1 The Drinfeld center

In the previous section we saw that BFCs can be put on a spectrum ranging from symmetric fusion categories to modular categories. Physically, symmetric fusion categories are degenerate theories of anyons as they exhibit no topological behaviour. Conversely modular categories are very well behaved anyons theories. In this section we introduce a general construction that turns braided fusion categories into modular categories, and we show its relationship with the Quantum double construction on a Hopf Algebra introduced in the first chapter. Topological dependencies between objects in fusion categories are captured by the braided structure.

Definition 1 (Half-braiding) *A half-braiding on some object X in a monoidal category $\mathcal{C}$ is a natural isomorphism*

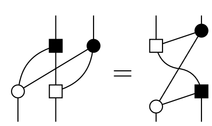
$$e^X : X \otimes (-) \Rightarrow (-) \otimes X$$

satisfying:

$$e^X_{Y\otimes Z} = (id_Y \otimes e^X_Z) \circ (e^X_Y \otimes id_Z)$$

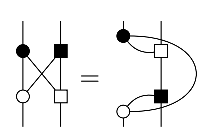
Definition 2 (Drinfeld center) *The braided (Drinfeld) center of a category $\mathcal{C}$ is the category $Z(\mathcal{C})$ with objects pairs (X, e^X) where $X \in \mathcal{C}$ and e_X is a half-braiding, and with morphisms given by the morphisms of $\mathcal{C}$ which commute with the half-braiding.*

Definition 3 (Yetter-Drinfeld modules) *Let H be a bialgebra, the category $\mathcal{D}^{lr}_H$ is the category of left-right Yetter-Drinfeld modules where objects are left H-modules which are simultaneously right H-comodules satisfying the following compatibility condition:*

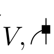


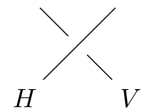
(1)

where the white box denotes the H coaction and the black box defines the right coaction. Morphisms of $\mathcal{D}^{lr}_H$ are both H-module and H-comodule morphisms. Left-left Yetter-Drinfeld modules are defined in the obvious way and form a category $\mathcal{D}^{ll}_H$. The compatibility condition then looks like this:

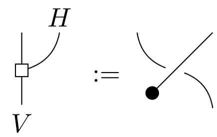


(2)

Fix a bialgebra H and suppose $\left(V,\right.$  $\left.\right) \in obj(Rep(H))$ and (V, e_V) is in $Z(RepG)$. Note that H has a natural H-module structure given by right multiplication. Consider the component of the half-braiding of V at H.

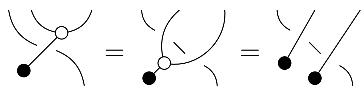


Define a right coaction of H on V by:



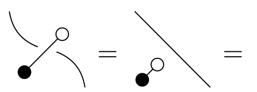
(3)

Note that from the bialgebra laws  and , seen as morphisms on the H-module H are intertwiners in $Rep(H)$. Therefore by naturality of the half braiding we get:



(4)

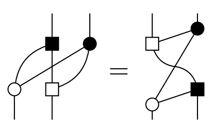
and



(5)

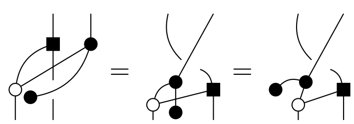
So that the coaction indeed defines a right H-comodule.

Claim 1

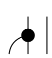


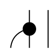
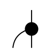
(6)

Proof As the braiding is an intertwiner, it commutes with the action of H on $V \otimes H$, therefore:



(7)

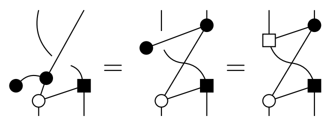
Now note that $\Big(H \otimes H,$  $\Big)$ is in $Rep(H)$ and

 $: \Big(H \otimes H,$  $\Big) \to \Big(H,$  $\Big)$

is an intertwiner by associativity. Also it is easy to check that the symmetry morphism lifted from $Vect$

$$\Big(H, \,|\,\Big) \otimes V \to V \otimes \Big(H, \,|\,\Big)$$

is an intertwiner. [But we don't know if it is the component of the half braiding on V...]   Therefore by naturality of the braid:

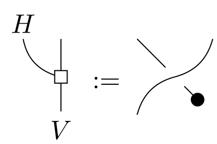


(8)

[This is only true if we are sure that the component of the half braiding on V at (H,trivial action) is the symmetry morphism right?]

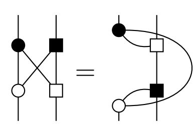
■

We have defined a functor $F_1 : Z(Rep(H)) \to \mathcal{D}_H^{lr}$ which is identity on arrows and sends (V, e_V) to the left-right Yetter-Drinfeld module with black H action and white H coaction. To see that it is well defined to say it is identity on arrows (and so faithful) note that if an H-module morphism f is in $Z(Rep(H))$ then it commutes with the half-brading, in particular it commutes with the H-component of the half-braiding and therefore it commutes with the H-coaction as defined.
Similarly we can define a functor $F_2 : Z(Rep(H)) \to \mathcal{D}_H^{ll}$ by considering the V component of the half braiding on the H-module H and defining the following left-coaction:



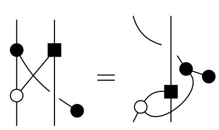
(9)

Claim 2



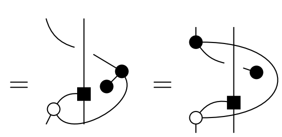
(10)

Proof The proof is very similar to that of the previous claim. Using the fact that the braid is an intertwiner we obtain



(11)

Then using the unit law and the same trick as before we see that



(12)

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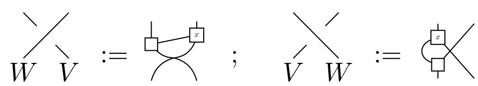
For the same reasons as for F_1, F_2 is faithful. To show F_1 and F_2 are equivalences of categories we still need to show they are full and essentially surjective.

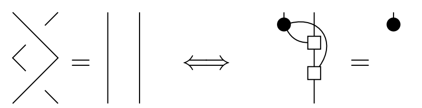
- F_1 is full: [The claims only work at the level of objects of the two categories, is it true that any morphism which commutes with the coaction also commute with the half braiding?] Suppose f is a morphism $V \to W$ in $\mathcal{D}_H^{lr}$,

[Is it true that the center of $Rep(H)$ is equivalent to left-right yetter drinfeld modules in general?] For any (W, x) define

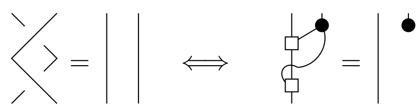


(13)

Claim 3

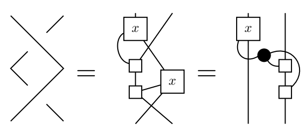


(14)



(15)

Proof We will only proove the first statement, the second proof is very similar. First note that from the definition and as W is a H-module:



(16)

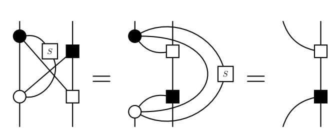
($\Longleftarrow$) is straightforward. To show ($\Longrightarrow$) set $W = H$ with the natural left module structure given by left multiplication, then inserting the unit state on the left of the tensor we obtain the required identity.

■

[This can be used to show that if a skew antipode exists then we get a braided structure? We are using both the right and left coaction on V...]

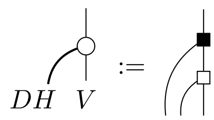
Proposition 1 *If H is a finite-dimensional Hopf Algebra then right-right Yetter Drinfeld modules are DH-modules.*

Proof The previous claims only assumed the bialgebra laws. Making use of the antipode and the compatibility condition we obtain:



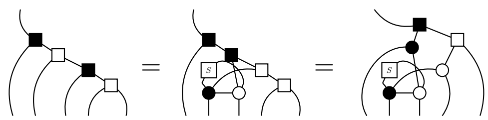
(17)

When H is finite dimensional, we can define the action of DH on V as follows (where thick wires carry DH and thin wires carry H)



(18)

This action gives V a DH-module structure as:



(19)

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Corollary 1 *If H is a finite dimensional Hopf algebra with invertible antipode $Z(RepH) \simeq RepDH$*

[haven't used invertible antipode...]