Quantum Field Theory 1

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TODO

- Check all the math notations and symbols used throughout the notes for consistency.
- Check if the introduction is coherent with the rest of the notes.

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Introduction

Quantum Field Theory (QFT) provides the modern framework for describing the dynamics of elementary particles and their interactions. It unifies the principles of quantum mechanics with those of special relativity, and it naturally incorporates particle creation and annihilation processes, which are absent in non-relativistic quantum mechanics.

The Poincaré Group and Relativistic Symmetry

The starting point of any relativistic theory is the invariance under the Poincaré group, the group of isometries of Minkowski spacetime. It consists of Lorentz transformations (rotations and boosts) together with spacetime translations.

The representations of the Poincaré group classify possible relativistic particles, characterized by two invariants: the mass m and the spin s.

Classical Field Theory

Before quantization, fields are introduced as classical dynamical variables defined over spacetime. Their dynamics are determined by the principle of stationary action, which leads to the **Euler–Lagrange equations** for fields.

Within this framework, different types of relativistic fields arise naturally: scalar fields, spinor fields, and vector fields, each described by an appropriate Lagrangian density.

A central result of the Lagrangian formalism is **Noether's theorem**, which establishes a direct correspondence between continuous symmetries of the action and conserved physical quantities. For instance, spacetime translation invariance implies conservation of energy and momentum, while internal phase symmetries give rise to conserved charges.

Canonical Quantization of Free Fields

Quantization promotes classical fields to operator-valued distributions acting on a Fock space. The canonical approach consists in imposing equal-time commutation or anticommutation relations, in accordance with the spin–statistics theorem.

- **Spin-0 (Klein–Gordon theory):** A real scalar field is quantized by expanding it in Fourier modes with creation and annihilation operators obeying bosonic commutation relations. This leads to the description of spinless relativistic particles.
- **Spin-1/2 (Dirac and Weyl theories):** Spinor fields are quantized by imposing fermionic anticommutation relations. The Dirac theory provides the framework for massive spin-$\frac{1}{2}$ particles such as electrons, while the Weyl theory describes massless chiral fermions.
- **Spin-1 (Maxwell and Proca theories):** Vector fields can be quantized only after addressing gauge redundancy. The Maxwell theory describes a massless gauge boson (the photon), whereas the Proca theory provides a consistent formulation for a massive spin-1 particle.

These procedures yield the free quantum field theories for the three basic types of relativistic particles: scalars, fermions, and gauge bosons.

Towards Interacting Theories

Free field theories provide the starting point of quantum field theory, but they describe particles without mutual influence. To account for the physical world, one must introduce interactions by adding non-linear terms to the Lagrangian density.

The analysis of interacting quantum fields relies on perturbation theory, systematically organized through Feynman diagrams. Within this framework one computes observable quantities such as decay rates, which measure the probability per unit time that an unstable particle decays, and cross sections, which quantify the likelihood of scattering processes. These observables form the bridge between the abstract formalism of quantum field theory and the experimental study of particle physics.

We will not cover exact solutions of interacting theories, which require advanced techniques and mathematical tools beyond the scope of these notes. Instead, practical approaches rely on approximation schemes. In particular, perturbative expansions reduce the dynamics of interacting fields to a collection of coupled harmonic oscillators, whose behavior is well understood from quantum mechanics. This analogy provides the foundation for treating interactions as small corrections to free theories, ultimately leading to the perturbative framework of Feynman diagrams.

1 A New Framework

The development of Quantum Field Theory was driven by the need to reconcile the principles of quantum mechanics with those of special relativity. Special relativity describes the structure of spacetime and the behavior of objects moving at high velocities, while quantum mechanics governs the behavior of particles at microscopic scales. However, neither theory alone could adequately describe phenomena involving both high energies and small distances, such as particle creation and annihilation.

Quantum mechanics does not include relativistic effects:

- The concept of a **limiting speed is absent**.
- The **energy expression** for a free particle is incompatible with the relativistic: $E_{\mathrm{QM}} = \frac{p^2}{2m}$ instead of $E_{\mathrm{SR}} = \sqrt{p^2c^2 + m^2c^4}$.

Special relativity, on the other hand, does not incorporate quantum principles:

- It does not account for **quantization** of physical observables, first amongst all energy.
- The **promotion of observables to operators** acting on a Hilbert space is missing.

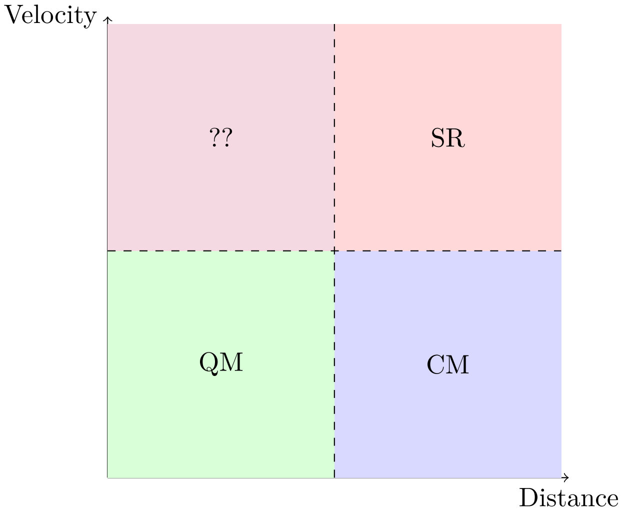

Figure 1.1: Schematic diagram of the four fundamental regimes of physics.

We need a new framework to describe the regime of small distances and high velocities, where both quantum and relativistic effects are significant. The idea to overcome the shown limitations is to use expressions from SR and incorporate them into a quantum framework: a **relativistic quantum mechanics** theory.

1.1 Relativistic Quantum Mechanics and its Limitations

The first attempt to construct a relativistic quantum theory was to modify the Schrödinger equation by replacing the non-relativistic energy-momentum relation with the relativistic one. We are basically in a quantum framework:

- Observables are promoted to operators acting on a Hilbert space by imposing *canonical commutation relations*:
  $[\hat{\mathbf{x}}, \hat{\mathbf{p}}] = i\hbar \implies \hat{\mathbf{p}} = -i\hbar\frac{\mathrm{d}}{\mathrm{d}\mathbf{x}}, \quad [\hat{t}, \hat{H}] = i\hbar \implies \hat{H} = i\hbar\frac{\mathrm{d}}{\mathrm{d}t}.$
- Operators act on a Hilbert space $\mathcal{H}$, where its states represent physical states of the system, with a **fixed number of particles**.
- Eigenvectors of a complete set of commuting observables form a basis for the Hilbert space; the eigenvalues correspond to the possible measurement outcomes:
  $\hat{\mathbf{p}} \left|p\right\rangle = p \left|p\right\rangle, \quad \int \mathrm{d}p \left|p\right\rangle \left\langle p\right| = 1.$
- The time evolution of states is governed by the **generalized Schrödinger equation**: $-i\hbar\frac{\partial}{\partial t}\psi(\mathbf{x}, t) = \hat{H}\psi(\mathbf{x}, t) = \sqrt{\hat{\mathbf{p}}^2c^2 + m^2c^4}\psi(\mathbf{x}, t)$.

However, this approach leads to several issues: it cannot account for particle creation and annihilation, which are essential in relativistic contexts. Additionally, the theory struggles with maintaining causality and Lorentz invariance, and predicts an infinite number of negative energy states, leading to an unstable vacuum.

Production/Annihilation of Particles

A picture in which the number of particles is fixed cannot describe processes where particles are created or destroyed, such as in high-energy collisions (LHC or other particle colliders) or in nuclear decays.

If we consider a particle with mass m at rest, its energy is given by $E = mc^2$. If we now add an energy to the system, as the particle acquires momentum, its mass becomes negligible and the system is energetically favorable to create new particles. But in order to preserve physical quantities such as charge, lepton number, baryon number, etc., particles must be created in pairs.

Example: particle in a box. Consider a particle of mass m confined in a three-dimensional box of length L. The *Heisenberg uncertainty principle* states that the uncertainty in position Δx and the uncertainty in momentum Δp satisfy the relation $\Delta x \Delta p \geq \frac{\hbar}{2}$. For a particle confined in a box, we can estimate $\Delta x \sim L$, leading to an uncertainty in momentum of at least $\Delta p \sim \frac{2\hbar}{L}$. If we take the particle to the ultrarelativistic regime,[1] its energy can be approximated as $E \approx pc$. Therefore, the uncertainty in energy will be:

$$\Delta E \geq \frac{2\hbar c}{L}.$$

If we want to avoid the production of particle-antiparticle pairs, we must ensure that the energy uncertainty is less than the energy required to create such a pair, which is $2mc^2$. But if we decrease the size of the box L to increase the precision in position, we increase the uncertainty in energy.

$$2mc^2 = \frac{2\hbar c}{L} \implies L = \frac{\hbar}{mc} = \lambda_c.$$

Here, λ_c is the Compton wavelength of the particle, representing a fundamental limit to the precision with which we can localize a particle without inducing pair production. If we try to confine the particle within a region smaller than its Compton wavelength, the energy uncertainty becomes sufficient to create particle-antiparticle pairs, making it impossible to describe the system with a fixed number of particles.

We need a framework that allows for a variable number of particles, accommodating the creation and annihilation processes inherent in relativistic quantum phenomena (as we will see, a Fock space formalism is required: $\mathcal{F} = \bigoplus_n \mathcal{H}_n$).

Violation of Causality

[1] In the ultrarelativistic regime, the particle's kinetic energy is much greater than its rest mass energy: $E^2 = p^2c^2 + m^2c^4 \approx p^2c^2$.

Appendices

A Notation and Conventions

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