~~Let $\bar{X}_I$ be the~~ mea

Let $I = \{a_1 \dots a_m\}$. Let $\mu = \frac{1}{n}\sum_{i=1}^{N} a_i$ $\qquad Y_i(a_i) = \frac{1}{n}$ (uniform distribution)

Let $X_i(a_i)$ be a random variable assigned to $a_i$, with $E[X_i(a_i)] = \mu$ because the points are uniformly distributed.

Then,

$$X_I = \{X_1, X_2, \dots, X_m\}, \quad \bar{X}_I = \frac{1}{m}\sum_{i=1}^{m} X_i$$

And we wish to prove:

$$E_I[\bar{X}_I] = \mu.$$

Then,

$$E_I[\bar{X}_I] = \frac{1}{m}\sum_{i=1}^{m} X_i$$

$$E_I[\bar{X}_I] = \frac{1}{m}\left(X_1 + X_2 + \dots + X_m\right)$$

$$E_I[\bar{X}_I] = \frac{1}{m}X_1 + \frac{1}{m}X_2 + \dots + \frac{1}{m}X_m$$

By linearity of expectation, regardless of $X_i$ and $X_j$ are independent or dependent, we have:

$$\begin{aligned}
E_I[\bar{X}_I] &= E\left[\frac{1}{m}X_1\right] + \dots + E\left[\frac{1}{m}X_m\right] \\
&= \frac{1}{m}E[X_1] + \dots + \frac{1}{m}E[X_m] \\
&= \frac{1}{m}\mu_{(1)} + \dots + \frac{1}{m}E\,\mu_{(m)} \\
&= \frac{m}{m}\mu \\
&= \mu \\
&= \frac{1}{n}\sum_{i=1}^{N} a_i
\end{aligned}$$

—

$\mathbb{E}_I[\nabla L_I(x,y,\theta)] = \nabla \mathbb{E}_I[L_I(x,y,\theta)]$

$\nabla$

$L_I(x,y,\theta) = \frac{1}{m}\sum_{i \in I} \ell(x^{(i)}, y^{(i)}, \theta)$

$L(x,y,\theta) = \frac{1}{n}\sum_{i=1}^{n} \ell(x^{(i)}, y^{(i)}, \theta)$

Thus

$$\begin{aligned}
\mathbb{E}_I[\nabla L_I(x,y,\theta)] &= \nabla \mathbb{E}_I[L_I(x,y,\theta)] \quad (\star) \\
&= \nabla \mathbb{E}_I\left[\frac{1}{m}\sum_{i\in I} \ell(x^{(i)}, y^{(i)}, \theta)\right] \\
&= \nabla\left(\mathbb{E}\left[\frac{1}{m}\ell(x^{(1)}, y^{(1)}, \theta)\right] + \cdots + \mathbb{E}\left[\frac{1}{m}\ell(x^{(m)}, y^{(m)}, \theta)\right]\right) \\
&= \nabla\left(\frac{1}{m}\mathbb{E}[\ell(x^{(1)}, y^{(1)}, \theta)] + \cdots + \frac{1}{m}\mathbb{E}[\ell(x^{(m)}, y^{(m)}, \theta)]\right) \\
&= \nabla\left(\frac{1}{m}\left[\frac{1}{n}\sum_{i=1}^{n} \ell(x^{(i)}, y^{(i)}, \theta)\right] + \cdots + \frac{1}{m}\left[\frac{1}{n}\sum_{i=1}^{n} \ell(x^{(i)}, y^{(i)}, \theta)\right]\right) \\
&= \nabla\left(\frac{\cancel{m}}{\cancel{m}}\left[\frac{1}{n}\sum_{i=1}^{n} \ell(x^{(i)}, y^{(i)}, \theta)\right]\right) \\
&= \nabla\left(\frac{1}{n}\sum_{i=1}^{n} \ell(x^{(i)}, y^{(i)}, \theta)\right) \\
&= \nabla\left(L(x,y,\theta)\right).
\end{aligned}$$

$(\star)$: Based on the Leibniz integral rule:

$$\frac{d}{dx}\left(\int_a^b f(x,t)\,dx\right) = \int_a^b \frac{\partial}{\partial x} f(x,t)\,dt$$

Where we treat expectation as the discrete form of the above integral (~~and thus can be ex~~) also

$$\frac{d}{dx}\left[E(f)\right] = \frac{\partial}{\partial x}\left[x_1 f(x_1) + \cdots + x f(x_n)\right] = \left[\frac{\partial}{\partial x} x_1 f(x_1) + \cdots + \frac{\partial}{\partial x} x_n f(x_n)\right].$$