EE2211 Lecture 7: Overfitting, Model Complexity, Feature Selection & Regularization, Bias-Variance Tradeoff

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Acknowledgements to

Xinchao, Helen, Thomas, Kar Ann, Chen Khong, Robby, and Haizhou

Outline

Review of Regression

- Goal: Given feature(s) $\mathbf{x} \in \mathbb{R}^d$, we want to predict target $y \in \mathbb{R}$.
  1. $\mathbf{x}$ can be one-dimensional (in which case we may write x);
  2. Or $\mathbf{x}$ may be high dimensional;
  3. y is often one-dimensional (trying to predict one output).

- Two types of input data
  1. Training Set $\{(\mathbf{x}_i, y_i)\}_{i=1}^m$;
  2. Test Set $\{\mathbf{x}_j\}_{j=m+1}^n$ (no targets) [We called one of these samples $\mathbf{x}_{\text{new}}$ previously];

- m training samples, n test samples, all in $\mathbb{R}^d$

- Learning/Training
  Training set used to estimate regression coefficients $\bar{\mathbf{w}}^*$.

- Testing/Prediction
  Prediction performed on test set to evaluate performance.

Review of Regression: Linear Case

- Suppose the feature is one-dimensional $x \in \mathbb{R}$.
- Want to learn the affine relationship between x and $y \in \mathbb{R}$.
- With $m = 4$ training samples, the design matrix and target vector are

$$\mathbf{X} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ 1 & x_4 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

- Learning/Training

$$\bar{\mathbf{w}}^* = (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{y}.$$

- Testing/Prediction on x_{new}

$$y_{\text{test}} = \begin{bmatrix} 1 \\ x_{\text{new}} \end{bmatrix}^\top \bar{\mathbf{w}}^*.$$

Regression Review: Polynomial

- Features $\mathbf{x} \in \mathbb{R}^d$ may be one-dimensional or high dimensional; $y \in \mathbb{R}$ is a one-dimensional target as usual.
- Want to learn a polynomial relationship between x and y
- Quadratic illustration ($m = 4$ training samples and x is one-dim) of design matrix and target vector

$$\mathbf{X} = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ 1 & x_4 & x_4^2 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

- Learning/Training

$$\bar{\mathbf{w}}^* = (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{y} \in \mathbb{R}^3.$$

- Testing/Prediction on x_{new}

$$y_{\text{test}} = \begin{bmatrix} 1 \\ x_{\text{new}} \\ x_{\text{new}}^2 \end{bmatrix}^\top \bar{\mathbf{w}}^*.$$

Polynomial Regression Review: Notation

- Sometimes, we use the notation $\mathbf{P}$ in place of $\mathbf{X}$ to emphasize that the features $\mathbf{x}$ are encoded in a polynomial way in the design matrix $\mathbf{P} = \mathbf{X}$.
- Learning/Training

$$\bar{\mathbf{w}}^* = (\mathbf{P}^\top \mathbf{P})^{-1} \mathbf{P}^\top \mathbf{y} \in \mathbb{R}^3.$$

- Testing/Prediction on x_{new}

$$y_{\text{new}} = \underbrace{\begin{bmatrix} 1 \\ x_{\text{new}} \\ x_{\text{new}}^2 \end{bmatrix}^\top}_{= \mathbf{p}_{\text{new}}^\top} \bar{\mathbf{w}}^*.$$

Note on Training & Test Sets

- Affine is special case of polynomial
- Use $\mathbf{P}$ instead of $\mathbf{X}$ from now on.
- Training/Learning (primal) on training set

$$\bar{\mathbf{w}}^* = (\mathbf{P}^\top \mathbf{P})^{-1} \mathbf{P}^\top \mathbf{y} \in \mathbb{R}^{d'}.$$

- Prediction/Testing on the entire test set

$$\mathbf{y}_{\text{new}} = \mathbf{P}_{\text{new}} \bar{\mathbf{w}}^* \in \mathbb{R}^n$$

where

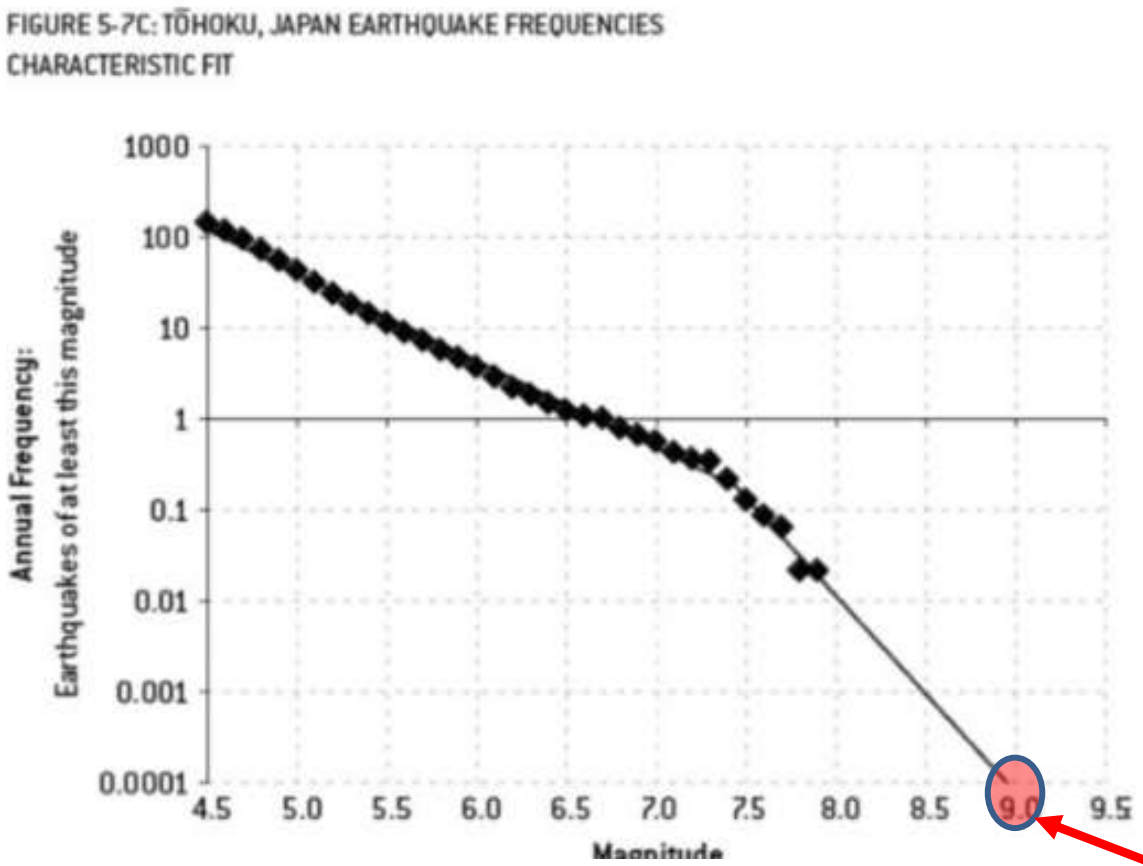
$$\mathbf{P}_{\text{new}} = \begin{bmatrix} \mathbf{p}_{m+1}^\top \\ \mathbf{p}_{m+2}^\top \\ \vdots \\ \mathbf{p}_{m+n}^\top \end{bmatrix} \in \mathbb{R}^{n \times d'}$$

Note on Training & Test Sets

- There should be zero overlap between training & test sets.
- Important goal of regression: prediction on new unseen data, i.e., the test set.
- Question: Why is test set important for evaluation?

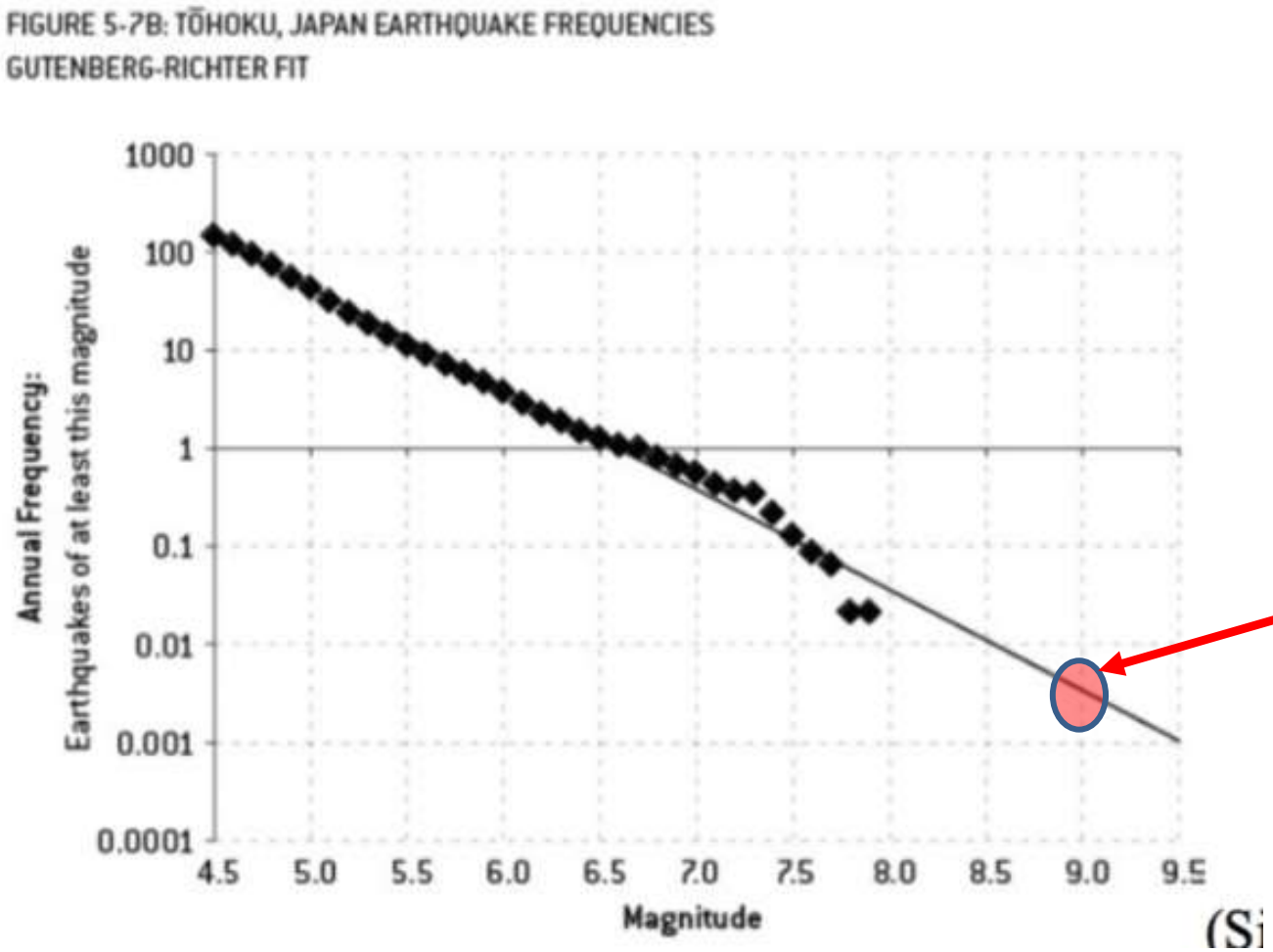
Questions?

Overfitting Motivation: Fukushima Disaster



- In March 2011, there was a large earthquake in Japan.
- Small earthquakes occur frequently while massive earthquakes occur rarely.
- Engineers fitted a polynomial model
- Earthquake of magnitude 9.0 occurs once every 10,000 years

Overfitting Motivation: Fukushima Disaster

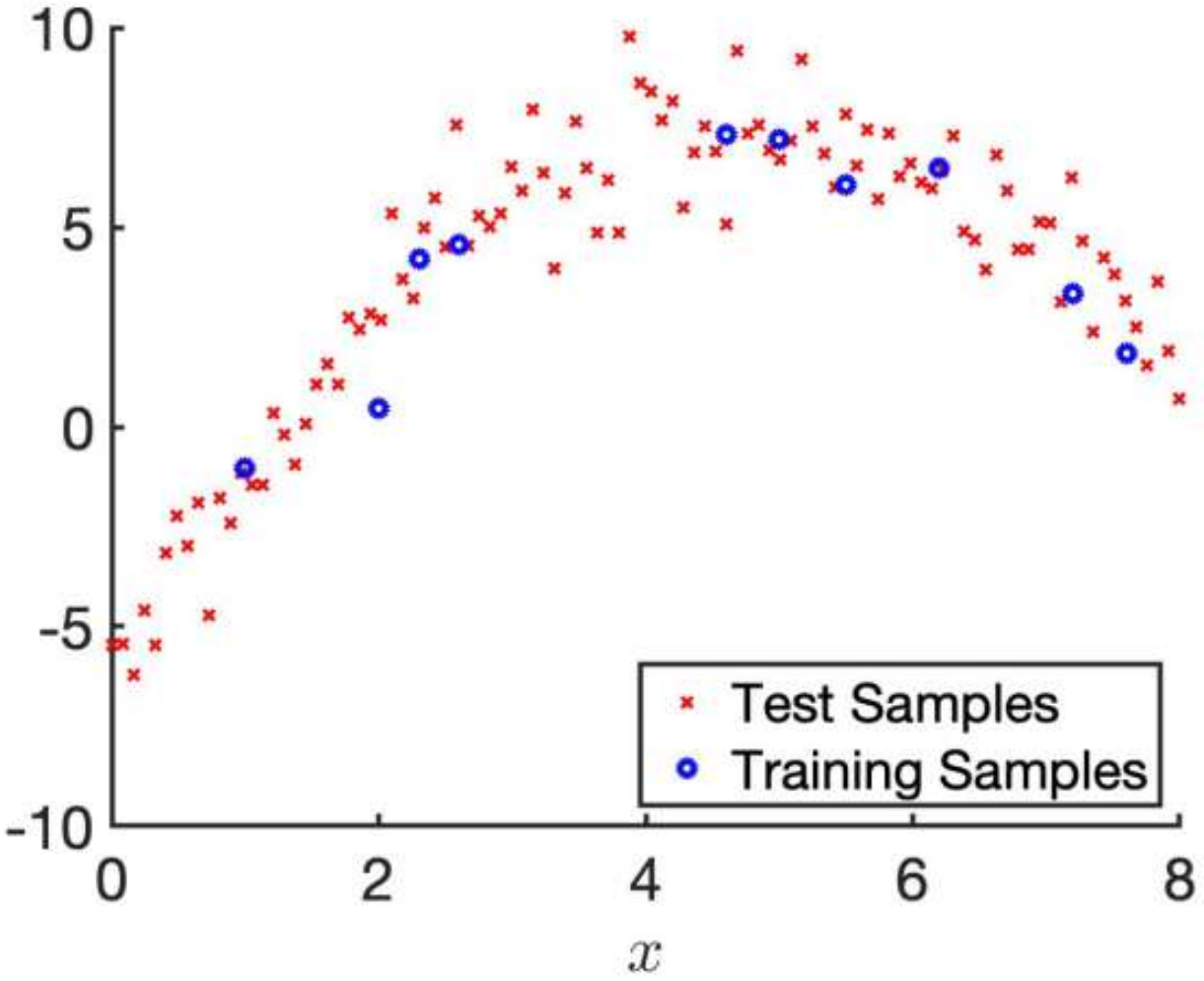


Might want to fit an affine model instead

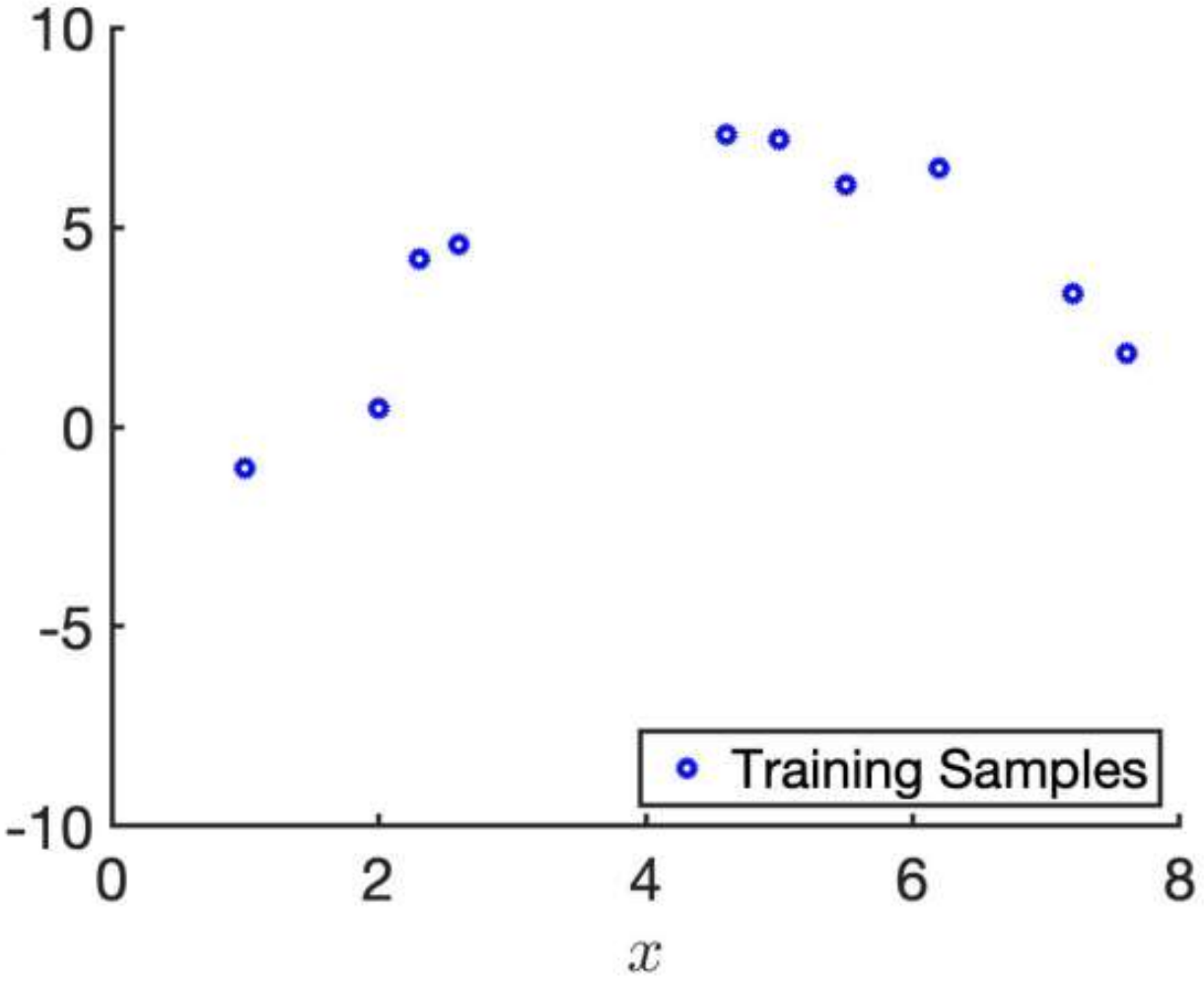
Predicts that an earthquake of magnitude 9.0 occurs once every 500 years

Could have reinforced the buildings

Overfitting Example

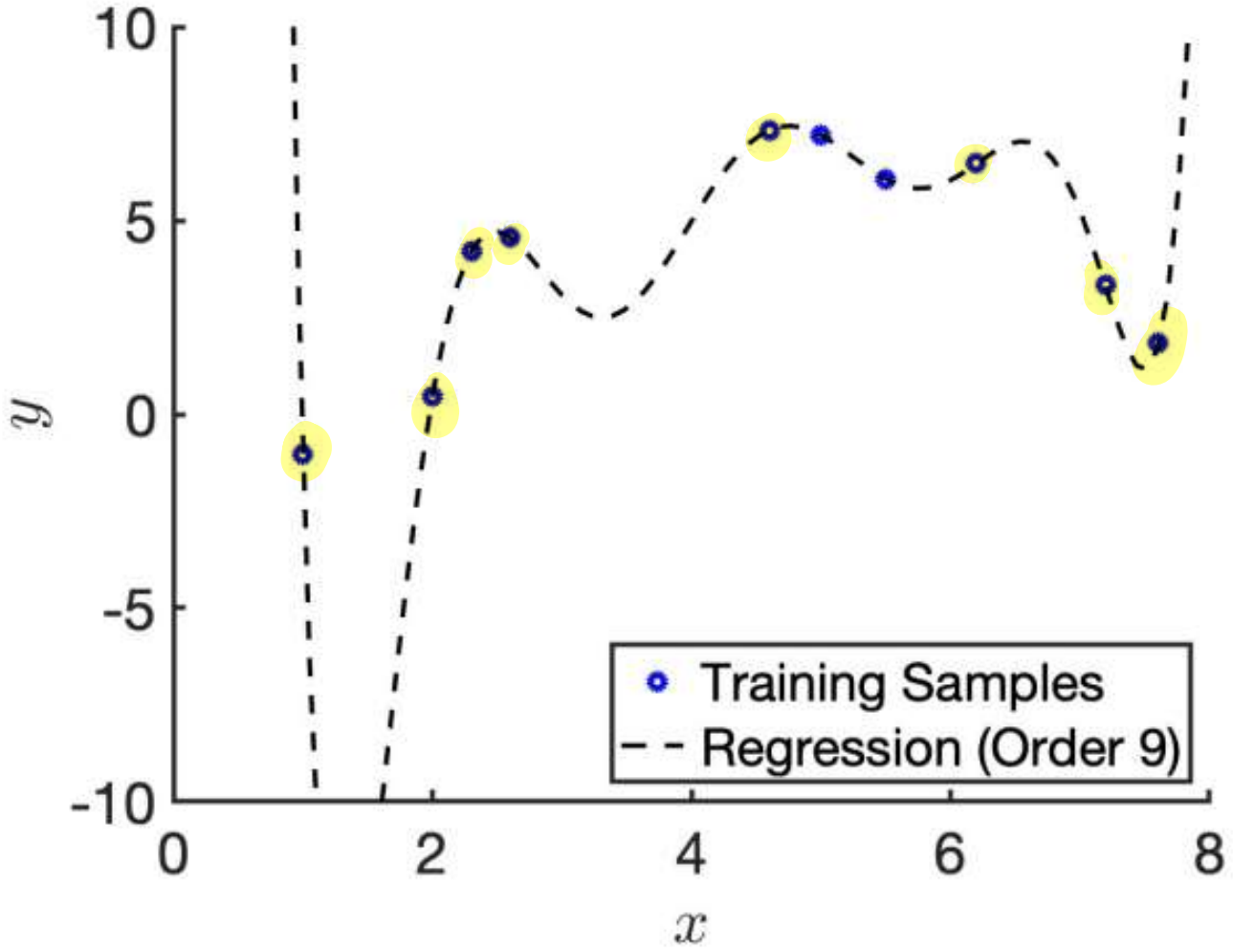


Overfitting Example



Overfitting Example

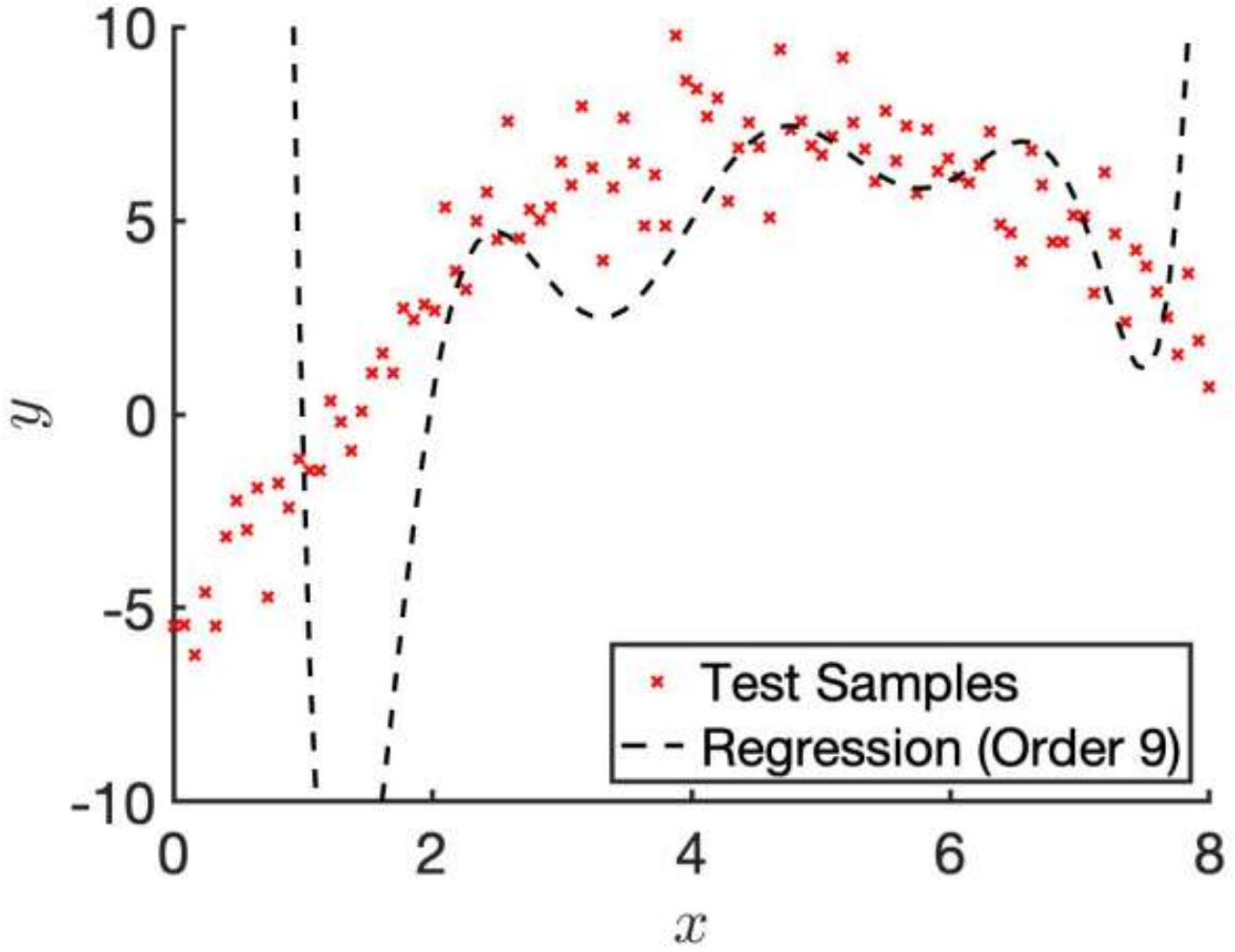
$$y = w_0 + w_1 x + w_2 x^2 + \ldots + w_9 x^9$$



Perfect

	Training Set Fit	
Order 9	Good	

Overfitting Example



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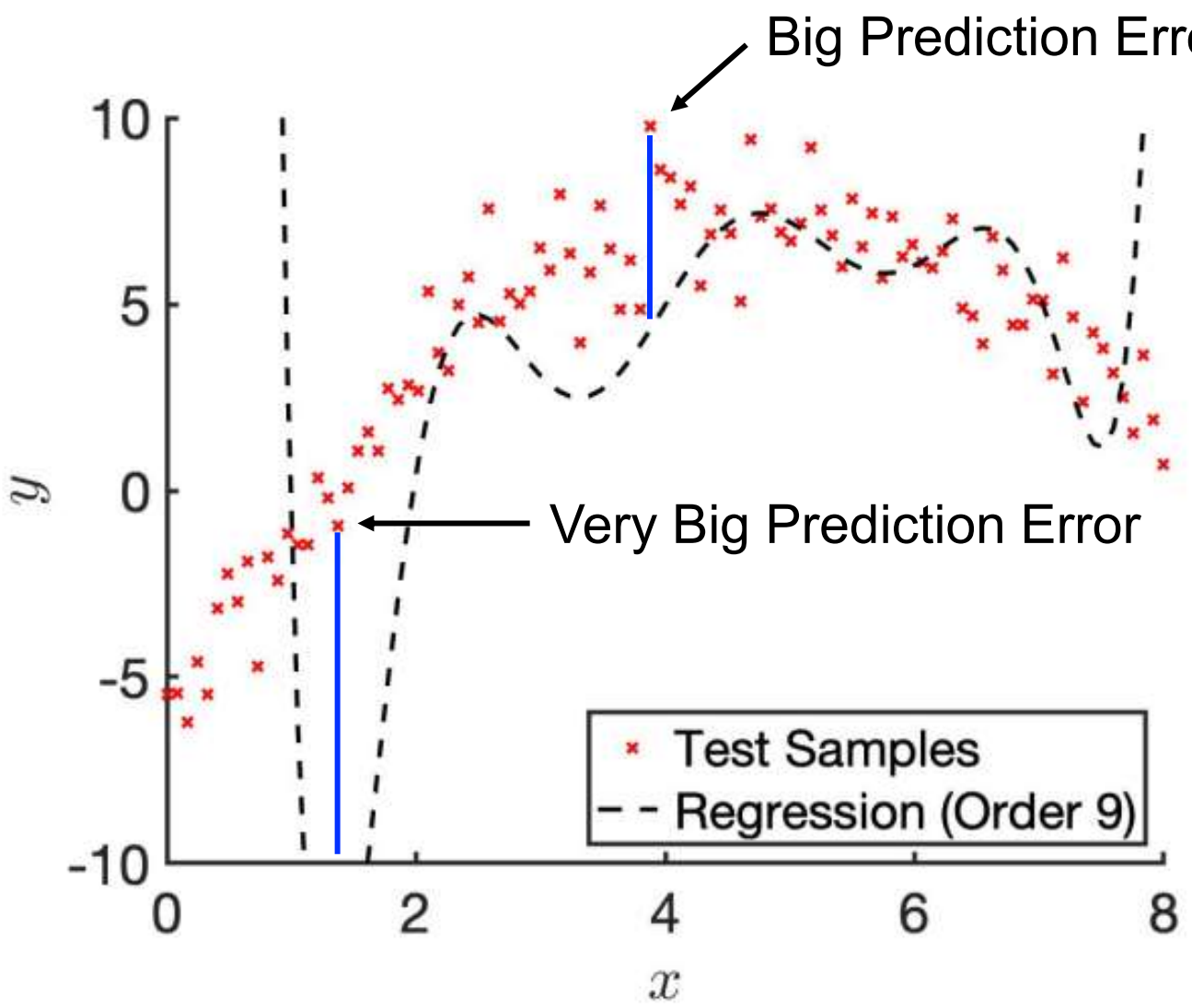
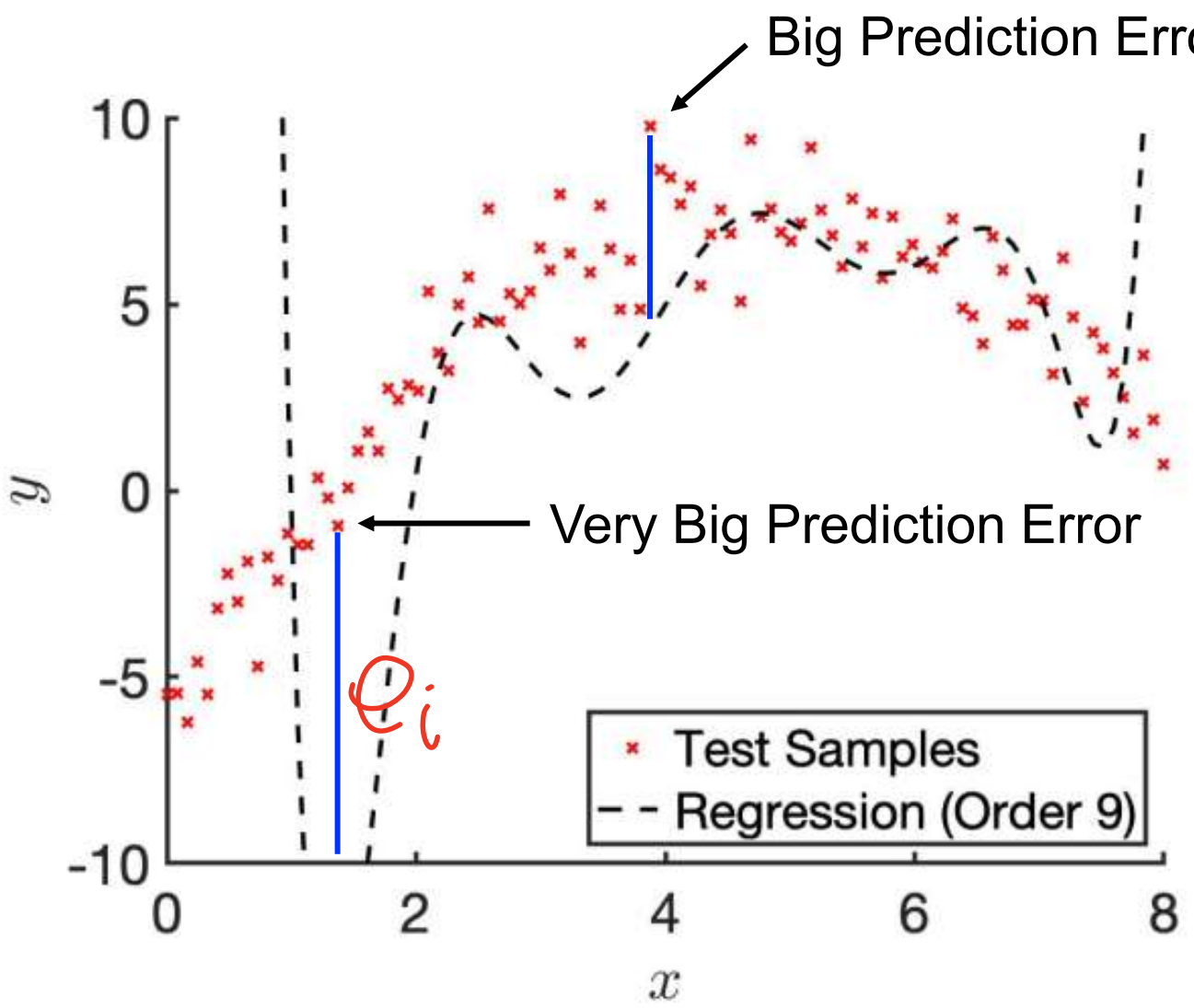


	Training Set Fit	Test Set Fit
Order 9	Good	Bad

Overfitting Example



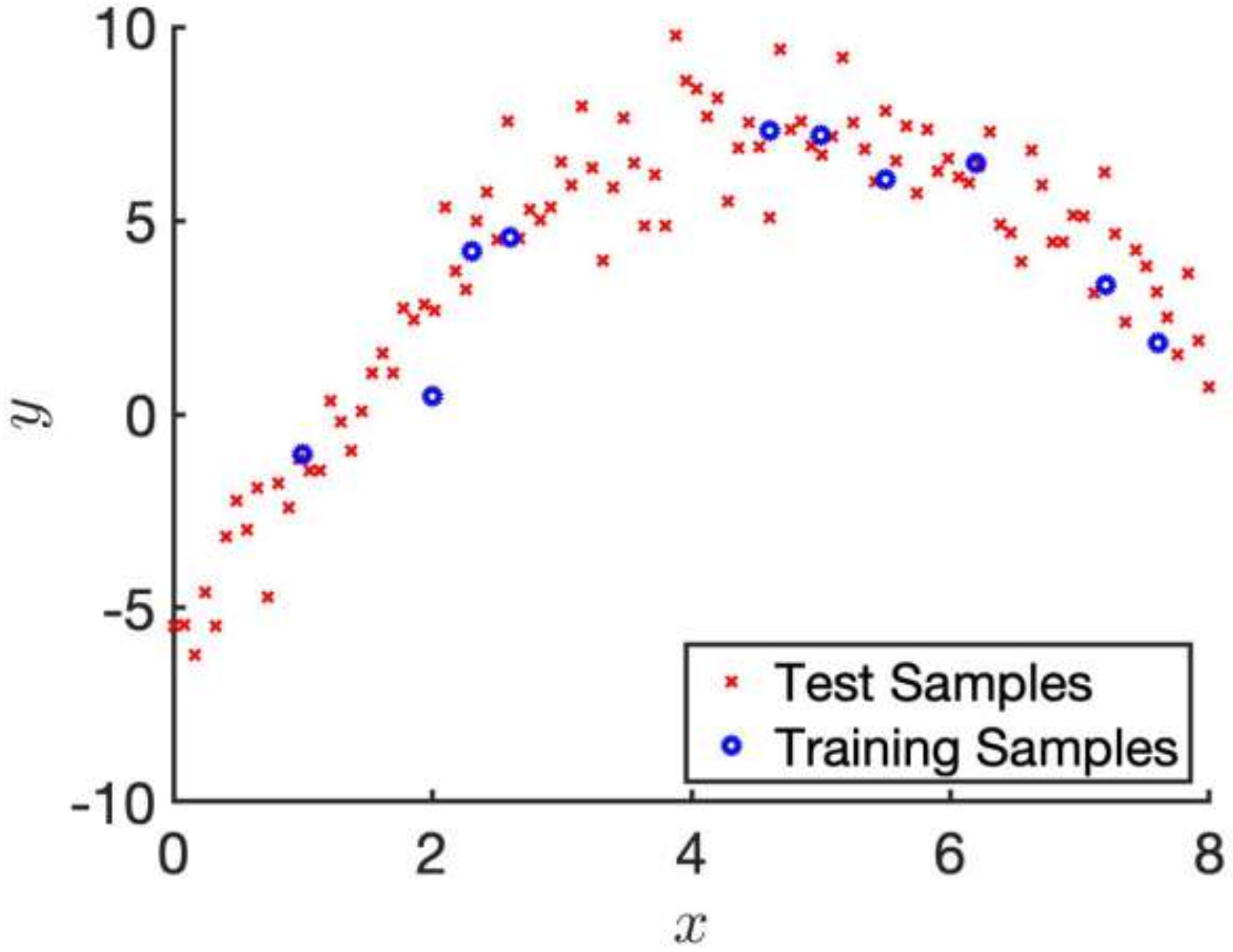
- If we take one of the blue lines and compute the square of its length, this is called **"squared error" for that particular data point**
- If we average squared errors across all the red crosses, it's called **mean squared error (MSE)** in the test set

e_i^2

$\frac{1}{n}\sum_{i=m+1}^{m+n} e_i^2$

test points

Underfitting Example



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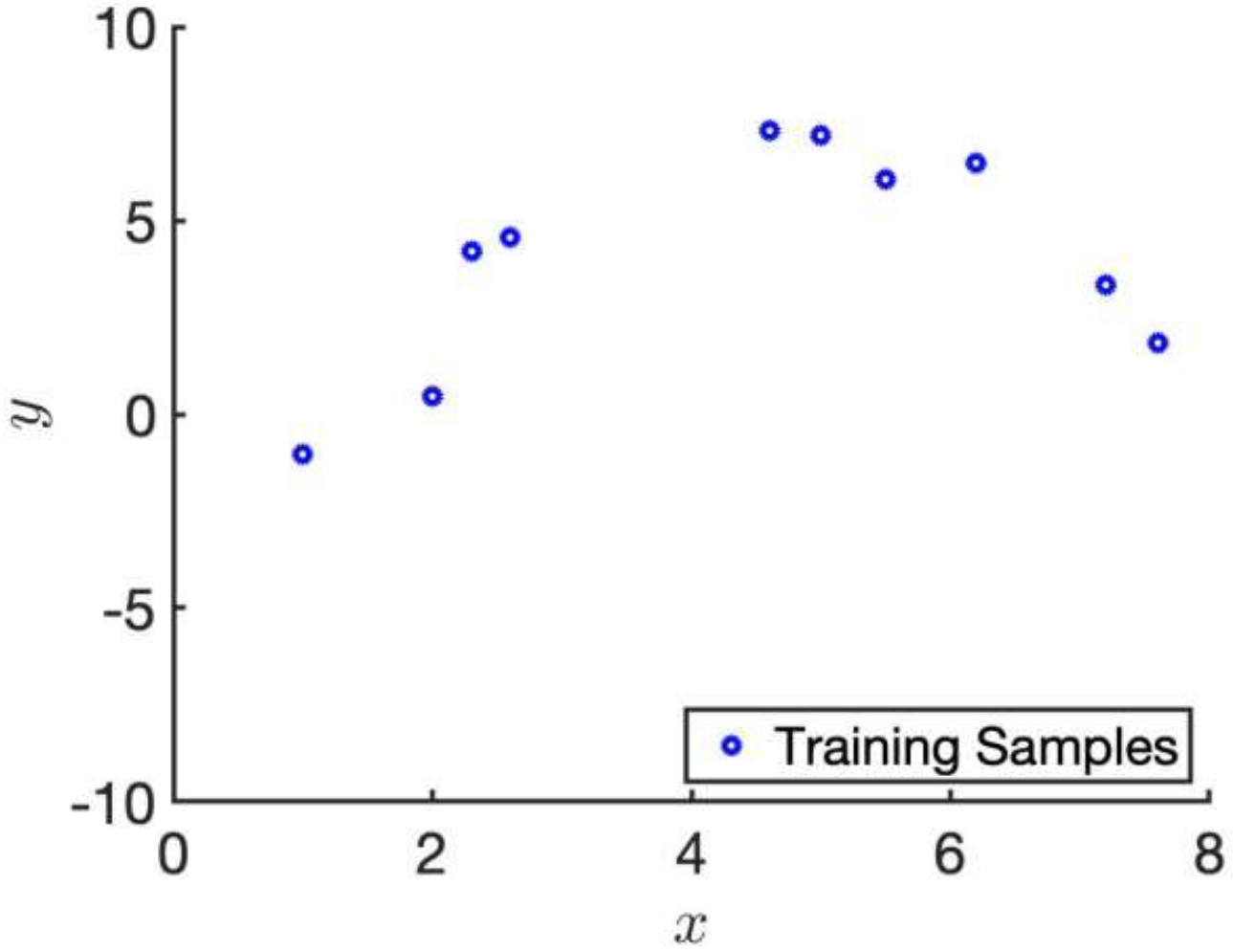

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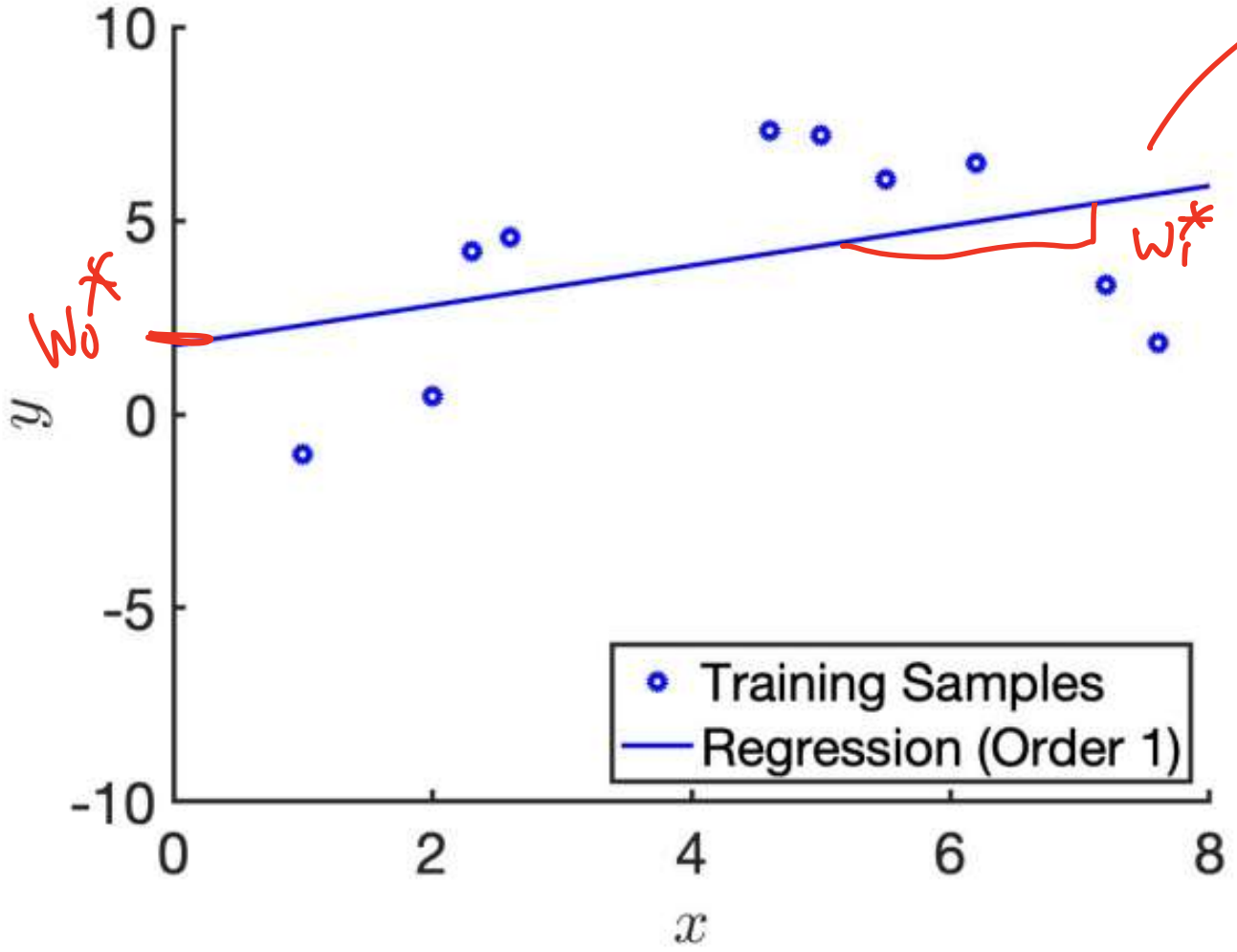

$y = w_0 + w_1 x$

affine.

w_0^*

w_1^*

	Training Set Fit	Test Set Fit
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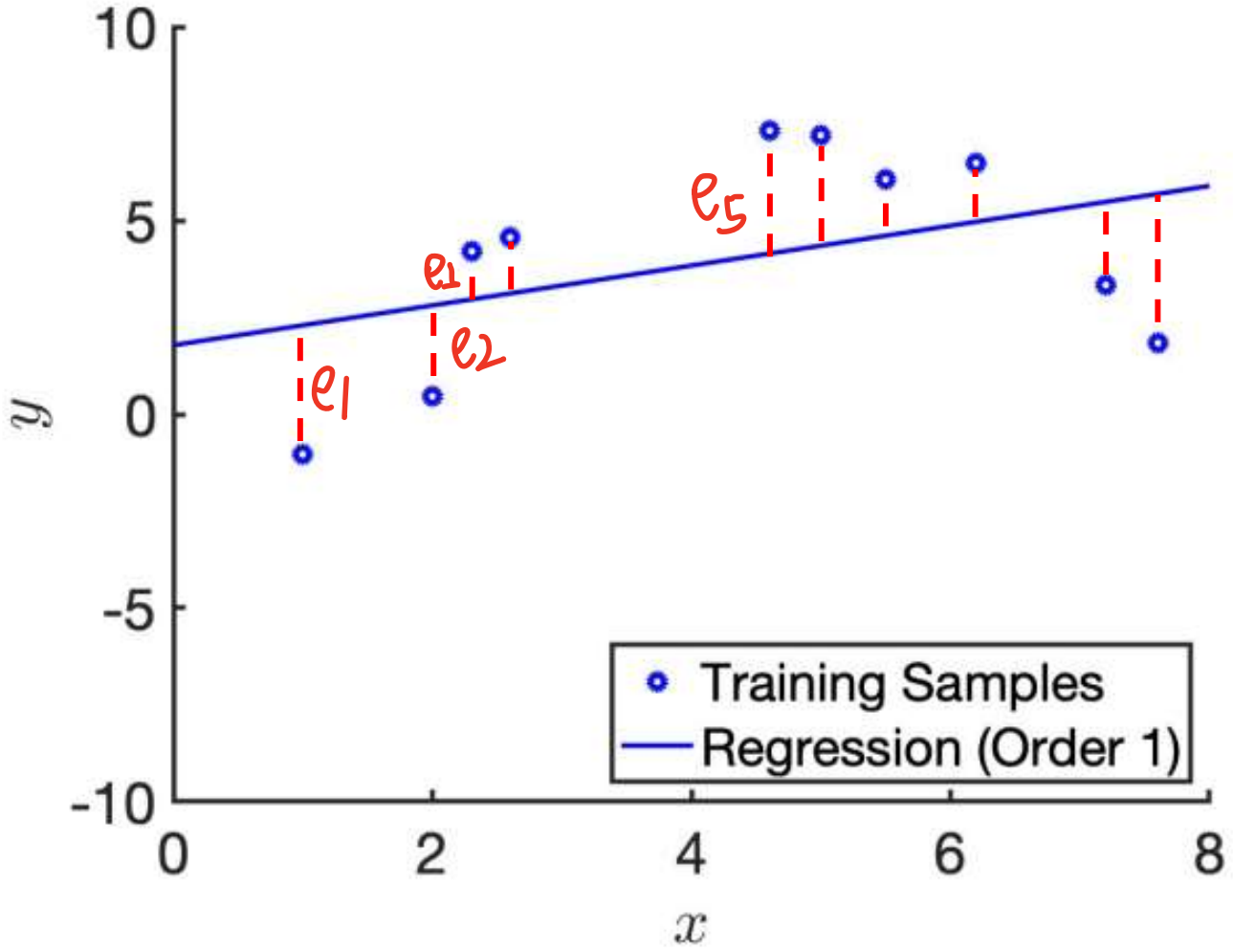
Order 1		

Underfitting Example



	Training Set Fit	Test Set Fit
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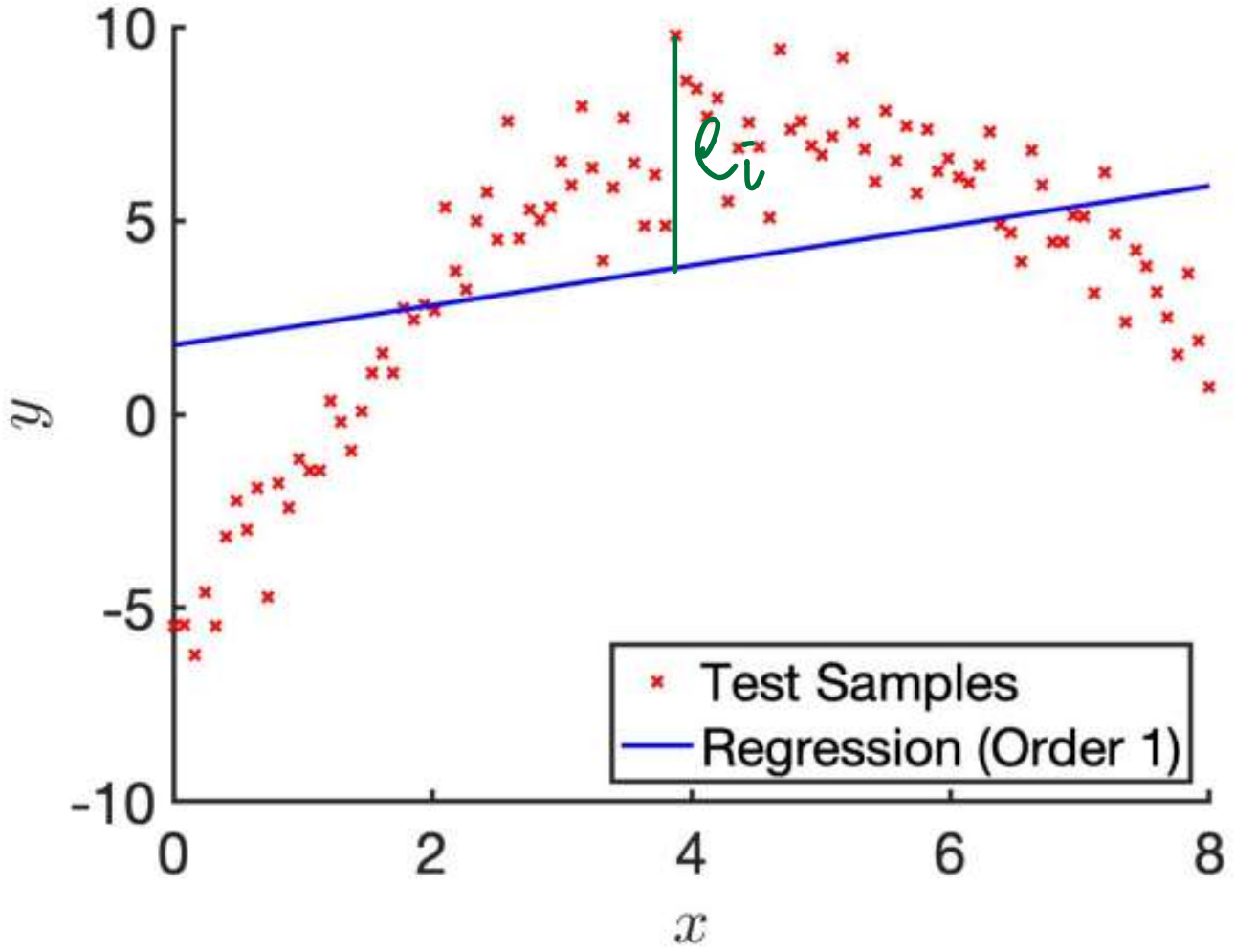
Underfitting Example

$$\frac{1}{n}\sum_{i=mf1}^{mf1} e_i^2 : \text{bad}$$



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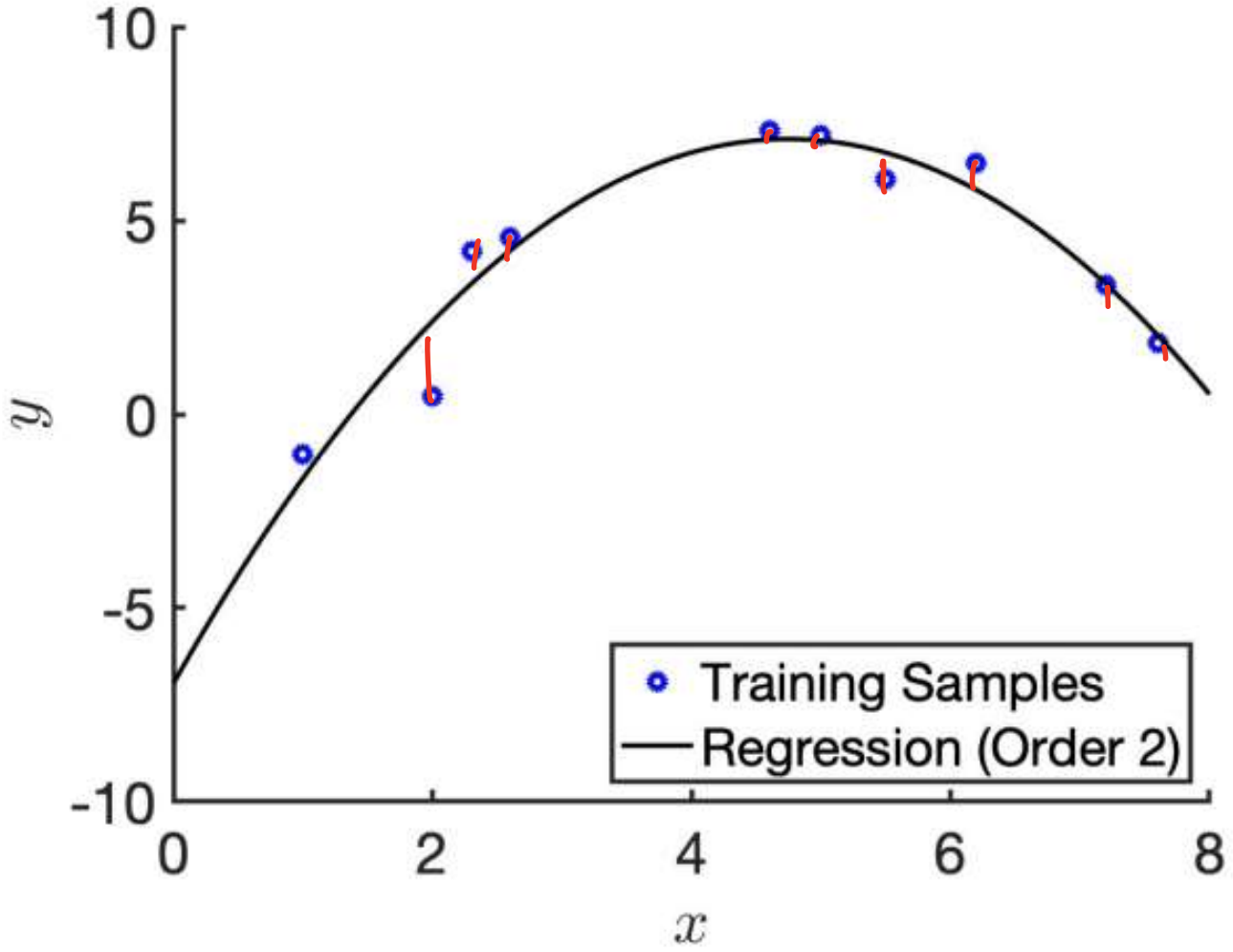
"Just Nice"

$y = w_0 + w_1 x + w_2 x^2$



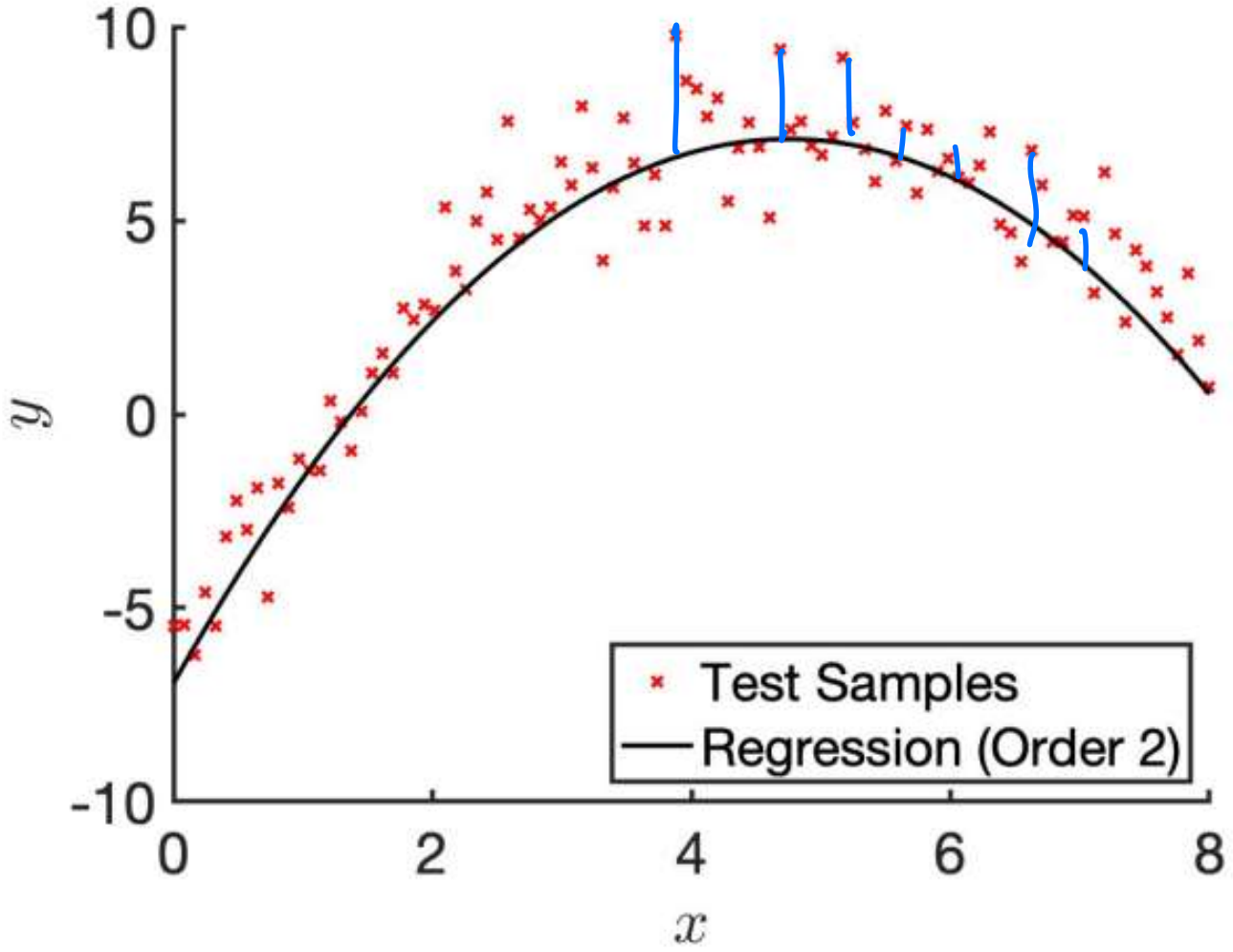
	Training Set Fit	Test Set Fit
Order 9	Good	Bad
Order 1	Bad	Bad
Order 2	Good	

"Just Nice"

$$\frac{1}{n}\sum_{i=m+1}^{m+n} e_i^2$$

q

test



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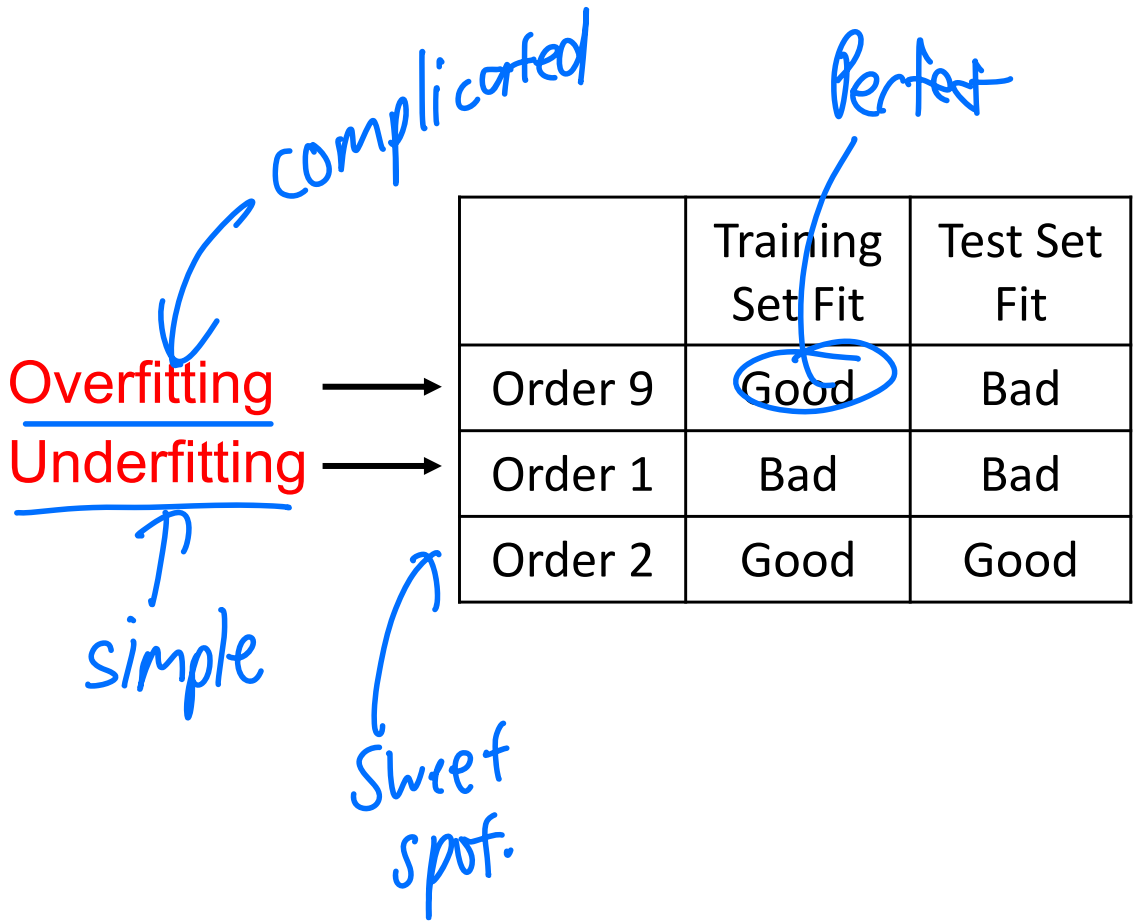
Overfitting & Underfitting



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- **Reason 2**
  - Too many features but number of training samples too small
  - Even linear model can overfit, e.g., linear model with 9 input features (i.e., w is 10-D) and 10 data points in training set => data might not be enough to estimate 10 unknowns well

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- **Solutions**
  - Use **simpler models** (e.g., lower order polynomial)
  - Use **regularization** (see next part of lecture)

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Overfitting & Underfitting

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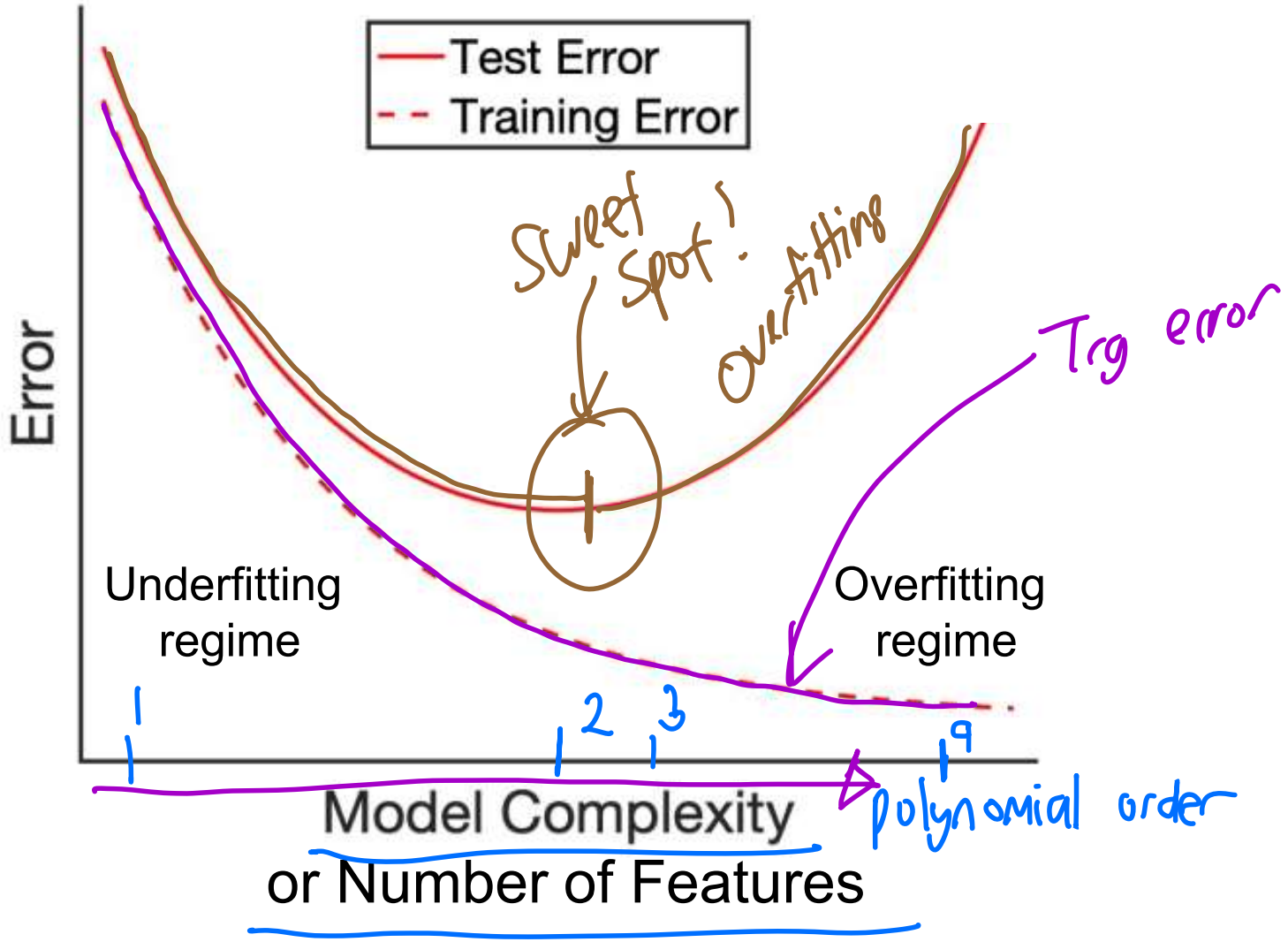
- **Reason 1**
  - Model is too simple for the data
  - Previous example: Fit order 1 polynomial to 10 data points that came from an order 2 polynomial
  - **Solution**: Try more complex model
- **Reason 2**
  - Features are not informative enough
  - **Solution**: Try to develop more informative features

data comes from quadratic $y = w_0 + w_1 x + w_2 x^2$.
but we only learned 2 parameter $y = w_0 + w_1 x$

Overfitting / Underfitting Schematic



Questions?

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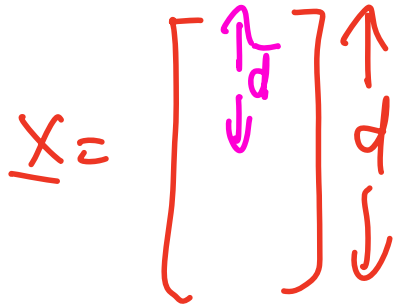
- Feature selection procedure
  - Step 1: feature selection in training set
  - Step 2: fit model using selected features in training set
  - Step 3: evaluate trained model using test set

Feature Selection

d: features.

- ~~Less~~ Fewer features might reduce overfitting
  - Want to discard useless features & keep good features, so perform feature selection
- Feature selection procedure
  - Step 1: feature selection in training set
  - Step 2: fit model using selected features in training set
  - Step 3: evaluate trained model using test set
- Very common mistake
  - Feature selection with test set (or full dataset) leads to inflated performance
  - Do not perform feature selection with test data



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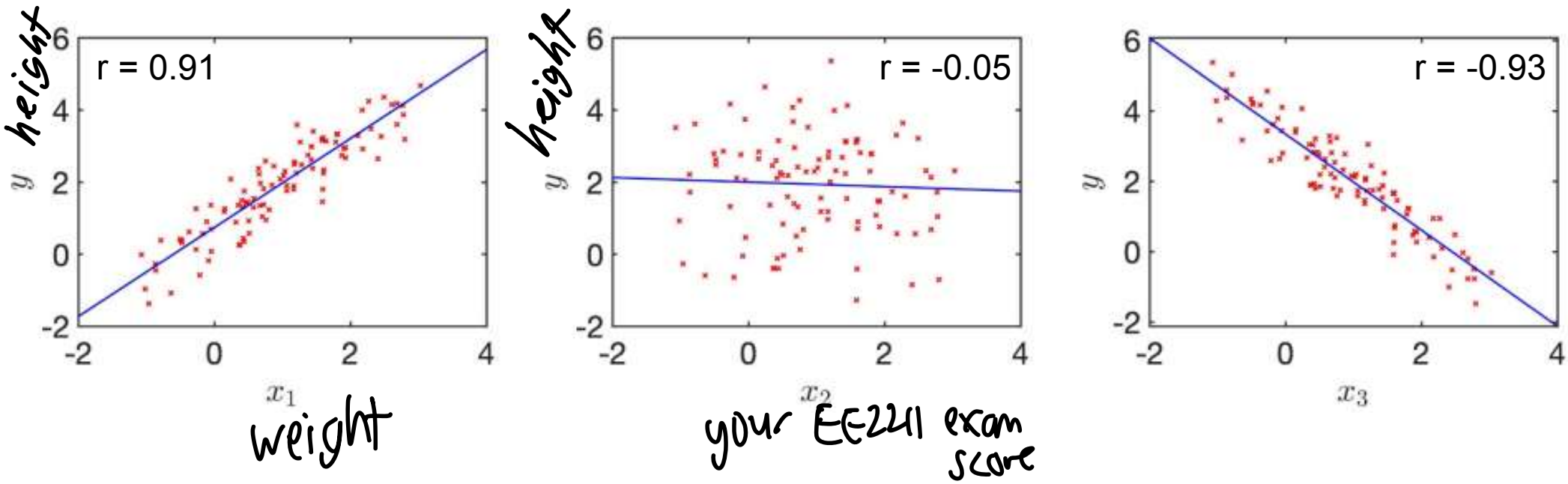

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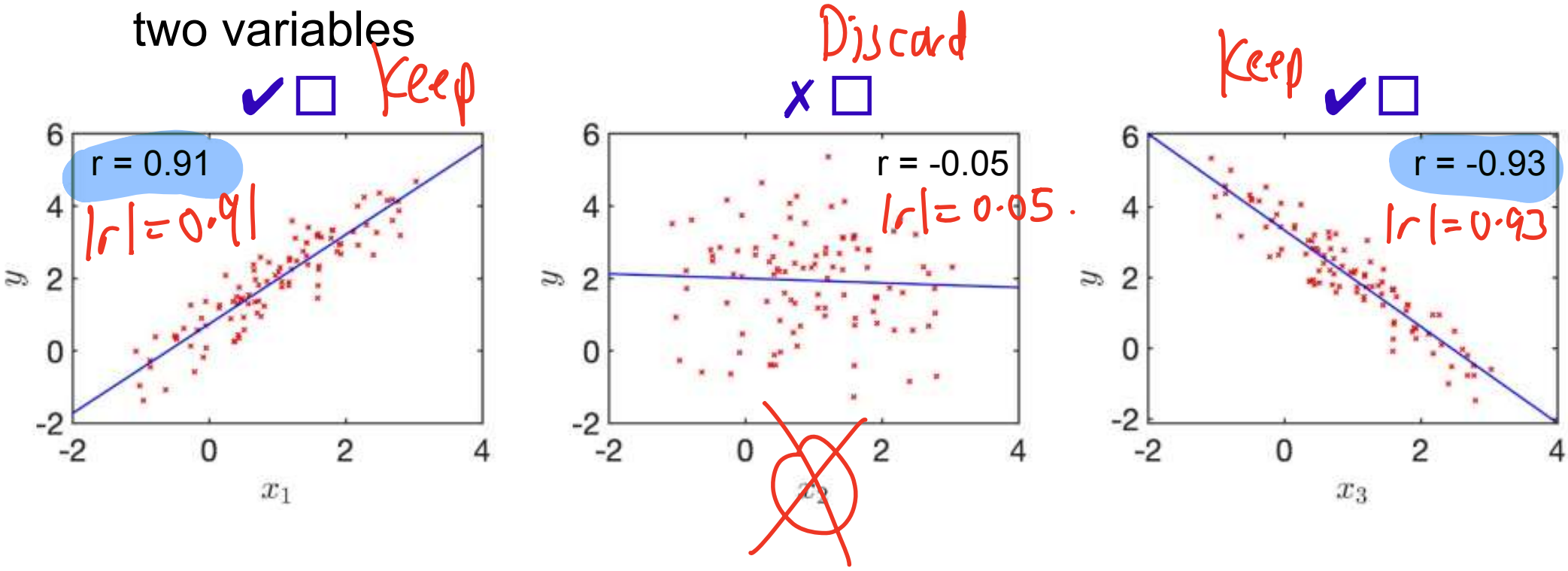

Selecting Features With Pearson's r

- Given features x, we want to predict target y
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- Two options
  - Option 1: Pick K features with largest absolute correlations
  - Option 2: Pick all features with absolute correlations > C
  - K & C are "magic" numbers set by the ML practitioner

$C \in [0, 1]$.

Selecting Features With Pearson's r

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- Two options
  - Option 1: Pick K features with largest absolute correlations
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  - K & C are "magic" numbers set by the ML practitioner
- Other metrics besides Pearson's correlation are possible

- Compute Pearson's r correlation for all features x_i against y (target)
- Pick the top "few."

Questions?

Regularization

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- For example, in previous lecture, we added $\lambda \mathbf{w}^T \mathbf{w}$:

$$\underset{\mathbf{w}}{\operatorname{argmin}} (\mathbf{P}\mathbf{w} - \mathbf{y})^T (\mathbf{P}\mathbf{w} - \mathbf{y}) + \lambda \mathbf{w}^T \mathbf{w}$$

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- Minimizing with respect to $\mathbf{w}$, primal solution is

$$\hat{\mathbf{w}} = (\mathbf{P}^T \mathbf{P} + \lambda \mathbf{I})^{-1} \mathbf{P}^T \mathbf{y}$$

Regularization

P: wide
does not have full col. rank

- Regularization is an umbrella term that includes methods forcing learning algorithm to build less complex models.
- Motivation 1: Solve an ill-posed problem

  $(P^TP)^{-1}$ doesn't exist.

  – For example, estimate 10th order polynomial with just 5 datapoints

  $d'=11$ $n=5$.
- Motivation 2: Reduce overfitting
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- Minimizing with respect to $\mathbf{w}$, primal solution is

$$\hat{\mathbf{w}} = (\mathbf{P}^T\mathbf{P}+\lambda\mathbf{I})^{-1}\mathbf{P}^T\mathbf{y} = P^T(PP^T+\lambda I)^{-1}y.$$

primal dual.

- For $\lambda > 0$, matrix becomes invertible (Motivation 1)

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- $\hat{\mathbf{w}}$ might also perform better in test set, i.e., reduces overfitting (Motivation 2) – will show example later

Regularization

- Consider minimization from previous slide

$$\underset{\mathbf{w}}{\operatorname{argmin}} (\mathbf{P}\mathbf{w} - \mathbf{y})^T (\mathbf{P}\mathbf{w} - \mathbf{y}) + \lambda \mathbf{w}^T \mathbf{w}$$

least squares obj. regularization

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Cost function quantifying data fitting error in training set

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Regularization

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$$\underset{\mathbf{w}}{\operatorname{argmin}}(\mathbf{Pw}-\mathbf{y})^T(\mathbf{Pw}-\mathbf{y})+\lambda\mathbf{w}^T\mathbf{w}$$

- $\mathbf{w}^T\mathbf{w}=w_0^2+w_1^2+\cdots+w_d^2$ $=\|w\|^2$

Regularization

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$$\underset{\mathbf{w}}{\operatorname{argmin}}(\mathbf{P}\mathbf{w}-\mathbf{y})^T(\mathbf{P}\mathbf{w}-\mathbf{y})+\lambda\mathbf{w}^T\mathbf{w}$$

- $\mathbf{w}^T\mathbf{w}=w_0^2+w_1^2+\cdots+w_d^2$ — L2 - Regularization

Regularization

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$$\underset{\mathbf{w}}{\operatorname{argmin}}(\mathbf{Pw}-\mathbf{y})^T(\mathbf{Pw}-\mathbf{y})+\lambda\mathbf{w}^T\mathbf{w}$$

- $\mathbf{w}^T\mathbf{w}=w_0^2+w_1^2+\cdots+w_d^2$ — L2 - Regularization
- Encourage $w_0,\cdots,w_d$ to be small (also called shrinkage or weight-decay)

Regularization

- Consider minimization from previous slide

$$\underset{\mathbf{w}}{\operatorname{argmin}}(\mathbf{Pw}-\mathbf{y})^T(\mathbf{Pw}-\mathbf{y})+\lambda\mathbf{w}^T\mathbf{w}$$

- $\mathbf{w}^T\mathbf{w}=w_0^2+w_1^2+\cdots+w_d^2$ — L2 - Regularization
- Encourage $w_0,\cdots,w_d$ to be small (also called shrinkage or weight-decay) => constrain model complexity

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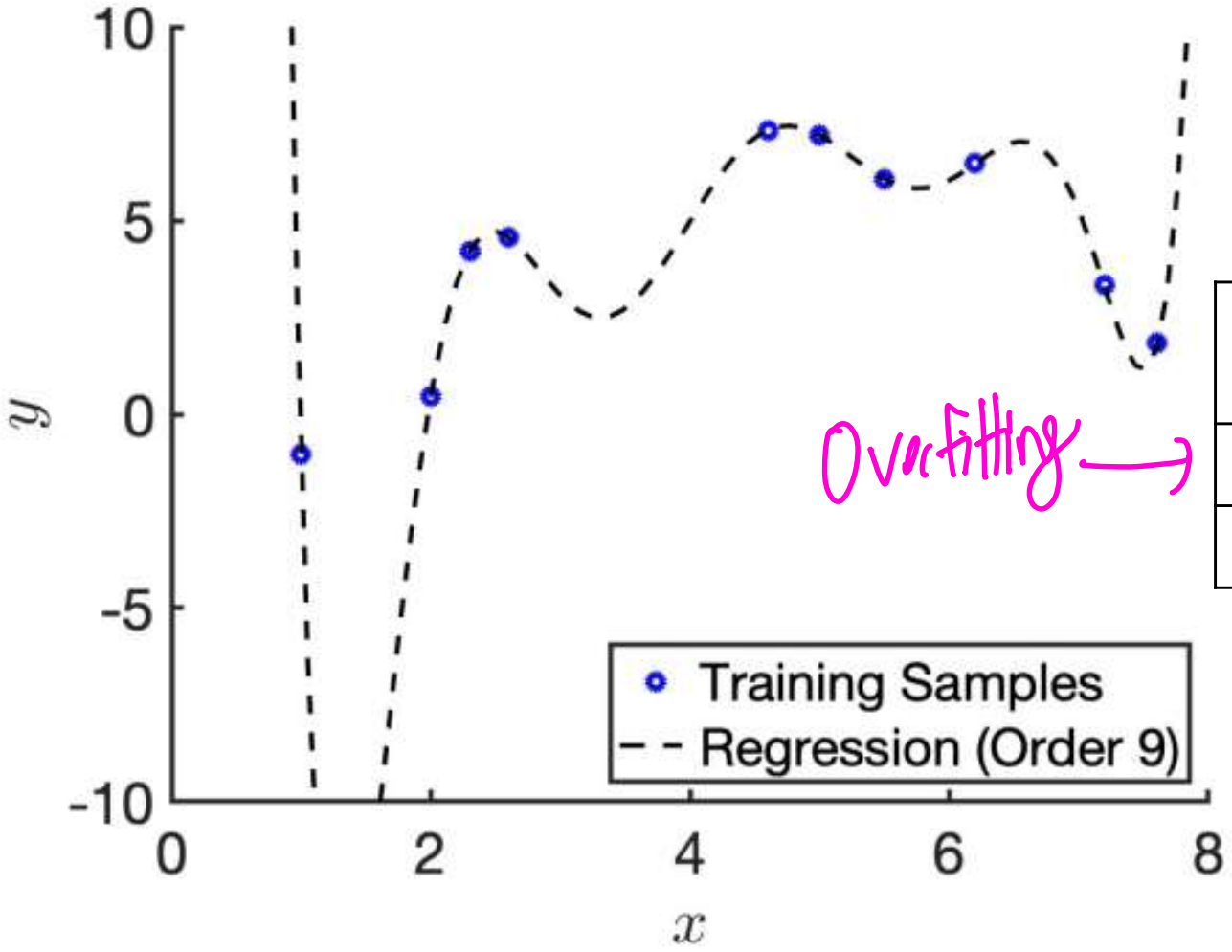
- **Data-Loss(w)** quantifies fitting error to training set given parameters **w:** smaller error => better fit to training data
- **Regularization(w)** penalizes more complex models

Regularization Example

$m = 10$ data points.

$$y = \sum_{i=0}^{9} w_i x^i$$

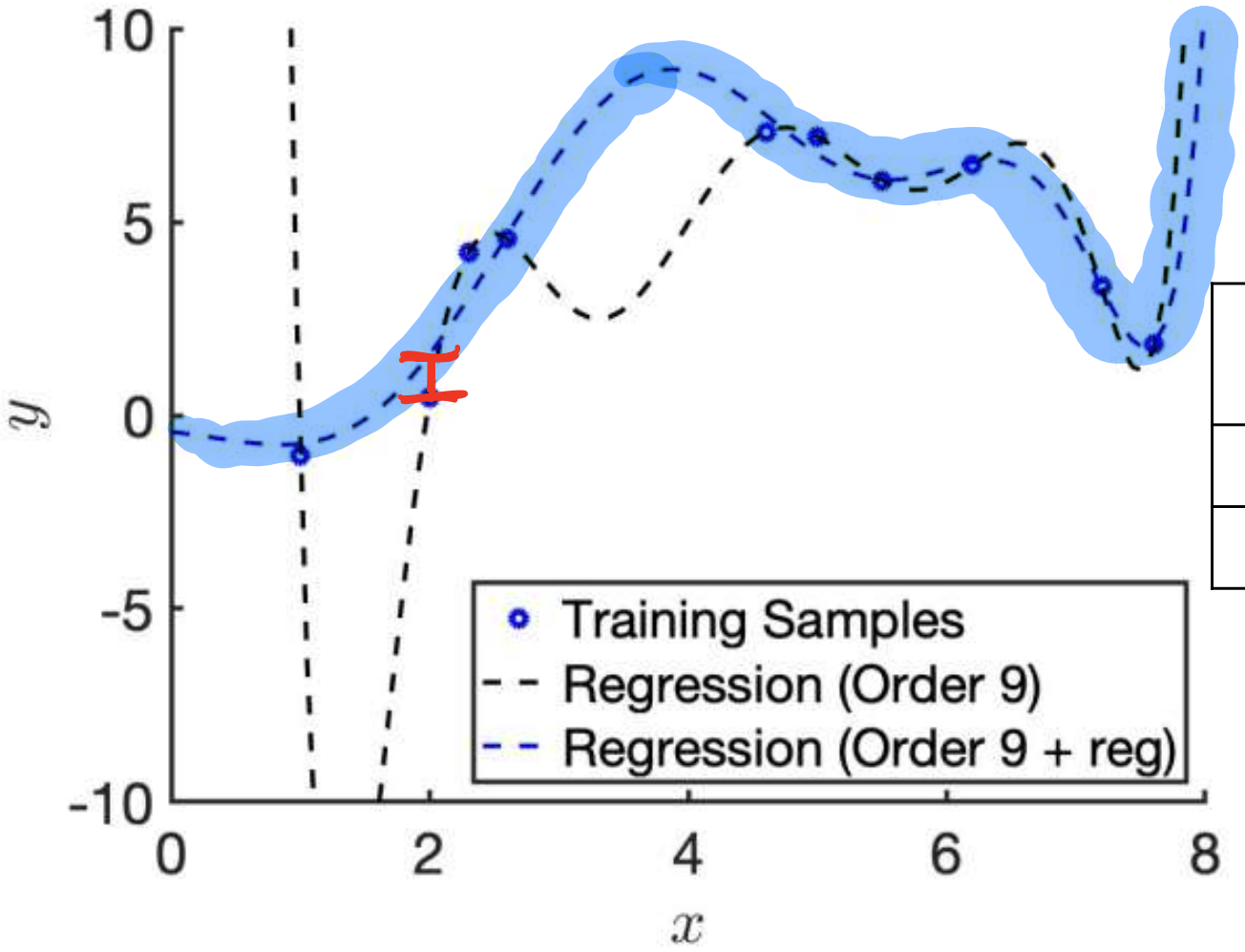


	Training Set Fit	Test Set Fit
Order 9	Good	Bad

Perfect

Horrible

Regularization Example



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Order 9, $\lambda = 1$	Good	

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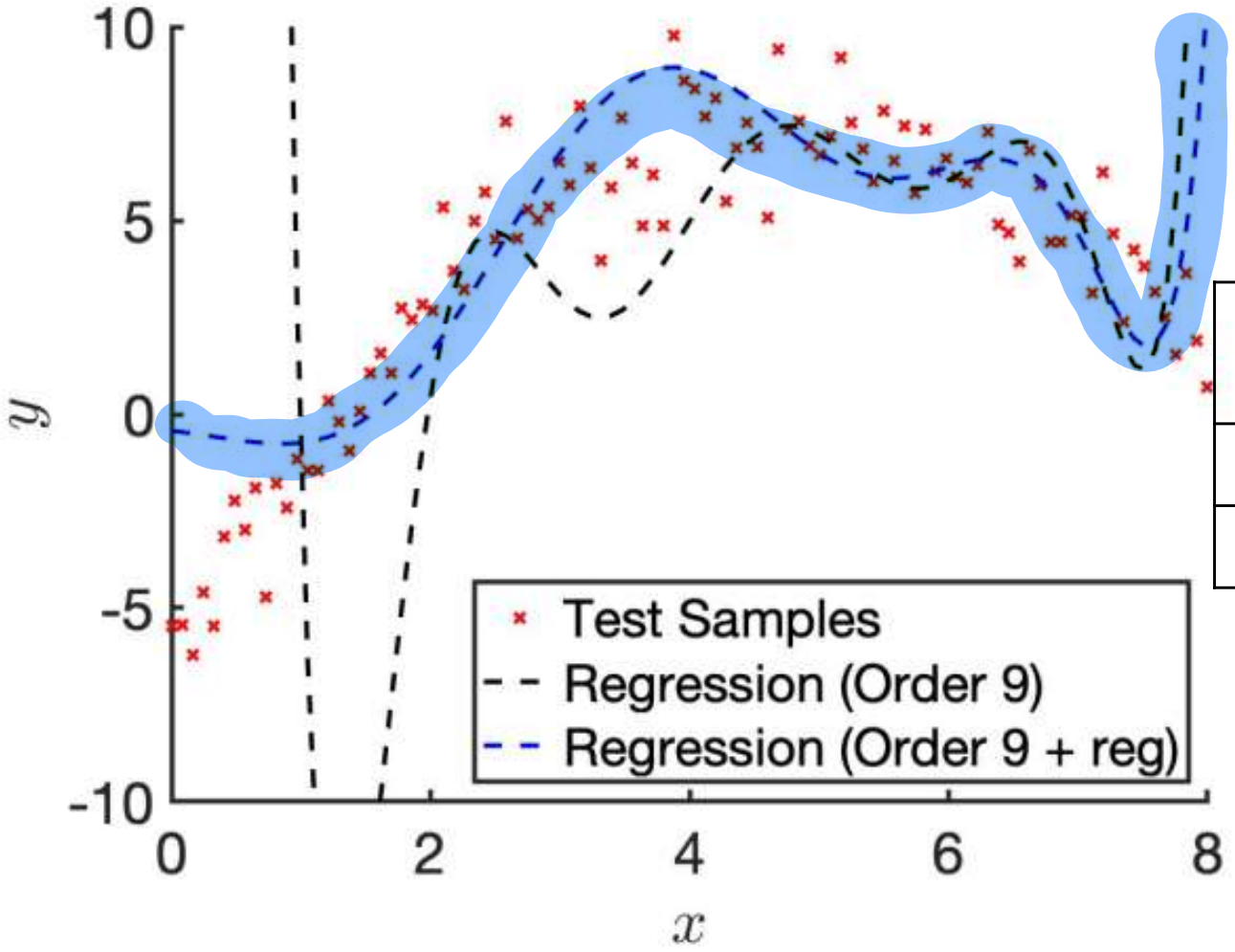
Horrible

$\lambda w^T w$ in the objective/cost function

Regularization Example



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Order 9	Good	Bad
Order 9, $\lambda = 1$	Good	Good

Overfitting

2 ways to mitigate this prob.

- Reduce model complexity (e.g. use lower-order poly)
- Use regularization (ridge regression with $\lambda > 0$).

Questions?

$$\underset{w}{\operatorname{argmin}} \cancel{\text{Data-Loss}(w)} + \lambda w^T w$$

dominate

$$w^T w$$

$$\begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix}^T \begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} w_0 & w_1 & w_2 \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix} = w_0^2 + w_1^2 + w_2^2$$

$$\hat{w} = (P^T P + \lambda I)^{-1} P^T y$$

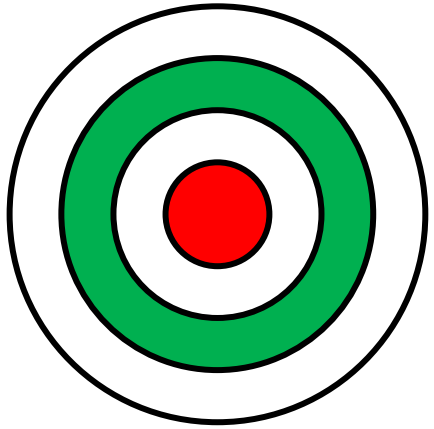
ρ scalar

Bias versus Variance

$\hat{w} = \frac{p}{p^2 + \lambda} y$

$\lambda \to \infty$

- Suppose we are trying to predict red target below:



Bias versus Variance

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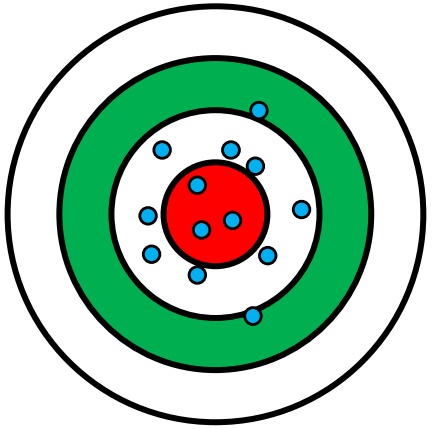
Low Bias

High Variance



Low Bias: blue predictions on average close to red target
High Variance: large variability among predictions

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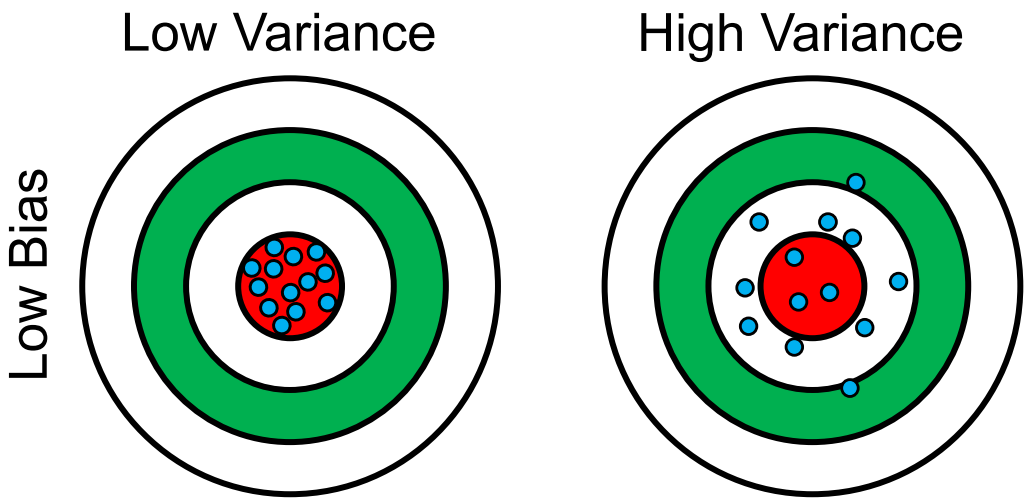

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High Variance

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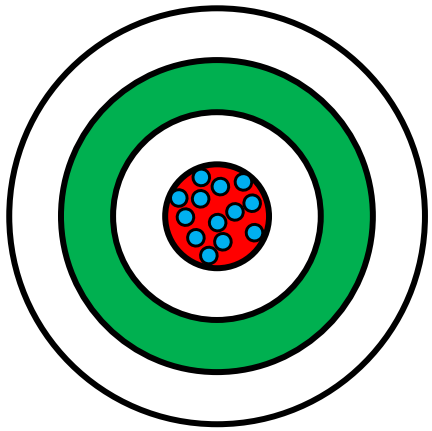

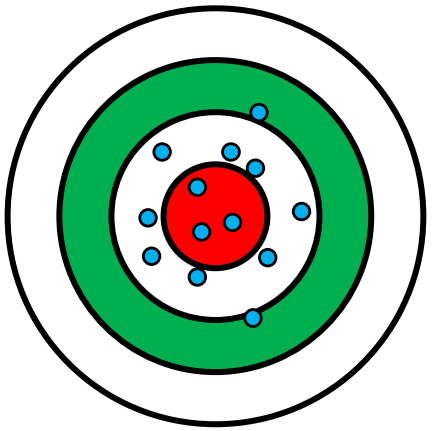

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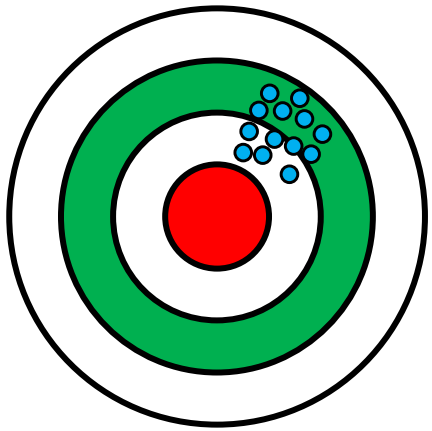
High Bias



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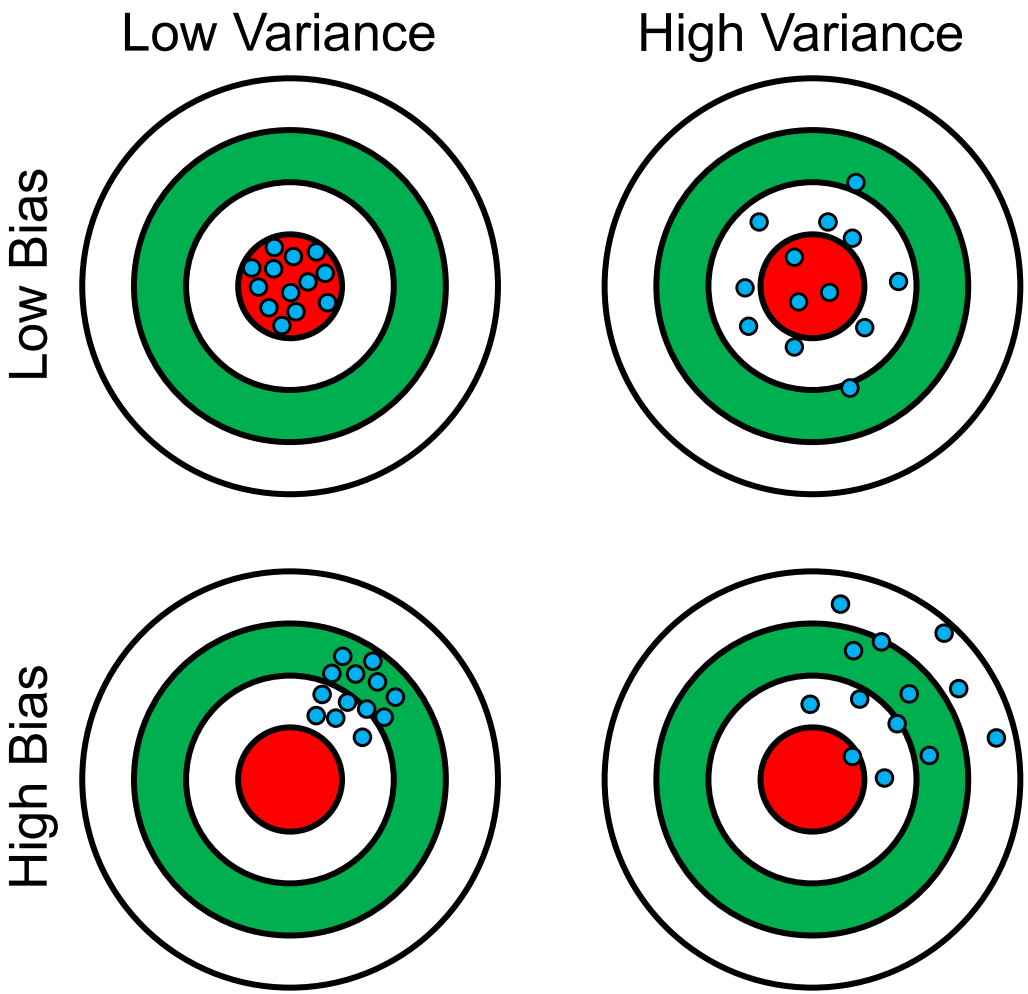

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Bias + Variance Trade off

- Test error = Bias Squared + Variance + Irreducible Noise

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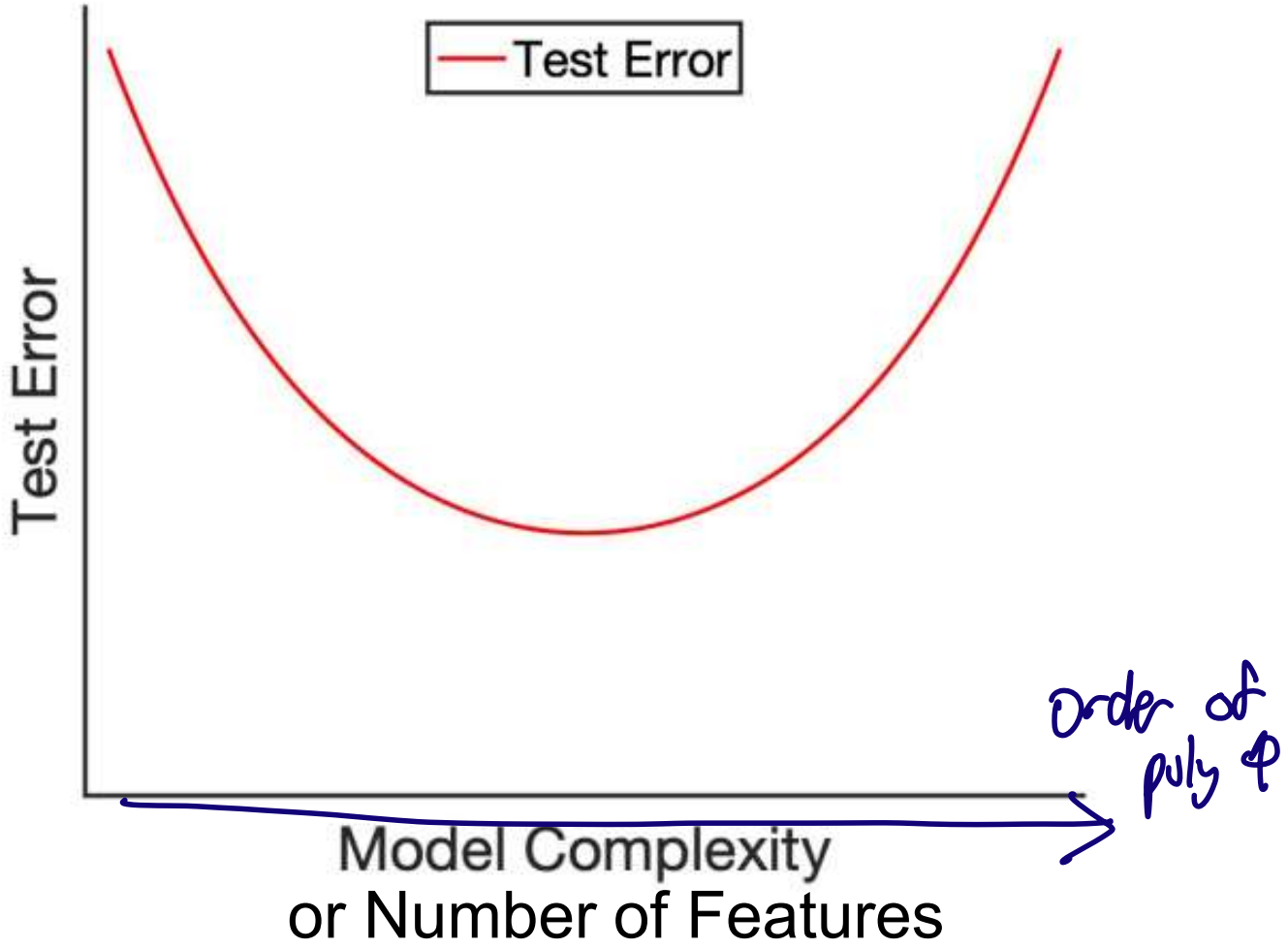

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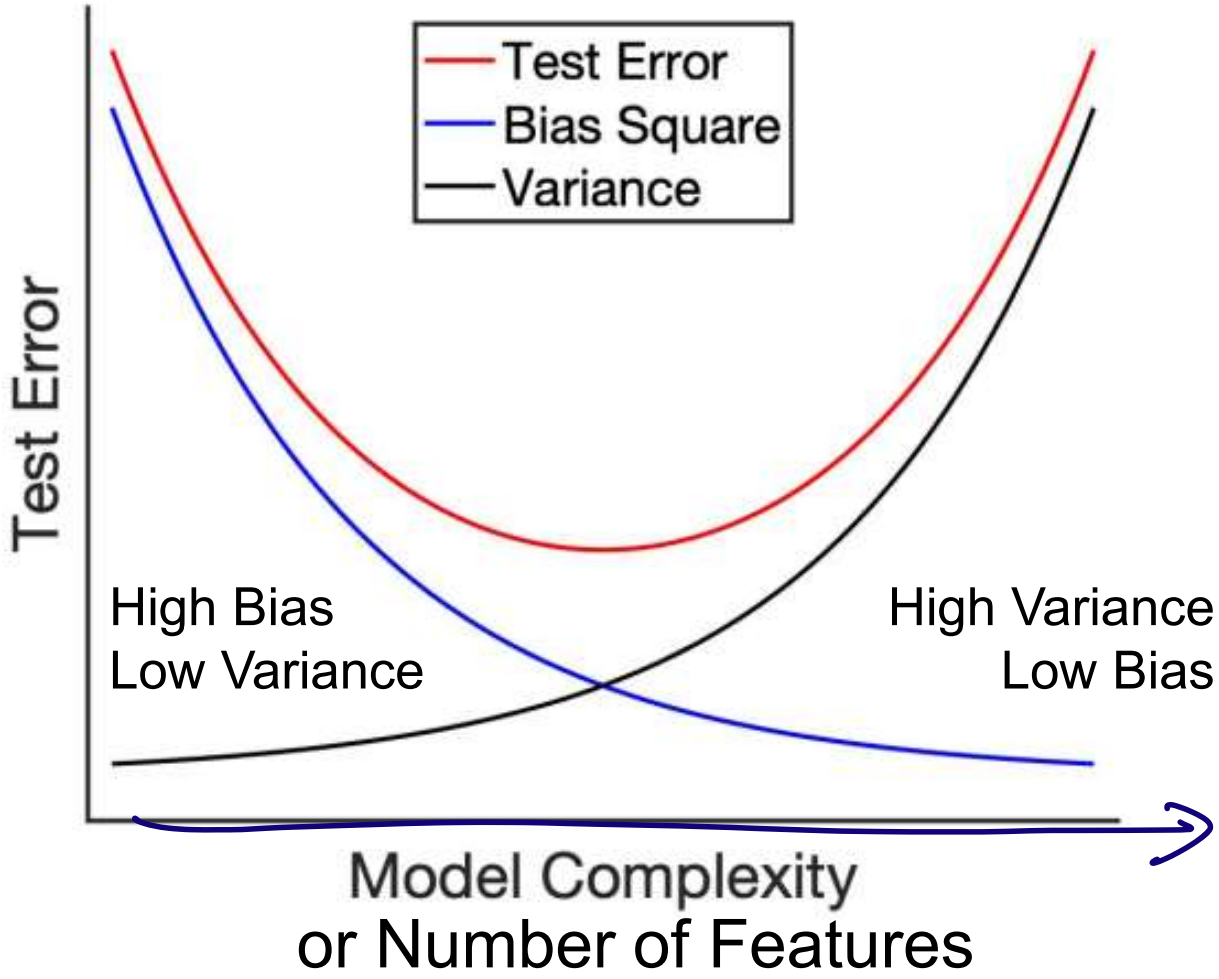
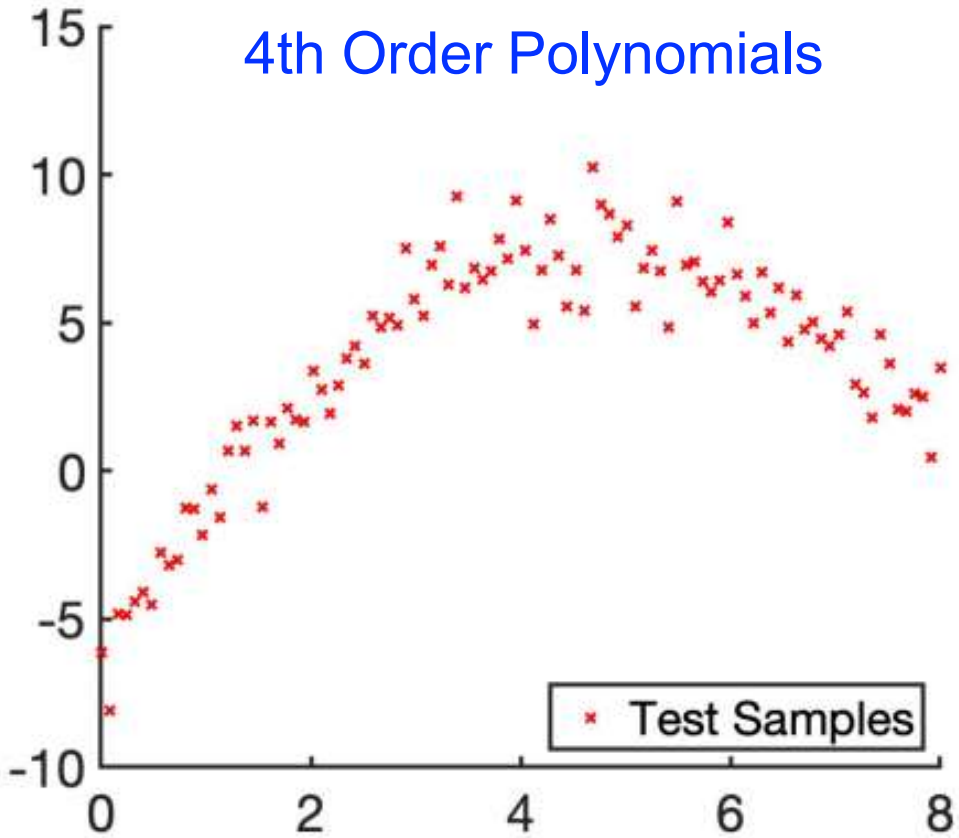
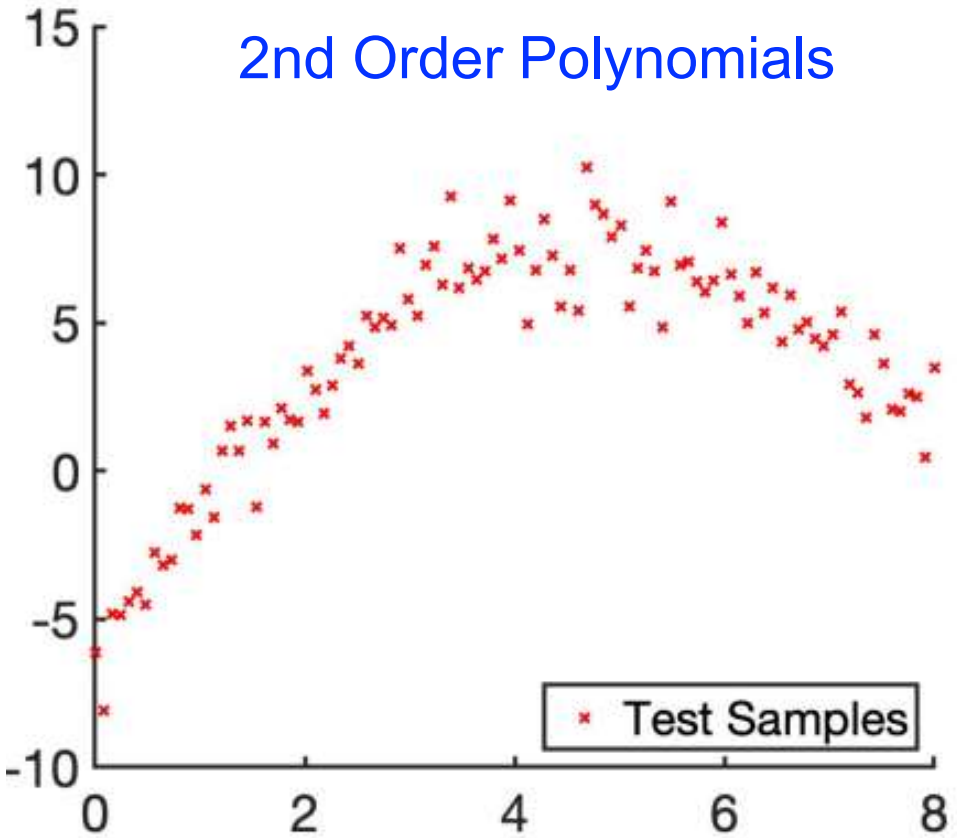
Mean-squared error



Bias + Variance Example

- Simulate data from order 2 polynomial (+ noise)
- Randomly sample 10 training samples each time





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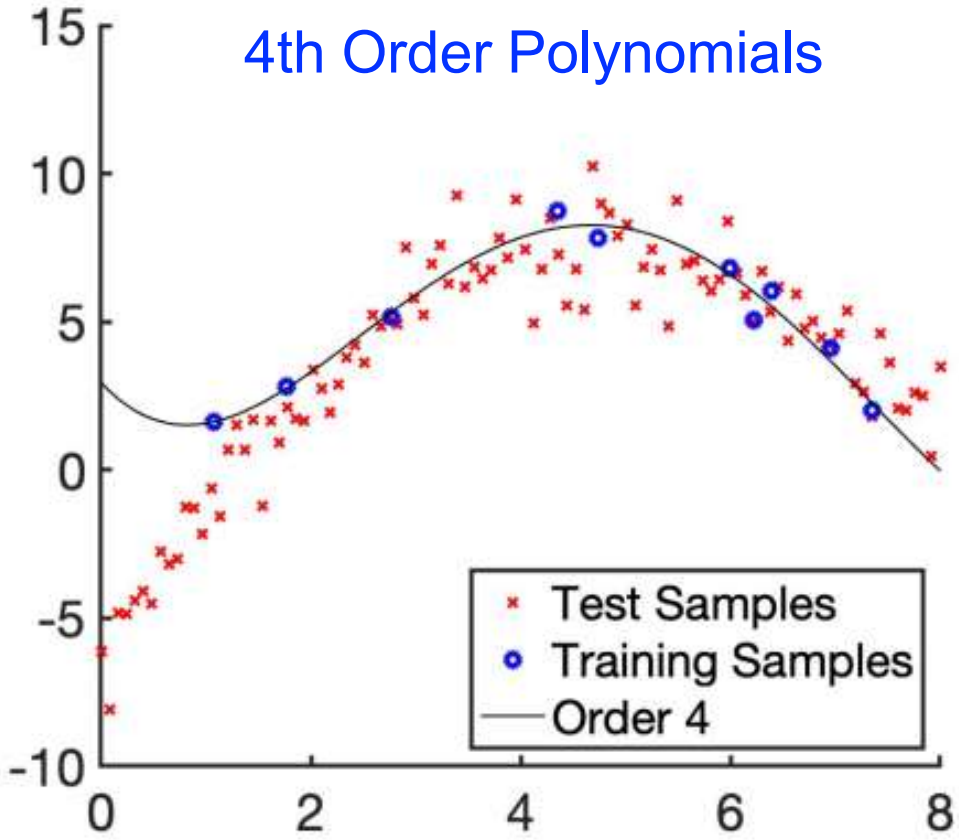

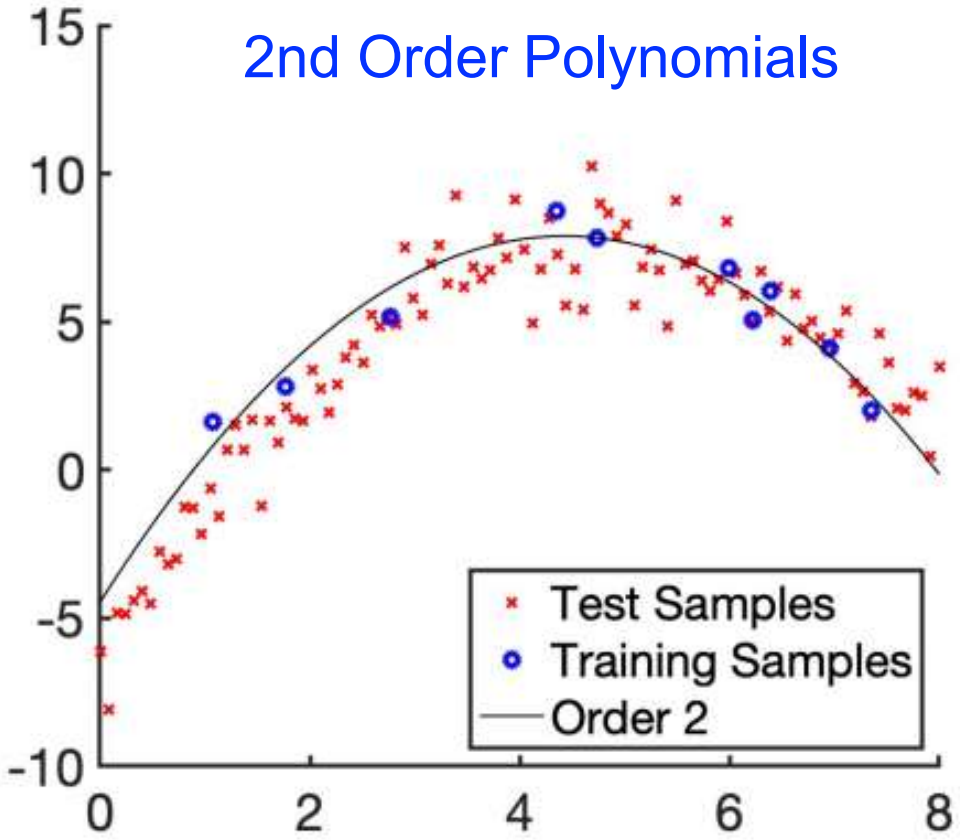

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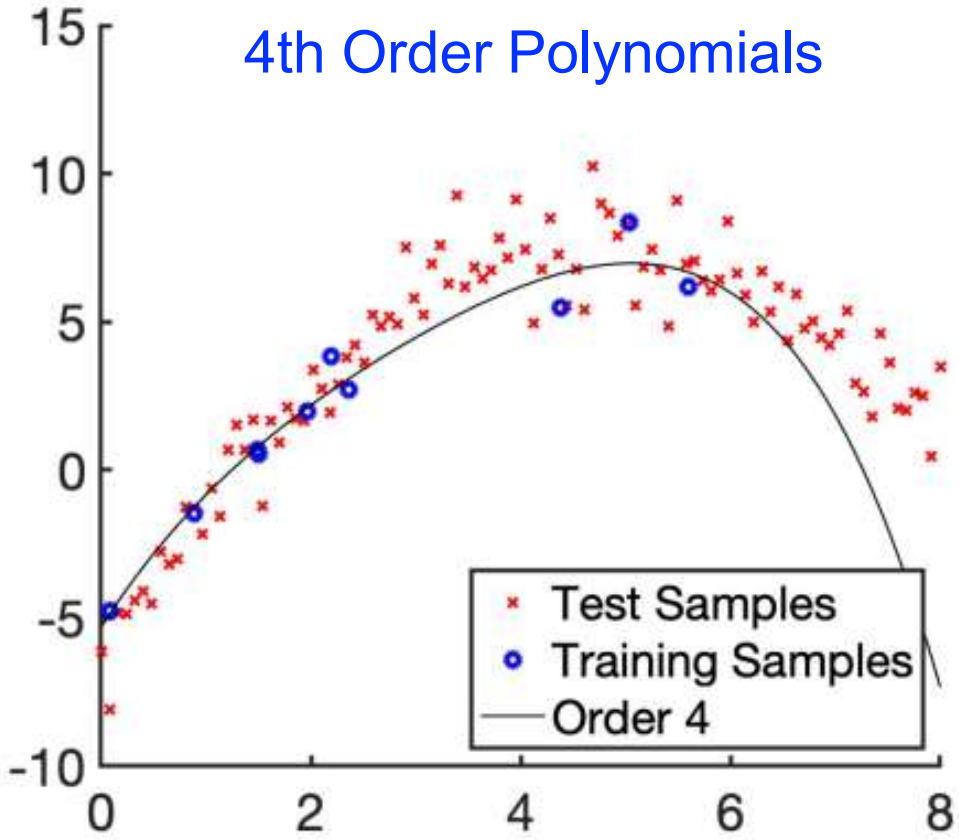

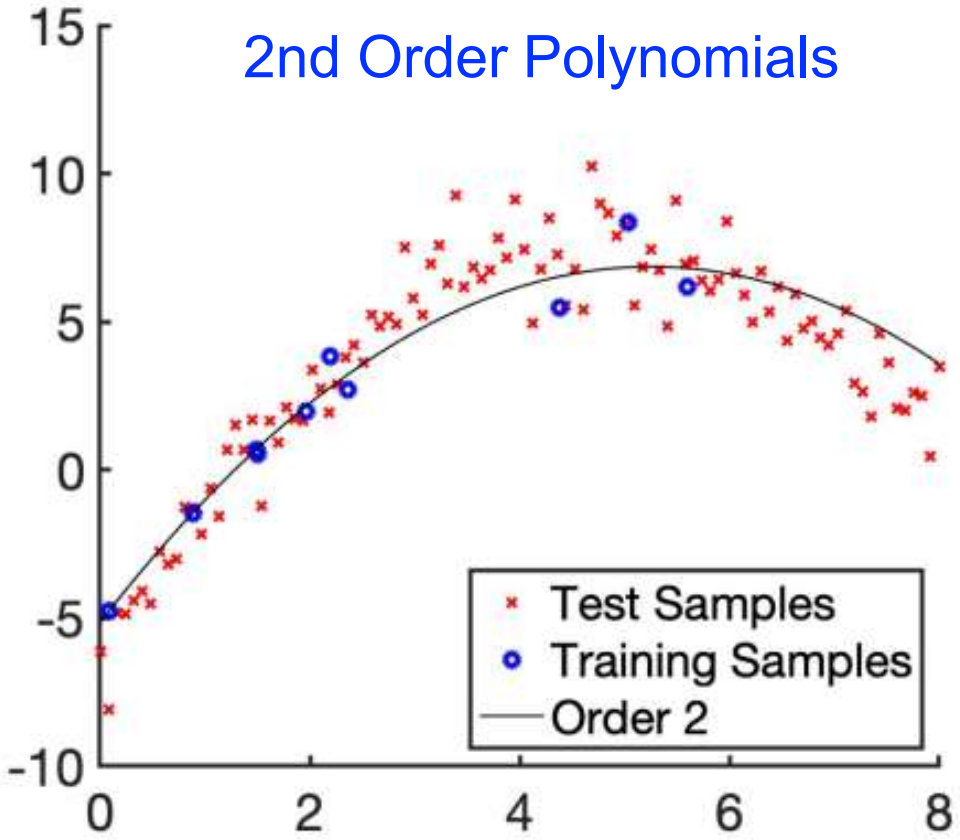

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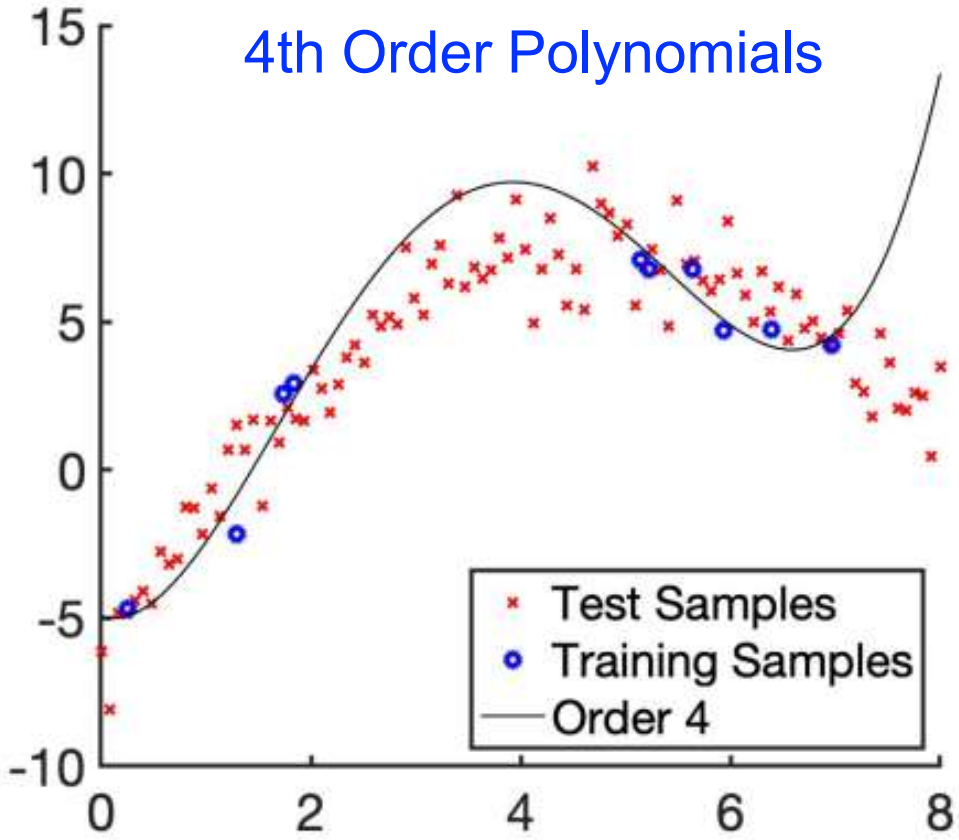

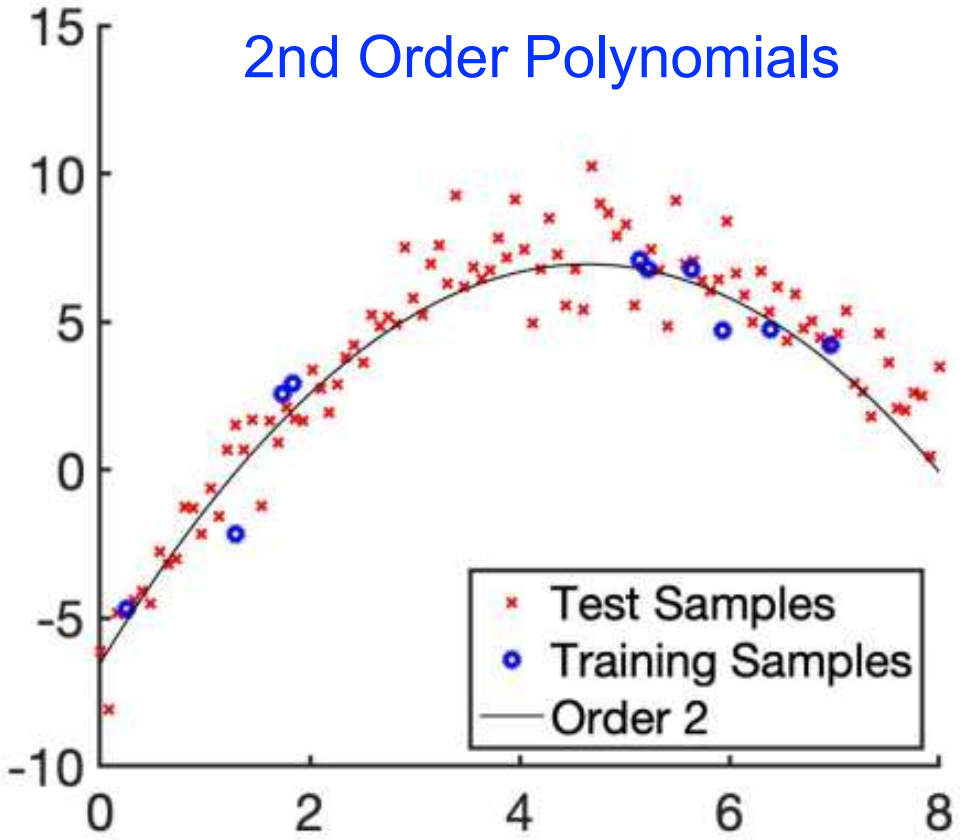

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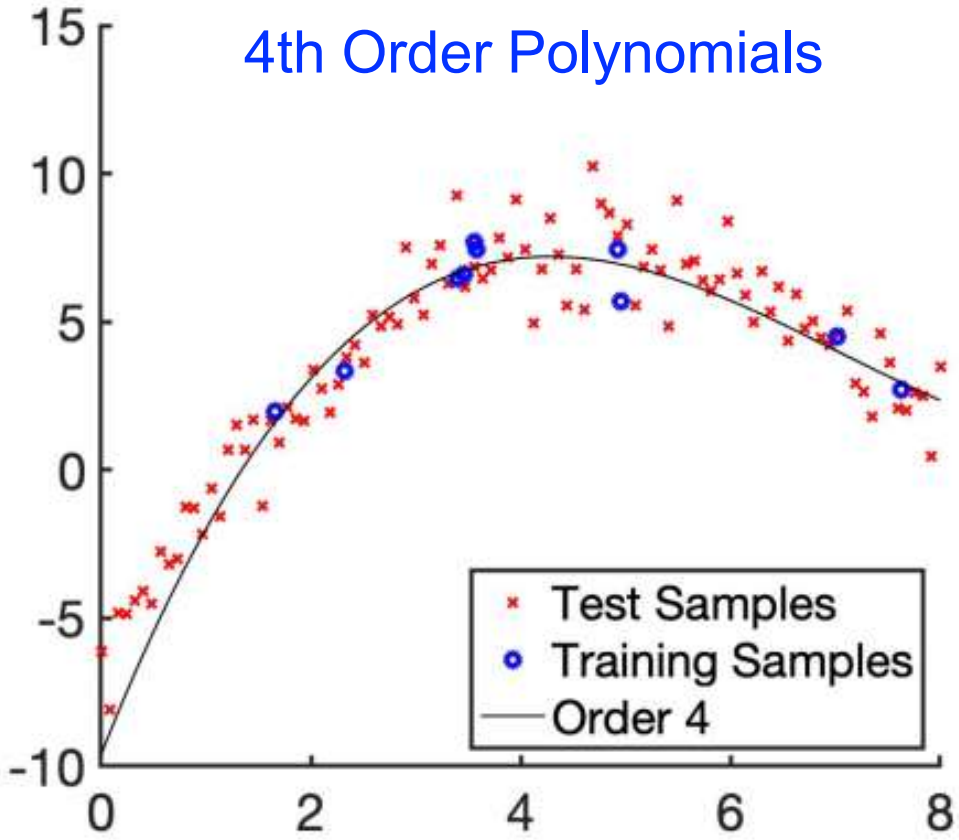

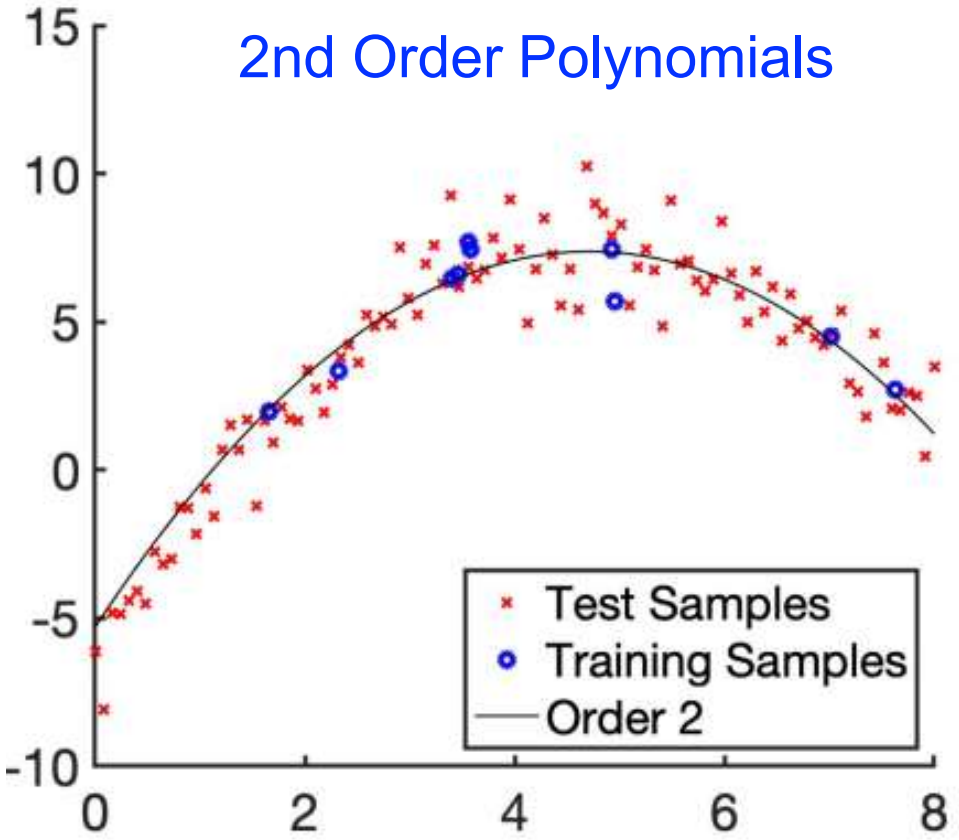

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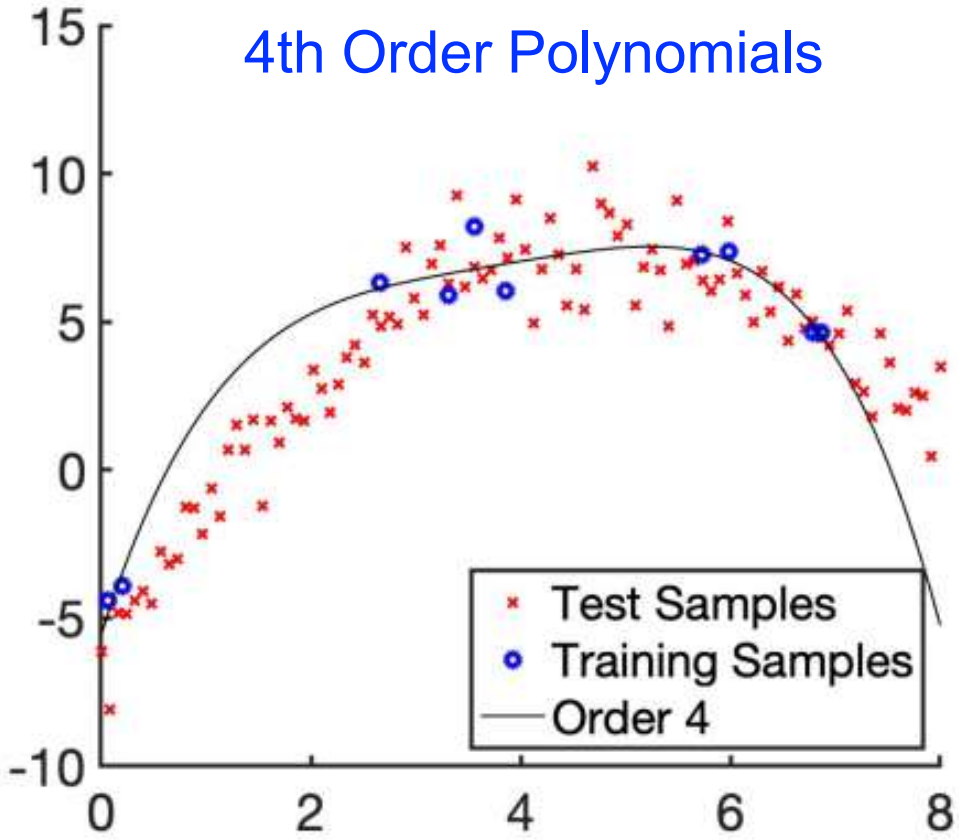

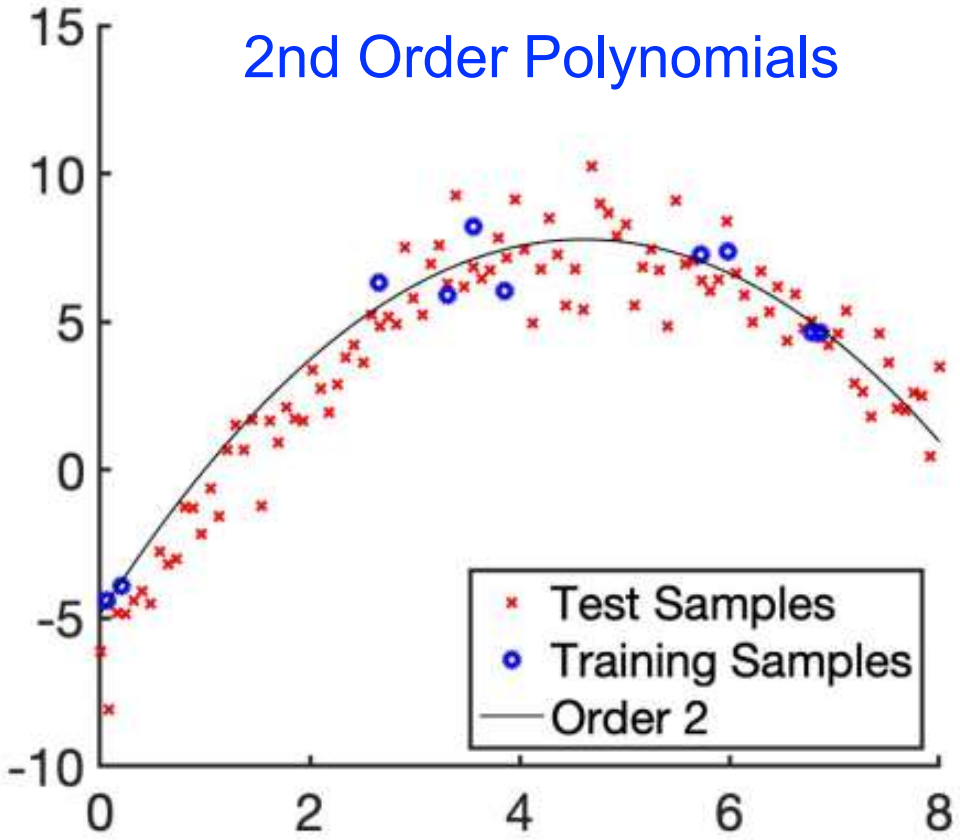

Bias + Variance Example

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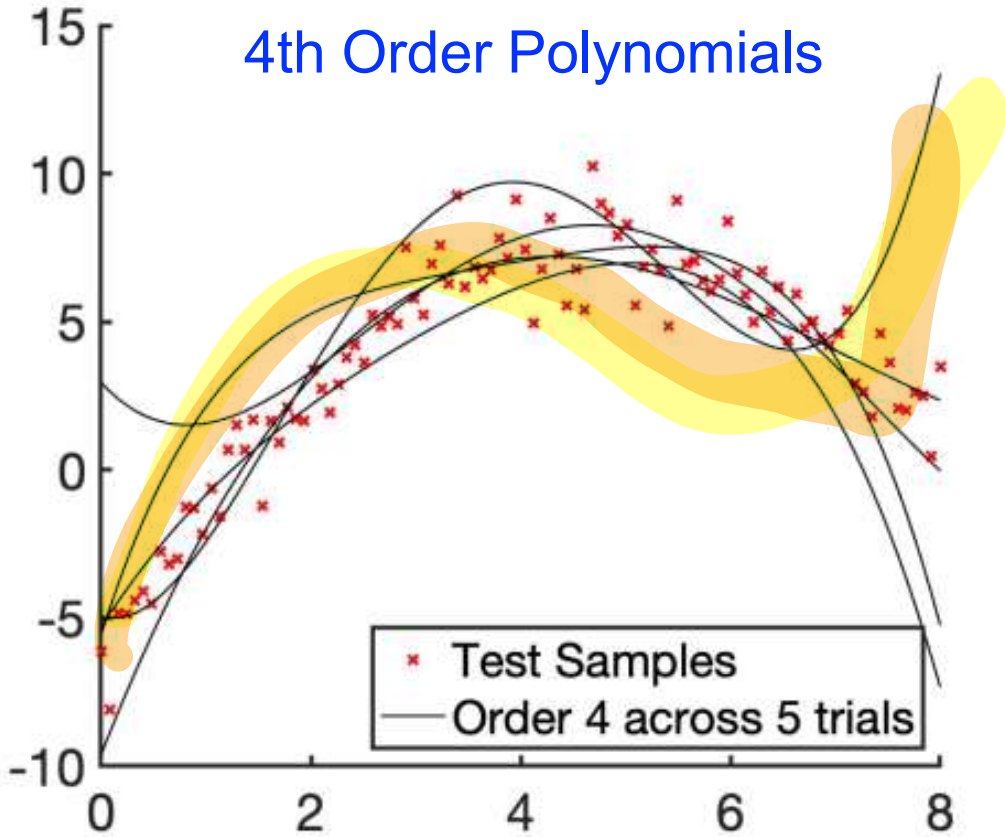
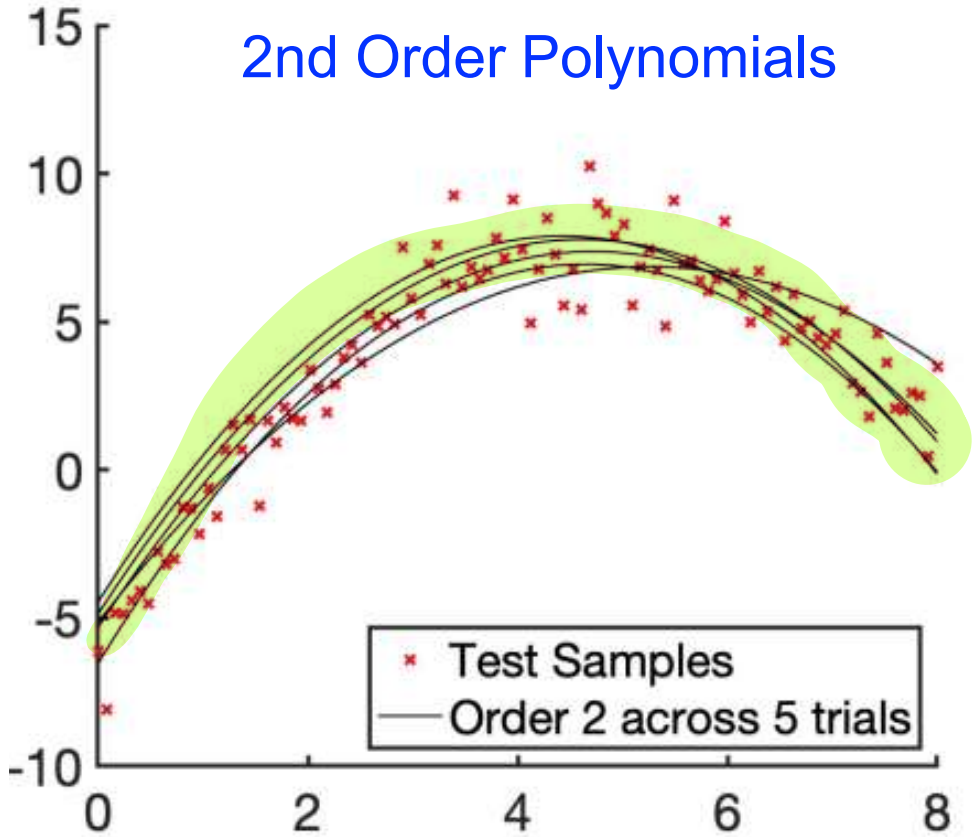
- Fit with order 2 polynomial: low variance
- Fit with order 4 polynomial: high variance





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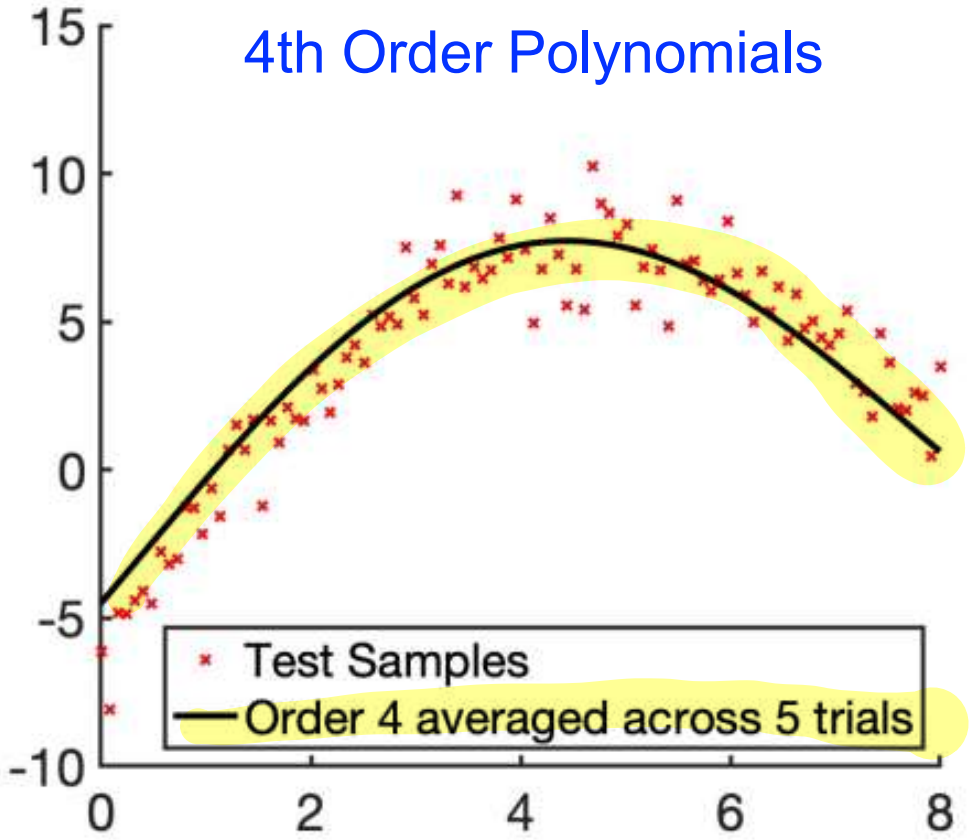

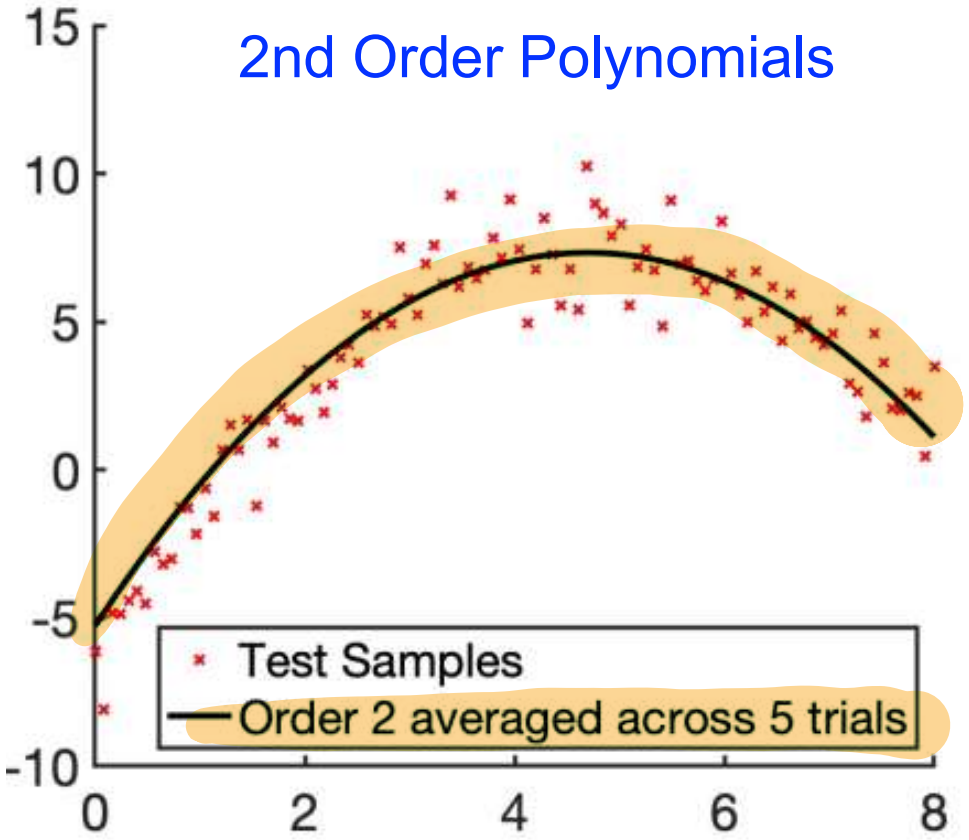

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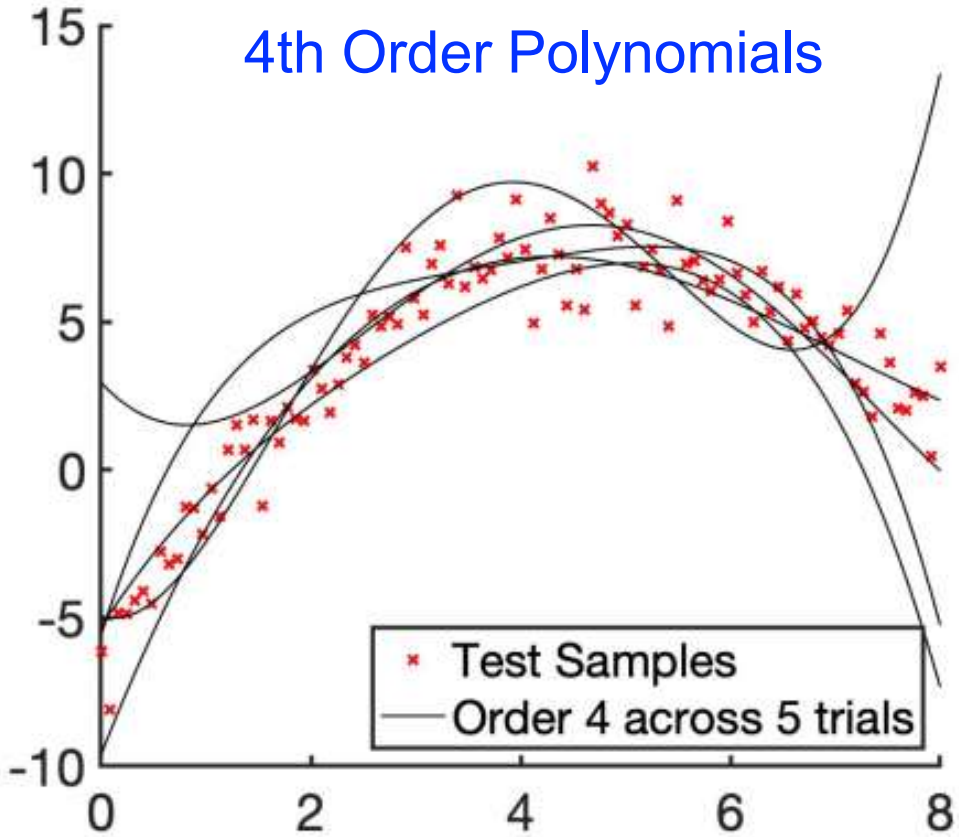
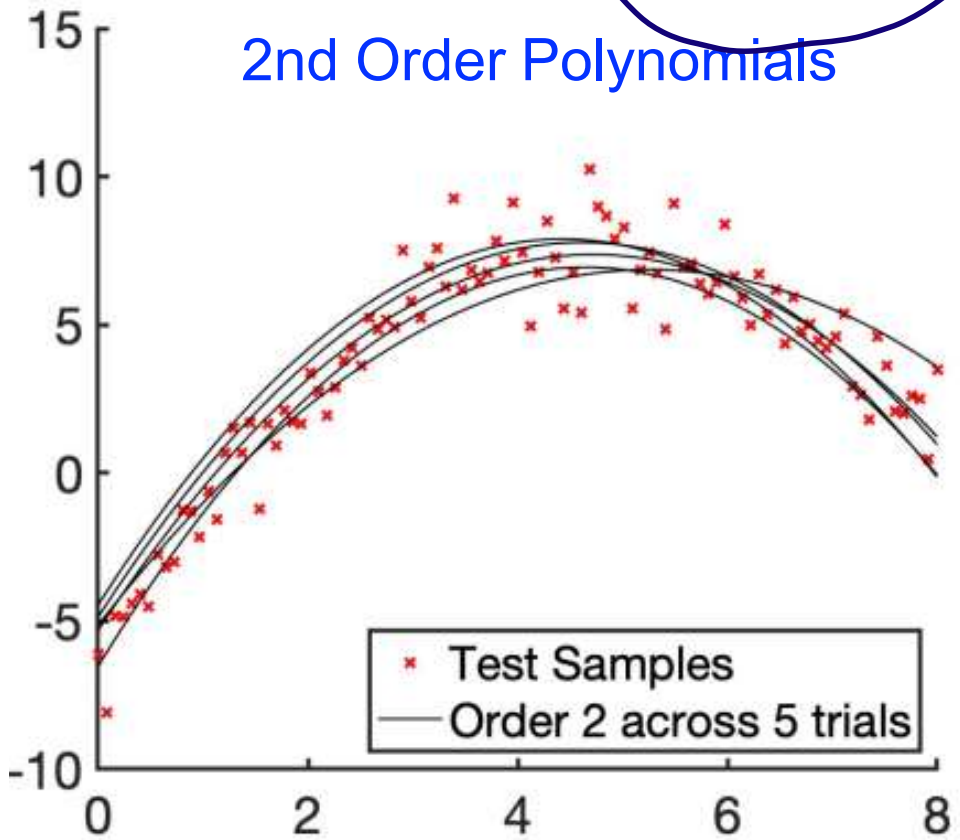
Prefer

Order 2 Achieves Lower Test Error





Bias-Variance Decomposition Theorem

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- E_D is expectation with respect to $p(D)$: think of this as averaging across ∞ trials

Bias-Variance Decomposition Theorem

- Think of E_D as averaging across ∞ trials

- See proof in extra uploaded material (won't be tested)

Bias-Variance Decomposition Theorem

- Think of E_D as averaging across ∞ trials
- According to the Bias-Variance Decomposition Theorem, the mean squared error for a new test sample x is given by

$$E_D\left[\left(y(x) - \hat{f}_D(x)\right)^2\right]$$

$\hat{y}_{test}$

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$$\text{Bias}(\hat{f}) = \hat{f}_{avg}(x) - f(x),$$

and

$$\text{Var}(\hat{f}) = E_D\left[\left(\hat{f}_D(x) - \hat{f}_{avg}(x)\right)^2\right]$$

- See proof in extra uploaded material (won't be tested)

Numerical Example of Bias and Variance Tradeoff I

- We have randomly sampled a training set $D = \{(\mathbf{x}_i, y_i)\}_{i=1}^m$ from a probability distribution $p(D)$.
- Using D, we training a regression model to predict y from $\mathbf{x}$.
- For simplicity, say that $\mathbf{x}$ and $\mathbf{x}_i$ are one-dimensional, so we write them as x and x_i for brevity.
- We repeat this process of sampling the dataset from $p(D)$ a total of 4 times, resulting in 4 trained models

  $$\hat{f}_{D_1}, \hat{f}_{D_2}, \hat{f}_{D_3}, \hat{f}_{D_4}$$

  based on the datasets D_1, D_2, D_3, D_4.

$\mathcal{E} = 0$

- We have a new test sample $x_{\text{new}} = 3$ whose (unknown) target is $y_{\text{new}} = f(x_{\text{new}}) = 7.5$.
- The function $f(x)$ represents the real/true relation between y and x.

Numerical Example of Bias and Variance Tradeoff II

- Suppose now the predictions of x_{new} from $\hat{f}_{D_1}, \hat{f}_{D_2}, \hat{f}_{D_3}, \hat{f}_{D_4}$ are $\hat{y}_{\text{new},1} = 6$, $\hat{y}_{\text{new},2} = 7$, $\hat{y}_{\text{new},3} = 7$, $\hat{y}_{\text{new},4} = 8$ respectively. That is

$$\begin{aligned}
\hat{y}_{\text{new},1} &= \hat{f}_{D_1}(x_{\text{new}}) = \hat{f}_{D_1}(3) = 6 \\
\hat{y}_{\text{new},2} &= \hat{f}_{D_2}(x_{\text{new}}) = \hat{f}_{D_2}(3) = 7 \\
\hat{y}_{\text{new},3} &= \hat{f}_{D_3}(x_{\text{new}}) = \hat{f}_{D_3}(3) = 7 \\
\hat{y}_{\text{new},4} &= \hat{f}_{D_4}(x_{\text{new}}) = \hat{f}_{D_4}(3) = 8
\end{aligned}$$

- What is the bias and variance based on these models?

Numerical Example of Bias and Variance Tradeoff III

- Average prediction

$$\hat{f}_{\text{avg}}(x_{\text{new}}) = \hat{f}_{\text{avg}}(3) = \frac{1}{4}(6+7+7+8) = 7.$$

- Bias

$$\text{Bias} = \hat{f}_{\text{avg}}(x_{\text{new}}) - f(x_{\text{new}}) = \hat{f}_{\text{avg}}(x_{\text{new}}) - y_{\text{new}} = 7 - 7.5 = -\frac{1}{2}.$$

- Variance

7 6,7,7,8

$$\text{Variance} = E_D\left[\left(\hat{f}_{\text{avg}}(x_{\text{new}}) - \hat{f}_D(x_{\text{new}})\right)^2\right]$$

$$= \frac{1}{4}\left((6-7)^2 + (7-7)^2 + (7-7)^2 + (8-7)^2\right)^2 = \frac{1}{2}$$

Numerical Example of Bias and Variance Tradeoff IV

- If there is no irreducible noise, using the bias-variance formula,

$$\text{MSE} = \text{Bias}^2 + \text{Variance} = \left(-\frac{1}{2}\right)^2 + \frac{1}{2} = \frac{3}{4}.$$

- Now, using actual definition of MSE

your predictions = true target

$$\begin{aligned}\text{MSE} &= E_D\left[\left(\hat{f}_D(x_{\text{new}}) - f(x_{\text{new}})\right)^2\right] \\ &= \frac{1}{4}\left((6-7.5)^2 + (7-7.5)^2 + (7-7.5)^2 + (8-7.5)^2\right) \\ &= \frac{3}{4}\end{aligned}$$

- Voila!

Bias-Variance Python Demo

```
import numpy as np
from sklearn.model_selection import train_test_split
from sklearn.linear_model import LinearRegression, Ridge
from sklearn import datasets
```

```
# preparing the dataset into inputs (feature matrix) and outputs (target vector)
data = datasets.load_boston()  # fetch the data
X = data.data  # feature matrix
y = data.target  # target vector

# split the data into training and test samples
X_train, X_test, y_train, y_test = train_test_split(X, y, test_size=0.3)
```

- Here, we are loading the Boston Housing dataset
- We split the raw dataset into 70% training and 30% testing in the last line
- Use .shape to check the dimensions of the various matrices!
- X_train.shape should give (354,13)

Bias-Variance Python Demo

```python
def draw_bootstrap_sample(rng, X, y):
    sample_indices = np.arange(X.shape[0])
    bootstrap_indices = rng.choice(
        sample_indices, size=sample_indices.shape[0], replace=True
    )
    return X[bootstrap_indices], y[bootstrap_indices]
```

- Here, we are drawing various datasets from the given training set
- We are using a technique known as bootstrapping
- **You don't need to know this!**
- This simulates the effect of …

$$\text{Think of } E_D \text{ as averaging across } \infty \text{ trials}$$

Bias-Variance Python Demo

```python
def bias_variance_decomp(estimator, X_train, y_train, X_test, y_test, num_rounds=200, random_seed=20):
    rng = np.random.RandomState(random_seed)

    all_pred = []
    for i in range(num_rounds):
        # do bootstrap sampling, i.e., sampling with replacement
        X_boot, y_boot = draw_bootstrap_sample(rng, X_train, y_train)

        # fit a model on bootstrap samples and make prediction on test samples
        pred = estimator.fit(X_boot, y_boot).predict(X_test)
        all_pred.append(pred)

    all_pred = np.array(all_pred)

    # calculate MSE
    avg_mse = ((all_pred - y_test[None,:])**2).mean() # y_test[None,:] will reshape y_test from (N,) to (1,N)

    # average prediction of all bootstrap models on test set
    avg_predictions = np.mean(all_pred, axis=0)

    # calculate bias squared
    avg_bias = np.sum((avg_predictions - y_test) ** 2) / y_test.size

    # calculate variance
    avg_var = np.sum((avg_predictions - all_pred) ** 2) / all_pred.size

    return avg_mse, avg_bias, avg_var
```

$$E_D\left[\left(y(x) - \hat{f}_D(x)\right)^2\right]$$

$$\text{Bias}(\hat{f}) = \hat{f}_{avg}(x) - f(x),$$

$$\text{Var}(\hat{f}) = E_D\left[\left(\hat{f}_D(x) - \hat{f}_{avg}(x)\right)^2\right]$$

- The different samples of the training dataset D are collected in all_pred
- The mean-squared error is calculated in avg_mse
- The bias squared is calculated in avg_bias
- The variance is calculated in avg_var

Bias-Variance Python Demo

```
# define the model
model = LinearRegression()

# estimating the bias and variance
avg_mse, avg_bias, avg_var = bias_variance_decomp(model, X_train,
                                                  y_train, X_test,
                                                  y_test,
                                                  num_rounds=500,
                                                  random_seed=0)


# summary of the results
print('Average mean-squared error: %.3f' % avg_mse)
print('Average bias: %.3f' % avg_bias)
print('Average variance: %.3f' % avg_var)
```

```
Average mean-squared error: 27.360
Average bias: 26.064
Average variance: 1.296
```

- Here, we train a linear regression model.
- The MSE is indeed the squared bias plus the variance. Proof by python!
- There is no residual noise in this model.

Bias-Variance Python Demo

```
# define the model
model = Ridge(alpha=1)

# estimating the bias and variance
avg_mse, avg_bias, avg_var = bias_variance_decomp(model, X_train,
                                                  y_train, X_test,
                                                  y_test,
                                                  num_rounds=500,
                                                  random_seed=0)


# summary of the results
print('Average mean-squared error: %.3f' % avg_mse)
print('Average bias: %.3f' % avg_bias)
print('Average variance: %.3f' % avg_var)
model.coef_
```

```
Average mean-squared error: 27.375
Average bias: 26.133
Average variance: 1.242
```

- Here, we train a ridge regression model with lambda = 1.0 (it is alpha in python)
- Bias goes up, variance goes down as expected.
- MSE may go up or down.
- Try experimenting with different alpha's (or lambda's)!