairs of neighbors of i that are connected}}{\text{total number of pairs of i}}$$

Similarity

between nodes i and j, σ_{ij}

Structural

have similar neighbors

$$n_{ij} = \sum_k A_{ik} A_{jk} = [A^2]_{ij}$$

To normalize this. we take Salton's cosine rule

$$\frac{\vec{x}.\vec{y}}{|\vec{x}||\vec{y}|} = \cos\theta_{xy}$$

$$\sigma_{ij} = \frac{\sum_k A_{ik} A_{jk}}{\sqrt{\sum_l A_{il}^2}\sqrt{\sum_k A_{jk}^2}} = \frac{n_{ij}}{\sqrt{k_i}\sqrt{k_j}}$$

Regular

▼ Initial thought process

two nodes are similar if their neighbors are similar

$$\sigma_{ij} = \alpha \sum_{k,l} A_{ik} A_{jl} \sigma_{kl} + \delta_{ij}$$

for i=j the value of σ_{ij} is significantly lower. Hence we add the δ_{ij}

$$\sigma = \alpha A \sigma A + I$$

To let the value converge, we iterate for different values of sigma

initially, $\sigma^{(0)} = 0$

$\sigma^{(1)} = I$

$\sigma^{(2)} = \alpha A^2 + I$

$\sigma^{(3)} = \alpha^2 A^4 + \alpha A^2 + I$

we are capturing even path length (even powers of A) which is not what we want

Redefining ⇒ i, j are similar if k=neigh(i) is similar to j

$$\sigma_{ij} = \alpha \sum_k A_{ik}\sigma_{kj} + \delta_{ij}$$

$$\sigma = \alpha A\sigma + I$$

To let the value converge, we iterate for different values of sigma

initially, $\sigma^{(0)} = 0$

$\sigma^{(1)} = I$

$\sigma^{(2)} = \alpha A + I$

$\sigma^{(3)} = \alpha^2 A^2 + \alpha A + I$

$\sigma^{(\infty)} = (I - \alpha A)^{-1} I$

$$\sigma = (I - \alpha A)^{-1} I$$

which is similar to Katz centrality

Node Moving Algorithm

for assigning nodes in a network into groups to maximize Q

we won't be getting the best solution globally. but we'll get the best solution locally

1. randomly assign communities
2. flip the group of each node at a time and recalculate Q
3. select the option which does max ΔQ (increases Q) or does min ΔQ (has no effect when in either group)
4. repeat from 2

$$S_i = \begin{cases} +1 & i \in g_1 \\ -i & i \in g_2 \end{cases}$$

$$
\begin{aligned}
Q &= \frac{1}{4m}\sum_{i,j}\left(A_{ij} - \frac{k_i k_j}{2m}\right)(S_i S_j + 1) \\
&= \frac{1}{4m}\sum_{i,j} B_{ij} S_i S_j + \sum_{i,j} B_{ij} \\
&= \frac{1}{4m}\sum_{i,j} B_{ij} S_i S_j \\
&= \frac{1}{4m} S^T B S
\end{aligned}
$$

if we relax $BS = \lambda S$

$$
Q = \frac{1}{4m} S^t(\lambda S) = \frac{\lambda \alpha}{4m}
$$

$$
\begin{gathered}
Q = S^T B S \\
\frac{\partial Q}{\partial S} = 0 \\
\frac{\partial^2 Q}{\partial S^2} < 0
\end{gathered}
$$

$\sum_i S_i^2 = N$

When you have a minimize a function $f(x)$ st it satisfies another function $h(x)$. The solution lies when the normal of both the function intersect

$$
\begin{gathered}
\nabla_x f(x) = \lambda \nabla_x h(x) \\
\nabla_x (f - \lambda h) = 0 \\
\nabla_x L = 0
\end{gathered}
$$

$$
\begin{gathered}
\frac{\partial}{\partial S}(S^T B S - \lambda(S^t S - n)) = 0 \\
2BS - 2\lambda S = 0 \\
BS = \lambda S
\end{gathered}
$$

$$
\frac{\partial^2}{\partial S^2}(L) < 0
$$

Louvain Algorithm

Flocking systems

There is a difference in the way the fish reacts

The old equation used to be

$$\theta_{i,t+1} =< \theta_{j,t} >_{j \in R_i} + \eta$$

The fish selects one of its neighbors and bases it's direction on that

$$\theta_{i,t+1} = \theta_{j \in R_i} + \eta$$

Ideally, interactions are:

- synchronous ⇒ they all happen at the same time
- additive ⇒ interactions from all agents add up

In real life, interactions are

- asynchronous ⇒ they don't all necessarily change at the same time (there is no single dt)
- They may spontaneously exchange values

  $Y_{1/2} \xrightarrow{S} Y_{2/1}$
- They may copy/align with neighbors instead of averaging

  $Y_1 + Y_2 \xrightarrow{C} 2Y_1$ or $Y_1 + Y_2 \xrightarrow{C} 2Y_2$

$X_1 \to X_1 \pm 1/N$

$X_2 \to X_2 \mp 1/N$

$$\begin{aligned} T_{12} = T(X_1 + \frac{1}{N}, X_2 - \frac{1}{N} \Big| X_1, X_2) \\ = CX_1X_2 + SX_2 \\ = CX_1X_2 + SX_1 \end{aligned}$$

$$\frac{\partial P(X,t)}{\partial t} = \sum_{X' \neq X} \Big(T\Big(X|X'\Big)P\Big(X',t\Big) - T\Big(X'|X\Big)P\Big(X,t\Big)\Big)$$

$$\frac{\partial P(X_1,X_2,t)}{\partial t} = \sum_{X_1' \neq X_1} \sum_{X_2' \neq X_2} \Big(T\Big(X_1X_2|X_1'X_2'\Big)P\Big(X_1',X_2',t\Big) - T\Big(X_1'X_2'|X_1X_2\Big)P\Big(X_1,X_2,t\Big)\Big)$$

$X_1 + X_2 = 1$

$(T_{12} + T_{21})P(X_1, X_2, t)$

$$T\Big(X_1, X_2|X_1 \mp \frac{1}{N}, X_2 \pm \frac{1}{N}\Big)P\Big(X_1 \mp \frac{1}{N}, X_2 \pm \frac{1}{N}, t\Big)$$

$$\epsilon_i^{\pm} f(X_i) = f(X_i \pm \frac{1}{N})$$

$$\epsilon_1^{-}\epsilon_2^{+}T_{12}(X_1, X_2|X_1 - \frac{1}{N}, X_2 + \frac{1}{N})P_{X_1X_2} + \epsilon_2^{-}\epsilon_1^{+}T_{21}P$$

$$\frac{\partial P}{\partial t}(X_1, X_2, t) = (\epsilon_1^{-}\epsilon_2^{+} - 1)T_{12} + (\epsilon_1^{+}\epsilon_2^{-} - 1)T_{21}$$

When N is large enough

$$\epsilon_i^{\pm} f(X_i) = f(X_i| \pm \frac{1}{N}) = f(X_i) + \frac{\partial f}{\partial X_i}(X_i)\frac{1}{N} + \frac{1}{2!}\frac{\partial^2 f}{\partial X_i^2}\frac{1}{N^2} + \dots$$

$$\frac{\partial P}{\partial t} = -\frac{1}{N}\Big(\frac{\partial}{\partial X_1} - \frac{\partial}{\partial X_2}\Big)\Big(SX_2 - SX_1\Big)P + \dots$$

$m = X_1 - X_2$

$$\frac{dm}{dt} = -2Sm + \sqrt{\frac{C(1-m^2) + 2S}{N}}$$

Two days worth of content