Hopf Algebras in Quantum Computation

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1 Introduction

Categories and diagrams
Symmetry, quantization and categorification
Categorification = replacing equalities with isomorphisms [1].
For an account of the relationships between categorification and quantization consider [2].
Mathematically: from groups to quasitriangular hopf algebras, G to DG
Categorically: from symmetric fusion categories to modular categories
Physically: fermions/bosons to anyons, local symmetries to topological symmetries, 3D to 2D
Computation: from PQC to TQC, complexity theory
Logic: mirror the relationship, all statements about RepDG are statements in RepG, a modality, programming language

2 Diagrams and Hopf Algebras

2.1 Monoidal categories

In this section, we set in place the basic definitions and diagrammatic intuitions which we will use throughout the thesis. The standard reference about basic category theory results is [3]. Many of the definitions are taken from [**?**]. A more detailed and up to date survey on monoidal categories can be found in [4]. For an introduction to diagrammatic reasoning in monoidal categories consider the first two chapters of [**?**]. Many of the results in this section and their relationship to quantum mechanics can be found in [8].
Recall the definition of a category.

Definition 1 *A category $\mathcal{C}$ consists of the data:*

- *a collection of objects $obj(\mathcal{C})$*
- *a collection of morphisms (or arrows) $arr(\mathcal{C})$*
- *domain and codomain assignments $dom, cod : arr(\mathcal{C}) \to obj(\mathcal{C})$. For any two objects $a, b \in obj(\mathcal{C})$ we define the hom-set*

$$\mathcal{C}(a,b) := \{f \in arr(\mathcal{C}) : a = dom(f), b = cod(f)\}$$

- *for any triple of objects a, b, c a composition map*

$$c_{a,b,c} : \mathcal{C}(a,b) \times \mathcal{C}(b,c) \to \mathcal{C}(a,c)$$

We denote the composition by $g \circ f$, diagrammatically:

$$\begin{tikzcd} & b \arrow[dr, "g"] & \\ a \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & & c \end{tikzcd}$$

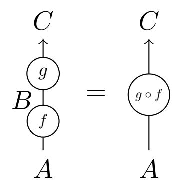
- *For any object a an identity morphism $id_a : a \to a$*

Satisfying the following axioms:

$$h \circ (g \circ f) = (h \circ g) \circ f \quad f \circ id_a = f = id_b \circ f$$

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & & C \end{tikzcd}$$

The commutativity of the above diagram is a statement about $\mathcal{C}$, and it has exactly the same information to its dual diagram. Where objects are one-dimensional wires and morphisms are (zero dimensional) boxes:



We will mainly use this second diagrammatic language in this work. When $\mathcal{C}$ is just a category we only have one way of composing morphisms and the language is one dimensional.

Example 1 *Examples of categories are: $Sets$ of sets and functions, $FSets$ of finite sets and functions, Rel of sets and relations, $Vect_k$ of vector spaces over k and linear maps and $FVect_k$ of finite dimensional vector spaces and linear maps.*

Category theory is a really good language for talking about equivalences and relationships between structures. This is achieved with the following tools.

Definition 2 (Functor) *A functor $F : \mathcal{C} \to \mathcal{D}$ is a mapping that*

- *associates an object $F(X)$ of $\mathcal{D}$ to each object X of $\mathcal{C}$.*
- *associates to each morphism $f : X \to Y$ a morphism $F(f) : F(X) \to F(Y)$ such that $F(id_X) = id_Y$ and $F(g \circ f) = F(g) \circ F(f)$ for all morphisms $f : X \to Y$ and $g : Y \to Z$.*

For instance there is a functor $Q : Sets \to Vect_k$ called '1st quantization' and taking a set to the free vector space generated by that set. Given two functors with matching source and target we can have natural transformations between them

Definition 3 (Natural Transformation) *Given categories $\mathcal{C}$ and $\mathcal{D}$ and functors $F, G : \mathcal{C} \to \mathcal{D}$ a natural transformation $\alpha : F \Rightarrow G$ is an assignment to every object a in $\mathcal{C}$ of a morphism $\alpha_a : F(a) \to G(a)$ in $\mathcal{D}$ such that for each morphism $f : a \to b$, the following commutes:*

$$
\begin{tikzcd}
G(a) \arrow[r, "G(f)"] & G(b) \\
F(a) \arrow[u, "\alpha_a"] \arrow[r, "F(f)"'] & F(b) \arrow[u, "\alpha_b"']
\end{tikzcd}
$$

A natural isomorphism is a natural transformation such that all components are isomorphisms.

Recall that a monoid is a triple $(X, \times, 1)$ where X is a set, $1 \in X$ and $\times$ is an associative and unital multiplication on X. The notion of a monoidal category is the categorification of a monoid. Elements of the set are replaced by objects in a category $\mathcal{C}$, multiplication by a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and the equalities in the unit and association axioms are replaced by natural isomorphisms. In order for this new structure to be well-behaved we will also need to impose compatibility conditions. We obtain the following definition:

Definition 4 (Monoidal category) *A monoidal category is a category $\mathcal{C}$ equipped with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called tensor product, an object 1 called unit object, a natural isomorphism*

$$a : - \otimes (- \otimes -) \overset{\sim}{\Rightarrow} (- \otimes -) \otimes -$$

called associator, a natural isomorphism

$$\lambda : 1 \otimes (-) \Rightarrow (-)$$

called left unitor and a natural isomorphism

$$\rho : (-) \otimes 1 \Rightarrow (-)$$

called right unitor. Subject to the following coherence conditions holding for all objects a, b, c, d in $\mathcal{C}$:

1. *Pentagon axiom: the following diagram commutes*

$$
\begin{tikzcd}
 & (a \otimes b) \otimes (c \otimes d) \arrow[dr, "\alpha_{a,b,c\otimes d}"] & \\
((a \otimes b) \otimes c) \otimes d \arrow[ur, "\alpha_{a\otimes b,c,d}"] \arrow[d, "\alpha_{a,b,c} \otimes id_d"'] & & a \otimes (b \otimes (c \otimes d)) \\
(a \otimes (b \otimes c)) \otimes d \arrow[rr, "\alpha_{a,b\otimes c,d}"'] & & a \otimes ((b \otimes c) \otimes d) \arrow[u, "id_a \otimes \alpha_{b,c,d}"']
\end{tikzcd}
$$

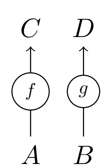
2. *Triangle identity: the following diagram commutes*

$$\begin{tikzcd}
(a \otimes 1) \otimes b \arrow[rr, "\alpha_{a,1,b}"] \arrow[dr, "\rho_a \otimes id_b"'] & & (a \otimes 1) \otimes b \arrow[dl, "id_a \otimes \lambda_b"] \\
& a \otimes b &
\end{tikzcd}$$

Let us give three important examples of monoidal categories.

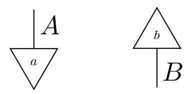
Example 2 *The category Sets of sets and functions is monoidal with the cartesian product $\times$ as bifunctor and the singleton set as unit object.*
The category $Vect_k$ of finite dimensional vector spaces over a field k is monoidal with the usual tensor product $\otimes$ and the one dimensional vector space k as unit object.
The category Rel of sets and relations is monoidal with the cartesian product $\times$ and the singleton as unit object.

The more structure comes with a category the more complicated diagrams we can draw. Monoidal categories have a two-dimensional diagrammatic language. The presence of unitors and associators and the conditions they satisfy make sure that this graphical language is well behaved. This is known as the coherence theorem for monoidal categories and can be found in [3]. It says that any well formed diagram in a monoidal category, made up of associators and unitors commutes. When the associators are trivial morphisms (i.e identity morphisms) we say the category is strict monoidal. It is known that every monoidal category is equivalent to a strict one [3], but it is sometimes useful to take associators into account as we will see in our discussion on permutational quantum computation. We write the tensor of two morphisms $f \otimes g : A \otimes B \to C \otimes D$ simply putting them side by side:



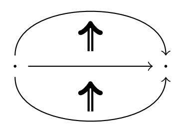
In our diagrams we can picture the unit I of the tensor as the plane on which we are drawing. Indeed we could imagine drawing as many copies as we wanted of id_I on the previous diagram to obtain obtain an equivalent diagram as $id_I \otimes f = f$ for any morphism f. So really the identity on I is just the empty diagram which we can stick next to any diagram we like.

Definition 5 (States and costates) *Given a system A, a state of is a morphism $1 \to A$. A costate (or effect) of A is a morphism $A \to 1$. In the diagrammatic language we draw states and costates respectively:*

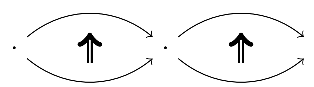


Remark It is perhaps useful to understand monoidal categories as degenerate 2-categories. Although this viewpoint requires one additional initial step of abstraction (the definition of a 2-category), it gives us the diagrammatic language for monoidal categories for free. For the rigorous definition of a 2-category we refer to [BAEZ], for our purposes we will only need the intuition. A 2-category is a collection of objects with 1-arrows between them and 2-arrows between the 1-arrows. Note that there are two ways of composing the 2-arrows:

- vertical composition:



- parallel composition:



Taking the dual of the above diagrams we obtain the diagrammatic language. Monoidal categories are 2-categories with only one 0-object called 1. We can think of the 0-object as the underlying plane, wires carry systems (1-arrows), boxes are morphisms (2-arrows). We recover the given definition of monoidal category by calling 1-arrow objects, and 2-arrows morphisms. The unit object 1 is then the identity 1-arrow $1 \to 1$ which is simply denoted 1.

Example 3 *The cartesian product in Sets $A \times B$ of sets A and B, satisfies the universal properties of a categorical product, in the sense that we have projections p_1 and p_2 such that if f and g are maps from some set C there is a unique function h making the following diagram commute:*

$$\begin{tikzcd}
 & A \times B \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
A & & B \\
 & C \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uu, "h"] &
\end{tikzcd}$$

Because of this property all states of $(Sets, \times)$ are separable. This category is the ambient Cartesian world of classical physics.

Example 4 *In $Vect_k$ states are vectors and costates are functionals. Note that the diagrammatic notation provides a two-dimensional generalisation of Dirac's notation. The category $Hilb$ of Hilbert spaces and linear maps is monoidal when equipped with the usual tensor product $\otimes$. Note that $\otimes$ is not a categorical product, and in fact we can have entangled states. Quantum mechanics is based on $(Hilb, \otimes)$ [8].*

Definition 6 (Scalars) *Scalars in a monoidal category are morphisms $1 \to 1$.*

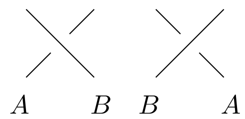
The category *Sets* has only one scalar. *Rel* has two scalars forming the cyclic group $\mathbb{Z}_2$ under composition. $Vect_k$ has scalars from k. Given a vector and a functional we obtain a scalar by composing them analogously to Dirac's formalism.

Definition 7 (BMC) *A braided monoidal category is a monoidal category $\mathcal{C}$ equipped with a natural isomorphism $B_{a,b} : a \otimes b \to b \otimes a$ called braiding, subject to the following compatibility conditions (called hexagon equations):*

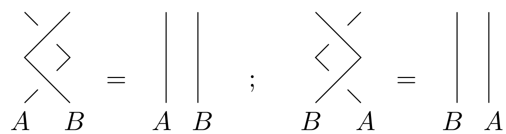
$$\begin{tikzcd}
 & a \otimes (b \otimes c) \arrow[r, "B_{a,b\otimes c}"] & (b \otimes c) \otimes a \arrow[dr, "\alpha_{b,c,a}"] & \\
(a \otimes b) \otimes c \arrow[ur, "\alpha_{a,b,c}"] \arrow[dr, "B_{a,b} \otimes id_c"'] & & & b \otimes (c \otimes a) \\
 & (b \otimes a) \otimes c \arrow[r, "\alpha_{b,a,c}"'] & b \otimes (a \otimes c) \arrow[ur, "id_b \otimes B_{a,c}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & (a \otimes b) \otimes c \arrow[r, "B_{a\otimes b,c}"] & c \otimes (a \otimes b) \arrow[dr, "\alpha_{c,a,b}"] & \\
a \otimes (b \otimes c) \arrow[ur, "\alpha_{a,b,c}"] \arrow[dr, "id_a \otimes B_{b,c}"'] & & & (c \otimes a) \otimes b \\
 & a \otimes (c \otimes b) \arrow[r, "\alpha_{a,c,b}"'] & (a \otimes c) \otimes b \arrow[ur, "B_{a,c} \otimes id_b"'] &
\end{tikzcd}$$

In the diagrammatic language this means we have braidings:

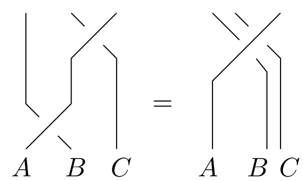


for any A and B, satifying:



(1)

The compatibility conditions are obvious statements in the diagrammatic calculus, for instance the first hexagon equation just says:



(2)

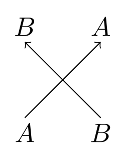
Both *Sets* and *Hilb* are examples of symmetric monoidal categories in the following sense.

Definition 8 (SMC) *A braided monoidal category is symmetric if the braiding* $B_{a,b}$ *satisfies*

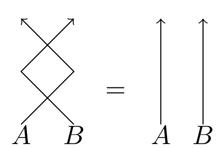
$$B_{a,b} \circ B_{b,a} = id_{a \otimes b}$$

For all objects a, b

In a SMC the braiding is called symmetry morphism and is denoted



It satisfies:



We will now describe some new classes of examples of monoidal categories. These are of a different nature to the categories we have seen so far.

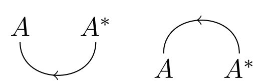
Definition 9 (PROPs) *A PROP (products and permutations category) is a strict symmetric monoidal category where every object is of the form* $x^{\otimes n}$ *for a single object* x *and* $n \geq 0$.

This means that we are only allowed one type of wire when drawing diagrams about $PROPs$ but we can use as many copies as we like and we can make swaps with them. Categories satisfying these properties are useful syntactic tools as we will see. One way to think of a $PROP$ A is as an abstract algebraic structure carrying some axioms, we can then instantiate those axioms in some other symmetric monoidal category $\mathcal{C}$ by considering symmetric monoidal functors $F : A \to \mathcal{C}$. We call such functors algebras or models of A in $\mathcal{C}$. If A is defined in terms of generators and relations (as is most often done), the choice of such functor corresponds to the choice of one object from $\mathcal{C}$ and morphisms on that object respecting the defining relations of A. On its own A has no clear interpretation, it just defines a syntax, but if $\mathcal{C}$ is a semantic category (i.e one with a clear interpretation) then F is a 'filling' of the syntax with meaning. This reasoning was first proposed in Lawvere's Phd thesis in 1963 [**?**].
It will sometimes be useful to drop the 'permutational' structure of $PROPs$.

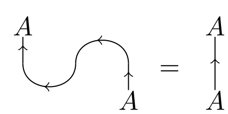
Definition 10 (PRO) *A PRO (products category) is a strict monoidal category where every object is of the form $x^{\otimes n}$ for a single object x and $n \geq 0$.*

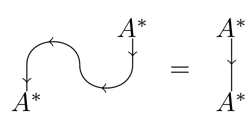
The semantic categories we will consider the most are *Sets* and *Hilb*. One important difference between them is that *Hilb* exhibits duality.

Definition 11 (Rigidity) *Let $\mathcal{C}$ be a monoidal category and $A \in obj(\mathcal{C})$. A left-dual of A is an object A^* with morphisms*



Satisfying the snake equations:



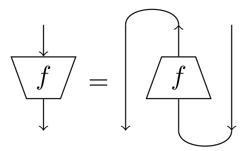


If every object has a left-dual, we say that $\mathcal{C}$ is left-rigid. Similarly we can define right-duals and right-rigid categories by interchanging the roles of A and $A^\star$ in the definition.

Given a (left/right) rigid structure we can define (left/right)transpose as follows.

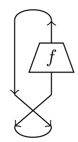
Definition 12 (Transpose) *Given a (left/right) rigid category $\mathcal{C}$ and any process $f : A \to B$ the (left/right) transpose f^* (or left transpose f^l, right transpose*

f^r if it is not clear from context) is:

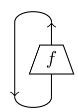


(3)

Definition 13 (Trace) *In a symmetric monoidal category $\mathcal{C}$, if A has a left dual $A^\star$, the trace of some morphism $f : A \to A$ is defined as the following scalar:*



A pivotal structure on a rigid monoidal category $\mathcal{C}$ is a natural isomorphism $id_{\mathcal{C}} \Rightarrow (-)^{\star\star}$. It allows to define traces without using the symmetry. Most categories we will consider have both sided duals, and therefore a trivial (identity) pivotal structure. Given a pivotal structure we can define left pivotal traces as:

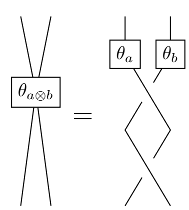


Where we have hidden the pivotal natural isomorphism. Similarly we can define right pivotal traces on endomorphisms in the obvious way.

Definition 14 *A rigid monoidal category with a pivotal strucutre is spherical if left and right traces coincide. In a spherical category, if a is an object, the trace $tr : End(a) \to End(1)$ is well defined and $tr(id_a)$ is called the categorical (or quantum) dimension of a.*

For a braided monoidal category, giving a spherical structure is equivalent to giving a ribbon structure [**?**] where:

Definition 15 *A ribbon (or twist) structure on a braided monoidal category with left duality $\star$ is a natural isomorphism $\theta : id_{\mathcal{C}} \Rightarrow id_{\mathcal{C}}$ satisfying:*



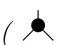
(4)

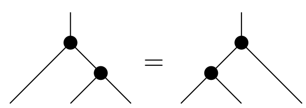
and compatible with the rigid structure $(\theta_a)^\star = \theta_{a^\star}$

Remark It can be shown that in the diagrammatic language, having a ribbon structure really corresponds to replacing wires by ribbons which we can twist. We won't use such diagrammatic language, but in the section on modular categories the twist will be an important process.

2.2 Hopf Algebras

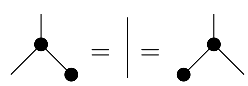
Now that we have set in place a diagrammatic machinery based on monoidal categories, let us make use of it. In this section we will meet some mathematical structures which have been used by mathematicians to describe symmetry. The notion of Hopf algebras is a powerful generalization of that of a group. Since their discovery in the 1940s, Hopf algebras have been used in various fields of pure mathematics (such as number theory, algebraic geometry, and representation theory) and have found applications in Quantum mechanics. Most of the results of this section can be found in [**?**].

Definition 16 (Monoid) Δ *is a PRO generated by morphisms (*  *,*  *) satisfying associativity:*



(5)

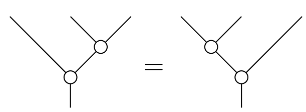
and the unit law:



(6)

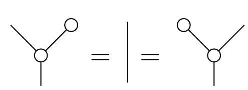
Models of Δ in monoidal categories are called monoids and they are very well known, examples include the natural numbers under addition, lists of some alphabet under concatenation and any group. Taking the opposite category Δ^{op} corresponds to flipping all the diagrams.

Definition 17 (Comonoid) Δ^{op} *is a PRO generated (*  *,*  *) satisfying coassociativity:*



(7)

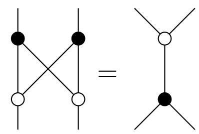
and the counit law:



(8)

Models of these are comonoids, the most common example is the copy map on any set with 'delete' as counit. Monoids and comonoids are simple structures that we can stick together to form more complicated ones. Bialgebras arise from one type of interaction of a monoid and comonoid.

Definition 18 (Bialg) *Bialg is a PROP generated by (, , ,), where and form a monoid, and a comonoid and the morphisms additionally satisfy the following laws:*



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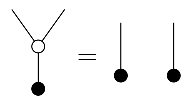
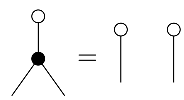


(10)



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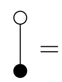
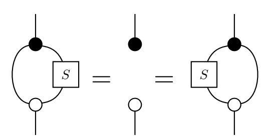


(12)

Where the empty diagram is the identity on the tensor unit.

Models of $Bialg$ in $Vect$ are bialgebras. We leave examples for later as we are now ready to introduce one of the main topics of this thesis.

Definition 19 (Hopf) *Hopf is a PROP generated by (, , , ,). Where (, , ,) is a bialgebra and the antipode S satisfies the Hopf law:*



(13)

We will argue that $Hopf$ is a good syntax to talk about symmetry. Let us start by instantiating $G : Hopf \to Sets$. This corresponds to choosing a set G, with a binary function $G \times G \to G$ (or multiplication) with a unit. Using the counit rule it is easy to see that the comultiplication in $Sets$ must be the copy map $g \mapsto (g, g)$ so that the antipode is the morphism $g \mapsto g^{-1}$ and G forms a group. Since the 19^{th} century groups have been used by mathematicians and physicists to describe symmetry.

Example 5 (Finite groups) *We will only make use of the following classes of finite groups:*

- $\mathbb{Z}_n$ *the cyclic group with n elements.*
- S_n *the symmetric group, can be seen as the group of permutations of a set with n elements, has order $n!$. S_3 is the smallest non-abelian group up to isomorphism.*

Example 6 (Groups of matrices) *Here we will fix some notation on the infinite groups of matrices we will meet. All matrices we will consider are over the complex numbers.* $GL(n)$ *is the group of invertible* n *by* n *complex matrices.* $U(n)$ *is the group of unitary* $n \times n$ *matrices (i.e such that* $U^\dagger U = UU^\dagger = I$*). The special unitary group* $SU(n)$ *is the subgroup of* $U(n)$ *consisting of matrices with determinant* 1*. The representation theory of* $SU(n)$ *is widely used in particle physics, for instance representations of* $SU(2)$ *model the behaviour of spin-*$\frac{1}{2}$ *particles.*

If we take a model of $H : Hopf \to Vect$ we obtain what is known as a Hopf Algebra.

Example 7 (Group algebras) *If* G *is a group with unit* e*, the group algebra* $\mathbb{C}G$ *(of dimension* $|G|$*) is a hopf algebra with multiplication linearly generated by* $|g\rangle \otimes |h\rangle \to |gh\rangle$*, unit* $|e\rangle$*, comultplication generated by* $|g\rangle \to |g\rangle \otimes |g\rangle$ *and counit* $\sum_g \langle g|$*.*

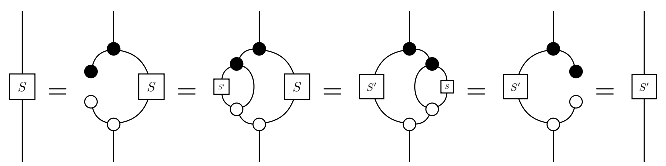
The previous example gives the usual definition of a group algebra which, for finite sets and finite dimensional vector spaces is just the composition $Q \circ G$ (as shown in the diagram) where Q is the 1st quantization functor. It is easy to see that Q preserves the monoidal structure as well as the symmetry morphisms (we say Q is a symmetric monoidal functor) so that the composition is also symmetric monoidal and $Q \circ G$ is a model of $Hopf$.

$$
\begin{tikzcd}
 & Hopf \arrow[dl, "G"'] \arrow[dr, "\mathbb{C}G"] & \\
FSets \arrow[rr, "Q"'] & & FVect
\end{tikzcd}
$$

In this case the comultiplication in $Hilb$ is the linearisation of the copy map (the copy map on some basis extended linearly to the whole Hilbert space) which is co-commutative. For a general $H : Hopf \to Vect$ this doesn't have to be the case. Hopf algebras provide a broader framework to talk about symmetry, as we can have non co-commutative Hopf algebras. We can see it as a quantization of the notion of symmetry, it will allow us to describe symmetries of quantum systems. Physically we will see that Hopf algebras allow to talk about local symmetries and exchange statistics on the same footing [**?**]. In particular if the Hopf algebra is not cocommutative the exchange statistics can be highly non-trivial, in which case they will describe the symmetries of anyons. The following two propositions are simple but important results about the antipode of a hopf algebra.

Proposition 1 *The antipode of a Hopf algebra is unique. It follows that being a Hopf algebra is a property of bialgebras.*

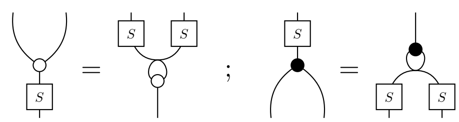
Proof Suppose S and S' are two antipodes for some Hopf algebra, then:



(14)

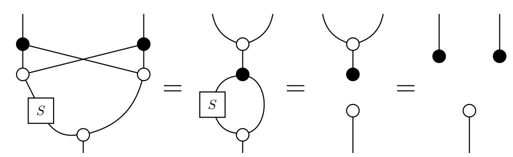
■

Proposition 2 *The antipode is an anti-(co)algebra homomorphism.*

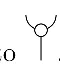


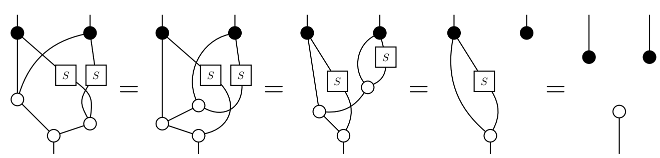
(15)

Proof First note that:

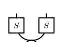


(16)

So that  is a left convolution inverse to .
Also:

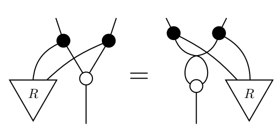


(17)

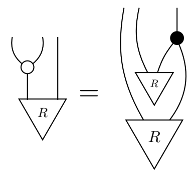
So that  is a right convolution inverse to . And it is easy to see using associativity and co-associativity that right and left convolution inverses must coincide. We deduce that the antipode is an anti-coalgebra homomorphism. For a proof that the antipode is an anti-algebra morphism simply flip all the diagrams and interchange white with black.

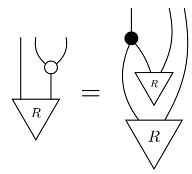
■

Definition 20 (Quasitriangularity) *A Hopf algebra H is quasitriangular if there is an invertible element $R \in H \otimes H$ satisfying the following equations:*



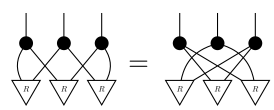
(18)

 (19)

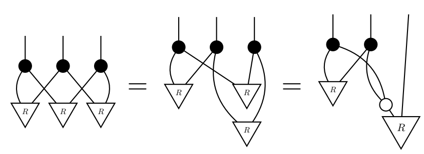
 (20)

R is called the 'universal R-matrix', and it can be thought as controlling the non-cocommutativity of the Hopf algebra. Quasitriangular hopf-algebras are sometimes called Quantum groups. We will see that they exhibit topological behaviour, as the following proposition hints to.

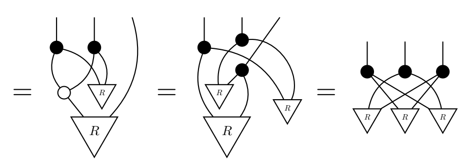
Proposition 3 *The universal R-matrix satisfies the Quantum Yang-Baxter equation:*

 (21)

Proof First using isotopy invariance and the second rule of quaitriangularity we get:

 (22)

Then using the first rule:

 (23)

■

Example 8 *The most trivial example of quasitriangular hopf algebras are the cocommutative ones. It is easy to check that if H is cocommutative, it is quasitriangular with*  *as R-matrix.*

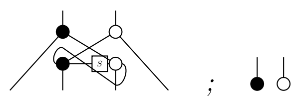
We will only be considering finite dimensional Hopf Algebras, as for finite dimensional vector spaces, these always have duals.

Definition 21 (Dual Hopf Algebra) *For a finite dimensional Hopf Algebra H the dual Hopf algebra is the vector space $H^\star$ of linear functionals on H with Hopf Algebra structure given by transposing all of the structure.*

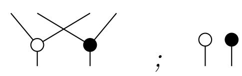
Given any finite dimensional hopf algebra H with invertible antipode there is a standard way of constructing a Quasitriangular Hopf Algebra first introduced by Drinfeld [**?**]. It will be implicit from now on that all Hopf algebras (and vector spaces) are finite-dimensional unless stated otherwise.

Definition 22 (Quantum double of a Hopf algebra) *The quantum double of a finite dimensional Hopf algebra $(H, \mu, 1, \Delta, \epsilon, S)$ with invertible antipode is the vector space $H \otimes H^\star$, with the following structure:*

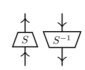
- *multiplication and unit:*



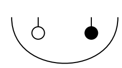
- *comultiplication and counit:*



- *antipode:*



It is easy to check this is indeed a Hopf algebra and that it is quasitriangular with universal R-matrix:

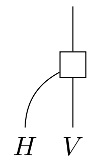


2.3 Representations of Hopf algebras

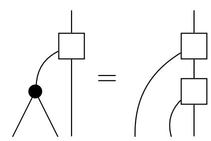
Recall that a group describes the symmetries of some space X when it acts on it (e.g crystals, classical symmetries= symmetries of sets). If we apply the same reasoning to Hopf Algebras we have to make H act on some quantum state space (i.e Hilbert space). So our object of study is not H on its own but rather a module (or representation) of H.

Where V is a finite dimensional vector space. Note that the above diagram represents a linear map, all diagrams we will be drawing in this section are diagrams in $Hilb$. In order for V to be a representation the following must hold.

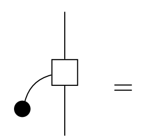
Definition 23 (Module) *Let H be bialgebra, a (left) H-module (or representation of H) is a vector space V together with a (left) action of H on V.*



Satisfying the following conditions:



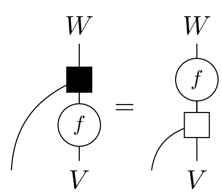
(24)



(25)

A right H-module is defined similarly with a right H-action.

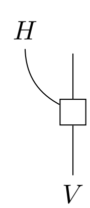
Suppose V and W are representations of H, then we say $f : V \to W$ (a linear map) is a H-module homomorphism (or intertwiner) if:



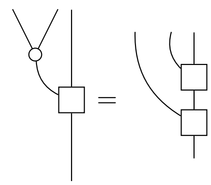
(26)

Where the black square denotes the action of H on W.
Dually we can define H-comodules and H-comodule homorphisms as follows.

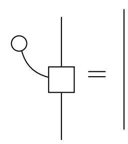
Definition 24 (Comodule) *Let H be bialgebra, an H-comodule is a vector space V together with a coaction of H on V.*



Satisfying the following conditions:



(27)



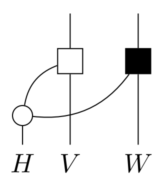
(28)

A right H-comodule is defined similarly with a right H-coaction. And H-comodule homomorphisms are linear maps which commute with the H-coaction.

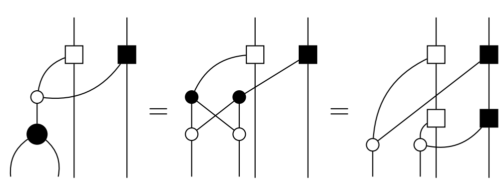
Let us consider, the category $Rep(H)$ where objects are representations of H and morphisms intertwiners. It is easy to see that the axioms of a category are satisfied, composition is just lifted from vector spaces. This category has really nice structure induced from the defining axioms of hopf algebras.

Proposition 4 *$Rep(H)$ is a monoidal category for any bialgebra H with tensor unit the trivial one-dimensional representation* $(\mathbb{C}, ⚲)$.

Proof Given H-modules V and W (with white and black actions respectively), $V \otimes W$ has natural H-module structure induced by the comultiplication:

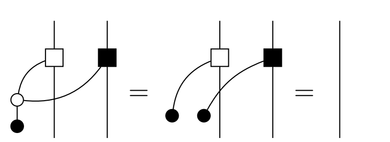


And $V \otimes W$ with this action is indeed a module as:



(29)

Also:



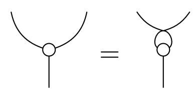
(30)

Using the bialgebra law and the fact that V and W are H-modules. Showing that $(\mathbb{C}, ⚲)$ is the tensor unit is a trivial application of the counit law.

■

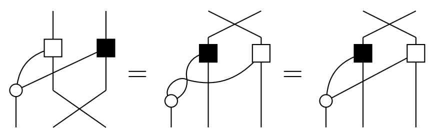
Proposition 5 *If H is cocommutative, then $Rep(H)$ is symmetric.*

Proof Cocommutativity means:



(31)

So the symmetry morphism on $V \otimes W$ from $Vect$ is an intertwiner:



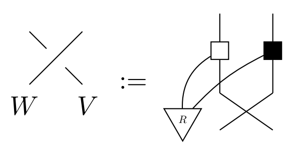
(32)

■

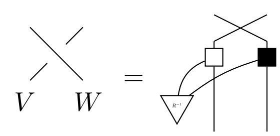
Recall that when H is cocommutative, it is trivially quasitriangular. The following is an important generalisation of the previous result.

Proposition 6 *If H is quasitriangular, then $Rep(H)$ is braided.*

Proof For any H-modules V and W, using the symmetry morphism from $Vect$ define:

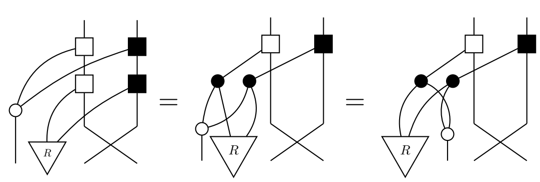


(33)



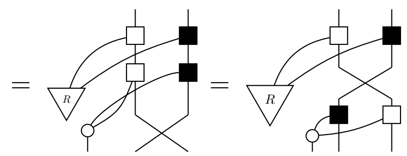
(34)

It is easy to see these are inverses of each other, we just need to check they are intertwiners.



(35)

Using H-module definition and the defining relation for R.



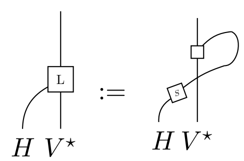
(36)

And a similar proof works for the inverse.

■

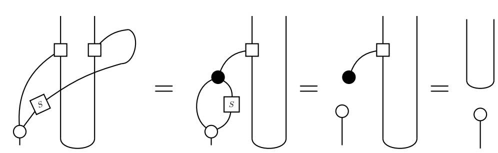
Proposition 7 *If H is a Hopf algebra, then $Rep(H)$ is left-rigid.*

Proof For any H-module V let $V^\star$ be its dual in $Vect$, we can define a dual H-action on $V^\star$ using the antipode:



(37)

Then the usual cups and caps from $Hilb$ are intertwiners.

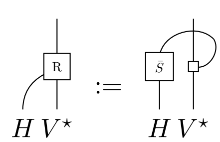


(38)

Using the module laws and the antipode law. A similar derivation holds for the cap.

■

We can see that the proof relies on the existence of the antipode. If a skew-antipode $\bar{S}$ exists, $Rep(H)$ is right-rigid, where the right dual is defined:



(39)

The proof that this choice works is very similar to the one above. In particular, $\bar{S}$ exists when the antipode is an invertible morphism as we can define $\bar{S} = S - S^{-1}$ If the antipode coincides with the skew antipode then $Rep(H)$ then left and right duals in $Rep(H)$ coincide, we say it is rigid.

2.4 Tannaka duality

This section is only meant as a motivation for the study of Hopf Algebras and we wont prove the reconstruction theorems, surveys on Tannaka reconstruction are given by [**?**] and [**?**].
Reconstruction results are recipes which produce all the examples of a class of categories (i.e categories with some fixed structure) from simpler mathematical objects. As we have seen in the previous sections, the structure of categories of H-modules is induced from the axioms of the Hopf Algebra H. It is surprising that Hopf algebras underly most of the categories with this structure.

Theorem 8 (Tannaka reconstruction)

- *Any monoidal category $\mathcal{C}$ equipped with a fiber functor (i.e a strict monoidal functor) $U : \mathcal{C} \to Vect$ is equivalent to Rep(B) where B is a bialgebra.*
- *Any (braided) rigid monoidal category equipped with a fiber functor (here this means strict (braided) rigid monoidal functor) to $Vect$ is equivalent to Rep(H) for some (quasitriangular) hopf algebra H.*

Note that any category can be seen as a process theory in the sense of [**?**] and [**?**]. Objects are systems and morphisms are their possible physical evolutions. The tensor product of a monoidal category can then be regarded as a way of forming composed systems. Quantum systems usually exhibit duality (particle, antiparticle pairs) and entanglement which are captured by the rigid structure of the category. From this perspective, this reconstruction result has an interesting physical interpretation. It says that any physical theory (monoidal category) is completely determined by the symmetries of the systems under consideration (the algebra structure). In the next section we will take this reasoning further to study physical theories of certain topological quantum systems.

3 The Algebra of Anyons

In this section we introduce the physics of Anyons and use the framework developed in the first section to define categorical models for theories of these particles. For an introduction to the physics of anyons consider the foundational paper [**?**] or Simon's notes [**?**], for a categorical presentation [**?**] and for a thorough survey of the mathematical aspects of anyons [2].
In the first section we introduce the physics and make the link with braided fusion categories, in the second part of we will develop the categorical formalism and the third part is dedicated to one result where quantification and categorical constructions play an important role.

To understand how anyons arise physically, let us consider n indistinguishable particles evolving in space. The quantum amplitude for a space-time evolution of the system will depend on the topology of the particle word-lines and not on the detailed geometry. This means that isotopic space-time evolution will yield the same amplitude.

To formalize the situation suppose we have n indistinguishable particles in D dimensions, the configuration space can be written as:

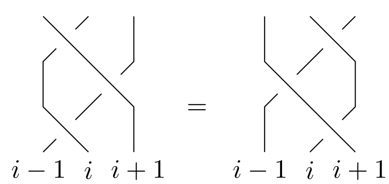
$$\mathrm{C} = (\mathbb{R}^{nD} - \Delta)/S_n$$

Where Δ is the space of coincidences (where at least two of the n particles occupy the same position in $\mathbb{R}^D$). We are quotienting the space by S_n to account for the indistinguishability of the particles (i.e we do not care about the order of the n coordinates in D dimensions). Let us fix the starting and endpoint in the configuration space, the space of paths from starting to endpoint C divides into topologically distinct classes, described by the fundamental group $\pi_1(\mathrm{C})$. These classes account for the different possible exchange statistics of the particles.
We can then describe the evolution of the wave function for the system via unitary transformations induced from the element of the fundamental group corresponding to particles word-lines. In mathematical terms this corresponds to a representation of $\pi_1(\mathrm{C})$.
If space-time has $D = 3 + 1$ dimensions, the topological class of paths is completely determined by the corresponding permutation of the particles, because there are no knots in 4 dimensions. Therefore the evolution of the system under particle exchanges will be described by a representation of the symmetric group S_n. In $2 + 1$ dimensions we have more exotic behaviour, as the paths in configuration space can braid. The time evolution of the wave function is then described by a representation of the braid group on n strands, denoted B_n.

Definition 25 (Braid group) *The braid group on n strands B_n is the group generated by $\{\sigma_i : i = 1, .., n-1\}$ satisfying the following relations:*

- $\sigma_i \sigma_j = \sigma_j \sigma_i$ *for* $i + 1 < j$
- $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$ *for* $1 < i < n$.

The second relation is called Yang-Baxter equation and can be drawn as follows:



(40)

- Abelian case

We say the system is abelian if the wave function lives in a one-dimensional representation of the group of paths in configuration space. In $3 + 1$ dimensions, this means we have to consider the one-dimensional representations of S_N. Note that there are only two possibilities (namely the trivial and the sign representations) corresponding to the two possible types of particle statistics in $3 + 1$ dimensions (Bose and Fermi statistics respectively). In $2 + 1$ dimensions we have many more possibilities as the evolution of the wave function will be described by a one-dimensional representation of the braid group B_N. There are infinitely many one dimensional representations of the braid group connecting the fermions and bosons case. These are described by a single parameter θ. Only one parameter because using Yang-Baxter we can show that all N phases have to be the same, also can show θ has to be a fraction from physical considerations. We obtain abelian anyons.

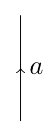
- Non-abelian case

In the non-abelian case, the wave function lives in a higher-dimensional representation of In $3+1$ dimensions we don't get anything more than bosons and fermions if we also want to consider creation, annihilation (splitting, fusion) of particles (Doplicher-Roberts theorem). In $2+1$ dimensions we obtain degeneracy, non-abelian anyons, braidings give all unitaries.

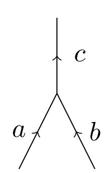
In the previous section, we only considered groups of symmetries of a Hamiltonian. In order take topological symmetries of a system into account, we need the more general framework of Hopf Algebras. In particular, as those symmetries arise from braids, we need quasitriangular hopf algebras (or quantum groups) to treat all symmetries on the same level. We will see that the universal R-matrix plays an important role in the description of topological dependencies.

3.1 Models of anyons

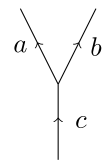
We want to construct a category $\mathcal{C}$ that models the behaviour of anyons. Ojects of $\mathcal{C}$ will correspond to quantum systems and morphisms to their possible evolutions, or to the processes we can perform on them.
Let us first set a finite set of labels $I = \{a, b, c...\}$ of distinct particle types, these will be objects of $\mathcal{C}$. In our theory we must be able to consider many particles at the same time, so $\mathcal{C}$ must be monoidal [**?**]. The unit of the tensor $\mathbf{1}$ corresponds to the vacuum particle type (or "no-particle") and must be within our labels. So for the moment our theory is a monoidal category $\mathcal{C}$ and we can already use the diagrammatic language. A particle of type a evolving trivially in time is denoted:



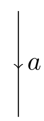
Where we have adopted the convention that time flows upwards.
Two particles of types a and b can fuse to a third particle of type c. So we have fusion morphisms:



Similarly a particle c can split to give two particles a and b. And $\mathcal{C}$ contains splitting morphism:



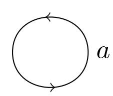
Any particle a in quantum physics comes with its antiparticle a^* which we can picture as a particle of type a travelling backwards in time.



It has the property of fusing to the vacuum when it encounters a. Dually the vacuum can yield a particle-antiparticle pair, so we have cups and caps morphisms

Categorically this corresponds to a rigid structure on $\mathcal{C}$, where we have assumed that every object has two-sided duals. We will also assume the category is well behaved: it is spherical and $1^* = 1$. This allows us to define the quantum numbers for each particle type a to be the following scalar:

$$d_a := tr(id_a) = \tag{41}$$



At this point we need to linearise the theory to take superpositions into account. This means we make $\mathcal{C}$ into a rigid tensor category (see appendix). We have biproducts $\oplus$ to account for superpositions. In order for the fusions to behave well with superpositions we must require that our labels for particle types be simple objects in the category and all objects to decompose as direct sums of simple ones, we say $\mathcal{C}$ is semisimple.

Definition 26 (Fusion category) *A fusion category is a finite semisimple k-linear tensor category with two sided duals*

At this point, our category $\mathcal{C}$ is a spherical fusion category and the fusion rules look like this:

$$a \otimes b \simeq \oplus_c N_{ab}^c c \tag{42}$$

Where $N_{ab}^c \in \mathbb{N}$. This defines a matrix for any a simple indexed by simple objects $i, j \in I$:

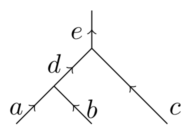
$$(N_a)_{i,j} = N_{ij}^a$$

We can also define the dimension of the theory $\mathcal{C}$ as the following scalar:

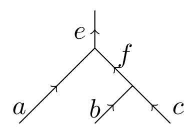
$$dim(\mathcal{C}) = \sum_{i\in I} d_i^2$$

Given the fusion structure we can define the F-matrix which contains the information of the fusions interacting with the quasi-associativity of the tensor product.

Definition 27 (F-matrix) *Given particles types a,b,c we have two ways of fusing them to obtain particle type e, the matrix F^e_{abc} is the change of basis matrix:*

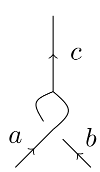


$$= \sum_{f\in I} (F^e_{abc})_{df} \tag{43}$$

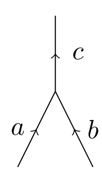


The possibilities for the F-matrices are constrained by the pentagon axiom of a monoidal category, it corresponds to a matrix representation of the associators. We still have one important question to ask to the theory, what happens when the position of two particles is exchanged? To answer this question the theory must have a braided structure and we obtain a braided fusion category. The braided structure determines the long-distance, topological interactions between particles. Braided fusion categories induce representations of the braid group B_n, given our discussion at the beginning of this chapter, we see that they are very good candidates for describing theories of anyons. The braided structure is captured by the following piece of data:

Definition 28 (R-matrix) *Given particle types a, b and c the matrix R^c_{ab} is defined by:*



$$= R^c_{ab} \tag{44}$$



From the first section we know that sphericality of the theory, interacting with the braided structure yields a ribbon structure. The twist θ has physical significance, it can be seen as a rotation of the particle and in most interesting cases it will be non-trivial.
In the case of abelian anyons, the twist is just a global phase, if we denote by h_a the topological spin of the particle then $\theta_a = e^{2\pi i h_a}$ is the twist factor of a. In this scenario, the R-matrices are scalars and it is easy to see, using the definition of the twist, that the R coefficients and the twist factors are related by:

$$R^c_{ab}R^c_{ba} = \frac{\theta_c}{\theta_a\theta_b}$$

These coefficients are also constrained by the hexagon axiom of braided monoidal categories. One way to build theories of abelian anyons, is to construct R matrices, twist factors and F matrices which satisfy both hexagon and pentagon axioms. However, these constraints do not fix R and F uniquely.

Example 9 (G-graded vector spaces) *Suppose we start from a set of labels and define the fusions to form a group.* 1 *is the identity particle type, for any particle type* a, a^* *will be its inverse. We have defined the skeleton of a spherical fusion category, which we obtain by linearising, i.e taking a fiber functor to* $Vect$. *We obtain the category* Vec_G, *of* G *graded vector spaces over* $\mathbb{C}$. *The category* Vec_G *for* G *a group is a symmetric spherical fusion category. Linearity and tensor are given by the underlying* $Vect$ *structure, simple objects* V_g *are one-dimensional and indexed by elements* $g \in G$, *duality is proved by using the group inverse and fusions are given by the group multiplication.*

$$V_g \otimes V_h \simeq V_{gh}$$

In this case both the F *and* R *matrices are trivial.*
Tannaka duality hints that this should be a category of representations and indeed it is easy to show that $Vec_G \simeq Rep(Func(G))$ *where* $Func(G)$ *is the function algebra on* G.
For $G = \mathbb{Z}_2$ *we have two irreducible representations* τ_+ *and* τ_-, *both one dimensional with the obvious fusion rules given by the cyclic group of order* 2.

Proposition 9 *If* H *is a finite dimensional, semisimple, quasitriangular Hopf algebra, then* $Rep(H)$ *is a braided fusion category and* $dim(Rep(H)) = dim(H)$.

Proof The proof is given by [find citation]

$\blacksquare$

This proposition gives us a way of building theories of anyons from hopf algebras. The first example that comes to mind is that of a group algebra $\mathbb{C}G$. So let us suppose the theory is described by the category $RepG$. First consider the object $V = \mathbb{C}G$. It is known that $\mathbb{C}G \simeq \oplus_{i \in I} X_i \otimes X_i^*$. Simple objects X_i correspond to particle types so V can be seen as the completely mixed state. This object (i.e the vector space with it's G-action) carries all the information of the theory (remember tannaka duality) and indeed we could study the theory by just considering this algebra. We can think of elements of G as particle subtypes, particle types correspond to conjugacy classes, a state $v \in V$ is a superposition of particle subtypes. The action of G permutes the basis vectors, and precisely corresponds to fusion. So acting with $g \in G$ on a state $v \in V$ corresponds to fusing a particle of type g with one that is in a superposition v of particle types.
In the case where G is abelian all irreducible representations are one dimensional, each corresponding to an element of the group. So really $Rep(G) \simeq Vec_G$ and behaves exactly like $\mathbb{C}G$ (without distinction between particle types and

subtypes). This case is perhaps interesting philosophically as the representations of our symmetries have the same structure as the symmetries themselves [cite majid self-duality]. From a computational perspective it is a trivial situation, as only classical processes can be performed (no entanglement is possible).
If G is not abelian we must have a higher dimensional irreducible representation of G. So we could obtain more interesting processes but from a topological quantum perspective it remains a trivial case as no computational power can be obtained from the braided structure. This is because $RepG$ is symmetric as we have seen in the first section. Physically, we have seen that symmetric exchange of particles applies to fermions and bosons, from a topological perspective those types of particles can be seen as degenerate cases of anyons. Groups are therefore not enough to describe interesting anyon theories. In the next section we pin down a smaller class of categories which correspond to non-degenerate theories of anyons.

3.2 Modular categories

In this section we define modular categories and state a few results that we will use in the next section. As we have seen, braided fusion categories are well suited to describe theories of anyons. These form a big class of categories, some of which are uninteresting from the physical point of view. To distinguish between them we can place braided fusion categories in a spectrum by asking what their symmetric center Z_2 is.

Definition 29 (Symmetric center) *If $\mathcal{C}$ is a monoidal category, the symmetric center $Z_2(\mathcal{C})$ is the full subcategory of $\mathcal{C}$ defined by:*

$$obj\ Z_2(\mathcal{C}) = \{X \in \mathcal{C} : c_{X,Y} \circ c_{Y,X} = id_{Y \otimes X} \quad \forall Y \in \mathcal{C}\}$$

It is easy to see that $\mathcal{C}$ is symmetric iff $Z_2(\mathcal{C}) = \mathcal{C}$.

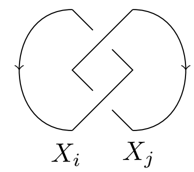
Definition 30 (Modular categories) *A braided fusion category is:*

- *pre-modular if it is spherical,*
- *non-degenerate if $Z_2(\mathcal{C})$ is trivial (i.e it only contains direct sums of the tensor unit as objects, i.e every simple object is isomorphic to the tensor unit)*
- *modular if it is pre-modular and non-degenerate.*

The two opposite ends of this spectrum are symmetric fusion categories on one side (such that $Z_2(\mathcal{C}) = \mathcal{C}$) and modular tensor categories (as defined). In the first case, we have only symmetric exchange of quantum systems which means all particles in the theory are either bosons or fermions. Such categories exhibit no topological behaviour. Modular categories are the opposite situation, the theory doesn't contain any bosons or fermions but only non-degenerate anyons (i.e anyons with non-trivial twist). Modular categories are very well-behaved theories as we can assign to them the so called modular S-matrix which will contain all the information on fusion rules as well as the braided structure.

Definition 31 (S-matrix) *Let $\mathcal{C}$ be a spherical braided fusion category and let I be the set of isomorphism classes of simple objects in $\mathcal{C}$. We define $S_{i,j}$ for $i,j \in I$ to be the following:*

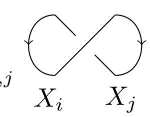
$$S_{i,j} := tr(B_{X_j,X_i} \circ B_{X_i,X_j}) = \tag{45}$$



Remark Note that it doesn't matter on which side we take the trace by sphericality.

Definition 32 (T-matrix) *Let $\mathcal{C}$ be a spherical braided fusion category, we define the T matrix (indexed by I) given by*

$$T_{i,j} := \delta_{i,j} tr(B_{X_i,X_j}) = \delta_{i,j} \tag{46}$$



Remark Categories of this type are called modular as it can be shown that S and T satisfy the same relations as the generators of the modular group $SL(2,\mathbb{Z})$, so that any modular category induces a representation of $SL(2,\mathbb{Z})$. It is a conjecture that the S ant T matrices determine modular categories up to ribbon equivalence.

Definition 33 (Mueger centralizer) *If $\mathcal{D}$ is a full (tensor) subcategory of $\mathcal{C}$ can define $C_{\mathcal{C}}(D)$ to be the full subcategory such that*

$$obj(C_{\mathcal{C}}(D)) = \{X \in obj(\mathcal{C}) : B_{X,Y} \circ B_{Y,X} = id_{X\otimes Y}\}$$

It is easy to check this is indeed a monoidal subcategory and it is replete (i.e closed under isomorphisms) [**?**]. Also note that $Z_2(\mathcal{C}) = C_{\mathcal{C}}(\mathcal{C})$. The following result is one of the most important pure category theoretic results on modular categories. We will need it for the discussion on the Drinfeld center.

Theorem 10 (Mueger decomposition) *Let $\mathcal{C}$ be a modular category and $\mathcal{K}$ a semisimple full tensor subcategory, then there is an equivalence of braided fusion categories:*

$$\mathcal{C} \simeq \mathcal{K} \boxtimes C_{\mathcal{C}}(\mathcal{K})$$

Proof This theorem was proved by Mueger in 2002 [**?**].

■

Theorem 11 *$\mathcal{C}$ is modular iff the S-matrix is invertible.*

Proof Suppose $\mathcal{C}$ is not modular, then $Z_2(\mathcal{C})$ is non-trivial $\implies$ there is a non-trivial simple object a such that its braiding is the symmetry. Therefore $S_{a,i} = d_a d_i$ for all $i \in I$, but also $S_{1,i} = d_i$ and so the first and ath rows of S are proportional $\implies$ S is not invertible.
The other direction is less easy and can be found in [?] and [?].

■

We also state the following result which is proved in [?].

Proposition 12 *The modular S-matrix diagonalises the N-matrix.*

Therefore the S-matrix contain all the information of the fusion rules, and with some algebraic manipulation (which can be found in [?]) we obtain the well-known Verlinde formula for the fusion coefficients:

$$N_{ij}^{k} = \sum_{r \in I} \frac{S_{ir} S_{jr} S_{kr}}{S_{1r}} \tag{47}$$

Remark Given any modular category we can use the Turaev-Viro construction [?], which yields a $2+1$ topological quantum field theory. For our purposes we only need to view the modular category as a process theory of anyons in the sense of [?] and we won't introduce the $TQFT$ formalism.

Example 10 (Fibonacci anyons) *The category Fib of Fibonacci anyons is one of the most popular examples of modular categories as it have a purely algebraic formulation. Anyons of this type are non-abelian and complete for topological quantum computation [?]. We will meet them again in the next chapter.*
Fib has only two simple objects: τ and the vacuum type 1. The fusion rules are given by:

$$1 \otimes \tau = \tau = \tau \otimes 1$$

$$\tau \otimes \tau = 1 \oplus \tau$$

It turns out that those equations together with the hexagon and pentagon constraints completely determine a modular category [?].

3.3 The Drinfeld center

In this section we introduce a general construction that turns braided fusion categories into modular categories, and we show its relationship with the Quantum double construction on a Hopf Algebra introduced in the first chapter.
Topological dependencies between objects in fusion categories are captured by the braided structure. Let us fix some definitions before discussing the Drinfeld construction.

Definition 34 (Half-braiding) *A half-braiding on some object X in a monoidal category $\mathcal{C}$ is a natural isomorphism*

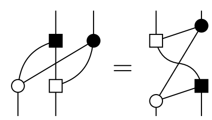
$$e^X : X \otimes (-) \Rightarrow (-) \otimes X$$

satisfying the compatibility condition:

$$e^X_{Y \otimes Z} = (id_Y \otimes e^X_Z) \circ (e^X_Y \otimes id_Z)$$

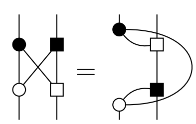
Definition 35 (Drinfeld center) *The braided (Drinfeld) center of a monoidal category $\mathcal{C}$ is the category $Z(\mathcal{C})$ with objects pairs (X, e^X) where $X \in \mathcal{C}$ and e^X is a half-braiding, and with morphisms given by the morphisms of $\mathcal{C}$ which commute with the half-braiding.*

Definition 36 (Yetter-Drinfeld modules) *Let H be a bialgebra, the category $\mathcal{D}^{lr}_H$ is the category of left-right Yetter-Drinfeld modules where objects are left H-modules which are simultaneously right H-comodules satisfying the following compatibility condition:*



(48)

where the white box denotes the H coaction and the black box defines the right coaction. Morphisms of $\mathcal{D}^{lr}_H$ are both H-module and H-comodule morphisms. Left-left Yetter-Drinfeld modules are defined in the obvious way and form a category $\mathcal{D}^{ll}_H$. The compatibility condition then looks like this:



(49)

Proposition 13 *Let $\mathcal{C}$ be a monoidal category, then $Z(\mathcal{C})$ is braided monoidal.*

Proof It is easy to check that defining the tensor as $(X \otimes Y, e^{X \otimes Y}_Z = (e^X_Z \otimes id_Y) \circ (id_X \otimes e^Y(Z))$ and the braiding as e^X_Y yields a braided monoidal structure on $Z(\mathcal{C})$.

■

The following proposition hints to the relationship between the Drinfeld center and the quantum double.

Proposition 14 *The Drinfeld center of a spherical fusion category is modular. And $dim(Z(\mathcal{C})) = dim(\mathcal{C})^2$*

Proof Proof is given by [**?**].

■

In general $Z(\mathcal{C})$ is not symmetric as we will see, but in the case of $Vect$ the Drinfeld construction is trivial.

Proposition 15 $Z(Vect) \simeq Vect$

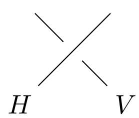
Proof Using the Mueger decomposition 10, note that $Z(Vect)$ is modular and $Vect$ is a full fusion subcategory of $Z(Vect)$, therefore

$$Z(Vect) \simeq Vect \boxtimes C_{Z(Vect)}(Vect)$$

But if (A, e^A) is an object of $C_{Z(Vect)}(Vect)$ then any component e^A_B must be the inverse of the symmetry morphism on $A \otimes B \implies$ it must be the symmetry morphism $\implies C_{Z(Vect)}(Vect) \simeq Vect$ and so $Z(Vect) \simeq Vect$

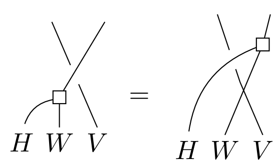
■

Fix a bialgebra H and suppose $\left(V, \right) \in obj(Rep(H))$ and (V, e_V) is in $Z(RepG)$. Note that H has a natural H-module structure given by right multiplication. Consider the component of the half-braiding of H at V.



In the arguments that follow we will use repeatedly the following trick which we state as a Lemma, it exploits the copy of $Vect$ which lives inside any category of representations.

Lemma 16 *For any W object of $Rep(H)$ with white action and V with half braiding.*



(50)

Proof Note that $\left(H \otimes W, \right)$ is in $Rep(H)$ and

$$: \left(H \otimes W, \right) \to \left(W, \right)$$

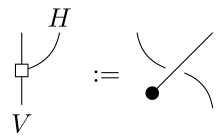
is an intertwiner by the module law. Also it is easy to check that the symmetry morphism lifted from $Vect$

$$\left(W, \right) \otimes V \to V \otimes \left(W, \right)$$

is an intertwiner. And it follows from $Z(Vect) = Vect$ that it must be the W-component (where W has the trivial action) of the half braiding on V as W lives in the copy of $Vect$ in $Rep(H)$.

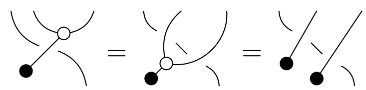
■

Define a right coaction of H on V:



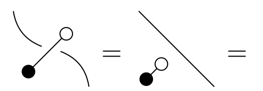
(51)

Note that, from the bialgebra laws,  and  (seen as morphisms on the H-module H) are intertwiners in $Rep(H)$. Therefore by naturality of the half braiding we get:



(52)

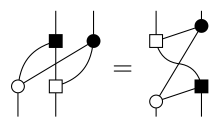
and



(53)

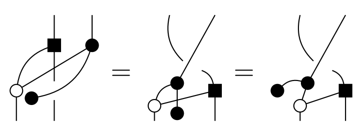
So that the coaction indeed defines a left H-comodule.

Claim 1



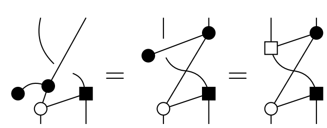
(54)

Proof As the braiding is an intertwiner, it commutes with the action of H on $V \otimes H$, therefore:



(55)

Therefore by Lemma 16 and naturality of the braid:

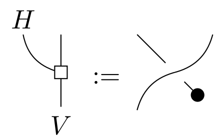


(56)

■

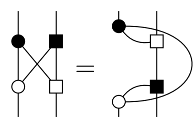
We have defined a functor $F_1 : Z(Rep(H)) \to \mathcal{D}_H^{lr}$ which is identity on arrows and sends (V, e_V) to the left-right Yetter-Drinfeld module with black H action and white H coaction. To see that it is well defined to say it is identity on arrows (and so faithful) note that if an H-module morphism f is in $Z(Rep(H))$ then it commutes with the half-brading, in particular it commutes with the H-component of the half-braiding and therefore it commutes with the H-coaction

as defined.
Similarly we can define a functor $F_2 : Z(Rep(H)) \to \mathcal{D}_H^{ll}$ by considering the H component of the half braiding on V and defining the following left-coaction:

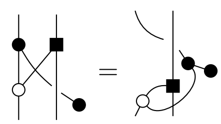


(57)

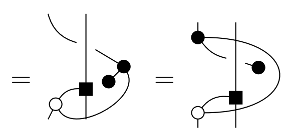
Claim 2


(58)

Proof The proof is very similar to that of the previous claim. Using the fact that the braid is an intertwiner we obtain


(59)

Then using the unit law and the same trick as before we see that

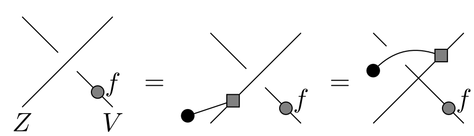

(60)

■

For the same reasons as for F_1, F_2 is faithful. To show F_1 and F_2 are equivalences of categories we still need to show they are full and essentially surjective.

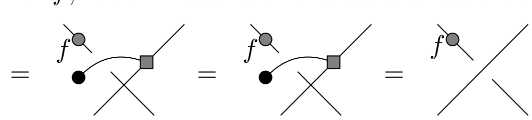
Proposition 17 *F_1 and F_2 are full.*

Proof Suppose f is a morphism $V \to W$ in $\mathcal{D}_H^{lr}$, then using the Lemma we see that for any Z in $Rep(H)$ with gray H-action:



(61)

And by definition of f, it commutes with the coaction so that:



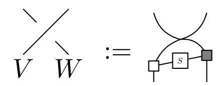
(62)

So f commutes with the half braiding $\implies$ it is a morphism in $Z(Rep(H))$. Therefore F_1 is full. And a similar proof applies to F_2.

■

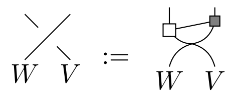
Proposition 18 *If H is a Hopf algebra, F_1 is essentially surjective.*

Proof To prove this we construct a half braiding for any object V of $\mathcal{D}_H^{lr}$ which yields the coaction of the form [cite equations].
Fix any object V with white right H-coaction and for any $(W, \ )$ define



(63)

(63) is an isomorphism as



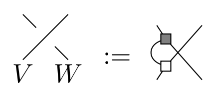
(64)

is an inverse by the hopf law. It is natural in W as all morphisms are intertwiners (so they commute with the H-action on W). And it satisfies the compatibility condition by definition of H-comodule. Clearly setting $W = H$ in (64) with the natural left-multiplication action, and inserting on the left of the tensor yields the H-coaction.

■

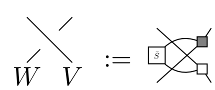
Proposition 19 *If H has a skew antipode, F_2 is essentially surjective.*

Proof Define



(65)

The same argument as the previous proposition applies defining the inverse using the skew antipode $\bar{S}$:

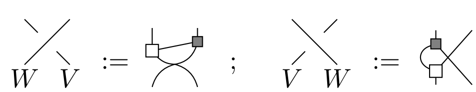


(66)

■

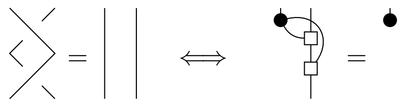
Corollary 1 *If H is a Hopf algebra then $Z(RepH) \simeq \mathcal{D}_H^{lr}$. If H has a skew antipode then $Z(RepH) \simeq \mathcal{D}_H^{ll}$.*

Let us see how the two kinds of Yetter Drinfeld modules interact with one another. For any $(W, \ )$ we have

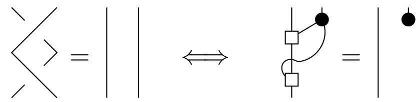


(67)

Claim 3

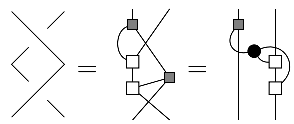


(68)



(69)

Proof We will only prove the first statement, the second proof is very similar. First note that from the definition and as W is a H-module:

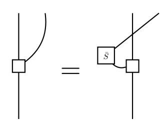


(70)

($\Longleftarrow$) is straightforward. To show ($\Longrightarrow$) set $W = H$ with the natural left module structure given by left multiplication, then inserting the unit state  on the left of the tensor we obtain the required identity.

■

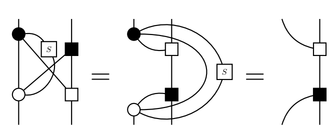
And we say that if H is a hopf algebra with a skew antipode we can make the right hand sides hold imposing:



(71)

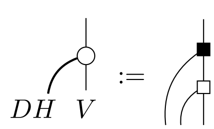
Proposition 20 *If H is a finite-dimensional Hopf Algebra with invertible antipode then left-left Yetter Drinfeld modules are DH-modules.*

Proof Note that defining DH requires the antipode to be invertible (it is used in the definition of the antipode for DH). Making use of the antipode and the compatibility condition we obtain:



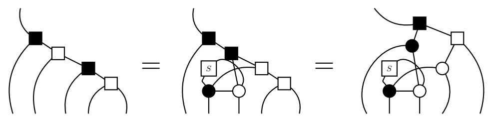
(72)

When H is finite dimensional, we can define the action of DH on V as follows (where thick wires carry DH and thin wires carry H)



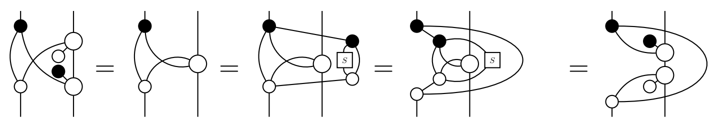
(73)

This action gives V a DH-module structure as:



(74)

Now it is easy to see that morphisms commute with the DH-action iff they commute with the H-action and H-coaction. So we have defined a fully faithful embedding of $\mathcal{D}_H^{ll}$ into $Rep(DH)$. To see it is essentially surjective note that, given a DH-action on V, we can recover the H-action by plugging the counit in the $H^\star$ component of the DH-action and the H-coaction by plugging the unit in the H-component and bending the $H^\star$ wire up. It remains to check that those indeed define a left-left Yetter-Drinfeld module in all cases, i.e that the compatibility condition is satisfied. And it is indeed the case:



(75)

■

Corollary 2 *If H is a finite dimensional Hopf algebra with invertible antipode $Z(RepH) \simeq RepDH$*

We have found many equivalent ways of constructing non-degenerate theories of anyons. In the next section we will use the simplest examples induced by groups, justified by the following proposition

Proposition 21 *If G is a finite non-abelian group then $Rep(D(G))$ is modular.*

Proof A direct proof is given in the third chapter of [?]. But this also follows from the fact that $Z(Rep(G))$ is modular.

■

4 Quantum Computation

4.1 Phases of Matter

Classical phases of matter: solid, liquid and gas.
Temperature is proportional to energy
Close to zero temperature, phases of matter are quantized. We obtain exotic behaviours of matter.
In order to talk about quantum phases of matter we need to consider many-body quantum systems. We will use the definitions of Zhenghan

Definition 37 (Many Body Quantum Systems) *A Many-body Quantum system (MQS), is a triple $(\mathcal{L}, b, \mathcal{H})$, where $\mathcal{L}$ is a Hilbert space with a distinguished ONB b and a hermitian operator $\mathcal{H} : \mathcal{L} \to \mathcal{L}$, called Hamiltonian.*

The eigenvalues of the Hamiltonian correspond to the energy levels of the system. The elements of the basis b are the initial classical states of the system. Many-body quantum systems will usually be obtained from spatial configurations of particles, which we will descibe by graphs.

Definition 38 *A Hamiltonian is k-local if...*

Definition 39 *An MQS on a graph $\mathrm{T} = (V, E)$ with $\mathbb{C}^d$ degrees of freedom (qudit space) is an MQS $(\mathcal{L}, b, \mathcal{H})$ where*

$$\mathcal{L} = \otimes_{e \in E} \mathbb{C}^d$$

b is obtained from the standard basis of $\mathbb{C}^d$ and $\mathcal{H}$ is a local Hamiltonian.

Many interesting spatial configurations of matter are obtained from triangulations of manifolds by taking their 1-skeleton graph.
Let us consider unitary operators which commute with the Hamiltonian $\mathcal{H}$. These are operators which leave the energy of the system unchanged. These transformations form a group G under composition and a particle subject to the Hamiltonian $\mathcal{H}$ will be described by irreducible representations of G. G is the group of symmetries of the system under $\mathcal{H}$.

Proposition 22 *Let $\mathcal{H} : V \to V$ be the Hamiltonian for some physical system described by a hilbert space V. Then the unitary operators which commute with $\mathcal{H}$ form a group G.*

Proof Suppose $R_1\mathcal{H} = \mathcal{H}R_1$ and $R_2\mathcal{H} = \mathcal{H}R_2$, then $R_1R_2\mathcal{H} = R_1\mathcal{H}R_2 = \mathcal{H}R_1R_2$. Also $R^{-1}\mathcal{H} = R^{-1}\mathcal{H}RR^{-1} = R^{-1}R\mathcal{H}R^{-1} = \mathcal{H}R^{-1}$. The unit is the identity operator $id_{V^* \otimes V}$.

■

Proposition 23 *If G is the group of symmetries of a Hamiltonian $\mathcal{H}$ then each energy eigenspace carries an irreducible representation of G.*

Proof Note that under the obvious G-action, V is a representation of G. The eigenvalues of the Hamiltonian, correspond to energy levels of the physical system which we previously called 'particle types'. Fix any eigenvalue E of $\mathcal{H}$, the allowed states of a particle with energy E live in the corresponding eigenspace V_E. Indeed these are invariant under the action of $\mathcal{H}$:

$$\mathcal{H} |\psi\rangle = E |\psi\rangle$$

Note that if $R \in G$ then

$$\mathcal{H}R |\psi\rangle = R\mathcal{H} |\psi\rangle = ER |\psi\rangle$$

So $R\left|\psi\right\rangle$ is an eigenvector with eigenvalue E. Therefore G acts on V_E for any energy level. We prove V_E is irreducible by showing that $End(V_E) \simeq \mathbb{C}$. Indeed suppose $f : V_E \to V_E$ is an intertwiner, then $fR = Rf \, \forall R \in G$ TO PROVE

■

Starting with a hamiltonian $\mathcal{H}$ we have shown that energy levels (or particle types) correspond to the irreducible representations of the group G of symmetries of $\mathcal{H}$. The particle theory correponding to the hamiltonian $\mathcal{H}$ has irreducible representations of G as objects and intertwiners, preserving the energy of the system, as processes. This is the definition of the category $Rep(G)$. Note there are no restrictions yet on the group of symmetries.

Example 11 *Suppose $\mathcal{H} = \sigma_Z$ be the Hamiltonian of a qubit living in $\mathbb{C}^2$. It has eigenvalues ± 1 and corresponding eigenvectors $\left|0\right\rangle$ and $\left|1\right\rangle$.. The group algebra of symmetries of $\mathcal{H}$ is generated by id and $\mathcal{H}$ and it is isomorphic to $\mathbb{C}\mathbb{Z}_2$. $\mathbb{Z}_2$ has two irreducible representations (the trivial and the sign) representations which correspond to the one-dimensional eigenspaces of $\mathcal{H}$. Note that*

4.2 Kitaev's model

Let us consider a two dimensional lattice of particles (situated on the edges of the lattice) under the influence of a magnetic field and an electric field. Anyonic behaviour is exhibited by excitations of the particles on the lattice. In our process theory the allowed processes are charge-flux composites, the states are lattice configurations (determined by the states of the particles, which are generally given by an element $g \in G$) . By mearuring the flux in certain regions of the lattice and acting on the charge of the corresponding flux sector we can create and control the behaviour of excitations on the material.

Example 12 *Let us consider the case where $G \simeq \mathbb{Z}_2$. We obtain a lattice of spins. Those particles we wont use as our qubits, indeed we will impose that their state is in a basis state $\{\left|0\right\rangle, \left|1\right\rangle\}$. One way to picture the states of the system is to draw the lattice and colour the edges red when the corresponding particle is in state $\left|0\right\rangle$. Note that the lattice can be embedded in any manifold (e.g. subsection quantum memory is a lattice on a torus, if we used a more layered lattice we could implement error correction). What we obtain is a picture of paths on the lattice which we call excitations.*

The magnetic flux of a particle is given by an element $h \in G$, this indexes the superselection sectors so that the charge lives in a unitary irreducible representation of the centralizer $N(h)$ of the flux h carried by the particle. We have two possible operations on our states: flux measurement and symmetry transformations on the charge. Flux measurements correspond to a projection $P_h \in \mathbb{C}G^*$ onto flux sector h. The residual global symmetry transformations are then implemented via some $g \in N(h)$.
Naturally the projectors form a Von Neumann family and satisfy

$$P_h P_{h'} = \delta_{h,h'} P_h.$$

A general element $g \in G$ is a global symmetry transformation and affects the fluxes via conjugation:

$$gP_h = P_{ghg^{-1}}g \tag{76}$$

The quantum double construction allows to capture both global symmetry transformations and projective measurements in one algebraic structure.

Definition 40 *Lagrangian, Noether's theorem*

Definition 41 *For any finite group G, its quantum double $D(G)$ is the algebra generated by $\{P_h g\}_{h,g\in G}$ with multiplication induced by (1). $D(G) \simeq \mathbb{C}G^* \otimes \mathbb{C}G$ and inherits their hopf algebra structure (comultiplication and antipode are given by tensoring).*

$D(G)$ has a natural quasi-triangular structure witnessed by the universal R-matrix $R = \sum_{g,h\in G} P_h e \otimes P_h g$, making $RepDG$ braided. Particle states then live in irreducible representations of $D(G)$. Let $\{C_i\}_{i=1}^n$ be the distinct conjugacy classes in G. To each of those conjugacy classes corresponds a centralizer subgroup N_i (two choices of representatives for C_i yield isomorphic centralizer subgroups). Then for any irreducible representation (α, V_α^i) of N_i with basis elements v_j^α, let $V_{i,\alpha} = \mathbb{C}C_i \otimes V_\alpha^i$, this has basis $\{|k, v_j^\alpha\rangle\}_{j=1,\ldots,dim\alpha}^{k\in C_i}$ and forms an irreducible representation of $D(G)$ under the action

$$P_h g\,|k, v_j^\alpha\rangle = \delta_{h,gkg^{-1}}\,|h, \alpha(h^{-1}gk)v_j^\alpha\rangle \tag{77}$$

and the $\{V_{i,\alpha}\}$ is the complete set of irreducible representations.

Remark Can we generalise the lattice construction? The following argument seems to work for abelian groups. Being a spherical fusion category, *RepG* is well suited to be a process theory of particles. As we pointed out earlier what is missing is the braided structure, but let us ignore this for the moment. We have simple objects corresponding to particle types and fusion rules determined by the group structure.

Now suppose we have particles whose fusion is described by a *RepG* and let us consider a lattice with those particles located at the edges. This lattice could be embedded in any space but let us assume it is a lattice on a torus. The states of our system are then configurations of particle types on the edges. So edges are coloured by simple representations from *RepG*. Now we can act on the lattice with vertex and plaquette operators which basically implement measurements at vertices and flips (fusion with other simple reps) of the particles on a plaquette. By now we have produced a theory where states are lattice configurations and processes are generated by vertex V_α and plaquette P_β operators. We want to extract the topological degrees of freedom of this theory. First we note that using vertex operators we can make sure that the product of all particles incident at all vertices is 1, i.e $V_\alpha = 1\ \forall\alpha$. We restrict our states to satisfy this local property and we drop vertex operators (in the sense that they are not allowed processes anymore), indeed we fixed their value at all points of the lattice. By also setting $P_\beta = 1$ at all plaquettes we quotient further the theory.

So we obtain a theory where all vertex and plaquette measurements have value 1. We finally declare two configurations to be equal if there is a sequence of plaquette operations taking us from one to the other. Noting that plaquette operators are local isometries we see this corresponds to quotienting out the category by an equivalence relation. This doesn't exaust the degrees of freedom of the theory because of the topology of the torus. Those topological degrees of freedom will be the simple objects our newly created category, their fusion rules are completely determined by the structure of $Rep(G)$. Indeed the category we obtained is $Z(Rep(G)) \simeq Rep(DG)$.

Example 13 *In the case $G = \mathbb{Z}_2$, recall $Rep\mathbb{Z}_2$ has two simple object of dimension one τ_+ and τ_- with fusions given by group structure. Let us draw the states of our system as colourings of the edges of the lattice. The condition on the vertex operators results in no endlines and no triple intersections on the lattice. The condition on plaquette operators only allows loop configurations. Quotienting out by plaquette operators relation we obtain 4 distinct classes, namely the vacuum* 1*, the first cycle of the torus X, the second cycle Z, and both cycles $X \otimes Z \simeq Y$ and we have the fusion rules. The theory we have obtained is $RepD\mathbb{Z}_2$ which has in fact 4 simple objects with same fusions. We are therefore treating the topological defects of a theory in $Rep\mathbb{Z}_2$ as particles particles in their own right, with their own theory. All those representations are one-dimensional and in fact we just formulated a theory of abelian anyons.*

Example 14 (Anyon Vacuum on a Torus and Quantum Memory) *If we consider the torus as our configuration space. Let C_1, C_2 be the two cycles. Consider the process T_i for $i = 1, 2$ which creates a particle-antiparticle pair, moves them in opposite directions around cycle C_i so that they meet on the other side of the torus and annihilate. Then we can show T_i do not commute with each other if the particles are abelian anyons with $\theta \neq 0, \pi$. We know θ must be a fraction p/q with p and q coprime. Then we can show that the system has degenerate ground states. We have q different ground state, so the vacuum state lives in a q dimensional space. If we initialise it in some superposition it will remain in that state unless a T_1 or T_2 operation is implemented. Because of their topological nature it is very unlikely that such processes occur spontaneously, and therefore the quantum information stored in the superposition is protected.*

4.3 Permutational Quantum Computing

This section is about a model of quantum computation introduced by Jordan [**?**]. We will give a categorical presentation of the model which is not present in the literature and will allow us to compare the model to other computational models.

Definition 42 *Let $\mathcal{J}$ be the symmetric fusion category with positive half integers as simple objects, fusions etc.*

The construction given by Jordan can be generalised. In this section we argue that Symmetric Fusion categories are models fo permutational quantum computation.

Theorem 24 *Any symmetric fusion category induces representations of the symmetric group S_n for any $n \in \mathbb{N}$.*

Theorem 25 *If $\mathcal{C}$ is a symmetric fusion categories, then $\mathcal{C}$ is symmetrically monoidally equivalent to Rep(G) for G some group (if the twist is trivial) or some supergroup (if the twist is -1).*

Proof Doplicher-Roberts theorem

■

Proposition 26 *Jordan's model $\mathcal{J}$*

Example 15 *Permutational quantum computation in Rep(S_3).*

Remark The model described by Jordan is implementable by defining a hamiltonian on a network of spins.
Generalised spin network systems for arbitrairy group G

Example 16 (Approximation of Dijkgraaf-Witten link invariants) *The link invariant essentially counts homomorphisms from the fundamental group of the link complement to the group G. (cite Zhenghan?)*

4.4 Topological Quantum Computation

Take the fusion space to be our topological hilbert space.
Topological qudits are usually encoded as fusion tree basis elements
Topologial gates: braids (can express as action of the braid group) + measurements (=fusions and associators).

4.5 A braided representation of Quantum computation

4.6 An adjunction between fPQC and TQC

This section is dedicated to the relationship between a category $\mathcal{C}$ and its braided centre $Z(\mathcal{C})$. In the first part we will talk about non-commutative logic and modalities. In the second part we will see

Free forgetful adjunction:

$$\Box : \mathcal{C} \rightleftarrows Z(\mathcal{C}) : U$$

$$\Box : RepG \to RepDG \simeq Z(RepG)$$

Theorem 27 *Let $\{X_i\}_{i\in I}$ be the set of representatives of the isomorphism classes of simple objects in RepG. Let $\mathbb{C}G$ be the regular representation, then*

$$\mathbb{C}G \simeq \oplus_{i\in I} X_i \otimes X_i^*$$

Theorem 28 $\Box V = \oplus_{i\in I} X_i^* \otimes V \otimes X_i$ *with action of DG given by ....*

Let G be a group, DG its quantum double, (π, V) a representation of G.
The induced representation $\Box V$ is the coequalizer of:

$$DG \otimes \mathbb{C}G \otimes V \rightrightarrows DG \otimes V$$

Where the top arrow is given by the right action of G on $DG \simeq \mathbb{C}G^* \otimes \mathbb{C}G$

$$(P_h g, k) \mapsto P_h(gk^{-1})$$

(this satisfies the axioms of an action but do we have to make the action conjugate the flux projection component?) and the bottom arrow is given by the π action on V.
To compute the coequalizer we consider the orbits of the action of G on DG, these form a partition of DG:

$$\{[P_k e] : k \in G\}$$

So, as a vector space $\Box V \simeq \mathbb{C}G \otimes V$ and the action of DG on $\Box V$ is given by:

$$P_h g [P_k e] v = \delta_{h,gkg-1} [P_h e] \pi(g) v \tag{78}$$

So that element $P_h g$ implements residual symmetry g and projects onto flux sector gkg^{-1}.
Note that if C_i are the conjugacy classes of G then $\mathbb{C}G \simeq \oplus_i \mathbb{C}C_i$ and we could try to decompose:

$$\Box V \simeq \oplus_i \mathbb{C}C_i \otimes V \tag{79}$$

The action (3) factors through the conjugacy classes, (4) gives us a decomposition of $\Box V$ into irreducibles if V is simple in $RepG$? (this hold for abelian groups, should be generalised given decomposition of V into Z_i modules).
$\Box$ is clearly not monoidal if we take $\otimes$ as tensor, is it monoidal under $\oplus$? i.e is it additive? (Note that the induced representation functor Ind_H^G for H subgroup of G is additive). If it is additive then it is left and right exact and could use this to find the decomposition of $\Box V$
Is the $\Box$ functor representable? In the sense $\Box \simeq Hom(\mathbb{C}G, \_)$?

Example 17 *If $G = \mathbb{Z}_2 = \{e, a\}$, irreducible representations are the trivial τ_+ and the one dimensional sign representation τ_-. $DG \simeq \mathbb{C}\mathbb{Z}_2^* \otimes \mathbb{C}\mathbb{Z}_2$ and the orbits of the right action of $\mathbb{Z}_2$ on $D\mathbb{Z}_2$ are given by*

$$\{[P_e e], [P_a e]\}$$

Recall that $D\mathbb{Z}_2$ has 4 irreducible one dimensional representations which we denoted 1,X,Z,Y. With fusion rules generated by $A \otimes A \simeq 1$ and $X \otimes Y \simeq Z$.
Now let us calculate $\Box\tau_-$, it has basis $\{[P_e e] =: w_e, [P_a e] =: w_a\}$ and

$$P_e e(xw_e + yw_a) = xw_e P_a e(xw_e + yw_a) = yw_a P_e a(xw_e + yw_a) = -xw_e P_a a(xw_e + yw_a) = -yw_a$$

And we see from the table (see Lahtinen "The Toric Code and the Quantum Double" for table of reps) that $\Box\tau_- \simeq X \oplus Y$. Similarly $\Box\tau_+ \simeq 1 \oplus Z$.
Consider the regular representation $V := \mathbb{C}\mathbb{Z}_2 \simeq \tau_+ \oplus \tau_-$, then $\Box V \simeq 1 \oplus X \oplus Z \oplus Y \simeq (1 \oplus X) \otimes (1 \oplus Z)$.
What happens if we braid the two components of $\Box V$? In $RepD\mathbb{Z}_2$ the braid is implemented by acting on the components with $R = \sum_{g,h \in \mathbb{Z}_2} P_g e \otimes P_g h \in D\mathbb{Z}_2 \otimes D\mathbb{Z}_2$ and swapping coordinates. I claim this implements a CNOT gate followed by a swap.

Example 18 *If $G = S_3 = \langle \sigma, \rho \rangle$ where σ is a reflection and ρ a rotation. S_3 has three irreducible representations: the trivial τ_+, the sign representation τ_- and the two dimensional τ_2.*

5 Conclusion

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A Fusion categories

Many of the results we will see in this section can be found in [5], [6] and [7].

Definition 43 *The category $\mathcal{C}$ is Ab if it is enriched over abelian groups. That is all hom-sets have abelian group structures and composition of morphisms is a group homomorphism.*

Definition 44 *An Ab-category $\mathcal{C}$ is additive if it has zero object and every pair of objects has a direct sum $\oplus$.*

Definition 45 *An abelian category is an additive category where every morphism has a kernel and a cokernel and every monic (epic) is a kernel (cokernel).*

Definition 46 *Let k be field, we say $\mathcal{C}$ is k-linear if all hom-sets are k-vector spaces and composition is bilinear.*

We will assume throughout the thesis that $k = \mathbb{C}$ so in particular the field is algebraically closed.

Definition 47 *An object X in a $\mathbb{C}$-linear category is called simple if $EndX = kid_X$.*

Definition 48 *$\mathcal{C}$ is semisimple if every object is isomorphic to a direct sum of simple objects. $\mathcal{C}$ is finite if there are finitely many isomorphism classes of simple objects.*

Definition 49 *A $\mathbb{C}$-linear tensor category is a fusion category if it has finite-dimensional hom-spaces, is semisimple with finitely many isomorphism classes of simple objects, the unit $\mathbf{1}$ is simple and all objects have duals.*

Theorem 29 *$Rep(H)$ is a fusion category*

Example 19 *Any group G is a hopf algebra (comonoid = copy).*
Therefore $RepG$ can also be made monoidal and rigid.

Example 20 *Recall the group $S_3 = \{e, g, g^2, \sigma, \sigma g, \sigma g^2\}$. The category $Rep(S_3)$ is a fusion category. By the known representation theory of S_3, $Rep(S_3)$ has three simple objects: the trivial representation 1, the sign representation -1 and the geometric two dimensional representation τ:*

$$\tau : \quad \sigma \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$g \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{pmatrix}$$

These satisfy the following fusion rules $\forall X$ simple object:

$$1 \otimes X \simeq X \simeq X \otimes 1 - 1 \otimes -1 \simeq 1 - 1 \otimes \tau \simeq \tau \simeq \tau \otimes -1 \tau \otimes \tau \simeq 1 \oplus -1 \oplus \tau \quad (80)$$

Example 21 (Graph invariants from spherical fusion categories)

We now explore an important class of categories. Braided fusion categories (BFCs) are very closely related to categories of representations via the Tannaka reconstruction theorem and its variations. BFCs have found numerous applications to Quantum computer science as we will see in the remaining sections. Braid group, Yang-Baxter, Quasitriangular Hopf algebras.

Definition 50 *Braided monoidal categories (with braid c)*

Definition 51 *Ribbon categories (twist θ)*

Definition 52 *The symmetric centre $Z_2(\mathcal{C})$ of a braided tensor category $\mathcal{C}$ is the full subcategory with the following objects:*

$$\{X \in \mathcal{C} : c_{X,Y} \circ c_{Y,X} = id_{Y \otimes X} \forall Y \in \mathcal{C}\}$$

Example 22 *Braid group B_n, free braid category, free construction Tensor categories $\to$ BTCs (2-adjunction)*

Example 23 *categories of tangles = free rigid braided categories.*
Tangle categories are not linear over a field but can be linearised by using the free vector space functor $Set \to Vect$. This gives still big categories, but can be quotiented out by an ideal defined in terms of link-invariants to give interesting cats.

Definition 53 *braided fusion category*

Definition 54 *Let $\mathcal{C}$ be a tensor category, $X \in \mathcal{C}$. A half-braiding e_X is a family $\{e_X(Y) : X \otimes Y \xrightarrow{\sim} Y \otimes X\}$ such that $e_X(\mathbf{1}) = id_X$ and*

$$e_X(Y \otimes Z) = id_Y \otimes e_X(Z) \circ e_X(Y) \otimes id_Z \quad \forall Y, Z \in \mathcal{C}$$

Theorem 30 *If $\mathcal{C}$ is k-linear, spherical or a *-category (k-linear dagger) then so is $Z_1(\mathcal{C})$*

Theorem 31 *If H is a quasitriangular Hopf Algebra then Rep(H) is a braided fusion category.*

N-matrix, R-matrix