Appendix C

Quaternions

In this appendix is a description on how to use Quaternions to represent rotations of different coordinate systems or reference frames.
This appendix starts with a description of the quaternion and some fundamental properties of the quaternion together with an example of the use of the quaternions. Finaly is the time derivative of a quaternion derived, which is used in this project to represent the rotational kinematic model of the aircraft.

Definitions

A quaternion is a hypercomplex number, which is an extension to the complex numbers. It is represented similar to complex numbers with a real part and imaginary parts. A quaternion (C.1) has four parameters called Euler parameters, three imaginary q_1, q_2, q_3 and one real q_4 defined in (C.1), representing a rotation of θ about an axis $\hat{\boldsymbol{e}}$,

$$\begin{aligned} \boldsymbol{q} &= q_4 + \boldsymbol{i}q_1 + \boldsymbol{j}q_2 + \boldsymbol{k}q_3 \\ &= \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \hat{\boldsymbol{e}}\sin(\frac{\theta}{2}) \\ \cos(\frac{\theta}{2}) \end{bmatrix} \end{aligned} \tag{C.1}$$

where $\hat{\boldsymbol{e}} = \begin{bmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \end{bmatrix}^{\mathrm{T}}$ is the unit vector representing Euler's eigenaxis[1] and θ is the rotation angle. The imaginary numbers $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are defined by the fundamental formula of quaternions (C.2) discovered by William Rowan Hamilton [1].

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{i}\boldsymbol{j}\boldsymbol{k} = -1 \tag{C.2}$$

Since the quaternion is a hypercomplex number the conjugated quaternion is defined as (C.3),[1]

$$\boldsymbol{q}^* = q_4 - \boldsymbol{i}q_1 - \boldsymbol{j}q_2 - \boldsymbol{k}q_3 \tag{C.3}$$

[1] Euler's eigenaxis is the axis which an object is rotated about [1].

One important property of quaternions are that they have the norm of 1 thus (C.4) is true [1].

$$|\boldsymbol{q}| \triangleq \sqrt{\boldsymbol{q} \otimes \boldsymbol{q}^*} = \sqrt{\boldsymbol{q}^* \otimes \boldsymbol{q}} = \sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2} \equiv 1 \tag{C.4}$$

where $\otimes$ denotes that quaternion multiplication is used see section C and the $\boldsymbol{q}^*$ is the conjugated quaternion. The norm (C.4) especially comes in handy in control application with regards to stability, because it bounds the size of the quaternion.

From (C.4) it can be derived that the inverse quaternion is defined as the complex conjugated quaternion thus giving (C.5)

$$\forall \boldsymbol{q}(|\boldsymbol{q}| = 1)|\boldsymbol{q}^{-1} \equiv \boldsymbol{q}^* \tag{C.5}$$

Quaternion Multiplication

This section contains the rules of multiplying two quaternions and an example of the usage of quaternion multiplication.

The quaternions are noncommutative and associative, thus usual multiplication is not possible. Instead it has to be carried out by quaternion multiplication (C.6) sometimes denoted by $\otimes$ or understod in the context of the equations[1].

$$\begin{aligned}
\boldsymbol{q}_C &= \boldsymbol{q}_A \boldsymbol{q}_B = \boldsymbol{q}_A \otimes \boldsymbol{q}_B \\
&= \begin{bmatrix} \boldsymbol{q}_{A_{1:3}} q_{B_4} + q_{A_4} \boldsymbol{q}_{B_{1:3}} + \boldsymbol{q}_{A_{1:3}} \times \boldsymbol{q}_{B_{1:3}} \\ q_{A_4} q_{B_4} - \boldsymbol{q}_{A_{1:3}}^{\mathrm{T}} \boldsymbol{q}_{B_{1:3}} \end{bmatrix} \\
&= \begin{bmatrix} \boldsymbol{S}(\boldsymbol{q}_{A_{1:3}}) + q_{A_4} \boldsymbol{I}_{3\times3} & \boldsymbol{q}_{A_{1:3}} \\ -\boldsymbol{q}_{A_{1:3}}^{\mathrm{T}} & q_{A_4} \end{bmatrix} \boldsymbol{q}_B && \text{(C.6)} \\
&= \begin{bmatrix} -\boldsymbol{S}(\boldsymbol{q}_{B_{1:3}}) + q_{B_4} \boldsymbol{I}_{3\times3} & \boldsymbol{q}_{B_{1:3}} \\ -\boldsymbol{q}_{B_{1:3}}^{\mathrm{T}} & q_{B_4} \end{bmatrix} \boldsymbol{q}_A && \text{(C.7)}
\end{aligned}$$

Where $\boldsymbol{S}(\boldsymbol{a})$ is a skew-symmetric matrix (C.9) of a vector $\boldsymbol{a} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}^{\mathrm{T}}$ defined as (C.9) and found through (C.8):

$$\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{S}(a)\boldsymbol{b} = -\boldsymbol{S}(b)\boldsymbol{a} \tag{C.8}$$

$$\boldsymbol{S}(\boldsymbol{a}) = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{C.9}$$

Another property of the quaternion is that it is a division algebra, thus (C.10) is true:

$$\boldsymbol{q}_C = \boldsymbol{q}_A \boldsymbol{q}_B \Leftrightarrow \boldsymbol{q}_B = \boldsymbol{q}_A^{-1} \boldsymbol{q}_C \tag{C.10}$$

Rotation by Quaternions

As described above a quaternion $\boldsymbol{q}$ represents a rotation in $\mathfrak{R}^3$, thus rotating a vector between two viewpoints or coordinate systems is done as in the following example.

Given two coordinate systems $\mathbb{R}$ and $\mathbb{B}$ and a vector ${}^{\mathbb{R}}\boldsymbol{\omega}_{\mathbb{B}}$ given in $\mathbb{R}$ representing the angular velocity of $\mathbb{B}$. To find the same vector given in $\mathbb{B}$ it is necessary to rotate the vector from $\mathbb{R}$ to $\mathbb{B}$. If the quaternion ${}^{\mathbb{B}}_{\mathbb{R}}\boldsymbol{q}$ represents the rotation from $\mathbb{R}$ to $\mathbb{B}$ then it is possible to rotate the vector ${}^{\mathbb{R}}\boldsymbol{\omega}_{\mathbb{B}}$ to $\mathbb{B}$ by (C.11) and back to $\mathbb{R}$ by (C.12).

$$\begin{bmatrix} {}^{\mathbb{B}}_{\mathbb{R}}\boldsymbol{\omega}_{\mathbb{B}} \\ 0 \end{bmatrix} = {}^{\mathbb{B}}_{\mathbb{R}}\boldsymbol{q} \otimes \begin{bmatrix} {}^{\mathbb{R}}\boldsymbol{\omega}_{\mathbb{B}} \\ 0 \end{bmatrix} \otimes {}^{\mathbb{B}}_{\mathbb{R}}\boldsymbol{q}^{*} \tag{C.11}$$

$$\begin{bmatrix} {}^{\mathbb{R}}\boldsymbol{\omega}_{\mathbb{B}} \\ 0 \end{bmatrix} = {}^{\mathbb{B}}_{\mathbb{R}}\boldsymbol{q}^{*} \otimes \begin{bmatrix} {}^{\mathbb{B}}_{\mathbb{R}}\boldsymbol{\omega}_{\mathbb{B}} \\ 0 \end{bmatrix} \otimes {}^{\mathbb{B}}_{\mathbb{R}}\boldsymbol{q} \tag{C.12}$$

Where $\begin{bmatrix} {}^{\mathbb{R}}\boldsymbol{\omega}_{\mathbb{B}} \\ 0 \end{bmatrix}$ is the original vector put on a quaternion form with the real part $q_4 = 0$, note that this is not a true quaternion since it doesn't have to have a norm of 1.

If another coordinate system $\mathbb{E}$ seen from $\mathbb{B}$ is rotated by ${}^{\mathbb{E}}_{\mathbb{B}}\boldsymbol{q}$, then ${}^{\mathbb{R}}\boldsymbol{\omega}_{\mathbb{B}}$ seen from $\mathbb{E}$ is given as:

$$\begin{bmatrix} {}^{\mathbb{E}}_{\mathbb{R}}\boldsymbol{\omega}_{\mathbb{B}} \\ 0 \end{bmatrix} = {}^{\mathbb{E}}_{\mathbb{B}}\boldsymbol{q} \otimes {}^{\mathbb{B}}_{\mathbb{R}}\boldsymbol{q} \otimes \begin{bmatrix} {}^{\mathbb{R}}\boldsymbol{\omega}_{\mathbb{B}} \\ 0 \end{bmatrix} \otimes {}^{\mathbb{B}}_{\mathbb{R}}\boldsymbol{q}^{*} \otimes {}^{\mathbb{E}}_{\mathbb{B}}\boldsymbol{q}^{*} \tag{C.13}$$

Thus showing that shifting between different coordinate systems is straight forward by using quaternion multiplication.

The Time Derivative of a Quaternion

This section contains the derivation of the time derivative of a quaternion. Which is usefull when predicting how object rotates or have rotated.

The method used is to solve (C.14) for a quaternion $\boldsymbol{q}(t)$ changing over time.

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} \tag{C.14}$$

Let $\boldsymbol{q}(t)$ be a quaternion changing continuously over time and let $\boldsymbol{q}(\Delta t)$ be a rotation over a very small amount of time, thus a very small rotation from this we get:

$$\boldsymbol{q}(t+\Delta t) = \boldsymbol{q}(t) \otimes \boldsymbol{q}(\Delta t) \tag{C.15}$$

Joining (C.14) with (C.15) gives:

$$\frac{\mathrm{d}\boldsymbol{q}(t)}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{\boldsymbol{q}(t+\Delta t) - \boldsymbol{q}(t)}{\Delta t} \tag{C.16}$$

Looking closer to the $\boldsymbol{q}(\Delta t)$ and the approximation, that for a very small amount of time the rotational angle is very small and $\lim_{\Delta\theta \to 0} e sin(\Delta\theta) = \Delta\theta$, where $e \leq 1$ thus giving:

$$\boldsymbol{q}(\Delta t) = \begin{bmatrix} \hat{\boldsymbol{e}}\sin(\frac{\Delta\theta}{2}) \\ \cos(\frac{\Delta\theta}{2}) \end{bmatrix} \approx \begin{bmatrix} \frac{\boldsymbol{\Delta\theta}}{2} \\ 1 \end{bmatrix} \tag{C.17}$$

Combining (C.17), (C.16) and (C.15) and rewriting using $1 = \frac{\Delta t}{\Delta t}$ and $\boldsymbol{\omega} = \frac{\Delta \theta}{\Delta t}$ gives:

$$
\begin{aligned}
\frac{\mathrm{d}\boldsymbol{q}(t)}{\mathrm{d}t} &= \lim_{\Delta t \to 0} \frac{\boldsymbol{q}(t+\Delta t) - \boldsymbol{q}(t)}{\Delta t} \\
&\approx \lim_{\Delta t \to 0} \frac{\begin{bmatrix} \frac{\boldsymbol{\Delta\theta}\Delta t}{2\Delta t} \\ 1 \end{bmatrix} \otimes \boldsymbol{q}(t) - \boldsymbol{q}(t)}{\Delta t} \\
&\approx \lim_{\Delta t \to 0} \frac{\begin{bmatrix} \frac{\boldsymbol{\omega}\Delta t}{2} \\ 1 \end{bmatrix} \otimes \boldsymbol{q}(t) - \boldsymbol{q}(t)}{\Delta t}
\end{aligned} \tag{C.18}
$$

Carrying out the quaternion multiplication (C.7) in (C.18) gives the final representation of the time derivative of a quaternion:

$$
\begin{aligned}
\frac{\mathrm{d}\boldsymbol{q}(t)}{\mathrm{d}t} &= \lim_{\Delta t \to 0} \frac{\begin{bmatrix} -\boldsymbol{S}(\frac{\boldsymbol{\omega}\Delta t}{2}) + \boldsymbol{I}_{3\times 3} & \frac{\boldsymbol{\omega}\Delta t}{2} \\ -\frac{\boldsymbol{\omega}^{\mathrm{T}}\Delta t}{2} & 1 \end{bmatrix} \boldsymbol{q}(t) - \boldsymbol{I}_{4\times 4}\boldsymbol{q}(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{\left(\frac{\Delta t}{2} \begin{bmatrix} -\boldsymbol{S}(\boldsymbol{\omega}) & \boldsymbol{\omega} \\ -\boldsymbol{\omega}^{\mathrm{T}} & 0 \end{bmatrix} + \boldsymbol{I}_{4\times 4} \right) \boldsymbol{q}(t) - \boldsymbol{I}_{4\times 4}\boldsymbol{q}(t)}{\Delta t} \\
&= \frac{1}{2} \begin{bmatrix} -\boldsymbol{S}(\boldsymbol{\omega}) & \boldsymbol{\omega} \\ -\boldsymbol{\omega}^{\mathrm{T}} & 0 \end{bmatrix} \boldsymbol{q}(t)
\end{aligned} \tag{C.19}
$$

Where $\boldsymbol{S(\omega)}$ is the skew-symetric matrix (C.9) and $\boldsymbol{\omega}$ is the angular velocity vector of which the object observed is rotating to a reference frame given in the rotating frame.

So if we have a coordinate system $\mathbb{B}$ and $\mathbb{R}$ then the change in rotation from $\mathbb{R}$ to $\mathbb{B}$ between the two coordinate systems is given as:

$$
\frac{\mathrm{d}{}^{\mathbb{B}}_{\mathbb{R}}\boldsymbol{q}(t)}{\mathrm{d}t} = \frac{1}{2} \begin{bmatrix} -\boldsymbol{S}({}^{\mathbb{B}}_{\mathbb{R}}\boldsymbol{\omega}_{\mathbb{B}}) & {}^{\mathbb{B}}_{\mathbb{R}}\boldsymbol{\omega}_{\mathbb{B}} \\ -{}^{\mathbb{B}}_{\mathbb{R}}\boldsymbol{\omega}_{\mathbb{B}}^{\mathrm{T}} & 0 \end{bmatrix} {}^{\mathbb{B}}_{\mathbb{R}}\boldsymbol{q}(t) \tag{C.20}
$$