

## syn.1 Subformulas

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sec

It is often useful to talk about the **formulas** that “make up” a given **formula**. explanation  
We call these its **subformulas**. Any **formula** counts as a **subformula** of itself; a subformula of  $\varphi$  other than  $\varphi$  itself is a *proper subformula*.

**Definition syn.1 (Immediate Subformula).** If  $\varphi$  is a **formula**, the *immediate subformulas* of  $\varphi$  are defined inductively as follows:

1. Atomic **formulas** have no immediate **subformulas**.
2.  $\varphi \equiv \neg\psi$ : The only immediate **subformula** of  $\varphi$  is  $\psi$ .
3.  $\varphi \equiv (\psi * \chi)$ : The immediate **subformulas** of  $\varphi$  are  $\psi$  and  $\chi$  ( $*$  is any one of the two-place connectives).
4.  $\varphi \equiv \forall x \psi$ : The only immediate **subformula** of  $\varphi$  is  $\psi$ .
5.  $\varphi \equiv \exists x \psi$ : The only immediate **subformula** of  $\varphi$  is  $\psi$ .

**Definition syn.2 (Proper Subformula).** If  $\varphi$  is a **formula**, the *proper subformulas* of  $\varphi$  are recursively as follows:

1. Atomic **formulas** have no proper **subformulas**.
2.  $\varphi \equiv \neg\psi$ : The proper **subformulas** of  $\varphi$  are  $\psi$  together with all proper **subformulas** of  $\psi$ .
3.  $\varphi \equiv (\psi * \chi)$ : The proper **subformulas** of  $\varphi$  are  $\psi$ ,  $\chi$ , together with all proper **subformulas** of  $\psi$  and those of  $\chi$ .
4.  $\varphi \equiv \forall x \psi$ : The proper **subformulas** of  $\varphi$  are  $\psi$  together with all proper **subformulas** of  $\psi$ .
5.  $\varphi \equiv \exists x \psi$ : The proper **subformulas** of  $\varphi$  are  $\psi$  together with all proper **subformulas** of  $\psi$ .

**Definition syn.3 (Subformula).** The **subformulas** of  $\varphi$  are  $\varphi$  itself together with all its proper **subformulas**.

Note the subtle difference in how we have defined immediate **subformulas** explanation and proper **subformulas**. In the first case, we have directly defined the immediate **subformulas** of a formula  $\varphi$  for each possible form of  $\varphi$ . It is an explicit definition by cases, and the cases mirror the inductive definition of the set of **formulas**. In the second case, we have also mirrored the way the set of all **formulas** is defined, but in each case we have also included the proper **subformulas** of the smaller **formulas**  $\psi$ ,  $\chi$  in addition to these **formulas** themselves. This makes the definition *recursive*. In general, a definition of a function on an inductively defined set (in our case, **formulas**) is recursive if the cases in the definition of the function make use of the function itself. To be well defined,

we must make sure, however, that we only ever use the values of the function for arguments that come “before” the one we are defining—in our case, when defining “proper **subformula**” for  $(\psi * \chi)$  we only use the proper **subformulas** of the “earlier” **formulas**  $\psi$  and  $\chi$ .

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## Bibliography