Mathematics

in Science, Engineering & Programming

SECTION

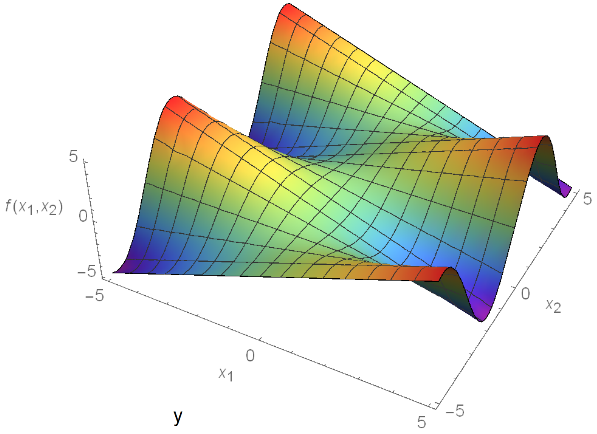
What you (don’t) know

What you know & What is new

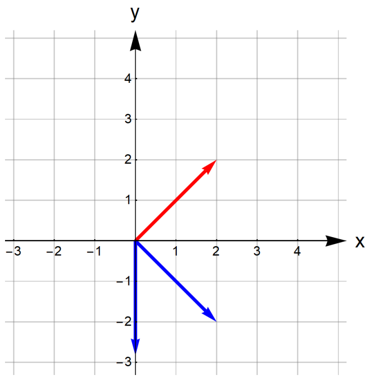
“Pilot section” – What you know from school is important but there is still much to learn

What you know

- Functions



- Exponential function



- Vectors

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \ldots$$

What is new

- Multidimensional functions
- Factorization
- Complex numbers

$$i^2 = -1 \qquad \sqrt{-1} = i$$

- Exponential function as a series
- Vectors in different coordinate systems
- Matrices

$$\underline{\underline{R}}_\alpha \mathbf{a} = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \mathbf{a}$$

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SECTION

What you (don't) know

LECTURE

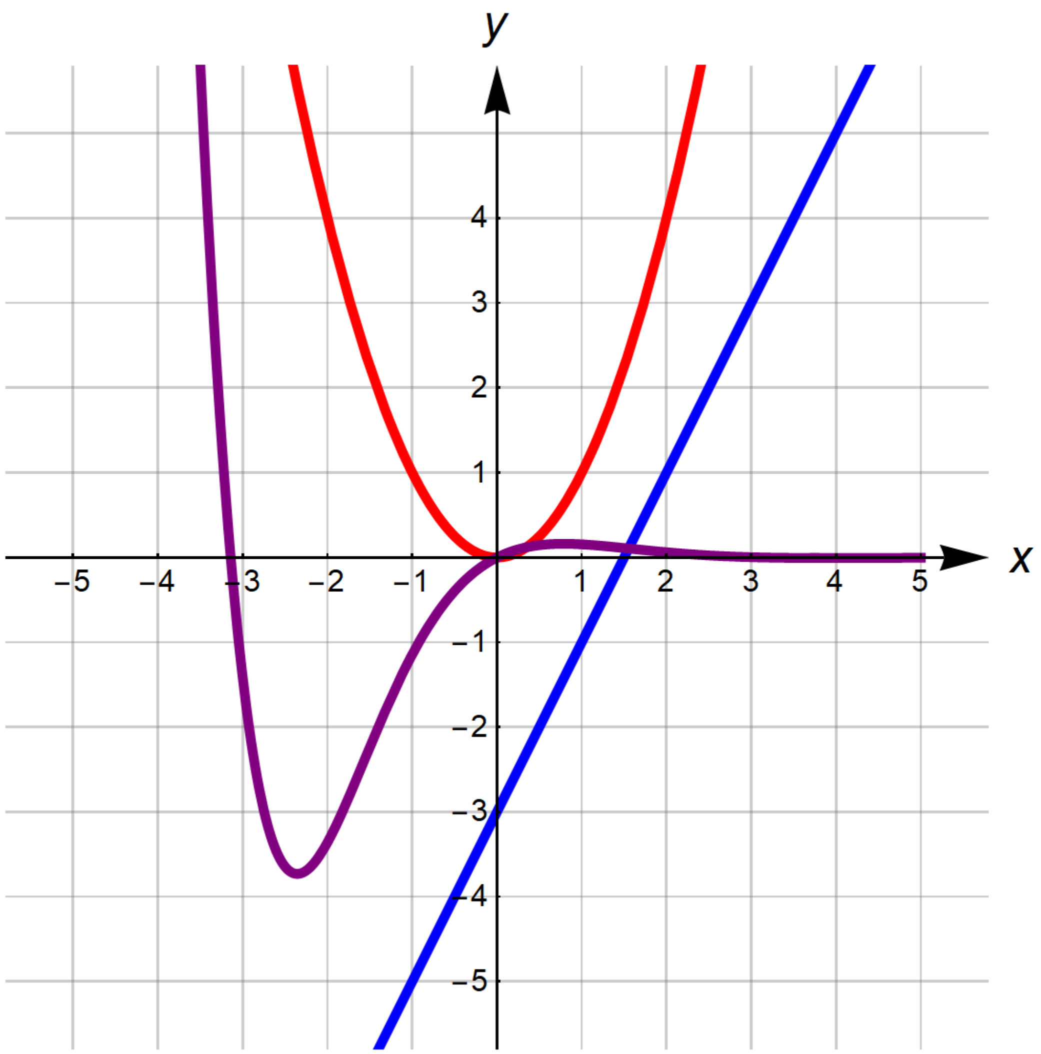
Functions

Functions

$$y = x^2$$

$$y = 2x - 3$$

$$y = \sin(x)e^{-x}$$



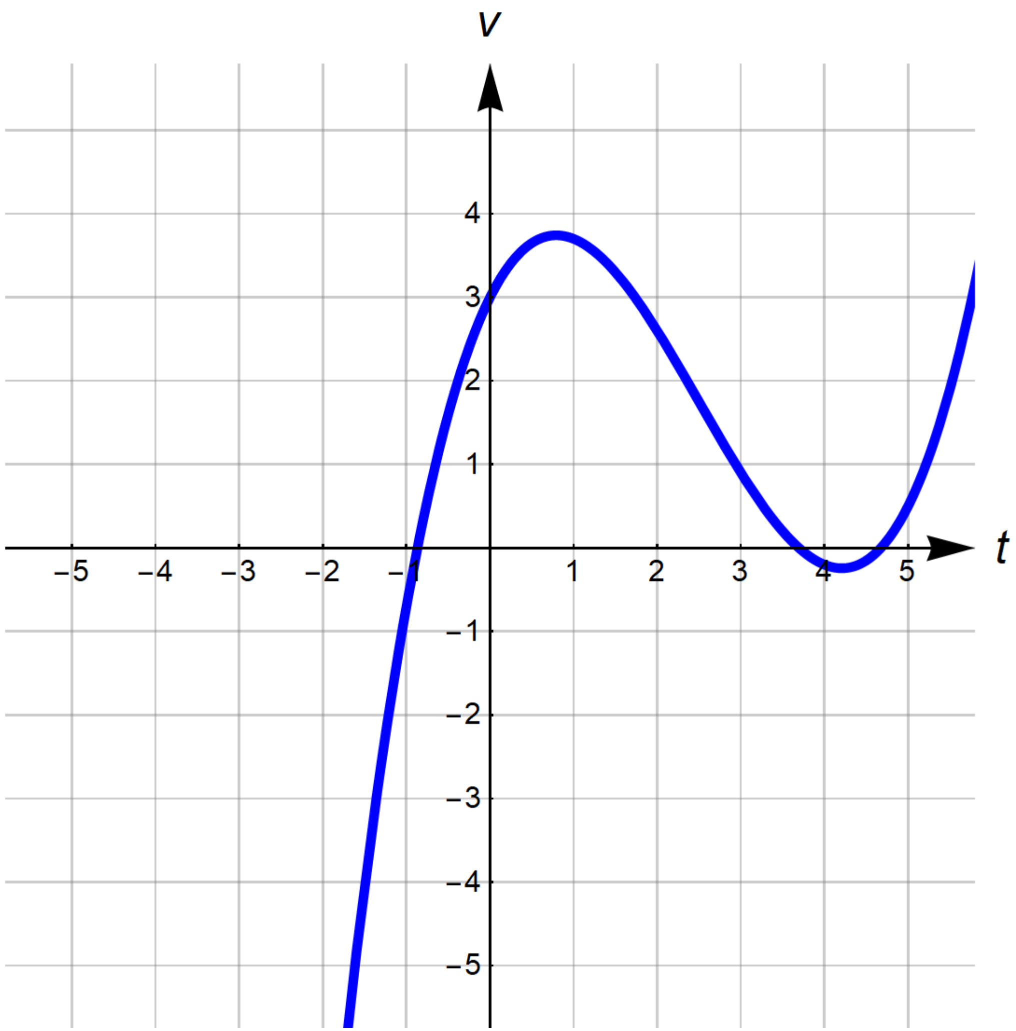
Functions

$$y = x^2$$

$$y = 2x - 3$$

$$y = \sin(x)e^{-x}$$

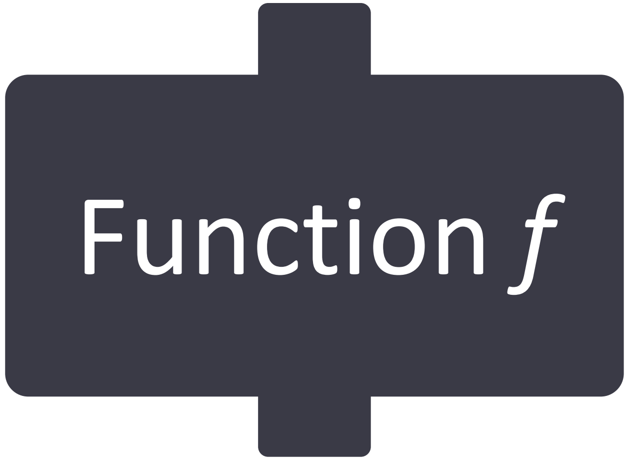
$$v(t) = \frac{1}{5}t^3 - \frac{3}{2}t^2 + 2t + 3$$



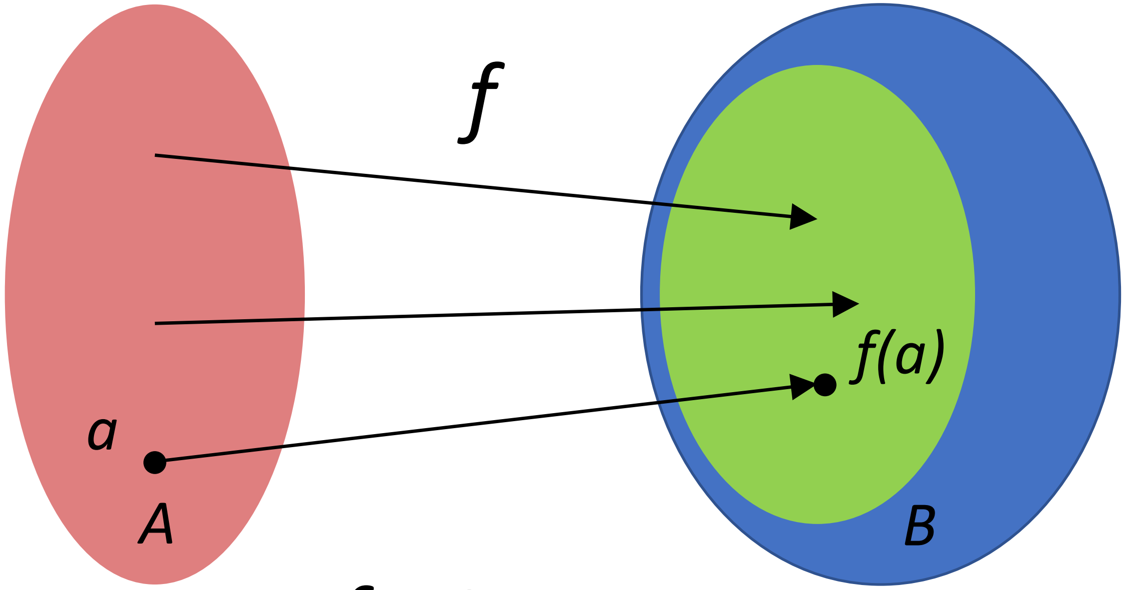
Function

$\cancel{y = \ldots x}$

Input a



Output $f(a)$



$$f : A \to B$$

A ... domain

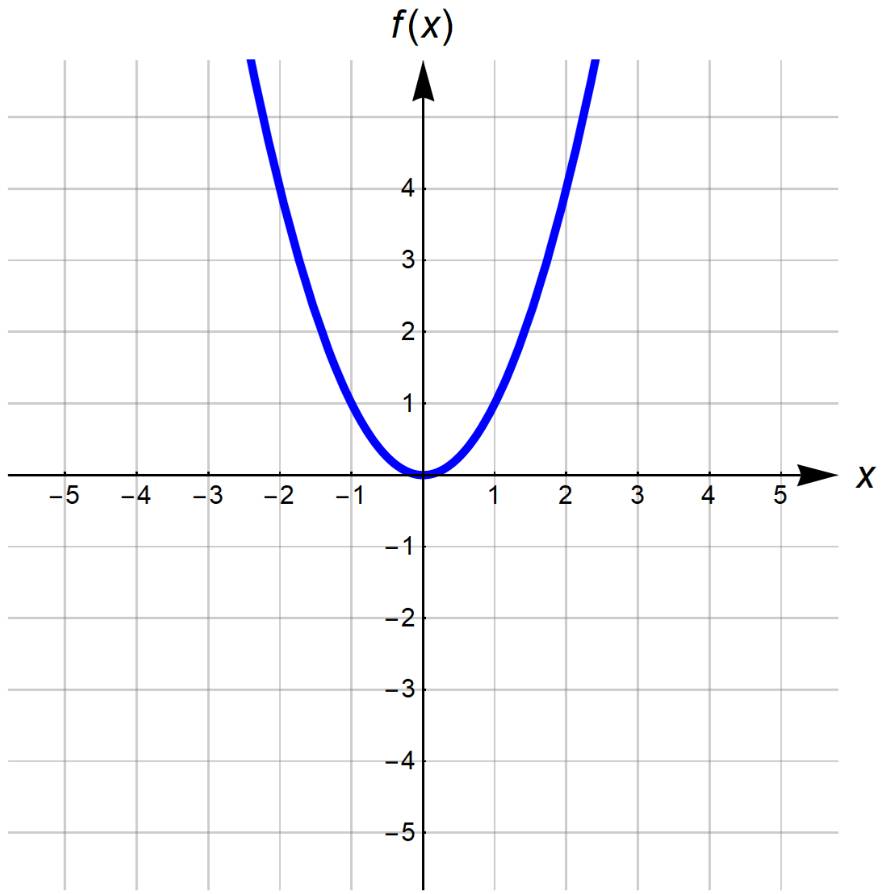
a ... argument

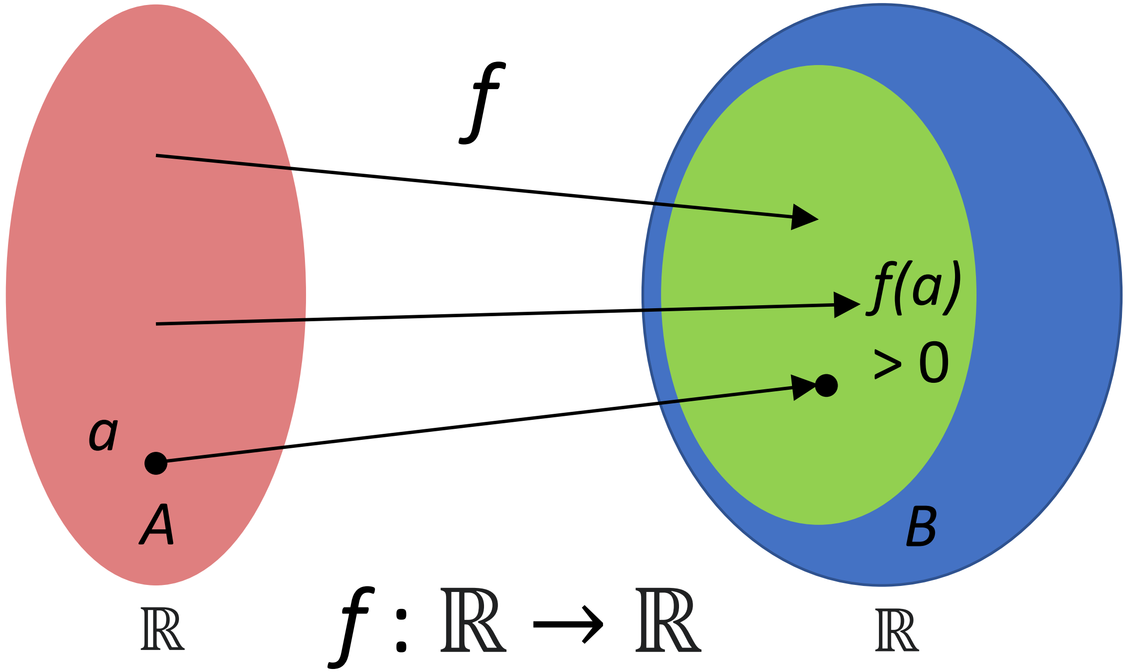
B ... codomain

$f(a)$ ... value

Function

$f(x) = x^2$





$f : \mathbb{R} \to \mathbb{R}$

A ... domain $\mathbb{R}$

B ... codomain $\mathbb{R}$

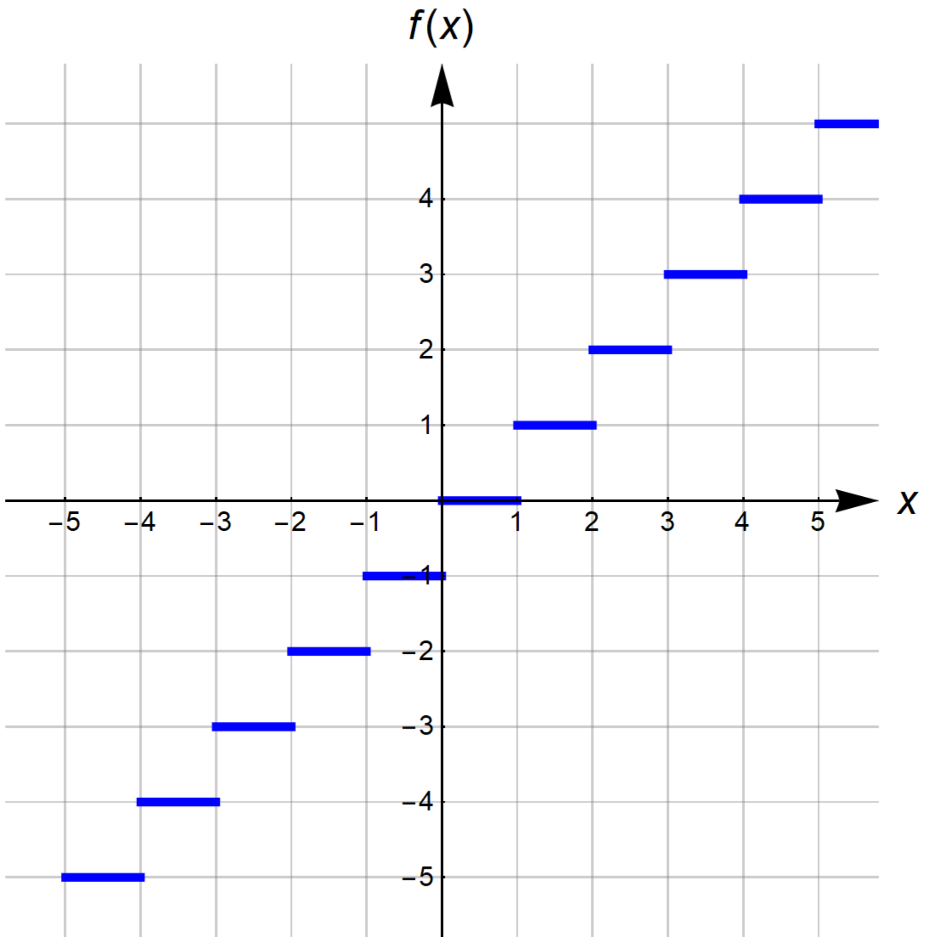
a ... argument

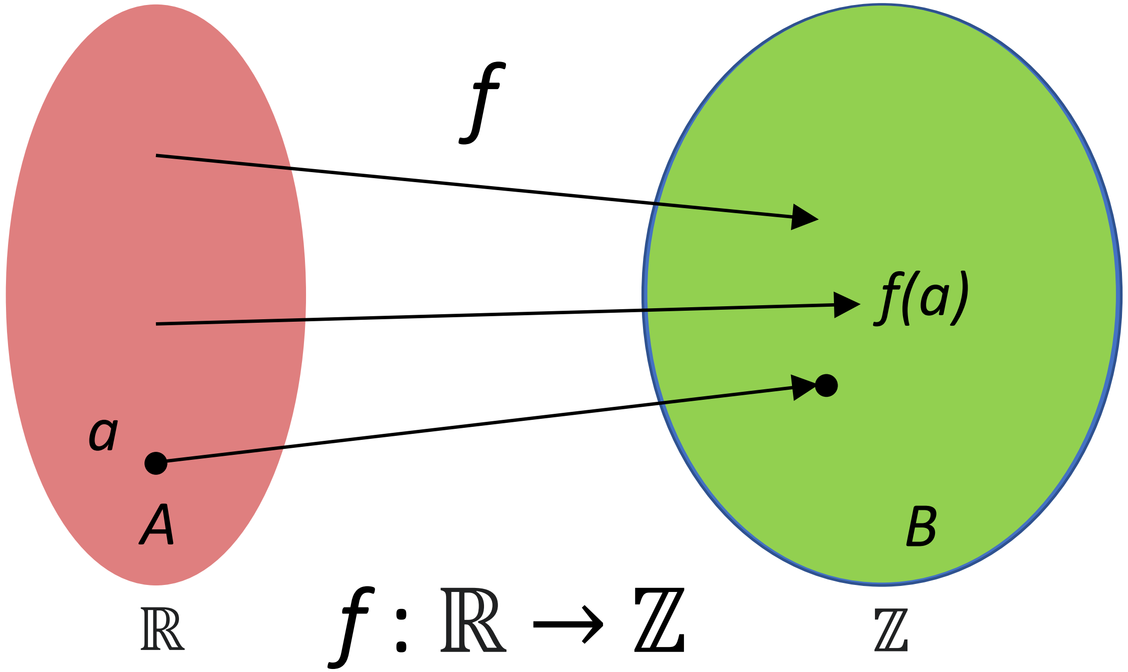
$f(a)$ ... value

Function

$$f(x) = \lfloor x \rfloor$$

Floor function





$$f : \mathbb{R} \to \mathbb{Z}$$

A ... domain $\mathbb{R}$

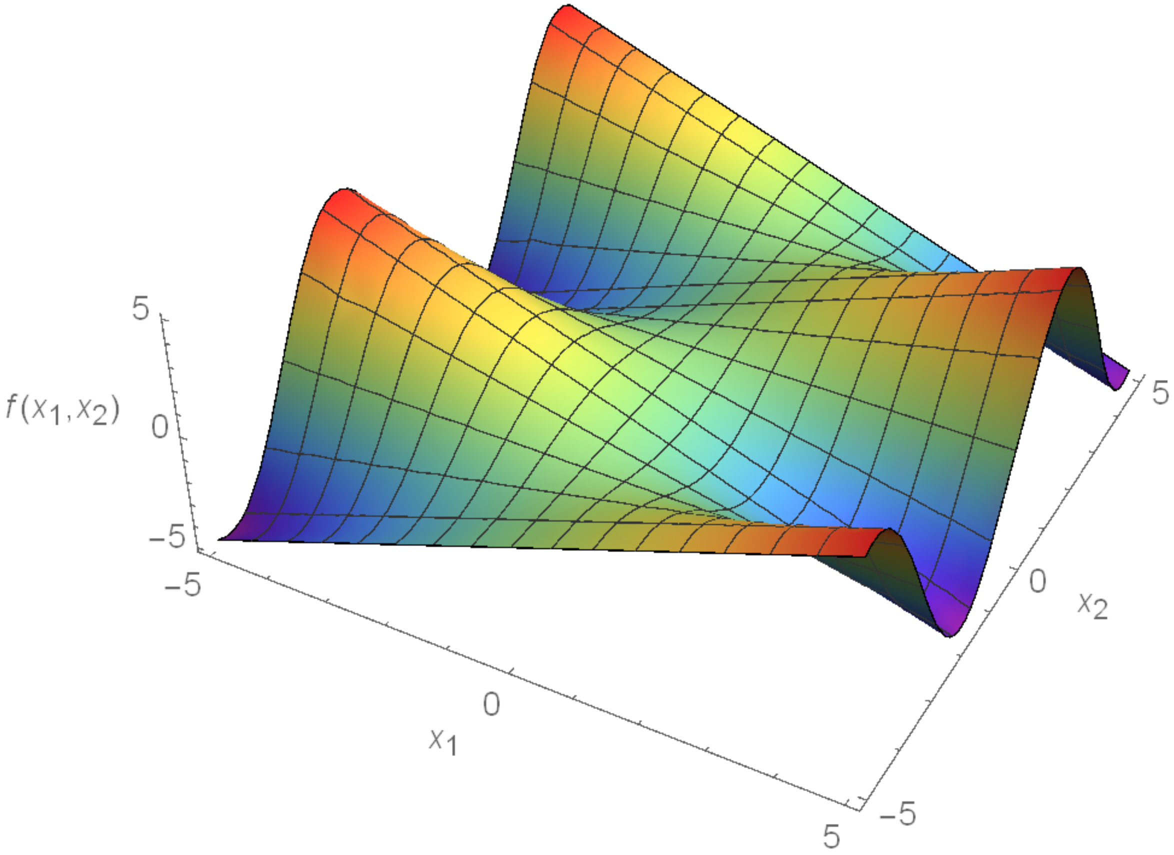
B ... codomain $\mathbb{Z}$

a ... argument

f(a) ... value

Outlook

$$f(x_1, x_2) = x_1 \sin(x_2)$$



- Functions can have multiple arguments
- Important for derivatives $f'(x) = \dfrac{\mathrm{d}f(x)}{\mathrm{d}x}$

What does $f'(x_1, x_2)$ mean?

$$\frac{\partial f(x_1, x_2)}{\partial x_1} \qquad \frac{\partial f(x_1, x_2)}{\partial x_2}$$

Take-home message $\cancel{y = \ldots x}$

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SECTION

What you (don’t) know

LECTURE

Linear functions

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Linear functions

$$y = mx + n$$

Intercept

Slope

$$m = \frac{\Delta y}{\Delta x}$$

- Determine function from two points

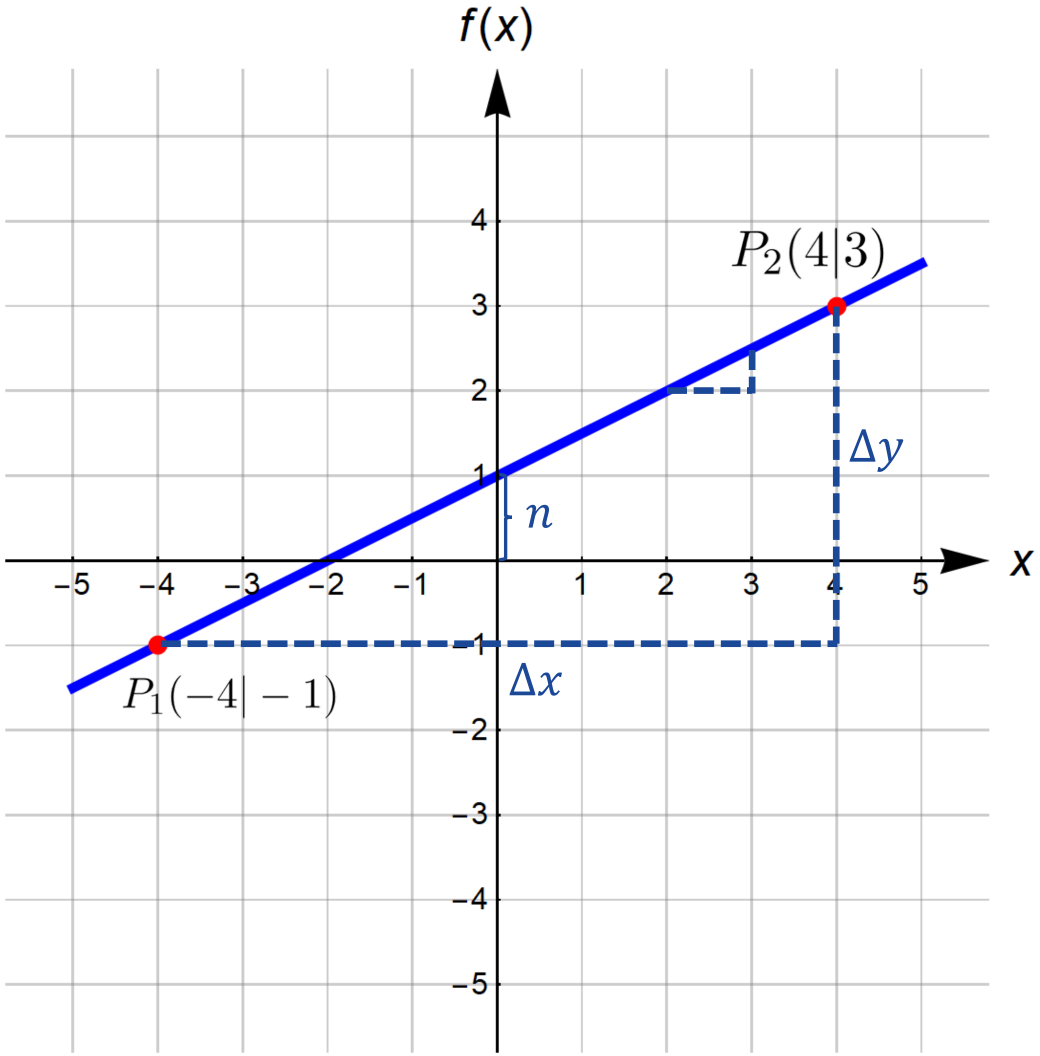
$$m = \frac{3-(-1)}{4-(-4)} = \frac{1}{2}$$

$$y = \frac{1}{2}x + n$$

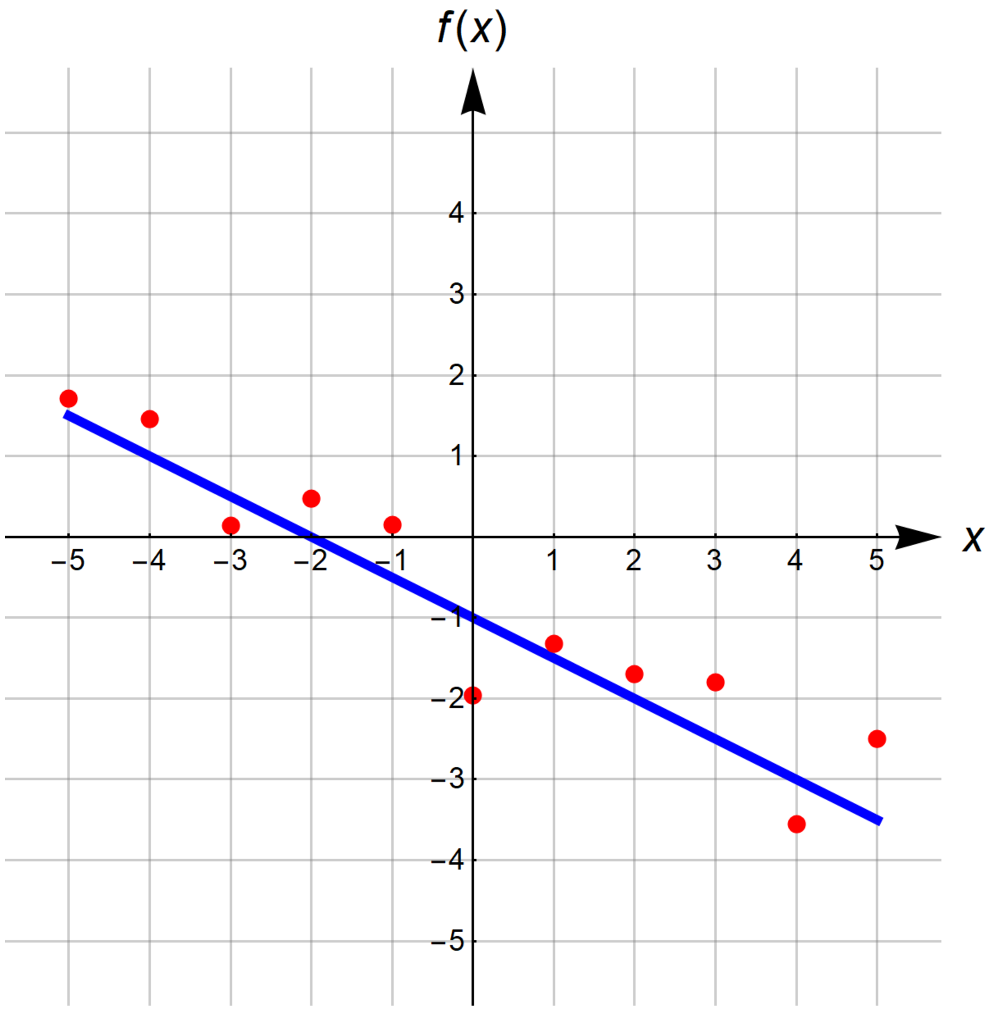
$$n = y - \frac{1}{2}x$$

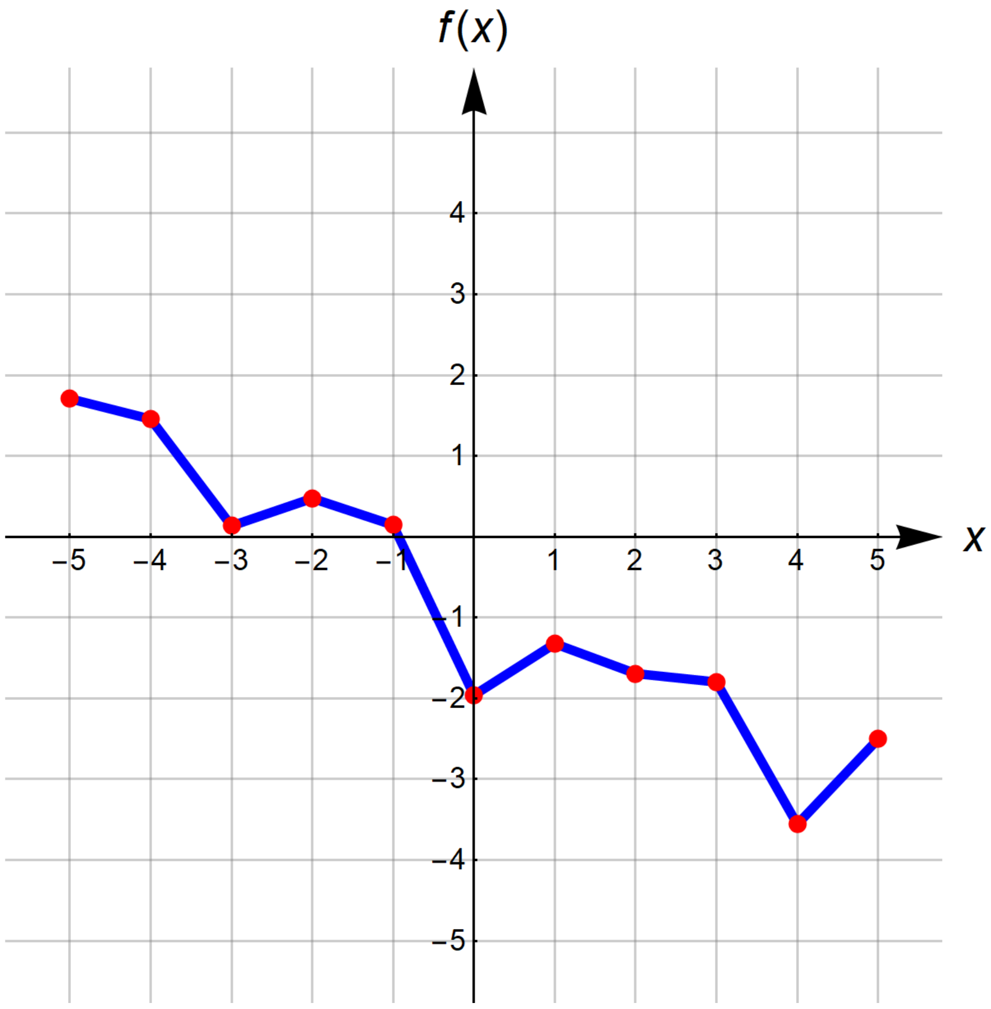
$$n = 3 - \frac{1}{2}4 = 1$$

$$y = \frac{1}{2}x + 1$$



Linear functions





Linear functions

Intercept

$$y = mx + n$$

Slope

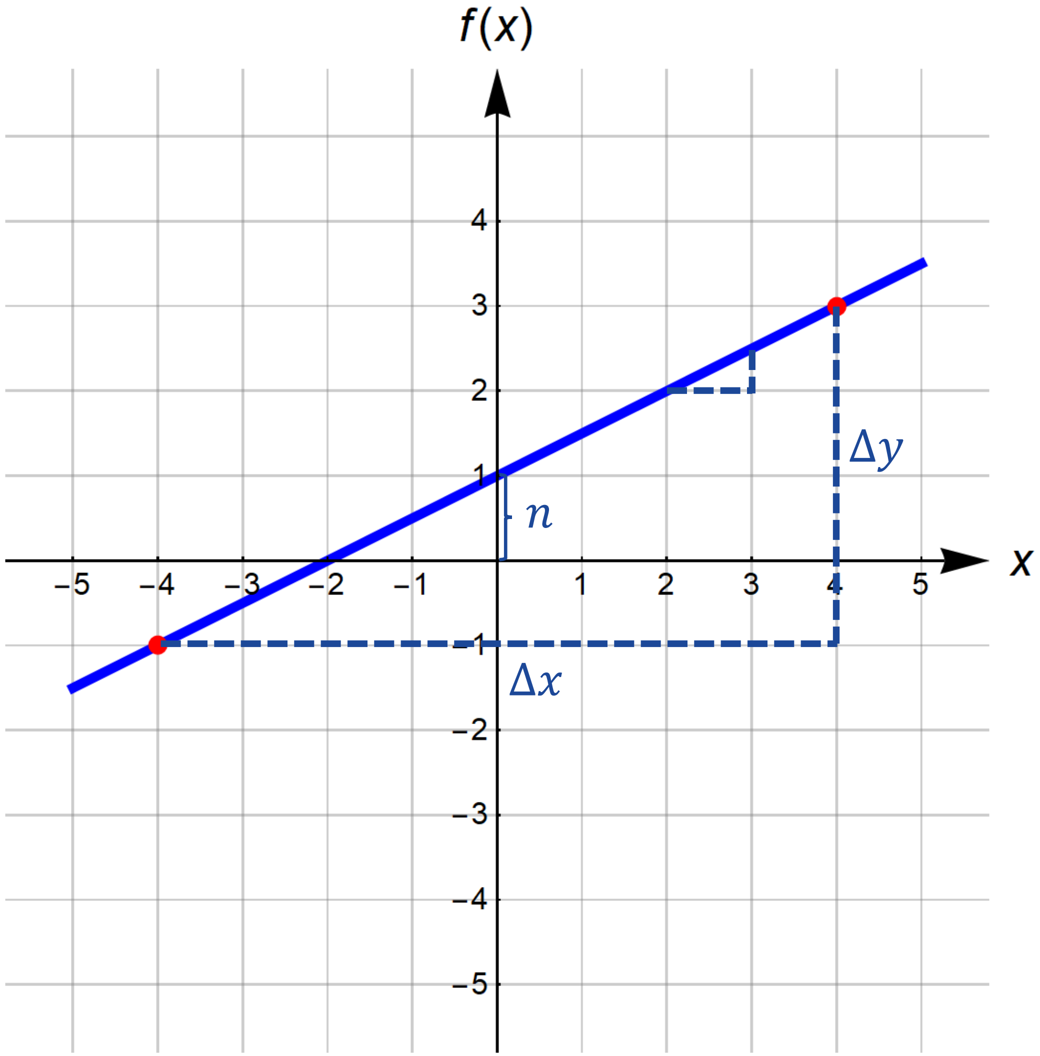
$$m = \frac{\Delta y}{\Delta x}$$

- First-order polynomial

$$f(x) = a_1 x + a_0$$

- Multidimensional generalization

$$f(\mathbf{x}) = a_0 + a_1 x_1 + \cdots + a_n x_n$$



Linear functions

$$y = mx + n$$

Intercept

Slope

$$m = \frac{\Delta y}{\Delta x}$$

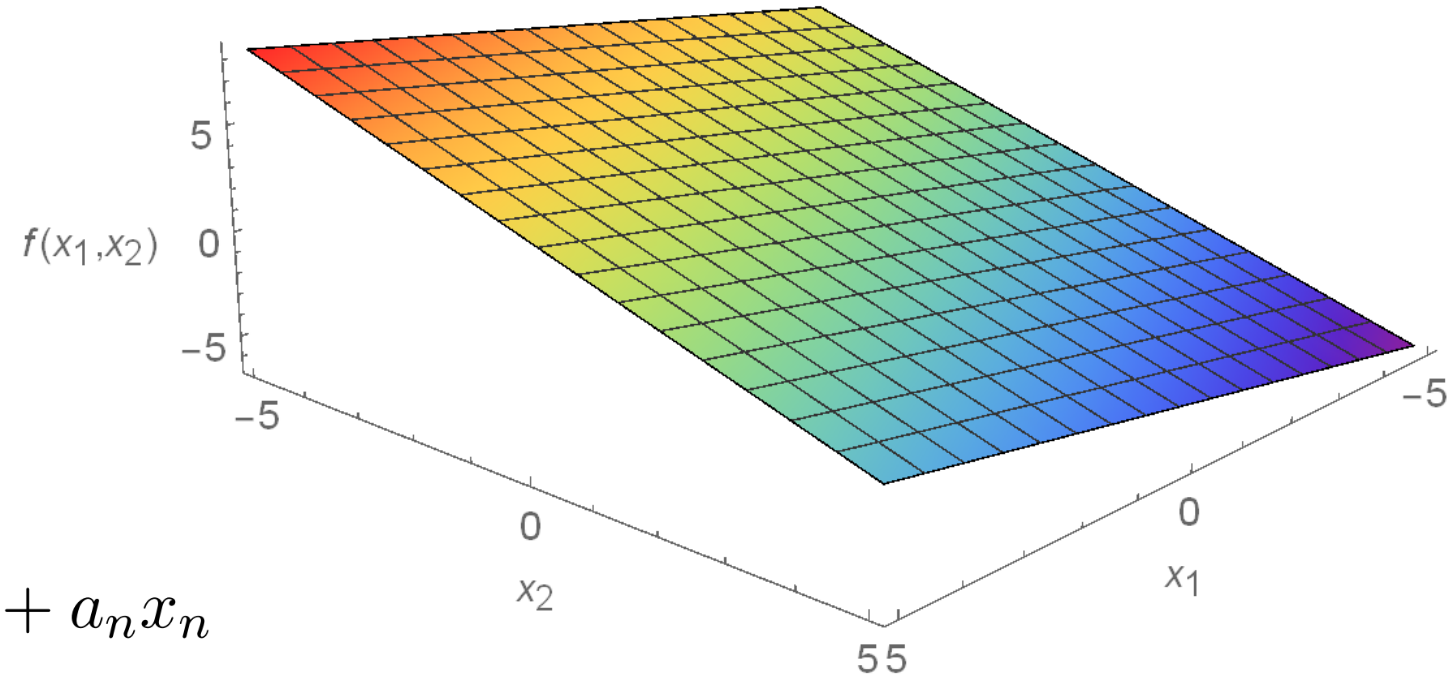
- First-order polynomial

$$f(x) = a_1 x + a_0$$

- Multidimensional generalization

$$f(\mathbf{x}) = a_0 + a_1 x_1 + \cdots + a_n x_n$$

e.g. $f(x_1, x_2) = 1 + \frac{1}{2} x_1 - x_2$



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SECTION

What you (don't) know

LECTURE

Quadratic functions & Polynomials

Polynomials

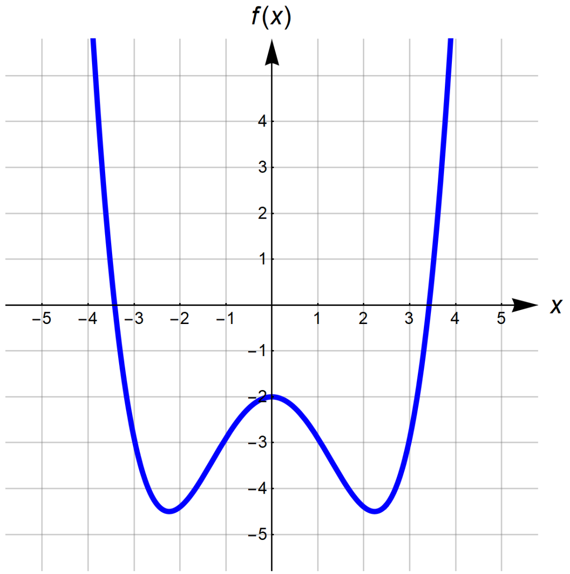
linear	$f(x) = a_0 + a_1 x$
quadratic	$f(x) = a_0 + a_1 x + a_2 x^2$
cubic	$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$
*n*th order	$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$

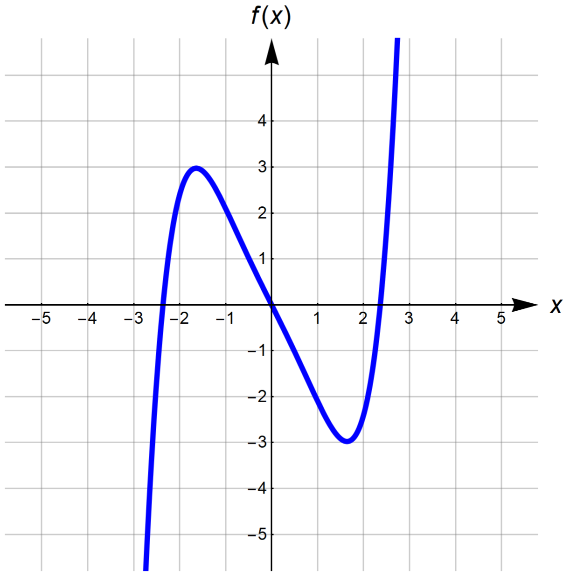
- Symmetry

symmetric (even)

$$f(x) = f(-x)$$

$$f(x) = \frac{1}{10}x^4 - x^2 - 2$$





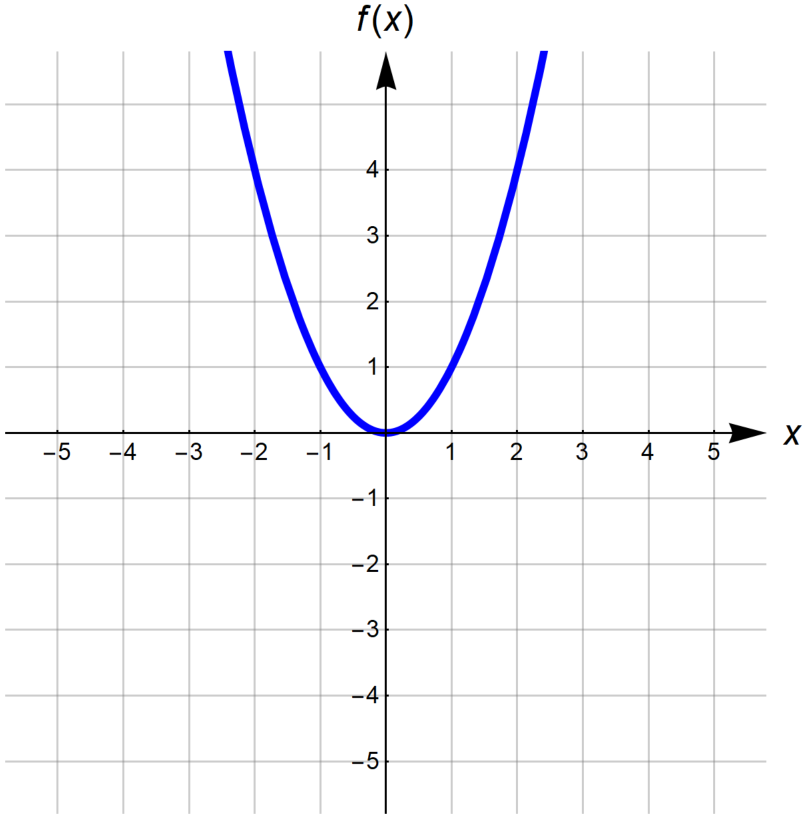
antisymmetric (odd)

$$-f(x) = f(-x)$$

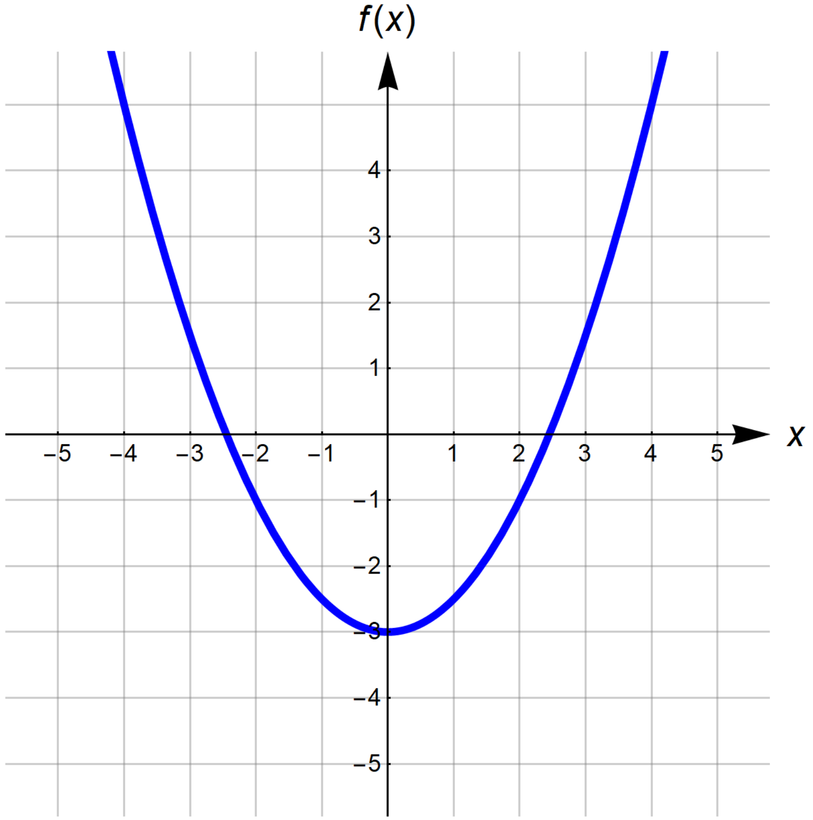
$$f(x) = \frac{1}{10}x^5 - \frac{1}{5}x^3 - 2x$$

Quadratic functions

Quadratic function $f(x) = a_0 + \cancel{a_1 x} + a_2 x^2$



$f(x) = x^2$



$f(x) = \frac{1}{2}x^2 - 3$

- Symmetry

symmetric (even)

$$f(x) = f(-x)$$

Quadratic functions

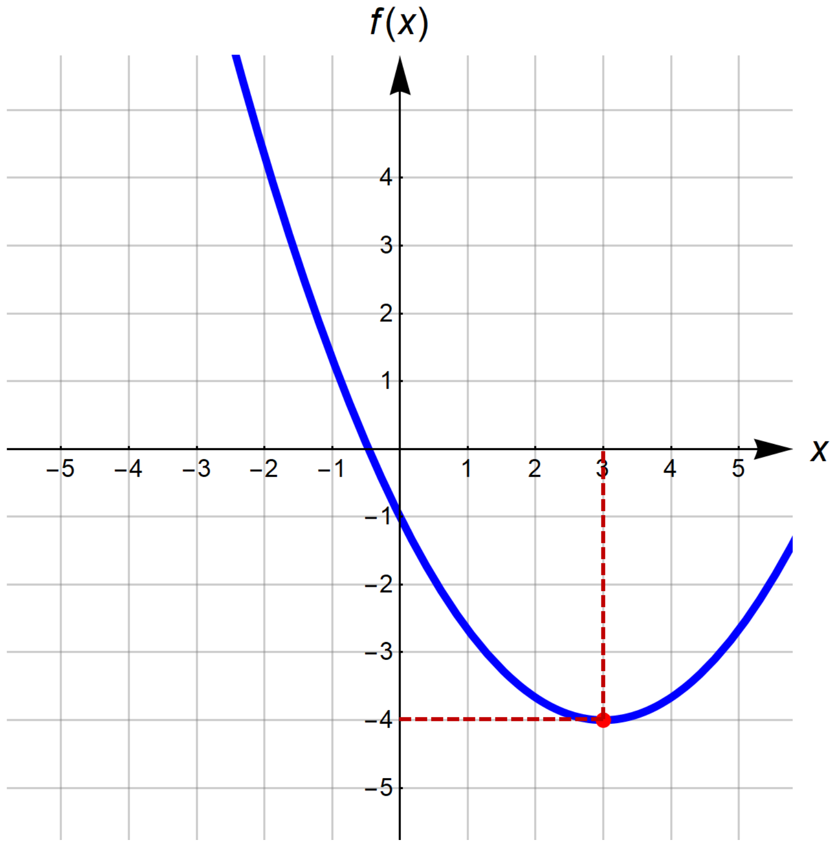
Quadratic function $f(x) = a_0 + a_1 x + a_2 x^2$

- Vertex form

$$f(x) = a(x-h)^2 + k$$

Extremum $P(h|k)$

$$f(x) = \frac{1}{3}(x-3)^2 - 4$$



$$f(x) = \frac{1}{3}x^2 - 2x - 1$$

- No symmetry

not symmetric (even)

$$f(x) \neq f(-x)$$

not antisymmetric (odd)

$$-f(x) \neq f(-x)$$

Quadratic functions

Quadratic function $$f(x) = a_0 + a_1 x + a_2 x^2$$

Vertex form $$f(x) = a(x-h)^2 + k$$

$$f(x) = a_0 + a_1 x + a_2 x^2$$

$$f(x) = a_2 \left(x^2 + \frac{a_1}{a_2} x + \frac{a_0}{a_2} \right)$$

$$f(x) = a_2 \left(x^2 + \frac{a_1}{a_2} x + \frac{1}{4}\left(\frac{a_1}{a_2}\right)^2 - \frac{1}{4}\left(\frac{a_1}{a_2}\right)^2 + \frac{a_0}{a_2} \right)$$

$$f(x) = a_2 \left(x + \frac{1}{2}\frac{a_1}{a_2} \right)^2 + \left(a_0 - \frac{1}{4}\frac{a_1^2}{a_2} \right)$$

Extremum $P\left(-\frac{1}{2}\frac{a_1}{a_2} \middle| a_0 - \frac{1}{4}\frac{a_1^2}{a_2}\right)$

Alternative

$$f'(x) = a_1 + 2a_2 x$$

$$0 = a_1 + 2a_2 x_m$$

$$x_m = -\frac{1}{2}\frac{a_1}{a_2}$$

$$f(x_m) = a_0 - a_1 \frac{1}{2}\frac{a_1}{a_2} + a_2 \frac{1}{4}\frac{a_1^2}{a_2^2}$$

$$f(x_m) = a_0 - \frac{1}{4}\frac{a_1^2}{a_2}$$

Quadratic equations

Find the roots of $f(x) = a_0 + a_1 x + a_2 x^2$

$$f(x) = a_2\left(x + \frac{1}{2}\frac{a_1}{a_2}\right)^2 + \left(a_0 - \frac{1}{4}\frac{a_1^2}{a_2}\right)$$

$$0 = a_0 + a_1 x + a_2 x^2$$

$$0 = \frac{a_0}{a_2} + \frac{a_1}{a_2}x + x^2$$

$$0 = x^2 + px + q$$

$$0 = \left(x + \frac{p}{2}\right)^2 + \left(q - \frac{p^2}{4}\right)$$

$$x + \frac{p}{2} = \pm\sqrt{\frac{p^2}{4} - q}$$

$$x = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$$

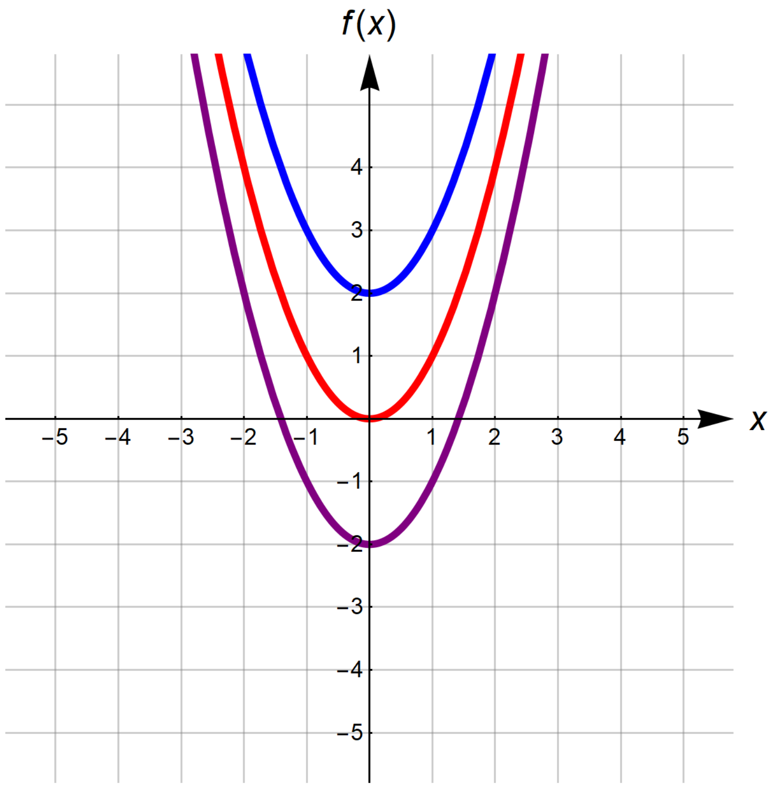
Exactly solvable

Quadratic equations

Find the roots of $f(x) = a_0 + a_1 x + a_2 x^2$

$$0 = x^2 + px + q$$

$$x_{1,2} = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$$



$\frac{p^2}{4} - q < 0$ No solution

$\frac{p^2}{4} - q = 0$ One degenerate solution

$\frac{p^2}{4} - q > 0$ Two solutions

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SECTION

What you (don’t) know

LECTURE

Factorization of polynomials

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Factorization of polynomials

Find the roots of $$f(x) = a_0 + a_1 x + a_2 x^2$$

$$0 = x^2 + px + q$$

$$x_{1,2} = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$$

Exactly solvable

$$f(x) = (x - x_1)(x - x_2)$$

$$f(x) = \left(x + \frac{p}{2} - \sqrt{\frac{p^2}{4} - q}\right)\left(x + \frac{p}{2} + \sqrt{\frac{p^2}{4} - q}\right)$$

$$f(x) = \left(x + \frac{p}{2}\right)^2 - \sqrt{\frac{p^2}{4} - q}^{\,2}$$

$$f(x) = \left(x^2 + px + \frac{p^2}{4}\right) - \left(\frac{p^2}{4} - q\right)$$

How can this work? Even if $\frac{p^2}{4} - q < 0$

(No solution x_1, x_2)

$$f(x) = x^2 + px + q$$

Factorization of polynomials

Find the roots of $f(x) = a_0 + a_1 x + a_2 x^2$

$$0 = x^2 + px + q$$

$$x_{1,2} = -\frac{p}{2} \pm \sqrt{\frac{p^2}{4} - q}$$

Exactly solvable

$$f(x) = (x - x_1)(x - x_2)$$

$$f(x) = \left(x + \frac{p}{2}\right)^2 - \sqrt{\frac{p^2}{4} - q}^{\,2}$$

How can this work? Even if $\frac{p^2}{4} - q < 0$

(No solution x_1, x_2)

Trick: We only need the square

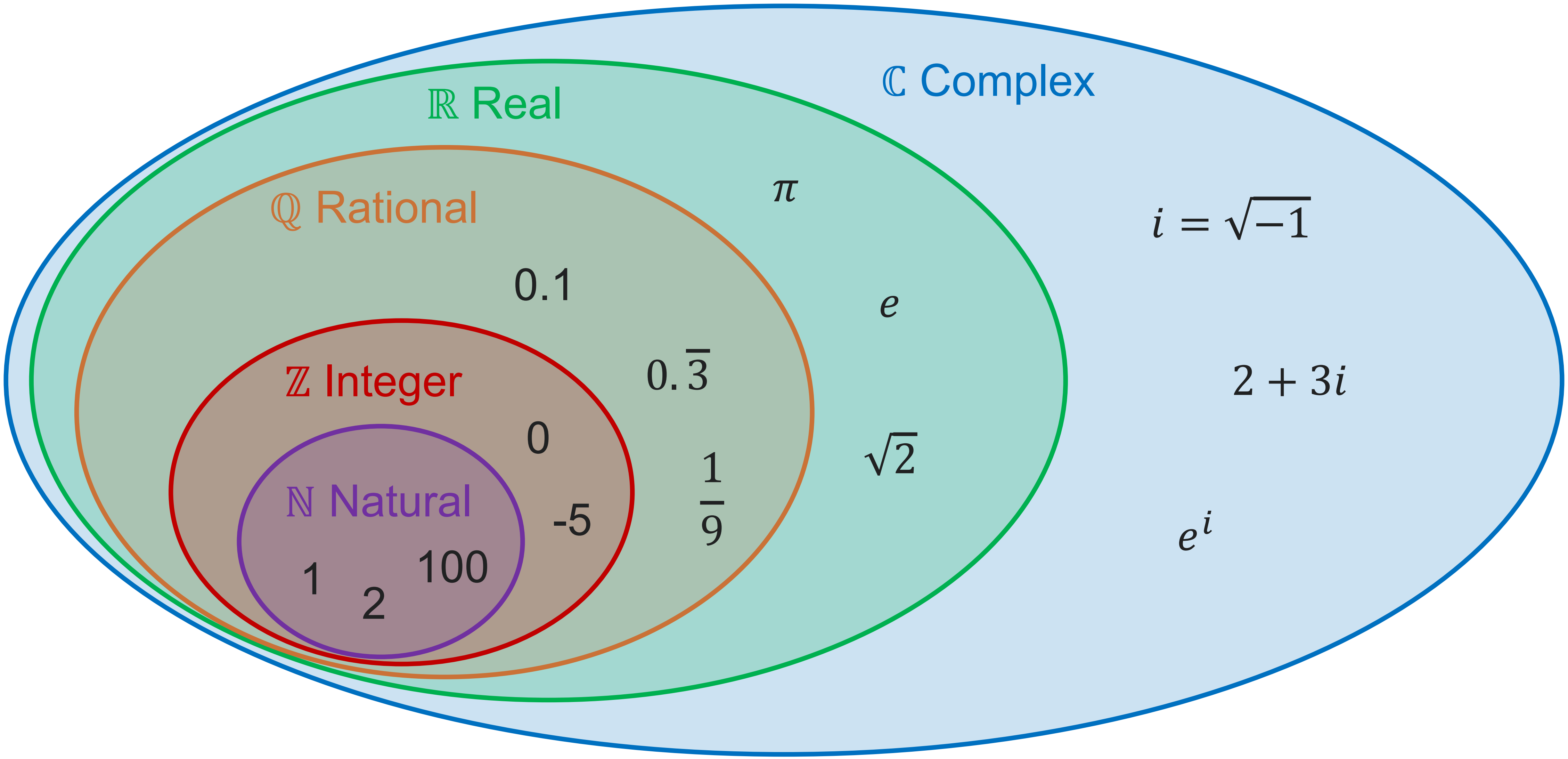
(which is a negative number)

Introduction of a new type of number: Complex numbers $\mathbb{C}$

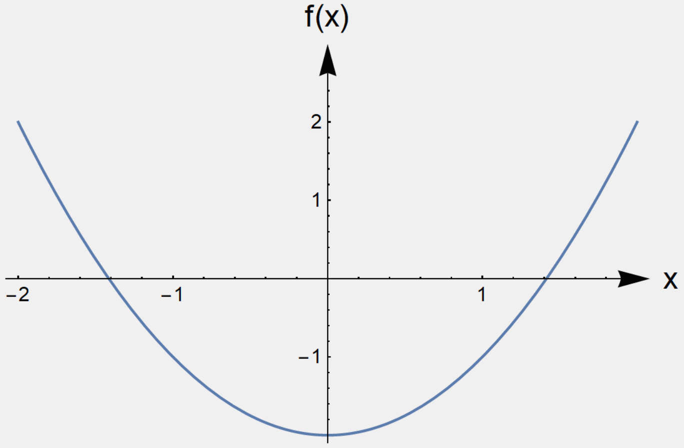
$$z = a + ib$$

$$i^2 = -1 \qquad \sqrt{-1} = i$$

Outlook: Complex numbers



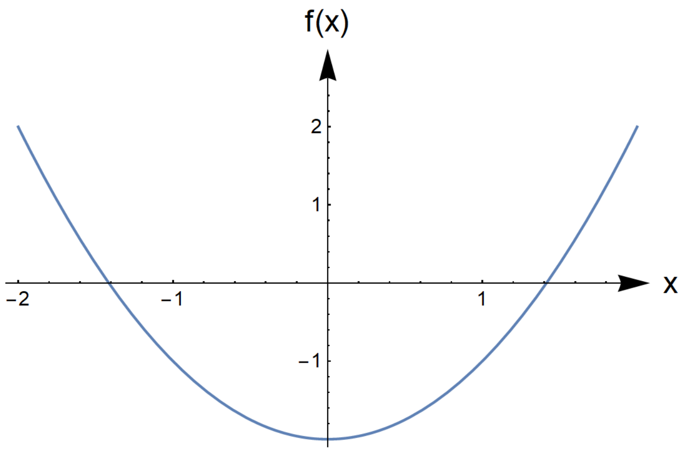
Quadratic functions



Real

$$y = x^2 - c$$

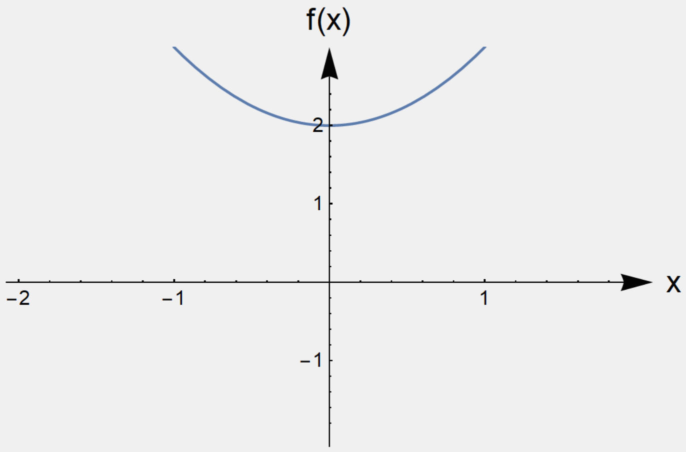
Quadratic functions



Real

$$y = x^2 - c$$
$$0 = x_0^2 - c$$
$$x_0^2 = c$$
$$x_0 = \pm\sqrt{c}$$

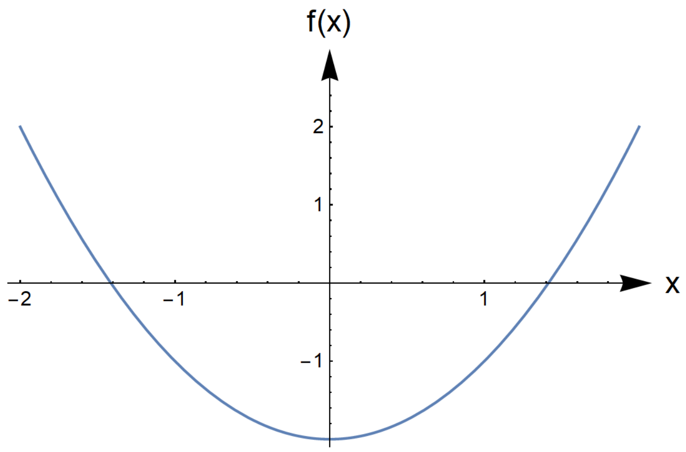
$$y = x^2 - 2$$
$$0 = x_0^2 - 2$$
$$x_0^2 = 2$$
$$x_0 = \pm\sqrt{2}$$



Imaginary $\boxed{i = \sqrt{-1}}$

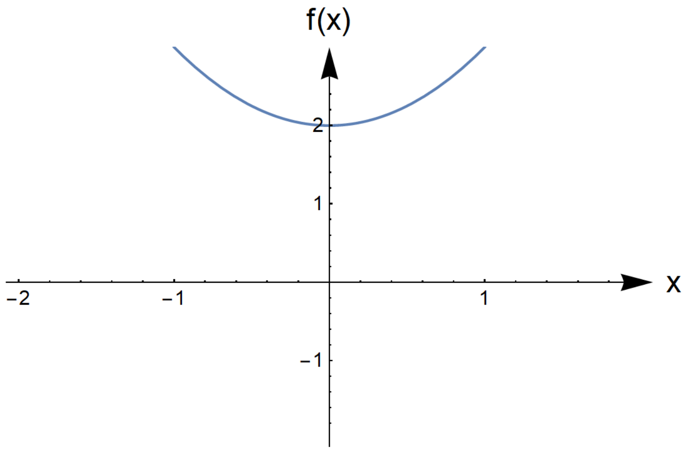
$$y = x^2 + c$$

Quadratic functions



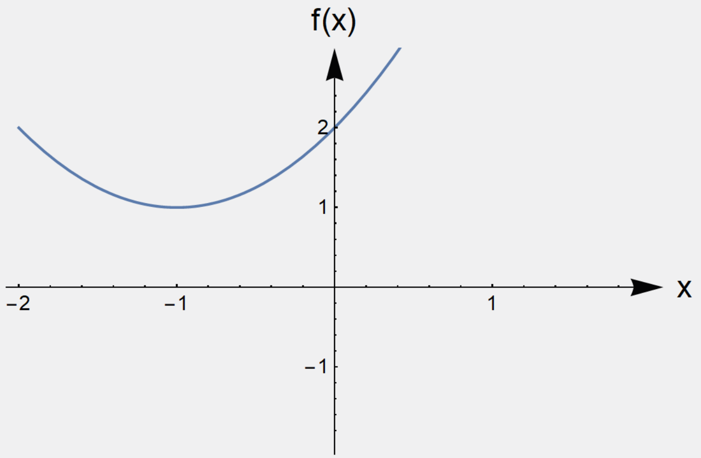
Real

$$y = x^2 - c \qquad y = x^2 - 2$$
$$0 = x_0^2 - c \qquad 0 = x_0^2 - 2$$
$$x_0^2 = c \qquad x_0^2 = 2$$
$$x_0 = \pm\sqrt{c} \qquad x_0 = \pm\sqrt{2}$$



Imaginary $\boxed{i = \sqrt{-1}}$

$$y = x^2 + c \qquad y = x^2 + 2$$
$$0 = x_0^2 + c \qquad 0 = x_0^2 + 2$$
$$x_0^2 = -c \qquad x_0^2 = -2$$
$$x_0 = \pm\sqrt{-c} \qquad x_0 = \pm\sqrt{-2}$$
$$x_0 = \pm\sqrt{c}\cdot\sqrt{-1} \qquad x_0 = \pm\sqrt{2}\cdot\sqrt{-1}$$
$$x_0 = \pm\sqrt{c}\,i \qquad x_0 = \pm\sqrt{2}\,i$$



Complex

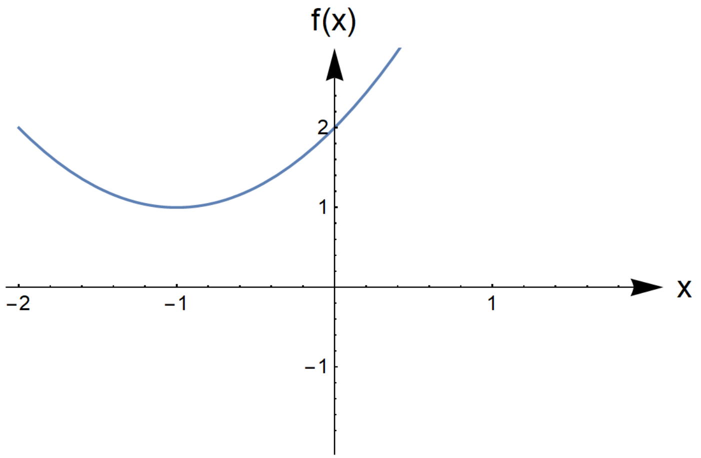
$$y = x^2 + bx + c$$

Quadratic functions

Factorization

$$[x-(-1+i)]\,[x-(-1-i)]$$

$$= [x+(1-i)]\,[x+(1+i)]$$
$$= x^2 + x(1+i) + x(1-i) + (1-i)(1+i)$$
$$= x^2 + 2x + 1 - i^2$$
$$= x^2 + 2x + 2$$



Complex

$$y = x^2 + bx + c \qquad y = x^2 + 2x + 2$$
$$0 = x_0^2 + bx + c \qquad 0 = x_0^2 + 2x + 2$$
$$x_0 = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - c} \qquad x_0 = -1 \pm \sqrt{1-2}$$
$$x_0 = -\frac{b}{2} \pm \sqrt{-1} \cdot \sqrt{c - \frac{b^2}{4}} \qquad x_0 = -1 \pm \sqrt{-1} \cdot \sqrt{2-1}$$
$$x_0 = -\frac{b}{2} \pm \sqrt{c - \frac{b^2}{4}}\, i \qquad x_0 = -1 \pm i$$

Quadratic functions

$$f(x) = \frac{1}{1000}(x-5)(x-4)(x-1)(x-1)(x+3)(x+i)(x-i)$$
$$= \frac{1}{1000}\left(x^7 - 8x^6 + 7x^5 + 60x^4 - 121x^3 + 128x^2 - 127x + 60\right)$$

Factorization

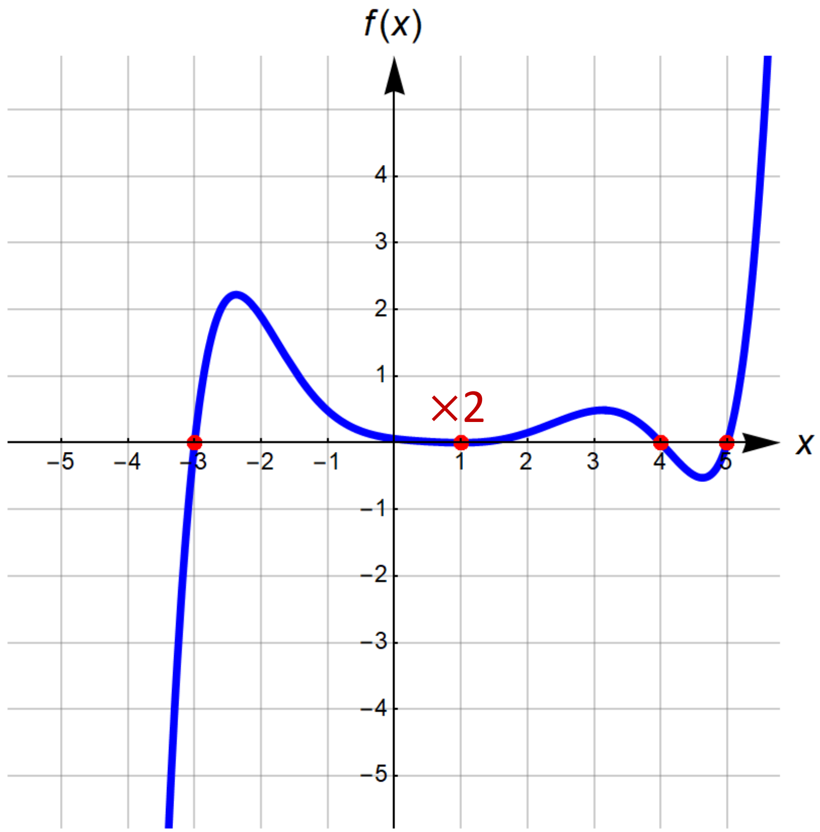
quadratic

$$f(x) = A(x - x_1)(x - x_2)$$

*n*th order

$$f(x) = A(x - x_1)(x - x_2)\ldots(x - x_n)$$

n (complex) roots x_i



Take-home message $\sqrt{-1}$ exists

Mathematics

in Science, Engineering & Programming

SECTION

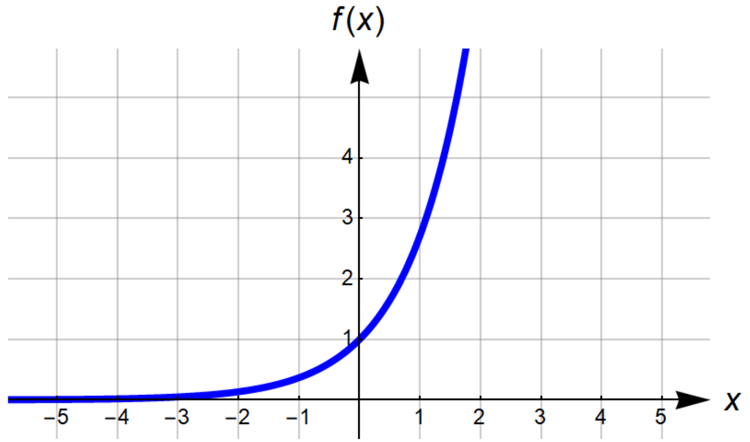
What you (don't) know

LECTURE

Exponential function

Exponential function

Motivation



(strictly) linear function $f(x) = mx$

$$f(ax) = af(x)$$
$$f(x+y) = f(x) + f(x)$$

$$f(ax) = m(ax) = amx = af(x)$$
$$f(x+y) = m(x+y) = mx + my = f(x) + f(y)$$

exponential function $f(x) = e^x$

$$f(x+y) = f(x)f(y)$$

$$f(x+y) = e^{x+y} = e^x e^y = f(x)f(y)$$

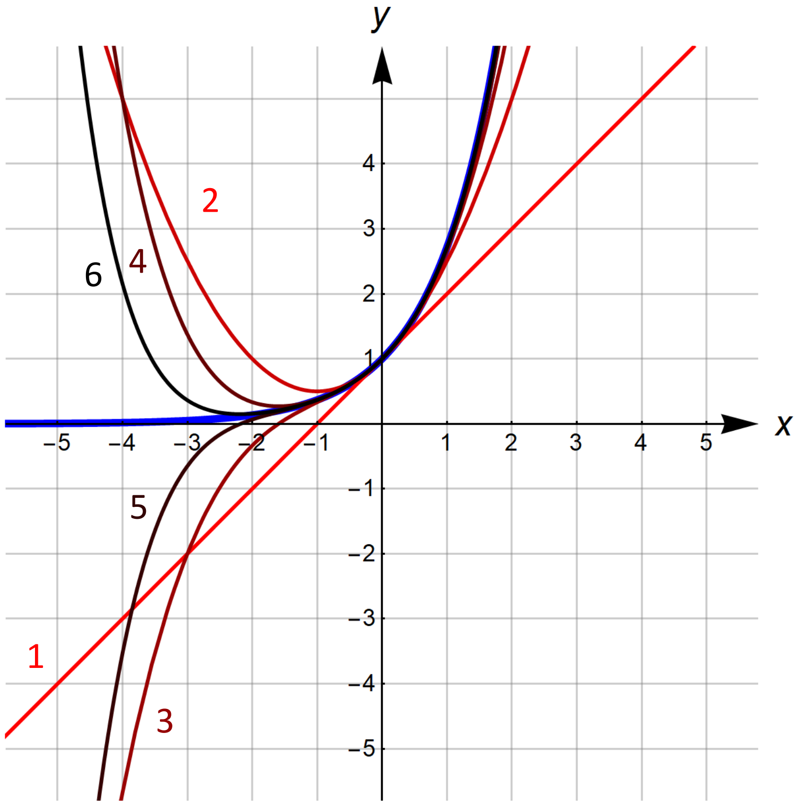
Exponential function

Motivation exponential function $f(x) = e^x$ $f(x+y) = f(x)f(y)$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots$$

$$e^y = \sum_{n=0}^{\infty} \frac{1}{n!} y^n = 1 + y + \frac{1}{2}y^2 + \frac{1}{6}y^3 + \frac{1}{24}y^4 + \dots$$

$$\begin{aligned} e^{x+y} &= \sum_{n=0}^{\infty} \frac{1}{n!}(x+y)^n \\ &= 1 + x + y + \frac{1}{2}(x+y)^2 + \frac{1}{6}(x+y)^3 + \frac{1}{24}(x+y)^4 + \dots \\ &= 1 + x + y + \frac{1}{2}x^2 + xy + \frac{1}{2}y^2 + \frac{1}{6}x^3 + \frac{1}{2}x^2y + \frac{1}{2}xy^2 + \frac{1}{6}y^3 \\ &\quad + \frac{1}{24}x^4 + \frac{1}{6}x^3y + \frac{1}{4}x^2y^2 + \frac{1}{6}xy^3 + \frac{1}{24}y^4 + \dots \end{aligned}$$



Exponential function

Motivation exponential function $f(x) = e^x$ $f(x+y) = f(x)f(y)$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots$$

$$\begin{aligned} e^{x+y} =& 1 + x + y + \frac{1}{2}x^2 + xy + \frac{1}{2}y^2 + \frac{1}{6}x^3 + \frac{1}{2}x^2y + \frac{1}{2}xy^2 + \frac{1}{6}x^3 \\ &+ \frac{1}{24}x^4 + \frac{1}{6}x^3y + \frac{1}{4}x^2y^2 + \frac{1}{6}xy^3 + \frac{1}{24}y^4 \dots \end{aligned}$$

$$\begin{aligned} e^x e^y &= \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots\right)\left(1 + y + \frac{1}{2}y^2 + \frac{1}{6}y^3 + \frac{1}{24}y^4 + \dots\right) \\ &= 1 + x + y + \frac{1}{2}x^2 + xy + \frac{1}{2}y^2 + \frac{1}{6}x^3 + \frac{1}{2}x^2y + \frac{1}{2}xy^2 + \frac{1}{6}y^3 \\ &\quad + \frac{1}{24}x^4 + \frac{1}{6}x^3y + \frac{1}{4}x^2y^2 + \frac{1}{6}xy^3 + \frac{1}{24}y^4 + \dots \end{aligned}$$

Derivative of the exponential function

Derivative exponential function $f(x) = e^x$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots$$

$$\frac{\mathrm{d}}{\mathrm{d}x} e^x = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\mathrm{d}}{\mathrm{d}x} x^n = 0 + 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$$

$$\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$$

$$f'(x) = f(x)$$

Mathematics

in Science, Engineering & Programming

SECTION

What you (don't) know

LECTURE

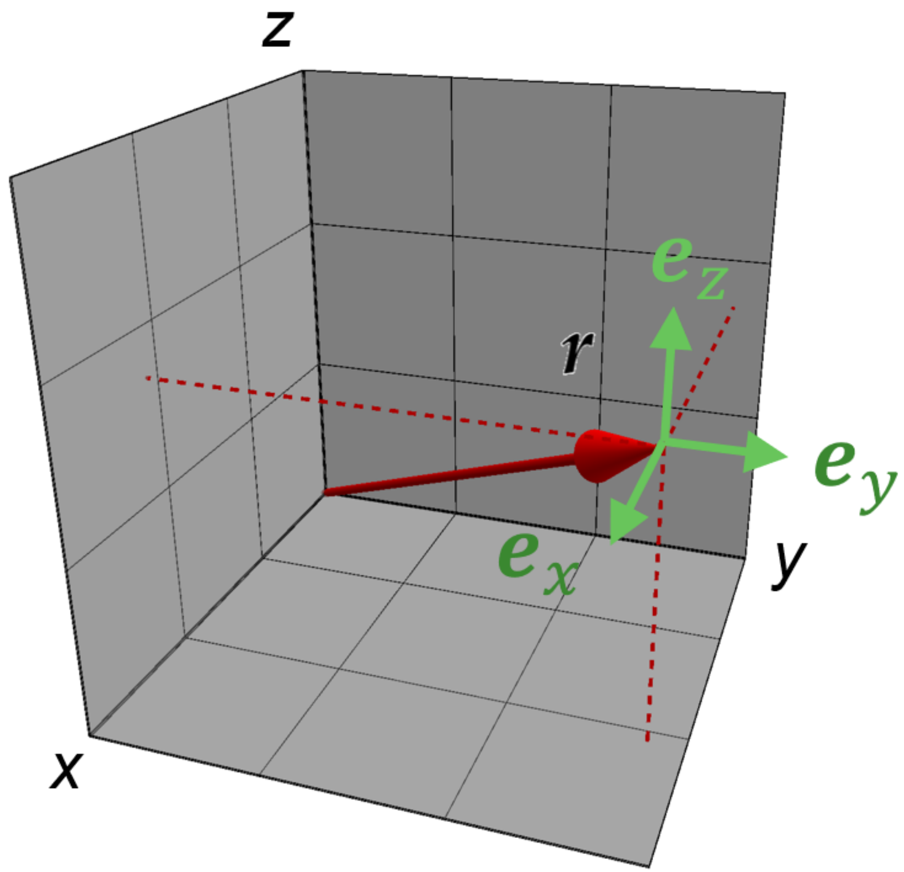
Vectors & Coordinate systems

BÖRGE GÖBEL

Cartesian coordinates

- Position vector $\boldsymbol{r} = x\boldsymbol{e}_x + y\boldsymbol{e}_y + z\boldsymbol{e}_z = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$
- Unit vectors

$$\boldsymbol{e}_x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \boldsymbol{e}_y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \boldsymbol{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$



Polar coordinates

- Position vector $\boldsymbol{r} = r\,\boldsymbol{e}_r = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix}$
- Unit vectors

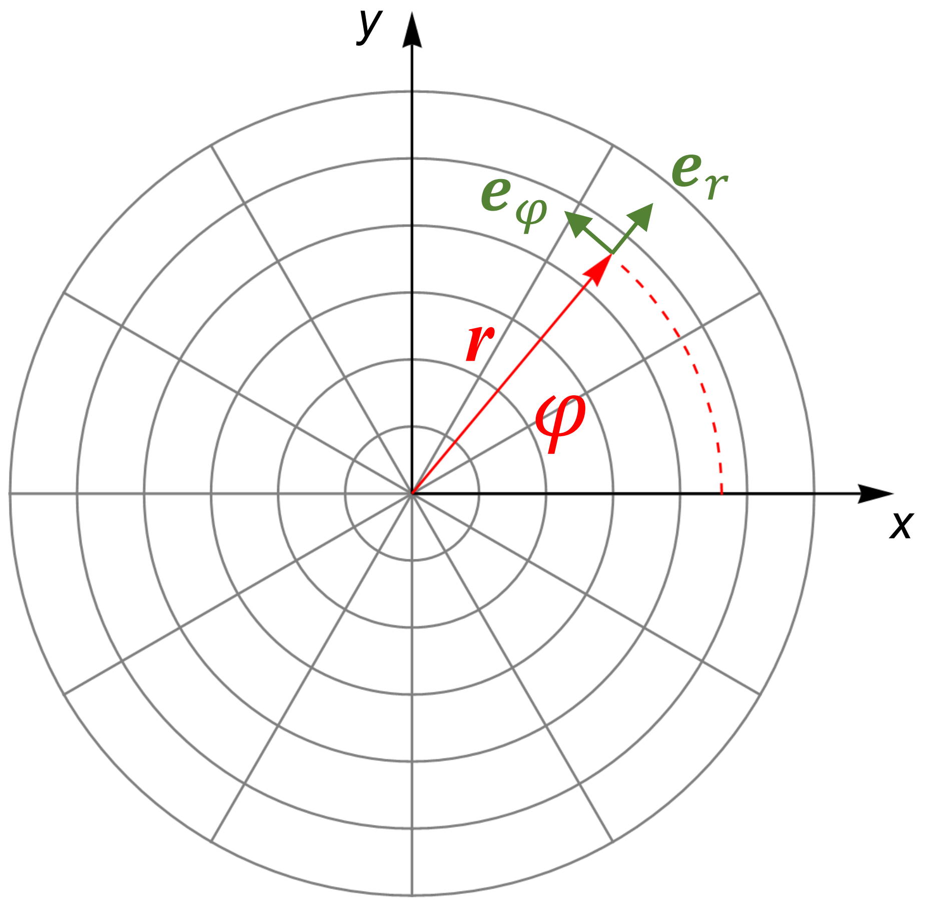
$$\boldsymbol{e}_r = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \qquad \boldsymbol{e}_\varphi = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix}$$

… perpendicular

- Transformation

$$x = r\cos\varphi \qquad r = \sqrt{x^2 + y^2}$$

$$y = r\sin\varphi \qquad \varphi = \operatorname{atan2}\frac{y}{x} = \arctan\frac{y}{x}\,(+\pi)$$



Cartesian vs. polar coordinates

Rotational motion

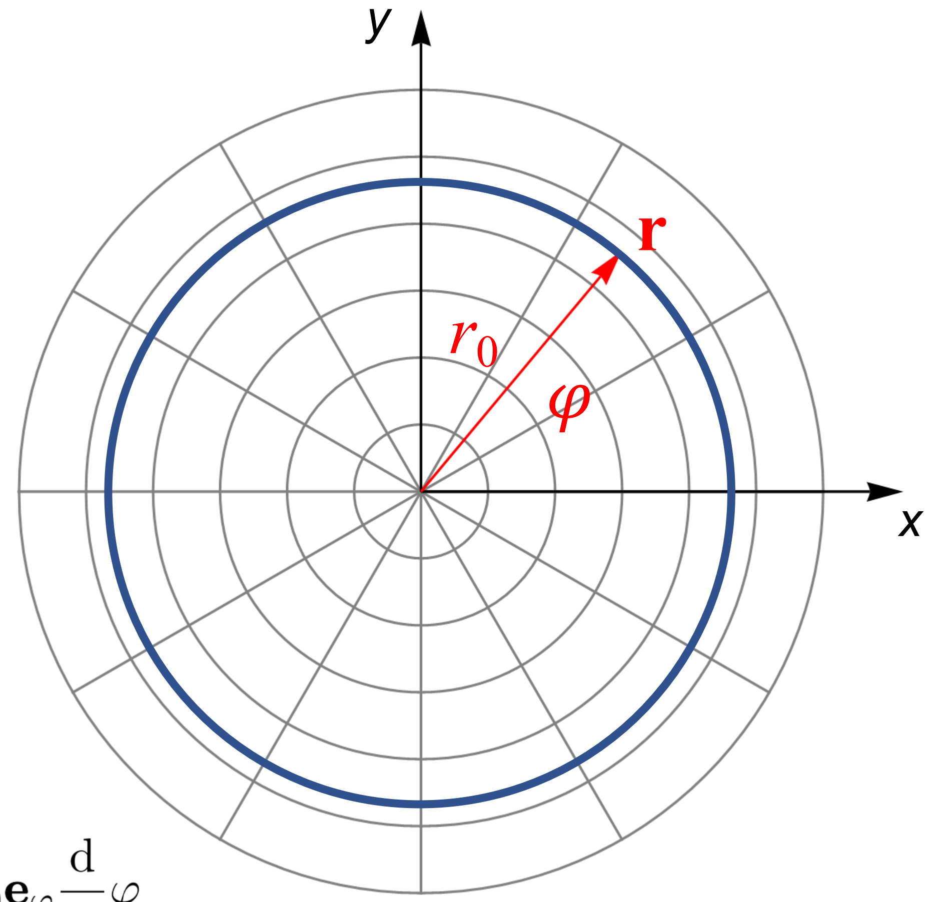
$$x(t) = r_0 \cos(\omega t)$$
$$y(t) = r_0 \sin(\omega t)$$

$$\mathbf{r}(t) = r_0 \cos(\omega t)\mathbf{e}_x + r_0 \sin(\omega t)\mathbf{e}_y$$
$$= r_0 \begin{pmatrix} \cos(\omega t) \\ \sin(\omega t) \end{pmatrix}$$

$$r = r_0$$
$$\varphi(t) = \omega t$$

$$\mathbf{r}(t) = r_0 \mathbf{e}_r(t)$$

only 1 variable
only 1 unit vector
(but time dependent)



- Calculate velocity

$$\mathbf{v} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r} = -r_0\omega \sin(\omega t)\mathbf{e}_x + r_0\omega \cos(\omega t)\mathbf{e}_y$$
$$= r_0\omega \begin{pmatrix} -\sin(\omega t) \\ \cos(\omega t) \end{pmatrix}$$

$$\mathbf{v} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r} = r_0 \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{e}_r = r_0 \mathbf{e}_\varphi \frac{\mathrm{d}}{\mathrm{d}t}\varphi$$
$$= r_0 \omega \mathbf{e}_\varphi$$

Take-home message

Vector is more than $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Mathematics

in Science, Engineering & Programming

SECTION

What you (don't) know

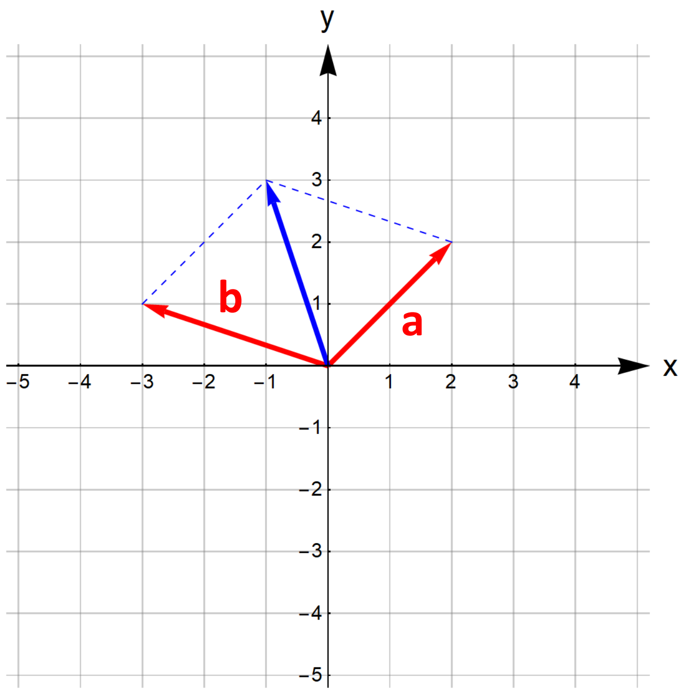
LECTURE

Vector operations

Typical vector operations

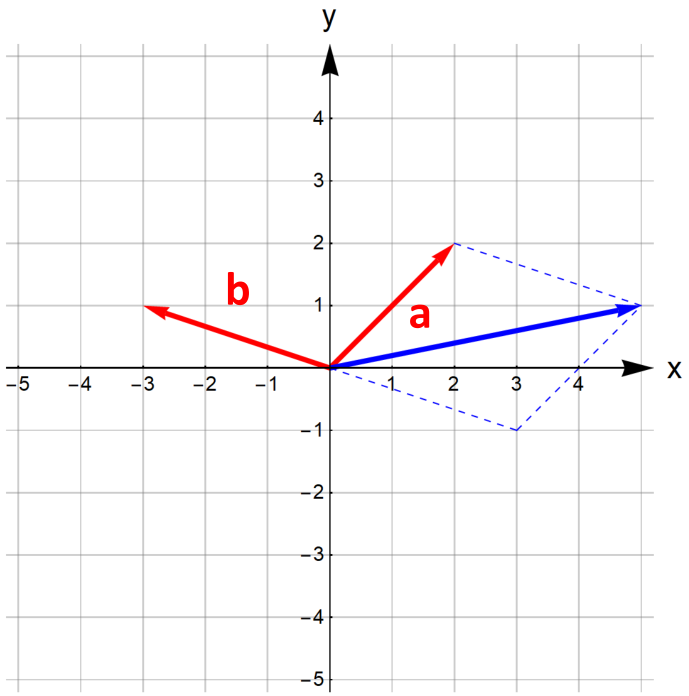
$$\mathbf{a} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

Addition



$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

Subtraction

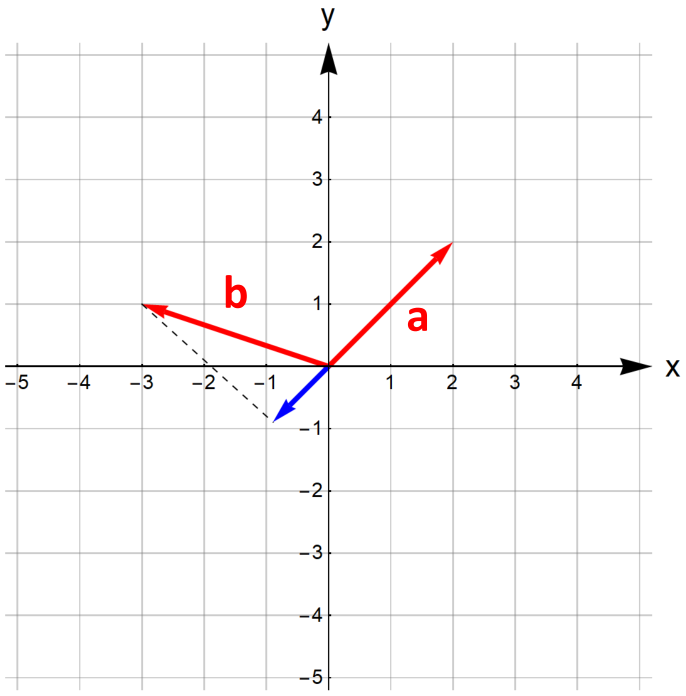


$$\mathbf{a} - \mathbf{b} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

Typical vector operations

$$\mathbf{a} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

Project

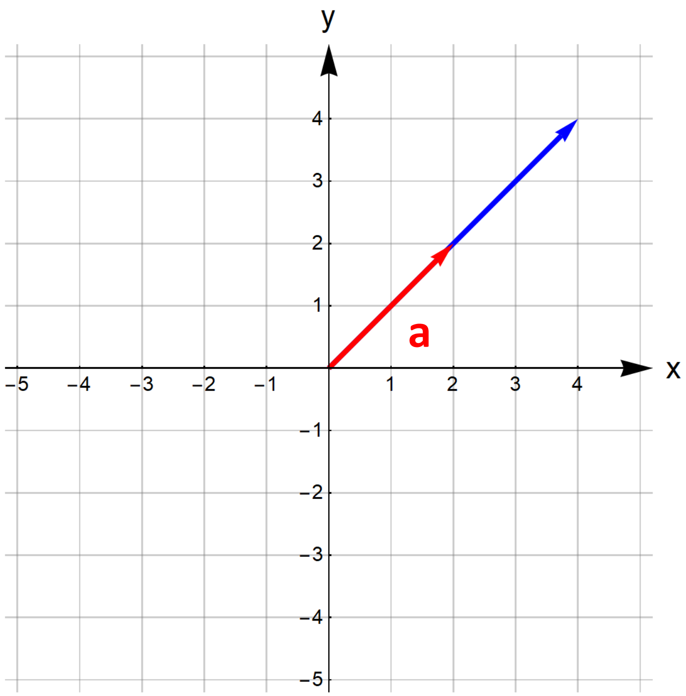


$$\mathbf{a} \cdot \mathbf{b} = 2 \cdot (-3) + 2 \cdot 1 = -4$$

$$|a| = \sqrt{2^2 + 2^2} = \sqrt{8}$$

$$\mathbf{a}\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

Scale



$$2\mathbf{a} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

Typical vector operations

$$\mathbf{a} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

Rotation

$$\underline{\underline{R}}_{\alpha}\mathbf{a} = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \mathbf{a}$$

for $\alpha = -90^\circ : \begin{pmatrix} 2 \\ -2 \end{pmatrix}$

for $\alpha = -135^\circ : \begin{pmatrix} 0 \\ -\sqrt{8} \end{pmatrix}$