1 Diagrams and Hopf Algebras

1.1 Monoidal categories

In this section, we set in place the basic definitions and diagrammatic intuitions which we will use throughout the thesis. The standard reference about basic category theory results is [1]. Many of the definitions are taken from [2]. A more detailed and up to date survey on monoidal categories can be found in [3]. For an introduction to diagrammatic reasoning in monoidal categories consider the first two chapters of [4]. Many of the results in this section and their relationship to quantum mechanics can be found in [5].
Recall the definition of a category.

Definition 1 *A category $\mathcal{C}$ consists of the data:*

- *a collection of objects $obj(\mathcal{C})$*
- *a collection of morphisms (or arrows) $arr(\mathcal{C})$*
- *domain and codomain assignments $dom, cod : arr(\mathcal{C}) \to obj(\mathcal{C})$. For any two objects $a, b \in obj(\mathcal{C})$ we define the hom-set*

$$\mathcal{C}(a,b) := \{f \in arr(\mathcal{C}) : a = dom(f), b = cod(f)\}$$

- *for any triple of objects a, b, c a composition map*

$$c_{a,b,c} : \mathcal{C}(a,b) \times \mathcal{C}(b,c) \to \mathcal{C}(a,c)$$

We denote the composition by $g \circ f$, diagrammatically:

$$\begin{tikzcd} & b \arrow[dr, "g"] & \\ a \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & & c \end{tikzcd}$$

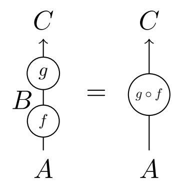
- *For any object a an identity morphism $id_a : a \to a$*

Satisfying the following axioms:

$$h \circ (g \circ f) = (h \circ g) \circ f \quad f \circ id_a = f = id_b \circ f$$

$$\begin{tikzcd} & B \arrow[dr, "g"] & \\ A \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & & C \end{tikzcd}$$

The commutativity of the above diagram is a statement about $\mathcal{C}$, and it has exactly the same information to its dual diagram. Where objects are one-dimensional wires and morphisms are (zero dimensional) boxes:



We will mainly use this second diagrammatic language in this work. When $\mathcal{C}$ is just a category we only have one way of composing morphisms and the language is one dimensional.

Example 1 *Examples of categories are: $Sets$ of sets and functions, $FSets$ of finite sets and functions, Rel of sets and relations, $Vect_k$ of vector spaces over k and linear maps and $FVect_k$ of finite dimensional vector spaces and linear maps.*

Category theory is a really good language for talking about equivalences and relationships between structures. This is achieved with the following tools.

Definition 2 (Functor) *A functor $F : \mathcal{C} \to \mathcal{D}$ is a mapping that*

- *associates an object $F(X)$ of $\mathcal{D}$ to each object X of $\mathcal{C}$.*
- *associates to each morphism $f : X \to Y$ a morphism $F(f) : F(X) \to F(Y)$ such that $F(id_X) = id_Y$ and $F(g \circ f) = F(g) \circ F(f)$ for all morphisms $f : X \to Y$ and $g : Y \to Z$.*

For instance there is a functor $Q : Sets \to Vect_k$ called '1st quantization' and taking a set to the free vector space generated by that set. Given two functors with matching source and target we can have natural transformations between them

Definition 3 (Natural Transformation) *Given categories $\mathcal{C}$ and $\mathcal{D}$ and functors $F, G : \mathcal{C} \to \mathcal{D}$ a natural transformation $\alpha : F \Rightarrow G$ is an assignment to every object a in $\mathcal{C}$ of a morphism $\alpha_a : F(a) \to G(a)$ in $\mathcal{D}$ such that for each morphism $f : a \to b$, the following commutes:*

$$\begin{tikzcd}
G(a) \arrow[r, "G(f)"] & G(b) \\
F(a) \arrow[u, "\alpha_a"] \arrow[r, "F(f)"'] & F(b) \arrow[u, "\alpha_b"']
\end{tikzcd}$$

A natural isomorphism is a natural transformation such that all components are isomorphisms.

Recall that a monoid is a triple $(X, \times, 1)$ where X is a set, $1 \in X$ and $\times$ is an associative and unital multiplication on X. The notion of a monoidal category is the categorification of a monoid. Elements of the set are replaced by objects in a category $\mathcal{C}$, multiplication by a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and the equalities in the unit and association axioms are replaced by natural isomorphisms. In order for this new structure to be well-behaved we will also need to impose compatibility conditions. We obtain the following definition:

Definition 4 (Monoidal category) *A monoidal category is a category $\mathcal{C}$ equipped with a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called tensor product, an object 1 called unit object, a natural isomorphism*

$$a : - \otimes (- \otimes -) \overset{\sim}{\Rightarrow} (- \otimes -) \otimes -$$

called associator, a natural isomorphism

$$\lambda : 1 \otimes (-) \Rightarrow (-)$$

called left unitor and a natural isomorphism

$$\rho : (-) \otimes 1 \Rightarrow (-)$$

called right unitor. Subject to the following coherence conditions holding for all objects a, b, c, d in $\mathcal{C}$:

1. *Pentagon axiom: the following diagram commutes*

$$\begin{tikzcd}
& (a \otimes b) \otimes (c \otimes d) \arrow[dr, "\alpha_{a,b,c\otimes d}"] & \\
((a \otimes b) \otimes c) \otimes d \arrow[ur, "\alpha_{a\otimes b,c,d}"] \arrow[dd, "\alpha_{a,b,c} \otimes id_d"'] & & a \otimes (b \otimes (c \otimes d)) \\
& & \\
(a \otimes (b \otimes c)) \otimes d \arrow[rr, "\alpha_{a,b\otimes c,d}"'] & & a \otimes ((b \otimes c) \otimes d) \arrow[uu, "id_a \otimes \alpha_{b,c,d}"']
\end{tikzcd}$$

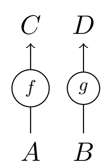
2. *Triangle identity: the following diagram commutes*

$$\begin{tikzcd}
(a \otimes 1) \otimes b \arrow[rr, "\alpha_{a,1,b}"] \arrow[dr, "\rho_a \otimes id_b"'] & & (a \otimes 1) \otimes b \arrow[dl, "id_a \otimes \lambda_b"] \\
& a \otimes b &
\end{tikzcd}$$

Let us give three important examples of monoidal categories.

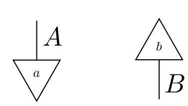
Example 2 *The category Sets of sets and functions is monoidal with the cartesian product $\times$ as bifunctor and the singleton set as unit object.*
The category $Vect_k$ of finite dimensional vector spaces over a field k is monoidal with the usual tensor product $\otimes$ and the one dimensional vector space k as unit object.
The category Rel of sets and relations is monoidal with the cartesian product $\times$ and the singleton as unit object.

The more structure comes with a category the more complicated diagrams we can draw. Monoidal categories have a two-dimensional diagrammatic language. The presence of unitors and associators and the conditions they satisfy make sure that this graphical language is well behaved. This is known as the coherence theorem for monoidal categories and can be found in [1]. It says that any well formed diagram in a monoidal category, made up of associators and unitors commutes. When the associators are trivial morphisms (i.e identity morphisms) we say the category is strict monoidal. It is known that every monoidal category is equivalent to a strict one [1], but it is sometimes useful to take associators into account as we will see in our discussion on permutational quantum computation. We write the tensor of two morphisms $f \otimes g : A \otimes B \to C \otimes D$ simply putting them side by side:



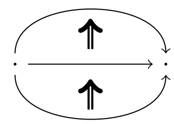
In our diagrams we can picture the unit I of the tensor as the plane on which we are drawing. Indeed we could imagine drawing as many copies as we wanted of id_I on the previous diagram to obtain obtain an equivalent diagram as $id_I \otimes f = f$ for any morphism f. So really the identity on I is just the empty diagram which we can stick next to any diagram we like.

Definition 5 (States and costates) *Given a system A, a state of is a morphism $1 \to A$. A costate (or effect) of A is a morphism $A \to 1$. In the diagrammatic language we draw states and costates respectively:*

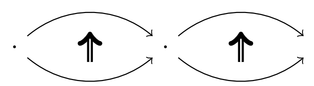


Remark It is perhaps useful to understand monoidal categories as degenerate 2-categories. Although this viewpoint requires one additional initial step of abstraction (the definition of a 2-category), it gives us the diagrammatic language for monoidal categories for free. For the rigorous definition of a 2-category we refer to [BAEZ], for our purposes we will only need the intuition. A 2-category is a collection of objects with 1-arrows between them and 2-arrows between the 1-arrows. Note that there are two ways of composing the 2-arrows:

- vertical composition:



- parallel composition:



Taking the dual of the above diagrams we obtain the diagrammatic language. Monoidal categories are 2-categories with only one 0-object called 1. We can think of the 0-object as the underlying plane, wires carry systems (1-arrows), boxes are morphisms (2-arrows). We recover the given definition of monoidal category by calling 1-arrow objects, and 2-arrows morphisms. The unit object 1 is then the identity 1-arrow $1 \to 1$ which is simply denoted 1.

Example 3 *The cartesian product in Sets $A \times B$ of sets A and B, satisfies the universal properties of a categorical product, in the sense that we have projections p_1 and p_2 such that if f and g are maps from some set C there is a unique function h making the following diagram commute:*

$$\begin{tikzcd}
& A \times B \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
A & & B \\
& C \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uu, "h"] &
\end{tikzcd}$$

Because of this property all states of $(Sets, \times)$ are separable. This category is the ambient Cartesian world of classical physics.

Example 4 *In $Vect_k$ states are vectors and costates are functionals. Note that the diagrammatic notation provides a two-dimensional generalisation of Dirac's notation. The category $Hilb$ of Hilbert spaces and linear maps is monoidal when equipped with the usual tensor product $\otimes$. Note that $\otimes$ is not a categorical product, and in fact we can have entangled states. Quantum mechanics is based on $(Hilb, \otimes)$ [5].*

Definition 6 (Scalars) *Scalars in a monoidal category are morphisms $1 \to 1$.*

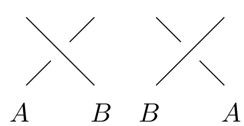
The category *Sets* has only one scalar. *Rel* has two scalars forming the cyclic group $\mathbb{Z}_2$ under composition. $Vect_k$ has scalars from k. Given a vector and a functional we obtain a scalar by composing them analogously to Dirac's formalism.

Definition 7 (BMC) *A braided monoidal category is a monoidal category $\mathcal{C}$ equipped with a natural isomorphism $B_{a,b} : a \otimes b \to b \otimes a$ called braiding, subject to the following compatibility conditions (called hexagon equations):*

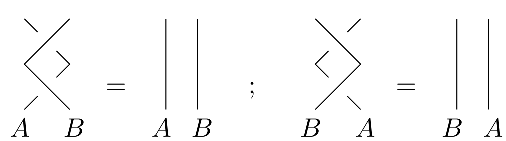
$$\begin{tikzcd}
 & a \otimes (b \otimes c) \arrow[r, "B_{a,b\otimes c}"] & (b \otimes c) \otimes a \arrow[dr, "\alpha_{b,c,a}"] & \\
(a \otimes b) \otimes c \arrow[ur, "\alpha_{a,b,c}"] \arrow[dr, "B_{a,b} \otimes id_c"'] & & & b \otimes (c \otimes a) \\
 & (b \otimes a) \otimes c \arrow[r, "\alpha_{b,a,c}"'] & b \otimes (a \otimes c) \arrow[ur, "id_b \otimes B_{a,c}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & (a \otimes b) \otimes c \arrow[r, "B_{a\otimes b,c}"] & c \otimes (a \otimes b) \arrow[dr, "\alpha_{c,a,b}"] & \\
a \otimes (b \otimes c) \arrow[ur, "\alpha_{a,b,c}"] \arrow[dr, "id_a \otimes B_{b,c}"'] & & & (c \otimes a) \otimes b \\
 & a \otimes (c \otimes b) \arrow[r, "\alpha_{a,c,b}"'] & (a \otimes c) \otimes b \arrow[ur, "B_{a,c} \otimes id_b"'] &
\end{tikzcd}$$

In the diagrammatic language this means we have braidings:

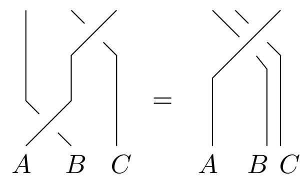


for any A and B, satifying:



(1)

The compatibility conditions are obvious statements in the diagrammatic calculus, for instance the first hexagon equation just says:



(2)

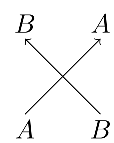
Both *Sets* and *Hilb* are examples of symmetric monoidal categories in the following sense.

Definition 8 (SMC) *A braided monoidal category is symmetric if the braiding* $B_{a,b}$ *satisfies*

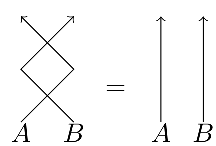
$$B_{a,b} \circ B_{b,a} = id_{a \otimes b}$$

For all objects a, b

In a SMC the braiding is called symmetry morphism and is denoted



It satisfies:



We will now describe some new classes of examples of monoidal categories. These are of a different nature to the categories we have seen so far.

Definition 9 (PROPs) *A PROP (products and permutations category) is a strict symmetric monoidal category where every object is of the form* $x^{\otimes n}$ *for a single object* x *and* $n \geq 0$.

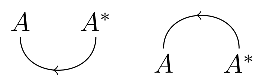
This means that we are only allowed one type of wire when drawing diagrams about *PROPs* but we can use as many copies as we like and we can make swaps with them. Categories satisfying these properties are useful syntactic tools as we will see. One way to think of a *PROP* A is as an abstract algebraic structure carrying some axioms, we can then instantiate those axioms in some other symmetric monoidal category $\mathcal{C}$ by considering symmetric monoidal functors $F : A \to \mathcal{C}$. We call such functors algebras or models of A in $\mathcal{C}$. If A is defined in terms of generators and relations (as is most often done), the choice of such functor corresponds to the choice of one object from $\mathcal{C}$ and morphisms on that object respecting the defining relations of A. On its own A has no clear interpretation, it just defines a syntax, but if $\mathcal{C}$ is a semantic category (i.e one with

a clear interpretation) then F is a 'filling' of the syntax with meaning. This reasoning was first proposed in Lawvere's Phd thesis in 1963 [6].
It will sometimes be useful to drop the 'permutational' structure of $PROPs$.

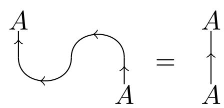
Definition 10 (PRO) *A PRO (products category) is a strict monoidal category where every object is of the form $x^{\otimes n}$ for a single object x and $n \geq 0$.*

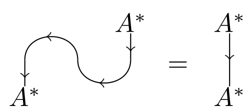
The semantic categories we will consider the most are $Sets$ and $Hilb$. One important difference between them is that $Hilb$ exhibits duality.

Definition 11 (Rigidity) *Let $\mathcal{C}$ be a monoidal category and $A \in obj(\mathcal{C})$. A left-dual of A is an object A^* with morphisms*



Satisfying the snake equations:

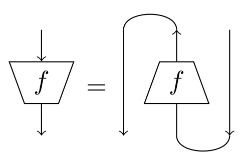




If every object has a left-dual, we say that $\mathcal{C}$ is left-rigid. Similarly we can define right-duals and right-rigid categories by interchanging the roles of A and $A^\star$ in the definition.

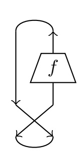
Given a (left/right) rigid structure we can define (left/right)transpose as follows.

Definition 12 (Transpose) *Given a (left/right) rigid category $\mathcal{C}$ and any process $f : A \to B$ the (left/right) transpose f^* (or left transpose f^l, right transpose f^r if it is not clear from context) is:*



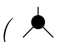
(3)

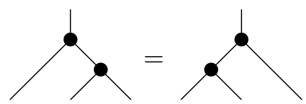
Definition 13 (Trace) *In a symmetric monoidal category $\mathcal{C}$, if A has a left dual $A^\star$, the trace of some morphism $f : A \to A$ is defined as the following scalar:*



1.2 Hopf Algebras

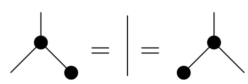
Now that we have set in place a diagrammatic machinery based on monoidal categories, let us make use of it. In this section we will meet some mathematical structures which have been used by mathematicians to describe symmetry. The notion of Hopf algebras is a powerful generalization of that of a group. Since their discovery in the 1940s, Hopf algebras have been used in various fields of pure mathematics (such as number theory, algebraic geometry, and representation theory) and have found applications in Quantum mechanics. Most of the results of this section can be found in [7].

Definition 14 (Monoid) Δ *is a PRO generated by morphisms (*  *,*  *) satisfying associativity:*



(4)

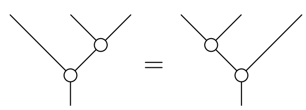
and the unit law:



(5)

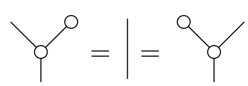
Models of Δ in monoidal categories are called monoids and they are very well known, examples include the natural numbers under addition, lists of some alphabet under concatenation and any group. Taking the opposite category Δ^{op} corresponds to flipping all the diagrams.

Definition 15 (Comonoid) Δ^{op} *is a PRO generated (*  *,*  *) satisfying coassociativity:*



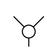
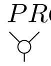
(6)

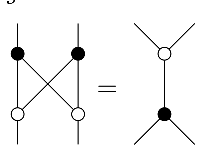
and the counit law:



(7)

Models of these are comonoids, the most common example is the copy map on any set with 'delete' as counit. Monoids and comonoids are simple structures that we can stick together to form more complicated ones. Bialgebras arise from one type of interaction of a monoid and comonoid.

Definition 16 (Bialg) *Bialg is a PROP generated by (*  *,*  *,*  *,*  *), where*  *and*  *form a monoid,*  *and*  *a comonoid and the morphisms additionally satisfy the following laws:*



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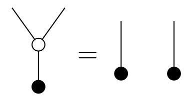

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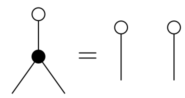
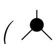
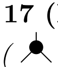
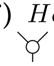
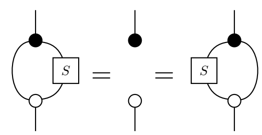


(10)

Models of $Bialg$ in $Vect$ are bialgebras. We leave examples for later as we are now ready to introduce one of the main topics of this thesis.

Definition 17 (Hopf) *$Hopf$ is a PROP generated by (*  *,*  *,*  *,*  *,*  *). Where (*  *,*  *,*  *,*  *) is a bialgebra and the antipode S satisfies the Hopf law:*



(11)

We will argue that $Hopf$ is a good syntax to talk about symmetry. Let us start by instantiating $G : Hopf \to Sets$. This corresponds to choosing a set G, with a binary function $G \times G \to G$ (or multiplication) with a unit. Using the counit rule it is easy to see that the comultiplication in $Sets$ must be the copy map $g \mapsto (g, g)$ so that the antipode is the morphism $g \mapsto g^{-1}$ and G forms a group. Since the 19^{th} century groups have been used by mathematicians and physicists to describe symmetry.

Example 5 (Finite groups) *We will only make use of the following classes of finite groups:*

- *$\mathbb{Z}_n$ the cyclic group with n elements.*
- *S_n the symmetric group, can be seen as the group of permutations of a set with n elements, has order $n!$. S_3 is the smallest non-abelian group up to isomorphism.*

Example 6 (Groups of matrices) *Here we will fix some notation on the infinite groups of matrices we will meet. All matrices we will consider are over the complex numbers. $GL(n)$ is the group of invertible n by n complex matrices. $U(n)$ is the group of unitary $n \times n$ matrices (i.e such that $U^\dagger U = UU^\dagger = I$). The special unitary group $SU(n)$ is the subgroup of $U(n)$ consisting of matrices with determinant 1. The representation theory of $SU(n)$ is widely used in particle physics, for instance representations of $SU(2)$ model the behaviour of spin-$\frac{1}{2}$ particles.*

If we take a model of $H : Hopf \to Hilb$ we obtain what is known as a Hopf Algebra.

Example 7 (Group algebras) *If G is a group with unit e, the group algebra $\mathbb{C}G$ (of dimension $|G|$) is a hopf algebra with multiplication linearly generated by $|g\rangle \otimes |h\rangle \to |gh\rangle$, unit $|e\rangle$, comultplication generated by $|g\rangle \to |g\rangle \otimes |g\rangle$ and counit $\sum_g \langle g|$.*

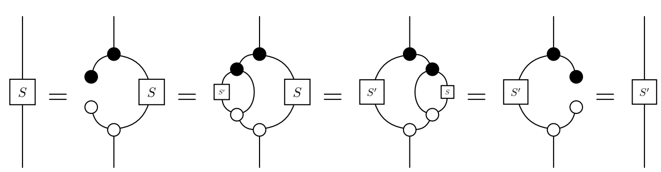
The previous example gives the usual definition of a group algebra which is really just the composition $Q \circ G$ (as shown in the diagram) where Q is the 1st quantization functor. It is easy to see that Q preserves the monoidal structure as well as the symmetry morphisms (we say Q is a symmetric monoidal functor) so that the composition is also symmetric monoidal and $Q \circ G$ is a model of $Hopf$.

$$
\begin{tikzcd}
 & Hopf \arrow[dl, "G"'] \arrow[dr, "\mathbb{C}G"] & \\
Sets \arrow[rr, "Q"'] & & Hilb
\end{tikzcd}
$$

In this case the comultiplication in $Hilb$ is the linearisation of the copy map (the copy map on some basis extended linearly to the whole Hilbert space) which is co-commutative. For a general $H : Hopf \to Hilb$ this doesn't have to be the case. Hopf algebras provide a broader framework to talk about symmetry, as we can have non co-commutative Hopf algebras. We can see it as a quantization of the notion of symmetry, it will allow us to describe symmetries of quantum systems. Physically we will see that Hopf algebras allow to talk about local symmetries and exchange statistics on the same footing [8]. In particular if the Hopf algebra is not cocommutative the exchange statistics can be highly non-trivial, in which case they will describe the symmetries of anyons. The following two propositions are simple but important results about the antipode of a hopf algebra.

Proposition 1 *The antipode of a Hopf algebra is unique. It follows that being a Hopf algebra is a property of bialgebras.*

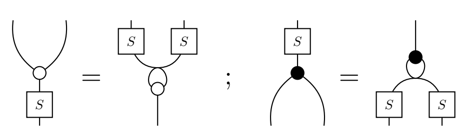
Proof Suppose S and S' are two antipodes for some Hopf algebra, then:



(12)

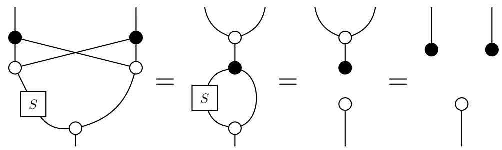
■

Proposition 2 *The antipode is an anti-(co)algebra homomorphism.*

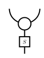
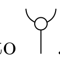


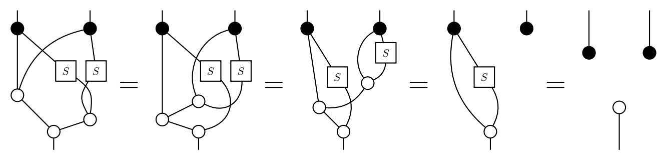
(13)

Proof First note that:

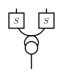
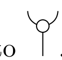


(14)

So that  is a left convolution inverse to .
Also:

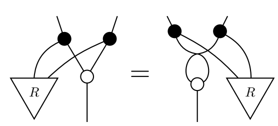


(15)

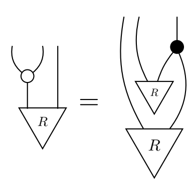
So that  is a right convolution inverse to . And it is easy to see using associativity and co-associativity that right and left convolution inverses must coincide. We deduce that the antipode is an anti-coalgebra homomorphism. For a proof that the antipode is an anti-algebra morphism simply flip all the diagrams and interchange white with black.

■

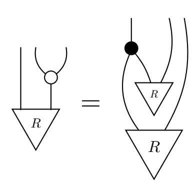
Definition 18 (Quasitriangularity) *A Hopf algebra H is quasitriangular if there is an invertible element $R \in H \otimes H$ satisfying the following equations:*



(16)



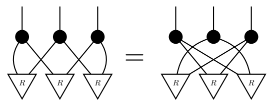
(17)



(18)

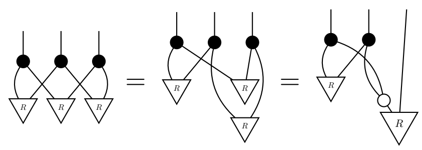
R is called the 'universal R-matrix', and it can be thought as controlling the non-cocommutativity of the Hopf algebra. Quasitriangular hopf-algebras are sometimes called Quantum groups. We will see that they exhibit topological behaviour, as the following proposition hints to.

Proposition 3 *The universal R-matrix satisfies the Quantum Yang-Baxter equation:*



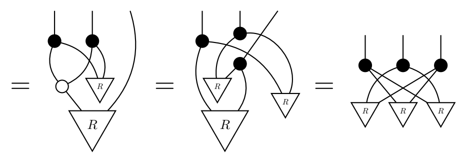
(19)

Proof First using isotopy invariance and the second rule of quaitriangularity we get:



(20)

Then using the first rule:



(21)

■

Example 8 *The most trivial example of quasitriangular hopf algebras are the cocommutative ones. It is easy to check that if H is cocommutative, it is quasi-triangular with*  *as R-matrix.*

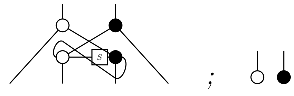
We will only be considering finite dimensional Hopf Algebras, as for finite dimensional vector spaces, these always have duals.

Definition 19 (Dual Hopf Algebra) *For a finite dimensional Hopf Algebra H the dual Hopf algebra is the vector space $H^\star$ of linear functionals on H with Hopf Algebra structure given by transposing all of the structure.*

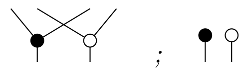
Given any finite dimensional hopf algebra H with invertible antipode there is a standard way of constructing a Quasitriangular Hopf Algebra first introduced by Drinfeld [9]. It will be implicit from now on that all Hopf algebras (and vector spaces) are finite-dimensional unless stated otherwise.

Definition 20 (Quantum double of a Hopf algebra) *The quantum double of a Hopf algebra $(H, \mu, 1, \Delta, \epsilon, S)$ with invertible antipode is the vector space $H^\star \otimes H$, with the following structure:*

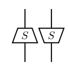
- *multiplication and unit:*



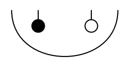
- *comultiplication and counit:*



- *antipode:*

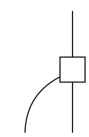


It is easy to check this is indeed a Hopf algebra and that it is quasitriangular with universal R-matrix:

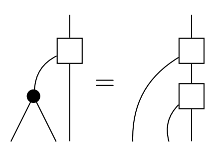


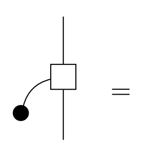
1.3 Representations of Hopf algebras

Recall that a group describes the symmetries of some space X when it acts on it (e.g crystals, classical symmetries= symmetries of sets). If we apply the same reasoning to Hopf Algebras we have to make H act on some quantum state space (i.e Hilbert space). So our object of study is not H on its own but rather a module (or representation) of H. In the diagrammatic language we depict it as follows:

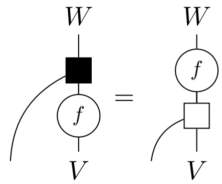


Where V is a finite dimensional vector space. Note that the above diagram represents a linear map, all diagrams we will be drawing in this section are diagrams in $Hilb$. In order for V to be a representation the following must hold.

 (22)

 (23)

Suppose V and W are representations of H, then we say $f : V \to W$ (a linear map)is an intertwiner if:

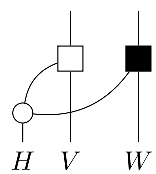


(24)

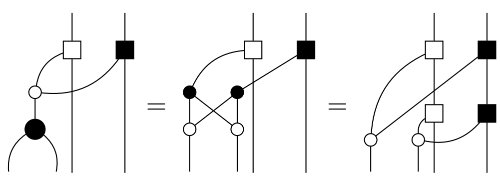
Where the black square denotes the action of H on W. Now consider the category *Rep*(H) where objects are representations of H and morphisms intertwiners. It is easy to see that the axioms of a category are satisfied, composition is just lifted from vector spaces. This category has really nice structure induced from the defining axioms of hopf algebras.

Proposition 4 *Rep*(H) *is a monoidal category for any bialgebra* H *with tensor unit the trivial one-dimensional representation* $(\mathbb{C}, ⚲)$.

Proof Given H-modules V and W (with white and black actions respectively), $V \otimes W$ has natural H-module structure induced by the comultiplication:

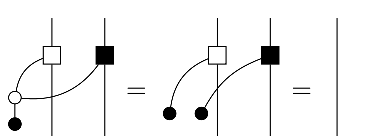


And $V \otimes W$ with this action is indeed a module as:



(25)

Also:



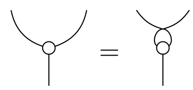
(26)

Using the bialgebra law and the fact that V and W are H-modules. Showing that $(\mathbb{C}, ⚲)$ is the tensor unit is a trivial application of the counit law.

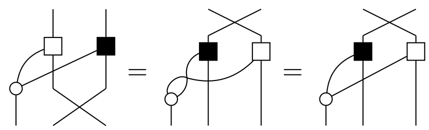
■

Proposition 5 *If* H *is cocommutative, then Rep*(H) *is symmetric.*

Proof Cocommutativity means:


(27)

So the symmetry morphism on $V \otimes W$ from $Vect$ is an intertwiner:

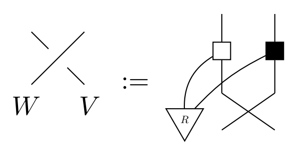

(28)

■

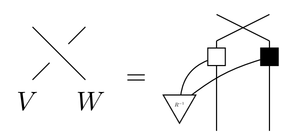
Recall that when H is cocommutative, it is trivially quasitriangular. The following is an important generalisation of the previous result.

Proposition 6 *If H is quasitriangular, then $Rep(H)$ is braided.*

Proof For any H-modules V and W, using the symmetry morphism from $Vect$ define:

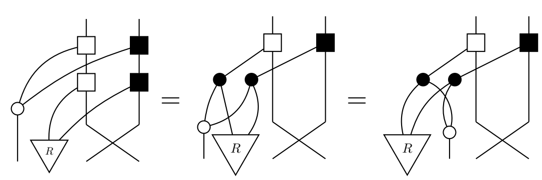


(29)



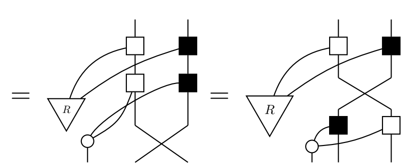
(30)

It is easy to see these are inverses of each other, we just need to check they are intertwiners.



(31)

Using H-module definition and the defining relation for R.



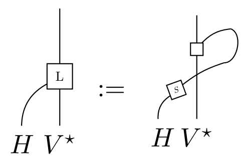
(32)

And a similar proof works for the inverse.

■

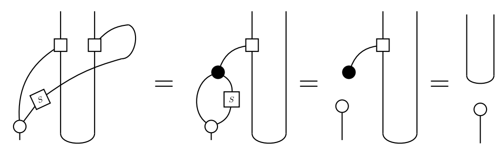
Proposition 7 *If H is a Hopf algebra, then $Rep(H)$ is left-rigid.*

Proof For any H-module V let $V^\star$ be its dual in $Vect$, we can define a dual H-action on $V^\star$ using the antipode:



(33)

Then the usual cups and caps from $Hilb$ are intertwiners.



(34)

Using the module laws and the antipode law. A similar derivation holds for the cap.

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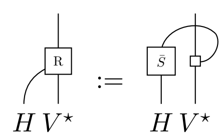
We can see that the proof relies on the existence of the antipode. If a skew-antipode $\bar{S}$ exists, $Rep(H)$ is right-rigid, where the right dual is defined:



(35)

The proof that this choice works is very similar to the one above. In particular, $\bar{S}$ exists when the antipode is an invertible morphism as we can define $\bar{S} = S - S^{-1}$ If the antipode coincides with the skew antipode then $Rep(H)$ then left and right duals in $Rep(H)$ coincide, we say it is rigid.m

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