$$X = \underbrace{\begin{bmatrix} & \vdots & \\ & \cdots & \\ & & \end{bmatrix}}_{d} \Big\} m, \quad \tilde{X} = [X \;\; y] = \underbrace{\begin{bmatrix} \cdots & y_1 \\ \cdots & y_2 \\ & \vdots \end{bmatrix}}_{d+1} \Big\} m$$

$m = d$ | $m > d$ | $m < d$

## Even-determined ($X$ is 'square')

case (i) : Unique solution (Usual Case)

$\text{rank}(X) = \text{rank}(\tilde{X}) = d$

$X = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \quad y = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$\text{rank}(X) = 2, \ \text{rank}(\tilde{X}) = 2$

$\hat{w} = X^{-1}y$

case (ii) : No solution

$\text{rank}(X) < \text{rank}(\tilde{X})$

$X = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \quad y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\text{rank}(X) = 1, \ \text{rank}(\tilde{X}) = 2$

After RREF $\Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

* Inconsistent system

$X$ is non-invertible

case (iii) : Infinitely many solutions

$\text{rank}(X) = \text{rank}(\tilde{X}) < d$

$X = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \ y = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$

$\text{rank}(X) = 1, \ \text{rank}(\tilde{X}) = 1, \ d = 2$

After RREF $\Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

## Over-determined ($X$ is 'tall')

case (ii) No solution (usual case)

$\text{rank}(X) < \text{rank}(\tilde{X})$

$X = \begin{bmatrix} 2 & 1 \\ 4 & 3 \\ 5 & 6 \end{bmatrix} \quad y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$\text{rank}(X) = 2, \ \text{rank}(\tilde{X}) = 3$

After RREF $\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Approximate solution by left-inverse:

$X^+ = (X^TX)^{-1}X^T$ (exists if $X$ has full column rank or $X^TX$ is invertible)

$w = X^+y$

case (i) : Unique solution

$\text{rank}(X) = \text{rank}(\tilde{X}) = d$

$X = \begin{bmatrix} 2 & 1 \\ 4 & 3 \\ 5 & 6 \end{bmatrix} \quad y = \begin{bmatrix} 4 \\ 10 \\ 17 \end{bmatrix}$

$\text{rank}(X) = \text{rank}(\tilde{X}) = 2$

After RREF $\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

case (iii) Infinitely many solutions

$\text{rank}(X) = \text{rank}(\tilde{X}) < d$

$X = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 4 \end{bmatrix}, \ y = \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix}$

$\text{rank}(X) = 1, \ \text{rank}(\tilde{X}) = 1, \ d = 2$

After RREF $\Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$

## Under-determined ($X$ is 'wide')

case (iii) Infinitely many solutions (usual case)

$\text{rank}(X) = \text{rank}(\tilde{X}) < d$

$X = \begin{bmatrix} 1 & 2 & 3 \\ 1 & -2 & 3 \end{bmatrix} \quad y = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\text{rank}(X) = 2, \ \text{rank}(\tilde{X}) = 2, \ d = 3$

After RREF $\Rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1/4 \end{bmatrix}$

$XX^T$ is invertible

($X$ full row rank)

Constrained solution by right-inverse

$\hat{w} = \underbrace{X^T(XX^T)^{-1}}_{\text{right-inverse}} y$

case (i) : Unique solution

$\text{rank}(X) = \text{rank}(\tilde{X}) = d$

Not possible since $m < d$

case (ii) : No solution

$\text{rank}(X) < \text{rank}(\tilde{X})$

$X = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 9 \end{bmatrix} \quad y = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\text{rank}(X) = 1, \ \text{rank}(\tilde{X}) = 2$

$d = 3$

After RREF $\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$X$ does not have full column/row rank

No least squares/least norm solution as $X^TX$ and $XX^T$ are both non-invertible