Force Directed Drawing

Vincent La

April 30, 2018

1 Introduction

2 Tutte's Barycenter Method

An early force directed drawing method was Tutte's Barycenter Method. In this method, the force on every vertex is given by

$$F(v) = \sum_{(u,v) \in E} (p_u - p_v)$$

Hence, we can ...

$$\sum_{(u,v) \in E} (x_u - x_v) = 0$$
$$\sum_{(u,v) \in E} (y_u - y_v) = 0$$

Which we may rewrite as

$$\deg(v) x_v - \sum_{u \in N_1(v)} x_u = \sum_{w \in N_0(v)} x_w^*$$
$$\deg(v) y_v - \sum_{u \in N_1(v)} y_u = \sum_{w \in N_0(v)} y_w^*$$

These equations are linear, and the resulting matrix is diagonally dominant (see Example 1.1). This is because the diagonal consists of vertex degrees, while the other entries a_{ij} are either -1's (if x_i and x_j are neighbors) or 0's if they aren't.

2.1 Example: Hypercube

A simple example for which Tutte's method gives aesthetically pleasing results is the hypercube.

In the image below (insert image later), the hypercube is placed in 500 x 500 pixel grid. The grid is governed by a simple Cartesian coordinate system, where the top left and bottom right corners have coordinates $(-250, 0)$ and $(250, 250)$ respectively. Four vertices are fixed and laid out into a circle of radius 250 centered at the origin. Hence, the bulk of the work performed algorithm is done in placing the center four free vertices. Labeling the free vertices as x_1, x_2, x_3, x_4, we may represent the task of laying out the free vertices with this matrix

$$\begin{bmatrix} 3 & -1 & 0 & -1 \\ -1 & 3 & -1 & 0 \\ 0 & -1 & 3 & -1 \\ -1 & 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 250 \\ 0 \\ -250 \end{bmatrix}$$

The solution to this matrix is given by $x_1 = x_3 = 0, x_2 = \frac{250}{3}, x_4 = -\frac{250}{3}$.

2.2 Resolution

One the the main drawbacks of this algorithm is potentially poor resolution. This is demonstrated best by the prism graph.

Theorem For any pair of adjacent "free" vertices in the prism graph drawn in the unit square, the distance between them is $O(\frac{1}{n})$.

Proof. Let P_n be the prism graph on n vertices and show that the distance between any two vertices is $O(n)$ by showing that $\text{dist}(p_i, p_{i+1}) \leq a \cdot \frac{1}{n}$. Let $a \geq \frac{2\sqrt{2}}{n}$.

First, for any p_i we have

$$p_i = \left(\frac{x_{i-1} + x_{i+1} + f_{ix}}{n}, \frac{y_{i-1} + y_{i+1} + f_{iy}}{n} \right)$$

and by extension

$$p_{i+1} = \left(\frac{x_i + x_{i+2} + f_{i+1x}}{n}, \frac{y_i + y_{i+2} + f_{i+1y}}{n} \right)$$

Thus,

$$\begin{aligned} &\text{dist}(p_i, p_{i+1}) \\ &= \sqrt{\left(\frac{x_i + x_{i+2} + f_{i+1} - x_{i-1} - x_{i+1} - f_i}{n} \right)^2 + \left(\frac{y_i + y_{i+2} + f_{i+1} - y_{i-1} - y_{i+1} - f_i}{n} \right)^2} \\ &= \sqrt{\left(\frac{x_{i+2} + f_{i+1}}{n} \right)^2 + \left(\frac{y_{i+2} + f_{i+1}}{n} \right)^2} \\ &= \sqrt{\left(\frac{2}{n} \right)^2 + \left(\frac{2}{n} \right)^2} && \text{Because } P \text{ is drawn} \\ &= \sqrt{\frac{8}{n^2}} \\ &= 2\sqrt{2}\sqrt{\frac{1}{n^2}} \\ &= \frac{2\sqrt{2}}{n} \leq a \cdot \frac{1}{n} \end{aligned}$$

as desired. $\square$