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Stochastic Integrals in the Plane

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The work described here was done in collaboration with R. Cairoli; it will be published in full elsewhere. In that article, however, we treat integration with respect to arbitrary square-integrable martingales, which requires the construction of some expensive machinery. In the present paper we will try to sketch some parts of the subject in their simplest terms.

I. White noise and line integrals. Let us begin with white noise in $\boldsymbol{R}^2_+$ (the positive quadrant of the plane). This is a finitely additive set function W defined on the Borel subsets of $\boldsymbol{R}^2_+$ such that

(i) $W(A)$ is a $N(0, |A|)$ random variable;

(ii) if $A \cap B = \varnothing$, then $W(A)$ and $W(B)$ are independent.

It is natural to consider stochastic integration with respect to W—this has in fact been done by numerous authors—but let us proceed slowly and first define a stochastic process W_{st} whose parameter set is $\boldsymbol{R}^2_+$. Let R_{st} denote the rectangle $[0, s] \times [0, t]$, and define $W_{st} = W(R_{st})$. (We use W to denote both the measure and the process; this will not cause confusion.)

W_{st}, $s, t \geqq 0$, is called the *two-parameter Wiener process*, or the *Brownian sheet* (to visualize its sample paths, picture a wrinkled bed sheet). It is a continuous mean zero Gaussian process; its covariance function is easily calculated from (i) and (ii). If t is fixed, the process $s \to W_{st}$ is a Brownian motion, as can be seen from its covariance function. Since the theory of stochastic integration with respect to Brownian motion is well known, we can have stochastic line integrals with respect to W along the lines t = constant. By symmetry, we can also integrate along the lines s = constant. Putting these two together, we can integrate along polygonal curves in $\boldsymbol{R}^2_+$ consisting of a finite number of horizontal and vertical segments. We call such curves *staircases*.

The above remarks tacitly assume we are integrating in the direction of increasing s and t, but we can define the integral in the direction of decreasing s and t simply by changing sign. We denote the line integral of ϕ over an oriented staircase Γ by $\int_\Gamma \phi\, \partial W$.

It is convenient to introduce an analogue of the integral of a differential form. This is totally trivial in the present setting but serves to simplify our notation. If Γ is a staircase, $\int_\Gamma \phi \partial_1 W$ is the line integral of ϕ over the *horizontal* segments of Γ, and $\int_\Gamma \phi\, \partial_2 W$ is the integral over the *vertical* segments. Then we have

$$\int_\Gamma \phi\, \partial W = \int_\Gamma \phi\, \partial_1 W + \int_\Gamma \phi\, \partial_2 W.$$

All three of the above line integrals can be extended to sufficiently regular curves by a limiting argment, but staircases will suffice here.

If $f(x, s, t)$ is twice continuously differentiable in its arguments, we can write Ito's formula in differential notation:

$$\begin{aligned} \partial_1 f(W_{st}, s, t) &= \frac{\partial f}{\partial x}(W_{st}, s, t)\, \partial_1 W \\ &\quad + \left(\frac{t}{2}\frac{\partial^2 f}{\partial x^2}(W_{st}, s, t) + \frac{\partial f}{\partial s}(W_{st}, s, t)\right) ds \end{aligned} \tag{1}$$

with the symmetric equation holding for $\partial_2 f$.

II. Martingales. Stochastic integration is inextricably mixed with martingale theory, and, before we go further, we should look at martingales in our context. We are working in the plane, so the processes we consider will have $\boldsymbol{R}_+^2$ as a parameter set. We give $\boldsymbol{R}_+^2$ the usual partial order: $(s, t) \prec (u, v) \Leftrightarrow s \leqq u$ and $t \leqq v$.

Define σ-fields $\mathscr{F}_z$, $z \in \boldsymbol{R}_+^2$, by $\mathscr{F}_z = \sigma\{W_\zeta, \zeta \prec z\}$. A stochastic process $\{M_z, z \in \boldsymbol{R}_+^2\}$ is a *martingale* relative to the fields $(\mathscr{F}_z)$ if

(i) $E\{|M_z|\} < \infty$, all z;

(ii) M_z is $\mathscr{F}_z$-measurable;

(iii) $z \prec z' \Rightarrow E\{M_{z'} | \mathscr{F}_z\} = M_z$.

Thus we are talking about martingales with a partially ordered two-dimensional parameter set. Cairoli [1] has proved versions of both the maximal inequality and the martingale convergence theorem in this setting, but on balance it seems that relatively little is known about such martingales, and almost nothing about the corresponding sub- and supermartingales. This is one of the principal difficulties of stochastic integration in higher dimensions. Indeed, extensions of the classical theory can be quite delicate, as is indicated by the following two facts:

1°. If $\{M_z, z \in \boldsymbol{R}_+^2\}$ is a separable martingale relative to $(\mathscr{F}_z)$ which is bounded in $L \log L$, its sample functions are a.s. continuous.

2°. There exists a separable, uniformly integrable—and hence L^1-bounded—martingale which is everywhere discontinuous with probability one.

REMARK. The continuity of $L \log L$-bounded martingales is a property of the particular fields $(\mathscr{F}_z)$. For more general fields it is not even known whether or not all bounded martingales have a right continuous version.

III. Surface integrals. Stochastic integration with respect to W can be defined, following Ito, exactly as in the classical case. We will outline this briefly.

If $A \subset \mathbf{R}_+^2$ is a closed rectangle with lower left-hand corner z_0, define ϕ by

$$\phi_z = \phi_0 I_A(z), \qquad z \in \mathbf{R}_+^2, \tag{2}$$

where ϕ_0 is $\mathscr{F}_{z_0}$-measurable and square-integrable. Then let

$$(\phi \cdot W)_z = \int_{R_z} \phi \, dW =_{\text{def}} \phi_0 W(R_z \cap A).$$

This defines a stochastic process $\phi \cdot W$ which is

(i) a.s. continuous in z,

(ii) a martingale, and

(iii) $E\{(\phi \cdot W)_z^2\} = \int_{R_z} E\{\phi_\zeta^2\} \, d\zeta$.

If ϕ is simple, i.e., a finite sum of processes of the form (2), its integral is defined by linearity. In general, if ϕ satisfies

(a) ϕ_z is $\mathscr{F}_z$-measurable,

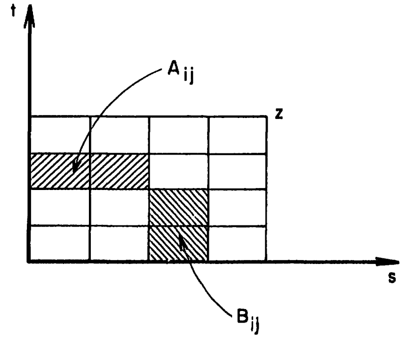
(b) $(z, \omega) \to \phi_z(\omega)$ is $\mathscr{B} \times \mathscr{F}$-measurable, and

(c) $\int_{R_z} E\{\phi_\zeta^2\} \, d\zeta < \infty$ for all $z \in \mathbf{R}_+^2$,

then we can find a sequence of simple ϕ_n for which $\int_{R_z} E\{(\phi_n - \phi)^2\} \, d\zeta \to 0$ for each z. The integrals $(\phi_n \cdot W)_z$ then converge in L^2, a subsequence converges a.s., and even uniformly for z in compacts thanks to Cairoli's maximal inequality. We define $\phi \cdot W = \lim \phi_n \cdot W$. Then $\phi \cdot W$ also satisfies (i), (ii), and (iii).

REMARK. In contrast to the classical case, there seem to be genuine difficulties in extending the integral very far beyond the square-integrable case.

IV. The measure J. It turns out that the integral we have just defined is not sufficient in this theory, and a second type of integral is needed. Wong and Zakai [3] have introduced a type of multiple integral which is exactly what is needed for a complete discussion, but we will be able to get by here with something simpler. The basic difficulty is that W, whatever else it may be, is not a product measure. We want to introduce a measure J which will in some sense satisfy $dJ \approx \partial_1 W \partial_2 W$, where $\partial_1 W$ and $\partial_2 W$ are respectively the "horizontal" and "vertical" increments of W_{st}. To be specific, let us fix a point $z \in \mathbf{R}_+^2$ and divide R_z into 2^{2n} congruent rectangles as in the diagram below.



Let A_{ij} and B_{ij} be the cross-hatched rectangles above. Note that if z_{ij} is the lower left-hand corner of the ijth rectangle, then $W(A_{ij}) = W_{z_{i,j+1}} - W_{z_{i,j}}$. We define J_z^n by

$$J_z^n = \sum_{i,j=1}^{2^n} W(A_{ij})W(B_{ij}),$$

and define J_z by $J_z = \lim_{n\to\infty} J_z^n$.

One can show this limit exists and defines a continuous martingale which is orthogonal to W, that is $E\{J_z W_{z'}\} = 0$ for all z, z'. We get the desired measure, again denoted by J, by setting $J(R_z) = J_z$, and extending it to all rectangles in the obvious way. Integrals with respect to J are defined exactly as in §III.

V. Green's formula. We need to introduce the notion of a stochastic partial derivative. We define this globally: A process $\{\Phi_{st},\ s, t \geqq 0\}$ has a *stochastic partial* (ϕ, ψ) *with respect to* (W, t) if there exist adapted measurable processes ϕ and ψ such that, for each s, $t \geqq 0$,

$$\Phi_{st} = \Phi_{0s} + \int_0^t \phi_{sv}\, \partial_2 W_{sv} + \int_0^t \psi_{sv}\, dv. \tag{3}$$

We will often simply speak of a stochastic partial with respect to t. We remark that if Φ is a martingale, then ψ vanishes a.e.

THEOREM 1 (GREEN'S FORMULA FOR RECTANGLES). *Let Φ be an adapted, measurable, L^2-bounded process which has stochastic partials (ϕ, ψ) with respect to (W, t). Suppose further that $\psi \equiv 0$. Let A be a rectangle whose boundary ∂A is oriented clockwise. Then*

$$\int_{\partial A} \Phi\, \partial_1 W = \int_A \Phi\, dW + \int_A \phi\, dJ. \tag{4}$$

Similarly, if Φ has stochastic partials (ϕ, ψ) with respect to (W, s) and if ψ vanishes, then

$$\int_{\partial A} \Phi\, \partial_2 W = -\int_A \Phi\, dW - \int_A \phi\, dJ. \tag{5}$$

REMARKS. 1°. The hypothesis $\psi \equiv 0$ serves purely to simplify (4) and (5). If ψ did not vanish, we would have to add the iterated integral $\int(\int \psi_{uv}\, \partial_1 W_{uv})\, dv$ to the right-hand side of (4). However, in our applications, Φ will be a martingale, so ψ will vanish.

2°. (4) and (5) are true for regions A with sufficiently regular boundaries.

For a quick application of the theorem, take $\Phi_{st} = W_{st}$, $\phi \equiv 1$, and apply (4) to $A = R_{st}$. Using Ito's formula (1) on the line integral, we get an expression for J:

$$J_{st} = \frac{1}{2}(W_{st}^2 - st) - \int_{R_{st}} W\, dW.$$

VI. Holomorphic processes. A process $\Phi = \{\Phi_z,\ z \in \boldsymbol{R}_+^2\}$ is said to be *holomorphic in* $\boldsymbol{R}_+^2$, or, more simply, holomorphic, if there exists an adapted measurable process $\phi = \{\phi_z,\ z \in \boldsymbol{R}_+^2\}$ such that $E\{\phi_z^2\}$ is bounded for z in compacts and such that, for all $z \in \boldsymbol{R}_+^2$ and any staircase $\Gamma \subset \boldsymbol{R}_+^2$ with initial point 0 and final point z,

$$\Phi_z = \Phi_0 + \int_\Gamma \phi\, \partial W \tag{6}$$

where ϕ_0 is constant. We call ϕ the *derivative* of Φ.

It is easily seen that the line integral of ϕ around any closed staircase vanishes, and we could treat this subject from the point of view of path independent integrals. But the structure of holomorphic processes bears a striking resemblance in some respects to that of classical holomorphic functions of a complex variable, and it seems worthwhile to bring this out here. We should emphasize, though, that our processes are real-, not complex-valued.

A holomorphic process is necessarily a martingale, being defined as a stochastic integral. The class of holomorphic processes is nontrivial, since W is holomorphic (its derivative is identically one). One might ask if W^2, W^3, etc., were holomorphic. They are not, but we should remember that the stochastic analogue of z^n is not W^n, but $H_n(W_{st}, st)$, where $H_n(x, t)$ is the nth Hermite polynomial.

Here, H_n is defined by

$$H_n(x, t) = \frac{(-t)^n}{n!} e^{x^2/2t} \frac{\partial^n}{\partial x^n} e^{-x^2/2t}. \tag{7}$$

Then $H_0 \equiv 1$, $H_1(x, t) = x$, $H_2(x, t) = \frac{1}{2}(x^2 - t)$ and $H_3(x, t) = (x^3 - 3xt)/3!$. For each fixed t, the set $\{H_n(x, t)\}_{n=0}^{\infty}$ is a complete set of orthogonal polynomials relative to the weight function $e^{-x^2/2t}\,dx$. In particular, since W_{st} is $N(0, st)$, we have

$$\begin{aligned} E\{H_n(W_{st}, st)H_m(W_{st}, st)\} &= 0 && \text{if } m \neq n, \\ &= (st)^n/n! && \text{if } m = n. \end{aligned} \tag{8}$$

Let us apply Ito's formula (1) to the process $H_n(W_{st}, st)$ along the line $t =$ constant. If we use the facts (derivable from (7)) that

$$\frac{\partial}{\partial x} H_n = H_{n-1} \quad \text{and} \quad \frac{1}{2}\frac{\partial^2}{\partial x^2} H_n + \frac{\partial}{\partial t} H_n = 0, \tag{9}$$

we see that

$$H_n(W_{st}, st) = \int_0^s H_{n-1}(W_{ut}, ut)\,\partial_1 W_{ut};$$

by symmetry, this is also

$$= \int_0^t H_{n-1}(W_{sv}, sv)\,\partial_2 W_{sv}.$$

We conclude from this that $\{H_n(W_{st}, st),\ s, t \geqq 0\}$ is a holomorphic process, with derivative $H_{n-1}(W_{st}, st)$. As sums of these processes are also holomorphic, the class of holomorphic processes is evidently relatively large. Now let us turn to the general case.

THEOREM 2. *If Φ is holomorphic and Γ is a closed staircase, then* $\int_\Gamma \Phi\,\partial W = 0$.

This is almost immediate. It reduces to the case where Γ is the boundary A of a rectangle, where it follows directly from (4) and (5) and the observations that Φ, being holomorphic, has a stochastic partial ϕ with respect to both s and t, and that

$$\int_{\partial A} \Phi\,\partial W = \int_{\partial A} \Phi\,\partial_1 W + \int_{\partial A} \Phi\,\partial_2 W.$$

It follows that we can define a holomorphic process Ψ with derivative Φ by $\Psi_z = \int_0^z \Phi \, \partial W$. Thus the integral of a holomorphic process is holomorphic. What about the derivative? This is a more delicate question, but it still has a positive answer.

THEOREM 3. *Let Φ be holomorphic. Then Φ admits a derivative ϕ which is also holomorphic.*

It follows from this that Φ has holomorphic derivatives of all orders. We denote the nth derivative of Φ by $\Phi^{(n)}$. We then have the analogue of Taylor's theorem, which gives the basic structure of holomorphic processes.

THEOREM 4. *If Φ is holomorphic, then*

$$\Phi_{st} = \sum_{n=0}^{\infty} \Phi_0^{(n)} H_n(W_{st}, st),$$

where the above series converges in L^2 for all $s, t \geqq 0$.

REMARKS. 1°. $\mathscr{F}_0$ is trivial, so the coefficients $\Phi_0^{(n)}$ are constant.

2°. Theorem 4 implies Theorem 3—indeed, the derivative of Φ is evidently

$$\Phi'_{st} = \sum_{n=1}^{\infty} \Phi_0^{(n)} H_{n-1}(W_{st}, st).$$

However, by far the most difficult part of the proof of Theorem 4 lies in establishing Theorem 3.

The most striking aspect of Theorem 4 is that we knew a priori only that Φ_{st} was $\mathscr{F}_{st}$-measurable. This is much weaker than what turns out to be true, namely that Φ_{st} is actually a function of W_{st}.

Let us close this article with one further result which indicates that the existence of stochastic partials is more demanding than one might think. In proving Theorem 3, it turns out that one constructs a holomorphic version of the derivative using only the fact that Φ is a martingale and has stochastic partials with respect to both s and t, but *without using the fact that they are equal.* Thus, applying Theorem 1 to this derivative, we get:

THEOREM 5. *Suppose $\{M_z, z \in \mathbf{R}_+^2\}$ is a square-integrable martingale which has stochastic partials with respect to both s and t. Then M is a holomorphic process.*

References

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Section 12

Complex Analysis