ALMA MATER STUDIORUM
UNIVERSITÀ DI BOLOGNA

Department of Physics and Astronomy "A. Righi"

Theoretical Physics

Quantum Field Theory I

Lectures held by
Prof. Michele Cicoli

Notes by
Gioele Mancino

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Introduction

Quantum Field Theory (QFT) provides the modern framework for describing the dynamics of elementary particles and their interactions. It unifies the principles of quantum mechanics with those of special relativity, and it naturally incorporates particle creation and annihilation processes, which are absent in non-relativistic quantum mechanics.

The Poincaré Group and Relativistic Symmetry

The starting point of any relativistic theory is the invariance under the Poincaré group, the group of isometries of Minkowski spacetime. It consists of Lorentz transformations (rotations and boosts) together with spacetime translations.

The representations of the Poincaré group classify possible relativistic particles, characterized by two invariants: the mass m and the spin s.

Classical Field Theory

Before quantization, fields are introduced as classical dynamical variables defined over spacetime. Their dynamics are determined by the principle of stationary action, which leads to the **Euler–Lagrange equations** for fields.

Within this framework, different types of relativistic fields arise naturally: scalar fields, spinor fields, and vector fields, each described by an appropriate Lagrangian density.

A central result of the Lagrangian formalism is **Noether's theorem**, which establishes a direct correspondence between continuous symmetries of the action and conserved physical quantities. For instance, spacetime translation invariance implies conservation of energy and momentum, while internal phase symmetries give rise to conserved charges.

Canonical Quantization of Free Fields

Quantization promotes classical fields to operator-valued distributions acting on a Fock space. The canonical approach consists in imposing equal-time commutation or anticommutation relations, in accordance with the spin–statistics theorem.

- **Spin-0 (Klein–Gordon theory):** A real scalar field is quantized by expanding it in Fourier modes with creation and annihilation operators obeying bosonic commutation relations. This leads to the description of spinless relativistic particles.

- **Spin-1/2 (Dirac and Weyl theories):** Spinor fields are quantized by imposing fermionic anticommutation relations. The Dirac theory provides the framework for massive spin-$\frac{1}{2}$ particles such as electrons, while the Weyl theory describes massless chiral fermions.
- **Spin-1 (Maxwell and Proca theories):** Vector fields can be quantized only after addressing gauge redundancy. The Maxwell theory describes a massless gauge boson (the photon), whereas the Proca theory provides a consistent formulation for a massive spin-1 particle.

These procedures yield the free quantum field theories for the three basic types of relativistic particles: scalars, fermions, and gauge bosons.

Towards Interacting Theories

Free field theories provide the starting point of quantum field theory, but they describe particles without mutual influence. To account for the physical world, one must introduce interactions by adding non-linear terms to the Lagrangian density.

The analysis of interacting quantum fields relies on perturbation theory, systematically organized through Feynman diagrams. Within this framework one computes observable quantities such as decay rates, which measure the probability per unit time that an unstable particle decays, and cross subsections, which quantify the likelihood of scattering processes. These observables form the bridge between the abstract formalism of quantum field theory and the experimental study of particle physics.

We will not cover exact solutions of interacting theories, which require advanced techniques and mathematical tools beyond the scope of these notes. Instead, practical approaches rely on approximation schemes. In particular, perturbative expansions reduce the dynamics of interacting fields to a collection of coupled harmonic oscillators, whose behavior is well understood from quantum mechanics. This analogy provides the foundation for treating interactions as small corrections to free theories, ultimately leading to the perturbative framework of Feynman diagrams.

1 A New Framework

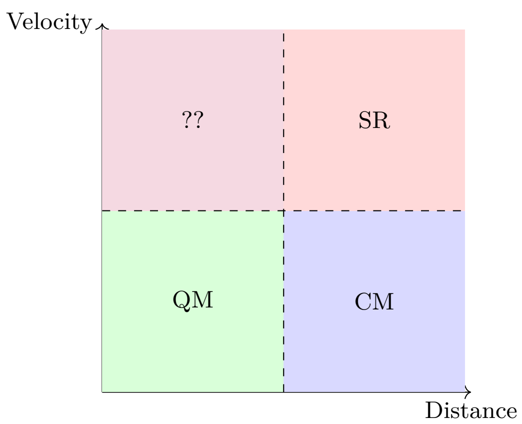
The development of Quantum Field Theory was driven by the need to reconcile the principles of quantum mechanics with those of special relativity. Special relativity describes the structure of spacetime and the behavior of objects moving at high velocities, while quantum mechanics governs the behavior of particles at microscopic scales. However, neither theory alone could adequately describe phenomena involving both high energies and small distances, such as particle creation and annihilation.

Quantum mechanics does not include relativistic effects:

- The concept of a **limiting speed is absent**.
- The **energy expression** for a free particle is incompatible with the relativistic one: $E_{\text{QM}} = \frac{p^2}{2m}$ instead of $E_{\text{SR}} = \sqrt{p^2c^2 + m^2c^4}$.

Special relativity, on the other hand, does not incorporate quantum principles:

- It does not account for **quantization** of physical observables, such as energy.
- The **promotion of observables to operators** acting on a Hilbert space is missing.



We need a new framework to describe the regime of small distances and high velocities, where both quantum and relativistic effects are significant. The idea to overcome the shown limitations is to use expressions from SR and incorporate them into a quantum framework: a **relativistic quantum mechanics** theory.

1.1 | Relativistic Quantum Mechanics and its Limitations

The first attempt to construct a relativistic quantum theory was to modify the Schrödinger equation by replacing the non-relativistic energy-momentum relation with the relativistic one. We are basically in a quantum framework:

- Observables are promoted to operators acting on a Hilbert space by imposing *canonical commutation relations*:
  $[\hat{\mathbf{x}}, \hat{p}] = i\hbar \implies \hat{p} = -i\hbar\frac{\mathrm{d}}{\mathrm{d}\mathbf{x}}, \quad [\hat{t}, \hat{H}] = i\hbar \implies \hat{H} = i\hbar\frac{\mathrm{d}}{\mathrm{d}t}$.
- Operators act on a Hilbert space $\mathcal{H}$, where its vectors represent physical states of the system, with a **fixed number of particles**.
- Eigenvectors of a complete set of commuting observables form a basis for the Hilbert space; the eigenvalues correspond to the possible measurement outcomes:
  $\hat{p}\,|p\rangle = p\,|p\rangle\,, \quad \int \mathrm{d}p\,|p\rangle\,\langle p| = 1$.
- The time evolution of states is governed by the **generalized Schrödinger equation**:
  $-i\hbar\frac{\partial}{\partial t}\psi(\mathbf{x}, t) = \hat{H}\psi(\mathbf{x}, t) = \sqrt{\hat{p}^2c^2 + m^2c^4}\,\psi(\mathbf{x}, t)$.

However, this approach leads to several issues: it cannot account for particle creation and annihilation, which are essential in relativistic contexts. Additionally, the theory struggles with maintaining causality and Lorentz invariance, and predicts an infinite number of negative energy states, leading to an unstable vacuum.

1.1.1 | Production/Annihilation of Particles

A picture in which the number of particles is fixed cannot describe processes where particles are created or destroyed, such as in high-energy collisions (LHC or other particle colliders) or in nuclear decays.

If we consider a particle with mass m at rest, its energy is given by $E = mc^2$. If we now start injecting energy into the system, as the particle acquires momentum, its mass becomes negligible and the system is energetically favorable to create new particles. But in order to preserve physical quantities such as charge, lepton number, baryon number, etc., particles must be created in pairs.

Example: particle in a box. Consider a particle of mass m confined in a three-dimensional box of length L. The *Heisenberg uncertainty principle* states that the uncertainty in position Δx and the uncertainty in momentum Δp satisfy the relation $\Delta x \Delta p \geq \frac{\hbar}{2}$. For a particle confined in a box, we can estimate $\Delta x \sim L$, leading to an uncertainty in momentum of at least $\Delta p \sim \frac{2\hbar}{L}$. If we take the particle to the ultrarelativistic regime,[1] its energy can be approximated as $E \approx pc$. Therefore, the uncertainty in energy will be:

$$\Delta E \geq \frac{2\hbar c}{L}.$$

If we want to avoid the production of particle-antiparticle pairs, we must ensure that the energy uncertainty is less than the energy required to create such a pair, which is $2mc^2$. But if we decrease

[1] In the ultrarelativistic regime, the particle's kinetic energy is much greater than its rest mass energy: $E^2 = p^2c^2 + m^2c^4 \approx p^2c^2$.

the size of the box L to increase the precision in position, we increase the uncertainty in energy.

$$2mc^2 = \frac{2\hbar c}{L} \implies L = \frac{\hbar}{mc} = \lambda_c.$$

Here, λ_c is the Compton wavelength of the particle, representing a fundamental limit to the precision with which we can localize a particle without inducing pair production. If we try to confine the particle within a region smaller than its Compton wavelength, the energy uncertainty becomes sufficient to create particle-antiparticle pairs, making it impossible to describe the system with a fixed number of particles.

We need a framework that allows for a variable number of particles, accommodating the creation and annihilation processes inherent in relativistic quantum phenomena (as we will see, a Fock space formalism is required: $\mathcal{F} = \bigoplus_n \mathcal{H}_n$).

1.1.2 | Violation of Causality

In order to be consistent with special relativity, a physical theory must respect the principle of causality, which states that cause precedes effect and that information cannot travel faster than the speed of light. In relativistic quantum mechanics, the wavefunction can exhibit non-local behavior, leading to potential violations of causality.

To start, let's consider a free particle in the position eigenstate $|\mathbf{x}\rangle$. The time evolution of the wavefunction is given by the operator $e^{-\frac{i}{\hbar}Ht}$. Now we can compute the amplitude for finding the particle at a different position $\mathbf{y}$ after a time t:

$$A_{\mathbf{x}\to\mathbf{y}}(t) = \langle \mathbf{y}| e^{-\frac{i}{\hbar}Ht} |\mathbf{x}\rangle \,.$$

We will show that this amplitude is non-zero even when the distance $|\mathbf{y}-\mathbf{x}|$ is greater than ct, which implies that the particle can be found outside the light cone, violating causality. This is true using QM as well as RQM, in the latter case the effect being less pronounced but still non-zero: the theory is refined, but still not right.

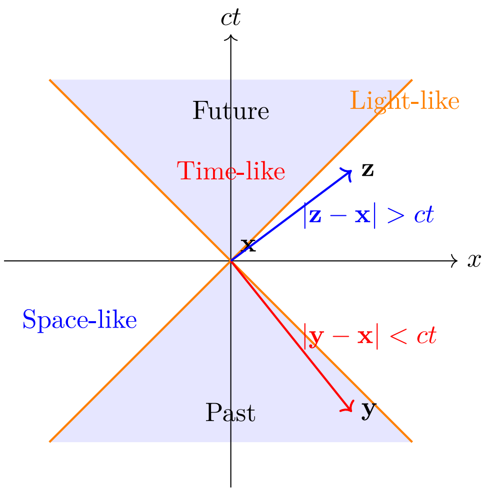


Figure 1.1: Spacetime diagram showing the light cone structure. The blue shaded regions represent the future and past light cones, while the orange lines indicate the paths of light rays. The red arrow represents a time-like separation, where causality is preserved, while the blue arrow represents a space-like separation, where causality is violated.

QM framework

In standard QM the expression for the hamiltonian of a free particle is $H = \frac{\hat{p}^2}{2m}$. The amplitude can be computed as follows:

$$A_{\mathbf{x}\to\mathbf{y}}(t) = \langle\mathbf{y}|\, e^{-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}t}\, |\mathbf{x}\rangle\,.$$

Using the *completeness relation* in the momentum basis $\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi\hbar)^3}\, |\mathbf{p}\rangle\langle\mathbf{p}| = \mathbb{I}$, we have:

$$A_{\mathbf{x}\to\mathbf{y}}(t) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi\hbar)^3}\, \langle\mathbf{y}|\mathbf{p}\rangle\, e^{-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}t}\, \langle\mathbf{p}|\mathbf{x}\rangle\,,$$

where $\langle\mathbf{p}|\mathbf{x}\rangle = \Psi^*_{\mathbf{p}}(\mathbf{x}) = e^{-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}}$ is the wavefunction of the momentum eigenstate in the position representation, basically a *plane wave*. Additionally, if we perform a Taylor expansion of the exponential operator, we get a polinomial expression for the momentum operator: using $\hat{p}^2|\mathbf{p}\rangle = p^2|\mathbf{p}\rangle$, we obtain a serie of powers of p, which can be resummed back into an exponential function. Therefore, we have:

$$A_{\mathbf{x}\to\mathbf{y}}(t) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi\hbar)^3} e^{\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{y}} e^{-\frac{i}{\hbar}\frac{p^2}{2m}t} e^{-\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{x}} = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi\hbar)^3} e^{-\frac{i}{\hbar}\frac{p^2}{2m}t} e^{\frac{i}{\hbar}\mathbf{p}\cdot(\mathbf{y}-\mathbf{x})}.$$

Now, we can perform the integral recognizing the *Gaussian integral* form $\int_{-\infty}^{\infty} e^{-ax^2+bx}\mathrm{d}x = \sqrt{\frac{\pi}{a}}e^{\frac{b^2}{4a}}$ for each component $\mathrm{d}p_1$, $\mathrm{d}p_2$, $\mathrm{d}p_3$ of the momentum vector:

$$A_{\mathbf{x}\to\mathbf{y}}(t) = \left(\frac{m}{2\pi i\hbar t}\right)^{3/2} e^{i\frac{m}{2\hbar}\frac{|\mathbf{y}-\mathbf{x}|^2}{t}} \neq 0 \quad \forall\, t,\, \mathbf{y},\, \mathbf{x}.$$

This result shows that the amplitude is non-zero for any values of t and $\mathbf{y}$ given $\mathbf{x}$, including cases where $|\mathbf{y}-\mathbf{x}| > ct$. This implies a non-zero probability of detecting the particle outside the light cone, which constitutes a violation of causality. However, this is not strictly a problem within quantum mechanics, as the theory does not incorporate a concept of limiting velocity and is fundamentally designed for low-speed regimes.

The next step is to see if the situation improves when we move to a relativistic framework.

RQM framework

In this framework, the hamiltonian of a free particle is given by $H = \sqrt{\hat{p}^2c^2 + m^2c^4}$. The amplitude can be computed as follows:

$$A_{\mathbf{x}\to\mathbf{y}}(t) = \langle\mathbf{y}|\, e^{-\frac{i}{\hbar}\sqrt{\hat{p}^2c^2+m^2c^4}\,t}\, |\mathbf{x}\rangle\,.$$

Again, using the completeness relation and the eigenvalues relation along with taylor expansion for the momentum operator, we get to:

$$A_{\mathbf{x}\to\mathbf{y}}(t) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi\hbar)^3} e^{-\frac{i}{\hbar}\sqrt{p^2c^2+m^2c^4}\,t} e^{\frac{i}{\hbar}\mathbf{p}\cdot(\mathbf{y}-\mathbf{x})}.$$

Let us rename $\mathbf{y}-\mathbf{x} = \mathbf{r}$ and, by using natural units $\hbar = c = 1$, $\sqrt{p^2+m^2} = \omega_p$. Now we can perform the integral using spherical coordinates in momentum space, where $\mathrm{d}^3\mathbf{p} = p^2\sin\theta\,\mathrm{d}p\,\mathrm{d}\theta\,\mathrm{d}\phi$

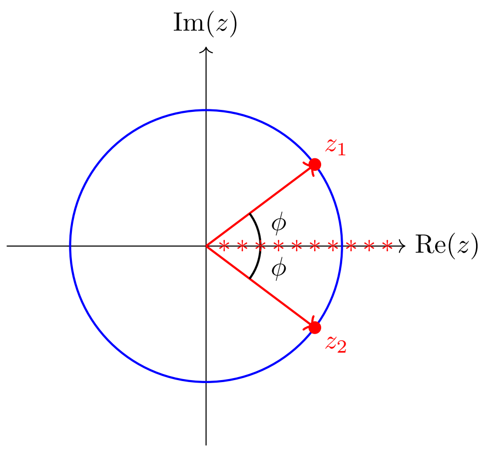
and the scalar product in the exponential becomes $\mathbf{p}\cdot\mathbf{r} = pr\cos\theta$:

$$\begin{aligned}
A_{\mathbf{x}\to\mathbf{y}}(t) &= \int_0^\infty \frac{p^2\mathrm{d}p}{(2\pi)^3}\int_0^\pi \sin\theta\,\mathrm{d}\theta \int_0^{2\pi}\mathrm{d}\phi e^{-i\omega_p t}e^{ipr\cos\theta}\\
&= \frac{1}{(2\pi)^2}\int_0^\infty \mathrm{d}p\, p^2 e^{-i\omega_p t}\int_{-1}^1 \mathrm{d}(\cos\theta)e^{ipr\cos\theta}\\
&= \frac{1}{(2\pi)^2}\int_0^\infty \mathrm{d}p\, p^2 e^{-i\omega_p t}\left[\frac{e^{ipr}-e^{-ipr}}{ipr}\right]\\
&= \frac{1}{(2\pi)^2}\frac{1}{ir}\int_0^\infty \mathrm{d}p\, p\left[e^{ipr}-e^{-ipr}\right]e^{-i\omega_p t}\\
&= \frac{-i}{(2\pi)^2 r}\int_{-\infty}^\infty \mathrm{d}p\, p e^{ipr}e^{-i\omega_p t}.
\end{aligned}$$

This integral is very complicated to solve analytically, needing a solution in form of *Bessel functions*. However, the important point is that the amplitude is still non-zero for $|\mathbf{y}-\mathbf{x}| = r > ct$, indicating that even in a relativistic quantum mechanics framework, there are still violations of causality; let us show then an approximated computation to highlight this.

Thus we promote p to be a complex variable z and use the *residue theorem* to solve the integral; but before that, let us remind some properties of the complex root:

$$z = \rho e^{i\phi} = x + iy, \quad \rho = |z| = \sqrt{x^2+y^2}, \quad \phi = \arg(z) = \tan^{-1}\left(\frac{y}{x}\right).$$



We have to consider the square root of a complex number $\sqrt{z}$, which is a multi-valued function: after completing a full rotation around the origin in the complex plane (i.e., increasing the argument ϕ by 2π), the value of the square root changes sign:

$$\sqrt{z_1} = \sqrt{\rho}e^{i\phi/2}, \quad \sqrt{z_2} = \sqrt{\rho}e^{i(\phi/2+\pi)} = -\sqrt{z_1},$$
$$z_1 = \rho e^{i\phi}, \quad z_2 = \rho e^{i(2\pi-\phi)}.$$

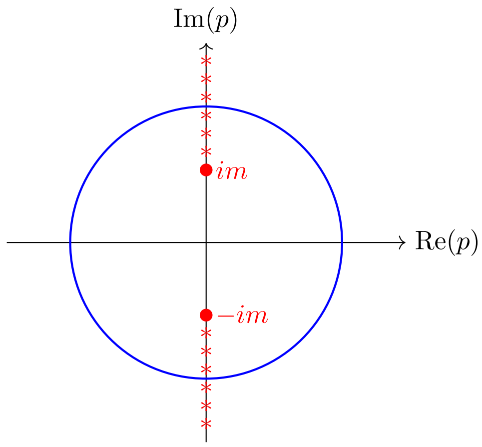
To ensure the function is single-valued, we introduce a branch cut along the positive real axis (indicated by red stars).

The two roots z_1 and z_2 take on different values in the limits $\phi\to 0$ and $\phi\to 2\pi$, with z_2 having a real part of opposite to that of z_1:

$$\sqrt{z_1} = \sqrt{\rho}\left[\cos\frac{\phi}{2} + i\sin\frac{\phi}{2}\right], \quad \sqrt{z_2} = \sqrt{\rho}\left[-\cos\frac{\phi}{2} + i\sin\frac{\phi}{2}\right].$$

Thus we can express our integral in a complex variable p, where we write $p^2+m^2 = (p+im)(p-im)$:

$$A_{\mathbf{x}\to\mathbf{y}}(t) = I = \frac{-i}{(2\pi)^2 r}\int_{-\infty}^\infty \mathrm{d}p\, p e^{ipr}e^{-i\sqrt{(p+im)(p-im)}\,t}.$$



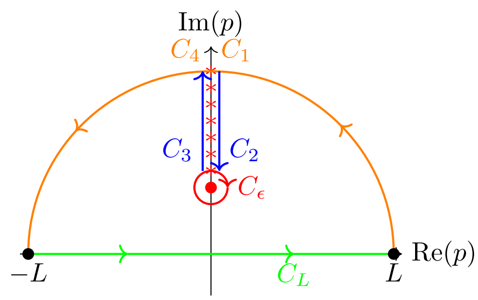
We have to consider the branch cuts starting from the branch points $z = \pm im$ since both $\sqrt{p+im}$ and $\sqrt{p-im}$ are multi-valued functions with branch cuts defined along the imaginary axis (this time the discontinuities of the functions take place while crossing the imaginary axis). The branch cuts are indicated by red stars. We can now close the integration contour in the upper half-plane (the integrand vanishes in the limit $|p| \to \infty$ in this half-plane).

Using the Cauchy residue theorem, we can evaluate our integral by considering a contour that encloses the upper half-plane, avoiding the branch cut.

$$\int f(p)dp = 2\pi i \sum \mathrm{Res}(f, p_k) = 0,$$

since there are no poles inside the contour. The integral along the real axis is equal to the negative of the integral along the branch cut (the integral along the arc at infinity vanishes indeed):

Let's denote the contour integral along the curve C as $\oint_C f(p)\mathrm{d}p$. The contour C consists of three parts: the integral along the real axis from $-\infty$ to ∞, the integral along the two arcs at infinity in the upper half-plane (left and right of the branch cut), the integrals along the branch cut and the infinitesimal circulation around im:



$$\int_{C_L} = -\left(\int_{C_1} + \int_{C_2} + \int_{C_\epsilon} + \int_{C_3} + \int_{C_4}\right).$$

We can now take the limit $L \to \infty$ and $\epsilon \to 0$. The integrals along the arcs at infinity vanish, and the integral around the infinitesimal circle C_ϵ also vanishes.

(i) **Integral on the right arc at infinity vanishes**, $\lim_{|p|\to\infty} \int_{C_1} f(p)\, dp = 0$:
Outside the light cone (which is the only regime we are interested in), we can write:

$$e^{ipr}e^{-it\sqrt{p^2+m^2}} = e^{i\overline{\theta}}e^{-\,\mathrm{Im}(p)r}e^{t\,\mathrm{Im}(\sqrt{p^2+m^2})} \to 0,$$

as $|p| \to \infty$ while mantaining $r > t$: after regrouping the real parts (which are multiplied by a i making the exponantial oscillate) in $\overline{\theta}$, we can se how the second exponential dominate the third when $r \gg t$ (space-like separated events).

(ii) **Integral on the left arc at infinity vanishes**, $\lim_{|p|\to\infty} \int_{C_4} f(p)\, dp = 0$:
since:

$$e^{ipr}e^{-it\sqrt{p^2+m^2}} = e^{i\left[\mathrm{Re}(p)r - t\,\mathrm{Re}(\sqrt{p^2+m^2})\right]}e^{-\left[\mathrm{Im}(p)r + \left|\mathrm{Im}(\sqrt{p^2+m^2})\right|t\right]} \to 0,$$

because we are on the left side of the branch cut, which imposes $\mathrm{Im}(\sqrt{p^2+m^2}) = -|\,\mathrm{Im}(\sqrt{p^2+m^2})| < 0$, thus the integral goes to zero as $|p| \to \infty$ (since again the real parts oscillate).

(iii) **Integral on infinitesimal circle vanishes**, $\lim_{\epsilon\to 0}\int_{C_\epsilon} f(p)\,dp = 0$:
From the Darboux inequality, we have:

$$|\int_{C_\epsilon} f(p)\mathrm{d}p| \le L_{C_\epsilon} \cdot \sup_{p\in C_\epsilon} |f(p)|,$$

where $L_{C_\epsilon} = 2\pi\epsilon$ and $p \in C_\epsilon \implies p = im + \epsilon e^{i\theta}$ for $\theta \in [0, 2\pi]$:

$$L_{C_\epsilon} \sup_{p\in C_\epsilon} |f(p)| = \frac{2\pi\epsilon}{(2\pi)^2 r} m e^{-mr} \sup_{\theta\in[0,2\pi]} \left| e^{-it\sqrt{2im\epsilon e^{i\theta}}} \right| \to 0 \quad \text{as } \epsilon \to 0.$$

Since $e^{-it\sqrt{2im\epsilon e^{i\theta_{\max}}}}$ is bounded for all $\theta \in [0, 2\pi]$, this integral goes to zero as $\epsilon \to 0$.

We are thus left with an expression for the integrals on the real axis and the two sides of the branch cut:

$$\int_{-\infty}^{\infty} f(p)\mathrm{d}p = -(\int_{C_2} + \int_{C_3}) f(p)\mathrm{d}p.$$

We are still considering the regime outside the light cone, $r > t$. We can parametrize the two sides of the branch cut as follows: on the right side $p = iy + \epsilon$ and on the left side $p = iy - \epsilon$, with $y \in [m, \infty)$ and $\epsilon \to 0^+$. The integrals along the two sides of the branch cut can be compressed into a single integral[2]:

$$\begin{aligned} A_{\mathbf{x}\to\mathbf{y}}(t) &= \frac{i}{(2\pi)^2 r} \int_m^\infty \mathrm{d}y\, y e^{-yr} \left[e^{t\sqrt{y^2-m^2}} - e^{-t\sqrt{y^2-m^2}} \right] \\ &= \frac{i}{(2\pi)^2 r} \int_m^\infty \mathrm{d}y\, y e^{-yr} 2\sinh\Big(t\sqrt{y^2-m^2}\Big). \end{aligned}$$

But we know that for $y \ge m$ the integrand becomes a sum of positive or zero terms, thus the integral is non-zero:

$$y e^{-yr} 2\sinh\Big(t\sqrt{y^2-m^2}\Big) \ge 0 \quad \forall y \ge m \qquad \Longrightarrow \qquad A_{\mathbf{x}\to\mathbf{y}}(t) \ne 0.$$

Therefore, we have shown that even in a relativistic quantum mechanics framework, there is a non-zero probability of finding a particle outside the light cone, indicating a violation of causality. This violation is less pronounced than in non-relativistic quantum mechanics (since it is exponentially suppressed), but it still exists. This suggests that a more comprehensive framework is needed to fully reconcile quantum mechanics with the principles of special relativity, leading us to the development of quantum field theory.

Upper bound on $|A_{\mathbf{x}\to\mathbf{y}}(t)|$. We can also derive an upper bound for the amplitude outside the light cone. Starting from the expression:

$$\begin{aligned} \sinh\Big(t\sqrt{y^2-m^2}\Big) &< e^{t\sqrt{y^2-m^2}} < e^{yt}, \\ \implies |A_{\mathbf{x}\to\mathbf{y}}(t)| &< \frac{1}{(2\pi)^2 r} \int_m^\infty \mathrm{d}y\, y e^{-y(r-t)} \\ &= \frac{1}{(2\pi)^2 r} \frac{m(r-t)+1}{(r-t)^2} e^{-m(r-t)}. \end{aligned}$$

[2]We use again the fact that on the left side of the branch cut, $\sqrt{p^2+m^2} = -\sqrt{y^2-m^2}$ while on the right side $\sqrt{p^2+m^2} = \sqrt{y^2-m^2}$.

This upper bound shows that the amplitude decreases exponentially with the distance r outside the light cone, modulated by a polynomial factor. This indicates that while causality violations are present, they become increasingly unlikely as one moves further away from the light cone, especially for larger values of the particle mass m.

1.2 | Towards Quantum Field Theory

The concept of a **field** is central to modern physics, describing how physical quantities vary over space and time. It was introduced in the 19th century to explain phenomena such as electromagnetism and gravity, in order to move away from the idea of *action at a distance*:

- The **gravitational field**, described by Newton's law of universal gravitation, which explains the attraction between masses.
- The **electromagnetic field**, unified by Maxwell's equations, which describes electric and magnetic phenomena and their interactions with charged particles.

In this framework, particles interact locally through fields, rather than instantaneously over a distance.

Photons, the quanta of the electromagnetic field, mediate electromagnetic interactions; they represent the field's excitations around its ground state, allowing for the exchange of energy and momentum between charged particles. This description seems to elect the field as the fundamental entity, with particles being secondary manifestations. However, matter particles, such as electrons and quarks, seem to be more fundamental, as they constitute the building blocks of matter. This duality raises the question: which is more fundamental, fields or particles?[3]

Quantum Field Theory provides a framework in which **fields are the fundamental quantities** and particles are viewed as excitations of underlying fields that permeate space and time, rather than independent entities. Each type of particle corresponds to a specific field:

- The **EM field** gives rise to photons, which mediate electromagnetic interactions.
- The **electron field** (Dirac) gives rise to electrons and positrons.
- The **quark fields** (Dirac) give rise to quarks, which combine to form protons and neutrons.
- The **Higgs field** (Klein-Gordon) is responsible for giving mass to particles through the Higgs mechanism.

Particles are thus seen as localized excitations or quanta of their respective fields around their ground states, and interactions between particles are understood as interactions among fields.

1.2.1 | Properties of the New Approach

The new framework has several important properties, which are granted by its formulation:

(i) **Locality**:
Interactions occur at specific points in space and time, ensuring that cause and effect are preserved. This is implemented by requiring that field operators at spacelike separated points either commute or anticommute, depending on whether they describe bosons or fermions, respectively. This ensures that measurements made at one point cannot instantaneously affect measurements made at another point outside its light cone.

[3] If we were to choose particles as fundamental, we would face the challenge of explaining how they interact at a distance, and fields like the EM one would be described as classical limits of a collection of photons.

(ii) **Causality**:
No information or influence can travel faster than the speed of light, preserving the causal structure of spacetime. *Intuitively*, consider a spacetime event where a particle-antiparticle pair is created, moves apart, and later annihilates (space-like separated events which could be seen in opposite causal order by an external observer). One description is that the particle travels forward in time and annihilates with the antiparticle. An equivalent view is that the antiparticle represents a particle propagating backward in time. In quantum field theory, what matters are the probability amplitudes for such processes, and this reinterpretation ensures that they remain causal: all contributions respect the light cone structure, and no amplitude allows information to propagate faster than light.

(iii) **Particle/antiparticle creation/annihilation**:
The framework naturally incorporates processes where particles can be created or destroyed, since fields can describe systems with infinite degrees of freedom. Let us consider how coordinates describe a system in different contexts:

- **QM**: $(\mathbf{x}, t) \to (\hat{\mathbf{x}}, \hat{t})$.
  The position and time of a particle are promoted to operators acting on a Hilbert space via *canonical quantization*.[4] The number of particles is fixed, and creation/annihilation processes cannot be described.
- **QFT**: $(\mathbf{x}, t) \to \phi(\mathbf{x}, t) \to \hat{\phi}(\mathbf{x}, t)$.
  The field configuration $\phi(\mathbf{x}, t)$ account for infinite degrees of freedom; it is promoted to an operator-valued distribution $\hat{\phi}(\mathbf{x}, t)$ acting on a Fock space, which can accommodate states with varying numbers of particles. Spacetime coordinates are interpreted as continuous labels for infinite field operators, not as operators themselves.

We can now describe processes such muon decay $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$, where a muon transforms into an electron, an electron antineutrino, and a muon neutrino. This process involves annihilation and creation of particles, varying even the number of particles involved, which is naturally described in QFT.

(iv) **Indistinguishability of particles of identical type**:
Particles of the same type are fundamentally indistinguishable, even if they were created in different processes and in distant spacetime regions. *Example*: protons produced in supernovae billions of light years away and in particle accelerators here on Earth are identical. This indistinguishability is naturally incorporated in QFT, where particles are excitations of the same underlying field which fills the entire universe.

(v) **Correct spin-statistics relations**:
Indistinguishability for identical particles leads to the requirement that the quantum states of a system of multiple identical particles must be either symmetric (for bosons) or antisymmetric (for fermions) under the exchange of any two particles. This requirement is known as the **spin-statistics theorem**, which states that particles with integer spin (bosons) obey Bose-Einstein statistics, while particles with half-integer spin (fermions) obey Fermi-Dirac statistics:

[4]This process involves promoting classical fields to operator-valued distributions by imposing canonical commutation relations.

- ψ is symmetric under exchange $\psi(\dots x_i \dots x_j \dots) = \psi(\dots x_j \dots x_i \dots)$:
  Bosons (integer spin) can occupy the same quantum state, leading to phenomena like **Bose-Einstein condensation**.
- ψ is antisymmetric under exchange $\psi(\dots x_i \dots x_j \dots) = -\psi(\dots x_j \dots x_i \dots)$:
  Fermions (half-integer spin) obey the **Pauli exclusion principle**, which prevents them from occupying the same quantum state.

In QM this relations are imposed as an additional postulate, while in QFT it emerges naturally from the commutation relations of field operators. While quantizing a field, one has to define strictly spin-statistics relations, imposing commutation relations for bosonic fields and anticommutation relations for fermionic fields:

$$\begin{aligned}
&\text{Bosons:} \quad [\hat{\phi}(\mathbf{x},t), \hat{\phi}(\mathbf{y},t)] = 0, \quad [\hat{\phi}(\mathbf{x},t), \hat{\pi}(\mathbf{y},t)] = i\delta^{(3)}(\mathbf{x}-\mathbf{y}),\\
&\text{Fermions:} \quad \{\hat{\psi}_\alpha(\mathbf{x},t), \hat{\psi}_\beta(\mathbf{y},t)\} = 0, \quad \{\hat{\psi}_\alpha(\mathbf{x},t), \hat{\pi}_\beta(\mathbf{y},t)\} = i\delta_{\alpha\beta}\delta^{(3)}(\mathbf{x}-\mathbf{y}).
\end{aligned}$$

This ensures **consistency**: $E > 0$ and no negative norm states for every particle; the correct statistics for particles emerge naturally from the field quantization process and the underlying symmetries of the theory.

Quantum Field Theory provides a consistent and comprehensive description of the fundamental particles and their interactions, not only in relativistic regimes but also in non-relativistic ones with variable number of particles. It has been remarkably successful in explaining a wide range of phenomena at the fundamental level, from the behavior of condensed matter systems to high energy particle collisions, from quantum gravity to cosmology. It also contributed to pure mathematical developments, in fields such as topology and geometry.

1.2.2 | Units and Scales

In nature we have three fundamental dimensionfull constants:

- The speed of light in vacuum c, which relates space and time.
- The reduced Planck constant $\hbar$, which relates energy and frequency.
- The gravitational constant G, which sets the strength of gravitational interactions.

Let us analyze their dimensions in terms of mass M, length L, and time T:

$$\begin{aligned}
[c] &= \frac{L}{T},\\
[\hbar] &= [E]T = \frac{ML^2}{T},\\
[G] &= \frac{[E]L}{M^2} = \frac{L^3}{MT^2}.
\end{aligned}$$

From these three constants we can construct a system of **natural units**, by setting $c = \hbar = 1$. This choice simplifies equations and calculations in theoretical physics, particularly in QFT and quantum gravity. In this system, all physical quantities can be expressed in terms of a single unit, typically mass (or energy), and even length and time have the same dimension $L = T$. For the

Compton wavelength in natural units we have:[5]

$$\lambda_c = \frac{\hbar}{mc} = \frac{1}{m} \implies L = \frac{1}{M}.$$

Thus since $L = T = \frac{1}{[E]} = \frac{1}{M}$, we can express the dimensions of the three constants as:

$$[c] = 1, \quad [\hbar] = 1, \quad [G] = \frac{L}{M} = \frac{1}{M^2},$$

The gravitational constant G has dimensions of inverse mass squared, indicating that gravitational interactions become significant at very high energy scales.

Sometimes *mass dimensions* will come in handy, especially when dealing with fields and coupling constants. They consists in the dimensions of a quantity expressed in terms of mass only. For example, in natural units:

$$[c] = [\hbar] = 0, \quad [G] = -2, \quad [E] = 1.$$

The **Planck scale** is the energy scale at which quantum effects of gravity become significant, and it is defined using the three fundamental constants:

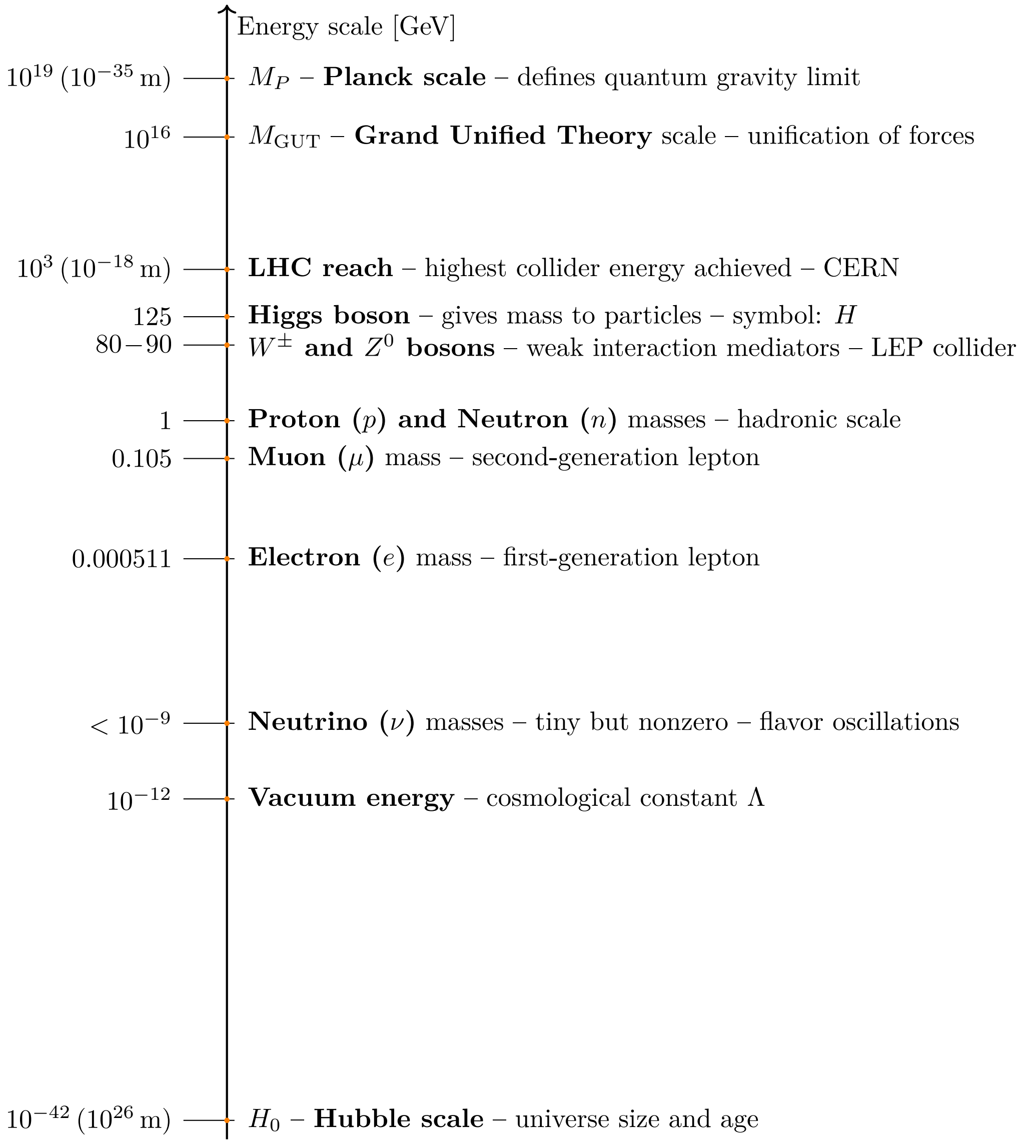
$$M_P = \sqrt{\frac{\hbar c}{G}} \approx 1.22 \times 10^{19}\,\text{GeV}.$$

It represents the scale where our current understanding of physics, based on QFT and General Relativity, breaks down, and a theory of quantum gravity is required. It is related to the **Planck length**:

$$l_P = \sqrt{\frac{\hbar G}{c^3}} \approx 1.62 \times 10^{-35}\,\text{m}.$$

In the following figure we summarize the energy scales of fundamental physics, from the Planck scale to the Hubble scale.

[5]The Compton wavelength λ_c of a particle with mass m is given by $\lambda_c = \frac{\hbar}{mc}$. It represents a fundamental limit to the precision with which we can localize a particle, as attempting to confine it to a region smaller than its Compton wavelength would require energies sufficient to create particle-antiparticle pairs, thus undermining the notion of a single, localized particle.


Energy scale [GeV]
10^{19} (10^{-35} m)
M_P – Planck scale – defines quantum gravity limit
10^{16}
M_{GUT} – Grand Unified Theory scale – unification of forces
10^3 (10^{-18} m)
LHC reach – highest collider energy achieved – CERN
125
Higgs boson – gives mass to particles – symbol: H
80 – 90
$W^\pm$ and Z^0 bosons – weak interaction mediators – LEP collider
1
Proton (p) and Neutron (n) masses – hadronic scale
0.105
Muon (μ) mass – second-generation lepton
0.000511
Electron (e) mass – first-generation lepton
$< 10^{-9}$
Neutrino (ν) masses – tiny but nonzero – flavor oscillations
10^{-12}
Vacuum energy – cosmological constant Λ
10^{-42} (10^{26} m)
H_0 – Hubble scale – universe size and age

1.3 | Mechanical Model of a Quantum Field

To build intuition for quantum fields, it is useful to begin with a simple mechanical system: a chain of coupled elastic strings. This model arises as the *continuum limit* of a one-dimensional lattice of N atoms, and already captures many of the essential features of field theories.

We consider a **meta-stable** system, meaning that its configuration does not change drastically under the action of a small perturbing force.[6] Thus, there must exist a restoring force that drives the system back toward equilibrium.

For a small displacement y from equilibrium, the restoring force can be expanded in a Taylor series:

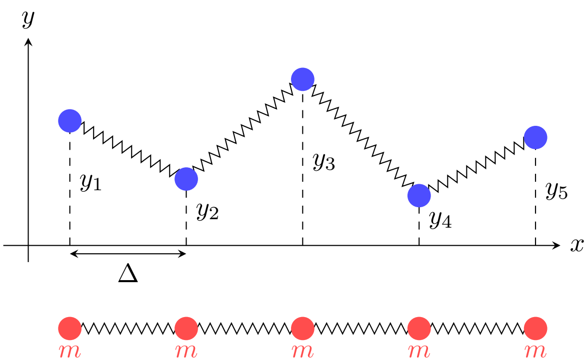
$$F(y) = F(0) + \left.\frac{dF}{dy}\right|_{y=0} y + \cdots = -|k|\, y + \mathcal{O}(y^2),$$

where we set $F(0) = 0$ (no force at equilibrium) and require $\left.\frac{dF}{dy}\right|_{y=0} = -|k| < 0$ (restoring behavior). The linear term dominates for sufficiently small displacements, showing that any meta-stable system can be modeled, to first approximation, as a collection of coupled harmonic oscillators.

The simplest realization is a chain of N identical atoms, each of mass m, connected by springs with spring constant k and arranged along a line with equilibrium spacing Δ. The total length of the chain is

$$L = N\Delta.$$

The displacement of the i-th atom from its equilibrium position will be denoted by $y_i(t)$. For simplicity, we restrict ourselves to transverse displacements along the y-axis, keeping the equilibrium positions along the x-axis fixed.



$y_i = 0$ is the equilibrium position of the i-th atom. The springs exert restoring forces proportional to the relative displacements of neighboring atoms: each atom is under the influence of the neighboring springs, describing a local interaction.

$y_i \neq 0$ describes the small oscillations around the equilibrium position: transverse **excitations** of the lattice.

In the limit $N \to \infty$ and $\Delta \to 0$ with L fixed, the discrete index i becomes a continuous spatial coordinate and the displacements $y_i(t)$ become a continuous field $\phi(x, t)$. This **continuum limit** transforms the system of coupled oscillators into a continuous field, allowing us to understand the dynamics of fields in terms of familiar mechanical concepts.

1.3.1 | Discrete case

To study the dynamics of the chain we employ the Euler–Lagrange equations. The Lagrangian of the system is constructed as the difference between the kinetic and potential energy contributions

[6] In fact, every physical system is meta-stable in this sense: if a small perturbation could radically alter its configuration, it could not persist in a stable state.

of each atom:

$$\begin{aligned} L &= \sum_{j=1}^{N} \left[\frac{1}{2} m \dot{y}_j^2 - \frac{1}{2} k \left(y_j - y_{j+1}\right)^2 \right] \\ &= \sum_{j=1}^{N} \left[\frac{1}{2} m \dot{y}_j^2 - \frac{1}{2} \kappa \left(\frac{y_j - y_{j+1}}{\Delta} \right)^2 \right], \end{aligned}$$

where we impose periodic boundary conditions $j \to j + N$[7] and we renamed the spring constant k as $\kappa = k\Delta^2$, which is now more of a *coupling constant* between neighboring atoms.

We are assuming small oscillations around the equilibrium configuration, such that the relative displacement between neighboring atoms is small compared to the natural lattice spacing Δ:

$$\frac{y_j - y_{j+1}}{\Delta} \ll 1.$$

It is worth noting that the coupling constant κ has the dimension of an energy, and can be written as

$$\kappa = mv^2,$$

where v is a characteristic velocity associated with the propagation of disturbances along the chain. With this identification, the Lagrangian becomes

$$L = \frac{1}{2} m \sum_{j=1}^{N} \left[\dot{y}_j^2 - v^2 \left(\frac{y_j - y_{j+1}}{\Delta} \right)^2 \right],$$

where the first term corresponds to the kinetic energy of the atoms, while the second encodes the elastic potential energy due to the coupling between neighbors.

The time evolution of the system is obtained by applying the principle of least action: the motion is such that the variation of the action vanishes,

$$S = \int L \, \mathrm{d}t, \qquad \delta S = 0,$$

which leads to the Euler–Lagrange equations governing the dynamics of the chain. Recalling the general form

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{y}_j} = \frac{\partial L}{\partial y_j}, \qquad j = 1, \ldots, N,$$

we obtain the governing equations of motion:

$$\ddot{y}_j(t) = -\, v^2 \left(\frac{2y_j - y_{j+1} - y_{j-1}}{\Delta^2} \right).$$

This result clearly describes a system of coupled harmonic oscillators: the displacement of the j-th atom is influenced not only by its own position but also by the relative positions of its nearest neighbors, $(j+1)$ and $(j-1)$. In other words, each atom is bound to oscillate around equilibrium under the restoring force arising from the springs that connect it to its neighbors.

Solving such a system directly is not convenient, since the equations are not independent. However, there is a natural way to simplify the problem by exploiting the translational symmetry of the lattice. The key idea is to perform a *discrete Fourier transform* of the displacements, which allows us to rewrite the dynamics in terms of normal modes of oscillation. Since $y_j(t)$ has to be periodic ($y_j(t) = y_{j+N}(t)$), we can write:

$$y_j(t) = \frac{1}{\sqrt{N}} \sum_{\sigma=1}^{N} e^{i \frac{2\pi}{N} \sigma j} \tilde{y}_\sigma(t).$$

[7] In other words, the first and the last atoms in the chain interact with each other.

In this new representation, each mode corresponds to a collective oscillation of the entire chain with a definite wavelength. From a mathematical perspective, this amounts to diagonalizing the interaction matrix: the coupling between neighbors is replaced by a set of independent equations for the Fourier modes. In physical terms, the Fourier transform identifies the proper "coordinates" in which the energy of the system can be expressed as a sum of independent contributions, one for each mode.[8]

Now, by substituting in the equation of motion, we can decouple the system:

$$\begin{aligned}
\frac{1}{\sqrt{N}}\sum_{\sigma=1}^{N} e^{i\frac{2\pi}{N}\sigma j}\ddot{\tilde{y}}_\sigma &= \frac{-1}{\sqrt{N}}\left(\frac{v}{\Delta}\right)^2\left[2\sum_{\sigma=1}^{N} e^{i\frac{2\pi}{N}\sigma j}\tilde{y}_\sigma - \sum_{\sigma=1}^{N} e^{i\frac{2\pi}{N}\sigma(j-1)}\tilde{y}_\sigma - \sum_{\sigma=1}^{N} e^{i\frac{2\pi}{N}\sigma(j+1)}\tilde{y}_\sigma\right]\\
&= \frac{-1}{\sqrt{N}}\left(\frac{v}{\Delta}\right)^2\sum_{\sigma=1}^{N} e^{i\frac{2\pi}{N}\sigma j}\tilde{y}_\sigma\left[2 - e^{-i\frac{2\pi}{N}\sigma} - e^{i\frac{2\pi}{N}\sigma}\right]\\
&= \frac{-1}{\sqrt{N}}\left(\frac{v}{\Delta}\right)^2\sum_{\sigma=1}^{N} e^{i\frac{2\pi}{N}\sigma j}\tilde{y}_\sigma\left[2 - 2\cos\left(\frac{2\pi}{N}\sigma\right)\right]\\
\iff \ddot{\tilde{y}}_\sigma(t) &= -2\frac{v^2}{\Delta^2}\left[1-\cos\left(\frac{2\pi}{N}\sigma\right)\right]\tilde{y}_\sigma(t) = -\left(\frac{2v}{\Delta}\sin\left(\frac{\pi\sigma}{N}\right)\right)^2\tilde{y}_\sigma(t),
\end{aligned}$$

where we have made use of the trigonometric identity $(1-\cos(2\theta)) = 2\sin^2(\theta)$. This leads to the following set of equations for the Fourier modes:

$$\begin{aligned}
\ddot{\tilde{y}}_\sigma(t) &= -\omega_\sigma^2\,\tilde{y}_\sigma(t), \qquad \forall\,\sigma = 1,2,\ldots,N,\\
\omega_\sigma &= \frac{2v}{\Delta}\,\sin\left(\frac{\pi\sigma}{N}\right).
\end{aligned}$$

Each Fourier component $\tilde{y}_\sigma(t)$ evolves independently and satisfies the equation of a simple harmonic oscillator with characteristic frequency ω_σ. In other words, the original coupled system of atoms has been diagonalized into a set of N decoupled oscillators, each associated with a normal mode of vibration.

The general solution for each mode is therefore given by a linear combination of oscillatory functions:

$$\tilde{y}_\sigma(t) = A_\sigma e^{-i\omega_\sigma t} + B_\sigma e^{+i\omega_\sigma t},$$

where the constants A_σ and B_σ are determined by the initial conditions. Physically, these modes correspond to standing waves propagating through the chain, each characterized by a discrete wave number and its corresponding frequency.

In practice, we will keep only the negative exponential,

$$\tilde{y}_\sigma(t) = A_\sigma e^{-i\omega_\sigma t},$$

since the positive-frequency solution is automatically recovered as the complex conjugate. This choice avoids redundancy and is particularly convenient when later quantizing the system, because creation and annihilation operators will naturally emerge from the decomposition into $e^{-i\omega t}$ and its conjugate.

[8] In other words, instead of tracking the motion of individual atoms, which are strongly coupled, we describe the system in terms of delocalized excitations (the normal modes), each evolving independently. This step paves the way to the field interpretation: in the continuum limit, these modes will be interpreted as excitations of a quantum field.

Now we can plug the solution for the Fourier modes back into the inverse transform in order to recover the displacement of the j-th atom:

$$\begin{aligned} y_j(t) &= \frac{1}{\sqrt{N}} \sum_{\sigma=1}^{N} e^{i\frac{2\pi}{N}\sigma j} \tilde{y}_\sigma(t) \\ &= \frac{1}{\sqrt{N}} \sum_{\sigma=1}^{N} A_\sigma e^{i\left(\frac{2\pi}{N}\sigma j - \omega_\sigma t\right)} \\ &= \frac{1}{\sqrt{N}} \sum_{\sigma=1}^{N} A_\sigma e^{i\left(\frac{2\pi}{N}\frac{\sigma}{\Delta}(\Delta j) - \omega_\sigma t\right)}, \end{aligned}$$

Here we have reintroduced the lattice spacing Δ, so that the quantity Δj corresponds exactly to the physical position x of the j-th atom along the chain.

This expression shows that the displacement of each atom can be written as a linear superposition of plane waves (like *sound waves*). In particular, the system supports N different wave numbers, each corresponding to one Fourier mode:

$$\kappa_\sigma = \frac{2\pi}{N}\frac{\sigma}{\Delta} = \frac{2\pi}{\lambda_\sigma}, \qquad \sigma = 1, 2, \dots, N,$$

with associated discrete wavelengths

$$\lambda_\sigma = \frac{N\Delta}{\sigma} = N\Delta, \frac{N\Delta}{2}, \dots, \Delta.$$

Remark. *The propagation speed of each mode is obtained as the ratio between its frequency and its wave number:*

$$v_\sigma = \frac{\omega_\sigma}{\kappa_\sigma} = \frac{\frac{2v}{\Delta}\sin\left(\frac{\pi\sigma}{N}\right)}{\frac{2\pi}{N}\frac{\sigma}{\Delta}} = v\,\frac{N}{\pi\sigma}\,\sin\left(\frac{\pi\sigma}{N}\right).$$

Introducing the parameter $\theta_\sigma = \frac{\pi\sigma}{N}$*, this can be written in the compact form*

$$v_\sigma = v\,\frac{\sin(\theta_\sigma)}{\theta_\sigma}.$$

This result shows that each Fourier mode propagates with a distinct phase velocity, depending on the ratio σ/N*. In the continuum limit* $N \to \infty$ *with* $\sigma/N \to 0$*, one has* $\theta_\sigma \to 0$ *and therefore*

$$\lim_{\theta_\sigma \to 0} \frac{\sin(\theta_\sigma)}{\theta_\sigma} = 1,$$

so all modes propagate with the same velocity $v_\sigma \sim v$*. This recovers the expected propagation speed of sound waves in the continuous chain.*

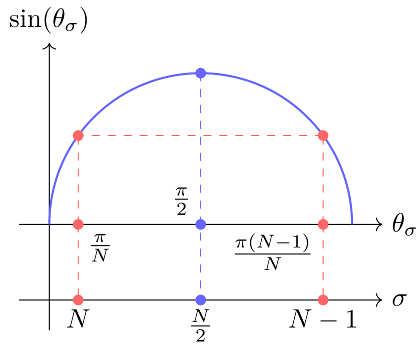
Remark. *The actual number of independent vibrational degrees of freedom is* $N-1$*, since the* N*-th mode corresponds to a zero frequency:*

$$\omega_{\sigma=N} = \frac{2v}{\Delta}\sin\left(\pi\frac{N}{N}\right) = 0.$$

This means that $\tilde{y}_N(t)$ *does not oscillate in time. Physically, this mode represents a uniform translation of the entire chain, where all atoms are displaced by the same amount. As such, it does not contribute to the internal vibrational dynamics, and the only genuine oscillatory modes are the first* $N-1$*.*

Moreover we can show that the waves with $\sigma = s$ and $\sigma = N-s$ in general have the same frequency ω_s, since:

$$\begin{aligned}\omega_{N-s} &= \frac{2v}{\Delta}\,\sin\left(\pi\frac{N-s}{N}\right)\\ &= \frac{2v}{\Delta}\,\sin\left(\pi-\frac{\pi s}{N}\right)\\ &= \frac{2v}{\Delta}\,\sin\left(\frac{\pi s}{N}\right)=\omega_s.\end{aligned}$$



We can interpret the two modes with the same frequency as forming a single complex degree of freedom. This interpretation is supported by the observation that the original variable y_j is real, even though it is expressed as a sum of complex exponentials:

$$\begin{aligned}y_j(t) &= \frac{1}{\sqrt{N}}\sum_{\sigma=1}^{N} e^{i\frac{2\pi}{N}\sigma j}\tilde{y}_\sigma(t),\\ y_j^*(t) &= \frac{1}{\sqrt{N}}\sum_{\sigma=1}^{N} e^{-i\frac{2\pi}{N}\sigma j}\tilde{y}_\sigma^*(t).\end{aligned}$$

Thus, by requiring that the two expressions coincide, we can compute:

$$\begin{aligned}\sum_{\sigma=1}^{N} e^{i\frac{2\pi}{N}\sigma j}\tilde{y}_\sigma(t) &= \sum_{\sigma=1}^{N} e^{-i\frac{2\pi}{N}\sigma j}\tilde{y}_\sigma^*(t),\\ \sum_{\sigma=N-1}^{0} e^{i\frac{2\pi}{N}(N-\sigma)j}\tilde{y}_{N-\sigma}(t) &= \sum_{\sigma=1}^{N} e^{-i\frac{2\pi}{N}\sigma j}\tilde{y}_\sigma^*(t),\\ \sum_{\sigma=N-1}^{0} e^{i2\pi j}e^{-i\frac{2\pi}{N}\sigma j}\tilde{y}_{N-\sigma}(t) &= \sum_{\sigma=1}^{N} e^{-i\frac{2\pi}{N}\sigma j}\tilde{y}_\sigma^*(t),\\ \sum_{\sigma=N-1}^{0} e^{-i\frac{2\pi}{N}\sigma j}\tilde{y}_{N-\sigma}(t) &= \sum_{\sigma=1}^{N} e^{-i\frac{2\pi}{N}\sigma j}\tilde{y}_\sigma^*(t),\\ \sum_{\sigma=1}^{N} e^{-i\frac{2\pi}{N}\sigma j}\tilde{y}_{N-\sigma}(t) &= \sum_{\sigma=1}^{N} e^{-i\frac{2\pi}{N}\sigma j}\tilde{y}_\sigma^*(t),\\ \sum_{\sigma=1}^{N}\left[\tilde{y}_{N-\sigma}(t)-\tilde{y}_\sigma^*(t)\right] &= 0,\end{aligned}$$

which lead us to conclude that $\tilde{y}_{N-\sigma}(t) = \tilde{y}_\sigma^*(t)$.[9]

Now we are in a position to compute the total number of independent vibrational degrees of freedom. To do so, we must distinguish two cases:

- **If N is even ($N = 2l$)**, the total number of pairs of harmonic oscillators with the same frequency (denoted by m) is

$$m = \frac{2l-2}{2} = l-1.$$

[9]In the first step, we rename the summation index on the right-hand side of the equation, replacing each σ with $N-\sigma$. We then use the periodicity condition $e^{i2\pi j} = 1$ to simplify the exponential factor. Next, we reverse the order of summation and, since the terms corresponding to $\sigma = 0$ and $\sigma = N$ are identical, we can safely rewrite the sum over the range $1 \le \sigma \le N$.

In this case we have to exclude the modes $\sigma = N$ and $\sigma = \frac{N}{2}$. The reason is that, when N is even, the mode with index $\sigma = \frac{N}{2}$ is self-conjugate, since it coincides with its complementary mode $\sigma = N - \frac{N}{2} = \frac{N}{2}$.

- **If** N **is odd** $(N = 2l + 1)$, the total number of such oscillator pairs is

$$m = \frac{(2l+1)-1}{2} = l.$$

Here the only index that must be removed is $\sigma = N$, since $\frac{N}{2}$ is not an integer and therefore does not correspond to any mode index.

The two oscillators associated with the same frequency can be regarded as components of a single complex degree of freedom. In this representation, the physical displacement corresponds to the real part of the complex variable, while the imaginary part encodes the redundant conjugate mode.

We can now exploit the solutions obtained in the new basis to diagonalize the Hamiltonian, since the transformation allows us to rewrite it in terms of independent normal modes, each of which behaves as an uncoupled harmonic oscillator. Let us temporarily neglect the contributions from the modes $\sigma = \frac{N}{2}, N$. Then the kinetic energy can be rewritten as

$$\begin{aligned}
T &= \frac{1}{2} m \sum_{j=1}^{N} \dot{y}_j^2 \\
&= \frac{1}{2} m \sum_{j=1}^{N} \frac{1}{N} \sum_{\sigma=1}^{\frac{N}{2}-1} \left(e^{i\frac{2\pi}{N}\sigma j} \dot{\tilde{y}}_\sigma + e^{-i\frac{2\pi}{N}\sigma j} \dot{\tilde{y}}_{N-\sigma} \right) \sum_{\xi=1}^{\frac{N}{2}-1} \left(e^{i\frac{2\pi}{N}\xi j} \dot{\tilde{y}}_\xi + e^{-i\frac{2\pi}{N}\xi j} \dot{\tilde{y}}_{N-\xi} \right) \\
&= \frac{1}{2} m \sum_{\sigma,\xi=1}^{\frac{N}{2}-1} \left[\dot{\tilde{y}}_\sigma \dot{\tilde{y}}_\xi \left(\frac{1}{N} \sum_{j=1}^{N} e^{i\frac{2\pi}{N}(\sigma+\xi)j} \right) + \dot{\tilde{y}}_{N-\sigma} \dot{\tilde{y}}_\xi \left(\frac{1}{N} \sum_{j=1}^{N} e^{i\frac{2\pi}{N}(\xi-\sigma)j} \right) \right. \\
&\qquad \left. + \dot{\tilde{y}}_\sigma \dot{\tilde{y}}_{N-\xi} \left(\frac{1}{N} \sum_{j=1}^{N} e^{i\frac{2\pi}{N}(\sigma-\xi)j} \right) + \dot{\tilde{y}}_{N-\sigma} \dot{\tilde{y}}_{N-\xi} \left(\frac{1}{N} \sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(\sigma+\xi)j} \right) \right] \\
&= \frac{1}{2} m \sum_{\sigma,\xi=1}^{\frac{N}{2}-1} \left(\dot{\tilde{y}}_\sigma \dot{\tilde{y}}_\xi \, \delta_{\sigma+\xi,0} + \dot{\tilde{y}}_{N-\sigma} \dot{\tilde{y}}_\xi \, \delta_{\sigma,\xi} + \dot{\tilde{y}}_\sigma \dot{\tilde{y}}_{N-\xi} \, \delta_{\sigma,\xi} + \dot{\tilde{y}}_{N-\sigma} \dot{\tilde{y}}_{N-\xi} \, \delta_{\sigma+\xi,0} \right) \\
&= \frac{1}{2} m \sum_{\sigma=1}^{\frac{N}{2}-1} \left(2 \dot{\tilde{y}}_\sigma \dot{\tilde{y}}_{N-\sigma} \right) = m \sum_{\sigma=1}^{\frac{N}{2}-1} \dot{\tilde{y}}_\sigma \dot{\tilde{y}}_\sigma^* = m \sum_{\sigma=1}^{\frac{N}{2}-1} |\dot{\tilde{y}}_\sigma|^2.
\end{aligned}$$

The sum over j enforce orthogonality, which manifests as Kronecker deltas. These deltas select only the diagonal terms, pairing each mode σ with its complex conjugate $N - \sigma$. The final expression is a sum over independent Fourier modes, where the kinetic energy is simply the mass times the squared modulus of the time derivative of each mode amplitude. Now for the potential energy:

$$\begin{aligned}V &= \frac{1}{2}m\left(\frac{v}{\Delta}\right)^2\sum_{j=1}^{N}\left(y_j - y_{j+1}\right)^2 = \\ &= \frac{1}{2}m\left(\frac{v}{\Delta}\right)^2\sum_{j=1}^{N}\left[\sum_{\sigma=1}^{\frac{N}{2}-1}\left[\left(e^{i\frac{2\pi}{N}\sigma j}\tilde{y}_\sigma + e^{-i\frac{2\pi}{N}\sigma j}\tilde{y}_{N-\sigma}\right)\right.\right. \\ &\qquad\qquad \left.\left. - \left(e^{i\frac{2\pi}{N}\sigma(j+1)}\tilde{y}_\sigma + e^{-i\frac{2\pi}{N}\sigma(j+1)}\tilde{y}_{N-\sigma}\right)\right]\right]^2 \\ &= \frac{1}{2}m\left(\frac{v}{\Delta}\right)^2\sum_{j=1}^{N}\left[\sum_{\sigma=1}^{\frac{N}{2}-1}\left[\tilde{y}_\sigma e^{i\alpha_\sigma j}\left(1-e^{i\alpha_\sigma}\right) + \tilde{y}_{N-\sigma}e^{-i\alpha_\sigma j}\left(1-e^{-i\alpha_\sigma}\right)\right]\right]^2 \\ &= \frac{1}{2}m\left(\frac{v}{\Delta}\right)^2\sum_{j=1}^{N}\left[\sum_{\sigma=1}^{\frac{N}{2}-1}2i\sin\left(\frac{\alpha_\sigma}{2}\right)\left(-\tilde{y}_\sigma e^{i\alpha_\sigma(j+\frac{1}{2})} + \tilde{y}_{N-\sigma}e^{-i\alpha_\sigma(j+\frac{1}{2})}\right)\right]^2,\end{aligned}$$

where $\alpha_\sigma = \frac{2\pi}{N}\sigma$. Now, using again the sum over j to enforce orthogonality (noting that the term $j+\frac{1}{2}$ on the exponential does not influence the result since it is canceled in cross-terms, while diagonal terms are zeroed by $\delta_{\sigma+\xi,0}$) and recalling the expression for ω_σ, we arrive to the diagonal form of the potential:

$$V = \frac{1}{2}m\left(\frac{v}{\Delta}\right)^2\sum_{\sigma=1}^{\frac{N}{2}-1}8\sin\left(\frac{\alpha_\sigma}{2}\right)^2\tilde{y}_\sigma\tilde{y}_{N-\sigma} = m\sum_{\sigma=1}^{\frac{N}{2}-1}\omega_\sigma^2|\tilde{y}_\sigma|^2.$$

Finally, by adding the contributions from the modes $\sigma = \frac{N}{2}, N$, we arrive to the diagonal form of our Lagrangian:

$$L = m\sum_{\sigma=1}^{\frac{N}{2}-1}\left(|\dot{\tilde{y}}_\sigma|^2 - \omega_\sigma^2|\tilde{y}_\sigma|^2\right) + \frac{m}{2}\left(\dot{\tilde{y}}_{\frac{N}{2}}^2 + \dot{\tilde{y}}_N^2 - \omega_{\frac{N}{2}}^2\tilde{y}_{\frac{N}{2}}^2\right),$$

since $\omega_N = 0$.[10] Thus, in the end, we have $\frac{N-2}{2}$ complex degrees of freedom from the first term, which become $N-2$ real degrees of freedom with $\frac{N-2}{2}$ different frequencies, while the second term contributes with 2 real degrees of freedom.

In practice, the diagonalization procedure eliminates the cross-terms that mix different coordinates, leaving a sum of quadratic contributions that can be interpreted as the energies of the individual modes. As a result, the system is reduced to a collection of independent harmonic oscillators, each characterized by its own frequency.

Quantization

In order to proceed with the quantization of the system, it is useful to briefly recall the main ingredients of the quantum harmonic oscillator.

We start from the Hamiltonian operator expressed in terms of the position and momentum operators $\hat{y}$ and $\hat{p}$:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{y}^2.$$

The canonical quantization rule imposes the commutation relation between $\hat{y}$ and $\hat{p}$:

$$[\hat{y}, \hat{p}] = i\hbar.$$

[10] Let us remember that the $\frac{N}{2}$-th mode is absent when N is odd, and the sum stops at the $\frac{N-1}{2}$-th mode.

It is then convenient to introduce the so–called ladder (or creation/annihilation) operators:

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\,\hat{y} + \frac{i}{\sqrt{2m\hbar\omega}}\,\hat{p}, \qquad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\,\hat{y} - \frac{i}{\sqrt{2m\hbar\omega}}\,\hat{p}.$$

These operators satisfy the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$. In terms of $\hat{a}$ and $\hat{a}^\dagger$, the Hamiltonian takes the following simple form:

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \tfrac{1}{2}\right).$$

Finally, the position and momentum operators can be expressed back in terms of the ladder operators as

$$\hat{y} = \sqrt{\frac{\hbar}{2m\omega}}\,\big(\hat{a} + \hat{a}^\dagger\big), \qquad \hat{p} = i\sqrt{\frac{m\hbar\omega}{2}}\,\big(\hat{a}^\dagger - \hat{a}\big).$$

We now examine the action of the ladder operators by considering their commutators with the Hamiltonian. To understand how the operators affect the energy states, let us compute the commutator with the annihilation operator $\hat{a}$:

$$[\hat{H}, \hat{a}] = \hbar\omega\,[\hat{a}^\dagger\hat{a}, \hat{a}].$$

Using the general identity $[AB, C] = A[B, C] + [A, C]B$, we obtain

$$[\hat{a}^\dagger\hat{a}, \hat{a}] = \hat{a}^\dagger[\hat{a}, \hat{a}] + [\hat{a}^\dagger, \hat{a}]\hat{a} = -\hat{a},$$

so that

$$[\hat{H}, \hat{a}] = -\hbar\omega\,\hat{a}.$$

Similarly, for the creation operator $\hat{a}^\dagger$, we have

$$[\hat{H}, \hat{a}^\dagger] = \hbar\omega\,[\hat{a}^\dagger\hat{a}, \hat{a}^\dagger] = \hbar\omega\,\hat{a}^\dagger.$$

From these commutators, one can immediately deduce the action of the Hamiltonian on the states $\hat{a}|n\rangle$ and $\hat{a}^\dagger|n\rangle$. Using $\hat{H}|n\rangle = \hbar\omega\left(n + \tfrac{1}{2}\right)|n\rangle$, we find

$$\hat{H}(\hat{a}|n\rangle) = (\hat{a}\hat{H} + [\hat{H}, \hat{a}])|n\rangle = \hbar\omega\left(n - \tfrac{1}{2}\right)(\hat{a}|n\rangle),$$

$$\hat{H}(\hat{a}^\dagger|n\rangle) = (\hat{a}^\dagger\hat{H} + [\hat{H}, \hat{a}^\dagger])|n\rangle = \hbar\omega\left(n + \tfrac{3}{2}\right)(\hat{a}^\dagger|n\rangle).$$

These results show explicitly that $\hat{a}$ lowers the energy of a state by one quantum $\hbar\omega$, while $\hat{a}^\dagger$ raises the energy by the same amount.

It is convenient to introduce the number operator, defined as

$$\hat{N} = \hat{a}^\dagger\hat{a}.$$

This operator counts the number of quanta in a given state, from its action on the energy eigenstates:

$$\hat{N}|n\rangle = n|n\rangle.$$

The ladder operators then have a simple interpretation in terms of the number operator: the annihilation operator $\hat{a}$ lowers the quantum number by one,

$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle,$$

while the creation operator $\hat{a}^\dagger$ raises it by one,

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle.$$

In this way, the states $|n\rangle$ can be constructed by successive application of $\hat{a}^\dagger$ starting from the vacuum state $|0\rangle$, and the number operator provides a direct measure of the excitation level of each state. These operators are the cornerstone for describing the spectrum and dynamics of the quantum harmonic oscillator.

We are now ready to quantize our *mechanical string* with Lagrangian:

$$L = m \sum_{\sigma=1}^{\frac{N}{2}-1} \left(|\dot{\tilde{y}}_\sigma|^2 + \omega_\sigma^2 |\tilde{y}_\sigma|^2\right) + \frac{m}{2}\left(\dot{\tilde{y}}^2_{\frac{N}{2}} + \dot{\tilde{y}}^2_N + \omega^2_{\frac{N}{2}} \tilde{y}^2_{\frac{N}{2}}\right),$$

since it is describing a system of decoupled harmonic oscillators, we can imagine to define the ladder operators for each Fourier mode independently, and describe the whole system as a set of excitations (particles) of different harmonic oscillators, thus phonons at different frequencies.

It is useful to stress that each complex mode can be decomposed into its real and imaginary parts. In particular,

$$\tilde{y}_\sigma = \frac{1}{\sqrt{2}}(\operatorname{Re}\tilde{y}_\sigma + i\operatorname{Im}\tilde{y}_\sigma) = \frac{\tilde{y}_\sigma^{(\mathrm{R})} + i\tilde{y}_\sigma^{(\mathrm{I})}}{\sqrt{2}},$$

so that the kinetic term can be written as

$$|\dot{\tilde{y}}_\sigma|^2 = \frac{1}{2}\left[(\dot{\tilde{y}}_\sigma^{(\mathrm{R})})^2 + (\dot{\tilde{y}}_\sigma^{(\mathrm{I})})^2\right].$$

For the real normal modes we introduce the usual ladder operators $\hat{a}_r, \hat{a}_r^\dagger$ and have[11]

$$\hat{\tilde{y}}_\sigma^{(\mathrm{R})} = \sqrt{\frac{\hbar}{2m\omega_\sigma}}\,(\hat{a}_\sigma^{(\mathrm{R})} + \hat{a}_\sigma^{\dagger(\mathrm{R})}), \qquad \hat{p}_\sigma^{(\mathrm{R})} = i\sqrt{\frac{m\hbar\omega_\sigma}{2}}\,(\hat{a}_\sigma^{(\mathrm{R})} - \hat{a}_\sigma^{\dagger(\mathrm{R})}),$$

$$\hat{\tilde{y}}_\sigma^{(\mathrm{I})} = \sqrt{\frac{\hbar}{2m\omega_\sigma}}\,(\hat{a}_\sigma^{(\mathrm{I})} + \hat{a}_\sigma^{\dagger(\mathrm{I})}), \qquad \hat{p}_\sigma^{(\mathrm{I})} = i\sqrt{\frac{m\hbar\omega_\sigma}{2}}\,(\hat{a}_\sigma^{(\mathrm{I})} - \hat{a}_\sigma^{\dagger(\mathrm{I})}),$$

$$\hat{\tilde{y}}_{\frac{N}{2}} = \sqrt{\frac{\hbar}{2m\omega_{\frac{N}{2}}}}\,(\hat{a}_{\frac{N}{2}} + \hat{a}^\dagger_{\frac{N}{2}}), \qquad \hat{p}_{\frac{N}{2}} = i\sqrt{\frac{m\hbar\omega_{\frac{N}{2}}}{2}}\,(\hat{a}_{\frac{N}{2}} - \hat{a}^\dagger_{\frac{N}{2}}),$$

The natural framework to describe our quantized mechanical string is the **Fock space**, which allows us to account for an arbitrary number of excitations in each mode.

Formally, the Fock space is constructed as the direct sum of n-particle Hilbert spaces:

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)} = \mathcal{H}^{(0)} \oplus \mathcal{H}^{(1)} \oplus \mathcal{H}^{(2)} \oplus \ldots,$$

where $\mathcal{H}^{(0)} \cong \mathbb{C}$ represents the vacuum (the state with no phonons), while for $n \geq 1$ we define

$$\mathcal{H}^{(n)} = \underbrace{\mathcal{H}^{(1)} \otimes \mathcal{H}^{(1)} \otimes \cdots \otimes \mathcal{H}^{(1)}}_{n \text{ times}}.$$

Here, $\mathcal{H}^{(1)}$ is the Hilbert space of a single phonon, $\mathcal{H}^{(2)}$ is the space of two phonons, and in general $\mathcal{H}^{(n)}$ describes n phonons. Each $\mathcal{H}^{(n)}$ is a subspace where the total number of excitations is fixed, and the tensor product structure reflects the fact that each phonon can occupy its own independent state within the mode basis.

[11] We have worked with the complex degrees of freedom since the Fourier transform, now we are reconducing the results to the real degrees of freedom, which are the double.

The Fock space $\mathcal{F}$ as a whole thus contains all possible states with any number of phonons. This construction is crucial because in a quantum harmonic system the number of excitations is not fixed: the system can fluctuate between states with zero, one, or arbitrarily many phonons. Using the Fock space, we can systematically describe the action of creation and annihilation operators, build eigenstates of the total Hamiltonian, and keep track of the occupation numbers of all modes simultaneously.

- $\mathcal{H}^{(1)}$ is spanned by:

$$\hat{a}_\sigma^{\dagger(\mathrm{R})}|0\rangle\,, \quad \hat{a}_\sigma^{\dagger(\mathrm{I})}|0\rangle\,, \quad \hat{a}_{\frac{N}{2}}^{\dagger}|0\rangle\,, \quad \sigma = 1, \dots, \frac{N}{2} - 1,$$

  which we can denote more compactly as $\hat{a}_i^\dagger|0\rangle$.

- $\mathcal{H}^{(2)}$ is spanned by all two-phonon states, obtained by applying any two creation operators to the vacuum:

$$\hat{a}_i^\dagger \hat{a}_j^\dagger |0\rangle\,, \qquad i, j = 1, 2, \dots, N-1,$$

  including the possibility $i = j$ (two phonons in the same mode).[12]

- In general, $\mathcal{H}^{(n)}$ is spanned by all n-phonon states:

$$(\hat{a}_{i_1}^\dagger)^{n_1}(\hat{a}_{i_2}^\dagger)^{n_2}\dots(\hat{a}_{i_l}^\dagger)^{n_l}|0\rangle\,, \qquad \text{with } n_1 + n_2 + \dots + n_l = n,$$

  which include all possible distributions of n phonons among the modes. Each $\mathcal{H}^{(n)}$ is thus the subspace of the Fock space with exactly n excitations.

Thus, we can define the vacuum more rigorously as

$$|0\rangle = |0\rangle_{\omega_1}^{(\mathrm{R})} \otimes |0\rangle_{\omega_1}^{(\mathrm{I})} \otimes \dots \otimes |0\rangle_{\omega_{N/2-1}}^{(\mathrm{R})} \otimes |0\rangle_{\omega_{N/2-1}}^{(\mathrm{I})} \otimes |0\rangle_{\omega_{N/2}} = |0, 0, \dots, 0\rangle\,,$$

so that the action of the creation operators gives

$$\hat{a}_1^{\dagger(\mathrm{R})}|0\rangle = |1, 0, \dots, 0\rangle\,, \qquad \hat{a}_1^{\dagger(\mathrm{I})}|0\rangle = |0, 1, \dots, 0\rangle\,,$$

meaning that we now have a single phonon with frequency ω_1 in either the real or imaginary part of the mode.

We can now express an arbitrary **base element** of our Fock space as

$$\left| n_1^{(\mathrm{R})}, n_1^{(\mathrm{I})}, \dots, n_{N/2-1}^{(\mathrm{R})}, n_{N/2-1}^{(\mathrm{I})}, n_{N/2} \right\rangle,$$

that is, by explicitly specifying the number of phonons in each independent mode of the system. Equivalently, such a state can be written in terms of creation operators as

$$C \left(\hat{a}_1^{\dagger(\mathrm{R})}\right)^{n_1^{(\mathrm{R})}} \left(\hat{a}_1^{\dagger(\mathrm{I})}\right)^{n_1^{(\mathrm{I})}} \dots \left(\hat{a}_{N/2}^{\dagger}\right)^{n_{N/2}} |0\rangle\,,$$

where the constant C ensures the correct normalization of the state. The integers $n_i^{(\mathrm{R})}$ and $n_i^{(\mathrm{I})}$ represent the occupation numbers of the real and imaginary components of the mode with frequency ω_i, their sum gives the total number of phonons in the mode ω_i.

Therefore, a **generic Fock state** encodes the excitation content of the string: although the total number of phonons in the system is finite for each specific state, every mode can host an arbitrarily

[12]One should properly symmetrize the two-phonon states, for instance: $\frac{1}{\sqrt{2}}\left(\hat{a}_i^\dagger \hat{a}_j^\dagger|0\rangle \pm \hat{a}_j^\dagger \hat{a}_i^\dagger|0\rangle\right)$, to ensure the correct bosonic/fermionic symmetry.

large number of excitations, which makes the Fock space infinite-dimensional. It can be expressed as:

$$|\Psi\rangle = \sum_{n_1^{(\mathrm{R})}}^{\infty}\sum_{n_1^{(\mathrm{I})}}^{\infty}\cdots\sum_{n_{N/2}}^{\infty} C_{n_1^{(\mathrm{R})}n_1^{(\mathrm{I})}\ldots n_{N/2}} \left|n_1^{(\mathrm{R})},\, n_1^{(\mathrm{I})}, \ldots,\, n_{N/2}\right\rangle,$$

where $|C_{n_1^{(\mathrm{R})}n_1^{(\mathrm{I})}\ldots n_{N/2}}|^2$ gives the probability of finding the system in the specific configuration with $n_1^{(\mathrm{R})} + n_1^{(\mathrm{I})}$ phonons in the first mode and so on, up to $n_{N/2}$ phonons in the last mode. Clearly, this probabilistic description is valid only if

$$\|\Psi\|^2 = \langle\Psi|\Psi\rangle = \sum_{n_1^{(\mathrm{R})}}^{\infty}\sum_{n_1^{(\mathrm{I})}}^{\infty}\cdots\sum_{n_{N/2}}^{\infty} |C_{n_1^{(\mathrm{R})}n_1^{(\mathrm{I})}\ldots n_{N/2}}|^2 = 1.$$

Now that we have obtained the explicit form of the energy eigenstates, the last step is to express the Hamiltonian in terms of the ladder operators. In this way, the full quantum description of the system becomes transparent: each mode of the string behaves as an independent quantum harmonic oscillator, whose excitations correspond to phonons.

$$\hat{H} = \sum_{\sigma=1}^{\frac{N}{2}-1}\left[\omega_\sigma\left(\hat{a}_\sigma^{\dagger(\mathrm{R})}\hat{a}_\sigma^{(\mathrm{R})} + \hat{a}_\sigma^{\dagger(\mathrm{I})}\hat{a}_\sigma^{(\mathrm{I})} + 1\right)\right] + \omega_{\frac{N}{2}}\left(\hat{a}^\dagger_{\frac{N}{2}}\hat{a}_{\frac{N}{2}} + \frac{1}{2}\right),$$

The Hamiltonian can be rewritten in terms of the number operators, making it clear that the total energy is simply the sum of the contributions of all the phonons. This parallels the case of free particles, where the energy is the sum of the individual excitations:

$$\hat{H} = \sum_{\sigma=1}^{\frac{N}{2}-1}\left[\omega_\sigma\left(\hat{N}_\sigma + 1\right)\right] + \omega_{\frac{N}{2}}\left(\hat{N}_{\frac{N}{2}} + \frac{1}{2}\right).$$

An important feature that emerges is the presence of a non-vanishing vacuum energy: even in the absence of phonons, the ground state carries a finite amount of energy due to the zero-point motion of each mode:

$$E_0 = \sum_{\sigma=1}^{\frac{N}{2}-1}\omega_\sigma + \frac{\omega_{\frac{N}{2}}}{2}.$$

This vacuum energy can become very large when summing over all modes, but in a theory without gravity it can be consistently shifted to zero, since only energy differences are physically relevant.

1.3.2 | Continuum Limit

We now consider the continuum limit of our discrete system. This is achieved by letting the natural spacing between points on the string, denoted by Δ, tend to zero, while keeping the total length L of the string fixed. To enforce this constraint, we impose the relation $L = N\Delta$ and take the limits $N \to \infty$ and $\Delta \to 0$ simultaneously.

In this limit, the discrete indices j and σ transition into continuous variables. As a result, quantities that were previously discrete—such as the wavelength λ_σ and the wave number κ_σ—become continuous:

$$\kappa_\sigma = \frac{2\pi\sigma}{N\Delta} = \frac{2\pi\sigma}{L}, \quad \kappa_\sigma \in [0,\, \infty),$$
$$\lambda_\sigma = \frac{N\Delta}{\sigma} = \frac{L}{\sigma}, \quad \lambda_\sigma \in [0,\, \infty).$$

As $N \to \infty$ and $\Delta \to 0$, both κ_σ and λ_σ sweep out continuous ranges: $\kappa_\sigma,\, \lambda_\sigma \in [0,\, \infty)$.

Simultaneously, the discrete spatial coordinate becomes a continuous variable. By defining $x = j\Delta$, the displacement variable y_j becomes a smooth field $\psi(x, t)$. Thus, the original system of N coupled harmonic oscillators is reformulated as an infinite set of decoupled harmonic oscillators, each labeled by a continuous wave number κ:

$$\omega_\sigma = 2\frac{v}{\Delta}\sin\left(\frac{\pi\sigma}{N}\right) \xrightarrow[N\to\infty]{} \frac{v}{\Delta}\cdot\frac{2\pi\sigma}{N} = v\kappa_\sigma$$

In this limit, all wave modes propagate with the same velocity:

$$v_\sigma = \frac{\omega_\sigma}{\kappa_\sigma} = 2\frac{v}{\Delta}\sin\left(\frac{\pi\sigma}{N}\right)\cdot\frac{N\Delta}{2\pi\sigma} \xrightarrow[N\to\infty]{} v$$

This result reflects a key feature of the continuum limit: **the dispersion relation becomes linear**, and all modes—regardless of their wavelength—travel at the same speed v. This uniform propagation speed is characteristic of non-dispersive media and simplifies the dynamics considerably.

We are thus left with an infinite continuum of independent harmonic oscillators, each labeled by a continuous wave number κ and oscillating with frequency $\omega_\kappa = v\kappa$. Physically, each oscillator corresponds to a traveling wave mode on the string, and the entire system can be viewed as a superposition of such modes, each propagating at speed v.

Upon quantization, these classical vibrational modes become *phonons*—the quantized excitations of the elastic medium. Each phonon carries energy and momentum, and the linear dispersion relation implies that their energy is directly proportional to their momentum.

If we now identify the propagation speed v with the speed of light c, the dispersion relation takes the form:

$$\omega_\kappa = c\kappa \iff \hbar\omega_\kappa = c\hbar\kappa \implies E = pc,$$

which is precisely the relativistic energy-momentum relation for a massless particle. This correspondence highlights a deep connection between the quantized excitations of a one-dimensional elastic medium and the behavior of relativistic particles in field theory. In this sense, the phonon becomes a prototype for more general quantum field excitations, including photons and other massless bosons.

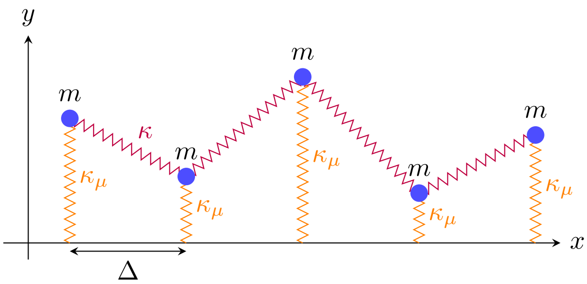
Particle Interpretation

The field $\psi(x, t)$ can be interpreted as describing a massless particle in the discrete chain. Upon quantization, excitations at frequency ω_κ of this field — the *phonons* — behave as particles of definite momentum $\hbar\kappa$ and energy $\hbar\omega_\kappa$. In this picture, particles of definite momentum emerge as quanta of excitation of the underlying field, created or annihilated by the corresponding creation and annihilation operators. These operators allow us to add or remove a particle of definite momentum from the system state.

Adding a mass term. In order to describe a massive particle, we need to introduce a term that tends to restore the system to its equilibrium position. Physically, this can be imagined as an additional spring attached to each atom in the chain, pulling it towards its rest position. This inertial interaction gives rise to a mass term in the continuum limit. Indeed, for a relativistic particle, the energy-momentum relation reads

$$E^2 = p^2c^2 + m^2c^4,$$

and the additional spring introduces the κ_μ term responsible for the m^2 contribution.



The additional elastic constant κ_μ that tends to restore each particle to its equilibrium position is directly related to the inertial mass of the corresponding particle field.
For the discrete chain, the Lagrangian can be written as:

$$L = \sum_{j=1}^{N} \left[\frac{1}{2} m \dot{y}_j^2 - \frac{1}{2} m v^2 \left(\frac{y_j - y_{j+1}}{\Delta} \right)^2 - \frac{1}{2} \kappa_\mu y_j^2 \right].$$

Passing to the continuum limit, the discrete sums and differences are replaced by integrals and derivatives:

$$\begin{aligned}
\sum_{j=1}^{N} &\longrightarrow \int \mathrm{d}x, \\
y_j^2 &\longrightarrow \psi(x,t)^2, \\
\dot{y}_j^2 &\longrightarrow \left(\frac{\partial \psi(x,t)}{\partial t} \right)^2, \\
\left(\frac{y_j - y_{j+1}}{\Delta} \right)^2 &\longrightarrow \left(\frac{\partial \psi(x,t)}{\partial x} \right)^2.
\end{aligned}$$

Thus, the continuum Lagrangian (with $v = c = 1$) reads:

$$L = \frac{1}{2} m \int \mathrm{d}x \left[\left(\frac{\partial \psi(x,t)}{\partial t} \right)^2 - \left(\frac{\partial \psi(x,t)}{\partial x} \right)^2 - \frac{\kappa_\mu}{m} \psi(x,t)^2 \right],$$

where the last term plays the role of a mass term, proportional to m^2 in natural units. This formulation makes clear how a restoring interaction at the discrete level leads, in the continuum limit, to a massive field with excitations corresponding to massive particles.

Let us now introduce the **Lagrangian density** $\mathcal{L}$, which allows us to describe the system in the continuum and relativistic framework. The total Lagrangian is obtained by integrating the Lagrangian density over all space:

$$L = \int_{-\infty}^{\infty} \mathrm{d}x \, \mathcal{L}, \quad \text{so that the action reads} \quad S = \int \mathrm{d}t \, L = \int \mathrm{d}t \, \mathrm{d}x \, \mathcal{L}.$$

Here we integrate along the entire real axis, reflecting the fact that we consider the chain to be infinitely long in the continuum limit.

For the discrete Lagrangian derived previously, the corresponding density is:

$$\mathcal{L} = \frac{m}{2} \left[(\partial_t \psi)^2 - (\partial_x \psi)^2 \right] - \frac{\kappa_\mu}{2} \psi^2.$$

Minkowski metric and derivatives To make contact with special relativity, we need to account for the Minkowski metric, using the *mostly-minus* convention to ensure the correct sign of the kinetic term:

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \eta^{\mu\nu}, \quad \mu, \nu = 0, 1, 2, 3.$$

In our 1-dimensional spatial model, the derivatives transform as:

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \eta_{\mu\nu}\partial^\nu = \eta_{\mu\nu}\frac{\partial}{\partial x_\nu}, \quad \text{so that} \quad \partial_0 = \frac{\partial}{\partial t} = \partial^0, \quad \partial_1 = \frac{\partial}{\partial x} = -\partial^1.$$

With this convention, the Minkowski contraction of derivatives reads:

$$\partial_\mu\psi\,\partial^\mu\psi = \eta^{\mu\nu}\partial_\mu\psi\,\partial_\nu\psi = (\partial_0\psi)^2 - (\partial_1\psi)^2 = (\partial_t\psi)^2 - (\partial_x\psi)^2.$$

Relativistic form of the Lagrangian density Finally, we can express the Lagrangian density in a manifestly Lorentz-covariant form:

$$\mathcal{L} = \frac{m}{2}\,\partial_\mu\psi\,\partial^\mu\psi - \frac{\kappa_\mu}{2}\psi^2,$$

where the second term plays the role of a mass term, endowing the field ψ with an effective rest energy proportional to m^2. This compact form makes the relativistic structure of the theory explicit and allows for a straightforward generalization to higher-dimensional field theories. Moreover, it provides a natural route to the corresponding equations of motion through the Euler–Lagrange formalism in relativistic notation.

Let us now derive the field equations explicitly. The Euler–Lagrange equations for a continuous field $\psi(x,t)$ read:

$$\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial(\partial_t\psi)} = \frac{\partial\mathcal{L}}{\partial\psi} - \frac{\partial}{\partial x}\frac{\partial\mathcal{L}}{\partial(\partial_x\psi)}.$$

Here, the spatial derivative term arises from the elastic contribution associated with the stiffness constant κ_μ. Computing these derivatives gives:

$$m\,\frac{\partial^2\psi(x,t)}{\partial t^2} = m\,\frac{\partial^2\psi(x,t)}{\partial x^2} - \kappa_\mu\psi(x,t),$$

which can be rearranged as

$$\partial_t^2\psi - \partial_x^2\psi = -\frac{\kappa_\mu}{m}\,\psi.$$

Introducing the *D'Alembert operator* (or box operator)

$$\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu = \partial_t^2 - \partial_x^2,$$

the field equation takes the compact relativistic form

$$\left(\Box + \frac{\kappa_\mu}{m}\right)\psi(x,t) = 0.$$

To analyze the dynamics more conveniently, we perform a Fourier transform that diagonalizes the spatial dependence:

$$\psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\kappa\, e^{i\kappa x}\,\tilde{\psi}(\kappa,t),$$

which represents the continuum limit of the discrete Fourier series

$$y_j(t) = \frac{1}{\sqrt{N}}\sum_{\sigma=1}^{N} e^{i\kappa_\sigma(\Delta j)}\tilde{y}_\sigma(t), \qquad \kappa_\sigma = \frac{2\pi\sigma}{\Delta N}, \qquad (\Delta j) = x.$$

As in the discrete case, we require the field $\psi(x,t)$ to be real, implying that

$$\tilde{\psi}(\kappa,t) = \tilde{\psi}^*(-\kappa,t) \implies \tilde{\psi}^*(\kappa,t) = \tilde{\psi}(-\kappa,t).$$

Differentiating under the integral sign, we obtain

$$\partial_t^2\psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\kappa\, e^{i\kappa x}\,\ddot{\tilde{\psi}}(\kappa,t),$$
$$\partial_x^2\psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} d\kappa\,(-\kappa^2)\,e^{i\kappa x}\,\tilde{\psi}(\kappa,t),$$

which leads to the following evolution equation for each Fourier mode:

$$\ddot{\tilde{\psi}}(\kappa,t) = -\left(\kappa^2 + \frac{\kappa_\mu}{m}\right)\tilde{\psi}(\kappa,t).$$

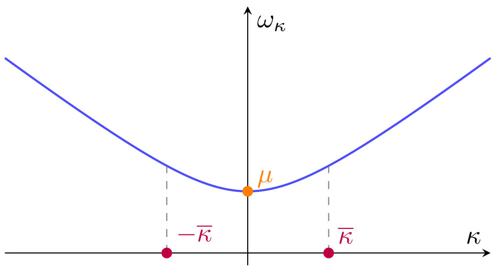
This is the equation of motion of a harmonic oscillator with frequency

$$\omega_\kappa = \sqrt{\kappa^2 + \frac{\kappa_\mu}{m}} = \sqrt{\kappa^2+\mu^2},$$

so that in momentum space we have

$$\ddot{\tilde{\psi}}(\kappa,t) + \omega_\kappa^2\tilde{\psi}(\kappa,t) = 0.$$

$$\omega_\kappa = \sqrt{\kappa^2+\mu^2} \text{ with } \mu = \sqrt{\tfrac{\kappa_\mu}{m}}$$



The field $\psi(x,t)$ is real because its Fourier transform satisfies $\tilde{\psi}(\kappa,t) = \tilde{\psi}^*(\kappa,t) = \tilde{\psi}(-\kappa,t)$, a direct consequence of the symmetry of ω_κ under $\kappa \to -\kappa$.
The parameter κ_μ sets the particle mass μ, while m may be absorbed in a redefinition of the field or set to unity in natural units.

We have thus described a system equivalent to an infinite collection of *decoupled harmonic oscillators*, each characterized by its own wavenumber κ and frequency ω_κ:

$$\omega_\kappa^2 = \kappa^2 + \frac{\kappa_\mu}{m},$$
$$\hbar^2\omega_\kappa^2 = \hbar^2\kappa^2 + \hbar^2\frac{\kappa_\mu}{m},$$
$$E^2 = p^2 + \mu^2.$$

Hence, the quanta of excitation of the field obey the relativistic *energy–momentum relation*, identifying μ as the particle's rest mass, determined by

$$\kappa_\mu = \frac{\mu^2 m}{\hbar^2}.$$

Returning to position space, we recover the celebrated **Klein–Gordon equation** in natural units:

$$\left(\Box + \mu^2\right)\psi(x,t) = 0,$$

which describes the classical dynamics of a scalar (spin-0) field of mass μ. Upon quantization, each mode $\tilde{\psi}(\kappa,t)$ corresponds to a relativistic particle satisfying the previously derived dispersion relation.

The associated Lagrangian densities can now be written both in configuration and in momentum space:

$$\mathcal{L}(x,t) = \frac{1}{2}\partial_\mu \psi(x,t)\,\partial^\mu \psi(x,t) - \frac{1}{2}\mu^2\psi^2, \qquad L = \int dx\,\mathcal{L},$$

and

$$\mathcal{L}(\kappa,t) = \frac{1}{2}\left(\dot{\tilde{\psi}}^2(\kappa,t) - \omega_\kappa^2\tilde{\psi}^2(\kappa,t)\right), \qquad L = \int d\kappa\,\mathcal{L}.$$

Remark. *This correspondence between a relativistic scalar field and a continuous set of harmonic oscillators is the cornerstone of quantum field theory: quantizing each mode $\tilde{\psi}(\kappa,t)$ gives rise to particle excitations, while Lorentz invariance ensures that all inertial observers describe the same energy–momentum relation.*

Quantization

In order to quantize the field, we promote each of the infinitely many decoupled harmonic oscillators to operators acting on a *Fock space*. This space can be written as the direct sum of Hilbert spaces corresponding to states with a definite number of particles:

$$\mathcal{F} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots,$$

where $\mathcal{H}_n$ denotes the Hilbert space of n-particle states.

In this framework, the mode amplitudes become operators satisfying canonical commutation relations, and can be expressed as

$$\hat{\tilde{\psi}}_\kappa = \frac{1}{\sqrt{2\omega_\kappa}}\left(\hat{a}_\kappa + \hat{a}_\kappa^\dagger\right), \qquad \hat{\tilde{\pi}}_\kappa = -i\sqrt{\frac{\omega_\kappa}{2}}\left(\hat{a}_\kappa - \hat{a}_\kappa^\dagger\right),$$

where $\hat{a}_\kappa$ and $\hat{a}_\kappa^\dagger$ are, respectively, the annihilation and creation operators for the κ-labeled mode.

A generic basis element of the Fock space can then be written as

$$(\hat{a}_{\kappa_1}^\dagger)^{n_1}(\hat{a}_{\kappa_2}^\dagger)^{n_2}\ldots(\hat{a}_{\kappa_l}^\dagger)^{n_l}\,|0\rangle, \qquad \text{with } n_1 + n_2 + \cdots + n_l = n,$$

which represents a state with n_1 particles of momentum κ_1, n_2 particles of momentum κ_2, and so on. Each state has a definite energy given by

$$E = \sum_{i=1}^{l} n_i\,\omega_{\kappa_i} + \frac{1}{2}\int \mathrm{d}\kappa\;\omega_\kappa.$$

The first term represents the sum of the energies of all excitations, while the second term is the **vacuum energy**, i.e. the sum of the zero-point energies $\frac{1}{2}\omega_\kappa$ of all the modes.

To compute the probability of finding the system in a particular Fock state with occupation numbers $(n_1,\ldots,n_l)$, one takes the modulus squared of the projection of the wave function onto that state:

$$\left|\langle\Psi|\,(\hat{a}_{\kappa_1}^\dagger)^{n_1}(\hat{a}_{\kappa_2}^\dagger)^{n_2}\ldots(\hat{a}_{\kappa_l}^\dagger)^{n_l}\,|0\rangle\right|^2.$$

The number operator $\hat{N}_\kappa = \hat{a}_\kappa^\dagger\hat{a}_\kappa$ acts on these states as

$$\hat{N}_{\kappa_i}\Big[(\hat{a}_{\kappa_1}^\dagger)^{n_1}(\hat{a}_{\kappa_2}^\dagger)^{n_2}\ldots(\hat{a}_{\kappa_l}^\dagger)^{n_l}\,|0\rangle\Big] = n_i\Big[(\hat{a}_{\kappa_1}^\dagger)^{n_1}(\hat{a}_{\kappa_2}^\dagger)^{n_2}\ldots(\hat{a}_{\kappa_l}^\dagger)^{n_l}\,|0\rangle\Big],$$

thus counting the number of excitations in the mode of frequency ω_{κ_i}.

The Hamiltonian of the quantized field can then be written as

$$\hat{H} = \int \mathrm{d}\kappa\ \omega_\kappa \left(\hat{a}^\dagger_\kappa \hat{a}_\kappa + \frac{1}{2} \right) = \int \mathrm{d}\kappa\ \omega_\kappa \left(\hat{N}_\kappa + \frac{1}{2} \right),$$

showing that the total energy is the sum of the energies of N independent, non-interacting excitations.

However, the vacuum energy term diverges:

$$E_0 = \frac{1}{2} \int \mathrm{d}\kappa\ \omega_\kappa = \frac{1}{2} \int \mathrm{d}\kappa\ \sqrt{\kappa^2 + \mu^2}.$$

This divergence is of ultraviolet nature, since it originates from the contribution of arbitrarily high-frequency modes (the continuum limit $\Delta \to 0$). In practice, this means that the theory cannot be valid at all length scales, and that our idealization breaks down beyond a certain energy cutoff.

In the absence of gravity — which would couple directly to the absolute energy density of the vacuum — this infinite constant can be safely neglected, as only energy *differences* have physical meaning in non-gravitational systems. We can therefore redefine the energy scale by setting $E_0 = 0$.

If we introduce a finite cutoff, e.g. imposing $\kappa\Delta \leq 1$ for a small but finite Δ, the vacuum energy becomes finite, representing the physically meaningful contribution of modes below that cutoff.

> **Remark.** *Different Fock states can correspond to the same total energy, since what matters is the combination of occupation numbers and mode frequencies. In this free-field framework, interactions are absent, so transitions between different states cannot occur. Nevertheless, the formalism provides a powerful description of systems with variable particle number: it allows us to represent, for instance, both the initial and final states of a decay or scattering process, even though their microscopic interaction dynamics lie beyond the scope of the free theory. Including interactions would require additional, non-linear terms in the Lagrangian density — corresponding to higher-order terms in its Taylor expansion around the equilibrium (vacuum) configuration.*

2 Spacetime Symmetries

Symmetry principles play a central role in modern theoretical physics, providing the mathematical foundation for conservation laws and the formulation of fundamental interactions. Group theory offers the natural language to describe these symmetries, both discrete and continuous. In particular, Lie groups and their corresponding algebras form the backbone of relativistic field theories. This chapter reviews the essential concepts of group theory and Lie algebras, culminating in the Lorentz and Poincaré groups, the fundamental symmetry group underlying the structure of spacetime in special relativity, and their representations.

2.1 | Definition of Group

A *group* is a set G equipped with a binary operation, denoted by $\circ$, satisfying four fundamental properties:

1. **Closure:** For any $a, b \in G$, the product $a \circ b \in G$.
2. **Associativity:** For any $a, b, c \in G$, one has $(a \circ b) \circ c = a \circ (b \circ c)$.
3. **Identity element:** There exists an element $e \in G$ such that $e \circ a = a \circ e = a$ for all $a \in G$.
4. **Inverse element:** For each $a \in G$, there exists $a^{-1} \in G$ such that $a \circ a^{-1} = a^{-1} \circ a = e$.

If the operation $\circ$ is commutative, i.e. $a \circ b = b \circ a$ for all $a, b \in G$, the group is called *Abelian.*

Typical important groups in physics can be classified as follows:

- **Abelian groups:** groups with commutative operations, often associated with conserved quantities via Noether's theorem.
  - $(\mathbb{R}, +)$: additive group of real numbers, e.g., translations in one dimension.
  - $(\mathbb{C}^\times, \cdot)$: multiplicative group of nonzero complex numbers.
  - $\mathrm{U}(1)$: phase rotations, fundamental in electromagnetism and quantum mechanics.
- **Rotation groups:** describe symmetries under rotations in space.
  - $\mathrm{SO}(2)$: rotations in a plane.
  - $\mathrm{SO}(3)$: rotations in three-dimensional space, relevant for angular momentum.
  - $\mathrm{SU}(2) \times \mathrm{SU}(2)$: cover of $\mathrm{SO}(3)$, used for spin-$\frac{1}{2}$ particles.
- **Lorentz and Poincaré groups:** spacetime symmetries in relativity.
  - $\mathrm{O}(1, 3)$: Lorentz transformations preserving the Minkowski metric.
  - $\mathrm{ISO}(1, 3)$ or Poincaré group: Lorentz transformations plus translations, fundamental in field theory.
- **Internal symmetry groups:** act on internal degrees of freedom of fields.
  - $\mathrm{SU}(3)$: color symmetry in quantum chromodynamics.
  - $\mathrm{SU}(2) \times \mathrm{U}(1)$: electroweak symmetry in the Standard Model.
- **Discrete groups:** describe symmetries with a finite number of elements.
  - Permutation groups S_n, relevant for identical particles and statistics.
  - Point groups in crystallography and molecular physics.

In physics, continuous groups — those depending smoothly on continuous parameters — play a central role, as they describe symmetries of space, time, and dynamical systems. These are known as *Lie groups.*

2.2 | Lie Groups and Lie Algebras

A **Lie group**, a group which is also a smooth differentiable manifold, has to be continuous, i.e. labelled by continuous indices, such that the group operations (multiplication and inversion) are smooth maps. Lie groups combine the structure of a continuous symmetry with the differentiability properties of manifolds, enabling the use of calculus to study symmetry transformations.

Each Lie group G is associated with a corresponding **Lie algebra** $\mathfrak{g}$, which captures its local (infinitesimal) structure. Formally, $\mathfrak{g}$ is defined as *the tangent space to the identity element* $e \in G$, endowed with a bilinear antisymmetric operation — the *Lie bracket*:

$$[\,\cdot\,,\,\cdot\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},$$

which satisfies the Jacobi identity:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0, \qquad \forall X, Y, Z \in \mathfrak{g}.$$

In a neighborhood of the identity, an element of the group is mapped by an exponential map from an element of the Lie algebra:

$$g(\epsilon) = e^{i\epsilon^a T_a}, \tag{2.2.1}$$

where $\{T_a\}$ are the **generators** of the Lie group, and ϵ^a are real infinitesimal parameters labeling the transformation. The commutation relations among generators define the algebraic structure:

$$[T_a, T_b] = i f_{ab}{}^c T_c, \tag{2.2.2}$$

where the real coefficients $f_{ab}{}^c$ are the **structure constants** of the algebra.

The dimension of a Lie algebra equals the number of independent generators — that is, the number of continuous parameters of the corresponding Lie group.

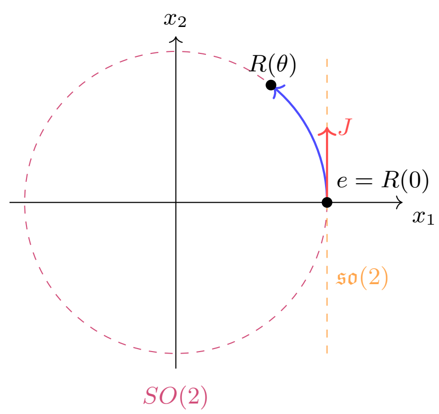
Example ($SO(2)$ and $\mathfrak{so}(2)$). The group $SO(2)$ represents planar rotations:

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad \theta \in [0, 2\pi).$$

Its Lie algebra $\mathfrak{so}(2)$ is the tangent space at the identity, spanned by the generator $J \equiv T_1$:

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad R(\theta) = e^{\theta J}.$$

The exponential map $\exp : \mathfrak{so}(2) \to SO(2)$ thus identifies each tangent vector θJ with a rotation of angle θ on the unit circle.



Common Lie groups:

- $\mathbf{O}(\mathbf{n}) = \{R \in \mathbb{R}^{n\times n} \mid R^T R = I\}$: orthogonal group, preserves Euclidean lengths.
- $\mathbf{SO}(\mathbf{n}) = \{R \in O(n) \mid \det R = 1\}$: special orthogonal group, proper rotations in n dimensions.

- $\mathbf{U(n)} = \{U \in \mathbb{C}^{n\times n} \mid U^\dagger U = I\}$: unitary group, preserves complex inner products.
- $\mathbf{SU(n)} = \{U \in U(n) \mid \det U = 1\}$: special unitary group, important for internal symmetries in particle physics.

2.2.1 | Representations

A **representation** of a Lie group or Lie algebra provides a concrete realization of its abstract elements as linear operators acting on a vector space. Mathematically, a representation of a Lie group G is a homomorphism, for example

$$D : G \to \mathrm{GL}(V),$$

where V is the vector space on which our representation is acting and $\mathrm{GL}(V)$ denotes the group of invertible linear transformations on V. This means that the group law is preserved:

$$D(g_1 g_2) = D(g_1)D(g_2), \qquad \forall g_1, g_2 \in G.$$

At the infinitesimal level, a representation of the associated Lie algebra $\mathfrak{g}$ assigns to each generator T_a a linear operator $D(T_a)$ such that

$$[D(T_a), D(T_b)] = i f_{ab}{}^c D(T_c),$$

mirroring the structure constants $f_{ab}{}^c$ of the algebra.

Representations are fundamental both in mathematics and in physics. Mathematically, they allow us to study abstract symmetry groups through their action on vector spaces, revealing their structure via matrices or operators. Physically, they describe how different objects transform under a given symmetry: scalars correspond to the trivial (one-dimensional) representation, vectors to the fundamental representation, and spinors or tensors to higher-dimensional or mixed representations.

Of particular importance are the **irreducible representations**, which cannot be decomposed into smaller invariant subspaces. These play the role of the "elementary building blocks" of a group—just as elementary particles of definite spin are the basic excitations transforming irreducibly under spacetime symmetries.

Different Lie groups admit representations of various **dimensions**, depending on how the group acts on the underlying vector space. For example, the group $SO(2)$ can be represented by 2×2 rotation matrices acting on vectors in the plane, but also by complex phase factors $e^{in\theta}$ acting on one-dimensional complex spaces, each labeled by an integer n. In general, higher-dimensional representations describe systems that transform with more internal components—such as vectors, tensors, or spinors—while lower-dimensional ones correspond to simpler transformation laws. The dimension of the representation thus reflects the number of independent degrees of freedom that transform under the symmetry.

2.3 | The Lorentz Group

The most fundamental symmetry group in relativistic field theory is the **Lorentz group**, denoted by $\mathrm{O}(1,3)$. It consists of all linear transformations $\Lambda : \mathbb{R}^{1,3} \to \mathbb{R}^{1,3}$ that leave the Minkowski metric invariant:

$$\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1),$$

i.e.

$$\eta_{\mu\nu}\,\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} = \eta_{\rho\sigma}.$$

We can write the Lorentz group as

$$\mathrm{O}(1,3) = \left\{\Lambda \in \mathrm{GL}(4,\mathbb{R}) \mid \eta_{\mu\nu}\,\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} = \eta_{\rho\sigma}\right\}. \tag{2.3.1}$$

These transformations preserve the spacetime interval

$$s^2 = \eta_{\mu\nu}x^{\mu}x^{\nu} = (x^0)^2 - \mathbf{x}^2,$$

or, infinitesimally,

$$\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^{\mu}\mathrm{d}x^{\nu}.$$

Indeed, under a Lorentz transformation $x^{\mu} \mapsto x'^{\mu} = \Lambda^{\mu}{}_{\rho}x^{\rho}$, we have

$$\begin{aligned}\mathrm{d}s'^2 &= \eta_{\mu\nu}\,\mathrm{d}x'^{\mu}\mathrm{d}x'^{\nu} = \eta_{\mu\nu}\,\Lambda^{\mu}{}_{\rho}\mathrm{d}x^{\rho}\,\Lambda^{\nu}{}_{\sigma}\mathrm{d}x^{\sigma}\\ &= \Lambda^{\mu}{}_{\rho}\,\eta_{\mu\nu}\,\Lambda^{\nu}{}_{\sigma}\,\mathrm{d}x^{\rho}\mathrm{d}x^{\sigma} = \eta_{\rho\sigma}\,\mathrm{d}x^{\rho}\mathrm{d}x^{\sigma} = \mathrm{d}s^2,\end{aligned}$$

confirming that Lorentz transformations leave the Minkowski interval invariant.

The determinant of a Lorentz transformation can be either $+1$ or -1, and $\Lambda^{0}{}_{0}$ can be greater or smaller than 1, corresponding to whether the transformation preserves or reverses spatial orientation and the direction of time: thus, the Lorentz group has four disconnected components. The subgroup with $\det\Lambda = +1$ and $\Lambda^{0}{}_{0} \geq 1$ is called the **proper orthochronous Lorentz group**, denoted by $\mathrm{SO}^{+}(1,3)$: it includes transformations that preserve both spatial orientation and the direction of time and is connected to the identity (i.e. any element can be continuously deformed to the identity transformation).

2.3.1 | Infinitesimal Transformations and Generators

An infinitesimal Lorentz transformation can be written as

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}, \qquad |\omega^{\mu}{}_{\nu}| \ll 1,$$

with the condition

$$\eta_{\mu\nu}\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} = \eta_{\rho\sigma} \quad \Rightarrow \quad \omega_{\mu\nu} = -\omega_{\nu\mu}.$$

Hence, $\omega_{\mu\nu}$ is an antisymmetric tensor containing six independent parameters, corresponding to the six generators of the Lorentz algebra.

We can introduce a set of generators $M_{\mu\nu}$ (antisymmetric in their indices) such that a general infinitesimal transformation acts on a four-vector x^{ρ} as:

$$x'^{\rho} = \left(\delta^{\rho}{}_{\sigma} + \frac{i}{2}\,\omega^{\mu\nu}(M_{\mu\nu})^{\rho}{}_{\sigma}\right)x^{\sigma}, \tag{2.3.2}$$

as taylor expansion of

$$\Lambda^\rho{}_\sigma = e^{\frac{i}{2}\,\omega^{\mu\nu}(M_{\mu\nu})^\rho{}_\sigma},$$

where the generators of the group are

$$(M_{\mu\nu})^\rho{}_\sigma = -i\left(\eta_{\mu\sigma}\,\delta^\rho{}_\nu - \eta_{\nu\sigma}\,\delta^\rho{}_\mu\right), \tag{2.3.3}$$

satisfying the antisymmetry property $M_{\mu\nu} = -M_{\nu\mu}$.

The commutation relations among the generators define the **Lorentz algebra** $\mathfrak{so}(1,3)$:

$$[M_{\mu\nu}, M_{\rho\sigma}] = i\left(\eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} + \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\sigma}M_{\mu\rho}\right). \tag{2.3.4}$$

2.3.2 | Generators: Rotations and Boosts

The six generators can be naturally separated into:

$$\begin{cases} J_i &= \dfrac{1}{2}\epsilon_{ijk}M_{jk} \quad \text{(spatial rotations)}, \\ K_i &= M_{0i} \quad \text{(Lorentz boosts)}. \end{cases} \tag{2.3.5}$$

In terms of these generators, the Lorentz algebra decomposes as:

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \qquad [J_i, K_j] = i\epsilon_{ijk}K_k, \qquad [K_i, K_j] = -i\epsilon_{ijk}J_k. \tag{2.3.6}$$

This structure reveals the non-compact nature of the Lorentz group: unlike spatial rotations, boosts do not form a compact subgroup, as rapidities are unbounded.[1] Furthermore, the commutation relations indicate that boosts and rotations do not commute, reflecting the intricate structure of spacetime symmetries: boosts in different directions generate rotations for example; but we will later see how to pass to a basis where the algebra decomposes into two commuting $\mathfrak{su}(2)$ algebras.

For finite transformations, one can write:

$$R(\boldsymbol{\theta}) = e^{-i\boldsymbol{\theta}\cdot\mathbf{J}}, \qquad B(\boldsymbol{\beta}) = e^{-i\boldsymbol{\beta}\cdot\mathbf{K}},$$

where $\boldsymbol{\theta} = (\theta_1,\, \theta_2,\, \theta_3)$ are the rotation angles and $\boldsymbol{\beta} = (\beta_1,\, \beta_2,\, \beta_3)$ the boost parameters. We can ultimately reconstruct the expression for $\omega_{\mu\nu}$:

$$\begin{cases} \omega_{ij} &= \epsilon_{ijk}\theta^k, \\ \omega_{0i} &= \beta_i. \end{cases} \tag{2.3.7}$$

Together, they parametrize any proper orthochronous Lorentz transformation as a combination of a boost and a rotation:

$$\Lambda = B(\boldsymbol{\beta})\,R(\boldsymbol{\theta}).$$

Thus we have six independent generators and six indipendent and continuous parameters to describe our Lie group, which topologically can be addressed to as $\mathbb{R}_3 \times S_3/\mathbb{Z}_2$, highlighting how the non compactness derive from the unbounded boosts subgroup.

[1] When we later discuss the relation between $\mathrm{SO}^+(1,3)$ and $\mathrm{SU}(2)\times\mathrm{SU}(2)$, this distinction will be important: $\mathrm{SU}(2)\times\mathrm{SU}(2)$ is compact, whereas the proper orthochronous Lorentz group is not.

2.3.3 | Representations

We are going to discuss now the representations of the Lorentz group, which are crucial for understanding how different physical fields transform under Lorentz transformations: we will see how scalars, spinors, and vectors arise naturally from these representations, and we will classify them as finite or infinite-dimensional. Let us start by introducing the concept of the double cover of the Lorentz group and some examples, then we will summarize the main representations used in QFT.

Double cover of Lorentz group. The Lorentz algebra $\mathfrak{so}(1,3)$ is locally isomorphic to the direct sum $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ (algebra of $SU(2)\times SU(2)$, equivalent up to different hermiticity relations arising because of compactess). This correspondence allows the classification of all finite-dimensional representations of the Lorentz group in terms of pairs (j_+, j_-) of SU(2) spins.

One can then construct the combinations

$$\mathbf{A} = \frac{1}{2}(\mathbf{J} + i\mathbf{K}), \qquad \mathbf{B} = \frac{1}{2}(\mathbf{J} - i\mathbf{K}), \tag{2.3.8}$$

which satisfy independent $\mathfrak{su}(2)$ algebras:

$$[A_i, A_j] = i\epsilon_{ijk}A_k, \quad [B_i, B_j] = i\epsilon_{ijk}B_k, \quad [A_i, B_j] = 0.$$

This shows explicitly the **algebra decomposition into a direct sum of two commuting algebras**. A concrete realization of the Lorentz algebra generators can be constructed using **spinor spaces**. Consider spin-$\frac{1}{2}$ representations, and let σ_i denote the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.3.9}$$

We can define the following operators:

$$\mathbf{A} = \frac{1}{2}\boldsymbol{\sigma} \otimes \mathbf{1}, \qquad \mathbf{B} = \frac{1}{2}\mathbf{1} \otimes \boldsymbol{\sigma},$$

which act on the four-dimensional space $\mathbb{C}^2 \otimes \mathbb{C}^2$, since each $\boldsymbol{\sigma}$ acts on a two-dimensional complex vector space where it respects its $\mathfrak{su}(2)$ algebra (the same algebra satisfied by the angular momentum operators in quantum mechanics).

In this language, the **left-handed** (LH) **spinors** transform under the $(\frac{1}{2}, 0)$ representation, corresponding to the action of $\boldsymbol{\sigma}$ (through $\mathbf{A}$) on the first factor, while the **right-handed** (RH) **spinors** transform under the $(0, \frac{1}{2})$ representation, corresponding to the action of $\boldsymbol{\sigma}$ (through $\mathbf{B}$) on the second factor.

The Dirac spinor arises as the direct sum $\psi = \psi_L \oplus \psi_R \in \mathbb{C}^2 \oplus \mathbb{C}^2$, and higher-dimensional representations of the Lorentz group can be constructed by taking tensor products of these spinor spaces, yielding (j_+, j_-) representations with arbitrary spins.

Trivial representation. In addition to the nontrivial representations, every Lie group admits a trivial representation, denoted by $(0, 0)$ in the $\mathrm{SU}(2)\times\mathrm{SU}(2)$ classification. In this representation, the group acts identically on the vector space: for any group element $g \in G$ and any vector v in the representation space,

$$g \cdot v = v.$$

The trivial representation is one-dimensional and corresponds to *scalars* under the symmetry: objects that remain invariant under all group transformations, such as the Higgs field under Lorentz transformations.

Although the underlying representation space $\mathbb{C}^2 \otimes \mathbb{C}^2$ is the same, different choices of the (j_+, j_-) representation allow us to classify distinct particles and fields according to their transformation properties under the Lorentz group, as shown in the following table.

Representation (j_+, j_-)	Field / Particle type	Total spin s
$(0,0)$	Scalar	0
$\left(\frac{1}{2},0\right)$	LH Weyl spinor	$\frac{1}{2}$
$\left(0,\frac{1}{2}\right)$	RH Weyl spinor	$\frac{1}{2}$
$\left(\frac{1}{2},0\right) \oplus \left(0,\frac{1}{2}\right)$	Dirac spinor	$\frac{1}{2}$
$\left(\frac{1}{2},\frac{1}{2}\right)$	Vector	1
$\left(1,\frac{1}{2}\right) \oplus \left(\frac{1}{2},1\right)$	Rarita–Schwinger field	$\frac{3}{2}$
$(1,1)$	Graviton	2

The representations listed above correspond to the basic field types used in relativistic quantum field theory. Scalars describe spinless particles such as the Higgs boson. Weyl spinors transform under chiral representations of the Lorentz group, and their direct sum forms the Dirac spinor, which accounts for massive spin-$\frac{1}{2}$ fermions like the electron. The vector representation corresponds to spin-1 gauge bosons, such as the photon or gluons. The Rarita–Schwinger field describes spin-$\frac{3}{2}$ particles, notably the gravitino, while the symmetric rank-2 tensor representation corresponds to the graviton, a hypothetical spin-2 quantum of the gravitational field.

This representation theory is the foundation for classifying relativistic fields and particles by their transformation properties under the Lorentz group.

2.3.4 | Finite-dimensional representations

Finite-dimensional representations of the Lorentz group play a central role in relativistic field theory, as they determine how different types of physical fields transform under Lorentz transformations. These representations encode the intrinsic angular momentum (or *spin*) of the fields and thus classify the fundamental particles in a relativistic framework.

The Lorentz group $\mathrm{SO}^+(1,3)$ is non-compact, which implies that all its non-trivial finite-dimensional representations are necessarily *non-unitary*. This is not a problem in classical or quantum field theory, since these representations act on field components rather than on the Hilbert space of physical states (where unitarity is instead required to preserve probabilities).

A key fact is that the proper orthochronous Lorentz group is locally isomorphic to the complex special linear group:

$$\mathrm{SO}^+(1,3) \simeq \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2.$$

Because $\mathrm{SL}(2,\mathbb{C})$ can be viewed as the complexification of $\mathrm{SU}(2)$, one can decompose its Lie algebra into two commuting copies of the $\mathfrak{su}(2)$ algebra:

$$\mathfrak{so}(1,3)_{\mathbb{C}} \simeq \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R.$$

Consequently, finite-dimensional irreducible representations of the Lorentz group are labeled by a pair of half-integers (j_L, j_R), which specify the dimensions of the two SU(2) factors:

$$\mathrm{SO}^+(1,3) \sim \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R.$$

Each field then transforms according to one such representation:

- Scalars correspond to $(0,0)$;
- Left-handed Weyl spinors to $(\frac{1}{2}, 0)$;
- Right-handed Weyl spinors to $(0, \frac{1}{2})$;
- Four-vectors to $(\frac{1}{2}, \frac{1}{2})$.

These representations can be combined to construct higher-spin fields and their tensor products. In this sense, finite-dimensional representations capture the internal (non-spacetime) transformation properties of fields and form the mathematical foundation for understanding spin in relativistic theories.

Trivial representation

The simplest Lorentz representation is the **trivial representation**, in which all generators vanish identically:

$$M^{\mu\nu} = 0 \quad \Longrightarrow \quad \Lambda = e^{-\frac{i}{2}\omega_{\mu\nu} M^{\mu\nu}} = \mathbb{I}.$$

It is a one-dimensional ($1D$) representation acting on scalar fields $\phi(x)$:

$$\phi(x) \;\longrightarrow\; \phi'(x') = \phi(x),$$

where the argument x transforms as usual, but the field value itself remains invariant. Such fields are therefore completely unchanged under Lorentz transformations—they possess no internal structure and no preferred direction in spacetime.

This representation is labeled by $(0,0)$ in the (j_L, j_R) classification. It corresponds to **spinless** particles, whose states are invariant under spatial rotations and boosts. Examples include the Higgs field in the Standard Model or the pion field in low-energy effective theories.

Physically, the trivial representation captures the idea of a *scalar quantity* that is the same for all inertial observers: the field value at a spacetime point does not depend on the reference frame, even though its argument x^μ does. Such fields are often the simplest starting point for constructing Lorentz-invariant Lagrangians.

Vectorial representation

The next non-trivial case is the **vectorial representation**, where the Lorentz generators act on four-vectors V^μ according to:

$$(M^{\rho\sigma})^{\mu}{}_{\nu} = -i\left(\eta^{\mu\sigma}\delta^{\rho}{}_{\nu} - \eta^{\mu\rho}\delta^{\sigma}{}_{\nu}\right). \tag{2.3.10}$$

These are 4×4 matrices generating infinitesimal transformations of the form:

$$V^\mu \;\longrightarrow\; V'^\mu = \Lambda^{\mu}{}_{\nu} V^\nu = \left(e^{-\frac{i}{2}\omega_{\rho\sigma}(M^{\rho\sigma})}\right)^{\mu}{}_{\nu} V^\nu.$$

This representation acts on four-vectors and mixes their components under rotations and boosts.

The vectorial representation corresponds to $(\frac{1}{2}, \frac{1}{2})$ in the (j_L, j_R) classification. It describes **spin–1 fields**, which carry one unit of intrinsic angular momentum. Such fields appear naturally in the description of gauge bosons (photon, gluons, and the weak vector bosons $W^\pm$ and Z^0), whose classical field components $A^\mu(x)$ transform as four-vectors.

Geometrically, this representation encodes how spacetime directions themselves transform under Lorentz transformations: the components of V^μ are projections of a geometric vector onto the observer's axes, and thus change when the observer is boosted or rotated. This is the fundamental representation used to describe tensors of rank one, from which higher-rank tensor representations can be built through tensor products.

Spinorial representation

The **spinorial representations** of the Lorentz group are more subtle and conceptually richer than the scalar or vector ones. This is because the proper orthochronous Lorentz group $\mathrm{SO}^+(1,3)$ is not simply connected: it contains closed paths that cannot be continuously deformed to the identity. As a result, certain representations—such as those describing spin–$\frac{1}{2}$ particles—cannot be defined consistently on $\mathrm{SO}^+(1,3)$ itself: after a 2π rotation, a spinor changes sign, and only after a full 4π rotation does it return to its original state. To obtain single-valued representations that can describe spin–$\frac{1}{2}$ particles, one must therefore pass to its universal covering group:

$$\mathrm{Spin}^+(1,3) \equiv \mathrm{SL}(2,\mathbb{C}), \qquad \mathrm{SO}^+(1,3) \simeq \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2.$$

This means that for each Lorentz transformation $\Lambda \in \mathrm{SO}^+(1,3)$ there exist two corresponding elements $\pm A \in \mathrm{SL}(2,\mathbb{C})$, related by a two-to-one homomorphism:

$$A,\, B \in \mathrm{SL}(2,\mathbb{C}) \quad \longrightarrow \quad \Lambda \in \mathrm{SO}^+(1,3), \qquad \Lambda(A)\Lambda(B) = \Lambda(AB).$$

The representations of $\mathrm{SL}(2,\mathbb{C})$ thus provide a natural way to construct the **half-integer spin representations** of the Lorentz group, which have no analogue in purely vectorial transformations.

To make this correspondence explicit, we introduce the hermitian matrices

$$\sigma^\mu = (\mathbb{I}, \boldsymbol{\sigma}), \quad \overline{\sigma}^\mu = (\mathbb{I}, -\boldsymbol{\sigma}), \tag{2.3.11}$$

where $\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ are the **Pauli matrices** in (2.3.9). A spacetime point x^μ can then be represented by a 2×2 Hermitian matrix:

$$X = x_\mu \sigma^\mu = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}.$$

Under the action of an element $N \in \mathrm{SL}(2,\mathbb{C})$, this object transforms as

$$X \longrightarrow X' = NXN^\dagger,$$

which preserves the Minkowski norm, since

$$\det X' = \det N\, \det X\, \det N^\dagger = \det X = x_\mu x^\mu = x_0^2 - |\mathbf{x}|^2.$$

Hence, N and $-N$ correspond to the same Lorentz transformation, illustrating the double-cover structure:

$$N = \pm \mathbb{I}_2 \quad \longrightarrow \quad \Lambda = \mathbb{I}_4.$$

This construction shows that spinors are mathematical objects transforming under the fundamental representation of $\mathrm{SL}(2,\mathbb{C})$, rather than under the vector representation of $\mathrm{SO}^+(1,3)$. Physically, this explains why a 2π rotation changes the sign of a spinor, while only a full 4π rotation brings it back to itself — a key feature distinguishing half-integer spin fields from bosonic ones.

Representations of $\mathrm{SL}(2,\mathbb{C})$**.** We can now define two inequivalent fundamental spinor representations, corresponding to the two possible chiralities of spin-$\frac{1}{2}$ particles. These are the building blocks of relativistic fermionic fields.

1. **Left-handed Weyl spinors** (fundamental representation $(\frac{1}{2},0)$): These are two-component complex spinors transforming under the fundamental representation of $\mathrm{SL}(2,\mathbb{C})$:
$$\psi_\alpha = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \longrightarrow \psi'_\alpha = N_\alpha{}^\beta \psi_\beta, \qquad N \in \mathrm{SL}(2,\mathbb{C}), \quad \alpha,\beta = 1,2. \tag{2.3.12}$$
Under spatial rotations they behave as left-handed objects, hence the name. They form the representation usually denoted by $\mathrm{SU}(2)_L$ when restricted to rotations.

2. **Right-handed Weyl spinors** (complex conjugate representation $(0,\frac{1}{2})$): Their complex conjugates transform under the conjugate representation:
$$\bar{\chi}^{\dot\alpha} = \begin{pmatrix} \bar{\chi}^{\dot 1} \\ \bar{\chi}^{\dot 2} \end{pmatrix} \longrightarrow \bar{\chi}'^{\dot\alpha} = N^{*\dot\alpha}{}_{\dot\beta} \bar{\chi}^{\dot\beta}, \qquad \dot\alpha, \dot\beta = 1,2. \tag{2.3.13}$$
These spinors transform as right-handed under spatial rotations and so thay form the representation $\mathrm{SU}(2)_R$.

Remark. *In the Lorentz group* $\mathrm{SO}^+(1,3)$*, indices are raised and lowered using the Minkowski metric* $\eta_{\mu\nu}$*. In contrast, within* $\mathrm{SL}(2,\mathbb{C})$ *spinor indices are manipulated using the antisymmetric invariant tensors* $\epsilon^{\alpha\beta}$ *and* $\epsilon^{\dot\alpha\dot\beta}$*:*
$$\epsilon^{\alpha\beta} = \epsilon^{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$
These tensors are invariant under $\mathrm{SL}(2,\mathbb{C})$ *transformations, as*
$$\epsilon'^{\,\alpha\beta} = N^T \epsilon N = N^\alpha_{\ \gamma} \epsilon^{\gamma\delta} N_\delta{}^\beta = \epsilon^{\alpha\beta} \det N = \epsilon^{\alpha\beta},$$
and they allow one to raise and lower spinor indices as follows:
$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta, \qquad \bar{\chi}^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar{\chi}_{\dot\beta}.$$

Generators. The infinitesimal generators of $\mathrm{SL}(2,\mathbb{C})$ correspond to those of the Lorentz algebra and are represented as[2]
$$\begin{cases} (\sigma^{\mu\nu})_\alpha{}^\beta = \dfrac{i}{4}(\sigma^\mu \bar\sigma^\nu - \sigma^\nu \bar\sigma^\mu)_\alpha{}^\beta, \\ (\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} = \dfrac{i}{4}(\bar\sigma^\mu \sigma^\nu - \bar\sigma^\nu \sigma^\mu)^{\dot\alpha}{}_{\dot\beta}. \end{cases}$$
Thus, under an infinitesimal Lorentz transformation parameterized by $\omega_{\mu\nu}$, the spinors transform as
$$\begin{aligned} \psi_\alpha &\longrightarrow e^{-\frac{i}{2}\omega_{\mu\nu}(\sigma^{\mu\nu})_\alpha{}^\beta}\psi_\beta, && \text{(Left-handed Weyl spinor)}, \\ \bar\chi^{\dot\alpha} &\longrightarrow e^{-\frac{i}{2}\omega_{\mu\nu}(\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}}\bar\chi^{\dot\beta}, && \text{(Right-handed Weyl spinor)}. \end{aligned}$$
These two representations are inequivalent but related by complex conjugation, and together they form the building blocks of a Dirac spinor.

[2]Where $\overline{\sigma}^\mu = (\mathbb{I}, -\boldsymbol{\sigma})$, as defined in eq. (2.3.11).

Dirac spinors. A four-component spinor obtained as the direct sum of a left-handed and a right-handed Weyl spinor is called a **Dirac spinor**:

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}.$$

This is a reducible representation of the Lorentz group, containing four complex components (or equivalently eight real components, hence eight degrees of freedom). Under a Lorentz transformation, it transforms as

$$\Psi \longrightarrow \Psi' = e^{-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}}\Psi,$$

where the generators in the *Weyl basis* are block-diagonal:

$$\Sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] = \begin{pmatrix} \sigma^{\mu\nu} & 0 \\ 0 & \bar{\sigma}^{\mu\nu} \end{pmatrix}, \tag{2.3.14}$$

and the Dirac gamma matrices are defined as

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}. \tag{2.3.15}$$

This construction allows one to describe spin-$\frac{1}{2}$ particles with both chiralities in a single relativistic framework, suitable for the Dirac equation.

Chirality operator. The **chirality operator** is defined as

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbb{I}_2 & 0 \\ 0 & \mathbb{I}_2 \end{pmatrix}, \qquad (\gamma^5)^2 = \mathbb{I}. \tag{2.3.16}$$

It allows one to separate the Dirac spinor into its left- and right-handed components:

$$\gamma^5\Psi = \begin{pmatrix} -\psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}.$$

The eigenvalues of γ^5 identify the chirality: -1 for left-handed spinors and $+1$ for right-handed spinors. For massless fermions, chirality coincides with **helicity**, the projection of the spin along the direction of motion, making γ^5 a useful operator in the description of Weyl and Dirac fermions.

Chiral projectors. The left- and right-handed components of a Dirac spinor can be isolated using the **chiral projection operators**:

$$P_L = \frac{\mathbb{I}_4 - \gamma^5}{2}, \qquad P_R = \frac{\mathbb{I}_4 + \gamma^5}{2}. \tag{2.3.17}$$

Applied to a Dirac spinor, they give

$$P_L\Psi = \begin{pmatrix} \psi_\alpha \\ 0 \end{pmatrix}, \qquad P_R\Psi = \begin{pmatrix} 0 \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix},$$

corresponding to the left- and right-handed Weyl components, respectively.

Dirac conjugate. The Dirac conjugate spinor is defined as[3]

$$\bar{\Psi} = \Psi^\dagger \gamma^0 = \begin{pmatrix} \overline{\chi}^{*\dot{\alpha}} & \psi^*_\alpha \end{pmatrix} = \begin{pmatrix} \chi^\alpha & \overline{\psi}_{\dot{\alpha}} \end{pmatrix}. \tag{2.3.18}$$

This definition ensures that the complex conjugate of a left-handed spinor behaves as a right-handed spinor under Lorentz transformations, and vice versa, which is essential for constructing Lorentz-invariant bilinear forms in quantum field theory.

Charge conjugation. Charge conjugation exchanges particles with their antiparticles. Using the charge conjugation matrix C, a Dirac spinor transforms as

$$\Psi^c = C\bar{\Psi}^T = \begin{pmatrix} \chi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}, \quad C = \begin{pmatrix} \epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix}.$$

Under this operation, a left-handed particle becomes a right-handed antiparticle, and vice versa, which is crucial for defining CPT transformations in relativistic quantum field theory.

Majorana spinors. A spinor that is its own charge conjugate (e.g. particle is its own antiparticle) is called a **Majorana spinor**:

$$\Psi^c_M = \Psi_M \quad \Longleftrightarrow \quad \psi_\alpha = \chi_\alpha.$$

Explicitly, a Majorana spinor can be written as

$$\Psi_M = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi^{*1} \\ \psi^{*2} \end{pmatrix}.$$

Any Dirac spinor can be decomposed into two real Majorana spinors:

$$\Psi = \Psi_{M_1} + i\Psi_{M_2}, \quad \Psi^c = \Psi_{M_1} - i\Psi_{M_2}.$$

Although no fundamental Majorana fermions have been experimentally confirmed, electrically neutral particles such as neutrinos are prime candidates, making Majorana spinors important in theoretical particle physics and models of neutrino masses.

2.3.5 | Infinite-dimensional representations

Infinite-dimensional representations of the Lorentz group are required when the objects being transformed are not just internal components, but *functions of spacetime points* — namely, fields $\Phi(x)$ themselves. In this case, the transformation acts simultaneously on the field's internal indices and on its spacetime dependence. The representation thus involves both the finite-dimensional matrices acting internally and the differential operators which generate coordinate transformations.

[3] The *bar* and the *dotted index* are distinct concepts exploiting the same property of the Weyl spinor: the bar denotes the conjugate Weyl representation ($\psi \in \mathrm{SU}(2)_L$, while $\overline{\psi} \in \mathrm{SU}(2)_R$), while the dotted index indicates transformation under the conjugate representation: when we are operating the complex conjugate of a left-handed spinor, we need to introduce dotted index and bar, while for the conjugation of a RH spinor we need the opposite.

Field representations

When a field $\Phi_a(x)$ is subjected to a Lorentz transformation, it transforms both through its internal indices and through its spacetime dependence. Each of these two aspects corresponds to a distinct representation of the Lorentz algebra:

$$\Phi_a(x) \xrightarrow{\mathrm{SO}^+(1,3)} \left(e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}\right)_a^{\ b} \left(e^{-\frac{i}{2}\omega_{\mu\nu}L^{\mu\nu}}\right)\Phi_b(x) = \left(e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}}\right)_a^{\ b}\Phi_b(x).$$

Here:

- $S^{\mu\nu}$ generates transformations on the **internal indices** of the field (a finite-dimensional representation);
- $L^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu)$ generates transformations on the **spacetime argument** x^μ (an infinite-dimensional representation);
- $J^{\mu\nu} = S^{\mu\nu} + L^{\mu\nu}$ generates the full transformation on the field.

Depending on the nature of the field, the representation $S^{\mu\nu}$ takes different explicit forms:

$$\begin{aligned} S^{\mu\nu} &= 0, && \text{for scalar fields,} \\ S^{\mu\nu} &= (M^{\mu\nu})^\rho{}_\sigma, && \text{for vector fields,} \\ S^{\mu\nu} &= \sigma^{\mu\nu},\, \bar{\sigma}^{\mu\nu},\, \Sigma^{\mu\nu}, && \text{for spinor fields.} \end{aligned}$$

The second exponential acts on the field's spacetime dependence, implementing the transformation $x^\mu \to \Lambda^\mu{}_\nu x^\nu$. Thus, field representations combine both finite- and infinite-dimensional actions, reflecting the double nature of fields: they carry internal degrees of freedom and are defined over spacetime.

One-particle Hilbert space representation

The above discussion applies to *classical fields*. Once the theory is quantized, fields become *operator-valued distributions* acting on a Fock space.[4] The Fock space $\mathcal{F}$ is constructed as a direct sum of multi-particle Hilbert spaces:

$$\mathcal{F} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots,$$

where $\mathcal{H}_1$ is the Hilbert space describing a single particle of the field.

The Lorentz group now acts as a unitary representation on the one-particle space $\mathcal{H}_1$. By **Wigner's theorem**, any symmetry preserving transition probabilities must be represented by a unitary (or antiunitary) operator on the Hilbert space:

Theorem 2.1 (Wigner's theorem)**.** *Let $\mathcal{H}$ be a complex Hilbert space, and let $\mathcal{S}(\mathcal{H})$ denote the set of rays (one-dimensional subspaces) of $\mathcal{H}$. Suppose that a transformation*

$$T : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$$

[4]In quantum field theory, fields are not ordinary operator functions of spacetime points, since products like $\hat{\phi}(x)^2$ would be ill-defined at coincident points. Instead, they are interpreted as *operator-valued distributions*: they only acquire a definite meaning when integrated (or "smeared") against smooth test functions $f(x)$, forming well-defined operators $\hat{\phi}(f) = \int \mathrm{d}^4x\, f(x)\,\hat{\phi}(x)$. This formulation ensures the mathematical consistency of field operators and reflects the fact that physical measurements always involve finite spacetime regions.

preserves transition probabilities, i.e.

$$\left|\langle\psi|\phi\rangle\right|^2 = \left|\langle T\psi|T\phi\rangle\right|^2, \qquad \forall\,|\psi\rangle, |\phi\rangle \in \mathcal{H}.$$

Then there exists a bijective map

$$U : \mathcal{H} \to \mathcal{H}$$

such that U is either ***unitary*** *or* ***antiunitary****, and*

$$T(|\psi\rangle) = \lambda_\psi\, U|\psi\rangle,$$

where λ_ψ is a phase factor with $|\lambda_\psi| = 1$. In particular, every symmetry transformation preserving quantum transition probabilities is represented on the Hilbert space by a unitary or antiunitary operator.

Hence, Lorentz transformations are implemented by unitary operators $U(\Lambda)$ whose infinitesimal form involves Hermitian generators:

$$U(\Lambda) = e^{-\frac{i}{2}\omega_{\mu\nu}\hat{J}^{\mu\nu}}, \qquad \hat{J}^{\mu\nu} = \text{Hermitian generators of the Lorentz algebra.}$$

Let us now see how this acts on one-particle states of a free field. The normalized one-particle state with momentum $\mathbf{p}$ and spin s is

$$|\mathbf{p}, s\rangle = \sqrt{2E_\mathbf{p}}\,\hat{a}^{\dagger\,s}_\mathbf{p}\,|0\rangle\,,$$

where $\hat{a}^{\dagger\,s}_\mathbf{p}$ creates a particle of momentum $\mathbf{p}$ and spin projection s from the vacuum.

Under a Lorentz transformation, this state transforms as

$$U(\Lambda)\,|\mathbf{p}, s\rangle = |\Lambda\mathbf{p}, s'\rangle = \sqrt{2E_{\Lambda\mathbf{p}}}\,\hat{a}^{\dagger\,s'}_{\Lambda\mathbf{p}}\,|0\rangle\,,$$

where the spin index s' may mix with others according to the representation of the little group associated with the particle's momentum.

In this way, the Lorentz group acts unitarily on the quantum states of the theory, providing an infinite-dimensional representation on the Fock space, built from the transformation properties of single-particle states.

Remark. *Finite-dimensional representations of the Lorentz group (such as those acting on spinor or tensor indices) suffice to describe how classical fields transform under Lorentz transformations. When quantizing the theory, however, one must consider infinite-dimensional representations:*

1. ***Field representation:*** *acts on the fields themselves as functions of spacetime points, allowing us to construct Lagrangians, field equations, and interactions in a relativistically covariant way.*
2. ***One-particle Hilbert space representation:*** *acts on the quantum states of single particles in Fock space via unitary operators $U(\Lambda)$, ensuring probability conservation and the relativistic invariance of the quantum theory.*

Both are essential for a consistent relativistic quantum field theory, but they serve complementary roles: one for describing fields, the other for describing particle states.

2.4 | The Poincaré Group

The **Poincaré group** is the fundamental symmetry group of relativistic spacetime. It extends the Lorentz group by including spacetime translations, and thus encodes the full structure of special relativity:

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu,$$

where Λ is a Lorentz transformation and a^μ a spacetime translation vector. Hence, it is also called the *inhomogeneous proper orthochronous Lorentz group*, often denoted as $ISO^+(1,3)$. In quantum field theory, the Poincaré group plays a central role: every isolated physical system must admit a unitary representation of this group on its Hilbert space, ensuring the invariance of the theory under rotations, boosts, and translations.

From an algebraic point of view, the Poincaré group is a **ten-parameter Lie group** generated by the four-momentum operators P_μ (associated with translations) and the six Lorentz generators $M_{\mu\nu}$ (associated with rotations and boosts), which satisfy the *Poincaré algebra*:

$$[P^\mu, P^\nu] = 0, \qquad [M^{\mu\nu}, P^\sigma] = i\left(P^\mu \eta^{\nu\sigma} - P^\nu \eta^{\mu\sigma}\right),$$

together with the usual Lorentz commutation relations for $M^{\mu\nu}$. The fact that the momentum operators commute among themselves shows that the translation subgroup is **abelian**. Physically, this reflects that momentum is additive and that free particles can be simultaneously diagonalized with respect to their energy and momentum. This also connects to gauge theories: interactions mediated by abelian gauge bosons (like the photon) do not self-interact, while non-abelian gauge bosons (like W and Z bosons) have self-interactions due to the non-commuting structure of their symmetry generators.

Field representation. In the *field representation*, the Poincaré generators act on fields as differential operators. In particular, spacetime translations are represented by

$$P^\mu = i\frac{\partial}{\partial x_\mu} = i\partial^\mu,$$

so that a translation of a field $\Phi(x)$ is realized as

$$\Phi(x) \to \Phi'(x) = e^{-ia_\mu P^\mu}\Phi(x) = \Phi(x+a).$$

This makes explicit how continuous spacetime symmetries are represented by *infinitesimal differential operators* acting on classical or quantum fields.

2.4.1 | Representations: one-particle Hilbert space

The representation theory of the Poincaré group on the *one-particle Hilbert space* provides the foundation for the relativistic description of quantum particles. Each irreducible unitary representation corresponds to a distinct type of elementary particle, characterized by the eigenvalues of the Casimir operators that we are about to introduce. This framework, originally developed by Wigner, unifies the treatment of spin and momentum in relativistic quantum mechanics and serves as the bridge between the abstract group-theoretic structure and the physical notion of particles.

Casimir operators

Recall that operators A that commute with a generator T of a symmetry represent conserved quantities under that transformation:

$$[A, T] = 0 \implies AT = TA.$$

Explicitly, if $|\psi\rangle$ is an eigenstate of A:

$$A |\psi\rangle = a |\psi\rangle ,$$

then under the transformation generated by T:

$$|\psi\rangle \to |\psi'\rangle = e^{i\alpha T} |\psi\rangle = \left(\mathbb{I} + i\alpha T + \mathcal{O}(\alpha^2)\right) |\psi\rangle ,$$

we have

$$\begin{aligned} A |\psi'\rangle &= A (\mathbb{I} + i\alpha T) |\psi\rangle = A |\psi\rangle + i\alpha AT |\psi\rangle \\ &= a |\psi\rangle + i\alpha TA |\psi\rangle = a(\mathbb{I} + i\alpha T) |\psi\rangle = a |\psi'\rangle . \end{aligned}$$

This shows that the eigenvalue a is invariant under the transformation: the conserved quantity remains the same.

An operator that commutes with all generators of a group is called a **Casimir operator**. Casimir operators are important because their eigenvalues are invariants that can be used to label irreducible representations, i.e., multiplets of states sharing the same conserved quantities.

For the Poincaré group, there are two independent Casimir operators:

$$C_1 = P^\mu P_\mu, \qquad C_2 = W^\mu W_\mu, \tag{2.4.1}$$

where W_μ is the **Pauli-Lubanski vector**:

$$W_\mu = +\frac{1}{2}\epsilon_{\mu\nu\rho\sigma} P^\nu M^{\rho\sigma}. \tag{2.4.2}$$

The physical interpretation is:

- $C_1 = P^\mu P_\mu$ corresponds to the squared mass of the particle. Since P_μ commutes with itself and with all Lorentz generators, all states in an irreducible representation have the same mass.
- $C_2 = W^\mu W_\mu$ is related to the intrinsic spin (for massive particles) or helicity (for massless particles). It provides a Lorentz-invariant characterization of the particle's rotational degrees of freedom.

Hence, different irreducible representations of the Poincaré group are classified by the eigenvalues of C_1 and C_2, giving a systematic way to organize particle types in relativistic quantum mechanics.

Massive representation

For **massive particles** ($m > 0$), the first Casimir operator of the Poincaré group satisfies

$$C_1 = P^\mu P_\mu = P^2, \quad \text{with eigenvalue } p^2 = E_{\mathbf{p}}^2 - |\mathbf{p}|^2 = m^2 \neq 0.$$

This defines a family of states all having the same mass m, obtained by applying the full set of Poincaré transformations to a chosen reference four-momentum p^μ. The resulting set of states can be denoted by $\{p^\mu\}$, and a generic state can be initially labeled as $|m,\, p^\mu\rangle$.

To classify these states further, we consider the second Casimir operator

$$C_2 = W^\mu W_\mu,$$

where W_μ is the Pauli-Lubanski vector defined in eq. (2.4.2). Its eigenvalues are associated with intrinsic spin degrees of freedom of the particle.

Little group. To understand the internal structure, we fix p^μ and look for all Poincaré generators that commute with P^μ. This subset of generators forms a subgroup of the Poincaré group called the *little group*: this subgroup leaves the four-momentum p^μ invariant and its representations classify the internal degrees of freedom of the particle, such as spin.

It is defined as the subgroup of Lorentz transformations Λ that leaves this vector unchanged:

$$\Lambda^\mu{}_\nu p^\nu = p^\mu.$$

Importantly, Wigner showed that the structure of the little group does not depend on the specific choice of p^μ within its equivalence class $\{p^\mu\}$, allowing us to select the simplest frame for calculations: the rest frame

$$p^\mu = (m, 0, 0, 0).$$

In the rest frame, it turns out that only the spatial rotation generators

$$J_i = \frac{1}{2}\epsilon_{ijk} M^{jk}, \quad i = 1, 2, 3,$$

commute with P^μ, so the little group is isomorphic to $SO(3)$, the group of spatial rotations. In more complicated situations, finding the little group may require evaluating nontrivial commutators, but for massive particles in the rest frame the identification is straightforward.

Pauli-Lubanski vector. The Pauli-Lubanski vector components in the rest frame are

$$\begin{aligned} W_0 &= \frac{1}{2}\epsilon_{0\nu\rho\sigma} P^\nu M^{\rho\sigma} = \frac{1}{2}\epsilon_{00\rho\sigma} m M^{\rho\sigma} = 0, \\ W_i &= \frac{1}{2}\epsilon_{i\nu\rho\sigma} P^\nu M^{\rho\sigma} = \frac{1}{2}\epsilon_{i0jk} m M^{jk} = -\frac{m}{2}\epsilon_{0ijk} M^{jk} = -mJ_i, \end{aligned}$$

where the antisymmetry of the Levi-Civita tensor ensures that terms with repeated indices vanish. We thus recover the familiar rotation generators J_i, so we can hypothesize that W_μ generates transformations which leave p^μ invariant:

$$\begin{aligned} [W_\mu,\, P_\nu] &= \frac{1}{2}\epsilon_{\mu\alpha\beta\gamma}[P^\alpha M^{\beta\gamma},\, P_\nu] = \frac{1}{2}\epsilon_{\mu\alpha\beta\gamma}\eta_{\nu\rho}[P^\alpha M^{\beta\gamma},\, P^\rho] \\ &= \frac{1}{2}\epsilon_{\mu\alpha\beta\gamma}\eta_{\nu\rho}\left(P^\alpha[M^{\beta\gamma},\, P^\rho] + [P^\alpha,\, P^\rho]M^{\beta\gamma}\right) \\ &= \frac{1}{2}\epsilon_{\mu\alpha\beta\gamma}\eta_{\nu\rho}\left(P^\alpha i(P^\beta\eta^{\gamma\rho} - P^\gamma\eta^{\beta\rho}) + 0M^{\beta\gamma}\right) \\ &= \frac{i}{2}\epsilon_{\mu\alpha\beta\gamma}P^\alpha\left(P^\beta\eta_{\nu\rho}\eta^{\gamma\rho} - P^\gamma\eta_{\nu\rho}\eta^{\beta\rho}\right) \\ &= \frac{i}{2}\epsilon_{\mu\alpha\beta\gamma}P^\alpha(P^\beta\delta_\nu{}^\gamma - P^\gamma\delta_\nu{}^\beta) = \frac{i}{2}\epsilon_{\mu\alpha\beta\nu}P^\alpha P^\beta - \frac{i}{2}\epsilon_{\mu\alpha\nu\gamma}P^\alpha P^\gamma \\ &= \frac{i}{2}\epsilon_{\mu\alpha\beta\nu}P^\alpha P^\beta + \frac{i}{2}\epsilon_{\mu\alpha\gamma\nu}P^\alpha P^\gamma = i\epsilon_{\mu\alpha\beta\nu}P^\alpha P^\beta = 0, \end{aligned}$$

since $P^\alpha P^\beta$ is symmetric while $\epsilon_{\mu\alpha\beta\nu}$ is antisymmetric for exchange of α and β. This results hold for both massive and massless particles.

Consequently, the second Casimir operator reduces to

$$C_2 = W^\mu W_\mu = -m^2 \mathbf{J}^2,$$

with $\mathbf{J}^2$ the squared angular momentum operator. Its eigenvalues are

$$j(j+1), \quad j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots,$$

while the eigenvalues of J_3 are

$$j_3 = -j, -j+1, \ldots, j-1, j.$$

Labeling the multiplet. Each state in the massive representation can thus be labeled as

$$|m, j, j_3, p^\mu\rangle,$$

where m is the mass, j the total spin, j_3 the spin projection along a chosen axis, and p^μ the four-momentum. The first three labels classify the multiplet, encoding internal spin degrees of freedom, while p^μ specifies the individual state within the multiplet.

This construction shows explicitly how internal symmetries of a particle, such as spin, are directly derived from spacetime symmetries, illustrating how the Poincaré group unifies the treatment of momentum and spin in relativistic quantum mechanics.

Massless representation

For **massless particles** ($m = 0$), the first Casimir operator of the Poincaré group satisfies

$$C_1 = P^\mu P_\mu = P^2, \quad \text{with eigenvalue } p^2 = E_{\mathbf{p}}^2 - |\mathbf{p}|^2 = m^2 = 0.$$

This condition identifies all possible states having vanishing invariant mass, i.e. those lying on the light cone of momentum space. Such states correspond to particles moving at the speed of light, for which energy and momentum are related by $E_{\mathbf{p}} = |\mathbf{p}|$.

As in the massive case, we must now understand how the internal structure of these states — their intrinsic degrees of freedom — emerges from the symmetries of spacetime. To this purpose, we again study the subgroup of the Poincaré group that leaves a reference four-momentum p^μ invariant: the so-called **little group**.

Little group. To identify the little group, we fix a representative momentum p^μ within the orbit of all lightlike four-momenta under Lorentz transformations. A convenient and symmetric choice is

$$p^\mu = (E, 0, 0, E),$$

which describes a massless particle propagating along the z-axis with energy E and momentum magnitude $|\mathbf{p}| = E$.

Geometrically, these transformations conserve the direction of motion of the particle (commute with p^μ), but can include both rotations around that direction and certain "null" boosts that act as translations in the plane transverse to it. Algebraically, this subgroup turns out to be isomorphic to the two-dimensional Euclidean group ISO(2), consisting of rotations and translations in a plane.

To see how this arises, we can exploit the fact that the Pauli–Lubanski vector W_μ commutes with the four-momentum operator $[W_\mu,\, P_\nu] = 0$. Therefore, the components of W_μ provide a convenient way to identify the generators of the little group that leave p^μ invariant. Let us explicitly compute these components in the chosen frame:

$$\begin{aligned} W_0 &= \frac{1}{2}\epsilon_{0\nu\rho\sigma}P^\nu M^{\rho\sigma} = \frac{1}{2}\epsilon_{00\rho\sigma}EM^{\rho\sigma} + \frac{1}{2}\epsilon_{03\rho\sigma}EM^{\rho\sigma} \\ &= \frac{1}{2}\left(\epsilon_{0312}EM^{12} + \epsilon_{0321}EM^{21}\right) \\ &= E\frac{1}{2}\left(M^{12} - M^{12}\right) = EJ_3. \end{aligned}$$

and similarly

$$\begin{aligned} W_1 &= -E\left(J_1 + K_2\right), \\ W_2 &= -E\left(J_2 - K_1\right), \\ W_3 &= -EJ_3 = -W_0. \end{aligned}$$

Thus, three independent combinations of the Lorentz generators appear naturally: J_3, $J_1 + K_2$, and $J_2 - K_1$. The first corresponds to rotations around the direction of propagation, while the other two mix boosts and rotations in such a way that they act as “translations” in the plane orthogonal to the momentum.

We can confirm that these generators indeed satisfy the commutation relations of the ISO(2) algebra:

$$\begin{aligned} [W_1,\, W_2] &= 0, \\ [W_3,\, W_1] &= -iEW_2, \\ [W_3,\, W_2] &= iEW_1. \end{aligned}$$

This algebra corresponds to the group of isometries of the two-dimensional Euclidean plane: W_1 and W_2 generate translations, while W_3 generates rotations about the axis perpendicular to that plane. From a group-theoretical point of view, this means that the internal symmetry space of a massless particle is not spherical (as in the massive case, where it was associated to SO(3)), but planar. The plane is the space orthogonal to the direction of motion, and the physical states are classified according to how they transform under rotations and translations within it.

If the eigenvalues of W_1 and W_2 are non-zero, the resulting representations are infinite-dimensional, because the translations generate a continuous family of states labeled by continuous parameters – a manifestation of the non-compactness of the subgroup generated by W_1 and W_2. In contrast, W_3 generates a compact subgroup corresponding to SO(2), whose eigenvalues are discrete.

However, in nature, only the latter type of representations are realized: physical massless particles correspond to the case in which the eigenvalues of W_1 and W_2 vanish. This restriction eliminates the continuous degrees of freedom associated with the non-compact part of the little group, leading to finite-dimensional representations that are fully characterized by the eigenvalues of W_3.

Such representations describe the internal rotational properties of massless particles — properties that will later be associated with *helicity*, the projection of the spin along the direction of motion.

Helicity. In the physically relevant case where the transverse components of the Pauli–Lubanski vector vanish, i.e. by setting $W_1 = W_2 = 0$, the remaining non-zero components simplify drastically:

$$W_0 = -W_3 = EJ_3,$$
$$W_0 = W^0 = EJ_3 = W^3 = -W_3,$$
$$W^\mu = J_3\,(E,\,0,\,0,\,E) = J_3 P^\mu.$$

This shows that W^μ becomes proportional to the four-momentum P^μ, with the proportionality factor given by the operator J_3, the generator of rotations around the direction of propagation. In this limit, the only remaining internal degree of freedom is the projection of the spin along the momentum direction — a quantity known as the **helicity**.

The helicity operator is thus defined as

$$h = \frac{W_\mu P^\mu}{P^\nu P_\nu}.$$

However, since for massless particles $P^\nu P_\nu = 0$, this expression must be interpreted carefully. In practice, one considers the proportionality relation $W_\mu = h\,P_\mu$, valid on the physical subspace of states with definite helicity. The operator h therefore measures how the intrinsic angular momentum (spin) is aligned or anti-aligned with the momentum of the particle. Its eigenvalues h are discrete and can take integer or half-integer values depending on the spin of the field under consideration.

Physically, $h > 0$ corresponds to a particle whose spin points in the same direction as its momentum (*right-handed* or positive helicity state), while $h < 0$ corresponds to the opposite alignment (*left-handed* or negative helicity state). These two possibilities represent the only independent polarization states available to a massless particle, since there is no rest frame in which to define a third (longitudinal) polarization component.

The second Casimir operator of the Poincaré group,

$$C_2 = W^\mu W_\mu,$$

vanishes identically in this case, $C_2 = 0$, because W^μ is proportional to the null vector P^μ. Thus, unlike the massive case where $C_2 = -m^2\mathbf{J}^2$ provided a discrete spin multiplet, here the intrinsic structure of the representation is fully captured by the helicity eigenvalue.

Labeling the multiplet. Each state in the massless representation can therefore be labeled as

$$|0,\,0,\,p^\mu,\,\pm h\rangle \equiv |p^\mu,\,\pm h\rangle\,,$$

where h denotes the helicity and p^μ the four-momentum of the state. The sign $\pm$ distinguishes the two helicity states corresponding to opposite spin alignments relative to the direction of motion.

Under a *parity* transformation, which reverses spatial orientation ($\mathbf{p} \to -\mathbf{p}$), the helicity changes sign:

$$h \xrightarrow{P} -h.$$

Hence, parity inversion exchanges right-handed and left-handed states. For this reason, a complete relativistic theory must generally include both helicities in order to be invariant under the combined CPT transformation[5].

[5]CPT stands for Charge conjugation (C), Parity inversion (P), and Time reversal (T).

In the Standard Model, different particle species correspond to different allowed helicities:

$$\begin{aligned} h &= 0 && \text{Higgs boson (scalar)}, \\ h &= \pm\tfrac{1}{2} && \text{leptons and quarks (fermions)}, \\ h &= \pm 1 && \text{photon and gluons (gauge bosons)}, \\ h &= \pm 2 && \text{graviton (tensor boson)}. \end{aligned}$$

The two helicity states $\pm h$ thus correspond to the two independent polarization states of a massless field. For instance, in the case of the photon, they represent the right-handed and left-handed circular polarizations of light, which are experimentally observable and fundamental to our understanding of electromagnetism in quantum field theory.

Remark. *It is conceptually illuminating to note that, at its most fundamental level, quantum field theory is intrinsically a* ***massless theory****. The full Poincaré symmetry—which combines Lorentz invariance and spacetime translations—is naturally realized on representations with* $m = 0$*. Introducing a nonzero mass "by hand" would in fact bend this symmetry, since a massive term explicitly selects a preferred reference frame (the rest frame) and breaks conformal invariance.*

From this viewpoint, all fundamental fields are most naturally described as massless excitations of underlying quantum fields, each transforming under irreducible representations of the Poincaré group. Mass then arises dynamically through interaction with the scalar Higgs field: the Higgs mechanism endows particles with an effective rest mass while preserving the local gauge and Lorentz symmetries of the theory. This elegant construction reconciles the need for mass with the fundamental symmetry principles that govern relativistic quantum dynamics.

3 Classical Field Theory

In classical field theory there is a fundamental shift of perspective: we move from describing a physical system through a *finite* set of discrete trajectories evolving in time to describing it through a *continuous* entity that takes different values at each point in space and evolves in time — the **field**. This transition reflects the passage from a mechanics of point-like objects to a mechanics of distributed quantities, where energy, momentum, or charge may be continuously spread over space.

- **Classical particle mechanics**

  In classical mechanics, the state of a system with N particles is determined by its *generalized coordinates*
  $$q_i(t),$$
  together with their time derivatives $\dot{q}_i(t)$. Each coordinate q_i specifies the position (or a suitable generalized coordinate) of the i-th degree of freedom at a given time t. The system therefore possesses a *finite* number of degrees of freedom, labelled by
  $$i = 1, \dots, N.$$
  The evolution of the system is then described by a set of ordinary differential equations in time — typically the Euler–Lagrange equations derived from a Lagrangian $L(q_i, \dot{q}_i, t)$.

- **Classical field theory**

  In classical field theory, the basic object is a *field*
  $$\psi_i(t, \mathbf{x}),$$
  which assigns a value (scalar, vector, or tensor) to each point in space and time. The field contains an *infinite* number of degrees of freedom: the discrete index $i = 1, \dots, N$ labels internal components (for example, the components of the electromagnetic or spinor field), while the continuous variable $\mathbf{x}$ plays the role of a label identifying each point in space. The dynamics are now governed by *partial differential equations* derived from a Lagrangian density $\mathcal{L}(\psi_i, \partial_\mu \psi_i, x^\mu)$, which generalizes the Lagrangian of particle mechanics.

In this framework, the notion of a particle *trajectory* loses its meaning. There is no single path to follow: the field as a whole evolves, deforming continuously in space and time. What acquires physical significance is therefore not the position of an individual object, but the *configuration of the field* — the collective state of the system at each instant. Classical field theory thus provides the natural language for describing systems where locality, continuity, and symmetry play a fundamental role, setting the conceptual foundation upon which modern relativistic and quantum field theories are built.

3.1 | Action and Lagrangian Density

In classical field theory, the dynamics of a field $\psi_i(t, \mathbf{x}) = \psi(x)$[1] are encoded in the **Lagrangian density** $\mathcal{L}$, which is a functional of the fields, their first derivatives, and possibly the spacetime coordinates:

$$\mathcal{L} = \mathcal{L}(\psi_i, \partial_\mu \psi_i, x^\mu),$$

where $\partial_\mu \psi_i = \frac{\partial \psi_i}{\partial x^\mu}$ denotes the spacetime derivative of the field. In this formulation, the Lagrangian density depends only on *first derivatives* of the fields — analogously to classical particle mechanics, where the Lagrangian depends on positions and velocities but not on accelerations. Allowing higher-order derivatives would, in general, lead to fourth-order equations of motion and to instabilities (known as Ostrogradsky instabilities), hence such terms are typically excluded from physically meaningful theories.

The **action** S is then defined as the spacetime integral of the Lagrangian density:

$$S[\psi_i] = \int \mathrm{d}^4x \, \mathcal{L}(\psi_i, \partial_\mu \psi_i, x^\mu),$$

where $\mathrm{d}^4x = \mathrm{d}t \, \mathrm{d}^3\mathbf{x}$ in Minkowski spacetime. The action is a scalar quantity under Lorentz transformations and encapsulates the full dynamics of the system. The physical evolution of the fields will be determined by the principle of stationary action, discussed later on.

3.1.1 | Units and Dimensional Analysis

It is often useful to analyze the Lagrangian density in terms of **mass dimensions**, especially when comparing different interaction terms. Working in natural units, where $\hbar = c = 1$, the action is dimensionless:

$$[S] = 0.$$

Since the differential measure in $3 + 1$ dimensions has dimension

$$[\mathrm{d}^4x] = [L^4] = [M^{-4}] := -4,$$

the Lagrangian density must have mass dimension

$$[\mathcal{L}] = [M^4] = 4.$$

This fact constrains the allowed forms of the Lagrangian and determines the mass dimensions of fields and coupling constants.

Example: Klein–Gordon Field. Consider the free scalar field described by the Klein–Gordon Lagrangian:

$$\mathcal{L} = \frac{1}{2} \partial_\mu \psi \, \partial^\mu \psi - \frac{1}{2} m^2 \psi^2.$$

Each term in $\mathcal{L}$ must have the same mass dimension, $[\mathcal{L}] = 4$. Since derivatives have dimension $[\partial_\mu] = 1$, we can deduce the dimension of the scalar field:

$$4 = [\mathcal{L}] = 2[\partial_\mu] + 2[\psi] = 2 + 2[\psi] \quad \Rightarrow \quad [\psi] = 1.$$

[1] We will use the compact 4-vector notation $x^\mu = (t, \mathbf{x})$.

Interaction Terms and Couplings. If we extend the Lagrangian by including interaction terms,

$$\mathcal{L}' = \mathcal{L} + g\,\psi^3 - \lambda\,\psi^4,$$

then the requirement $[\mathcal{L}'] = 4$ implies:

$$[g] = 4 - 3[\psi] = 1, \qquad [\lambda] = 4 - 4[\psi] = 0.$$

Couplings with positive or zero mass dimension ($[g] \geq 0$) correspond to **renormalizable** interactions, whereas those with negative mass dimension would lead to non-renormalizable theories, whose predictive power breaks down at high energies.

In summary, dimensional analysis plays a crucial role in identifying which interaction terms yield consistent and physically meaningful field theories. Renormalizable theories — those with interaction terms up to fourth order in the fields in $3+1$ dimensions — are the backbone of modern particle physics, while higher-order terms typically appear as effective corrections suppressed by powers of a large mass scale.

3.1.2 | The Variational Principle

The evolution of a classical field is determined by the **principle of stationary action** (or **principle of least action**). According to this principle, the physical configuration of the field $\psi_i(x)$ is the one that makes the action $S[\psi_i]$ stationary under small variations $\delta\psi_i(x)$ that vanish at the boundaries:

$$\delta S = 0. \tag{3.1.1}$$

In other words, among all possible field configurations that interpolate between two fixed states at times t_1 and t_2, the physical field follows the path in configuration space that extremizes the action functional. This generalizes the least-action principle of classical mechanics to systems with infinitely many degrees of freedom.

Starting from the definition of the action,

$$S[\psi_i] = \int \mathrm{d}^4x\, \mathcal{L}(\psi_i, \partial_\mu\psi_i, x^\mu), \tag{3.1.2}$$

we consider an infinitesimal variation of the fields:

$$\psi_i(x) \longrightarrow \psi_i(x) + \delta\psi_i(x),$$

with $\delta\psi_i(x)$ assumed to vanish at the spacetime boundaries:

$$\delta\psi_i(t_1, \mathbf{x}) = \delta\psi_i(t_2, \mathbf{x}) = 0, \qquad \delta\psi_i(x) \to 0 \text{ as } |x| \to \infty.$$

These boundary conditions express the physical idea that there is no relevant dynamics infinitely far away in space or time — equivalently, that the field configuration is fixed at the initial and final times.

The variation of the action then reads:

$$\delta S = \int \mathrm{d}^4x \left[\frac{\partial\mathcal{L}}{\partial\psi_i}\,\delta\psi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\,\delta(\partial_\mu\psi_i) \right].$$

Since the variation and the derivative commute, we can rewrite:

$$\delta(\partial_\mu\psi_i) = \partial_\mu(\delta\psi_i),$$

so that

$$\delta S = \int \mathrm{d}^4x \left[\frac{\partial \mathcal{L}}{\partial \psi_i} \, \delta\psi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_i)} \, \partial_\mu(\delta\psi_i) \right].$$

To isolate the variation $\delta\psi_i$, we integrate the second term by parts using the product rule:

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_i)} \, \partial_\mu(\delta\psi_i) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_i)} \, \delta\psi_i \right) - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_i)} \right) \delta\psi_i.$$

The first term is a total divergence and can be converted, via Gauss's theorem, into a surface integral over the boundary $\partial\mathcal{R}$ of the integration region $\mathcal{R}$ (the region where $\mathbf{x} \in]-\infty, \infty[$, $t \in [t_1, t_2]$):

$$\int_{\mathcal{R}} \mathrm{d}^4x \, \partial_\mu A^\mu = \oint_{\partial\mathcal{R}} \mathrm{d}\sigma_\mu \, A^\mu, \qquad A^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_i)} \, \delta\psi_i.$$

Because the variations $\delta\psi_i$ vanish at the boundary, this surface term gives no contribution. Therefore, the variation of the action reduces to:

$$\delta S = \int \mathrm{d}^4x \left[\frac{\partial \mathcal{L}}{\partial \psi_i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_i)} \right) \right] \delta\psi_i.$$

Since the variations $\delta\psi_i$ are arbitrary within the integration region, the only way for δS to vanish is for the quantity in brackets to be zero everywhere. This yields the **Euler–Lagrange equations for fields**:

$$\boxed{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_i)} \right) - \frac{\partial \mathcal{L}}{\partial \psi_i} = 0.} \tag{3.1.3}$$

These are the fundamental **equations of motion** of classical field theory. They generalize the Euler–Lagrange equations of particle mechanics to continuous systems with infinitely many degrees of freedom, describing the local evolution of the field at every point in spacetime. The beauty of this formulation lies in its compactness: all of the dynamics — from wave equations to Maxwell's equations and beyond — can be derived from a single scalar quantity, the action.

Hamiltonian formalism. While the Lagrangian framework is sufficient to formulate the *Path Integral quantization scheme*, the *Hamiltonian formalism* is required in order to impose **canonical commutation relations** among fields and their conjugate momenta.

We begin by defining the **momentum density** $\pi^i(x)$ conjugate to the field $\psi^i(x)$:

$$\pi^i(x) = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_i(x)}. \tag{3.1.4}$$

The Hamiltonian is then obtained through a *Legendre transformation* of the Lagrangian density:

$$H = \int \mathrm{d}^3x \, \mathcal{H}, \qquad \mathcal{H} = \pi^i \dot{\psi}_i - \mathcal{L}, \tag{3.1.5}$$

where the time derivatives $\dot{\psi}_i(x)$ are expressed in terms of the conjugate momenta $\pi^i(x)$.

The goal of this construction is to recast the dynamics in terms of canonical variables (ψ^i, π^i), so that the classical Poisson brackets can later be promoted to **quantum commutation relations**. In this framework, fields become operators acting on the Fock space that defines the quantum states of the theory.

3.2 | Noether's Theorem

Symmetries play a central role in the formulation of physical theories: they represent the transformations that leave the form of the equations of motion—or equivalently, the action—unchanged. In both Lagrangian mechanics and field theory, the existence of a continuous symmetry implies the conservation of a physical quantity, a deep correspondence established by **Noether's theorem**.

To clarify the different kinds of symmetries that may appear in a physical system, it is useful to distinguish between those acting on the *spacetime coordinates* and those acting on the *dynamical variables* of the field.

Category	Type	Nature	Character	Examples
	Translational	Global	Continuous	$x^\mu \to x^\mu + a^\mu$
	Lorentz	Global	Continuous	Rotations and boosts, $\Lambda^\mu{}_\nu$
Spacetime	Poincaré	Global	Continuous	Translations + Lorentz transformations
	General coordinate	Local	Continuous	$x^\mu \to x'^\mu(x)$ (curved spacetime invariance)
	Discrete (C, P, T)	Global	Discrete	$P : \mathbf{x} \to -\mathbf{x},\ T : t \to -t,\ C : \psi \to \psi^c$
	Discrete $P(x)$	Local	Discrete	$P(x) : \mathbf{x} \to -\mathbf{x}$
	Supersymmetry (SUSY)	Global	Continuous	$Q_\alpha, \bar{Q}_{\dot\alpha}$: mix bosons and fermions
	Phase symmetry	Global	Continuous	$U(1)_F : \psi \to e^{i\alpha}\psi$ (fermion number)
	Flavor symmetry	Global	Continuous	$U(1)_F,\ SU(3)_F$: fermion flavor rotations
Internal	Gauge symmetry	Local	Continuous	$U(1)$, $SU(2)$, $SU(3)$ (QED, weak, strong)
	Standard Model	Local	Continuous	$SU(3)_c \times SU(2)_L \times U(1)_Y$
	Discrete internal	Global	Discrete	$\mathbb{Z}_2$, $\mathbb{Z}_N$ (Ising, parity sectors)
	Local discrete	Local	Discrete	$\mathbb{Z}_2(x)$: position-dependent sign flip

Table 3.1: Classification of spacetime and internal symmetries in classical and quantum field theory.

Remark. *Spacetime symmetries act on the coordinates x^μ and define the kinematical structure of the theory, while internal symmetries act on field components at fixed spacetime points, relating distinct internal degrees of freedom. Continuous global symmetries give rise to conserved currents via Noether's theorem, while local symmetries require the introduction of gauge fields to preserve invariance.*

Such symmetries can be **continuous**, like rotations or translations, or **discrete**, such as parity or charge conjugation. In the case of continuous transformations, infinitesimal variations of the action lead naturally to conserved quantities, whose explicit form depends on the invariance properties of the Lagrangian density. The precise relation between symmetry and conservation law is established by Noether's theorem.

Symmetries and constraints on the Lagrangian. Requiring that the Lagrangian be invariant under a given symmetry transformation imposes nontrivial constraints on its structure. Invariance means that

$$\mathcal{L}(\phi, \partial_\mu \phi) = \mathcal{L}(\phi', \partial_\mu \phi'),$$

so only specific combinations of fields and their derivatives are allowed. In other words, symmetries restrict the admissible terms that can appear in the Lagrangian. For instance, a global $U(1)$ phase invariance forbids terms like ϕ^2 but allows $|\phi|^2$. When the symmetry is promoted from global to local, i.e. the transformation parameter becomes space–time dependent, $\alpha = \alpha(x)$, the original Lagrangian typically loses its invariance. To restore it, one must introduce new compensating fields—gauge fields—whose dynamics and couplings are dictated by the symmetry itself. This mechanism naturally induces interaction terms between matter and gauge fields, such as the electromagnetic coupling $e\bar{\psi}\gamma^\mu A_\mu \psi$ in Quantum Electrodynamics. Thus, symmetries not only constrain the form of the Lagrangian but also determine the possible interactions among fields.

Let us now formulate Noether's theorem in the context of quantum field theory.

Theorem 3.1 (Noether's theorem)**.** *Every continuous symmetry of the Lagrangian density $\mathcal{L}$ gives rise to a* ***conserved current*** *$J^\mu(x)$. The Euler–Lagrange equations of motion imply the local continuity equation:*

$$\partial_\mu J^\mu(x) = 0, \tag{3.2.1}$$

or equivalently

$$\frac{\mathrm{d}}{\mathrm{d}t} J^0 + \nabla \cdot \mathbf{J} = 0,$$

which expresses the conservation of a physical quantity associated with the symmetry.

Corollary 3.2 (Conserved charge)**.** *Given a conserved current $J^\mu(x)$ satisfying eq.* (3.2.1)*, one can define the corresponding* ***conserved charge****:*

$$Q = \int_{\mathbb{R}^3} \mathrm{d}^3\mathbf{x}\, J^0(x), \tag{3.2.2}$$

which is constant in time provided the current vanishes sufficiently fast at spatial infinity.

Proof. Taking the time derivative of Q, we have:

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \int_{\mathbb{R}^3} \mathrm{d}^3\mathbf{x}\, \frac{\mathrm{d}}{\mathrm{d}t} J^0 = -\int_{\mathbb{R}^3} \mathrm{d}^3\mathbf{x}\, \nabla \cdot \mathbf{J} = -\oint_{\partial\mathbb{R}^3} \mathrm{d}\mathbf{s} \cdot \mathbf{J}.$$

If the spatial current $\mathbf{J}$ vanishes at infinity, the surface term disappears and $\dot{Q} = 0$. Therefore, Q is conserved in time. □

This conservation law can also be interpreted as a **local conservation**: within a finite volume V,

$$\frac{\mathrm{d}Q_V}{\mathrm{d}t} = \int_V \mathrm{d}^3\mathbf{x}\, \frac{\mathrm{d}}{\mathrm{d}t} J^0 = -\oint_{\partial V} \mathrm{d}\mathbf{s} \cdot \mathbf{J},$$

meaning that any variation of the charge inside V is exactly compensated by the flux of $\mathbf{J}$ across its boundary.

3.2.1 | Proof of Noether's theorem

To prove Noether's theorem, we start by considering a *continuous infinitesimal transformation* of the field variables:

$$\psi_i \to \psi_i' = \psi_i + \delta\psi_i, \tag{3.2.3}$$

where $\delta\psi_i$ is assumed to be infinitesimally small. Under this transformation, the Lagrangian density changes as

$$\mathcal{L} \to \mathcal{L}' = \mathcal{L} + \delta\mathcal{L}.$$

If the theory is invariant under the transformation, the variation of the Lagrangian must be expressible as the *total derivative* of a four-vector function $K^\mu(\psi_i)$:

$$\delta\mathcal{L} = \partial_\mu K^\mu(\psi_i),$$

so that the corresponding action

$$S = \int \mathrm{d}^4x\, \mathcal{L}$$

remains unchanged:

$$S' = S + \int \mathrm{d}^4x\, \delta\mathcal{L} = S + \int \mathrm{d}^4x\, \partial_\mu K^\mu(\psi_i) = S,$$

since the surface term vanishes at infinity. This expresses the physical requirement that $\mathcal{L}$ and $\mathcal{L}'$ describe the same dynamics.

Now, let us compute the variation of the Lagrangian explicitly. Using the chain rule, we have

$$\begin{aligned}
\delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\psi_i}\,\delta\psi_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\,\delta(\partial_\mu\psi_i) \\
&= \frac{\partial\mathcal{L}}{\partial\psi_i}\,\delta\psi_i + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\,\delta\psi_i\right) - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\right)\delta\psi_i \\
&= \delta\psi_i\left[\frac{\partial\mathcal{L}}{\partial\psi_i} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\right)\right] + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\,\delta\psi_i\right).
\end{aligned}$$

In the second line we used the product rule for derivatives, and in the third we isolated a total divergence term. By the Euler–Lagrange equations for fields,

$$\frac{\partial\mathcal{L}}{\partial\psi_i} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\right) = 0,$$

the first term in brackets vanishes, leaving

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\,\delta\psi_i\right) = \partial_\mu K^\mu(\psi_i).$$

Since both expressions for $\delta\mathcal{L}$ represent total divergences, we can equate them and obtain

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\,\delta\psi_i - K^\mu(\psi_i)\right) = 0.$$

Thus, the quantity inside the parentheses defines a **conserved current**:

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\,\delta\psi_i - K^\mu(\psi_i), \tag{3.2.4}$$

which satisfies $\partial_\mu J^\mu = 0$.

Finally, for strictly invariant Lagrangians ($\delta\mathcal{L} = 0$), the term K^μ vanishes identically, yielding this particular expression for the conserved current:

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\,\delta\psi_i.$$

Remark. *The essential mathematical step of Noether's theorem is recognizing that a continuous symmetry corresponds to an infinitesimal field transformation under which the Lagrangian changes by a total derivative. This guarantees the existence of a four-vector J^μ whose divergence vanishes identically, establishing a one-to-one correspondence between continuous symmetries and conserved quantities.*

3.3 | Energy–Momentum Tensor

In classical mechanics, invariance under spatial translations leads to the conservation of linear momentum, while invariance under time translations yields the conservation of energy. We now explore how this idea generalizes in the framework of quantum field theory.

3.3.1 | Infinitesimal spacetime translations

Let us consider an *infinitesimal spacetime translation.* Starting from a general Poincaré transformation,

$$x^\nu \to x'^\nu = \Lambda^\nu{}_\mu x^\mu + a^\nu,$$

we restrict to the case in which there is no Lorentz transformation ($\Lambda^\nu{}_\mu = \delta^\nu{}_\mu$), and $a^\nu = \epsilon^\nu$ is an infinitesimal constant displacement. In this case, the transformation acts on the field as a **scalar transformation**:

$$\psi_i(x) \to \psi'_i(x) = \psi_i(x^\nu + \epsilon^\nu) = \psi_i(x) + \epsilon^\nu \partial_\nu \psi_i(x), \tag{3.3.1}$$

where we expanded to first order in ϵ^ν.

Active and passive viewpoints

It is important to distinguish between two equivalent but conceptually different interpretations of a transformation: the *active* and the *passive* viewpoints. Although they lead to the same mathematical result, they differ in what is considered to be "moved" — the field or the coordinate system — and this can be counterintuitive at first.

- **Active transformation:** the field configuration itself is displaced in spacetime, while the coordinates remain fixed. In this picture,

  $$\psi'_i(x') = \psi_i(x),$$

  meaning that what changes is the *physical field*, which is "pushed forward" along the translation vector ϵ^ν. One can imagine taking the same field profile and sliding it through spacetime without altering its shape.

- **Passive transformation:** here, we view the change as a relabeling of the coordinate system rather than a motion of the field itself:

  $$\psi'_i(x) = \psi_i(x').$$

  The field configuration is left untouched, but the coordinate grid used to describe it has been shifted by ϵ^ν. Consequently, the field *appears* to change its functional form when expressed in the new coordinates.

Although the two interpretations seem opposite — one moves the field, the other moves the coordinates — they are mathematically equivalent. The difference lies only in our point of view: in one case we deform the object within a fixed frame, in the other we deform the frame itself. This equivalence is crucial in field theory, where symmetries are often expressed in one language or the other depending on convenience.

3.3.2 | Variation of the Lagrangian

Since the Lagrangian density $\mathcal{L}$ is a scalar function of the fields and their derivatives, $\mathcal{L} = \mathcal{L}(\psi_i(x), \partial_\mu \psi_i(x))$, its transformation under (3.3.1) follows as

$$\mathcal{L}(x) \to \mathcal{L}'(x) = \mathcal{L}(x) + \epsilon^\nu \partial_\nu \mathcal{L}(x).$$

Therefore, its infinitesimal variation reads

$$\delta\mathcal{L} = \mathcal{L}' - \mathcal{L} = \epsilon^\nu \partial_\nu \mathcal{L}(x).$$

It is often convenient to rewrite this expression as a total divergence:

$$\delta\mathcal{L} = \epsilon^\nu \partial_\mu (\delta^\mu{}_\nu \mathcal{L}) = \partial_\mu (\epsilon^\nu \delta^\mu{}_\nu \mathcal{L}).$$

The introduction of the Kronecker symbol $\delta^\mu{}_\nu$ has no numerical effect, since $\partial_\mu(\delta^\mu{}_\nu \mathcal{L}) = \partial_\nu \mathcal{L}$; its purpose is purely formal, allowing us to display explicitly the divergence index μ. This step makes the expression ready for application of Noether's theorem, since a variation that can be written as a total divergence directly corresponds to a conserved current.

The four independent spacetime translations (one temporal and three spatial) therefore yield four conserved quantities, which we will identify shortly as the components of the **four-momentum**.

3.3.3 | Noether current and charges

Using the general expression for the Noether current derived in eq. (3.2.4), we have

$$(J^\mu)_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_i)}\, \delta\psi_i - K^\mu(\psi_i) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_i)}\, \epsilon^\nu \partial_\nu \psi_i - \epsilon^\nu \delta^\mu{}_\nu \mathcal{L} = \epsilon^\nu T^\mu{}_\nu.$$

Factoring out the infinitesimal ϵ^ν, we recognize the expression of the **energy–momentum tensor** (also called the *stress–energy tensor*):

$$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_i)}\, \partial_\nu \psi_i - \delta^\mu{}_\nu \mathcal{L}. \tag{3.3.2}$$

The tensor $T^\mu{}_\nu$ has the same physical dimensions as the Lagrangian density, namely that of an energy density, $[\mathcal{L}] = [T] = 4$. Each component of $T^\mu{}_\nu$ corresponds to a conserved current associated with one of the four spacetime translations.

The conserved charges follow from Eq. (3.2.2) as

$$Q_\nu = \int_{\mathbb{R}^3} \mathrm{d}^3\mathbf{x}\, T^0{}_\nu.$$

Raising the index with the Minkowski metric gives the **four-momentum**:

$$P^\nu = \int_{\mathbb{R}^3} \mathrm{d}^3\mathbf{x} \left(\frac{\partial \mathcal{L}}{\partial(\partial_0 \psi_i)}\, \partial^\nu \psi_i - \eta^{0\nu} \mathcal{L} \right).$$

Let us compute its components explicitly:

$$P^0 = \int_{\mathbb{R}^3} \mathrm{d}^3\mathbf{x} \left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}_i}\, \dot{\psi}_i - \mathcal{L} \right) = \int_{\mathbb{R}^3} \mathrm{d}^3\mathbf{x} \left(\pi^i \dot{\psi}_i - \mathcal{L} \right) = \int_{\mathbb{R}^3} \mathrm{d}^3\mathbf{x}\, \mathcal{H} = H,$$

where $\pi^i = \partial\mathcal{L}/\partial\dot{\psi}_i$ are the canonical momenta, and $\mathcal{H}$ is the Hamiltonian density. Hence, the temporal component of the four-momentum is the system's total energy.

For the spatial components we obtain:

$$P^j = \int_{\mathbb{R}^3} \mathrm{d}^3\mathbf{x} \left(\frac{\partial \mathcal{L}}{\partial \dot{\psi}_i} \partial^j \psi_i - \eta^{0j} \mathcal{L} \right) = \int_{\mathbb{R}^3} \mathrm{d}^3\mathbf{x}\, \pi^i \partial^j \psi_i = - \int_{\mathbb{R}^3} \mathrm{d}^3\mathbf{x}\, \pi^i \partial_j \psi_i .$$

Thus, the spatial components correspond to the total linear momentum of the field configuration. For a single particle, the linear momentum is defined as $p = \frac{\partial L}{\partial \dot{q}}$, and the total momentum of a continuous system is obtained by integrating this density over space. In field theory, the quantity

$$\pi^i(x) = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_i(x)}$$

plays the role of the conjugate momentum to the field value $\psi_i(x)$. The integrand $\pi^i \nabla \psi_i$ acts as a local momentum density: it quantifies how the field's temporal variation (via π^i) couples to its spatial variation (via $\nabla \psi_i$). The minus sign reflects that a positive spatial derivative corresponds to momentum flowing in the opposite direction.

Remark. *The energy–momentum tensor $T^\mu{}_\nu$ encodes how energy and momentum flow through spacetime. Its conservation law,*

$$\partial_\mu T^\mu{}_\nu = 0,$$

expresses the local conservation of energy ($\nu = 0$) and momentum ($\nu = 1, 2, 3$). This tensor will later play a central role in both field theory and general relativity, where it acts as the source of spacetime curvature.

3.4 | Electrodynamics

Electromagnetism admits a very compact and powerful description in terms of a four-potential and a Lagrangian density. In this section we introduce the basic electromagnetic fields and charges, express the fields in terms of the four-potential A^μ, and present the standard field-theory Lagrangian whose Euler–Lagrange equations reproduce Maxwell's equations (inhomogeneous ones). We work in natural units $c = \hbar = 1$ and use the metric signature $\eta_{\mu\nu} = \mathrm{diag}(+, -, -, -)$.

The electric and magnetic fields $\mathbf{E}(t, \mathbf{x})$ and $\mathbf{B}(t, \mathbf{x})$ and the sources (charge and current densities) $\rho(t, \mathbf{x})$ and $\mathbf{j}(t, \mathbf{x})$ satisfy Maxwell's equations (in vacuum, in differential form):

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0, \\ \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= 0, \\ \nabla \cdot \mathbf{E} &= \rho, \\ \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} &= \mathbf{j}. \end{aligned} \tag{3.4.1}$$

The first two are the *homogeneous* equations (no sources) and are kinematic identities once the fields are written in terms of a potential; the last two are the *inhomogeneous* Maxwell equations (sources on the right-hand side).

3.4.1 | Four-potential and homogeneous Maxwell equations

We introduce the electromagnetic four-potential as

$$A^\mu(x) = (\phi(t, \mathbf{x}),\ \mathbf{A}(t, \mathbf{x})), \tag{3.4.2}$$

where ϕ denotes the scalar potential and $\mathbf{A}$ the vector potential. The electric and magnetic fields can be expressed in terms of A^μ as

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla \times \mathbf{A}.$$

This representation makes it clear that both fields originate from derivatives of the same underlying potential, a fact that has deep geometrical meaning and automatically enforces part of Maxwell's equations.

In particular, the homogeneous Maxwell equations $(3.4.1)_{1,2}$ are identically satisfied. Indeed, the magnetic field is divergenceless:

$$\nabla \cdot \mathbf{B} = \nabla \cdot (\nabla \times \mathbf{A}) = \partial_i \epsilon_{ijk} \partial_j A_k = \epsilon_{ijk} \partial_i \partial_j A_k = 0,$$

because the contraction between the totally antisymmetric Levi-Civita tensor ϵ_{ijk} and the symmetric derivatives $\partial_i \partial_j$ vanishes identically. Similarly, the second homogeneous equation follows:

$$\begin{aligned} \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= \nabla \times \left(-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \right) + \frac{\partial}{\partial t}(\nabla \times \mathbf{A}) \\ &= \epsilon_{ijk} \partial_j (-\partial_k \phi - \partial_0 A_k) + \partial_0 \epsilon_{ijk} \partial_j A_k = -\epsilon_{ijk} \partial_j \partial_k \phi = 0. \end{aligned}$$

This result shows that the potential formulation automatically encodes the structure of the electromagnetic field so that the homogeneous Maxwell equations

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

are identically fulfilled, independently of the specific field configuration.

Following this formalism, it is also convenient to collect the charge and current densities into a single four-vector known as the *four-current*:

$$J^\mu(x) = (\rho, \mathbf{j}),$$

which satisfies the local charge conservation law, or *continuity equation*,

$$\partial_\mu J^\mu = 0.$$

This compact relation expresses the conservation of electric charge in covariant form, showing that the temporal and spatial components of J^μ are intrinsically linked as parts of a single relativistic entity that sources the electromagnetic field.

3.4.2 | Lagrangian and inhomogeneous Maxwell equations

The standard gauge-invariant Lagrangian density for the free electromagnetic field coupled to an external four-current J^μ reads

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}(\partial_\mu A^\mu)^2 - A_\mu J^\mu. \tag{3.4.3}$$

The quadratic terms contain the kinetic terms for the gauge field while the second term is the minimal coupling to sources. This Lagrangian remain invariant under the gauge transformation

$$A_\mu(x) \mapsto A_\mu(x) + \partial_\mu \Lambda(x),$$

provided the scalar function Λ vanishes suitably at the boundary, ensuring that no surface terms arise in the variation.

Let us do some observations:

- The mass dimension of the Lagrangian density is $[\mathcal{L}] = 4$, while that of the derivative operator is $[\partial_\mu] = 1$. From these we infer the dimensions of the potential and the current:
$$[A^\mu] = 1, \quad [J^\mu] = 3.$$
- The temporal component A_0 of the four potential is not a dynamical variable, as the Lagrangian contains no quadratic term in its time derivative $\dot{A}_0$. In fact, the term $\frac{1}{2}(\partial_0 A_0)(\partial^0 A^0)$ in the first kinetic contribution cancels exactly with $\frac{1}{2}(\partial_0 A^0)^2$ from the second one.
- Since $A_i = -A^i$, the overall minus sign in the first kinetic term guarantees the correct positive sign for the kinetic energy of the vector potential:
$$-\tfrac{1}{2}(\partial_0 A_i)(\partial^0 A^i) = \tfrac{1}{2}(\dot{A}_i)^2 = \tfrac{1}{2}(\dot{A}^i)^2.$$

Hence, if A^0 is non-dynamical, the field initially possesses three independent components, but *gauge symmetry* will allow us to eliminate one more degree of freedom, leaving only two physical ones — corresponding to the **two transverse polarizations** of electromagnetic waves.

Treating A_σ as the dynamical field, we apply the Euler–Lagrange equations:

$$\partial_\rho \left(\frac{\partial \mathcal{L}}{\partial(\partial_\rho A_\sigma)} \right) - \frac{\partial \mathcal{L}}{\partial A_\sigma} = 0.$$

one easily computes

$$\frac{\partial \mathcal{L}}{\partial A_\sigma} = -J^\sigma .$$

For the other term we should write the Lagrangian explicitly with respect to $\partial_\mu A_\nu$: we use the metric tensor in order to lower all the indices of similar terms

$$\mathcal{L} = -\frac{1}{2} \underbrace{(\partial_\mu A_\nu)\eta^{\mu\alpha}\eta^{\nu\beta}(\partial_\alpha A_\beta)}_{(\partial_\mu A_\nu)(\partial^\mu A^\nu)} + \frac{1}{2} \underbrace{\eta^{\mu\alpha}(\partial_\mu A_\alpha)\eta^{\nu\beta}(\partial_\nu A_\beta)}_{(\partial_\mu A^\mu)(\partial_\nu A^\nu)} - A_\mu J^\mu ,$$

so that now it's easier to compute the derivative with respect to $\partial_\mu A_\nu$ (knowing that $\frac{\partial x^\nu}{\partial x^\mu} = \delta^\mu_\nu$), so that

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial(\partial_\rho A_\sigma)} &= -\frac{1}{2}\eta^{\mu\alpha}\eta^{\nu\beta}\left[\delta^\rho{}_\mu \delta^\sigma{}_\nu (\partial_\alpha A_\beta) + (\partial_\mu A_\nu)\delta^\rho{}_\alpha \delta^\sigma{}_\beta\right] \\
&\quad + \frac{1}{2}\eta^{\mu\alpha}\eta^{\nu\beta}\left[\delta^\rho{}_\mu \delta^\sigma{}_\alpha (\partial_\nu A_\beta) + (\partial_\mu A_\alpha)\delta^\rho{}_\nu \delta^\sigma{}_\beta\right] \\
&= -\frac{1}{2}\left[\eta^{\rho\alpha}\eta^{\sigma\beta}(\partial_\alpha A_\beta) + \eta^{\mu\rho}\eta^{\nu\sigma}(\partial_\mu A_\nu)\right] \\
&\quad + \frac{1}{2}\left[\eta^{\rho\sigma}\eta^{\nu\beta}(\partial_\nu A_\beta) + \eta^{\mu\alpha}\eta^{\rho\sigma}(\partial_\mu A_\alpha)\right] \\
&= -(\partial^\rho A^\sigma) + \eta^{\rho\sigma}(\partial_\alpha A^\alpha) ,
\end{aligned}$$

and applying now the derivative ∂_ρ from EL equations, we get to:

$$\begin{aligned}
\partial_\rho \left(\frac{\partial \mathcal{L}}{\partial(\partial_\rho A_\sigma)}\right) &= \partial_\rho \left[-(\partial^\rho A^\sigma) + \eta^{\rho\sigma}(\partial_\alpha A^\alpha)\right] \\
&= -\partial_\rho \partial^\rho A^\sigma + \partial^\sigma \partial_\alpha A^\alpha = -\partial_\rho \partial^\rho A^\sigma + \partial^\sigma \partial_\rho A^\rho \\
&= -\partial_\rho(\partial^\rho A^\sigma - \partial^\sigma A^\rho).
\end{aligned}$$

Thus the equation of motion reads:

$$J^\sigma = \partial_\rho(\partial^\rho A^\sigma - \partial^\sigma A^\rho).$$

This compact relation, as we will now see, is equivalent to the inhomogeneous Maxwell equations.

Field strength tensor. Introducing the *field strength tensor*

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu ,$$

the equation of motion simplifies and acquires a direct physical interpretation. Its components naturally contain the electric and magnetic fields:

$$F^{i0} = -F^{0i} = \partial^i A^0 - \partial^0 A^i = (-\nabla\phi - \partial_t \mathbf{A})^i = E^i ,$$

and

$$F^{ij} = -F^{ji} = \partial^i A^j - \partial^j A^i = -\epsilon^{ijk} B^k ,$$

with diagonal components $F^{00} = F^{ii} = 0$.[2]

Using $F_{\mu\nu}$, the Lagrangian (3.4.3) can be rewritten in the elegant and manifestly Lorentz-invariant form (see Appendix B for the detailed derivation):

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - J_\mu A^\mu ,$$

[2]When computing the magnetic terms, note that $-\partial_i A^j + \partial_j A^i = -\epsilon^{ijk}\partial_i A^j = -(\nabla \times \mathbf{A})^k = -B^k$, thus the Levi-Civita tensor is put in to explicit the relation $F^{ji} = -F^{ij}$ and the antisymmetry of the field strength tensor (the minus sign comes from the lowered indices for nabla definition).

and the equations of motion become

$$\partial_\mu F^{\mu\nu} = J^\nu.$$

This equation compactly represents the *inhomogeneous* Maxwell equations. Let us show this explicitly:

- ($\nu = 0$) **Gauss's law**

$$\partial_\mu F^{\mu 0} = \rho,$$
$$\partial_0 F^{00} + \partial_i F^{i0} = \rho,$$

  but since $\partial_0 F^{00} = 0$ and $F^{i0} = E^i$, we recovered equation $(3.4.1)_3$:

$$\partial_i E^i = \nabla \cdot \mathbf{E} = \rho.$$

- ($\nu = i$) **Ampère–Maxwell's law**

$$\partial_\mu F^{\mu i} = J^i,$$
$$\partial_0 F^{0i} + \partial_j F^{ji} = J^i,$$

  but since $F^{ji} = \epsilon^{ijk} B^k$ and $\partial_0 F^{0i} = -\frac{\partial E^i}{\partial t}$, we can write

$$-\frac{\partial E^i}{\partial t} + \partial_j \epsilon^{ijk} B^k = \left(-\frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B}\right)^i = J^i,$$

  we have recovered equation $(3.4.1)_4$:

$$-\frac{\partial \mathbf{E}}{\partial t} + \nabla \times \mathbf{B} = \mathbf{j}.$$

These are exactly the *inhomogeneous* Maxwell equations for a free field coupled with a source J^μ in covariant form.

Bianchi identity. Together with the antisymmetric definition of $F_{\mu\nu}$, the cyclic identity

$$\partial_{[\lambda} F_{\mu\nu]} \equiv \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \tag{3.4.4}$$

is automatically satisfied. This is known as the *Bianchi identity*, and when expanded in components it reproduces the homogeneous Maxwell equations in $(3.4.1)_{1,2}$. Hence, the Lagrangian formalism not only yields the inhomogeneous equations of motion but also encodes the homogeneous ones as geometric identities following from the antisymmetry of $F_{\mu\nu}$.

Gauge fixing and wave equation. If one imposes the Lorenz gauge condition $\partial_\mu A^\mu = 0$, the inhomogeneous equations simplify because the term proportional to $\partial^\sigma \partial_\rho A^\rho$ vanishes. In this gauge the equations of motion reduce to the manifestly relativistic, decoupled wave equations for the components of the potential:

$$\Box A^\nu \equiv \partial_\mu \partial^\mu A^\nu = J^\nu.$$

This form is convenient both for solving classical radiation problems and for quantization, since each component A^ν satisfies a standard sourced wave equation. Gauge fixing removes some redundancies associated with gauge transformations among descriptions of equivalent systems, therefore eliminating non-physical degrees of freedom. We will see in section 6.2 that after imposing the Lorentz gauge condition on the EM field, considerations can be made on the system in order to add restriction on the residual gauge freedom, removing effectively further unphysical components to isolate the two transverse physical polarizations of the photon.

3.4.3 | Energy–momentum tensor

For the free electromagnetic field (no external sources), the Lagrangian becomes

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}.$$

Using the canonical expression (3.3.2) with A^μ as the dynamical variable, we compute the canonical energy–momentum tensor:

$$\begin{aligned} T^{\mu\nu} &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\rho)}\,\partial^\nu A_\rho - \eta^{\mu\nu}\mathcal{L} \\ &= -\partial^\mu A^\rho\,\partial^\nu A_\rho + \eta^{\mu\rho}(\partial_\sigma A^\sigma)\,\partial^\nu A_\rho + \eta^{\mu\nu}\frac{1}{4}F_{\lambda\sigma}F^{\lambda\sigma} \\ &= -\partial^\mu A^\rho\,\partial^\nu A_\rho + \underbrace{\partial^\nu A^\mu(\partial_\sigma A^\sigma)}_{\text{not symmetric under }\mu\leftrightarrow\nu} + \frac{1}{4}\eta^{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma}. \end{aligned}$$

So the energy-momentum tensor is not symmetric for exchange of μ and ν, but is that a problem?

Einstein field equation. In general relativity the dynamics of spacetime itself are determined by the distribution of energy and momentum of matter and fields through the celebrated *Einstein field equation*:

$$R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}\,T_{\mu\nu}, \tag{3.4.5}$$

where:

- $R_{\mu\nu}$ is the *Ricci tensor*, obtained by contracting the Riemann curvature tensor, and it is symmetric under the exchange $\mu \leftrightarrow \nu$;
- $R = g^{\rho\sigma}R_{\rho\sigma}$ is the *Ricci scalar*, which measures the curvature of spacetime as a whole;
- $g_{\mu\nu}$ is the metric tensor of the curved spacetime, encoding its geometric structure;
- Λ is the *cosmological constant*, originally introduced by Einstein to allow for a static universe, now interpreted as the energy density of the vacuum;
- G is Newton's gravitational constant and c the speed of light in vacuum.

The left-hand side represents the purely geometric content of spacetime curvature, while the right-hand side contains the *energy–momentum tensor* $T_{\mu\nu}$, describing the distribution of matter and energy that act as the source of gravity. For consistency, since the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu}$ is symmetric, the energy–momentum tensor must also satisfy $T_{\mu\nu} = T_{\nu\mu}$.

Belinfante-Rosenfeld tensor. In order to make $T^{\mu\nu}$ symmetric for exchange of indices, the idea is to redefine it as a sum of two contributions:

$$\tilde{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda \Xi^{\lambda\mu\nu},$$

where $\Xi^{\lambda\mu\nu}$ has to be a functions of the fields antysimmetric in the first two indices

$$\Xi^{\lambda\mu\nu} = -\Xi^{\mu\lambda\nu},$$

so that we can preserve the condition $\partial_\mu T^{\mu\nu} = 0$ even for the new $\tilde{T}^{\mu\nu}$:

$$\partial_\mu \tilde{T}^{\mu\nu} = \partial_\mu T^{\mu\nu} + \partial_\mu\partial_\lambda \Xi^{\lambda\mu\nu} = 0,$$

since the first term is already zero, and the second is the product of a symmetric quantity for an antisymmetric one in $\mu \leftrightarrow \nu$. In the EM case, this function can be defined as

$$\Xi^{\lambda\mu\nu} = F^{\mu\lambda}A^{\nu},$$

since $F^{\mu\lambda} = -F^{\lambda\mu}$, thus

$$\partial_\lambda F^{\mu\lambda}A^{\nu} = (\partial_\lambda F^{\mu\lambda})A^{\nu} + F^{\mu\lambda}(\partial_\lambda A^{\nu}),$$

but since $J^{\mu} = 0 = \partial_\lambda F^{\mu\lambda}$, then only the second term survives. We should now recast the expression of the energy-momentum tensor in the new form (full computation in appendix B)

$$T^{\mu\nu} = -F^{\mu\lambda}(\partial^{\nu}A_{\lambda}) + \frac{1}{4}\eta^{\mu\nu}\left(F_{\rho\sigma}F^{\rho\sigma}\right),$$

so that the new tensor take the following form:

$$\begin{aligned}
\tilde{T}^{\mu\nu} &= -F^{\mu\lambda}(\partial^{\nu}A_{\lambda}) + \frac{1}{4}\eta^{\mu\nu}\left(F_{\rho\sigma}F^{\rho\sigma}\right) + F^{\mu\lambda}(\partial_\lambda A^{\nu}) \\
&= F^{\mu\lambda}\left(\partial_\lambda A^{\nu} - \partial^{\nu}A_{\lambda}\right) + \frac{1}{4}\eta^{\mu\nu}\left(F_{\rho\sigma}F^{\rho\sigma}\right) \\
&= F^{\mu\lambda}F_{\lambda}{}^{\nu} + \frac{1}{4}\eta^{\mu\nu}\left(F_{\rho\sigma}F^{\rho\sigma}\right),
\end{aligned}$$

which is now symmetric under exchange $\mu \leftrightarrow \nu$. We notice indeed how $\eta^{\mu\nu}$ is symmetric, while

$$F^{\mu\lambda}F_{\lambda}{}^{\nu} = F^{\mu}_{\ \lambda}F^{\lambda\nu} = (-F_{\lambda}{}^{\mu})(-F^{\nu\lambda}) = F^{\nu\lambda}F_{\lambda}{}^{\mu}.$$

This is the true form of the energy-momentum tensor which appears in Einstein's field equation, and it is called the **Belinfante-Rosenfeld tensor**:

$$\tilde{T}^{\mu\nu} = F^{\mu\lambda}F_{\lambda}{}^{\nu} + \frac{1}{4}\eta^{\mu\nu}\left(F_{\rho\sigma}F^{\rho\sigma}\right). \tag{3.4.6}$$

Noether current and charges. Since we can identify our Belinfante-Rosenfeld tensor as the Noether current, we know it contains 4 conserved charges. We expect the *energy density* $\mathcal{H}$ to be the first consered quantity:

$$\mathcal{H} = \tilde{T}^{00}, \quad H = \int \mathrm{d}^3\mathbf{x}\,\mathcal{H} = \int \mathrm{d}^3\mathbf{x}\,\tilde{T}^{00};$$

we can compute the anergy density by computing the right term of (3.4.6):

$$\begin{aligned}
\mathcal{H} = \tilde{T}^{00} &= F^{0i}F_{i}{}^{0} + \frac{1}{4}\eta^{00}\left(F_{0i}F^{0i} + F_{i0}F^{i0} + F_{ij}F^{ij}\right) \\
&= F^{0i}F_{i}{}^{0} + \frac{1}{2}F_{0i}F^{0i} + \frac{1}{4}F_{ij}F^{ij} \\
&= \eta_{ij}F^{0i}F^{j0} + \frac{1}{2}\eta_{ij}F^{0j}F^{0i} + \frac{1}{4}\eta_{ia}\eta_{jb}F^{ab}F^{ij} \\
&= \eta_{ij}\left(\frac{1}{2}F^{0j}F^{0i} - F^{0i}F^{0j}\right) + \frac{1}{4}\eta_{ia}\eta_{jb}F^{ab}F^{ij} \\
&= \frac{1}{2}F^{0i}F^{0i} + \frac{1}{4}F^{ij}F^{ij} = \frac{1}{2}\left(|\mathbf{E}|^2 + |\mathbf{B}|^2\right),
\end{aligned}$$

where we have used the fact that for raising or lowering time-like indices the sign does not change, while $\eta^{ij} = -\delta^{ij}$ on the space-like indices. We have therefore found the energy density. For the *momentum density* $\mathcal{P}$ we have:

$$\mathcal{P}^{i} = \tilde{T}^{0i}, \quad P^{i} = \int \mathrm{d}^3\mathbf{x}\,\mathcal{P}^{i} = \int \mathrm{d}^3\mathbf{x}\,\tilde{T}^{0i};$$

we can compute it again from (3.4.6) as follows

$$\begin{aligned}\mathcal{P}^i &= \tilde{T}^{0i} = F^{0j}F_j{}^i + \frac{1}{4}\eta^{0i}\left(F_{\mu\nu}F^{\mu\nu}\right) = F^{0j}F_j{}^i \\ &= -F^{j0}(-F^{ji}) = (-E^j)(-\epsilon^{ijk}B^k) = (\mathbf{E}\times\mathbf{B})^i,\end{aligned}$$

since $\eta^{0i} = 0$ and $F_j{}^i = -F^{ji}$. Therefore we found the momentum density.

4 Klein Gordon Theory

We have now developed the Hamiltonian formulation of **Classical Field Theory**, where the dynamical degrees of freedom are represented by the fields $\phi_a(\mathbf{x}, t)$ and their conjugate momenta $\pi_a(\mathbf{x}, t)$, which satisfy canonical Poisson brackets. The next step is to promote this classical description to a **Quantum Field Theory (QFT)**, in which fields become operators acting on a Hilbert space, and classical observables become non-commuting operators.

From Classical Mechanics to Quantum Mechanics. The passage from classical to quantum mechanics provides a useful template. In Classical Mechanics (CM), a system with n degrees of freedom is described by canonical variables (q_i, p_i), whose dynamics is determined by Hamilton's equations:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \tag{4.0.1}$$

where $H(q, p)$ is the Hamiltonian function governing the time evolution of the system. The canonical variables satisfy Poisson brackets encoding the symplectic structure:

$$\{q_i, p_j\} = \delta_{ij}.$$

In Quantum Mechanics (QM), the canonical variables are promoted to Hermitian operators $(\hat{q}_i, \hat{p}_i)$ on a Hilbert space, with the replacement

$$\{\cdot, \cdot\} \longrightarrow \frac{1}{i\hbar}[\cdot, \cdot],$$

leading to the canonical commutation relations

$$[\hat{q}_i, \hat{p}_j] = i\hbar\, \delta_{ij}.$$

Observables evolve according to the Heisenberg equation

$$i\hbar \frac{d\hat{A}}{dt} = [\hat{A}, \hat{H}]. \tag{4.0.2}$$

From Classical Field Theory to Quantum Field Theory. The same canonical quantization idea extends to continuous systems. Here we adopt the **Schrödinger picture**, in which operators are constant in time while states evolve.

In Classical Field Theory, the generalized coordinates are fields $\phi_a(\mathbf{x}, t)$, and the conjugate momenta are (as defined in (3.1.4)):

$$\pi_a(\mathbf{x}, t) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a(\mathbf{x}, t)},$$

with canonical Poisson brackets

$$\{\phi_a(\mathbf{x},t), \pi_b(\mathbf{y},t)\} = \delta_{ab}\,\delta^{(3)}(\mathbf{x}-\mathbf{y}). \tag{4.0.3}$$

The Hamiltonian functional

$$H[\phi,\pi] = \int d^3x\,\mathcal{H}(\phi_a, \pi_a, \nabla\phi_a)$$

generates the time evolution of the fields.

In **Quantum Field Theory**, the fields become operators on a Hilbert (or Fock) space:

$$\phi_a(\mathbf{x},t) \;\longrightarrow\; \hat{\phi}_a(\mathbf{x}), \qquad \pi_a(\mathbf{x},t) \;\longrightarrow\; \hat{\pi}_a(\mathbf{x}),$$

and the Poisson brackets are replaced by equal-time commutation relations:

$$[\hat{\phi}_a(\mathbf{x}), \hat{\pi}_b(\mathbf{y})] = i\hbar\,\delta_{ab}\,\delta^{(3)}(\mathbf{x}-\mathbf{y}), \qquad [\hat{\phi}_a(\mathbf{x}), \hat{\phi}_b(\mathbf{y})] = [\hat{\pi}_a(\mathbf{x}), \hat{\pi}_b(\mathbf{y})] = 0, \tag{4.0.4}$$

defining operators acting on the Fock space which can be written in terms of *ladder operators* (annihilation and creation operators) acting on the vacuum.

In the Schrödinger picture, the states $|\psi(t)\rangle$ evolve according to the Schrödinger equation:

$$i\hbar\frac{d}{dt}\,|\psi(t)\rangle = \hat{H}\,|\psi(t)\rangle\,, \tag{4.0.5}$$

where the wave functional $\psi[\phi, t]$ encodes the probability amplitude for the field configuration $\phi(\mathbf{x})$ at time t. In this equation, we find the field inside the ket, where it just represents the configuration on which the wave functional is evaluated, but the field operator acts on the wave functional itself as creating or annihilating quanta of the field (which ultimately change the field configuration thus the state).

Solving the theory requires diagonalizing the Hamiltonian $\hat{H}$, which is generally difficult due to the infinite number of degrees of freedom and the possible discrete or continuous field labels. An important exception is **free theories** with quadratic Lagrangians, where the equations of motion are linear and can be solved exactly (e.g., the continuum limit of an elastic string).

Thus, canonical quantization provides a direct bridge from the Hamiltonian structure of classical fields to the operator formalism of quantum theory, where particles naturally arise as quantized excitations of the underlying fields.

4.1 | The Klein-Gordon Field as a Set of Harmonic Oscillators

The simplest relativistic free field theory is the Klein-Gordon theory for a real scalar field $\psi(\mathbf{x}, t)$. Its **Lagrangian density** is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\psi\,\partial^\mu\psi - \frac{1}{2}m^2\psi^2, \tag{4.1.1}$$

and the corresponding **classical equations of motion**, obtained via the Euler-Lagrange procedure, read

$$(\Box + m^2)\psi(x) = 0, \qquad \Box = \partial_\mu\partial^\mu. \tag{4.1.2}$$

Remark. *Although the Klein-Gordon equation appears as a single field equation in coordinate space, it is effectively a* coupled system *of infinitely many degrees of freedom: the Laplacian term* $-\nabla^2\psi$ *couples the field at different spatial points. By performing a spatial Fourier transform, one can* decouple *the system, obtaining independent harmonic oscillators for each momentum mode* $\mathbf{k}$*. In momentum space, each mode evolves independently and can be quantized separately.*

We perform a spatial Fourier transform of the field:

$$\psi(t, \mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\, e^{i\mathbf{p}\cdot\mathbf{x}}\tilde{\psi}(t, \mathbf{p}).$$

The Klein-Gordon equation in momentum space becomes

$$\left[\frac{\partial^2}{\partial t^2} + \omega_{\mathbf{p}}^2\right]\tilde{\psi}(t, \mathbf{p}) = 0, \qquad \omega_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}.$$

This shows explicitly that the system reduces to an infinite set of *decoupled harmonic oscillators*, one for each momentum mode $\mathbf{p}$.

4.1.1 | Canonical Quantization in Momentum Space

Following the analogy with the quantum harmonic oscillator in quantum mechanics, each momentum mode can be quantized independently. Recall that for a single harmonic oscillator with mass $m = 1$ and frequency ω, the canonical operators satisfy

$$\hat{q} = \frac{1}{\sqrt{2\omega}}(\hat{a} + \hat{a}^\dagger), \qquad \hat{p} = -i\sqrt{\frac{\omega}{2}}(\hat{a} - \hat{a}^\dagger),$$

where $\hat{a}, \hat{a}^\dagger$ are the annihilation and creation operators, obeying $[\,\hat{a}, \hat{a}^\dagger\,] = 1$.

We can generalize this construction to the field operators: since we have seen in the mechanical model of a quantum field (section 1.3) that in the Fourier space the KG equation is equivalent to a system of decoupled harmonic oscillators, we can perform a Fourier transform of the field and use the ladder operators (which create and annihilate excitations of the quantum harmonic oscillator) as Fourier coefficient. A general solution to the Klein-Gordon equation (satisfying reality of the field) can be written as

$$\hat{\psi}(\mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left[\hat{a}_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} + \hat{a}_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right],$$

with conjugate momentum operator

$$\hat{\pi}(\mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\,(-i)\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\left[\hat{a}_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} - \hat{a}_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}}\right],$$

where the ladder operators acts on the Fock space creating and destroying particles of momentum $\mathbf{p}$ (or $\mathbf{q}$). Furthermore, they satisfy the canonical commutation relations:

$$[\,\hat{a}_{\mathbf{p}}, \hat{a}_{\mathbf{q}}\,] = [\,\hat{a}^{\dagger}_{\mathbf{p}}, \hat{a}^{\dagger}_{\mathbf{q}}\,] = 0, \qquad [\,\hat{a}_{\mathbf{p}}, \hat{a}^{\dagger}_{\mathbf{q}}\,] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}).$$

Our solutions now have to respect similar canonical commutation relations. In fact we know that:

$$\begin{aligned}
&[\,\hat{\psi}(\mathbf{x}), \hat{\pi}(\mathbf{y})\,] = i\delta^{(3)}(\mathbf{x}-\mathbf{y}), \text{ while} \\
&[\,\hat{\psi}(\mathbf{x}), \hat{\psi}(\mathbf{y})\,] = [\,\hat{\pi}(\mathbf{x}), \hat{\pi}(\mathbf{y})\,] = 0.
\end{aligned}$$

Let us show this results:

$$\begin{aligned}
[\,\hat{\psi}(\mathbf{x}), \hat{\pi}(\mathbf{y})\,] &= \int \frac{\mathrm{d}^3\mathbf{p}\mathrm{d}^3\mathbf{q}}{(2\pi)^6}\,\frac{(-i)}{2}\sqrt{\frac{\omega_{\mathbf{q}}}{\omega_{\mathbf{p}}}}\,[\,\hat{a}_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} + \hat{a}^{\dagger}_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}\,,\,\hat{a}_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{y}} - \hat{a}^{\dagger}_{\mathbf{q}}e^{-i\mathbf{q}\cdot\mathbf{y}}\,] \\
&= \int \frac{\mathrm{d}^3\mathbf{p}\mathrm{d}^3\mathbf{q}}{(2\pi)^6}\,\frac{(-i)}{2}\sqrt{\frac{\omega_{\mathbf{q}}}{\omega_{\mathbf{p}}}}\left[-[\,\hat{a}_{\mathbf{p}}, \hat{a}^{\dagger}_{\mathbf{q}}\,]e^{i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})} + [\,\hat{a}^{\dagger}_{\mathbf{p}}, \hat{a}_{\mathbf{q}}\,]e^{-i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})}\right] \\
&= \int \frac{\mathrm{d}^3\mathbf{p}\mathrm{d}^3\mathbf{q}}{(2\pi)^6}\,\frac{(-i)}{2}\sqrt{\frac{\omega_{\mathbf{q}}}{\omega_{\mathbf{p}}}}(2\pi)^3\left[-\delta^{(3)}(\mathbf{p}-\mathbf{q})e^{i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})} + (-\delta^{(3)}(\mathbf{q}-\mathbf{p}))e^{-i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})}\right] \\
&= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\,\frac{i}{2}\sqrt{\frac{\omega_{\mathbf{q}}}{\omega_{\mathbf{p}}}}\left[\delta^{(3)}(\mathbf{p}-\mathbf{q})\left(e^{i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})} + e^{-i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})}\right)\right] \\
&= \frac{i}{2}\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\sqrt{\frac{\omega_{\mathbf{p}}}{\omega_{\mathbf{p}}}}\left(e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} + e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\right) \\
&= \frac{i}{2}\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\,e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} + \frac{i}{2}\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\,e^{i\mathbf{p}\cdot(\mathbf{y}-\mathbf{x})} \\
&= \frac{i}{2}\delta^{(3)}(\mathbf{x}-\mathbf{y}) + \frac{i}{2}\delta^{(3)}(\mathbf{y}-\mathbf{x}) = i\delta^{(3)}(\mathbf{x}-\mathbf{y}),
\end{aligned}$$

where in the first step we simplified the null commutators of the ladder operators, while in the last steps we recognized the integral representation of the Dirac delta function

$$\delta^{(3)}(\mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\,e^{\pm i\mathbf{p}\cdot\mathbf{x}}$$

which is by definition the Fourier transform of a constant and it respects the property

$$\delta(\alpha\mathbf{x}) = \frac{\delta(\mathbf{x})}{|\alpha|}, \quad \rightarrow \quad \delta(-\mathbf{x}) = \delta(\mathbf{x}).$$

For the other commutators we do not have to report the full computation, we need only to pay attention to the change in the signs with respect to the previous computation:

$$\begin{aligned}
[\,\hat{\psi}(\mathbf{x}), \hat{\psi}(\mathbf{y})\,] &= \int \frac{\mathrm{d}^3\mathbf{p}\mathrm{d}^3\mathbf{q}}{(2\pi)^6}\,\frac{1}{2\sqrt{\omega_{\mathbf{q}}\omega_{\mathbf{p}}}}\,[\,\hat{a}_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} + \hat{a}^{\dagger}_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}\,,\,\hat{a}_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{y}} + \hat{a}^{\dagger}_{\mathbf{q}}e^{-i\mathbf{q}\cdot\mathbf{y}}\,] \\
&= \int \frac{\mathrm{d}^3\mathbf{p}\mathrm{d}^3\mathbf{q}}{(2\pi)^6}\,\frac{1}{2\sqrt{\omega_{\mathbf{q}}\omega_{\mathbf{p}}}}\left[[\,\hat{a}_{\mathbf{p}}, \hat{a}^{\dagger}_{\mathbf{q}}\,]e^{i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})} + [\,\hat{a}^{\dagger}_{\mathbf{p}}, \hat{a}_{\mathbf{q}}\,]e^{-i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})}\right] \\
&= \frac{1}{2}\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{\omega_{\mathbf{p}}}\left(e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} - e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\right) = 0,
\end{aligned}$$

since in the second integral we can change the integration variable $\mathbf{p} \rightarrow -\mathbf{p}$: $\omega_{\mathbf{p}}$ is an even function and the subtraction among the exponantials remains given the reordering of the integration limits. Similarly:

$$\begin{aligned}
[\,\hat{\pi}(\mathbf{x}), \hat{\pi}(\mathbf{y})\,] &= \int \frac{\mathrm{d}^3\mathbf{p}\mathrm{d}^3\mathbf{q}}{(2\pi)^6}\,\frac{(-i)^2}{2}\sqrt{\omega_{\mathbf{q}}\omega_{\mathbf{p}}}\,[\,\hat{a}_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} - \hat{a}^{\dagger}_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}\,,\,\hat{a}_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{y}} - \hat{a}^{\dagger}_{\mathbf{q}}e^{-i\mathbf{q}\cdot\mathbf{y}}\,] \\
&= \int \frac{\mathrm{d}^3\mathbf{p}\mathrm{d}^3\mathbf{q}}{(2\pi)^6}\,\frac{(-1)}{2}\sqrt{\omega_{\mathbf{q}}\omega_{\mathbf{p}}}\left[-[\,\hat{a}_{\mathbf{p}}, \hat{a}^{\dagger}_{\mathbf{q}}\,]e^{i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})} - [\,\hat{a}^{\dagger}_{\mathbf{p}}, \hat{a}_{\mathbf{q}}\,]e^{-i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})}\right] \\
&= \frac{1}{2}\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\,\omega_{\mathbf{p}}\left(e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} - e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\right) = 0.
\end{aligned}$$

4.1.2 | Hamiltonian and Ladder Operators

Let us examine the classical Klein-Gordon Hamiltonian:

$$H = \int \mathrm{d}^3\mathbf{x}\, T^{00} = \int \mathrm{d}^3\mathbf{x}\, (\pi\dot{\psi} - \mathcal{L}),$$

where we have used the *Legendre* transform to express the Hamiltonian density T^{00}. Here, π is the conjugate momentum of the field, and we recall that the Lagrangian density $\mathcal{L}$ for a real scalar field is:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\psi\partial^\mu\psi - \frac{1}{2}m^2\psi^2 = \frac{1}{2}\left(\dot{\psi}^2 - |\nabla\psi|^2 - m^2\psi^2\right).$$

Proceeding with the computation, and utilizing the relationship $\pi = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = \dot{\psi}$, we simplify the total Hamiltonian to:

$$\begin{aligned} H &= \int \mathrm{d}^3\mathbf{x}\left(\pi^2 - \frac{1}{2}\pi^2 + \frac{1}{2}|\nabla\psi|^2 + \frac{m^2}{2}\psi^2\right) \\ &= \frac{1}{2}\int \mathrm{d}^3\mathbf{x}\left(\pi^2 + |\nabla\psi|^2 + m^2\psi^2\right). \end{aligned}$$

Now, to quantize the system, we promote the classical fields ψ and π to operators acting on a Fock Space within the **Schrödinger picture** (where operators have no explicit time dependence). We quantize via Fourier transform, which introduces the convenient definitions of the ladder operators:

$$\hat{\psi}(x) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{p}}}\left[\hat{a}_\mathbf{p}e^{i\mathbf{p}\cdot\mathbf{x}} + \hat{a}^\dagger_\mathbf{p}e^{-i\mathbf{p}\cdot\mathbf{x}}\right],$$

where $\omega_\mathbf{p} = \sqrt{|\mathbf{p}|^2 + m^2}$, and

$$\hat{\pi}(x) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}(-i)\sqrt{\frac{\omega_\mathbf{p}}{2}}\left[\hat{a}_\mathbf{p}e^{i\mathbf{p}\cdot\mathbf{x}} - \hat{a}^\dagger_\mathbf{p}e^{-i\mathbf{p}\cdot\mathbf{x}}\right].$$

The promoted Hamiltonian operator now takes the form:

$$\hat{H} = \frac{1}{2}\int \mathrm{d}^3\mathbf{x}\left(\hat{\pi}^2 + |\nabla\hat{\psi}|^2 + m^2\hat{\psi}^2\right).$$

We compute the integrands of the individual squared terms (implicitly involving a double momentum integral $\int \mathrm{d}^3\mathbf{p}\mathrm{d}^3\mathbf{q}$):

$$\begin{aligned} \hat{\pi}^2 &= -\frac{\sqrt{\omega_\mathbf{p}\omega_\mathbf{q}}}{2}\left[\hat{a}_\mathbf{p}e^{i\mathbf{p}\cdot\mathbf{x}} - \hat{a}^\dagger_\mathbf{p}e^{-i\mathbf{p}\cdot\mathbf{x}}\right]\left[\hat{a}_\mathbf{q}e^{i\mathbf{q}\cdot\mathbf{x}} - \hat{a}^\dagger_\mathbf{q}e^{-i\mathbf{q}\cdot\mathbf{x}}\right], \\ |\nabla\hat{\psi}|^2 &= \frac{1}{2\sqrt{\omega_\mathbf{p}\omega_\mathbf{q}}}i\mathbf{p}\left[\hat{a}_\mathbf{p}e^{i\mathbf{p}\cdot\mathbf{x}} - \hat{a}^\dagger_\mathbf{p}e^{-i\mathbf{p}\cdot\mathbf{x}}\right]i\mathbf{q}\left[\hat{a}_\mathbf{q}e^{i\mathbf{q}\cdot\mathbf{x}} - \hat{a}^\dagger_\mathbf{q}e^{-i\mathbf{q}\cdot\mathbf{x}}\right], \\ m^2\hat{\psi}^2 &= \frac{m^2}{2\sqrt{\omega_\mathbf{p}\omega_\mathbf{q}}}\left[\hat{a}_\mathbf{p}e^{i\mathbf{p}\cdot\mathbf{x}} + \hat{a}^\dagger_\mathbf{p}e^{-i\mathbf{p}\cdot\mathbf{x}}\right]\left[\hat{a}_\mathbf{q}e^{i\mathbf{q}\cdot\mathbf{x}} + \hat{a}^\dagger_\mathbf{q}e^{-i\mathbf{q}\cdot\mathbf{x}}\right]. \end{aligned}$$

By substituting these expressions into the Hamiltonian, performing the spatial integration, and simplifying (full computation detailed in Appendix B), we arrive at the final form:

$$\hat{H} = \frac{1}{2}\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\omega_\mathbf{p}\left(\hat{a}_\mathbf{p}\hat{a}^\dagger_\mathbf{p} + \hat{a}^\dagger_\mathbf{p}\hat{a}_\mathbf{p}\right), \tag{4.1.3}$$

which depends solely on the ladder operators. Manipulating this expression slightly by recalling the canonical commutation relation for the operators:

$$[\,\hat{a}_\mathbf{p}, \hat{a}^\dagger_\mathbf{q}\,] = (2\pi)^3\delta^{(3)}(\mathbf{p} - \mathbf{q}),$$

we find that the Hamiltonian contains a fundamental divergence:

$$\hat{H} = \frac{1}{2}\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\,\omega_{\mathbf{p}}\left(2\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} + [\,\hat{a}_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{p}}\,]\right) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\,\omega_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} + \frac{1}{2}\int \mathrm{d}^3\mathbf{p}\,\omega_{\mathbf{p}}\delta^{(3)}(0).$$

This Hamiltonian diverges for two principal reasons, corresponding to the limits of high and low momentum integration:

- **Ultraviolet (UV) Divergence**
  This results from the high-momentum (short-distance) contribution. When we integrate $\omega_{\mathbf{p}}$ over $\mathrm{d}^3\mathbf{p}$, the contributions from infinitely many harmonic oscillators with arbitrarily large momenta cause the integral to diverge. Since for large $|\mathbf{p}|$ the dispersion relation is $\omega_{\mathbf{p}} \sim |\mathbf{p}|$, it is expected that $\int \mathrm{d}^3\mathbf{p}\,\omega_{\mathbf{p}} \to \infty$.

- **Infrared (IR) / Volume Divergence**
  This divergence is mathematically due to the Dirac delta function: $\delta^{(3)}(0) \to \infty$. Physically, this provides insight into the nature of the problem: we are summing the **zero-point energy** of each of our harmonic oscillators. Since there are infinitely many oscillators filling a virtually infinite volume of space, the total energy is infinite.

Vacuum Structure and Regularization. In order to fully comprehend this infinity, we must study the structure of the **vacuum state** $|0\rangle$, which by definition is annihilated by all annihilation operators:

$$\hat{a}_{\mathbf{p}}\,|0\rangle = 0 \quad \forall\mathbf{p}.$$

Its energy can be computed by applying the Hamiltonian operator to this state:

$$\hat{H}\,|0\rangle = E_0\,|0\rangle = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\,\omega_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}}\,|0\rangle + \frac{1}{2}\int \mathrm{d}^3\mathbf{p}\,\omega_{\mathbf{p}}\delta^{(3)}(0)\,|0\rangle\,.$$

Since the first term vanishes due to the action of the annihilation operator, the vacuum energy is entirely given by the second term:

$$E_0 = \frac{1}{2}\int \mathrm{d}^3\mathbf{p}\,\omega_{\mathbf{p}}\delta^{(3)}(0) \to \infty.$$

As discussed, the energy of the vacuum diverges due to the **IR/Volume divergence**. The connection to the infinite volume V is explicit in the definition:

$$(2\pi)^3\delta^{(3)}(0) = \lim_{L\to\infty}\int_{-L/2}^{L/2} \mathrm{d}^3\mathbf{x}e^{i\mathbf{p}\cdot\mathbf{x}}\|_{\mathbf{p}=0} = \lim_{L\to\infty}\int_{-L/2}^{L/2} \mathrm{d}^3\mathbf{x} = V \to \infty.$$

We typically resolve this volume dependence by shifting our focus from pure energy E_0 to the energy density $\mathcal{E}_0$. By renormalizing properly by the volume, the dependence on $\delta^{(3)}(0)$ is removed:

$$\mathcal{E}_0 = \frac{E_0}{V} = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\,\frac{\omega_{\mathbf{p}}}{2}.$$

This quantity still diverges to infinity because of the **Ultraviolet divergence**. This remaining problem implies we have implicitly considered our theory to be valid at all energy scales. To proceed, we must introduce a momentum *cut-off* Λ to regularize the integral. This cut-off restricts the available momenta, acknowledging that our model breaks down above the Planck scale (M_p), where gravitational effects become non-negligible (as cited in section 1.2.2). The consequent **UV cutoff** imposes the restriction:

$$|\mathbf{p}| \leq \Lambda \leq M_p \sim 10^{18}\,\mathrm{GeV}.$$

Normal Ordering. We've established that the system's absolute energy value is problematic. Since we are typically interested only in *relative energy values* (such as differences between energy levels or energies measured relative to the vacuum), the simplest physical choice is to define the vacuum energy as zero: $E_0 = 0$. We achieve this formally by introducing the **Normal Ordering** of an operator $\hat{O}$:

$$: \hat{O} := \hat{O} - \langle 0| \hat{O} |0\rangle .$$

Applying this definition to the Hamiltonian $\hat{H}$ immediately ensures a null vacuum energy:

$$: \hat{H} : |0\rangle = \hat{H} |0\rangle - E_0 |0\rangle = 0.$$

The difference between the original Hamiltonian $\hat{H}$ and the normal ordered version $: \hat{H} :$ is solely due to the **ordering ambiguity** that arises when transitioning from a classical theory to a quantum field theory. Consider the classical Hamiltonian for a single harmonic oscillator, which can be expressed in two classically equivalent forms:

$$H = \frac{1}{2}p^2 + \frac{\omega^2}{2}q^2 \quad \text{or} \quad H = \frac{1}{2}(\omega q - ip)(\omega q + ip).$$

These expressions become non-equivalent when promoted to operators $\hat{q}$ and $\hat{p}$ due to non-zero commutators. The first expression yields the familiar quantum Hamiltonian:

$$\begin{aligned} \hat{H} &= \frac{1}{2}\hat{p}^2 + \frac{\omega^2}{2}\hat{q}^2 = \frac{\omega}{2}(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}) = \frac{\omega}{2}([\hat{a},\, \hat{a}^\dagger] + 2\hat{a}^\dagger\hat{a}) \\ &= \frac{\omega}{2}(1 + 2\hat{a}^\dagger\hat{a}) = \omega\hat{N} + \frac{\omega}{2}, \end{aligned}$$

where the final term $\omega/2$ accounts for the non-zero vacuum energy. The term $\omega\hat{N}$ multiplies the number of excitations $\hat{N}$ by the energy of a single oscillator (the same principle applies to counting particles in QFT). If we compute the second, factorized expression, we find:

$$\begin{aligned} \hat{H}' &= \frac{1}{2}(\omega\hat{q} - i\hat{p})(\omega\hat{q} + i\hat{p}) \\ &= \frac{1}{2}\left[\omega\frac{1}{\sqrt{2\omega}}(\hat{a} + \hat{a}^\dagger) - i\left(-i\sqrt{\frac{\omega}{2}}(\hat{a} - \hat{a}^\dagger)\right)\right]\left[\omega\frac{1}{\sqrt{2\omega}}(\hat{a} + \hat{a}^\dagger) + i\left(-i\sqrt{\frac{\omega}{2}}(\hat{a} - \hat{a}^\dagger)\right)\right] \\ &= \frac{\omega}{4}\left(\hat{a} + \hat{a}^\dagger - \hat{a} + \hat{a}^\dagger\right)\left(\hat{a} + \hat{a}^\dagger + \hat{a} - \hat{a}^\dagger\right) = \frac{\omega}{4}(2\hat{a}^\dagger)(2\hat{a}) = \omega\hat{a}^\dagger\hat{a} = \omega\hat{N}, \end{aligned}$$

where the vacuum energy term is naturally absent. This demonstrates the core rule of normal ordering: in any product of creation and annihilation operators, all annihilation operators ($\hat{a}$) must be placed to the right of all creation operators ($\hat{a}^\dagger$).

Applying normal ordering to the full Klein-Gordon Hamiltonian we found previously:

$$\hat{H} = \frac{1}{2}\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\,\omega_{\mathbf{p}}\left(\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}}\right),$$

the normal ordered version, $: \hat{H} :$, is directly obtained by moving $\hat{a}_{\mathbf{p}}$ to the right of $\hat{a}^\dagger_{\mathbf{p}}$ in the first term:

$$: \hat{H} := \frac{1}{2}\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\,\omega_{\mathbf{p}}\left(\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}}\right) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\,\omega_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}}. \tag{4.1.4}$$

From now on, we will safely set the vacuum energy to zero by implicitly using the normal ordering convention and dropping the cumbersome notation $: \hat{O} :$.

Remark. *Although we set the vacuum energy to zero here, it is not always negligible. When introducing interactions or considering gravity, the absolute value of the vacuum energy density (the cosmological constant) becomes physically crucial.*

4.1.3 | Particle States

For a single harmonic oscillator, the commutation relations between the Hamiltonian and the ladder operators define how the operators shift the energy levels:

$$[\hat{H},\, \hat{a}] = -\omega\hat{a}, \quad [\hat{H},\, \hat{a}^\dagger] = \omega\hat{a}^\dagger.$$

Similarly, in Klein-Gordon (KG) theory, the creation and annihilation operators $\hat{a}^\dagger_{\mathbf{p}}$ and $\hat{a}_{\mathbf{p}}$ shift the total energy by the energy of a single momentum mode, $\omega_{\mathbf{p}}$:

$$[\hat{H},\, \hat{a}_{\mathbf{p}}] = -\omega_{\mathbf{p}}\hat{a}_{\mathbf{p}}, \quad [\hat{H},\, \hat{a}^\dagger_{\mathbf{p}}] = \omega_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}}.$$

The derivation of these relations uses the normal ordered Hamiltonian and the fact that $\hat{a}_{\mathbf{p}}$ commutes with $\hat{a}_{\mathbf{q}}$ (the same also for the creation operator):

$$\begin{aligned}
[\hat{H},\, \hat{a}_{\mathbf{p}}] &= \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\, \omega_{\mathbf{q}} \left(\hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{q}}\hat{a}_{\mathbf{p}} - \hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{q}}\right) = \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\, \omega_{\mathbf{q}}[\hat{a}^\dagger_{\mathbf{q}},\, \hat{a}_{\mathbf{p}}]\hat{a}_{\mathbf{q}} \\
&= \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\, \omega_{\mathbf{q}}[-(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})]\hat{a}_{\mathbf{q}} = -\omega_{\mathbf{p}}\hat{a}_{\mathbf{p}}; \\
[\hat{H},\, \hat{a}^\dagger_{\mathbf{p}}] &= \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\, \omega_{\mathbf{q}} \left(\hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{q}}\hat{a}^\dagger_{\mathbf{p}} - \hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{q}}\right) = \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\, \omega_{\mathbf{q}}\hat{a}^\dagger_{\mathbf{q}}[\hat{a}_{\mathbf{q}},\, \hat{a}^\dagger_{\mathbf{p}}] \\
&= \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\, \omega_{\mathbf{q}}\hat{a}^\dagger_{\mathbf{q}}[(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})] = \omega_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}}.
\end{aligned}$$

We can build one particle **energy eigenstates** by acting on the vacuum state with a creation operator $\hat{a}^\dagger_{\mathbf{p}}$:

$$\begin{aligned}
\hat{a}^\dagger_{\mathbf{p}} \left|0\right\rangle &= \left|\mathbf{p}\right\rangle, \\
\hat{H}\left|\mathbf{p}\right\rangle &= E_{\mathbf{p}}\left|\mathbf{p}\right\rangle = \omega_{\mathbf{p}}\left|\mathbf{p}\right\rangle, \\
\text{with } \omega_{\mathbf{p}} &= \sqrt{|\mathbf{p}|^2 + m^2}.
\end{aligned}$$

The state $|\mathbf{p}\rangle$ represents a single scalar particle of mass m with momentum $\mathbf{p}$. The momentum operator $\hat{\mathbf{P}}$ confirms this. Classically, the momentum operator is given by the spatial components of the energy-momentum tensor integral:

$$P^i = \int \mathrm{d}^3\mathbf{x}\, T^{0i} = \int \mathrm{d}^3\mathbf{x}\, \pi\partial^i\psi = -\int \mathrm{d}^3\mathbf{x}\, \pi\partial_i\psi.$$

The corresponding quantum operator, after substituting the field expansions and simplifying, yields (using the normal ordering convention):

$$\begin{aligned}
\hat{P}^i &= -\int \mathrm{d}^3\mathbf{x}\, \hat{\pi}\partial_i\hat{\psi} \qquad \left[\partial_i(q^i x^i) = q^i\right] \\
&= -\frac{1}{2}\int \mathrm{d}^3\mathbf{x} \int \frac{\mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{p}}{(2\pi)^6}(-i)\left[\hat{a}_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} - \hat{a}^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}\right]\partial_i\left[\hat{a}_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{x}} + \hat{a}^\dagger_{\mathbf{q}}e^{-i\mathbf{q}\cdot\mathbf{x}}\right] \\
&= -\frac{1}{2}\int \mathrm{d}^3\mathbf{x}\, \frac{1}{2}\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^6}(-i)\left[\hat{a}_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} - \hat{a}^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}\right](iq^i)\left[\hat{a}_{\mathbf{q}}e^{i\mathbf{q}\cdot\mathbf{x}} - \hat{a}^\dagger_{\mathbf{q}}e^{-i\mathbf{q}\cdot\mathbf{x}}\right] \\
&= \frac{(-1)^2}{2}\int \frac{\mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{q}}{(2\pi)^6}q^i\left[\delta^{(3)}(\mathbf{p}+\mathbf{q})\left(\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{q}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\right) - \delta^{(3)}(\mathbf{p}-\mathbf{q})\left(\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{q}}\right)\right] \\
&= -\frac{1}{2}\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}p^i\left[-\left(\hat{a}_{\mathbf{p}}\hat{a}_{-\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{-\mathbf{p}}\right) - \left(\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}}\right)\right],
\end{aligned}$$

where we can notice how the first term vanishes due to symmetry considerations (integral of an odd function). Thus, the final expression for the momentum operator is (after removing the divergence via normal ordering):

$$\hat{P}^i = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} p^i \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}} \quad \text{or, more compactly} \quad \hat{\mathbf{P}} = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \mathbf{p}\, \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}}.$$

We verify that the state $|\mathbf{p}\rangle$ is an eigenstate of $\hat{\mathbf{P}}$ with eigenvalue $\mathbf{p}$:

$$\begin{aligned}
\hat{P}\,|\mathbf{p}\rangle &= \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \mathbf{q}\, \hat{a}^\dagger_{\mathbf{q}} \hat{a}_{\mathbf{q}} \hat{a}^\dagger_{\mathbf{p}} \,|0\rangle \\
&= \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \mathbf{q}\, \hat{a}^\dagger_{\mathbf{q}} \left([\hat{a}_{\mathbf{q}} \hat{a}^\dagger_{\mathbf{p}}] + \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{q}} \right) |0\rangle \\
&= \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \mathbf{q}\, \hat{a}^\dagger_{\mathbf{q}} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q})\,|0\rangle = \mathbf{p} \hat{a}^\dagger_{\mathbf{p}}\,|0\rangle = \mathbf{p}\,|\mathbf{p}\rangle\,.
\end{aligned}$$

This confirms that the state $|\mathbf{p}\rangle$ correctly possesses a momentum of $\mathbf{p}$ and an energy of $E_{\mathbf{p}} = \omega_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}$.

Multiparticle States. Since the commutator of creation operators associated with different momenta is null, i.e., $[\hat{a}^\dagger_{\mathbf{p}_1},\, \hat{a}^\dagger_{\mathbf{p}_2}] = 0$, multiparticle states are constructed similarly to the single-particle state, by sequential application of creation operators onto the vacuum $|0\rangle$:

$$|\mathbf{p}_1,\, \mathbf{p}_2,\, \ldots,\, \mathbf{p}_n\rangle = \hat{a}^\dagger_{\mathbf{p}_1}\, \hat{a}^\dagger_{\mathbf{p}_2}\, \cdots\, \hat{a}^\dagger_{\mathbf{p}_n}\,|0\rangle\,.$$

This definition immediately shows that the state $|\mathbf{p}_1,\, \mathbf{p}_2,\, \ldots,\, \mathbf{p}_n\rangle$ is *symmetric under the exchange of any two particles*. Explicitly, since creation operators commute:

$$|\mathbf{p},\, \mathbf{q}\rangle = \hat{a}^\dagger_{\mathbf{p}} \hat{a}^\dagger_{\mathbf{q}}\,|0\rangle = \hat{a}^\dagger_{\mathbf{q}} \hat{a}^\dagger_{\mathbf{p}}\,|0\rangle = |\mathbf{q},\, \mathbf{p}\rangle\,.$$

Consequently, the particles described by the Klein-Gordon field are **Bosons**: particles with integer spin that obey Bose-Einstein statistics, which let them occupy the same quantum state. The correct spin-statistics relationship is a direct consequence of the Quantum Field Theory framework, arising here from the imposition of canonical commutation relations during the quantization process.

The Fock space $\mathcal{F}$, is defined as the direct sum of all n-particle Hilbert spaces ($\mathcal{H}_n$): $\mathcal{F} = \bigoplus_{n\geq 0} \mathcal{H}_n$. This space is spanned by the set of all possible combinations of creation operators acting on the vacuum:

$$|0\rangle\,, \quad \hat{a}^\dagger_{\mathbf{p}}\,|0\rangle\,, \quad \hat{a}^\dagger_{\mathbf{p}} \hat{a}^\dagger_{\mathbf{q}}\,|0\rangle\,, \quad \hat{a}^\dagger_{\mathbf{p}} \hat{a}^\dagger_{\mathbf{q}} \hat{a}^\dagger_{\mathbf{r}}\,|0\rangle \ldots$$

The **number operator** $\hat{N}$ counts the total number of particles in a given state:

$$\hat{N} = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}}, \qquad \Longrightarrow \qquad \hat{N}\,|\mathbf{p}_1,\, \mathbf{p}_2,\, \ldots,\, \mathbf{p}_n\rangle = n\,|\mathbf{p}_1,\, \mathbf{p}_2,\, \ldots,\, \mathbf{p}_n\rangle\,.$$

Crucially, the number operator commutes with the Hamiltonian, $[\hat{H},\, \hat{N}] = 0$, meaning that particle number is a conserved quantity in this free theory. If the system is prepared in an n-particle state, it will remain in that sector indefinitely. This conservation property is exclusive to free theories; the introduction of interactions allows for the creation and annihilation of particles, causing the system to transition between different sectors of the Fock space.

4.1.4 | Lorentz Invariance

We now check that the quantized theory is Lorentz invariant. First, the vacuum state is normalized trivially:

$$\langle 0|0\rangle = 1.$$

The inner product of single-particle states is calculated using the canonical commutation relation (CCR) $[\hat{a}_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{q}}] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})$:

$$\begin{aligned}\langle \mathbf{p}|\mathbf{q}\rangle &= \langle 0|\,\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\,|0\rangle = \langle 0|\,[\hat{a}_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{q}}] + \hat{a}^\dagger_{\mathbf{q}}\hat{a}_{\mathbf{p}}\,|0\rangle \\ &= \langle 0|\,(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\,|0\rangle + 0 = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}).\end{aligned}$$

From this, we define the **Identity Operator** ($\hat{\mathbf{1}}$) on the one-particle Hilbert space via the *completeness relation*:

$$\hat{\mathbf{1}} = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\,|\mathbf{p}\rangle\,\langle\mathbf{p}|\,.$$

This operator acts correctly on momentum states:

$$\hat{\mathbf{1}}\,|\mathbf{q}\rangle = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\,|\mathbf{p}\rangle\,\langle\mathbf{p}|\mathbf{q}\rangle = \int \mathrm{d}^3\mathbf{p}\,|\mathbf{p}\rangle\,\delta^{(3)}(\mathbf{p}-\mathbf{q}) = |\mathbf{q}\rangle\,.$$

While the full identity operator $\hat{\mathbf{1}}$ is Lorentz invariant, its components—the integration measure $\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}$ and the projection operator $|\mathbf{p}\rangle\,\langle\mathbf{p}|$—are not Lorentz invariant when taken separately. To achieve *manifest* invariance, we must re-normalize the states such that both the measure and the states transform correctly.

We start by considering the manifestly Lorentz invariant measure in four-momentum space:

$$\int \mathrm{d}^4 p.$$

To relate this to our three-momentum measure, we must enforce the on-shell condition $p^2 = m^2$ and select only positive energy states ($p_0 = E_{\mathbf{p}}$). This is done using a Dirac delta function and a Heaviside theta function:

$$\int \mathrm{d}^4 p\,\delta(p^2 - m^2)\Theta(p_0) = \int \mathrm{d}^4 p\,\delta(p_0^2 - (|\mathbf{p}|^2 + m^2))\Theta(p_0).$$

To proceed, we utilize the property of the delta function $\delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|}$ where x_i are the roots. Here, $f(p_0) = p_0^2 - E_{\mathbf{p}}^2$, with roots $p_0 = \pm E_{\mathbf{p}}$. Since $f'(p_0) = 2p_0$, and $\Theta(p_0)$ selects only the positive root ($p_0 = +E_{\mathbf{p}}$), the integral over p_0 yields:

$$\implies \int \mathrm{d}^3\mathbf{p} \int \mathrm{d}p_0\,\delta(p_0^2 - E_{\mathbf{p}}^2) = \int \mathrm{d}^3\mathbf{p}\,\frac{1}{|2p_0|}\Big|_{p_0 = E_{\mathbf{p}}} = \int \frac{\mathrm{d}^3\mathbf{p}}{2E_{\mathbf{p}}}.$$

We now use this result to reconstruct the manifestly Lorentz invariant completeness relation by manipulating the original identity operator $\hat{\mathbf{1}}$:

$$\begin{aligned}\hat{\mathbf{1}} &= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\,|\mathbf{p}\rangle\,\langle\mathbf{p}| = \int \frac{\mathrm{d}^3\mathbf{p}}{2E_{\mathbf{p}}}\frac{2E_{\mathbf{p}}}{(2\pi)^3}\,|\mathbf{p}\rangle\,\langle\mathbf{p}| \\ &= \int \frac{\mathrm{d}^4 p}{(2\pi)^3}\delta(p^2 - m^2)\Theta(p_0)\,(2E_{\mathbf{p}}\,|\mathbf{p}\rangle\,\langle\mathbf{p}|)\,.\end{aligned}$$

This suggests a definition for a new, Lorentz-invariant state normalization:

$$|p\rangle = \sqrt{2E_{\mathbf{p}}}\,|\mathbf{p}\rangle = \sqrt{2E_{\mathbf{p}}}\,\hat{a}^\dagger_{\mathbf{p}}\,|0\rangle\,.$$

With this new normalization, the inner product becomes:

$$\langle p|q\rangle = \langle \mathbf{p}|\sqrt{2E_{\mathbf{p}}}\sqrt{2E_{\mathbf{q}}}|\mathbf{q}\rangle = 2E_{\mathbf{p}}(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}),$$

where the factor $2E_{\mathbf{p}}$ (or $2E_{\mathbf{q}}$) is known as the Lorentz boost factor, making the entire inner product behave correctly under Lorentz transformations.

4.2 | Two Real Klein Gordon Fields

Since our final goal is to describe all particles and their interactions, we can start to study a more complex model: the quantization of a system with more than one field.

The simplest generalization is the following: two real KG fields ψ_1 and ψ_2 with different masses $m_1 \neq m_2$. We are assuming the fields to be free, non interacting, so we can write the Lagrangian as the sum of two independent KG Lagrangian:

$$\mathcal{L} = \sum_{i=1,2} \frac{1}{2}\partial_\mu \psi_i \partial^\mu \psi_i - \frac{m_i^2}{2}\psi_i^2. \tag{4.2.1}$$

Since the lagrangian is a sum of decoupled and independently Lagrangians, we can derive the KG equation as a system of two independent equations:

$$\begin{cases} (\Box + m_1^2)\psi_1(x) = 0; \\ (\Box + m_2^2)\psi_2(x) = 0. \end{cases}$$

Following this procedure, other foundamental quantities will be defined as the sum of the corresponding quantities defined for the single scalar field: we are interested in Hamiltonian, momentum and number of particles:

$$H = H_1 + H_2, \quad \mathbf{P} = \mathbf{P}_1 + \mathbf{P}_2, \quad N = N_1 + N_2.$$

Now we can use ladder operators in normal ordering to promote those quantities to operators acting on the fields:

$$\begin{aligned} \hat{H}_i &= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\, \omega_{i,\,\mathbf{p}} \hat{a}^\dagger_{i,\,\mathbf{p}} \hat{a}_{i,\,\mathbf{p}}; \\ \hat{P}_i &= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\, \mathbf{p} \hat{a}^\dagger_{i,\,\mathbf{p}} \hat{a}_{i,\,\mathbf{p}}; \\ \hat{N}_i &= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\, \hat{a}^\dagger_{i,\,\mathbf{p}} \hat{a}_{i,\,\mathbf{p}}. \end{aligned}$$

Notice how we are basically spanning all the space (with the integral over all possible momenta) and counting the particles in it, then multiplying them by the actual eigenvalue in which we are interested: $\omega_{i,\,\mathbf{p}}$ for the energy or $\mathbf{p}$ for the momentum. The ladder operators create or destroy a particle of mass m_i and momentum $\mathbf{p}$ accordingly to the previous definitions:

$$\left|\mathbf{p}_{(1)}\right\rangle = \hat{a}^\dagger_{1,\,\mathbf{p}} \left|0\right\rangle, \quad \left|\mathbf{p}_{(2)}\right\rangle = \hat{a}^\dagger_{2,\,\mathbf{p}} \left|0\right\rangle,$$

where these particles have different masses and energies

$$\hat{H}\left|\mathbf{p}_{(1)}\right\rangle = \omega_{1,\,\mathbf{p}} \left|\mathbf{p}_{(1)}\right\rangle, \quad \omega_{1,\,\mathbf{p}} \neq \omega_{2,\,\mathbf{p}} \text{ since } m_1 \neq m_2;$$

but they are degenerate as far as number of particles and momentum are concerned:

$$\begin{aligned} \hat{P}\left|\mathbf{p}_{(i)}\right\rangle &= \mathbf{p}\left|\mathbf{p}_{(i)}\right\rangle, \quad \mathbf{p}_1 = \mathbf{p}_2 = \mathbf{p}; \\ \hat{N}\left|\mathbf{p}_{(i)}\right\rangle &= 1\left|\mathbf{p}_{(i)}\right\rangle, \quad n = 1 \text{ for both fields.} \end{aligned}$$

Thus these three operators are simultaneously diagonalizable, having the same set of eigenstates (forming a complete ON basis) but different eigenvalues.

4.2.1 | Equal Mass Scalar Fields

The more interesting case is when both fields share the same mass $m_1 = m_2 = m$: what happens to the system?[1] To start now we have equal eigenvalues for all the three previously considered operators. Furthermore, if we imagine the two KG fields as two axis of a plane we can deduce a new property of the system: if the state is a vector on this $(\psi_1 - \psi_2)$ plane, we have aquired a **rotational symmetry** in the space of fields. Since the Lagrangian is unvariant under a new group of transformations, we have aquired also a new Noether's current and charge.

We can write the field in a vectorial representation as

$$\vec{\psi}(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} \in \mathbb{R}^2, \quad \vec{\psi}^T(x) = \begin{pmatrix} \psi_1(x) & \psi_2(x) \end{pmatrix};$$

then we can express the Lagrangian on this new field plane as:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\psi}^T(x))(\partial^\mu \vec{\psi}(x)) - \frac{m^2}{2}\vec{\psi}^T(x)\vec{\psi}(x). \tag{4.2.2}$$

Thus this Lagrangian goes to itself after a SO(2) transformation:

$$\mathcal{L} \xrightarrow{\mathrm{SO}(2)} \mathcal{L}' = \mathcal{L},$$

where we have to pay attention to consider a **global** SO(2) transformation, thus rotating every point of the *field space* $(\psi_1 - \psi_2)$ under the same transformation. **Local** transformations (such as *gauge transformations*) change the Lagrangian, but if we add suitable interactions, we may result in canceling out the asymmetries of the Lagrangian making it invariant even under local transformations.

Remaining in free theories, we aim to find the Noether current and consequently its charge. Let's consider a transformation $R \in \mathrm{SO}(2)$, then $R^T = R^{-1}$ and $\det R = 1$:

$$\vec{\psi} \to \vec{\psi}' = R\vec{\psi}.$$

R is a continuous global symmetry for our Lagrangian, then let us study the infinitesimal transformation:

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \sim \begin{pmatrix} 1 & \theta \\ -\theta & 1 \end{pmatrix},$$

so that the we can compute the infinitesimal variation of the fields for the expression of the Noether current. Let us begin from the transformed field

$$\vec{\psi}' = \begin{pmatrix} \psi_1' \\ \psi_2' \end{pmatrix} = \begin{pmatrix} 1 & \theta \\ -\theta & 1 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \implies \begin{cases} \psi_1' = \psi_1 + \theta\psi_2; \\ \psi_2' = \psi_2 - \theta\psi_1. \end{cases}$$

from which is now easy to compute the variation of the fields under the infinitesimal transformation

$$\begin{cases} \delta\psi_1 = \psi_1' - \psi_1 = \theta\psi_2; \\ \delta\psi_2 = \psi_2' - \psi_2 = -\theta\psi_1. \end{cases}$$

Now we can compute the expression for the Noether current by recalling its definition in (3.2.4):

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_1)}\delta\psi_1 + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_2)}\delta\psi_2 = K^\mu = 0,$$

[1] We drop for now the index i on the energy $\omega_i, \mathbf{p}$ since they are equal for both fields, but in the following refinements we will reintroduce them to avoid confusion.

where we have set $K^\mu = 0$ since the Lagrangian is exactely invariant under the transformation. Plugging in the variations of the fields we have just computed, we get:[2]

$$J^\mu = (\partial^\mu \psi_1)(\theta\psi_2) + (\partial^\mu \psi_2)(-\theta\psi_1) = \theta\left[(\partial^\mu \psi_1)\psi_2 - (\partial^\mu \psi_2)\psi_1\right].$$

Ignoring the constant θ (which can be absorbed in the definition of the current since it's just an overall normalization) we have finally found the expression for the Noether current:

$$J^\mu = (\partial^\mu \psi_1)\psi_2 - (\partial^\mu \psi_2)\psi_1.$$

Now we can compute the conserved charge associated to this current:

$$Q = \int \mathrm{d}^3\mathbf{x}\, J^0 = \int \mathrm{d}^3 x\left[(\partial^0 \psi_1)\psi_2 - (\partial^0 \psi_2)\psi_1\right].$$

We can recognize the canonical momenta $\pi_i = \partial^0 \psi_i$ in the expression above, thus we can rewrite the charge as

$$Q = \int \mathrm{d}^3\mathbf{x}\,[\pi_1\psi_2 - \pi_2\psi_1].$$

Now it is time to promote this quantity to an operator acting on the quantum fields: we substitute the classical fields with the corresponding expressions in terms of ladder operators:

$$\begin{aligned}
\hat{Q} &= \int \mathrm{d}^3\mathbf{x}\,[\hat{\pi}_1\hat{\psi}_2 - \hat{\pi}_2\hat{\psi}_1] \\
\hat{\pi}_i &= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}(-i)\sqrt{\frac{\omega_\mathbf{p}}{2}}\left[\hat{a}_{i,\,\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} - \hat{a}^\dagger_{i,\,\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}\right], \\
\hat{\psi}_i &= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{p}}}\left[\hat{a}_{i,\,\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} + \hat{a}^\dagger_{i,\,\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}\right].
\end{aligned}$$

computing a final expression for the charge operator, which has to be normal ordered:[3]

$$\begin{aligned}
\int \mathrm{d}^3\mathbf{x}\,\hat{\pi}_1\hat{\psi}_2 &= \frac{-i}{2}\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\sqrt{\frac{\omega_\mathbf{p}}{\omega_\mathbf{q}}}\Big[\delta^{(3)}(\mathbf{p}+\mathbf{q})\left(\hat{a}_{1,\,\mathbf{p}}\hat{a}_{2,\,\mathbf{q}} - \hat{a}^\dagger_{1,\,\mathbf{p}}\hat{a}^\dagger_{2,\,\mathbf{q}}\right) \\
&\quad+\delta^{(3)}(\mathbf{p}-\mathbf{q})\left(\hat{a}_{1,\,\mathbf{p}}\hat{a}^\dagger_{2,\,\mathbf{q}} - \hat{a}^\dagger_{1,\,\mathbf{p}}\hat{a}_{2,\,\mathbf{q}}\right)\Big] \\
&= \frac{-i}{2}\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\left[\hat{a}_{1,\,\mathbf{p}}\hat{a}_{2,\,-\mathbf{p}} - \hat{a}^\dagger_{1,\,\mathbf{p}}\hat{a}^\dagger_{2,\,-\mathbf{p}} + \hat{a}_{1,\,\mathbf{p}}\hat{a}^\dagger_{2,\,\mathbf{p}} - \hat{a}^\dagger_{1,\,\mathbf{p}}\hat{a}_{2,\,\mathbf{p}}\right],
\end{aligned}$$

where after an integration over $\mathrm{d}^3\mathbf{x}$ (in the charge expression) we found the delta function, which we can apply performing one of the momentum integrals, obtaining also $\omega_\mathbf{p} = \omega_\mathbf{q} = \omega_{-\mathbf{p}}$ since $m_1 = m_2 = m$. Similarly we can compute the second term, which is identical if we swap the indices $1 \leftrightarrow 2$ and subtracting it:[4]

$$\hat{Q} = i\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\left[\hat{a}^\dagger_{1,\,\mathbf{p}}\hat{a}_{2,\,\mathbf{p}} - \hat{a}^\dagger_{2,\,\mathbf{p}}\hat{a}_{1,\,\mathbf{p}}\right].$$

Note that there is an ambiguity: $\hat{Q}$ is an hermitian operator (good, physical quantity), but if it is conserved and we take $c_1\hat{Q} + c_2$ is also conserved:

$$\frac{\mathrm{d}}{\mathrm{d}t}(c_1\hat{Q} + c_2) = c_1\frac{\mathrm{d}}{\mathrm{d}t}\hat{Q} = 0,$$

[2]Remember that $\partial_\mu\vec{\psi}^T\partial^\mu\vec{\psi} = \partial_\mu\psi_1\partial^\mu\psi_1 + \partial_\mu\psi_2\partial^\mu\psi_2$, thus the derivative of the Lagrangian with respect to $\partial_\mu\psi_1$ is just $\partial^\mu\psi_1$ and similarly for ψ_2, after a brief computation which starts lowering all the partial derivatives indices (notice how the 1/2 factor has been canceled out during the derivative).

[3]We would require the index i on $\omega_\mathbf{p}$ since in general the we are considering different energies associated to different fields, but for now this is not a problem (since $m_1 = m_2 = m$).

[4]Note that $\hat{a}_1$ and $\hat{a}_2$ with or without daggers commute since they act on different fields: the term $\propto \delta^{(3)}(\mathbf{p}-\mathbf{q})$ doubles itself, while the term $\propto \delta^{(3)}(\mathbf{p}+\mathbf{q})$ cancels out after making the change of variable $\mathbf{p} \to -\mathbf{p}$ in the second integral.

so c_1 controls *units* in which Q is measured (not so crucial, like a renormalization factor, but influence the $\hat{Q}$ eigenvalues: we could set it to a generic q and use it as eigenvalue); c_2 instead is just a constant which can be removed by normal ordering (similar to what we did for the Hamiltonian and the vacuum energy).

$$\begin{aligned}
&\hat{Q}\,|0\rangle = i\int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\left[\hat{a}^\dagger_{1,\,\mathbf{p}}\hat{a}_{2,\,\mathbf{p}} - \hat{a}^\dagger_{2,\,\mathbf{p}}\hat{a}_{1,\,\mathbf{p}}\right]|0\rangle = 0,\\
&\hat{\hat{Q}}\,|0\rangle = (\hat{Q}+c_2)\,|0\rangle = c_2\,|0\rangle\,, \iff \langle 0|\,\hat{\hat{Q}}\,|0\rangle = c_2\langle 0|0\rangle = c_2,\\
&:\hat{\hat{Q}}: = \hat{\hat{Q}} - \langle 0|\,\hat{\hat{Q}}\,|0\rangle = \hat{Q}.
\end{aligned}$$

Let us now determine the spectrum of the theory by exploiting the properties of the ladder operators: let us define:

$$\begin{aligned}
\hat{a}_{\pm,\,\mathbf{p}} &= \frac{1}{\sqrt{2}}\left(\hat{a}_{1,\,\mathbf{p}} \pm i\hat{a}_{2,\,\mathbf{p}}\right),\\
\hat{a}^\dagger_{\pm,\,\mathbf{p}} &= \frac{1}{\sqrt{2}}\left(\hat{a}^\dagger_{1,\,\mathbf{p}} \mp i\hat{a}^\dagger_{2,\,\mathbf{p}}\right).
\end{aligned}$$

It can be shown (full computation in appendix B) that the following commutators relate the new ladder operators to the charge operator:

$$\begin{aligned}
[\hat{Q},\,\hat{a}_{\pm,\,\mathbf{p}}] &= \mp\hat{a}_{\pm,\,\mathbf{p}},\\
[\hat{Q},\,\hat{a}^\dagger_{\pm,\,\mathbf{p}}] &= \pm\hat{a}^\dagger_{\pm,\,\mathbf{p}},
\end{aligned}$$

Let us now consider the state $|s\rangle$ with charge $q_s \longrightarrow \hat{Q}\,|s\rangle = q_s\,|s\rangle$, then:

$$\hat{Q}(\hat{a}^\dagger_{\pm,\,\mathbf{p}}\,|s\rangle) = [\hat{Q},\,\hat{a}^\dagger_{\pm,\,\mathbf{p}}]\,|s\rangle + \hat{a}^\dagger_{\pm,\,\mathbf{p}}\hat{Q}\,|s\rangle = (q_s \pm 1)(\hat{a}^\dagger_{\pm,\,\mathbf{p}}\,|s\rangle),$$

thus we recognize $\hat{a}^\dagger_{\pm,\,\mathbf{p}}$ to be a *ladder operator*[5] for $\hat{Q}$, since after its application onto $|s\rangle$ eiqenvalue of $\hat{Q}$, it went from the eigenspace of q_s to the eigenspace of $q_s \pm 1$; they are ladder operators even for $\hat{H}$ and $\hat{P}$ (linear combinations of $\hat{a}^\dagger_{1,\,\mathbf{p}}$ and $\hat{a}^\dagger_{2,\,\mathbf{p}}$ are ladder operators for $\hat{H}$ and $\hat{P}$).

Our goal is now to find *common eigenstates* of $\hat{H}$, $\hat{P}$ and $\hat{Q}$:

$$\begin{aligned}
&\hat{Q}\,|0\rangle = 0,\ \text{from normal ordering},\\
&\left|s_n^\pm\right\rangle = \prod_{i=1}^{n}\hat{a}^\dagger_{\pm,\,\mathbf{p}_i}\,|0\rangle\,,
\end{aligned}$$

so we have n particle states with positive ($|s_n^+\rangle$) or negative charge ($|s_n^-\rangle$), for a total charge of either $\pm nq$:[6]

$$\begin{aligned}
\hat{H}\left|s_n^\pm\right\rangle &= \left(\sum_{i=1}^{n}\omega_{\mathbf{p}_i}\right)\left|s_n^\pm\right\rangle,\\
\hat{P}\left|s_n^\pm\right\rangle &= \left(\sum_{i=1}^{n}\mathbf{p}_i\right)\left|s_n^\pm\right\rangle,\\
\hat{N}\left|s_n^\pm\right\rangle &= \pm n\left|s_n^\pm\right\rangle,\\
\hat{Q}\left|s_n^\pm\right\rangle &= \pm nq\left|s_n^\pm\right\rangle.
\end{aligned}$$

[5] In the more general sense of an operator that let you go up and down in eigenstates referring to a particular eigenvalue: usually we use the Hamiltonian ladder operators, in this case they are acting on charge eigenstates (but soon they will prove to act equally on Hamiltonian eigenstates).

[6] The computations for these secolar equations are tedious and not instructive.

This symmetry exists only for $m_1 = m_2 = m$, otherwise we have no internal symmetries and cannot find a Noether's charge. Thus a particle described by two real KG fields with the same mass can be characterized by 3 quantum numbers, i.e. the eigenvalues of three operators: energy, momentum and charge.

$\hat{Q}$ is the **electric charge** and it can describe *particles/antiparticles* with **positive** energy: $|s_n^{\pm}\rangle$ describes the sign of the charge: $|s_n^{+}\rangle$ for **particle** states and $|s_n^{-}\rangle$ for **antiparticle** states.

If we were, for example, to create a state with both particle and antiparticle, we would have:

$$\left(\hat{a}^\dagger_{+,\mathbf{p}}\hat{a}^\dagger_{-,\mathbf{p}}\right)|0\rangle = |S\rangle\,, \quad \hat{Q}\,|S\rangle = 0, \quad \hat{H}\,|S\rangle = 2\omega_{\mathbf{p}}\,|S\rangle\,,$$

so a state with particle and antiparticle with same momentum has zero total charge but nonzero energy.

Note that one real KG field can only describe a $Q = 0$ particle

$$Q = \int \mathrm{d}^3\mathbf{x}\,[\dot{\psi}_1\psi_2 - \dot{\psi}_2\psi_1] = 0, \text{ for } \psi_1 = \psi_2 = \psi,$$

if $\psi_1 = \psi_2 = \psi$, we need at least 2 degrees of freedom if we want to get a more complete description of a picture with particles and antiparticles with nonzero electric charge: **complex KG field**.

4.2.2 | Complex Klein Gordon Field

The initial Lagrangian can be rewritten also as follows

$$\mathcal{L} = \partial_\mu\psi^*\partial^\mu\psi - m^2\psi^*\psi,$$

with

$$\psi(x) = \frac{1}{\sqrt{2}}(\phi_1(x) + i\phi_2(x)), \quad \psi^*(x) = \frac{1}{\sqrt{2}}(\phi_1(x) - i\phi_2(x));$$

this is the **complex Klein Gordon field**, which can be seen as two real KG fields ϕ_1 and ϕ_2 combined into a single complex field. Obviously the equations of motion derived from this Lagrangian are the same KG equations for both ϕ_1 and ϕ_2, and one could prove that substituting back ϕ_1 and ϕ_2 into the Lagrangian we would recover the previous expression.

In this form, it is easy to recognize that Lagrangian is invariant under a global U(1) transformation:[7]

$$\psi(x) \xrightarrow[\mathrm{U}(1)]{\text{global}} \psi'(x) = e^{iq\theta}\psi(x),$$

where it's important that $\theta \neq \theta(x)$ so that the Lagrangian remains invariant (remember that $\mathcal{L}' = \mathcal{L} \implies K^\mu = 0$ when computing the Noether current). In fact, under this transformation we notice how $\partial^\mu\theta = 0$ is needed to recover the invariance:

$$\partial_\mu\psi(x) \xrightarrow[\mathrm{U}(1)]{\text{global}} \partial_\mu\psi'(x) = e^{iq\theta}\partial_\mu\psi(x) + iq(\partial_\mu\theta)\psi(x) = e^{iq\theta}\partial_\mu\psi(x),$$

hence $\mathcal{L} \xrightarrow{\mathrm{U}(1)} \mathcal{L}' = \mathcal{L}$ and we can now consider the infinitesimal transformation:

$$\psi(x) \to \psi'(x) = (1 + iq\theta)\psi(x) \implies \delta\psi(x) = iq\theta\psi(x);$$

[7] Note that the factor q in the exponential transformation is inserted only to get the correct charge normalization in the end.

similarly for $\psi^*(x)$; thus we can compute the infinitesimal variations of the fields:

$$\begin{cases} \delta\psi(x) = iq\theta\psi(x); \\ \delta\psi^*(x) = -iq\theta\psi^*(x). \end{cases}$$

Now we can compute the Noether current associated to this symmetry:

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^*)}\delta\psi^* = iq\theta[(\partial^\mu\psi^*)\psi - (\partial^\mu\psi)\psi^*],$$

where, as we expected, the current is very similar to the one we found for the two real KG fields, just rewritten in terms of ψ and ψ^*, and if we were to consider a real KG field ($\psi = \psi^*$) then we would have obtained zero current.

Finally, for the total charge (which is conserved) we can compute:

$$Q = \int \mathrm{d}^3\mathbf{x}\, J^0 = iq\int \mathrm{d}^3\mathbf{x}\,[(\partial^0\psi^*)\psi - (\partial^0\psi)\psi^*] = iq\int \mathrm{d}^3\mathbf{x}\,[\pi\psi - \pi^*\psi^*].$$

As we did before, we can promote this quantity to an operator acting on the quantum fields, by substituting the classical fields with the corresponding expressions in terms of ladder operators:

$$\hat{\psi}(x) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{p}}}\left[\hat{a}_{+,\,\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} + \hat{a}^\dagger_{-,\,\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}\right],$$
$$\hat{\psi}^*(x) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{p}}}\left[\hat{a}_{-,\,\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} + \hat{a}^\dagger_{+,\,\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}\right],$$

and since the new $\hat{a}_{\pm,\mathbf{p}}$ are ladder operators, they should satisfy the usual commutation relations among themselves, thus we can compute the final expression for the charge operator in the same way as before, obtaining (after normal ordering):

$$\hat{Q} = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} q\left[\hat{a}^\dagger_{+,\,\mathbf{p}}\hat{a}_{+,\,\mathbf{p}} - \hat{a}^\dagger_{-,\,\mathbf{p}}\hat{a}_{-,\,\mathbf{p}}\right].$$

This expression is very intuitive: the charge operator counts the number of particles created by $\hat{a}^\dagger_{+,\,\mathbf{p}}$ (particles with positive charge) minus the number of particles created by $\hat{a}^\dagger_{-,\,\mathbf{p}}$ (particles with negative charge, i.e. antiparticles):

$$\hat{Q} = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} q\left[\hat{N}_{+,\,\mathbf{p}} - \hat{N}_{-,\,\mathbf{p}}\right] = q\left(\hat{N}_+ - \hat{N}_-\right).$$

Since we are considering a free theory, $\hat{Q}$, $\hat{N}_+$, and $\hat{N}_-$ are separately conserved quantities, meaning they commute with the Hamiltonian and do not change with time. In interacting theories, however, only the total charge $\hat{Q}$ remains conserved, while the individual particle numbers $\hat{N}_+$ and $\hat{N}_-$ may vary due to particle-antiparticle creation and annihilation processes: interactions can create or destroy particle-antiparticle pairs, but the net charge remains constant.

4.3 | Heisenberg Picture and Propagators

Is our quantum theory Lorentz invariant? Does it preserve causality? These questions arise because when we quantized the KG field, we imposed commutation relations at equal times:

$$[\hat{\psi}(\mathbf{x},t),\,\hat{\pi}(\mathbf{y},t)] = i\delta^{(3)}(\mathbf{x}-\mathbf{y}),$$

which are not manifestly Lorentz invariant since they treat time and space differently. We were working in the Schrödinger picture, where states evolve in time

$$i\frac{\mathrm{d}}{\mathrm{d}t}\left|\mathbf{p}(t)\right\rangle = \hat{H}\left|\mathbf{p}(t)\right\rangle, \quad \left|\mathbf{p}(t)\right\rangle = e^{-i\hat{H}t}\left|\mathbf{p}(0)\right\rangle = e^{-i\omega_{\mathbf{p}}t}\left|\mathbf{p}(0)\right\rangle,$$

while operators remain constant, thus our field operators as well as the all the observables are defined at a fixed time t

$$\begin{aligned}\hat{\psi} &= \hat{\psi}(\mathbf{x}),\\ \hat{\pi} &= \hat{\pi}(\mathbf{x}).\end{aligned}$$

To answer these questions, we need to move to the Heisenberg picture, where operators evolve in time while states remain constant. In this picture, the time evolution of an operator $\hat{O}_H$ is given by:[8]

$$\hat{O}_H(t) = e^{i\hat{H}t}\hat{O}_S e^{-i\hat{H}t},$$

and the time derivative of the operator is given by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{O}_H(t) = i\hat{H}e^{i\hat{H}t}\hat{O}_S e^{-i\hat{H}t} + e^{i\hat{H}t}\hat{O}_S e^{-i\hat{H}t}(-i\hat{H}) = -i[\hat{O}_H(t),\,\hat{H}].$$

Thus if in the Schrödinger picture we had canonical commutation relations (computed at the same istant by definition of time independent operators)

$$\begin{cases}\left[\hat{\psi}(\mathbf{x}),\,\hat{\psi}(\mathbf{y})\right] = [\hat{\pi}(\mathbf{x}),\,\hat{\pi}(\mathbf{y})] = 0,\\ \left[\hat{\psi}(\mathbf{x}),\,\hat{\pi}(\mathbf{y})\right] = i\delta^{(3)}(\mathbf{x}-\mathbf{y}),\end{cases}$$

in the Heisenberg picture we have **equal-time canonical commutation relations**:

$$\begin{cases}\left[\hat{\psi}(\mathbf{x},t),\,\hat{\psi}(\mathbf{y},t)\right] = [\hat{\pi}(\mathbf{x},t),\,\hat{\pi}(\mathbf{y},t)] = 0,\\ \left[\hat{\psi}(\mathbf{x},t),\,\hat{\pi}(\mathbf{y},t)\right] = i\delta^{(3)}(\mathbf{x}-\mathbf{y}).\end{cases}$$

When two events are separated by a spacelike interval (outside each other's light cones), there exists a reference frame in which they occur at the same time. Therefore, to ensure causality in our quantum field theory, we require that operators corresponding to measurements at spacelike-separated points commute. This means that measurements performed at these points cannot influence each other (they can be measured with arbitrary precision independently at the same time), preserving the principle of causality.

$\hat{\psi}(t,\mathbf{x})$ and $\hat{\psi}(t,\mathbf{y})$ are certainly spacelike separated if $\mathbf{x}\neq\mathbf{y}$ at equal times, since they are describing events that occur at different spatial locations simultaneously.

[8] The pedices indicates if we are referring to the Schrödinger (S) or Heisenberg (H) picture.

KG in Heisemberg picture. We want now to see how the field operators evolve in time in the Heisenberg picture:[9]

(?) Why partial derivative wrt time? Shouldn't it be total derivative? (I wrote total, prof notes say partial.)

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\hat{\psi}(\mathbf{x},t) = i\left[\hat{H}(t),\, \hat{\psi}(\mathbf{x},t)\right] &= \frac{i}{2}\left[\int \mathrm{d}^3\mathbf{y}\, \left(\hat{\pi}^2(\mathbf{y},t) + |\nabla_{\mathbf{y}}\hat{\psi}(\mathbf{y},t)|^2 + m^2\hat{\psi}^2(\mathbf{y},t)\right),\, \hat{\psi}(\mathbf{x},t)\right] \\
&= \frac{i}{2}\int \mathrm{d}^3\mathbf{y}\, \left\{\left[\hat{\pi}^2(\mathbf{y},t),\, \hat{\psi}(\mathbf{x},t)\right] + \left[|\nabla_{\mathbf{y}}\hat{\psi}(\mathbf{y},t)|^2,\, \hat{\psi}(\mathbf{x},t)\right] + m^2\left[\hat{\psi}^2(\mathbf{y},t),\, \hat{\psi}(\mathbf{x},t)\right]\right\} \\
&\qquad \left[\hat{\pi}^2(\mathbf{y},t),\, \hat{\psi}(\mathbf{x},t)\right] = 2\hat{\pi}(\mathbf{y},t)[\hat{\pi}(\mathbf{y},t),\, \hat{\psi}(\mathbf{x},t)] = -2i\delta^{(3)}(\mathbf{x}-\mathbf{y})\hat{\pi}(\mathbf{y},t), \\
&\qquad \left[|\nabla_{\mathbf{y}}\hat{\psi}(\mathbf{y},t)|^2,\, \hat{\psi}(\mathbf{x},t)\right] = 2\nabla_{\mathbf{y}}\hat{\psi}(\mathbf{y},t)\nabla_{\mathbf{y}}\left[\hat{\psi}(\mathbf{y},t),\, \hat{\psi}(\mathbf{x},t)\right] = 0, \\
&\qquad \left[\hat{\psi}^2(\mathbf{y},t),\, \hat{\psi}(\mathbf{x},t)\right] = 2\hat{\psi}(\mathbf{y},t)\left[\hat{\psi}(\mathbf{y},t),\, \hat{\psi}(\mathbf{x},t)\right] = 0, \\
&= \frac{i}{2}\int \mathrm{d}^3\mathbf{y}\, (-2i)\delta^{(3)}(\mathbf{x}-\mathbf{y})\hat{\pi}(\mathbf{y},t) = \int \mathrm{d}^3\mathbf{y}\, \delta^{(3)}(\mathbf{x}-\mathbf{y})\hat{\pi}(\mathbf{y},t) = \hat{\pi}(\mathbf{x},t).
\end{aligned}
$$

where we have used the commutator of a product expression $[AB,\, C] = A[B,\, C] + [A,\, C]B$ and we were able to take the spatial derivative out of the commutator since it was cating on functions of $\mathbf{y}$ and not of $\mathbf{x}$. We found that $\frac{\mathrm{d}}{\mathrm{d}t}\hat{\psi}(\mathbf{x},t) = \hat{\pi}(\mathbf{x},t)$ as expected, and similarly we can compute the time derivative of $\hat{\pi}(\mathbf{x},t)$:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\hat{\pi}(\mathbf{x},t) = i[\hat{H}(t),\, \hat{\pi}(\mathbf{x},t)] &= \frac{i}{2}\left[\int \mathrm{d}^3\mathbf{y}\, \left(\hat{\pi}^2(\mathbf{y},t) + |\nabla_{\mathbf{y}}\hat{\psi}(\mathbf{y},t)|^2 + m^2\hat{\psi}^2(\mathbf{y},t)\right),\, \hat{\pi}(\mathbf{x},t)\right] \\
&= \frac{i}{2}\int \mathrm{d}^3\mathbf{y}\, \left\{\left[\hat{\pi}^2(\mathbf{y},t),\, \hat{\pi}(\mathbf{x},t)\right] + \left[|\nabla_{\mathbf{y}}\hat{\psi}(\mathbf{y},t)|^2,\, \hat{\pi}(\mathbf{x},t)\right] + m^2\left[\hat{\psi}^2(\mathbf{y},t),\, \hat{\pi}(\mathbf{x},t)\right]\right\} \\
&\qquad \left[\hat{\pi}^2(\mathbf{y},t),\, \hat{\pi}(\mathbf{x},t)\right] = 2\hat{\pi}(\mathbf{y},t)\,[\hat{\pi}(\mathbf{y},t),\, \hat{\pi}(\mathbf{x},t)] = 0, \\
&\qquad \left[|\nabla_{\mathbf{y}}\hat{\psi}(\mathbf{y},t)|^2,\, \hat{\pi}(\mathbf{x},t)\right] = 2\nabla_{\mathbf{y}}\hat{\psi}(\mathbf{y},t)\nabla_{\mathbf{y}}\left[\hat{\psi}(\mathbf{y},t),\, \hat{\pi}(\mathbf{x},t)\right] = 2i\nabla_{\mathbf{y}}\hat{\psi}(\mathbf{y},t)\nabla_{\mathbf{y}}\delta^{(3)}(\mathbf{x}-\mathbf{y}), \\
&\qquad \left[\hat{\psi}^2(\mathbf{y},t),\, \hat{\pi}(\mathbf{x},t)\right] = 2\hat{\psi}(\mathbf{y},t)\left[\hat{\psi}(\mathbf{y},t),\, \hat{\pi}(\mathbf{x},t)\right] = 2i\hat{\psi}(\mathbf{y},t)\delta^{(3)}(\mathbf{x}-\mathbf{y}), \\
&= \frac{i}{2}\int \mathrm{d}^3\mathbf{y}\, \left\{2i\nabla_{\mathbf{y}}\hat{\psi}(\mathbf{y},t)\nabla_{\mathbf{y}}\delta^{(3)}(\mathbf{x}-\mathbf{y}) + 2im^2\hat{\psi}(\mathbf{y},t)\delta^{(3)}(\mathbf{x}-\mathbf{y})\right\} \\
&= -\int \mathrm{d}^3\mathbf{y}\, \left\{-\nabla^2_{\mathbf{y}}\hat{\psi}(\mathbf{y},t) + m^2\hat{\psi}(\mathbf{y},t)\right\}\delta^{(3)}(\mathbf{x}-\mathbf{y}) = \nabla^2_{\mathbf{x}}\hat{\psi}(\mathbf{x},t) - m^2\hat{\psi}(\mathbf{x},t),
\end{aligned}
$$

where we find $\frac{\mathrm{d}}{\mathrm{d}t}\hat{\pi}(\mathbf{x},t) = (\nabla^2 - m^2)\hat{\psi}(\mathbf{x},t)$. Combining these two results, we find that the field operator $\hat{\psi}(\mathbf{x},t)$ satisfies the Klein-Gordon equation in the Heisenberg picture:

$$
(\partial_t^2 - \nabla^2 + m^2)\hat{\psi}(\mathbf{x},t) = 0, \quad \text{or equivalently} \quad (\partial_\mu\partial^\mu + m^2)\hat{\psi}(x) = 0.
$$

Thus the field operators in the Heisenberg picture evolve according to the same equations of motion (KG equation) as the Schrödinger picture operators, ensuring consistency between the two quantum descriptions.

Ladder operators in Heisemberg picture. Let us now write the field operator in the Heisenberg picture explicitly, by Fourier expanding it in terms of creation and annihilation operators: starting from the Schrödinger picture expression

$$
\hat{\psi}(\mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left[\hat{a}_{\mathbf{p}}e^{i\mathbf{p}\cdot\mathbf{x}} + \hat{a}^\dagger_{\mathbf{p}}e^{-i\mathbf{p}\cdot\mathbf{x}}\right],
$$

[9] Note that in the commutator we are using Heisenberg oprators, even if we do not know their expression yet; we can since the relations we will use are all well defined and applied at equal times, there will be no problems. The expression of the Hamiltonian comes from the Schrödinger picture Legendre transform of equation (3.1.5) (after quantization), and then promoted to the Heisenberg picture.

which is clearly time independent, due to the definition of the ladder operators. We can promote it to the Heisenberg picture by finding the time-dependent definition of the ladder operators:

$$(\hat{a}_{\mathbf{p}})_H = e^{i\hat{H}t}(\hat{a}_{\mathbf{p}})_S e^{-i\hat{H}t}, \quad (\hat{a}^\dagger_{\mathbf{p}})_H = e^{i\hat{H}t}(\hat{a}^\dagger_{\mathbf{p}})_S e^{-i\hat{H}t}.$$

If we recall the commutators between the ladder operators and the Hamiltonian, we can compute:[10]

$$[\hat{H},\, \hat{a}_{\mathbf{p}}] = -\omega_{\mathbf{p}}\hat{a}_{\mathbf{p}}, \quad [\hat{H},\, \hat{a}^\dagger_{\mathbf{p}}] = +\omega_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}},$$

thus if we apply the Hamiltonian multiple times we get:

$$\begin{aligned}
\hat{H}\hat{a}_{\mathbf{p}} &= [\hat{H},\, \hat{a}_{\mathbf{p}}] + \hat{a}_{\mathbf{p}}\hat{H} = \hat{a}_{\mathbf{p}}\left(\hat{H} - \omega_{\mathbf{p}}\right), \\
\hat{H}^2\hat{a}_{\mathbf{p}} &= \hat{H}\hat{a}_{\mathbf{p}}\left(\hat{H} - \omega_{\mathbf{p}}\right) = \hat{a}_{\mathbf{p}}\left(\hat{H} - \omega_{\mathbf{p}}\right)^2, \\
&\vdots \\
\hat{H}^n\hat{a}_{\mathbf{p}} &= \hat{a}_{\mathbf{p}}\left(\hat{H} - \omega_{\mathbf{p}}\right)^n.
\end{aligned}$$

Thus if we now expand the exponentials in the definition of the Heisenberg picture ladder operators, we find:

$$\begin{aligned}
e^{i\hat{H}t}\hat{a}_{\mathbf{p}} &= \sum_n \frac{(i\hat{H}t)^n}{n!}\hat{a}_{\mathbf{p}} = \sum_n \frac{(it)^n}{n!}\hat{H}^n\hat{a}_{\mathbf{p}} = \sum_n \frac{(it)^n}{n!}\hat{a}_{\mathbf{p}}\left(\hat{H} - \omega_{\mathbf{p}}\right)^n \\
&= \hat{a}_{\mathbf{p}}\sum_n \frac{(it)^n}{n!}(\hat{H} - \omega_{\mathbf{p}})^n = \hat{a}_{\mathbf{p}}e^{i(\hat{H}-\omega_{\mathbf{p}})t} \\
&= e^{-i\omega_{\mathbf{p}}t}\hat{a}_{\mathbf{p}}e^{i\hat{H}t}.
\end{aligned}$$

If we now plug this result back into the definition of the Heisenberg picture ladder operator, we find:

$$(\hat{a}_{\mathbf{p}})_H = e^{-i\omega_{\mathbf{p}}t}\hat{a}_{\mathbf{p}}e^{i\hat{H}t}e^{-i\hat{H}t} = e^{-i\omega_{\mathbf{p}}t}\hat{a}_{\mathbf{p}},$$

and similarly for the creation operator, so that

$$\begin{cases} (\hat{a}_{\mathbf{p}})_H = e^{-i\omega_{\mathbf{p}}t}\hat{a}_{\mathbf{p}}, \\ \left(\hat{a}^\dagger_{\mathbf{p}}\right)_H = e^{i\omega_{\mathbf{p}}t}\hat{a}^\dagger_{\mathbf{p}}. \end{cases} \tag{4.3.1}$$

Thus, substituting these expressions, the field operator in the Heisenberg picture is given by:

$$\hat{\psi}(\mathbf{x}, t) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left[\hat{a}_{\mathbf{p}}e^{-i\omega_{\mathbf{p}}t}e^{i\mathbf{p}\cdot\mathbf{x}} + \hat{a}^\dagger_{\mathbf{p}}e^{i\omega_{\mathbf{p}}t}e^{-i\mathbf{p}\cdot\mathbf{x}}\right].$$

But is this lorentz invariant? Yes, since we can notice how the measure can be treated as done in section 4.1.4 without complications and the contraction of the lorentz scalar $p_\mu x^\mu$ can be exploited to make the invariance explicit:

$$p_\mu x^\mu = p_0 x^0 - \mathbf{p}\cdot\mathbf{x} = \omega_{\mathbf{p}}t - \mathbf{p}\cdot\mathbf{x},$$

so that we can rewrite the field operator as:

$$\hat{\psi}(x) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left[\hat{a}_{\mathbf{p}}e^{-ip_\mu x^\mu} + \hat{a}^\dagger_{\mathbf{p}}e^{ip_\mu x^\mu}\right],$$

and similarly for $\hat{\pi}(x)$:

$$\hat{\pi}(x) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\left[\hat{a}_{\mathbf{p}}e^{-ip_\mu x^\mu} - \hat{a}^\dagger_{\mathbf{p}}e^{ip_\mu x^\mu}\right].$$

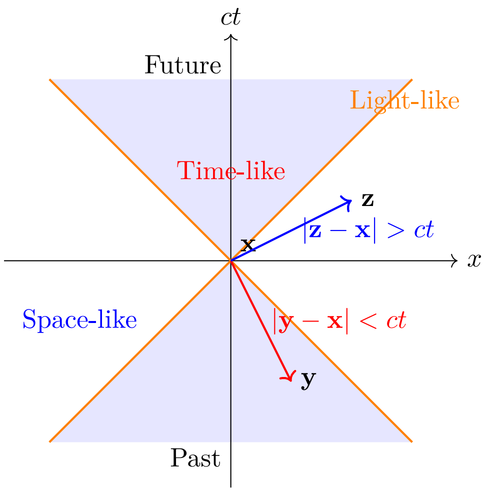
This expression is manifestly Lorentz invariant, as it depends only on the scalar product $p_\mu x^\mu$.

[10] Since $\hat{H}\left|0\right\rangle = 0 = \hat{H}\hat{a}_{\mathbf{p}}\left|\mathbf{p}\right\rangle = \left([\hat{H},\, \hat{a}_{\mathbf{p}}] + \hat{a}_{\mathbf{p}}\hat{H}\right)\left|\mathbf{p}\right\rangle$ leads to $[\hat{H},\, \hat{a}_{\mathbf{p}}]\left|\mathbf{p}\right\rangle = -\hat{a}_{\mathbf{p}}\hat{H}\left|\mathbf{p}\right\rangle = -\hat{a}_{\mathbf{p}}\omega_{\mathbf{p}}\left|\mathbf{p}\right\rangle$, and similarly for the creation operator $\hat{H}\hat{a}^\dagger_{\mathbf{p}}\left|0\right\rangle = \left([\hat{H},\, \hat{a}^\dagger_{\mathbf{p}}] + \hat{a}^\dagger_{\mathbf{p}}\hat{H}\right)\left|0\right\rangle$ leads to $[\hat{H},\, \hat{a}^\dagger_{\mathbf{p}}]\left|0\right\rangle = \hat{H}\hat{a}^\dagger_{\mathbf{p}}\left|0\right\rangle = \omega_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}}\left|0\right\rangle$.

4.3.1 | Causality

We are getting closer to a manifestly Lorentz invariant KG theory, but we still have to check if causality is preserved. The Heisemberg picture equal-time commutation relations ensure that operators at spacelike-separated points commute, preserving causality. However, we need to verify this explicitly for our field operators in generic time and space coordinates: we have to understand how observables at different spacetime points relate to each other with arbitrary time commutation relations.

We already know that at equal times the commutator is zero for different points:



$$\left[\hat{O}_1(\mathbf{x}, t),\, \hat{O}_2(\mathbf{z}, t)\right] = 0, \quad \mathbf{x} \neq \mathbf{z},$$

since it is a spacelike separation $(x_\mu - z_\mu)(x^\mu - z^\mu) < 0$: a measurement at $\mathbf{x}$ cannot influence a measurement at $\mathbf{z}$ if they are not causally connected.
If we recall the Heisenberg uncertainty principle, we know that two observables $\hat{O}_1$ and $\hat{O}_2$ can be measured simultaneously with arbitrary precision if they commute:

$$\Delta O_1 \Delta O_2 \geq \frac{1}{2} \left| \langle \left[\hat{O}_1,\, \hat{O}_2\right] \rangle \right|.$$

If two quantities commute, they can be measured simultaneously with arbitrary precision, meaning that a measurement of one does not disturb the other. This is crucial for preserving causality in quantum field theory, as it ensures that measurements at spacelike-separated points do not influence each other. We have to check if our theory respects this prescriptions.

To answer these questions, we have to compute the commutator between two field operators at different spacetime points: let us define the **Klein-Gordon propagator** as

$$\Delta(x - y) = \left[\hat{\psi}(x),\, \hat{\psi}(y)\right],$$

where $x = (x_0, \mathbf{x})$ and $y = (y_0, \mathbf{y})$ are two generic spacetime points. We have to check three things:

1. $\Delta(x - y)$ is Lorentz invariant;
2. $\Delta(x - y) \neq 0$ for $(x - y)^2 > 0$ (timelike separation);
3. $\Delta(x - y) = 0$ for $(x - y)^2 < 0$ (spacelike separation).

Let us first compute the propagator, which phisically describes the amplitude for a particle created at point y to be annihilated at point x:

$$\begin{aligned}\Delta(x-y) = \left[\hat{\psi}(x),\, \hat{\psi}(y)\right] &= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{q}}}} \\ &\times \left[\hat{a}_{\mathbf{p}}e^{-ip_\mu x^\mu} + \hat{a}^\dagger_{\mathbf{p}}e^{ip_\mu x^\mu},\, \hat{a}_{\mathbf{q}}e^{-iq_\mu y^\mu} + \hat{a}^\dagger_{\mathbf{q}}e^{iq_\mu y^\mu}\right] \\ &= \int \frac{\mathrm{d}^3\mathbf{p}\mathrm{d}^3\mathbf{q}}{(2\pi)^6}\frac{1}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}}\left\{[\hat{a}_{\mathbf{p}},\, \hat{a}^\dagger_{\mathbf{q}}]\left(e^{-i(p_\mu x^\mu - q_\mu y^\mu)}\right) + [\hat{a}^\dagger_{\mathbf{p}},\, \hat{a}_{\mathbf{q}}]\left(e^{i(p_\mu x^\mu - q_\mu y^\mu)}\right)\right\} \\ &= \int \frac{\mathrm{d}^3\mathbf{p}\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\frac{1}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}}\delta^{(3)}(\mathbf{p}-\mathbf{q})\left(e^{-i(p_\mu x^\mu - q_\mu y^\mu)} - e^{i(p_\mu x^\mu - q_\mu y^\mu)}\right),\end{aligned}$$

thus integrating over $\mathbf{q}$ we get the final expression for the KG propagator:

$$\Delta(x-y) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\left(e^{-ip_\mu(x^\mu - y^\mu)} - e^{ip_\mu(x^\mu-y^\mu)}\right), \tag{4.3.2}$$

Now we can check the three considerations:

1. **Lorentz invariance**: the exponentials $e^{\pm ip_\mu(x^\mu-y^\mu)}$ are manifestly Lorentz invariant since they depend on the contraction of two four-vectors; the measure $\mathrm{d}^3\mathbf{p}/(2\pi)^3 2E_{\mathbf{p}}$ is Lorentz invariant, as we have previously shown in section 4.1.4; thus the whole expression is Lorentz invariant. This means that if we compute $\Delta(x-y)$ in a reference frame where x and y are simultaneous, the result will hold for any other spacetime events which is related to y by a Lorentz transformation. But since we cannot relate events inside and outside the light cone with a Lorentz transformation, we can chack that the commutator vanishes for only one value of y outside of x light-cone, then it will be true for events throughout the whole region.

2. **Timelike separation**: $\Delta(x-y) \neq 0$. Let us consider the case where x and y are separated only in time, i.e. $\mathbf{x} = \mathbf{y}$ and $t_x \neq t_y$, thus $(x-y)^2 > 0$. We want to prove, in the easiest case where $x = (t,\, \mathbf{0})$ and $y = (0, \mathbf{0})$, that the commutator do not vanish:

$$\Delta(x-y) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\left(e^{-iE_{\mathbf{p}}t} - e^{iE_{\mathbf{p}}t}\right).$$

If we move to spherical coordinates in momentum space, we get:

$$\Delta(x-y) = \frac{1}{4\pi^2}\int_0^\infty \mathrm{d}|\mathbf{p}|\frac{|\mathbf{p}|^2}{\sqrt{|\mathbf{p}|^2+m^2}}\left(e^{-i\sqrt{|\mathbf{p}|^2+m^2}t} - e^{i\sqrt{|\mathbf{p}|^2+m^2}t}\right).$$

Now we can perform a change of variable from $|\mathbf{p}|$ to $E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2+m^2}$, so that $\mathrm{d}|\mathbf{p}| = \frac{E_{\mathbf{p}}}{\sqrt{E^2_{\mathbf{p}}-m^2}}\mathrm{d}E_{\mathbf{p}}$, thus we get:

$$\mathrm{d}|\mathbf{p}||\mathbf{p}|^2 = E_{\mathbf{p}}\sqrt{E^2_{\mathbf{p}}-m^2}\mathrm{d}E_{\mathbf{p}},$$

and the integral becomes

$$\Delta(x-y) = \frac{1}{4\pi^2}\int_m^\infty \mathrm{d}E_{\mathbf{p}}\sqrt{E^2_{\mathbf{p}}-m^2}\left(e^{-iE_{\mathbf{p}}t} - e^{iE_{\mathbf{p}}t}\right).$$

The solution of this integral can be computed through *Bessel functions*, such that the two contributions can be computed separately:

$$\frac{1}{4\pi^2}\int \mathrm{d}E_{\mathbf{p}}\sqrt{E^2_{\mathbf{p}}-m^2}e^{-iE_{\mathbf{p}}t} = \frac{m}{8\pi t}\left(Y_1(mt) + iJ_1(mt)\right)$$

where Y_1 is a Bessel function of the II kind while J_1 is of the I kind. Now let's take a look at the asymptotic behaviour:

$$J_1(x) \xrightarrow{x\to\infty} \sqrt{\frac{2}{\pi x}}\cos x,$$
$$Y_1(x) \xrightarrow{x\to\infty} \sqrt{\frac{2}{\pi x}}\sin x.$$

thus when we take $t \to \infty$ we get something like $\sin x + i\cos x$[11] so we get

$$Y_1(mt) + iJ_1(mt) \xrightarrow{t\to\infty} i\sqrt{\frac{2}{\pi mt}}e^{-imt},$$

so that

$$\Delta(x-y) = \frac{1}{4\pi^2}\int_m^\infty \mathrm{d}E_{\mathbf{p}}\sqrt{E_{\mathbf{p}}^2 - m^2}\left(e^{-iE_{\mathbf{p}}t} - e^{iE_{\mathbf{p}}t}\right) \xrightarrow{t\to\infty} \left(e^{-imt} - e^{imt}\right) \neq 0.$$

3. **Spacelike separation**: $\Delta(x-y) = 0$. Let us now consider the case where x and y are separated only in space, i.e. $t_x = t_y$ and $\mathbf{x} \neq \mathbf{y}$, thus $(x-y)^2 < 0$. We want to prove, in the easiest case where $x = (t, \mathbf{x})$ and $y = (t, \mathbf{y})$, that the commutator vanishes:

$$\begin{aligned}\Delta(x-y) &= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{p}}}\left(e^{-ip_\mu(x^\mu - y^\mu)} - e^{ip_\mu(x^\mu - y^\mu)}\right)\\ &= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{p}}}e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} - \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{p}}}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\\ &= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{p}}}\left(e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} - e^{i(-\mathbf{p})\cdot(\mathbf{x}-\mathbf{y})}\right) = 0,\end{aligned}$$

since $x_0 = y_0 = t$, only the space variables survive in the computation, and the sum of the last two integrals (performing a change of variable from $\mathbf{p}$ to $-\mathbf{p}$ on the second integral) we are left with two identical terms subtracted to each other, so we get 0. Since this guy is Lorentz invariant, we found this result true for any $(x-y)^2 = -(x-y)^2 < 0$, since the time can be different in x and y, but we are speaking of spacelike vectors (as we said, we can generalize via Lorentz transformations).

We have proven that KG theory mathematically preserve causality (remember that in RQMPI we looked at how there was a non zero probability of finding a particle propagating at $v > c$), let's check this instance phisically: the probability of finding a particle at $v > c$ should be null.

4.3.2 | Klein Gordon Correlators

Let us consider particle at a spacetime point $y = (t, \mathbf{y})$; how do we describe it in QFT? It is an exhited state of the vacuum $|0\rangle$, i.e. there is some kind of exhitation localized in our quantum field at the point y, like a localized and quantized wave; we recreate it with a creation operator, like $\hat{a}_{\mathbf{p}}|0\rangle$, but if we create a particle with definite momentum we could not possibly now its position, so we imagine a linear combination of all possible creation operators with all possible momenta: we apply the **field operator** on the vacuum

$$\hat{\psi}(y)|0\rangle .$$

Then let's look at the probability amplitude (with a scalar product) to find it in $x = (t, \mathbf{x})$:

$$D(x-y) = \langle 0|\hat{\psi}(x)\hat{\psi}(y)|0\rangle$$

[11]computed with exponentials it becomes $\sin x + i\cos x = \frac{1}{2}(-i(e^{ix} - e^{-ix}) + i(e^{ix} + e^{-ix})) = ie^{-ix}$.

Note that it is different from RQM, where we computed $\langle \mathbf{x}| e^{-i\hat{H}t} |\mathbf{y}\rangle$ with $\hat{H} = \sqrt{|\mathbf{p}|^2 + m^2}$. Lets compute the amplitude:

$$
\begin{aligned}
D(x-y) &= \langle 0| \hat{\psi}(x)\hat{\psi}(y) |0\rangle = \left(\hat{\psi}(x) |0\rangle\right)^\dagger \hat{\psi}(y) |0\rangle \qquad \left[\text{ KG scalar field: } \hat{\psi}^\dagger(x) = \hat{\psi}(x)\right], \\
&= \int \frac{\mathrm{d}^3\mathbf{p}\mathrm{d}^3\mathbf{q}}{(2\pi)^6} \frac{1}{2\sqrt{\omega_\mathbf{p}\omega_\mathbf{q}}} \langle 0| \left(\hat{a}_\mathbf{p} e^{-ip_\mu x^\mu} + \hat{a}^\dagger_\mathbf{p} e^{ip_\mu x^\mu}\right) \left(\hat{a}_\mathbf{q} e^{-iq_\mu y^\mu} + \hat{a}^\dagger_\mathbf{q} e^{iq_\mu y^\mu}\right) |0\rangle \\
& \quad \langle 0| \hat{a}_\mathbf{p}\hat{a}_\mathbf{q} |0\rangle = 0, \quad \langle 0| \hat{a}_\mathbf{p}\hat{a}^\dagger_\mathbf{q} |0\rangle \neq 0, \quad \langle 0| \hat{a}^\dagger_\mathbf{p}\hat{a}_\mathbf{q} |0\rangle = 0, \quad \langle 0| \hat{a}^\dagger_\mathbf{p}\hat{a}^\dagger_\mathbf{q} |0\rangle = 0, \\
&= \int \frac{\mathrm{d}^3\mathbf{p}\mathrm{d}^3\mathbf{q}}{(2\pi)^6} \frac{1}{2\sqrt{\omega_\mathbf{p}\omega_\mathbf{q}}} \langle 0| \hat{a}_\mathbf{p}\hat{a}^\dagger_\mathbf{q} |0\rangle e^{-ip_\mu x^\mu} e^{iq_\mu y^\mu} \\
&= \int \frac{\mathrm{d}^3\mathbf{p}\mathrm{d}^3\mathbf{q}}{(2\pi)^6} \frac{1}{2\sqrt{\omega_\mathbf{p}\omega_\mathbf{q}}} \left(\langle 0| \left[\hat{a}_\mathbf{p}, \hat{a}^\dagger_\mathbf{q}\right] |0\rangle + \langle 0| \hat{a}^\dagger_\mathbf{q}\hat{a}_\mathbf{p} |0\rangle\right) e^{-ip_\mu x^\mu} e^{iq_\mu y^\mu} \\
&= \int \frac{\mathrm{d}^3\mathbf{p}\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \frac{1}{2\sqrt{\omega_\mathbf{p}\omega_\mathbf{q}}} \delta^{(3)}(\mathbf{p}-\mathbf{q}) e^{-ip_\mu x^\mu} e^{iq_\mu y^\mu} = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{2\omega_\mathbf{p}} e^{-ip_\mu(x^\mu - y^\mu)},
\end{aligned}
$$

where we have used the fact that $\hat{a}_\mathbf{q}$ acting on the vacuum $|0\rangle$ is zero, which is the same to the application of $\hat{a}^\dagger_\mathbf{q}$ on $\langle 0|$ (it is only the conjugate of the previous expression); furthermore, when we see a product of operators, the idea is to use commutators $\hat{a}_\mathbf{p}\hat{a}^\dagger_\mathbf{q} = \left[\hat{a}_\mathbf{p}\, , \, \hat{a}^\dagger_\mathbf{q}\right] + \hat{a}^\dagger_\mathbf{q}\hat{a}_\mathbf{p}$ to simplify the expression, since the commutator is known while the second term acting on the vacuum is zero.

Thus we have found that this amplitude, the KG correlator, is proportional to the KG propagator in (4.3.2) (since the square modulus of this amplitude gives us the probability of the propagation of the particle from $\mathbf{y}$ to $\mathbf{x}$, should this be called propagator? No, and we will see why at the end of thus section), and is given by:

$$
D(x-y) = \langle 0| \hat{\psi}(x)\hat{\psi}(y) |0\rangle = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{2\omega_\mathbf{p}} e^{-ip_\mu(x^\mu - y^\mu)}, \tag{4.3.3}
$$

which, as we have seen before, is Lorentz invariant.

If we now evaluate this amplitude for propagations outside the light cone (among spacelike separated events x and y, happening at the same time in different places: $x - y = (0, \mathbf{r})$, and $r = |\mathbf{r}|$) it should be zero to preserve causality: using spherical coordinates in momentum space with the polar axis along $\mathbf{r}$ we have

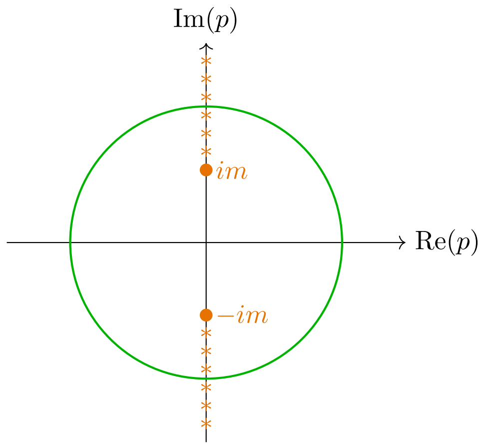
$$
\begin{aligned}
D(x-y) &= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{2\omega_\mathbf{p}} e^{-ip_\mu(x^\mu - y^\mu)} = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{2\omega_\mathbf{p}} e^{-i\mathbf{p}\cdot\mathbf{r}} \\
&= \frac{2\pi}{(2\pi)^3} \int_0^\infty \mathrm{d}|\mathbf{p}| \frac{|\mathbf{p}|^2}{2E_\mathbf{p}} \int_{-1}^1 \mathrm{d}(\cos\theta) e^{-i|\mathbf{p}|r\cos\theta},
\end{aligned}
$$

since $\mathrm{d}^3\mathbf{p} = 2\pi|\mathbf{p}|^2\mathrm{d}|\mathbf{p}|\mathrm{d}(\cos\theta)\mathrm{d}\phi$, and by computing the integral in $\mathrm{d}(\cos\theta)$ we get to

$$
\begin{aligned}
D(x-y) &= \frac{-1}{(2\pi)^2} \int_0^\infty \mathrm{d}|\mathbf{p}| \frac{|\mathbf{p}|^2}{2E_\mathbf{p}} \left[\frac{e^{i|\mathbf{p}|r} - e^{-i|\mathbf{p}|r}}{-i|\mathbf{p}|r}\right] \\
&= -\frac{i}{2(2\pi)^2 r} \left(\int_0^\infty \mathrm{d}|\mathbf{p}| \frac{|\mathbf{p}|}{E_\mathbf{p}} e^{i\mathbf{p}r} - \int_0^\infty \mathrm{d}|\mathbf{p}| \frac{|\mathbf{p}|}{E_\mathbf{p}} e^{-i\mathbf{p}r}\right),
\end{aligned}
$$

with $E_\mathbf{p} = \sqrt{|\mathbf{p}|^2 + m^2}$. Notice how the two integrals are identical except for the sign in the exponential, so we can combine them into a single integral from $-\infty$ to ∞ of a single exponential term:

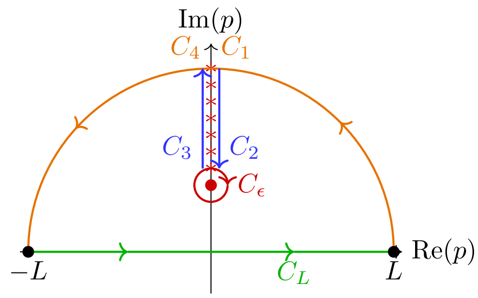
$$
D(x-y) = -\frac{i}{2(2\pi)^2 r} \left(\int_{-\infty}^\infty \mathrm{d}|\mathbf{p}| \frac{|\mathbf{p}|}{E_\mathbf{p}} e^{i\mathbf{p}r}\right).
$$



The idea is now to promote $|\mathbf{p}|$ to a complex variable and perform the integral with Cauchy theorem; the branch cuts are as shown in figure, and we recognize them to be the same as we have seen in section 1.1.2, having a complex square root. We want to get the integral on the real axis, so we compute the path as an integral from $-L$ to L on $\mathbb{R}$ and then we take $L \to \infty$, then we look at the contributions from the rest of the closed path (excluding the singularities) and put their sum to zero.

We have then to compute the two integrals on the imaginary axis (which will prove identical), with $|\mathbf{p}| = iy$ (such that $|\mathbf{p}|\mathrm{d}|\mathbf{p}| = -y\mathrm{d}y$ and $\sqrt{-y^2+m^2} = \sqrt{-1}\sqrt{y^2-m^2} = i\sqrt{y^2-m^2}$, paying attention on the sign change of the branch cut), since all the other integrals on the path are null, such that

$$D(x-y) = \frac{1}{(2\pi)^2 r}\int_m^\infty \mathrm{d}y\, \frac{ye^{-yr}}{\sqrt{y^2-m^2}}.$$



This integral is real and do not present mathematical issues, but the exact solution given in terms of *modified Bessel functions*, in the limit for $r \sim \infty$ then $D(x-y) \sim e^{-mr} \neq 0$, is clearly different from zero. We can arrive at the same conclusion with approximated considerations: all the terms inside the integral are positive, so the integral cannot be zero; furthermore, the exponential term e^{-yr} suppresses the contributions for large y, so the main contribution comes from $y \sim m$, but they are nonetheless different from zero, so the whole integral is non zero.

This seems similar to what we obtained in RQM, but it is not: if we indeed consider the previous result

$$\Delta(x-y) = \left[\hat{\psi}(x),\, \hat{\psi}(y)\right] = 0$$

for spacelike separations, so we can write

$$\begin{aligned}\Delta(x-y) &= \langle 0|\, \Delta(x-y)\, |0\rangle = \langle 0| \left[\hat{\psi}(x),\, \hat{\psi}(y)\right] |0\rangle \\ &= \langle 0|\, \hat{\psi}(x)\hat{\psi}(y)\, |0\rangle - \langle 0|\, \hat{\psi}(y)\hat{\psi}(x)\, |0\rangle = D(x-y) - D(y-x) = 0;\end{aligned}$$

Note that KG theory cannot distinguish particles from antiparticles, since it does not account for charge degrees of freedom; but if we **consider the complex KG field** we can interpret this result as the *probability amplitude for a particle to go from y to x minus the probability of the antiparticle going from x to y*, which is an interpratation of antiparticles going backward in time to preserve the causality outside the light cone.

If we consider the expressions of the field operators for complex KG fields:

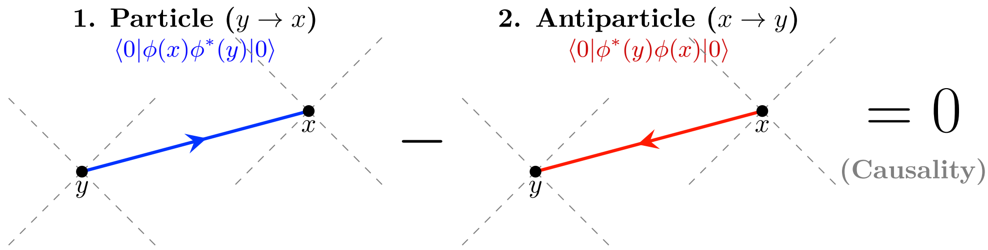
$$\begin{aligned}\hat{\psi}^*(y) &\propto \hat{a}_- + \hat{a}_+^\dagger, \\ \hat{\psi}(x) &\propto \hat{a}_+ + \hat{a}_-^\dagger,\end{aligned}$$

where $\hat{a}_+$ and $\hat{a}_+^\dagger$ are the annihilation and creation operators for particles, while $\hat{a}_-$ and $\hat{a}_-^\dagger$ are the annihilation and creation operators for antiparticles, then we can understand the two terms in the previous expression for the propagator outside the light cone:

$$\Delta(x-y) = D(x-y) - D(y-x) = [\psi(x),\, \psi^*(y)]\,;$$

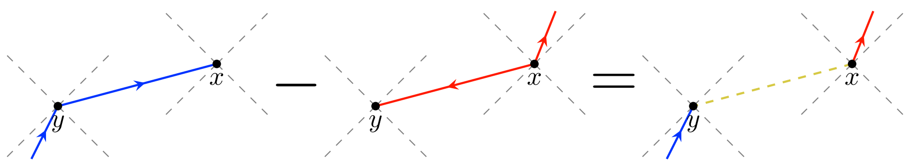
the first term $D(x-y)$ is the amplitude for a particle created in y to be annihilated in x, while the second term $D(y-x)$ is the amplitude for an antiparticle created in x to be annihilated in y, which we can read as an antiparticle propagating backwards in time to be annihilated with the first particle (this because if we apply consequently the operators we see that states of particle created in y and antiparticle created in x and viceversa are canceled in the final product with $\langle 0|$).

We do not need the amplitude of this event to be null, since we **cannot measure it** anyway: the propagator is the quantity we need to be zero, since it is proportional to the commutator of the fields, which has to vanish to preserve causality. We need **measurements at spacelike separated events to be independent**, not the individual amplitudes of particles and antiparticles propagating outside the light cone.



If we want a state in which we have a particle in the past and then in the future, we can also read this as a particle in y with a non zero probability of propagating outside its light cone, which is annihilated before this can happen from the creation of a couple of particle / antiparticle in x, where the new particle can propagate in the future while the antiparticle travels backwards in time to annihilate with the initial particle. We were computing just one half of the amplitude. All this comes from intrepretation of the mathematical results of our equations: we are looking at an individual phenomenon made up by two distinct contributions.

We will never measure the same particle propagating faster then light, but we could eventually measure a particle in y and then another identical particle in x outside the light cone of y, which is a new one created there, together with its antiparticle responsible for the annihilation of the first one before it could reach x. This physical intuition is mostly due to Feynman and his diagrams.



This kind of phenomena are called *virtual phenomena*, since they cannot be measured observed, but mathematic tells us that this happens and saves our theory; even in Feynman diagrams, the virtual particle does not respect an equation of motion, there is its existence for a really brief moment, and due to uncertainty in energy we can justify its existence and our computing methods and interpretations.

4.3.3 | Green Functions

4.3.4 | Feynman Propagator

5 Dirac theory

From now on, we aim to describe particles with spin different from zero. The simplest case is spin $\frac{1}{2}$ fermions, which are described by the Dirac equation. The starting point will be the Lagrangian, then we will show the most general solution to the equations of motion and finally proceed to quantize the system and promote the observables to operators on the Fock Space.

The lagrangian has to be Lorentz invariant, so we recall that the Lorentz group admits a $4D$ spinor representation, which for Dirac translates into

$$\psi_D \to \psi'_D = e^{-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}}\psi_D,$$

the **Dirac spinor**, where the 4×4 antisymmetric generators of the Lorentz group in the Dirac representation[1] are given by

$$\Sigma_{\mu\nu} = \frac{i}{4}\gamma^{\mu\nu}, \quad \gamma^{\mu\nu} = [\gamma^\mu, \gamma^\nu] = \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu,$$

where γ^μ are the **Dirac gamma matrices**, respecting the clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{I}_4, \tag{5.0.1}$$

of 4×4 complex matrices.

We adopt the **Weyl representation** (or chiral representation, eq. (2.3.15)) of the Dirac gamma matrices, which assumes the form

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$

where σ^i are the Pauli matrices (as given in eq. (2.3.9)):

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The generators of Lorentz transformations in the Dirac representation can be written also as

$$S^{\mu\nu} = -i\Sigma^{\mu\nu} = \frac{1}{4}\gamma^{\mu\nu}, \quad \psi'_D = e^{\frac{1}{2}\omega_{\mu\nu}S^{\mu\nu}}\psi_D.$$

If we introduce now the **spinorial indices**, we can write the transformation law as:

$$\psi^\alpha_D(x) \to \psi'^\alpha_D(x') = S^\alpha{}_\beta(\Lambda)\psi^\beta_D(x) = \left(e^{\frac{1}{2}\omega_{\mu\nu}S^{\mu\nu}}\right)^\alpha{}_\beta \psi^\beta_D(x), \tag{5.0.2}$$

where Λ is the Lorentz transformation, which also acted on the coordinates (they got transformed along with the Dirac spinor). From now on we will drop the subscript D for Dirac spinors, since we will only deal with them.

[1] $\Sigma^{\mu\nu}$ has only six independent components, corresponding to the six generators of the Lorentz group (as given by eq. (2.3.14)): it is antisymmetric in μ and ν and traceless, thus $(4 \times 4 - 4)/2 = 6$.

5.1 | Action and Lagrangian

Now the idea is to derive a Lorentz invariant Lagrangian dependent upon the Dirac spinor ψ, remembering that ψ has four complex components, so eight real degrees of freedom, and

$$\psi^\dagger(x) = (\psi^*)^T(x).$$

The lagrangian must be a Lorentz scalar, so we need to find building blocks which are lorentz scalars, vectors, tensors (whih will be contracted into scalars since the legrangian has to be one) built from the spinor ψ and its derivatives. We will consider different **spinor bilinears**, i.e. quantities built from two spinors, since a single spinor cannot build a Lorentz scalar alone.

5.1.1 | Building Block for the Mass Term

The simplest bilinear we can consider as a candidate for the mass term building block is

$$\psi^\dagger(x)\psi(x), \tag{5.1.1}$$

which is going to be a real number, but we have to check its transformation properties under Lorentz transformations to prove wether it is a Lorentz scalar.[2] We know that

$$\psi \to \psi' = S\psi, \quad \psi^\dagger \to \psi'^\dagger = \psi^\dagger S^\dagger,$$

hence their product

$$\psi^\dagger\psi \to \psi'^\dagger\psi' = \psi^\dagger S^\dagger S\psi,$$

which could be Lorentz scalar if and only if $S^\dagger = S^{-1}$, which is not the case, since the Dirac representation is not unitary:

$$S^\dagger S \neq \mathbb{I}_4 \implies \psi'^\dagger\psi' \neq \psi^\dagger\psi,$$

and therefore the spinor bilinear is not a good building block for our Lagrangian. Let's check explicitly why this representation $S^\dagger \neq S^{-1}$ is not unitary:

$$S = e^{\frac{1}{2}\omega_{\mu\nu}S^{\mu\nu}} \qquad \implies \begin{cases} S^{-1} & = e^{-\frac{1}{2}\omega_{\mu\nu}S^{\mu\nu}}; \\ S^\dagger & = e^{\frac{1}{2}\omega_{\mu\nu}(S^{\mu\nu})^\dagger}. \end{cases}$$

Thus for the representation to be unitary, we are requiring the generators $S^{\mu\nu}$ to be *anti-hermitian*

$$(S^{\mu\nu})^\dagger = -S^{\mu\nu} \implies (-i\Sigma^{\mu\nu})^\dagger = -(-i\Sigma^{\mu\nu});$$

in other words we are requiring the generators $\Sigma^{\mu\nu}$ to be hermitian

$$(\Sigma^{\mu\nu})^\dagger = \Sigma^{\mu\nu}.$$

But we can explicitly verify that this is never the case:

$$(S^{\mu\nu})^\dagger = \frac{1}{4}\left[\gamma^\mu, \gamma^\nu\right]^\dagger = \frac{1}{4}(\gamma^\nu)^\dagger(\gamma^\mu)^\dagger - \frac{1}{4}(\gamma^\mu)^\dagger(\gamma^\nu)^\dagger = -\frac{1}{4}\left[(\gamma^\mu)^\dagger, (\gamma^\nu)^\dagger\right],$$

which would be equal to $-S^{\mu\nu}$ only if

$$(S^{\mu\nu})^\dagger = -\frac{1}{4}\left[(\gamma^\mu)^\dagger, (\gamma^\nu)^\dagger\right] = -\frac{1}{4}\left[\gamma^\mu, \gamma^\nu\right] = -S^{\mu\nu}.$$

[2] By parallelism with the KG lagrangian we know that the mass term is made of Lorentz scalars built with fields, while the kinetic term (as we will soon see) should be made of two Lorentz vectors contracted into a scalar.

Thus the representation is unitary $\iff$ $(\gamma^\mu)^\dagger = \pm\gamma^\mu$, i.e. the gamma matrices are all hermitian or all anti-hermitian; we can check from the Clifford algebra (5.0.1) that this is not possible for all μ:

$$\{\gamma^\mu,\, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{I}_4 \implies (\gamma^0)^2 = \mathbb{I}_4 \quad (\gamma^i)^2 = -\mathbb{I}_4,$$

where we have computed the anticommutator for $\mu = \nu = 0$ and $\mu = \nu = i$. If we apply those squared matrices to a generic complex four-vector v, we get

$$(\gamma^0)^2 v = \lambda_0^2 v = v, \quad (\gamma^i)^2 v = \lambda_i^2 v = -v,$$

thus γ^0 has real eigenvalues $\lambda_0 = \pm 1$, while γ^i have imaginary eigenvalues $\lambda_i = \pm i$; hence γ^0 is hermitian, while γ^i are anti-hermitian. Therefore the gamma matrices are not all hermitian or anti-hermitian, so the generators $S^{\mu\nu}$ are not hermitian, and the Dirac representation is not unitary:

$$(\gamma^0)^\dagger = \gamma^0, \quad (\gamma^i)^\dagger = -\gamma^i \implies (S^{\mu\nu})^\dagger \neq -S^{\mu\nu} \implies S^\dagger \neq S^{-1}.$$

We then need to find another building block for our Lagrangian.

——

We can start from the following gamma matrices property: $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$ for all μ. Lets check it:

$$\begin{aligned} \mu = 0) \quad (\gamma^0)^\dagger &= \gamma^0\gamma^0\gamma^0 = \gamma^0 = (\gamma^0)^\dagger, \\ \mu = i) \quad (\gamma^i)^\dagger &= \gamma^0\gamma^i\gamma^0 = -\gamma^i = (\gamma^i)^\dagger, \end{aligned}$$

where in the first case we used the idempotency of γ^0 and its hermitianity, while in the second case we used the anticommutation relations among gamma matrices.[3] So this relation holds for all μ. Using this we can compute $(S^{\mu\nu})^\dagger$ as:

$$\begin{aligned} (S^{\mu\nu})^\dagger &= \frac{1}{4}\,[\gamma^\mu,\, \gamma^\nu]^\dagger = \frac{1}{4}(\gamma^\nu)^\dagger(\gamma^\mu)^\dagger - \frac{1}{4}(\gamma^\mu)^\dagger(\gamma^\nu)^\dagger \\ &= \frac{1}{4}\gamma^0\gamma^\nu\gamma^0\gamma^0\gamma^\mu\gamma^0 - \frac{1}{4}\gamma^0\gamma^\mu\gamma^0\gamma^0\gamma^\nu\gamma^0 \\ &= \gamma^0\left(\frac{1}{4}[\gamma^\nu,\, \gamma^\mu]\right)\gamma^0 = -\gamma^0\left(\frac{1}{4}[\gamma^\mu,\, \gamma^\nu]\right)\gamma^0 = -\gamma^0 S^{\mu\nu}\gamma^0, \end{aligned}$$

where we have expanded the commutator and simplified the $(\gamma^0)^2$. Since we are looking for the scalar building block for the lagrangian, we are lead to follow this result and compute $S^\dagger$:

$$S^\dagger = e^{\frac{1}{2}\omega_{\mu\nu}(S^{\mu\nu})^\dagger} = e^{-\frac{1}{2}\omega_{\mu\nu}\gamma^0 S^{\mu\nu}\gamma^0}.$$

Let's factor γ^0 out again, Taylor expanding the exponential:

$$\begin{aligned} S^\dagger &\sim \mathbb{I}_4 - \frac{1}{2}\omega_{\mu\nu}\gamma^0 S^{\mu\nu}\gamma^0 + \frac{1}{4}\omega_{\mu\nu}\omega_{\rho\sigma}\gamma^0 S^{\mu\nu}\gamma^0\gamma^0 S^{\rho\sigma}\gamma^0 - \frac{1}{8}\cdots \\ &= \gamma^0\gamma^0 - \frac{1}{2}\omega_{\mu\nu}\gamma^0 S^{\mu\nu}\gamma^0 + \frac{1}{4}\omega_{\mu\nu}\omega_{\rho\sigma}\gamma^0 S^{\mu\nu}\mathbb{I}_4 S^{\rho\sigma}\gamma^0 - \frac{1}{8}\cdots \\ &= \gamma^0\left(\mathbb{I}_4 - \frac{1}{2}\omega_{\mu\nu}S^{\mu\nu} + \frac{1}{4}\omega_{\mu\nu}\omega_{\rho\sigma}S^{\mu\nu}S^{\rho\sigma} - \frac{1}{8}\cdots\right)\gamma^0 \\ &= \gamma^0 e^{-\frac{1}{2}\omega_{\mu\nu}S^{\mu\nu}}\gamma^0 = \gamma^0 S^{-1}\gamma^0. \end{aligned}$$

Now we have definitive proof that the right expression for $S^\dagger$ is

$$S^\dagger = \gamma^0 S^{-1}\gamma^0,$$

[3] Indeed, computing it explicitly from the Clifford algebra we get $\{\gamma^i,\, \gamma^0\} = 2\eta^{i0} = 0 \implies \gamma^i\gamma^0 = -\gamma^0\gamma^i$.

since we already knew that $S^{-1} = e^{-\frac{1}{2}\omega_{\mu\nu}S^{\mu\nu}}$.

Thus finally we have insight on how to build a lorentz scalar from spinors: instead of $\psi^\dagger\psi$ we can consider the following bilinear

$$\overline{\psi}(x)\psi(x), \tag{5.1.2}$$

built using the **adjoint Dirac Spinor** $\overline{\psi}(x) = \psi^\dagger\gamma^0$ (more details in (2.3.18)). Let's check its transformation properties:

$$\begin{aligned}\overline{\psi}(x)\psi(x) \to \overline{\psi}'(x')\psi'(x') &= \psi'^\dagger(x')\gamma^0\psi'(x') = \psi^\dagger(x)S^\dagger\gamma^0 S\psi(x)\\ &= \psi^\dagger(x)\gamma^0 S^{-1}\gamma^0\gamma^0 S\psi(x) = \psi^\dagger(x)\gamma^0 S^{-1}S\psi(x)\\ &= \psi^\dagger(x)\gamma^0\psi(x) = \overline{\psi}(x)\psi(x).\end{aligned}$$

which is lorentz invariant indeed. Thus we could write the transormation of the adjoint spinor as

$$\overline{\psi} \to \overline{\psi}' = \psi^\dagger S^\dagger\gamma^0 = \psi^\dagger\gamma^0 S^{-1} = \overline{\psi}S^{-1}. \tag{5.1.3}$$

Thus $\overline{\psi}\psi$ is the lorentz scalar we will use to build our lagrangian, in particular for the mass term, where there is no derivative involved:

$$\mathcal{L}_{mass} = -m\overline{\psi}(x)\psi(x).$$

Recalling the expression for the KG Lagrangian

$$\mathcal{L}_{KG} = \frac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2\right),$$

we have indeed found the analog of the last term: $m\overline{\psi}(x)\psi(x)$; the first one instead, where the indices has to be contracted into a Lorentz scalar, involves derivatives. Thus we have to look for another spinor bilinear for the kinetic term, preserving Lorentz invariance with derivatives.

5.1.2 | Building Block for the Kinetic Term

Considering the following spinor bilinear

$$\overline{\psi}(x)\gamma^\mu\psi(x), \tag{5.1.4}$$

which is a vector of four components, we have to understand if that is a Lorentz vector or not, So that

$$\overline{\psi}'\gamma^\mu\psi' = \Lambda^\mu{}_\nu\overline{\psi}\gamma^\nu\psi.$$

If this is true, we can contract that μ indices in order to obtain a lorentz scalar (invariant)

$$\overline{\psi}\gamma^\mu\partial_\mu\psi.$$

So, summarizing, if we verify that $\overline{\psi}\gamma^\mu\psi$ transforms as a lorentz vector, we can use it to build the kinetic term of the lagrangian. Let's check how it transforms:

$$\overline{\psi}\gamma^\mu\psi \to \overline{\psi}'\gamma^\mu\psi' = \overline{\psi}S^{-1}\gamma^\mu S\psi,$$

where we have used the results of the previous section to write the expression for the adjoint Dirac spinor transformation of eq. (5.1.3). Thus if we now use the infinitesimal form of the Lorentz transformation in the foundamental representation (meaning vectorial representation, which generators are given by (2.3.10), since with foundamental is usually addressed the scalar transformations representation) of the Lorentz group SO$(1, 3)$

$$\Lambda = e^{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}},$$

and in parallel its spinorial representation

$$S = e^{-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}},$$

we are studying the 4×4 matrices realizing the Lorentz transformation, in two different representations; also $\Sigma^{\mu\nu}$ and $M^{\mu\nu}$ are generators of lorentz transformation in the Dirac and foundamental representation.

We can even make the parallel substitution of $S^{\mu\nu} = i\Sigma^{\mu\nu}$ in the foundamental representation, so that

$$\mathcal{M}^{\mu\nu} = -iM^{\mu\nu},$$

so that now it is possible to write both infinitesimal transformations in the same form:

$$\begin{aligned}
\Lambda &= e^{\frac{1}{2}\omega_{\mu\nu}\mathcal{M}^{\mu\nu}} \sim 1 + \frac{1}{2}\omega_{\mu\nu}\mathcal{M}^{\mu\nu} + O(\omega_{\mu\nu}^2),\\
S &= e^{\frac{1}{2}\omega_{\mu\nu}S^{\mu\nu}} \sim 1 + \frac{1}{2}\omega_{\mu\nu}S^{\mu\nu} + O(\omega_{\mu\nu}^2).
\end{aligned}$$

Now if we stick to the foundamental representation (vectorial representation of Lorentz group, which generators are given by (2.3.10)), we have that

$$(\mathcal{M}^{\rho\sigma})^{\mu}{}_{\nu}\gamma^{\nu} = (\eta^{\rho\mu}\delta^{\sigma}{}_{\nu} - \eta^{\sigma\mu}\delta^{\rho}{}_{\nu})\gamma^{\nu} = \eta^{\rho\mu}\gamma^{\sigma} - \eta^{\sigma\mu}\gamma^{\rho}.$$

Now we want some kind of relation among the two representations, since this last expression is practically the corresponding of $\Lambda^{\mu}{}_{\nu}\overline{\psi}\gamma^{\nu}\psi$. The idea is to expand the expression for $S^{\rho\sigma}$ in order to find a way to relate the previous expression to a commutator with gamma matrices. We have

$$\begin{aligned}
S^{\rho\sigma} &= \frac{1}{4}[\gamma^{\rho}, \gamma^{\sigma}] = \frac{1}{4}(\gamma^{\rho}\gamma^{\sigma} - \gamma^{\sigma}\gamma^{\rho})\\
&= \frac{1}{4}(2\gamma^{\rho}\gamma^{\sigma} - \{\gamma^{\rho}, \gamma^{\sigma}\}) = \frac{1}{2}(\gamma^{\rho}\gamma^{\sigma} - \eta^{\rho\sigma}),
\end{aligned}$$

where we have used the clifford algebra to rewrite the commutator in terms of the anticommutator. Now if we compute the following commutator among gamma matrices and the generators in the Dirac representation

$$\begin{aligned}
[S^{\rho\sigma}, \gamma^{\mu}] &= \frac{1}{2}[\gamma^{\rho}\gamma^{\sigma}, \gamma^{\mu}] - \frac{1}{2}\eta^{\rho\sigma}[\mathbb{I}, \gamma^{\mu}]\\
&= \frac{1}{2}(\gamma^{\rho}\gamma^{\sigma}\gamma^{\mu} - \gamma^{\mu}\gamma^{\rho}\gamma^{\sigma}) - 0,
\end{aligned}$$

and using again the Clifford algebra to compute $\gamma^{\sigma}\gamma^{\rho} = 2\eta^{\rho\sigma} - \gamma^{\rho}\gamma^{\sigma}$, we get to

$$\begin{aligned}
[S^{\rho\sigma}, \gamma^{\mu}] &= \frac{1}{2}\gamma^{\rho}\left(2\eta^{\mu\sigma} - \gamma^{\mu}\gamma^{\sigma}\right) - \frac{1}{2}\left(2\eta^{\rho\mu} - \gamma^{\rho}\gamma^{\mu}\right)\gamma^{\sigma}\\
&= \eta^{\mu\sigma}\gamma^{\rho} - \eta^{\rho\mu}\gamma^{\sigma} = -(\eta^{\rho\mu}\gamma^{\sigma} - \eta^{\sigma\mu}\gamma^{\rho}).
\end{aligned}$$

Recalling in the end the previous computation, it is clear that[4]

$$(\mathcal{M}^{\rho\sigma})^{\mu}{}_{\nu}\gamma^{\nu} = -[S^{\rho\sigma}, \gamma^{\mu}].$$

[4]Since the Minkowski metric $\eta^{\mu\sigma} = \eta^{\sigma\mu}$ is symmetric, we can exchange the indices in the last term.

And now finally we can compare the two infinitesimal transformations:

$$\Lambda^\mu{}_\nu \gamma^\nu = \left[\delta^\mu{}_\nu + \frac{1}{2}\omega_{\rho\sigma}(\mathcal{M}^{\rho\sigma})^\mu{}_\nu + O(\omega^2_{\rho\sigma})\right]\gamma^\nu = \gamma^\mu - \frac{1}{2}\omega_{\rho\sigma}[S^{\rho\sigma},\, \gamma^\mu] + O(\omega^2_{\rho\sigma}),$$

which is identical to

$$\begin{aligned} S^{-1}\gamma^\mu S &= \left(1 - \frac{1}{2}\omega_{\rho\sigma}S^{\rho\sigma} + O(\omega^2_{\rho\sigma})\right)\gamma^\nu \left(1 + \frac{1}{2}\omega_{\rho\sigma}S^{\rho\sigma} + O(\omega^2_{\rho\sigma})\right) \\ &= \gamma^\mu - \frac{1}{2}\omega_{\rho\sigma}S^{\rho\sigma}\gamma^\mu + \frac{1}{2}\omega_{\rho\sigma}\gamma^\mu S^{\rho\sigma} + O(\omega^2_{\rho\sigma}) \\ &= \gamma^\mu - \frac{1}{2}\omega_{\rho\sigma}[S^{\rho\sigma},\, \gamma^\mu] + O(\omega^2_{\rho\sigma}). \end{aligned}$$

So we finally proved the identity

$$S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu \gamma^\nu, \tag{5.1.5}$$

which implies that the Dirac bilinear $\overline{\psi}\gamma^\mu\psi$ transforms as a Lorentz four vector, since

$$\overline{\psi}\gamma^\mu\psi \to \overline{\psi}'\gamma^\mu\psi' = \overline{\psi}S^{-1}\gamma^\mu S\psi = \Lambda^\mu{}_\nu\overline{\psi}\gamma^\nu\psi.$$

We can now obtain other Lorentz scalars, vectors or tensors by contracting this four vector with other objects transforming in the vectorial representation of the Lorentz group, for example ∂_μ or a gauge field A_μ:

- $\overline{\psi}\gamma^\mu\partial_\mu\psi$, which is a contraction with the derivative operator, this term represents the kinetic term of the lagrangian and it is a Lorentz scalar;
- $\overline{\psi}\gamma^\mu A_\mu\psi$, which is a contraction with a field in vectorial representation, this term represents interaction among spin $\frac{1}{2}$ particles and spin 1 gauge bosons (e.g. photons);
- $\overline{\psi}\gamma^{\mu\nu}\psi$, where there is a Lorentz tensor, i.e. $\overline{\psi}'\gamma^{\mu\nu}\psi' = \Lambda^\mu{}_\rho\Lambda^\nu_\sigma\overline{\psi}\gamma^{\rho\sigma}\psi$ (every index transforming accordingly with a Lorentz transformation).

We can introduce the **slash notation** for indicating an object contracted with a gamma matrix:

$$\not{A} = \gamma^\mu A_\mu. \tag{5.1.6}$$

5.1.3 | Dirac Lagrangian

We can finally write an expression for the manifestly Lorentz invariant Dirac lagrangian:

$$\mathcal{L} = \overline{\psi}i\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi = \overline{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi. \tag{5.1.7}$$

It is worth to notice some features of this lagrangian:

- the i factor ensures the lagrangian to be real: we have to check the realness of the two terms separately. For the mass term we have

  $$(\overline{\psi}\psi)^\dagger = \psi^\dagger(\psi^\dagger\gamma^0)^\dagger = \psi^\dagger\gamma^0\psi = \overline{\psi}\psi,$$

  for hermitianity of γ^0. Instead for the kinetic term we have

  $$\begin{aligned} (\overline{\psi}\gamma^\mu\partial_\mu\psi)^\dagger &= (\partial_\mu\psi)^\dagger(\gamma^\mu)^\dagger\overline{\psi}^\dagger = (\partial_\mu\psi)^\dagger(\gamma^\mu)^\dagger\gamma^0\psi = (\partial_\mu\psi)^\dagger\gamma^0\gamma^\mu\psi \\ &= \partial_\mu(\psi^\dagger\gamma^0\gamma^\mu\psi) - \psi^\dagger\gamma^0\gamma^\mu\partial_\mu\psi = -\psi^\dagger\gamma^0\gamma^\mu\partial_\mu\psi = -\overline{\psi}\gamma^\mu\partial_\mu\psi. \end{aligned}$$

where we have used the usual trick of integrating by parts and then neglecting the boundary term computed at infinite times and distances, since this lagrangian is integrated to obtain the action. As we have shown, the kinetic term is anti-hermitian, so multiplying it by i we get a hermitian expression, ensuring the lagrangian to be real.

- Since the action is dimensionless, we can deduce the mass dimension of the field ψ:

$$\begin{aligned}[S] = 0, \quad [\mathrm{d}^4 x] = -4 \quad &\implies [\mathcal{L}] = 4, \\ [\partial_\mu] = 1, \quad [m] = 1 \quad &\implies [\psi] = [\overline{\psi}] = \frac{3}{2}.\end{aligned}$$

  So this field has a different mass dimention from KG scalar field, which had: $[\psi] = 1$.

- KG Lagrangian contains two derivatives $\partial_\mu \psi \partial^\mu \psi$ in the kinetic term, while Dirac's only one $\overline{\psi} i \gamma^\mu \partial_\mu \psi$: The KG lagrangian was of the second order, while Dirac's of the first; it changes the order of the equations of motion.

- Upon quantization the Dirac theory will describe particles/antiparticles (for whose description we needed a complex field for the charge dof, which is our case) with spin $\frac{1}{2}$ and mass m; in principle we have four complex dof, which are LH/RH and spin up/down (doubled because we need 4 real dof for the particle description and 4 real dof for the antiparticle one).

5.2 | Dirac Equation

The Dirac Lagrangian is

$$\mathcal{L} = \overline{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi,$$

where $\overline{\psi} = \psi^\dagger \gamma^0$ is the Dirac conjugate. We can find the equations of motion using the Euler-Lagrange equations for fields (3.1.3), treating ψ and $\overline{\psi}$ as independent fields:

$$\frac{\partial \mathcal{L}}{\partial \psi} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\right), \quad \frac{\partial \mathcal{L}}{\partial \overline{\psi}} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\psi})}\right);$$

in the second equation we have to notice that $\frac{\partial}{\partial \overline{\psi}}\mathcal{L} = 0$, since $\frac{\partial}{\partial(\partial_\mu \overline{\psi})}\mathcal{L} = 0$. This implies that if we differentiate the lagrangian with respect to $\overline{\psi}$ we get

$$\frac{\partial \mathcal{L}}{\partial \overline{\psi}} = \left(i\gamma^\mu \partial_\mu - m\right)\psi = 0,$$

which is the **Dirac equation**:

$$\left(i\gamma^\mu \partial_\mu - m\right)\psi = 0, \tag{5.2.1}$$

which is a first order partial differential equation for the spinor field $\psi(x)$.

We can also derive the equation of motion by considering the derivatives with respect to ψ, in order to get the conjugate equation:

$$\frac{\partial \mathcal{L}}{\partial \psi} = -m\overline{\psi}, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = i\overline{\psi}\gamma^\mu \implies \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\right) = i(\partial_\mu \overline{\psi})\gamma^\mu,$$

so that the Euler-Lagrange equation gives

$$-m\overline{\psi} = i(\partial_\mu \overline{\psi})\gamma^\mu \implies \overline{\psi}(-i\overleftarrow{\partial}_\mu \gamma^\mu - m) = 0,$$

which is the **conjugate Dirac equation**:

$$\overline{\psi}(-i\overleftarrow{\partial}_\mu \gamma^\mu - m) = 0, \tag{5.2.2}$$

where the arrow on the derivatives tells us it is to be applied to the left.

We can build such an equation of motion just thanks to the presence of γ^μ which grants lorentz invariance. Dirac equation is a **first order differential equation**, while for KG we could only get a second order one. Furthermore KG is a scalar equation, while Dirac is a spinor equation (vectorial in spinor space, with 4 components). The presence of gamma matrices in Dirac equation is crucial: they ensure lorentz invariance with their transformation properties, and they allow to have a first order equation, since they contract with the derivative index μ in $\partial_\mu \psi$.

Lorentz invariance determines the form of Dirac equation: if we want to describe a scalar particle, we need a second order equation (KG), while if we want to describe a spin $\frac{1}{2}$ particle (spinor) a first order equation (Dirac) is sufficient.

Notice that dirac equation mixes components of the spinor, since γ^μ are 4×4 matrices:

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.$$

So in components in the spinor space, Dirac equations reads:

$$i\begin{pmatrix}\partial_t\psi_3\\ \partial_t\psi_4\\ \partial_t\psi_1\\ \partial_t\psi_2\end{pmatrix}+i\begin{pmatrix}\partial_x\psi_4\\ \partial_x\psi_3\\ -\partial_x\psi_2\\ -\partial_x\psi_1\end{pmatrix}+i\begin{pmatrix}-\partial_y\psi_4\\ \partial_y\psi_3\\ \partial_y\psi_2\\ -\partial_y\psi_1\end{pmatrix}+i\begin{pmatrix}\partial_z\psi_3\\ -\partial_z\psi_4\\ -\partial_z\psi_1\\ \partial_z\psi_2\end{pmatrix}-m\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}=0.$$

In general it becomes a system of four coupled first order differential equations, mixing the four components of the spinor: for example the first equation is

$$i\partial_t\psi_3+i\partial_x\psi_4-i\partial_y\psi_4+i\partial_z\psi_3-m\psi_1=0.$$

However each component $\psi_\alpha(x)$ satosfy the KG equation, since in effect each components describe a degree of freedom of a relativistic scalar particle with mass m: if we multiply Dirac equation by $(i\gamma^\mu\partial_\mu+m)$ from the left (it is still valid since zero multiplied by anything remains zero), we get

$$\begin{aligned}(i\gamma^\mu\partial_\mu+m)(i\gamma^\nu\partial_\nu-m)\psi(x)&=0,\\ \left(-\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu-m^2\right)\psi(x)&=0,\\ \left(-\frac{1}{2}\{\gamma^\mu,\,\gamma^\nu\}\partial_\mu\partial_\nu-m^2\right)\psi(x)&=0,\\ \left(-\eta^{\mu\nu}\partial_\mu\partial_\nu-m^2\right)\psi(x)&=0,\\ \implies(\Box+m^2)\psi(x)&=0,\end{aligned}$$

where we have used the **clifford algebra** to simplify the product of gamma matrices, since we can insert the anticommutator for symmetry of $\partial_\mu\partial_\nu$:

$$\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu=\frac{1}{2}\gamma^\mu\gamma^\nu\left\{\partial_\mu,\partial_\nu\right\}=\frac{1}{2}\{\gamma^\mu,\,\gamma^\nu\}\partial_\mu\partial_\nu=\eta^{\mu\nu}\partial_\mu\partial_\nu,$$

just renaming indces. Thus each component of the Dirac spinor satisfies the KG equation:

$$(\Box+m^2)\psi_\alpha(x)=0\quad\forall\alpha=1,\,2,\,3,\,4.$$

5.2.1 | Chiral Spinors

Chirality means that Dirac representation can be decomposed into two irreducible representations, **Weyl representation** $(\frac{1}{2},\,0)\oplus(0,\,\frac{1}{2})$, of the Lorentz group (for more information on the representations see section 2.3.4). Weyl or Chiral spinors are two-component objects (complex dof) with different transformation properties:

- **Left-Handed** weyl spinors: $\psi_L\sim(\frac{1}{2},\,0)$, transform under S_L only (as in eq. (2.3.12));
- **Right-Handed** weyl spinors: $\psi_R\sim(0,\,\frac{1}{2})$, transform under S_R only (as in eq. (2.3.13)).

We can write the Dirac spinor as a combination of two weyl spinors:

$$\psi_D=\begin{pmatrix}\psi_L^{(w)}\\ \psi_R^{(w)}\end{pmatrix},\quad\psi_L^{(w)}\xleftrightarrow{\text{Parity}}\psi_R^{(w)}.$$

Chirality operator. In order to project out the two chiral components from a Dirac spinor we can introduce the chirality operator

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbb{I}_2 & 0 \\ 0 & \mathbb{I}_2 \end{pmatrix}, \quad (\gamma^5)^2 = \mathbb{I}_4,$$

using which we can define projectors able to select the desired chiral component from the Dirac spinor (as we did in eq. (2.3.17) for spinors):

$$\mathbb{P}_L\psi = \frac{1-\gamma^5}{2}\psi = \psi_L = \begin{pmatrix} \psi_L^{(w)} \\ 0 \end{pmatrix},$$
$$\mathbb{P}_R\psi = \frac{1+\gamma^5}{2}\psi = \psi_R = \begin{pmatrix} 0 \\ \psi_R^{(w)} \end{pmatrix}.$$

Thus the dirac spinor is the sum of the two chiral components $\psi_D = \psi_L + \psi_R$ and the projectors satisfy the usual properties:

$$\begin{cases} \mathbb{P}_L^2 = \mathbb{P}_L, \\ \mathbb{P}_L^\dagger = \mathbb{P}_L, \end{cases} \quad \begin{cases} \mathbb{P}_R^2 = \mathbb{P}_R, \\ \mathbb{P}_R^\dagger = \mathbb{P}_R, \end{cases} \quad \begin{cases} \mathbb{P}_L\mathbb{P}_R = \mathbb{P}_R\mathbb{P}_L = 0, \\ \mathbb{P}_L + \mathbb{P}_R = \mathbb{I}_4. \end{cases}$$

Note that the eigenvalues of γ^5 are ± 1, so the chirality operator measures the chirality of a spinor:

$$\begin{cases} \gamma^5\psi_L = (-1)\psi_L, \\ \gamma^5\psi_R = (+1)\psi_R. \end{cases}$$

We will see how chirality is related to helicity in the massless limit. This decomposition is very useful in the standard model, since weak interactions only involve LH particles and RH antiparticles; so we can use chiral projectors to select the interacting components.

Lagrangian and Chirality

To understand better, let's write the Dirac lagrangian

$$\mathcal{L} = \overline{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi,$$

in terms of chiral components (which is very useful for *statistical field theory* and the *standard model*, also for understanding the *massless limit*): we will exploit the vector and axial vector currents, defined as $\overline{\psi}\gamma^\mu\psi$ and $\overline{\psi}\gamma^\mu\gamma^5\psi$ respectively, but we will need only the first one to rewrite the kinetic term; then we will use the properties of the chirality operator to rewrite the mass term. Let's proceed step by step:

1. Starting from the **vector current** which is contracted with the derivative in the kinetic term:

$$\begin{aligned} \overline{\psi}\gamma^\mu\psi &= \psi^\dagger\gamma^0\gamma^\mu\left(\psi_L + \psi_R\right) = \psi^\dagger\gamma^0\gamma^\mu\left(\mathbb{P}_L^2 + \mathbb{P}_R^2\right)\psi \\ &= \psi^\dagger\gamma^0\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)^2\psi + \psi^\dagger\gamma^0\gamma^\mu\left(\frac{1+\gamma^5}{2}\right)^2\psi, \end{aligned}$$

   where, since the chirality matrix γ^5 anticommutes with all gamma matrices $\{\gamma^5, \gamma^\mu\} = 0$, we can use the identity $\gamma^0\gamma^\mu\gamma^5 = -\gamma^0\gamma^5\gamma^\mu = \gamma^5\gamma^0\gamma^\mu$ to obtain:

$$\overline{\psi}\gamma^\mu\psi = \psi^\dagger\left(\frac{1-\gamma^5}{2}\right)\gamma^0\gamma^\mu\left(\frac{1-\gamma^5}{2}\right)\psi + \psi^\dagger\left(\frac{1+\gamma^5}{2}\right)\gamma^0\gamma^\mu\left(\frac{1+\gamma^5}{2}\right)\psi.$$

Now exploiting hermitianity of the gamma matrices we can rewrite the projectors acting on $\psi^\dagger$ as $\psi^\dagger \mathbb{P}_{R/L} = \left(\mathbb{P}_{R/L}\psi\right)^\dagger$ to finally get (after reapplying γ^0 matrices to get $\overline{\psi}$):

$$\overline{\psi}\gamma^\mu\psi = \overline{\psi}_L\gamma^\mu\psi_L + \overline{\psi}_R\gamma^\mu\psi_R.$$

This found vector current does not mix chiral components, it is parity invariant and we will use this to rewrite the lagrangian. However, this current is not good for electroweak interactions and interpretations of $\mathrm{SU}(2)_L$ interactions: electroweak currents violate parity, this vector current is invariant under its action.

2. Moving on, we can try to insert a γ^5 in the vector current: in this way maybe we can get a parity violation and obtain a better candidate for weak interactions. So we consider

$$\overline{\psi}\gamma^\mu\gamma^5\psi = \psi^\dagger\gamma^0\gamma^\mu\gamma^5\left(\psi_L + \psi_R\right) = \psi^\dagger\gamma^0\gamma^\mu\gamma^5\left(\mathbb{P}_L^2 + \mathbb{P}_R^2\right)\psi.$$

We can easily compute that $\gamma^5\mathbb{P}_{L/R} = \mp\mathbb{P}_{L/R}$ (the first sign is for the left projector, the second for the right), so that we can write

$$\overline{\psi}\gamma^\mu\gamma^5\psi = \psi^\dagger\gamma^0\gamma^\mu\left(-\frac{1-\gamma^5}{2}\right)\psi_L + \psi^\dagger\gamma^0\gamma^\mu\left(\frac{1+\gamma^5}{2}\right)\psi_R = \overline{\psi}_R\gamma^\mu\psi_R - \overline{\psi}_L\gamma^\mu\psi_L,$$

which now, following the same steps as before, bring us to the expression for another current, called **axial vector current**: it changes sign under parity, but experiments show that weak interactions *violate parity maximally*, so this is not enough.

3. Now we can combine the two previous currents to get the **V-A current**:

$$\frac{1}{2}(V-A) = \overline{\psi}\gamma^\mu\frac{1-\gamma^5}{2}\psi = \overline{\psi}_L\gamma^\mu\psi_L.$$

V-A current violates parity maximally, since it involves only left-handed components, and it makes explicit the fact that only LH components enter the interaction: it is a good candidate for EW interactions.

4. Expanding the **mass term** instead:

$$\begin{aligned}\overline{\psi}\psi &= \psi^\dagger\gamma^0(P_R^2 + P_L^2)\psi = \psi^\dagger\gamma^0\left(\frac{1+\gamma^5}{2}\right)^2\psi + \psi^\dagger\gamma^0\left(\frac{1-\gamma^5}{2}\right)^2\psi \\ &= \psi^\dagger\left(\frac{1-\gamma^5}{2}\right)\gamma^0\left(\frac{1+\gamma^5}{2}\right)\psi + \psi^\dagger\left(\frac{1+\gamma^5}{2}\right)\gamma^0\left(\frac{1-\gamma^5}{2}\right)\psi,\end{aligned}$$

where we have used the identity $\gamma^0\mathbb{P}_{L/R} = \mathbb{P}_{R/L}\gamma^0$ to find that the mass term mixes chiral components:

$$\overline{\psi}\psi = \overline{\psi}_L^\dagger\gamma^0\psi_R + \overline{\psi}_R^\dagger\gamma^0\psi_L = \overline{\psi}_L\psi_R + \overline{\psi}_R\psi_L.$$

Now we have everything to write the Dirac lagrangian in terms of chiral components:

$$\mathcal{L} = \overline{\psi}_L i\gamma^\mu\partial_\mu\psi_L + \overline{\psi}_R i\gamma^\mu\partial_\mu\psi_R - m\left(\overline{\psi}_L\psi_R + \overline{\psi}_R\psi_L\right). \tag{5.2.3}$$

We can see that *in the massless limit the two chiral components decouple*, and we get two independent Weyl equations for each chiral component: this is important, since kinetic terms evolves independently the two chiral components, while the mass term allow us to perform a boost and mix the two chiralities: since the particle is massive there will be frames where the particle is RH and others where it is seen as LH.

If you take an electron whith both the chiral components, only the LH component will interact weakly, while the RH will not; but since the electron is massive you can always boost to a frame where the electron appears as RH, so both components are needed to describe a massive fermion. When we see only the RH component, the electron will not seem to interact weakly.

Dirac Equation in Weyl Components

If we write the Dirac equation in terms of chiral components we get:

$$\left(i\gamma^0\partial_0 + i\gamma^i\partial_i - m\right)\psi = 0, \quad \gamma^0 = \begin{pmatrix} 0 & \mathbb{I}_2 \\ \mathbb{I}_2 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$

which reads

$$i\begin{pmatrix} 0 & \partial_t + \boldsymbol{\sigma}\cdot\nabla \\ \partial_t - \boldsymbol{\sigma}\cdot\nabla & 0 \end{pmatrix}\begin{pmatrix} \psi_L^{(w)} \\ \psi_R^{(w)} \end{pmatrix} - m\begin{pmatrix} \psi_L^{(w)} \\ \psi_R^{(w)} \end{pmatrix} = 0,$$

which is a system of two coupled first order coupled differential equations

$$\begin{cases} i(\partial_t + \boldsymbol{\sigma}\cdot\nabla)\psi_R^{(w)} - m\psi_L^{(w)} = 0, \\ i(\partial_t - \boldsymbol{\sigma}\cdot\nabla)\psi_L^{(w)} - m\psi_R^{(w)} = 0. \end{cases}$$

In the massless limit the two equations decouple and we get two Weyl equations:

$$\begin{cases} i(\partial_t + \boldsymbol{\sigma}\cdot\nabla)\psi_R^{(w)} = 0, \\ i(\partial_t - \boldsymbol{\sigma}\cdot\nabla)\psi_L^{(w)} = 0, \end{cases} \tag{5.2.4}$$

and we call this last set of equations the **Weyl equations** for massless fermions. Now it is clearer the meaning of Left and Right-Handed Weyl spinors: in terms of operators (in natural units) we have

$$i\partial_t = \hat{H}, \quad -i\nabla = \hat{\mathbf{p}}, \quad \boldsymbol{\sigma} = \hat{\mathbf{S}},$$

so that we can see the last two equations as

$$\begin{cases} (\hat{H} - \hat{\mathbf{S}}\cdot\hat{\mathbf{p}})\psi_R^{(w)} = 0, \\ (\hat{H} + \hat{\mathbf{S}}\cdot\hat{\mathbf{p}})\psi_L^{(w)} = 0. \end{cases}$$

Now, dropping momentaneously the hat notation for operators (so that we can use it to indicate versors), we find clearly that the weyl components are eigenstate of the **helicity**: for massless particles indeed $E = |\mathbf{p}|$ so that

$$\begin{cases} |\mathbf{p}|\psi_R^{(w)} = \mathbf{S}\cdot\mathbf{p}\,\psi_R^{(w)}, \\ |\mathbf{p}|\psi_L^{(w)} = -\mathbf{S}\cdot\mathbf{p}\,\psi_L^{(w)}, \end{cases} \iff \begin{cases} \mathbf{S}\cdot\hat{\mathbf{p}}\,\psi_R^{(w)} = (+1)\psi_R^{(w)}, \\ \mathbf{S}\cdot\hat{\mathbf{p}}\,\psi_L^{(w)} = (-1)\psi_L^{(w)}, \end{cases}$$

where the helicity operator is defined as

$$h = \frac{\mathbf{S}\cdot\mathbf{p}}{|\mathbf{p}|} = \mathbf{S}\cdot\hat{\mathbf{p}}. \tag{5.2.5}$$

Since the eigenvalues are ± 1, we can affirm that in the massless limit chirality and helicity coincide: they have same eigenstates and eigenvalues.

If the neutrinos were massless, only the LH component would exist, since only that interacts weakly; but since neutrinos have a small mass, both chiral components exist, even if the RH component has never been observed (it interacts only gravitationally, so it is very difficult to detect it).

We now want to prove that under parity we can pass from one weyl component to the other:

$$\psi_L^{(w)} \xleftrightarrow{\text{Parity}} \psi_R^{(w)}.$$

Let's start from the Dirac equation:

$$(i\gamma^\mu \partial_\mu - m)\,\psi(x) = 0,$$
$$\left(i\gamma^0\partial_0 + i\gamma^i\partial_i - m\right)\psi(t,\,\mathbf{x}) = 0,$$

which, given the action of the parity:

$$\partial_t \xrightarrow{\text{Parity}} \partial_t, \quad \nabla \xrightarrow{\text{Parity}} -\nabla,$$

then transforms the Dirac equation as

$$\left(i\gamma^0\partial_0 - i\gamma^i\partial_i - m\right)\psi'(t,\,\mathbf{x}') = 0, \quad \mathbf{x}' = -\mathbf{x}.$$

But how does ψ transform under parity? We want to write the transformed spinor $\psi'(t,\,\mathbf{x}')$ in terms of the original one $\psi(t,\,\mathbf{x})$. Since gamma matrices satisfy

$$(\gamma^0)^2 = \mathbb{I}_4, \quad \{\gamma^0,\,\gamma^i\} = 0,$$

we can insert $(\gamma^0)^2$ in the transformed Dirac equation to get

$$\left(i\gamma^0\partial_0 - i\gamma^0\gamma^0\gamma^i\partial_i - \gamma^0\gamma^0 m\right)\psi'(t,\,\mathbf{x}') = 0,$$

which we can rewrite as

$$\gamma^0\left(i\partial_0 + i\gamma^i\gamma^0\partial_i - \gamma^0 m\right)\psi'(t,\,\mathbf{x}') = 0.$$

where we have grouped γ^0 on the left and changed the sign of the second term swapping γ^0 and γ^i. Now, inserting $(\gamma^0)^2$ one last time in the first term, we can group another γ^0 on the right:

$$\gamma^0\left(i\gamma^0\partial_0 + i\gamma^i\partial_i - m\right)\gamma^0\psi'(t,\,\mathbf{x}') = 0,$$

which, simplifying γ^0 on the left, and comparing with the original Dirac equation, tells us that the transformed spinor must respect

$$\psi'(t,\,\mathbf{x}') = \gamma^0\psi(t,\,\mathbf{x}).$$

Thus under parity the Dirac spinor transforms as

$$\psi = \begin{pmatrix}\psi_L^{(w)}\\ \psi_R^{(w)}\end{pmatrix} \xrightarrow{\text{Parity}} \psi' = \begin{pmatrix}\psi_L'^{(w)}\\ \psi_R'^{(w)}\end{pmatrix} = \begin{pmatrix}0 & \mathbb{I}_2\\ \mathbb{I}_2 & 0\end{pmatrix}\begin{pmatrix}\psi_L^{(w)}\\ \psi_R^{(w)}\end{pmatrix},$$

so that

$$\begin{cases}\psi_L'^{(w)} = \psi_R^{(w)},\\ \psi_R'^{(w)} = \psi_L^{(w)}.\end{cases}$$

As we wanted to prove, under parity the two weyl components are indeed exchanged:

$$\psi_L^{(w)} \xleftrightarrow{\text{Parity}} \psi_R^{(w)}.$$

5.2.2 | Solutions of the Dirac Equation

Each component of the dirac spinor $\psi(x)$ satisfies the KG equation, so we can look for plane wave solutions of the form

$$\psi_\alpha(x) = u_\alpha(\mathbf{p})e^{-ip_\mu x^\mu} = u_\alpha(\mathbf{p})e^{-i(E_\mathbf{p}t - \mathbf{p}\cdot\mathbf{x})}, \quad E_\mathbf{p} = \sqrt{\mathbf{p}^2 + m^2},$$

where $\alpha = 1, 2, 3, 4$ and $u_\alpha(\mathbf{p})$ are complex coefficients depending on the momentum $\mathbf{p}$ (it's a 4-component vector). Plugging this ansatz into the Dirac equation we get

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0, \quad \Longrightarrow \quad (\gamma^\mu p_\mu - m)u(\mathbf{p}) = 0.$$

which in spinorial representation reads

$$\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} p_0 + \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} p_i - m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] u(\mathbf{p}) = \begin{pmatrix} -m & p^\mu \sigma_\mu \\ p^\mu \bar{\sigma}_\mu & -m \end{pmatrix} u(\mathbf{p}) = 0,$$

where we have defined $\sigma^\mu = (\mathbb{I}, \sigma^i)$ and $\bar{\sigma}^\mu = (\mathbb{I}, -\sigma^i)$. Writing the spinor $u(\mathbf{p})$ in terms of its two-component chiral parts

$$u(\mathbf{p}) = \begin{pmatrix} u_L(\mathbf{p}) \\ u_R(\mathbf{p}) \end{pmatrix},$$

we get the system of equations

$$\begin{cases} (p^\mu \sigma_\mu) u_R(\mathbf{p}) = m u_L(\mathbf{p}), \\ (p^\mu \bar{\sigma}_\mu) u_L(\mathbf{p}) = m u_R(\mathbf{p}). \end{cases}$$

It's a system of two coupled equations for the two chiral components of the spinor. We can solve for one component in terms of the other; for example, solving for $u_R(\mathbf{p})$ from the second equation and plugging it into the first, we get

$$(p^\mu \sigma_\mu)(p^\nu \bar{\sigma}_\nu) = (p_0 + p_i \sigma^i)(p_0 - p_j \sigma^j) = p_0^2 - p_i p_j \sigma^i \sigma^j = p_0^2 - \mathbf{p}^2 = m^2,$$

where we have used the algebra of the Pauli matrices $\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k$[5] and we have found the relativistic dispersion relation $p_0^2 - \mathbf{p}^2 = m^2$. Thus we can write

$$\begin{aligned} u_L(\mathbf{p}) &= A(p^\mu \sigma_\mu)\chi, \\ (p^\mu \bar{\sigma}_\mu) u_L(\mathbf{p}) &= (p^\mu \bar{\sigma}_\mu)(p^\mu \sigma_\mu) A\chi = m^2 A\chi = m u_R(\mathbf{p}), \\ \Longrightarrow u_R(\mathbf{p}) &= mA\chi, \end{aligned}$$

and getting for $u_L(\mathbf{p})$:

$$\begin{aligned} u_L(\mathbf{p}) = (p^\mu \sigma_\mu) u_R(\mathbf{p}) \frac{1}{m} &= (p^\mu \sigma_\mu)(mA\chi)\frac{1}{m} = A(p^\mu \sigma_\mu)\chi, \\ \Longrightarrow u_L(\mathbf{p}) &= A(p^\mu \sigma_\mu)\chi, \end{aligned}$$

where χ is an arbitrary two-component spinors and A a normalization constant (thus the first equation is automatically satisfied). Thus the general solution for $u(\mathbf{p})$ can be written as

$$u(\mathbf{p}) = A \begin{pmatrix} (p^\mu \sigma_\mu)\chi \\ m\chi \end{pmatrix},$$

where now the idea is to symmetrize the solution by choosing $A = \frac{1}{m}$ and $\chi = \sqrt{p^\mu \bar{\sigma}_\mu}\xi$, where ξ is a constant two-component spinor respecting

$$\xi^\dagger \xi = 1,$$

so that

$$A(p^\mu \sigma_\mu)\chi = \frac{1}{m}\sqrt{(p^\mu \sigma_\mu)(p^\nu \sigma_\nu)}\sqrt{p^\mu \bar{\sigma}_\mu}\xi = \sqrt{p^\mu \sigma_\mu}\xi,$$

[5] The term proportional to $\epsilon^{ijk}\sigma^k$ goes to zero due to antisymmetry when contracted with symmetric indices.

since we have already computed that $(p^\mu\sigma_\mu)(p^\nu\bar{\sigma}_\nu) = m^2$. Thus we get the final expression for the **positive frequency solution** of the Dirac equation:

$$\begin{cases} u_L(\mathbf{p}) = \sqrt{p^\mu\sigma_\mu}\xi, \\ u_R(\mathbf{p}) = \sqrt{p^\mu\bar{\sigma}_\mu}\xi, \end{cases} \implies u(\mathbf{p}) = \begin{pmatrix} \sqrt{p^\mu\sigma_\mu}\xi \\ \sqrt{p^\mu\bar{\sigma}_\mu}\xi \end{pmatrix},$$

with plane wave solution in the final form

$$\psi(x) = u(\mathbf{p})e^{-ip_\mu x^\mu} = \begin{pmatrix} \sqrt{p^\mu\sigma_\mu}\xi \\ \sqrt{p^\mu\bar{\sigma}_\mu}\xi \end{pmatrix} e^{-i(E_\mathbf{p}t-\mathbf{p}\cdot\mathbf{x})}. \tag{5.2.6}$$

We can also find **negative frequency solutions** of the Dirac equation by considering

$$\psi(x) = v(\mathbf{p})e^{ip_\mu x^\mu},$$

where $v(\mathbf{p})$ satisfies

$$(\gamma^\mu p_\mu + m)v(\mathbf{p}) = \begin{pmatrix} m & p^\mu\sigma_\mu \\ p^\mu\bar{\sigma}_\mu & m \end{pmatrix} v(\mathbf{p}) = 0,$$

and following the same steps as before we get

$$v(\mathbf{p}) = \begin{pmatrix} \sqrt{p^\mu\sigma_\mu}\eta \\ -\sqrt{p^\mu\bar{\sigma}_\mu}\eta \end{pmatrix}, \quad \eta^\dagger\eta = 1,$$

with plane wave solution

$$\psi(x) = v(\mathbf{p})e^{ip_\mu x^\mu} = \begin{pmatrix} \sqrt{p^\mu\sigma_\mu}\eta \\ -\sqrt{p^\mu\bar{\sigma}_\mu}\eta \end{pmatrix} e^{i(E_\mathbf{p}t-\mathbf{p}\cdot\mathbf{x})}. \tag{5.2.7}$$

If we were to apply the Hamiltonian on these ansatzs we would get

$$\hat{H}\psi(x) = i\partial_t(u(\mathbf{p})e^{-ip_\mu x^\mu}) = E_\mathbf{p}(u(\mathbf{p})e^{-ip_\mu x^\mu}) = +E_\mathbf{p}\psi(x),$$
$$\hat{H}\psi(x) = i\partial_t(v(\mathbf{p})e^{ip_\mu x^\mu}) = -E_\mathbf{p}(v(\mathbf{p})e^{ip_\mu x^\mu}) = -E_\mathbf{p}\psi(x).$$

Thus $u(\mathbf{p})$ are positive energy solutions while $v(\mathbf{p})$ are negative energy solutions of the Dirac equation. This problem of negative energy solutions will be solved in QFT interpreting them as antiparticles, but for now $\psi(x)$ is just a classical field; after quantization $\hat{\psi}(x)$ will be an operator acting on the Fock space and we will see how to interpret these solutions. Both particles and antiparticles will be needed to build a consistent quantum theory, with positive definite energy.

Example (Particle rest frame and spinor transformations). In the rest frame of a massive particle

$$\mathbf{p} = 0, \quad E_\mathbf{p} = p^0 = m.$$

Thus the positive frequency solutions read

$$u(\mathbf{0}) = \begin{pmatrix} \sqrt{p^\mu\sigma_\mu}\xi \\ \sqrt{p^\mu\bar{\sigma}_\mu}\xi \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \sqrt{p^0} & 0 \\ 0 & \sqrt{p^0} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} \sqrt{p^0} & 0 \\ 0 & \sqrt{p^0} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \sqrt{p^0}\begin{pmatrix} \xi \\ \xi \end{pmatrix} = \sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix},$$

so that the plane wave solution is

$$\psi(x) = \sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix} e^{-imt}.$$

Now recalling the Lorentz transformation for spinors

$$\psi'^{\alpha}(x') = S(\Lambda)^{\alpha}{}_{\beta}\psi^{\beta}(x), \quad S(\Lambda)^{\alpha}{}_{\beta} = (e^{-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}})^{\alpha}{}_{\beta},$$

where the algebra generators of the Lorentz transformation are

$$S^{\mu\nu} = \frac{1}{4}[\gamma^{\mu}, \gamma^{\nu}] = i\Sigma^{\mu\nu},$$

with parameters $\omega_{\mu\nu}$ depending on the boost/rotation we are performing. For a pure spatial rotation we have

$$S^{ij} = -\frac{i}{2}\epsilon_{ijk}\begin{pmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix}, \quad (i \neq j),$$

after computing the commutator and exploiting the Pauli matrices algebra; here k is the axis of rotation, and the three parameters of the rotation

$$\omega_{ij} = -\epsilon_{ijk}\theta^k, \quad (i \neq j)$$

are related to the rotation angles around the three spatial axis. Thus the spinor transformation under a spatial rotation reads

$$e^{\frac{1}{2}\omega_{ij}S^{ij}} = \begin{pmatrix} e^{i\frac{\theta^k\sigma^k}{2}} & 0\\ 0 & e^{i\frac{\theta^k\sigma^k}{2}}\end{pmatrix},$$

and if we apply this transformation to the rest-frame spinor we get

$$\psi(x) = \sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix}e^{-imt} \to \psi'(x') = \begin{pmatrix} e^{i\frac{\theta^k\sigma^k}{2}} & 0\\ 0 & e^{i\frac{\theta^k\sigma^k}{2}}\end{pmatrix}\psi(x) = \sqrt{m}\begin{pmatrix} e^{i\frac{\theta^k\sigma^k}{2}}\xi\\ e^{i\frac{\theta^k\sigma^k}{2}}\xi\end{pmatrix}e^{-imt},$$

so that both chiral components transform in the same way under spatial rotations, as expected since in the rest frame chirality and helicity are not defined

$$\xi \to \xi' = e^{i\frac{\theta^k\sigma^k}{2}}\xi.$$

This is the representation of the standard SU(2) transformation for spin $\frac{1}{2}$ objects

$$\mathbf{S} = \frac{\hbar}{2}\boldsymbol{\sigma}.$$

We are considering particles at rest with spin $\frac{1}{2}$, so we can choose the basis where the spin is aligned along the z-axis:

$$\xi_+ = \begin{pmatrix}1\\0\end{pmatrix}, \quad \xi_- = \begin{pmatrix}0\\1\end{pmatrix},$$

which are eigenstates of the spin operator $S_z = \frac{\hbar}{2}\sigma^3$ with eigenvalues $\pm\frac{\hbar}{2}$:

$$\sigma_3\xi_+ = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = +1\begin{pmatrix}1\\0\end{pmatrix}, \quad \sigma_3\xi_- = -1\begin{pmatrix}0\\1\end{pmatrix}.$$

Example (Massless limit and chirality). Now consider a particle with spin up along the z-axis moving with momentum $\mathbf{p} = (0, 0, p_z) = (0, 0, p)$ along the z-axis

$$p^{\mu} = (E_{\mathbf{p}}, 0, 0, p), \quad E_{\mathbf{p}} = \sqrt{p^2 + m^2} = E.$$

We have the positive frequency solution as

$$\psi(x) = \begin{pmatrix}\sqrt{p^{\mu}\sigma_{\mu}}\xi_+\\ \sqrt{p^{\mu}\bar{\sigma}_{\mu}}\xi_+\end{pmatrix}e^{-ip_{\mu}x^{\mu}},$$

where computing the square roots we get[6]

$$\psi(x) = \begin{pmatrix} \begin{pmatrix} \sqrt{E-p} & 0 \\ 0 & \sqrt{E+p} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} \sqrt{E+p} & 0 \\ 0 & \sqrt{E-p} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} e^{-i(Et-pz)} = \begin{pmatrix} \sqrt{E-p} \\ 0 \\ \sqrt{E+p} \\ 0 \end{pmatrix} e^{-i(Et-pz)}.$$

where we have a right handed helicity state since both momentum and spin are aligned along the z-axis, but both chiral components since for massive particles chirality and helicity do not coincide. Now we can consider the massless limit $m \to 0$, so that $E = p$ and we get that the left chiral component vanishes

$$\psi(x) = \sqrt{2p} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{-ip(t-z)},$$

which is a purely right-handed spinor, as expected since in the massless limit chirality and helicity coincide: we were considering a particle with moentum and spin aligned along the z-axis, so it must be right-handed both in helicity and chirality. Thus in the massless limit we have

$$\begin{aligned} \psi_L^{(w)}(x) &= 0, \\ \psi_R^{(w)}(x) &= \sqrt{2p} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned}$$

5.2.3 | Useful Formulae

We want to introduce some useful formulae for manipulating Dirac spinors and gamma matrices, in order to comprehend better the structure of the quantum theory. We will focus on the positive and negative frequency solutions $u(\mathbf{p})$ and $v(\mathbf{p})$ of the Dirac equation defined in equations (5.2.6) and (5.2.7), which can be decomposed in a chosen basis of two-component spinors ξ and η:

$$\xi^{n\,\dagger}\xi^m = \delta^{nm}, \quad \eta^{n\,\dagger}\eta^m = \delta^{nm},$$

with trivial example

$$\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus we have defined a basis ξ^n and η^n for the two-component spinors to build the positive and negative frequency solutions of the Dirac equation (with $n = 1,\, 2$).

Inner products. We can label solutions of the Dirac equation with this index, in order to distinguish the two different spin states; thus we can start computing inner products between these solutions. For example for the positive frequency solutions we have

$$u^{n\,\dagger}(\mathbf{p})u^m(\mathbf{p}) = \begin{pmatrix} \xi^{n\,\dagger}\sqrt{p^\mu\sigma_\mu} & \xi^{n\,\dagger}\sqrt{p^\mu\bar{\sigma}_\mu} \end{pmatrix} \begin{pmatrix} \sqrt{p^\mu\sigma_\mu}\xi^m \\ \sqrt{p^\mu\bar{\sigma}_\mu}\xi^m \end{pmatrix}$$

[6] Pay attention that $p = |\mathbf{p}|$, thus we have to explicit the minus sign in the spatial part of the four-momentum when contracting with $\bar{\sigma}^\mu$ or σ^μ.

which can be computed as

$$\begin{aligned} u^{n\,\dagger}(\mathbf{p})u^m(\mathbf{p}) &= \xi^{n\,\dagger}(p^\mu\sigma_\mu)\xi^m + \xi^{n\,\dagger}(p^\mu\bar{\sigma}_\mu)\xi^m \\ &= 2p^0\xi^{n\,\dagger}\xi^m + p^i\sigma_i\xi^{n\,\dagger}\xi^m - p^i\sigma_i\xi^{n\,\dagger}\xi^m \\ &= 2p^0\xi^{n\,\dagger}\xi^m = 2E_{\mathbf{p}}\delta^{nm}, \end{aligned}$$

which is not a Lorentz invariant quantity since it involves the time component of the four-momentum $p^0 = E_{\mathbf{p}}$, but it will prove useful in the quantization procedure.

Instead if we have

$$\overline{u}^n(\mathbf{p})u^m(\mathbf{p}) = \begin{pmatrix} \xi^{n\,\dagger}\sqrt{p^\mu\sigma_\mu} & \xi^{n\,\dagger}\sqrt{p^\mu\bar{\sigma}_\mu} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{p^\mu\sigma_\mu}\xi^m \\ \sqrt{p^\mu\bar{\sigma}_\mu}\xi^m \end{pmatrix},$$

where we know the definition for the Dirac conjugate $\overline{u}^n = u^{n\,\dagger}\gamma^0$, we get

$$\begin{aligned} u^{n\,\dagger}(\mathbf{p})\gamma^0 u^m(\mathbf{p}) &= \xi^{n\,\dagger}\sqrt{p^\mu\sigma_\mu}\sqrt{p^\mu\bar{\sigma}_\mu}\xi^m + \xi^{n\,\dagger}\sqrt{p^\mu\bar{\sigma}_\mu}\sqrt{p^\mu\sigma_\mu}\xi^m \\ &= \xi^{n\,\dagger}\sqrt{m^2}\xi^m + \xi^{n\,\dagger}\sqrt{m^2}\xi^m \\ &= 2m\xi^{n\,\dagger}\xi^m = 2m\delta^{nm}, \end{aligned}$$

which is instead a Lorentz invariant quantity since it depends only on the mass m of the particle. Now we can summarize these two results as

$$\begin{cases} u^{n\,\dagger}(\mathbf{p})u^m(\mathbf{p}) = 2E_{\mathbf{p}}\delta^{nm}, \\ \overline{u}^n(\mathbf{p})u^m(\mathbf{p}) = 2m\delta^{nm}. \end{cases} \tag{5.2.8}$$

Similarly for the negative frequency solutions we have

$$\begin{cases} v^{n\,\dagger}(\mathbf{p})v^m(\mathbf{p}) = 2E_{\mathbf{p}}\delta^{nm}, \\ \overline{v}^n(\mathbf{p})v^m(\mathbf{p}) = -2m\delta^{nm}. \end{cases} \tag{5.2.9}$$

Indeed we can compute explicitly

$$\begin{aligned} v^{n\,\dagger}(\mathbf{p})v^m(\mathbf{p}) &= \eta^{n\,\dagger}(p^\mu\sigma_\mu)\eta^m + \eta^{n\,\dagger}(p^\mu\bar{\sigma}_\mu)\eta^m \\ &= 2p^0\eta^{n\,\dagger}\eta^m + p^i\sigma_i\eta^{n\,\dagger}\eta^m - p^i\sigma_i\eta^{n\,\dagger}\eta^m \\ &= 2p^0\eta^{n\,\dagger}\eta^m = 2E_{\mathbf{p}}\delta^{nm}, \end{aligned}$$

and

$$\begin{aligned} \overline{v}^n(\mathbf{p})v^m(\mathbf{p}) &= \eta^{n\,\dagger}\sqrt{p^\mu\sigma_\mu}(-\sqrt{p^\mu\bar{\sigma}_\mu})\eta^m + \eta^{n\,\dagger}(-\sqrt{p^\mu\bar{\sigma}_\mu})\sqrt{p^\mu\sigma_\mu}\eta^m \\ &= -\eta^{n\,\dagger}\sqrt{m^2}\eta^m - \eta^{n\,\dagger}\sqrt{m^2}\eta^m \\ &= -2m\eta^{n\,\dagger}\eta^m = -2m\delta^{nm}. \end{aligned}$$

Now we can compute the mixed products

$$\begin{aligned} \overline{u}^n(\mathbf{p})v^m(\mathbf{p}) = \begin{pmatrix} \xi^{n\,\dagger}\sqrt{p^\mu\sigma_\mu} & \xi^{n\,\dagger}\sqrt{p^\mu\bar{\sigma}_\mu} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{p^\mu\sigma_\mu}\eta^m \\ -\sqrt{p^\mu\bar{\sigma}_\mu}\eta^m \end{pmatrix} &= \\ =\xi^{n\,\dagger}\sqrt{p^\mu\sigma_\mu}(-\sqrt{p^\mu\bar{\sigma}_\mu})\eta^m + \xi^{n\,\dagger}\sqrt{p^\mu\bar{\sigma}_\mu}\sqrt{p^\mu\sigma_\mu}\eta^m &= \\ = -\,\xi^{n\,\dagger}\sqrt{m^2}\eta^m + \xi^{n\,\dagger}\sqrt{m^2}\eta^m = 0, \end{aligned}$$

and for the other combination (with the same cancellations) we get to the same result, thus we can write

$$\begin{cases} \overline{u}^n(\mathbf{p})v^m(\mathbf{p}) = 0, \\ \overline{v}^n(\mathbf{p})u^m(\mathbf{p}) = 0. \end{cases} \tag{5.2.10}$$

Thus we derived the **orthogonality relations**, which ensure that positive and negative frequency solutions are orthogonal to each other in the Dirac inner product sense.

We can also compute the mixed inner products with opposite momentum but without the Dirac conjugate, which will be useful in the quantization procedure:

$$
\begin{aligned}
u^{n\,\dagger}(\mathbf{p})v^m(-\mathbf{p}) &= \begin{pmatrix}\xi^{n\,\dagger}\sqrt{p^\mu\sigma_\mu} & \xi^{n\,\dagger}\sqrt{p^\mu\bar{\sigma}_\mu}\end{pmatrix}\begin{pmatrix}\sqrt{p'^\mu\sigma_\mu}\eta^m \\ -\sqrt{p'^\mu\bar{\sigma}_\mu}\eta^m\end{pmatrix} \\
&= \xi^{n\,\dagger}\left(\sqrt{p^\mu\sigma_\mu p'^\nu\sigma_\nu} - \sqrt{p'^\mu\bar{\sigma}_\mu p^\nu\bar{\sigma}_\nu}\right)\eta^m \\
&= \xi^{n\,\dagger}\left(\sqrt{p_0^2 - p_i p_j\sigma^i\sigma^j} - \sqrt{p_0^2 - p_i p_j\sigma^i\sigma^j}\right)\eta^m \\
&= \xi^{n\,\dagger}\,(m-m)\,\eta^m = 0,
\end{aligned},
$$

and similarly for the other combination, so that we have

$$
\begin{cases}
u^{n\,\dagger}(\mathbf{p})v^m(-\mathbf{p}) = 0, \\
v^{n\,\dagger}(\mathbf{p})u^m(-\mathbf{p}) = 0.
\end{cases}
\tag{5.2.11}
$$

Outer products. We can compute the outer products of the spinors, starting from the positive frequency solutions:

$$
\sum_n u^n(\mathbf{p})\overline{u}^n(\mathbf{p}) = \sum_n \begin{pmatrix}\sqrt{p^\mu\sigma_\mu}\xi^n \\ \sqrt{p^\mu\bar{\sigma}_\mu}\xi^n\end{pmatrix}\begin{pmatrix}\xi^{n\,\dagger}\sqrt{p^\mu\sigma_\mu} & \xi^{n\,\dagger}\sqrt{p^\mu\bar{\sigma}_\mu}\end{pmatrix}\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix},
$$

where we have used the definition of $\overline{u}^n$. Computing the sum over the basis we get

$$
\sum_n u^n(\mathbf{p})\overline{u}^n(\mathbf{p}) = \sum_n \begin{pmatrix}\sqrt{p^\mu\sigma_\mu}\xi^n\xi^{n\,\dagger}\sqrt{p^\mu\bar{\sigma}_\mu} & \sqrt{p^\mu\sigma_\mu}\xi^n\xi^{n\,\dagger}\sqrt{p^\mu\sigma_\mu} \\ \sqrt{p^\mu\bar{\sigma}_\mu}\xi^n\xi^{n\,\dagger}\sqrt{p^\mu\bar{\sigma}_\mu} & \sqrt{p^\mu\bar{\sigma}_\mu}\xi^n\xi^{n\,\dagger}\sqrt{p^\mu\sigma_\mu}\end{pmatrix}.
$$

Now if we use the completeness relation for the basis

$$
\sum_n \xi^n\xi^{n\,\dagger} = \begin{pmatrix}1 \\ 0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} + \begin{pmatrix}0 \\ 1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} = \begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix} = \mathbb{I}_2,
$$

then we find

$$
\sum_n u^n(\mathbf{p})\overline{u}^n(\mathbf{p}) = \begin{pmatrix}\sqrt{(p\sigma)(p\overline{\sigma})} & (p\sigma) \\ (p\overline{\sigma}) & \sqrt{(p\overline{\sigma})(p\sigma)}\end{pmatrix} = \begin{pmatrix}m & p^\mu\sigma_\mu \\ p^\mu\bar{\sigma}_\mu & m\end{pmatrix} = \gamma^\mu p_\mu + m\mathbb{I}_4.
$$

Similarly for the negative frequency solutions we have

$$
\begin{aligned}
\sum_n v^n(\mathbf{p})\overline{v}^n(\mathbf{p}) &= \sum_n \begin{pmatrix}\sqrt{p^\mu\sigma_\mu}\eta^n \\ -\sqrt{p^\mu\bar{\sigma}_\mu}\eta^n\end{pmatrix}\begin{pmatrix}\eta^{n\,\dagger}\sqrt{p^\mu\sigma_\mu} & -\eta^{n\,\dagger}\sqrt{p^\mu\bar{\sigma}_\mu}\end{pmatrix}\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix} \\
&= \sum_n \begin{pmatrix}\sqrt{p^\mu\sigma_\mu}\eta^n\eta^{n\,\dagger}\sqrt{p^\mu\sigma_\mu} & -\sqrt{p^\mu\sigma_\mu}\eta^n\eta^{n\,\dagger}\sqrt{p^\mu\bar{\sigma}_\mu} \\ -\sqrt{p^\mu\bar{\sigma}_\mu}\eta^n\eta^{n\,\dagger}\sqrt{p^\mu\sigma_\mu} & +\sqrt{p^\mu\bar{\sigma}_\mu}\eta^n\eta^{n\,\dagger}\sqrt{p^\mu\bar{\sigma}_\mu}\end{pmatrix}\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix} \\
&= \begin{pmatrix}-\sqrt{(p\sigma)(p\overline{\sigma})} & (p\sigma) \\ (p\overline{\sigma}) & -\sqrt{(p\overline{\sigma})(p\sigma)}\end{pmatrix} = \begin{pmatrix}-m & p^\mu\sigma_\mu \\ p^\mu\bar{\sigma}_\mu & -m\end{pmatrix} = \gamma^\mu p_\mu - m\mathbb{I}_4.
\end{aligned}
$$

Thus we can summarize these results as

$$
\begin{cases}
\displaystyle\sum_n u^n(\mathbf{p})\overline{u}^n(\mathbf{p}) = \gamma^\mu p_\mu + m\mathbb{I}_4, \\
\displaystyle\sum_n v^n(\mathbf{p})\overline{v}^n(\mathbf{p}) = \gamma^\mu p_\mu - m\mathbb{I}_4,
\end{cases}
\tag{5.2.12}
$$

remembering that the result is a 4×4 matrix.

5.3 | Quantizing Dirac Theory

We aim to quantize the theory as we did for the scalar KG field, promoting the classical fields to operators acting on a suitable Hilbert space. Starting from Lagrangian formalism, we aim to compute the conjugate momenta, Hamiltonian density, and Hamiltonian operator for the Dirac field.

We know $\pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}}$, with the Dirac lagrangian density

$$\mathcal{L} = \overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi,$$

we have

$$\pi = \frac{\partial}{\partial(\partial_0 \psi)} \overline{\psi}(i\gamma^0 \partial_0 + i\gamma^i \partial_i - m)\psi = \overline{\psi} i\gamma^0 = i\psi^\dagger.$$

Note that, since the Dirac equation is first order in time derivatives (as well as spatial derivatives), the conjugate momentum does not depend on $\dot{\psi}$ itself: we need to know only the field ψ, and not its time derivative, as initial condition to solve the equations of motion.

Now, exactely as for the scalar field, we have to impose canonical commutation relations between the field and its conjugate momentum at equal times, but now we have to take into account that the Dirac field is a spinor, so we have to consider each component separately; thus in Schrödinger picture we impose

$$[\hat{\psi}_\alpha(\mathbf{x}),\, \hat{\pi}_\beta(\mathbf{y})] = i\delta_{\alpha\beta}\delta^{(3)}(\mathbf{x}-\mathbf{y}),$$
$$[\hat{\psi}_\alpha(\mathbf{x}),\, \hat{\psi}_\beta(\mathbf{y})] = [\hat{\pi}_\alpha(\mathbf{x}),\, \hat{\pi}_\beta(\mathbf{y})] = 0.$$

We will see how this choice leads to inconsistencies, so we will have to modify it. In fact we will obtain either negative energy states or negative norm states, both unphysical; thus we will have to impose anticommutation relations instead of commutation relations, in order to obtain a consistent quantum theory for spin $\frac{1}{2}$ particles. This is related to the spin-statistics theorem, which states that particles with half-integer spin are fermions and must obey Fermi-Dirac statistics, while particles with integer spin are bosons and obey Bose-Einstein statistics.

5.3.1 | Quantizing with canonical commutation relations

Let us write the general solution of the Dirac equation in terms of linear combinations of positive and negative frequency solutions:

$$\hat{\psi}(\mathbf{x}) = \sum_{s=1}^{2} \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} \left(\hat{b}^s_\mathbf{p} u^s(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}} + \hat{c}^{s\,\dagger}_\mathbf{p} v^s(\mathbf{p}) e^{-i\mathbf{p}\cdot\mathbf{x}} \right),$$

and its adjoint

$$\hat{\psi}^\dagger(\mathbf{x}) = \sum_{s=1}^{2} \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} \left(\hat{b}^{s\,\dagger}_\mathbf{p} u^{s\,\dagger}(\mathbf{p}) e^{-i\mathbf{p}\cdot\mathbf{x}} + \hat{c}^s_\mathbf{p} v^{s\,\dagger}(\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}} \right).$$

We understand that the operators $\hat{b}^s_\mathbf{p}$ and $\hat{b}^{s\,\dagger}_\mathbf{p}$ are annihilation and creation operators for what we will call for now $b-$**type particles** (associated to positive frequency planar waves), with momentum $\mathbf{p}$ and spin s, while $\hat{c}^s_\mathbf{p}$ and $\hat{c}^{s\,\dagger}_\mathbf{p}$ are annihilation and creation operators for the so called $c-$**type particles** (associated to negative frequency planar waves) with momentum $\mathbf{p}$ and spin s. We will prove in the end that $b-$type particles correspond to particles, while $c-$type particles correspond to antiparticles, but for now we do not make this assumption.

If we now impose the canonical commutation relations on the field operators, we find that the commutation relations for the creation and annihilation operators are

$$\begin{aligned}[\hat{b}^s_{\mathbf{p}},\, \hat{b}^{r\,\dagger}_{\mathbf{q}}] &= (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\delta^{sr},\\ [\hat{c}^s_{\mathbf{p}},\, \hat{c}^{r\,\dagger}_{\mathbf{q}}] &= -(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\delta^{sr},\end{aligned} \tag{5.3.1}$$

with all other commutators vanishing. The minus sign in the second commutation relation is problematic, since it leads to negative norm states when we compute the norm of single particle states, as we will see.

It is easy to use these commutation relations to compute the fields commutators $\left[\hat{\psi}_\alpha(\mathbf{x}),\, \hat{\psi}^\dagger_\beta(\mathbf{y})\right] = \delta_{\alpha\beta}\delta^{(3)}(\mathbf{x}-\mathbf{y})$:[7]

$$\begin{aligned}\left[\hat{\psi}(\mathbf{x}),\, \hat{\psi}^\dagger(\mathbf{y})\right] &= \sum_{r,s}\int \frac{\mathrm{d}^3\mathbf{p}\mathrm{d}^3\mathbf{q}}{(2\pi)^6}\frac{1}{2\sqrt{E_{\mathbf{p}}E_{\mathbf{q}}}}\Big([\hat{b}^s_{\mathbf{p}},\, \hat{b}^{r\,\dagger}_{\mathbf{q}}]u^s(\mathbf{p})u^{r\,\dagger}(\mathbf{q})e^{i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})}\\ &\qquad +[\hat{c}^{s\,\dagger}_{\mathbf{p}},\, \hat{c}^r_{\mathbf{q}}]v^s(\mathbf{p})v^{r\,\dagger}(\mathbf{q})e^{-i(\mathbf{p}\cdot\mathbf{x}-\mathbf{q}\cdot\mathbf{y})}\Big)\\ &= \sum_s\int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\Big(u^s(\mathbf{p})u^{s\,\dagger}(\mathbf{p})e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}+v^s(\mathbf{p})v^{s\,\dagger}(\mathbf{p})e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\Big)\\ &= \sum_s\int\frac{\mathrm{d}\mathbf{p}}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}}}\Big(u^s(\mathbf{p})\overline{u}^s(\mathbf{p})\gamma^0e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}+v^s(\mathbf{p})\overline{v}^s(\mathbf{p})\gamma^0e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\Big),\end{aligned}$$

where we inseted identities ($(\gamma^0)^2 = \mathbb{I}$) to build Dirac conjugates and use the outer product formulae for the spinors of (5.2.12), reported below,

$$\begin{cases}\displaystyle\sum_{s=1}^2 u^s(\mathbf{p})\overline{u}^s(\mathbf{p}) = \gamma^\mu p_\mu + m,\\ \displaystyle\sum_{s=1}^2 v^s(\mathbf{p})\overline{v}^s(\mathbf{p}) = \gamma^\mu p_\mu - m,\end{cases}$$

to obtain

$$\left[\hat{\psi}(\mathbf{x}),\, \hat{\psi}^\dagger(\mathbf{y})\right] = \int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\left[\left(p_0\gamma^0+p_i\gamma^i+m\right)e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}+\left(p_0\gamma^0+p_i\gamma^i-m\right)e^{-i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\right]\frac{\gamma^0}{2E_{\mathbf{p}}},$$

where now we can split the integrals and perform a change of variable on the second term $\mathbf{p}\to-\mathbf{p}$, so that $p_i\to-p_i$ and $p_0\to p_0$, and we get

$$\left[\hat{\psi}(\mathbf{x}),\, \hat{\psi}^\dagger(\mathbf{y})\right] = \int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\left[\frac{2p_0}{2E_{\mathbf{p}}}(\gamma^0)^2e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\right] = \delta^{(3)}(\mathbf{x}-\mathbf{y}),$$

which is exactly what we wanted to prove.

If we consider the vacuum state $|0\rangle$ such that

$$\hat{c}^r_{\mathbf{p}}\,|0\rangle = 0,\quad \forall\mathbf{p},\, r,$$

and

$$\hat{c}^{s\,\dagger}_{\mathbf{p}}\,|0\rangle = |\mathbf{p},\, s\rangle,$$

but if we now compute the norm of this one-particle state, after inserting the commutator and simplifying the action of the annihilation operator on the void, we get

$$\langle\mathbf{p},\, s|\mathbf{p},\, s\rangle = \langle 0|\,\hat{c}^s_{\mathbf{p}}\hat{c}^{s\,\dagger}_{\mathbf{p}}\,|0\rangle = \langle 0|\,[\hat{c}^s_{\mathbf{p}},\, \hat{c}^{s\,\dagger}_{\mathbf{p}}]\,|0\rangle = -(2\pi)^3\delta^{(3)}(0)\,\langle 0|0\rangle < 0,$$

[7] Since the second term in the commutator is proportional to $\hat{\pi}$ as we have seen: we are computing the same commutators presented in the preamble of this section.

which is less than zero, leading to negative norm states, which are unphysical. Thus we have to modify our initial assumption, since maybe we have made a wrong choice in imposing the dagger on the creation operator; maybe the creation operator should be defined without the dagger, so that

$$\hat{c}^r_{\mathbf{p}} \left|0\right\rangle \neq 0 \rightarrow \hat{c}^r_{\mathbf{p}} \left|0\right\rangle = \left|\mathbf{p},\, r\right\rangle,$$

and if we swap the dagger in the commutation relations we get the opposite sign indeed:

$$\left[\hat{c}^{r\,\dagger}_{\mathbf{p}},\, \hat{c}^s_{\mathbf{q}}\right] = (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\delta^{rs}.$$

But this interpretation is not consistent, since when we consider the energy solutions they will be negative. As we will see, the solution to this problem is to impose anticommutation relations instead of commutation relations, so that we can have positive norm states for both type of particle, both with positive defined energy.

Let us ignore the probability interpretation for now and compute the Hamiltonian operator for the Dirac field, to show that this is not the solution of the problem. Starting from the classical Hamiltonian density

$$\begin{aligned}\mathcal{H} = \pi\dot{\psi} - \mathcal{L} &= i\psi^\dagger\partial_t\psi - \overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi \\ &= i\overline{\psi}\gamma^0\partial_0\psi - \overline{\psi}(i\gamma^0\partial_0 + i\gamma^i\partial_i - m)\psi \\ &= \overline{\psi}(-i\gamma^i\partial_i + m)\psi,\end{aligned}$$

where we have used the Legendre transformation, but we could have obtained the same result by computing explicity the invariance under spacetime translations: the associated conserved Noether charge is the energy density, expressed by the Hamiltonian density. Now we can write the Hamiltonian operator by promoting the fields to operators and starting to compute the right side of the expression:

$$\begin{aligned}(-i\gamma^i\partial_i + m)\psi(x) &= \sum_s\int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{(-i\gamma^i\partial_i + m)}{\sqrt{2E_{\mathbf{p}}}}\left(\hat{b}^s_{\mathbf{p}}u^s(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}} + \hat{c}^{s\,\dagger}_{\mathbf{p}}v^s(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}}\right) \\ &= \sum_s\int\frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(\hat{b}^s_{\mathbf{p}}(-\gamma^i p_i + m)u^s(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}} + \hat{c}^{s\,\dagger}_{\mathbf{p}}(\gamma^i p_i + m)v^s(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}}\right),\end{aligned}$$

where we had to pay attention on the momentum index in the derivative: $-i\gamma^i\partial_i e^{ip^jx^j} = -i\gamma^i(ip^i) = \gamma^i p^i = -\gamma^i p_i$, since lowering it means that the spatial components change sign. Thus it is now possible use the Dirac equations for the spinors to simplify further

$$(\gamma^\mu p_\mu - m)u^s(\mathbf{p}) = 0, \quad (\gamma^\mu p_\mu + m)v^s(\mathbf{p}) = 0,$$

and get finally to

$$\begin{aligned}(-\gamma^i p_i + m)u^s(\mathbf{p}) &= \gamma^0 p_0 u^s(\mathbf{p}) = E_{\mathbf{p}}\gamma^0 u^s(\mathbf{p}), \\ (\gamma^i p_i + m)v^s(\mathbf{p}) &= -\gamma^0 p_0 v^s(\mathbf{p}) = -E_{\mathbf{p}}\gamma^0 v^s(\mathbf{p}).\end{aligned}$$

Using these results and inserting them back into the last expression to get

$$(-i\gamma^i\partial_i + m)\psi(x) = \sum_s\int\frac{\mathrm{d}^3p}{(2\pi)^3}\sqrt{\frac{E_{\mathbf{p}}}{2}}\gamma^0\left(\hat{b}^s_{\mathbf{p}}u^s(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}} - \hat{c}^{s\,\dagger}_{\mathbf{p}}v^s(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}}\right).$$

Finally we can write the Hamiltonian operator $\hat{H} = \int \mathrm{d}^3x\,\hat{\overline{\psi}}(x)\left(-i\gamma^i\partial_i + m\right)\hat{\psi}(x)$ so that it takes the following form

$$\sum_{r,s}\int\mathrm{d}^3\mathbf{x}\frac{\mathrm{d}^3\mathbf{p}\mathrm{d}^3\mathbf{q}}{(2\pi)^6}\frac{1}{2}\sqrt{\frac{E_{\mathbf{p}}}{E_{\mathbf{q}}}}\left(\hat{b}^{r\,\dagger}_{\mathbf{q}}u^{r\,\dagger}(\mathbf{q})e^{-i\mathbf{q}\cdot\mathbf{x}} + \hat{c}^r_{\mathbf{q}}v^{r\,\dagger}(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{x}}\right)\gamma^0\gamma^0\left(\hat{b}^s_{\mathbf{p}}u^s(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}} - \hat{c}^{s\,\dagger}_{\mathbf{p}}v^s(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}}\right),$$

since $\overline{\psi} = \psi^\dagger \gamma^0$. Now, after simplifying $(\gamma^0)^2$, we can perform the integration over $\mathbf{x}$ which gives us a delta function $(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})$ or $(2\pi)^3\delta^{(3)}(\mathbf{p}+\mathbf{q})$, depending on the exponentials, so that we get

$$\hat{H} = \sum_{r,s} \int \frac{\mathrm{d}^3\mathbf{p}\mathrm{d}^3\mathbf{q}}{(2\pi)^3} \frac{1}{2}\sqrt{\frac{E_\mathbf{p}}{E_\mathbf{q}}} \left(\hat{b}^{r\,\dagger}_\mathbf{q} \hat{b}^s_\mathbf{p} u^{r\,\dagger}(\mathbf{q}) u^s(\mathbf{p}) \delta^{(3)}(\mathbf{p}-\mathbf{q}) + \dots \right),$$

where we can now use the delta functions to perform the integration over $\mathbf{q}$:[8]

$$\hat{H} = \sum_{r,s} \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{2} \left(\hat{b}^{r\,\dagger}_\mathbf{p} \hat{b}^s_\mathbf{p} u^{r\,\dagger}(\mathbf{p}) u^s(\mathbf{p}) - \hat{c}^r_\mathbf{p} \hat{c}^{s\,\dagger}_\mathbf{p} v^{s\,\dagger}(\mathbf{p}) v^s(\mathbf{p}) - (\cdot) u^{r\,\dagger}(-\mathbf{p}) v^s(\mathbf{p}) + (\cdot) v^{r\,\dagger}(-\mathbf{p}) u^s(\mathbf{p}) \right),$$

where we had to relabel the momentum derived from $\delta(\mathbf{p}+\mathbf{q})$ so that we have opposite sign momenta in the mixed terms. But remembering the inner product formulae for the mixed spinors in (5.2.11),

$$\begin{cases} u^{n\,\dagger}(\mathbf{p}) v^m(-\mathbf{p}) = u^{n\,\dagger}(-\mathbf{p}) v^m(\mathbf{p}) = 0, \\ v^{n\,\dagger}(\mathbf{p}) u^m(-\mathbf{p}) = v^{n\,\dagger}(-\mathbf{p}) u^m(\mathbf{p}) = 0, \end{cases}$$

those last terms vanish, while for the first two we can simplify their expression using the other inner products in $(5.2.8)_1$ and $(5.2.9)_1$, reported below:

$$\begin{cases} u^{n\,\dagger}(\mathbf{p}) u^m(\mathbf{p}) = 2E_\mathbf{p}\delta^{nm}, \\ v^{n\,\dagger}(\mathbf{p}) v^m(\mathbf{p}) = 2E_\mathbf{p}\delta^{nm}, \end{cases}$$

Thus the Hamiltonian takes the final form:

$$\hat{H} = \sum_s \int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_\mathbf{p} \left(\hat{b}^{s\,\dagger}_\mathbf{p} \hat{b}^s_\mathbf{p} - \hat{c}^s_\mathbf{p} \hat{c}^{s\,\dagger}_\mathbf{p} \right).$$

After this long computation, we check the physical consistency of this Hamiltonian operator: we are spanning spins and momenta, then we have an energy term $E_\mathbf{p} = \sqrt{\mathbf{p}^2 + m^2} > 0$ multiplied by number operators for $b-$type and $c-$type particles. The problem is that the $c-$particle term $c^s_\mathbf{p} c^{s\,\dagger}$ has a minus sign, associeted than to negative energy contribution. Thus we have fixed the norm problem, but now we have negative energy states, which are also unphysical.

We try to solve it by reordering the operators in the $c-$particle term, so that we have

$$\hat{c}^s_\mathbf{p} \hat{c}^{s\,\dagger}_\mathbf{p} = \hat{c}^{s\,\dagger}_\mathbf{p} \hat{c}^s_\mathbf{p} + \left[\hat{c}^s_\mathbf{p}, \hat{c}^{s\,\dagger}_\mathbf{p} \right] = \hat{c}^{s\,\dagger}_\mathbf{p} \hat{c}^s_\mathbf{p} - (2\pi)^3 \delta^{(3)}(0),$$

such that

$$\hat{H} = \sum_s \int \frac{\mathrm{d}^3 p}{(2\pi)^3} E_\mathbf{p} \left(\hat{b}^{s\,\dagger}_\mathbf{p} \hat{b}^s_\mathbf{p} - \hat{c}^{s\,\dagger}_\mathbf{p} \hat{c}^s_\mathbf{p} + (2\pi)^3 \delta^{(3)}(0) \right),$$

where now both $b-$type and $c-$type particles have positive energy, and the infinite constant term proportional to $\delta^{(3)}(0)$, which diverges, can be removed by redefining the zero point of energy and normal ordering, as we did for the scalar field; but we have already seen that $c-$particles defined with $c^{s\,\dagger}_\mathbf{p}$ as creation operator have negative norm.

Let us verify explicitly that both $b-$type and $c-$type particles have positive energy while the latter

[8] Remember that we are working with four dimensional vectors, but the Hamiltonian is either a number or an operator: in the end we ought to have products among row vectors (daggered) and column ones.

have negative norm, by computing the commutator of the Hamiltonian with the creation operators:

$$\begin{aligned}\left[\hat{H},\,\hat{b}^{s\,\dagger}_{\mathbf{p}}\right] &= \sum_r \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} E_{\mathbf{q}} \left(\left[\hat{b}^{r\,\dagger}_{\mathbf{q}}\hat{b}^{r}_{\mathbf{q}},\,\hat{b}^{s\,\dagger}_{\mathbf{p}}\right] - \left[\hat{c}^{r\,\dagger}_{\mathbf{q}}\hat{c}^{r}_{\mathbf{q}},\,\hat{b}^{s\,\dagger}_{\mathbf{p}}\right]\right)\\ &= \sum_r \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} E_{\mathbf{q}} \left(\hat{b}^{r\,\dagger}_{\mathbf{q}}\left[\hat{b}^{r}_{\mathbf{q}},\,\hat{b}^{s\,\dagger}_{\mathbf{p}}\right] + \left[\hat{b}^{r\,\dagger}_{\mathbf{q}},\,\hat{b}^{s\,\dagger}_{\mathbf{p}}\right]\hat{b}^{r}_{\mathbf{q}} - 0\right)\\ &= \sum_r \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} E_{\mathbf{q}} \left(\hat{b}^{r\,\dagger}_{\mathbf{q}}\delta^{rs}\delta^{(3)}(\mathbf{p}-\mathbf{q}) + 0\right) = E_{\mathbf{p}}\hat{b}^{s\,\dagger}_{\mathbf{p}},\end{aligned}$$

while for the $c-$type particles (negative norm) we have similarly

$$\begin{aligned}\left[\hat{H},\,\hat{c}^{s\,\dagger}_{\mathbf{p}}\right] &= \sum_r \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} E_{\mathbf{q}} \left(0 - \hat{c}^{r\,\dagger}_{\mathbf{q}}\left[\hat{c}^{r}_{\mathbf{q}},\,\hat{c}^{s\,\dagger}_{\mathbf{p}}\right] - \left[\hat{c}^{r\,\dagger}_{\mathbf{q}},\,\hat{c}^{s\,\dagger}_{\mathbf{p}}\right]\hat{c}^{r}_{\mathbf{q}}\right)\\ &= \sum_r \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} E_{\mathbf{q}} \left[-\hat{c}^{r\,\dagger}_{\mathbf{q}}\left(-\delta^{rs}\delta^{(3)}(\mathbf{p}-\mathbf{q})\right) - 0\right] = E_{\mathbf{p}}\hat{c}^{s\,\dagger}_{\mathbf{p}}.\end{aligned}$$

Now the action of the Hamiltonian on one-particle states can be resumed here as

$$\begin{aligned}\hat{H}\left(\hat{b}^{s\,\dagger}_{\mathbf{p}}\left|0\right\rangle\right) &= E_{\mathbf{p}}\left(\hat{b}^{s\,\dagger}_{\mathbf{p}}\left|0\right\rangle\right) + \hat{b}^{s\,\dagger}_{\mathbf{p}}\hat{H}\left|0\right\rangle = E_{\mathbf{p}}\left(\hat{b}^{s\,\dagger}_{\mathbf{p}}\left|0\right\rangle\right),\\ \hat{H}\left(\hat{c}^{s\,\dagger}_{\mathbf{p}}\left|0\right\rangle\right) &= E_{\mathbf{p}}\left(\hat{c}^{s\,\dagger}_{\mathbf{p}}\left|0\right\rangle\right) + \hat{c}^{s\,\dagger}_{\mathbf{p}}\hat{H}\left|0\right\rangle = E_{\mathbf{p}}\left(\hat{c}^{s\,\dagger}_{\mathbf{p}}\left|0\right\rangle\right),\end{aligned}$$

so that both types of particles have positive energy excitations, but we still have the negative norm problem for $c-$type particles.

Trying to use the other definition for $c-$type particles, where $\hat{c}^{s}_{\mathbf{p}}$ is the creation operator, we return to the expression of the Hamiltonian before reordering the operators:

$$\hat{H} = \sum_s \int \frac{\mathrm{d}^3p}{(2\pi)^3} E_{\mathbf{p}} \left(\hat{b}^{s\,\dagger}_{\mathbf{p}}\hat{b}^{s}_{\mathbf{p}} - \hat{c}^{s}_{\mathbf{p}}\hat{c}^{s\,\dagger}_{\mathbf{p}}\right),$$

and we compute again the commutator with the creation operator $\hat{c}^{s}_{\mathbf{p}}$:

$$\begin{aligned}\left[\hat{H},\,\hat{c}^{s}_{\mathbf{p}}\right] &= \sum_r \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} E_{\mathbf{q}} \left(0 - \hat{c}^{r}_{\mathbf{q}}\left[\hat{c}^{r\,\dagger}_{\mathbf{q}},\,\hat{c}^{s}_{\mathbf{p}}\right] - \left[\hat{c}^{r}_{\mathbf{q}},\,\hat{c}^{s}_{\mathbf{p}}\right]\hat{c}^{r\,\dagger}_{\mathbf{q}}\right)\\ &= \sum_r \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3} E_{\mathbf{q}} \left[-\hat{c}^{r}_{\mathbf{q}}\delta^{rs}\delta^{(3)}(\mathbf{p}-\mathbf{q}) - 0\right] = -E_{\mathbf{p}}\hat{c}^{s}_{\mathbf{p}},\end{aligned}$$

so that the action of the Hamiltonian on one-particle states is

$$\hat{H}\left(\hat{c}^{s}_{\mathbf{p}}\left|0\right\rangle\right) = -E_{\mathbf{p}}\left(\hat{c}^{s}_{\mathbf{p}}\left|0\right\rangle\right) + \hat{c}^{s}_{\mathbf{p}}\hat{H}\left|0\right\rangle = -E_{\mathbf{p}}\left(\hat{c}^{s}_{\mathbf{p}}\left|0\right\rangle\right),$$

which is negative definite, leading to negative energy states, unphysical as well, since one could create more and more negative energy states by adding more and more $c-$type particles, leading to an unbounded from below energy spectrum and an unstable vacuum.

To summarize, if we try quantizing the Dirac field by imposing commutation relations we end up with either negative norm states or negative energy states, both unphysical:

- $b-$**type** particles are associated to positive frequency solutions, have positive norm and positive energy;
- $c-$**type** particles, associated to negative frequency solutions, give us some problems:
  1. if we define them with positive energy states ($\hat{c}^{s\,\dagger}_{\mathbf{p}}$ defined as creation operator) they have negative norm: ill-defined Hilbert space;

2. if we define them with positive norm ($\hat{c}^s_{\mathbf{p}}$ defined as creation operator) they have negative energy states: energy unbounded from below.[9]

We thus have to change paradigm of quantization.

5.3.2 | Quantizing with canonical anticommutation relations

If we recall what happened for **bosons** in Klein-Gordon theory, we imposed canonical **commutation** relations between the field and its conjugate momentum at equal times, leading to commutation relations for creation and annihilation operators

$$|\mathbf{p},\, \mathbf{q}\rangle = \hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\,|0\rangle = \hat{a}^\dagger_{\mathbf{q}}\hat{a}^\dagger_{\mathbf{p}}\,|0\rangle\,,$$

since the creation operators commute

$$\left[\hat{a}^\dagger_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{q}}\right] = 0.$$

This means that we can create multiple particles in the same state, leading to *Bose-Einstein statistics.*

We do not find this acceptable for **fermions**, since they obey the *Pauli exclusion principle*, which states that no two fermions can occupy the same quantum state simultaneously. We need a minus sign when swapping two fermionic operators, thus we have to impose **canonical anticommutation relations** between the field and its conjugate momentum at equal times, leading to anticommutation relations for creation and annihilation operators:

$$\begin{aligned} \{\hat{\psi}_\alpha(\mathbf{x}),\, \hat{\psi}_\beta(\mathbf{y})\} = \{\hat{\psi}^\dagger_\alpha(\mathbf{x}),\, \hat{\psi}^\dagger_\beta(\mathbf{y})\} &= 0, \\ \{\psi_\alpha(\mathbf{x}),\, \hat{\psi}^\dagger_\beta(\mathbf{y})\} &= \delta_{\alpha\beta}\delta^{(3)}(\mathbf{x}-\mathbf{y}). \end{aligned} \tag{5.3.2}$$

This leads to anticommutation relations for creation and annihilation operators:

$$\begin{aligned} \{\hat{b}^s_{\mathbf{p}},\, \hat{b}^{r\,\dagger}_{\mathbf{q}}\} &= (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\delta^{sr}, \\ \{\hat{c}^s_{\mathbf{p}},\, \hat{c}^{r\,\dagger}_{\mathbf{q}}\} &= (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q})\delta^{sr}, \end{aligned} \tag{5.3.3}$$

while all other anticommutators vanish.

Now, as last time, we have to compute the commutator of the Hamiltonian operator with the creation operators to check if the energy excitations are positive and associated to positive norm states. If we were to repeat the previous computation of the Hamiltonian operator starting from the Hamiltonian density, we would end up with the same expression as before, since we had used only the field expansion and inner products of spinors, until normal ordering. Thus we have again

$$\hat{H} = \sum_s \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} E_{\mathbf{p}} \left(\hat{b}^{s\,\dagger}_{\mathbf{p}}\hat{b}^s_{\mathbf{p}} - \hat{c}^s_{\mathbf{p}}\hat{c}^{s\,\dagger}_{\mathbf{p}}\right),$$

where we can now reorder the operators in the $c-$type particle term, this time introducing a minus sign from the anticommutation relations:

$$\hat{H} = \sum_s \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} E_{\mathbf{p}} \left(\hat{b}^{s\,\dagger}_{\mathbf{p}}\hat{b}^s_{\mathbf{p}} + \hat{c}^{s\,\dagger}_{\mathbf{p}}\hat{c}^s_{\mathbf{p}} - (2\pi)^3\delta^{(3)}(0)\right),$$

[9]This is similar to what happened for the classical Dirac equation, where negative frequency solutions had to be reinterpreted as antiparticles going backward in time to avoid negative energy states; we say unbounded from below because we could create more and more negative energy states by adding more and more $c-$type particles, leading to an unstable vacuum.

and now again both types of particles have positive energy excitations, if we interpret $\hat{c}_{\mathbf{p}}^{\dagger}$ as the creation operators and thus the second term as the $c-$type particle number operator; in the end we have a divergent constant term which we can remove by normal ordering as before.[10] Thus we have finally reached a consistent quantum theory for spin $\frac{1}{2}$ particles, by imposing anticommutation relations instead of commutation relations and a normal ordered Hamiltonian defined as

$$\hat{H} = \sum_s \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} E_{\mathbf{p}} \left(\hat{b}_{\mathbf{p}}^{s\,\dagger} \hat{b}_{\mathbf{p}}^{s} + \hat{c}_{\mathbf{p}}^{s\,\dagger} \hat{c}_{\mathbf{p}}^{s} \right). \tag{5.3.4}$$

We can verify explicitly that both $b-$type and $c-$type particles have positive energy excitations, looking at the vacuum state with no particles

$$\hat{b}_{\mathbf{p}}^{s} \left|0\right\rangle = 0, \quad \hat{c}_{\mathbf{p}}^{s} \left|0\right\rangle = 0, \quad \forall \mathbf{p},\, s,$$

and by computing the commutator of the Hamiltonian with the creation operators, with the idea to apply it to one-particle states:

$$\begin{aligned} \left[\hat{H},\, \hat{b}_{\mathbf{p}}^{s\,\dagger}\right] &= E_{\mathbf{p}} \hat{b}_{\mathbf{p}}^{s\,\dagger}, \quad \left[\hat{H},\, \hat{c}_{\mathbf{p}}^{s\,\dagger}\right] = E_{\mathbf{p}} \hat{c}_{\mathbf{p}}^{s\,\dagger}, \\ \left[\hat{H},\, \hat{b}_{\mathbf{p}}^{s}\right] &= -E_{\mathbf{p}} \hat{b}_{\mathbf{p}}^{s}, \quad \left[\hat{H},\, \hat{c}_{\mathbf{p}}^{s}\right] = -E_{\mathbf{p}} \hat{c}_{\mathbf{p}}^{s}. \end{aligned}$$

Now on **one-particle states** we have

$$\begin{aligned} \left|\mathbf{p}, s\right\rangle_b &= \hat{b}_{\mathbf{p}}^{s\,\dagger} \left|0\right\rangle, \\ \left|\mathbf{p}, s\right\rangle_c &= \hat{c}_{\mathbf{p}}^{s\,\dagger} \left|0\right\rangle, \end{aligned}$$

so that the norm of these states is positive:

$$\begin{aligned} {}_b\left\langle \mathbf{q}, r | \mathbf{p}, s\right\rangle_b &= \left\langle 0\right| \hat{b}_{\mathbf{q}}^{r} \hat{b}_{\mathbf{p}}^{s\,\dagger} \left|0\right\rangle = \left\langle 0\right| \{\hat{b}_{\mathbf{q}}^{r},\, \hat{b}_{\mathbf{p}}^{s\,\dagger}\} \left|0\right\rangle = (2\pi)^3 \delta^{(3)}(0) \left\langle 0|0\right\rangle > 0, \\ {}_c\left\langle \mathbf{q}, r | \mathbf{p}, s\right\rangle_c &= \left\langle 0\right| \hat{c}_{\mathbf{q}}^{r} \hat{c}_{\mathbf{p}}^{s\,\dagger} \left|0\right\rangle = \left\langle 0\right| \{\hat{c}_{\mathbf{q}}^{r},\, \hat{c}_{\mathbf{p}}^{s\,\dagger}\} \left|0\right\rangle = (2\pi)^3 \delta^{(3)}(0) \left\langle 0|0\right\rangle > 0, \end{aligned}$$

wher the difference with respect to the commutation relation case is the anticommutator used to swap the operators, which does not introduce a minus sign as the commutator did.

Finally, we can compute the action of the Hamiltonian on one-particle states:

$$\begin{aligned} \hat{H} \left(\hat{b}_{\mathbf{p}}^{s\,\dagger} \left|0\right\rangle \right) &= \left[\hat{H},\, \hat{b}_{\mathbf{p}}^{s\,\dagger}\right] \left|0\right\rangle + \hat{b}_{\mathbf{p}}^{s\,\dagger} \hat{H} \left|0\right\rangle = E_{\mathbf{p}} \left(\hat{b}_{\mathbf{p}}^{s\,\dagger} \left|0\right\rangle \right), \\ \hat{H} \left(\hat{c}_{\mathbf{p}}^{s\,\dagger} \left|0\right\rangle \right) &= \left[\hat{H},\, \hat{c}_{\mathbf{p}}^{s\,\dagger}\right] \left|0\right\rangle + \hat{c}_{\mathbf{p}}^{s\,\dagger} \hat{H} \left|0\right\rangle = E_{\mathbf{p}} \left(\hat{c}_{\mathbf{p}}^{s\,\dagger} \left|0\right\rangle \right). \end{aligned}$$

Thus both $b-$type and $c-$type particles have positive norm states and positive energy excitations, solving all the problems we had before and leading to a consistent quantum theory for spin $\frac{1}{2}$ particles.

The question we are now left is: what is the difference between $b-$type and $c-$type particles? They are both spin $\frac{1}{2}$ particles with mass m and positive energy excitations; they are degenerate eigenstates of the Hamiltonian, momentum and spin. The answer is that there has to be another invariance of the system, associated to a noether charge symmetry, which distinguishes this two types of particles: we are speaking about the **electric charge**, and $b-$type particles are associated to particles with charge $+e$ (like electrons), while $c-$type particles are associated to antiparticles with charge $-e$ (like positrons), as we will see in the next section.

[10] This constant term is related to the vacuum energy, which in KG theory came with a plus sign.

5.3.3 | Internal Global Symmetry

We are looking for a conserved charge which distinguishes between $b-$type and $c-$type particles. To find it, we can search for a global (i.e., spacetime-independent) and internal (i.e., acting on the fields but not on spacetime coordinates) symmetry of the Dirac Lagrangian. It is easy to see that the Dirac Lagrangian is invariant under the global phase transformation

$$\psi(x) \to \psi'(x) = e^{-iq\theta}\psi(x), \quad \overline{\psi}(x) \to \overline{\psi}'(x) = e^{iq\theta}\overline{\psi}(x),$$

where q is a constant associated to the charge of the field and θ is a constant parameter. This is a symmetry since the Lagrangian density

$$\mathcal{L} = \overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi,$$

depends on ψ and $\overline{\psi}$ only in the combinations $\overline{\psi}\psi$ and $\overline{\psi}\gamma^\mu\partial_\mu\psi$, which are invariant under this transformation.

$$\mathcal{L} \to \mathcal{L}' = \overline{\psi}'(i\gamma^\mu\partial_\mu - m)\psi' = e^{iq\theta}\overline{\psi}(i\gamma^\mu\partial_\mu - m)e^{-iq\theta}\psi = \overline{\psi}(i\gamma^\mu\partial_\mu - m)\psi = \mathcal{L}.$$

This is a global U(1) symmetry, thus associated to a conserved noether current and a one dimensional charge.

> **Remark.** *If we were to consider a coordinate-dependent phase $\theta(x)$ we would have a local* U(1) *symmetry, which is the gauge symmetry of quantum electrodynamics (QED) and the starter point for introducing interactions between the Dirac field and the electromagnetic field. We thus need a global* U(1) *symmetry to have a conserved charge distinguishing between particles and antiparticles, since the derivatives in the Lagrangian would break the local symmetry.*

Using noether's theorem we can compute the conserved current with the expression in (3.2.4) (with the K^μ term vanishing since the Lagrangian is invariant):

$$\begin{aligned} J^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi_i)}\delta\psi_i = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\psi})}\delta\overline{\psi}, \\ &= \overline{\psi}i\gamma^\mu(-iq\psi) + 0\cdot(iq\overline{\psi}) = q\overline{\psi}\gamma^\mu\psi, \end{aligned}$$

since $\delta\psi = -iq\psi$ and $\delta\overline{\psi} = iq\overline{\psi}$ for an infinitesimal transformation. The current J^μ is a conserved vector, and if we differentiate it we get

$$\begin{aligned} \partial_\mu J^\mu &= q\left((\partial_\mu\overline{\psi})\gamma^\mu\psi + \overline{\psi}\gamma^\mu(\partial_\mu\psi)\right) \\ &= q\left(-i\overline{\psi}m\psi + i\overline{\psi}m\psi\right) = 0, \end{aligned}$$

where in the last step we have made use of the equations of motion

$$i\gamma^\mu\partial_\mu\psi = m\psi, \quad \partial_\mu\overline{\psi}i\gamma^\mu = -m\overline{\psi},$$

to prove that the current is indeed conserved.

The associated conserved charge is given by the integral over space of the time component of the current:

$$\hat{Q} = \int \mathrm{d}^3\mathbf{x} J^0 = q\int \mathrm{d}^3\mathbf{x}\,\hat{\overline{\psi}}(x)\gamma^0\hat{\psi}(x) = q\int \mathrm{d}^3\mathbf{x}\,\hat{\psi}^\dagger(x)\hat{\psi}(x),$$

where we can recognize a very similar expression to what we had for the Hamiltonian density:[11] the procedure is thus very similar to what we did for the Hamiltonian operator, and we end up

[11] The only differences are a changed sign in the expression of $\hat{\psi}(x)$ (which manifest itself in the middle of the two terms) and a missing energy factor $E_{\mathbf{p}}$.

with the **normal ordered charge operator**

$$\hat{Q} = q \sum_s \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \left(\hat{b}_{\mathbf{p}}^{s\,\dagger} \hat{b}_{\mathbf{p}}^{s} - \hat{c}_{\mathbf{p}}^{s\,\dagger} \hat{c}_{\mathbf{p}}^{s} \right) , \tag{5.3.5}$$

where we have used the field expansion in terms of creation and annihilation operators and the inner product relations for the spinors to simplify the expression. It is easy to see that the charge operator counts the number of $b-$type particles minus the number of $c-$type particles, multiplied by the charge q. Thus we can now compute the associated charge of one-particle states, knowing that it is exactly the same computation as for the Hamiltonian but with a minus sign in front of the $c-$type particle term:

$$\hat{Q} \left(\hat{b}_{\mathbf{p}}^{s\,\dagger} |0\rangle \right) = q \sum_r \int \frac{\mathrm{d}^3 \mathbf{q}}{(2\pi)^3} \left(\hat{b}_{\mathbf{q}}^{r\,\dagger} \hat{b}_{\mathbf{q}}^{r} - \hat{c}_{\mathbf{q}}^{r\,\dagger} \hat{c}_{\mathbf{q}}^{r} \right) \hat{b}_{\mathbf{p}}^{s\,\dagger} |0\rangle = q \left(\hat{b}_{\mathbf{p}}^{s\,\dagger} |0\rangle \right) ,$$

$$\hat{Q} \left(\hat{c}_{\mathbf{p}}^{s\,\dagger} |0\rangle \right) = q \sum_r \int \frac{\mathrm{d}^3 \mathbf{q}}{(2\pi)^3} \left(\hat{b}_{\mathbf{q}}^{r\,\dagger} \hat{b}_{\mathbf{q}}^{r} - \hat{c}_{\mathbf{q}}^{r\,\dagger} \hat{c}_{\mathbf{q}}^{r} \right) \hat{c}_{\mathbf{p}}^{s\,\dagger} |0\rangle = -q \left(\hat{c}_{\mathbf{p}}^{s\,\dagger} |0\rangle \right) ,$$

so that $b-$type states are particles with charge $+q$ while $c-$type states are antiparticles with charge $-q$, so that we can summarize the results of our quantization procedure as follows:

- $\hat{b}_{\mathbf{p}}^{s\,\dagger}$ create **particles** with energy $E_{\mathbf{p}}$, spin s, momentum $\mathbf{p}$ and charge $+q$;
- $\hat{c}_{\mathbf{p}}^{s\,\dagger}$ create **antiparticles** with energy $E_{\mathbf{p}}$, spin s, momentum $\mathbf{p}$ and charge $-q$.

Spin-statistics relations. We have seen that to quantize the Dirac field we had to impose anticommutation relations between the field and its conjugate momentum, leading to anticommutation relations for creation and annihilation operators. This is related to the *spin-statistics theorem*, which states that particles with integer spin (bosons) obey Bose-Einstein statistics and thus commutation relations, while particles with half-integer spin (fermions) obey Fermi-Dirac statistics and thus anticommutation relations.

We can see its effect by looking at two-particle states: for two fermions we have

$$|\mathbf{p},\, s;\; \mathbf{q},\, r\rangle = \hat{b}_{\mathbf{p}}^{s\,\dagger} \hat{b}_{\mathbf{q}}^{r\,\dagger} |0\rangle = -\hat{b}_{\mathbf{q}}^{r\,\dagger} \hat{b}_{\mathbf{p}}^{s\,\dagger} |0\rangle = - |\mathbf{q},\, r;\; \mathbf{p},\, s\rangle ,$$

which is antisymmetric under the exchange of the two particles, exactly as required by the **Pauli exclusion principle** and described by Fermi-Dirac statistics. In particular, if we try to create two fermions in the same state we have indeed $|\mathbf{p},\, s;\; \mathbf{p},\, s\rangle = -|\mathbf{p},\, s;\; \mathbf{p},\, s\rangle = 0$, showing that no two fermions can occupy the same quantum state simultaneously.

In the beginning, when Dirac was thinking about his equation in the RQM setup, he was not aware of this spin-statistics relation and he tried to interpret the negative energy solutions of his equation as physical states, leading to an unstable vacuum. He then proposed the *Dirac sea* idea, where all negative energy states are filled in the vacuum, and only **holes** in this sea (i.e., absence of negative energy electrons) can be interpreted as positrons, thus solving the negative energy problem. This creative idea was later abandoned in favor of the quantum field theory approach we have seen, where antiparticles arise naturally from the quantization procedure and the imposition of anticommutation relations for fermionic fields.

5.4 | Propagators

If we now move to the Heisenberg picture, we know that the time evolution of an operator $\hat{\psi}(x)$, with $x = (t, \mathbf{x})$, is given by

$$\partial_0 \hat{\psi}(x) = i \left[\hat{H},\, \hat{\psi}(x)\right],$$

which is the Heisenberg equation of motion, and it is solved by the time following operators

$$\hat{\psi}(x) = \sum_s \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} \left(\hat{b}^s_\mathbf{p} u^s(\mathbf{p}) e^{-ip_\mu x^\mu} + \hat{c}^{s\,\dagger}_\mathbf{p} v^s(\mathbf{p}) e^{ip_\mu x^\mu}\right),$$

and

$$\hat{\psi}^\dagger(x) = \sum_s \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} \left(\hat{b}^{s\,\dagger}_\mathbf{p} u^{s\,\dagger}(\mathbf{p}) e^{ip_\mu x^\mu} + \hat{c}^s_\mathbf{p} v^{s\,\dagger}(\mathbf{p}) e^{-ip_\mu x^\mu}\right),$$

where now the creation and annihilation operators are time-independent, while the time dependence is carried by the exponential factors: this is the Heisenberg picture analogue of the field expansion in the Schrödinger picture (we wrote the ladder operators in Heisenberg picture as we did for the KG field in section 4.3) and we did nothing but take the linear combination of solutions of the Dirac equation as before.

Now to ensure that there is no measurable effect outside the light cone, we have to impose that the anticommutator of the field at spacelike separated points vanishes:[12] we define the **fermionic propagator** as

$$\begin{aligned} iS_{\alpha\beta} &= \{\hat{\psi}_\alpha(x),\, \hat{\overline{\psi}}_\beta(y)\}, \\ iS(x-y) &= \{\hat{\psi}(x),\, \hat{\overline{\psi}}(y)\}, \end{aligned} \tag{5.4.1}$$

with suppressed spinor indices in the second expression. Now by substituting the field expansions into this expression and using the anticommutation relations for creation and annihilation operators, we find

$$\begin{aligned} iS(x-y) &= \sum_{r,s} \int \frac{\mathrm{d}^3\mathbf{p}\mathrm{d}^3\mathbf{q}}{(2\pi)^6} \frac{1}{2\sqrt{E_\mathbf{p}E_\mathbf{q}}} \Big(\{\hat{b}^s_\mathbf{p},\, \hat{b}^{r\,\dagger}_\mathbf{q}\} u^s(\mathbf{p}) \overline{u}^r(\mathbf{q}) e^{-i(p_\mu x^\mu - q_\mu y^\mu)} \\ &\qquad + \{\hat{c}^{s\,\dagger}_\mathbf{p},\, \hat{c}^r_\mathbf{q}\} v^s(\mathbf{p}) \overline{v}^r(\mathbf{q}) e^{i(p_\mu x^\mu - q_\mu y^\mu)}\Big) \\ &= \sum_s \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_\mathbf{p}} \left(u^s(\mathbf{p}) \overline{u}^s(\mathbf{p}) e^{-ip_\mu(x^\mu - y^\mu)} + v^s(\mathbf{p}) \overline{v}^s(\mathbf{p}) e^{ip_\mu(x^\mu - y^\mu)}\right) \end{aligned}$$

where we have used the anticommutation relations to simplify the expression; if we now apply the outer products of spinors in (5.2.12), we can rewrite the propagator as

$$iS(x-y) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_\mathbf{p}} \left((\not{p} + m) e^{-ip_\mu(x^\mu - y^\mu)} + (\not{p} - m) e^{ip_\mu(x^\mu - y^\mu)}\right),$$

recalling that $\not{p} = \gamma^\mu p_\mu$.

Now, this expression resembles the sum of two scalar field correlators for KG theory, which were defined in (4.3.3) as

$$D(x-y) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_\mathbf{p}} \left(e^{-ip_\mu(x^\mu - y^\mu)}\right),$$

[12] This expression is not clear as it is, but we will soon show that this condition implies the vanishing of all commutators of observables outside each other's light cone.

if we differenciate it with respect to x^μ we get

$$\partial_\mu^{(x)} D(x-y) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_\mathbf{p}} (-ip_\mu) e^{-ip_\mu(x^\mu - y^\mu)},$$

so that we can rewrite the fermionic propagator as

$$iS(x-y) = (i\gamma^\mu \partial_\mu^{(x)} + m)\left(D(x-y) - D(y-x)\right),$$

or equivalently

$$S(x-y) = -i(i\gamma^\mu \partial_\mu^{(x)} + m)\Delta(x-y), \tag{5.4.2}$$

where we have made use of the definition of the scalar propagator in (4.3.2).

This propagator now manifestly satisfies the following properties:

- For spacelike propagated points $(x-y)^2 < 0$:

  $$D(x-y) - D(y-x) = 0, \implies S(x-y) = 0,$$

  meaning that there is no measurable effect outside the light cone, preserving causality for the Dirac theory (which is still not clear and will be enforced in a moment).

  For bosons this point was ensured by commutation relations

  $$[\hat{\varphi}(x), \hat{\varphi}(y)] = 0 \quad \text{for} \quad (x-y)^2 < 0,$$

  while for fermions it is ensured by anticommutation relations:

  $$\{\hat{\psi}_\alpha(x), \hat{\psi}_\beta(y)\} = 0 \quad \text{for} \quad (x-y)^2 < 0;$$

- There is a problem: we want **observables to commute at spacelike separations**, while fermionic fields anticommute. The solution is that observables are constructed as *bilinear combinations of fermionic fields*, like the Hamiltonian density or the current density, which do commute at spacelike separations, thus preserving causality: the hamiltonian for example

  $$\hat{H} = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} E_\mathbf{p} \left(\hat{b}_\mathbf{p}^{s\,\dagger} \hat{b}_\mathbf{p}^s + \hat{c}_\mathbf{p}^{s\,\dagger} \hat{c}_\mathbf{p}^s\right),$$

  where the two main terms are fermionic bilinears respecting anti-commutation relations, thus their product commutes at spacelike separations:

  $$\hat{O}_1 = A_1 B_1, \quad \hat{O}_2 = A_2 B_2,$$
  $$\hat{O}_1 \hat{O}_2 = A_1 B_1 A_2 B_2 = -A_1 A_2 B_1 B_2 = -A_2 A_1 B_2 B_1 = A_2 B_2 A_1 B_1 = \hat{O}_2 \hat{O}_1,$$

  if each of the operators A_i, B_i respect the anticommutation relation of our fermionic fields; thus

  $$\left[\hat{O}_1,\, \hat{O}_2\right] = 0,$$

  since $\{A_1,\, B_1\} = \{A_2,\, B_2\} \neq 0$ while the other anticommutators are null.

- The last thing we can check is that **the fermionic propagator satisfies the Dirac equation** as a matrix:

  $$(i\gamma^\mu \partial_\mu^{(x)} - m) S(x-y) = 0,$$

since we can write the propagator in terms of the scalar propagator as

$$
\begin{aligned}
&-i(i\gamma^\mu\partial^{(x)}_\mu - m)(i\gamma^\mu\partial^{(x)}_\mu + m)\Delta(x-y)\\
&= -i\left(-\gamma^\mu\gamma^\nu\partial^{(x)}_\mu\partial^{(x)}_\nu - m^2\right)\Delta(x-y),
\end{aligned}
$$

where now we can use the symmetry of the derivatives to derive the anticommutation relations of gamma matrices

$$
\begin{aligned}
\gamma^\mu\gamma^\nu\partial^{(x)}_\mu\partial^{(x)}_\nu &= \frac{1}{2}\gamma^\mu\gamma^\nu\left(\partial^{(x)}_\mu\partial^{(x)}_\nu + \partial^{(x)}_\nu\partial^{(x)}_\mu\right)\\
&= \frac{1}{2}\{\gamma^\mu,\, \gamma^\nu\}\partial^{(x)}_\mu\partial^{(x)}_\nu = \eta^{\mu\nu}\partial^{(x)}_\mu\partial^{(x)}_\nu;
\end{aligned}
$$

thus we have

$$
i(\partial^{(x)}_\mu\partial^{\mu\,(x)} + m^2)\Delta(x-y) = i\left(\Box^{(x)} + m^2\right)\Delta(x-y) = 0,
$$

where in the last step we have used the definition of the scalar propagator in (4.3.2) and the fact that it satisfies the KG equation. Explicitly we have:

$$
\begin{aligned}
\left(\Box^{(x)} + m^2\right)\Delta(x-y) &= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}\left(\Box^{(x)} + m^2\right)\left(e^{-ip_\mu(x^\mu-y^\mu)} - e^{ip_\mu(x^\mu-y^\mu)}\right)\\
&= -\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}\left(p_\mu p^\mu - m^2\right)\left(e^{-ip_\mu(x^\mu-y^\mu)} - e^{ip_\mu(x^\mu-y^\mu)}\right) = 0,
\end{aligned}
$$

since on shell $p_\mu p^\mu = m^2$ and $\partial^{(x)}_\mu e^{-ip_\mu(x^\mu-y^\mu)} = -ip_\mu e^{-ip_\mu(x^\mu-y^\mu)}$, while $\partial^{\mu\,(x)}e^{-ip_\mu(x^\mu-y^\mu)} = -ip^\mu e^{-ip_\mu(x^\mu-y^\mu)}$ and similarly for the other exponential.

Thus we have defined the fermionic propagator, which ensures causality for the Dirac theory and satisfies the Dirac equation as expected. Note how the imposition of anticommutation relations for fermionic fields was crucial to ensure causality, since it led to the vanishing of the propagator at spacelike separations and the commutation of observables at spacelike separations. Furthermore we have seen how the fermionic propagator is related to the scalar propagator of KG theory, since at the foundamental level the components of the Dirac field satisfy the KG equation as well.

6 Spin 1 Particles

We are going to describe the Electromagnetic field as a quantum field theory of spin 1 particles, the photons. We will see that the photon is **massless** and has only **two polarization states**, which makes it different from the massive spin 1 particles we will study later, such as the W and Z bosons.

We have alredy gone through the quantization procedure for scalar fields (spin 0 particles) and spin $\frac{1}{2}$ fields (fermions), so we will focus here on the peculiarities of spin 1 fields, which arise mainly due to the gauge invariance of the electromagnetic field.

6.1 | The Classical Electromagnetic Field

The lagrangian density for the classical electromagnetic field is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu},$$

where the field strength tensor is defined as

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \quad A^\mu = (\phi, \mathbf{A}).$$

We can study the equations of motion using the Euler-Lagrange equations:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) = \frac{\partial \mathcal{L}}{\partial A_\nu} = 0, \implies \partial_\mu F^{\mu\nu} = 0.$$

It can be shown that the field strength tensor $F^{\mu\nu}$ respects the **Bianchi identity** (as we have already seen in section 3.4.2):

$$\partial_{[\lambda} F_{\mu\nu]} \equiv \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0.$$

The electric and magnetic fields can be expressed in terms of the potentials as

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A},$$

as the spatial and temporal components of the field strength tensor:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}, \quad F^{\mu\nu} = \begin{pmatrix} 0 & -E^x & -E^y & -E^z \\ E^x & 0 & -B^z & B^y \\ E^y & B^z & 0 & -B^x \\ E^z & -B^y & B^x & 0 \end{pmatrix}, \tag{6.1.1}$$

and thus

$$F^{i0} = E^i, \quad F^{ij} = -\epsilon_{ijk} B^k.$$

Finally we can see how the Maxwell equations are encapsulated in this formalism, since they arise from the equations of motion and the Bianchi identity:

$$\begin{cases} \nabla \cdot \mathbf{E} = 0, \\ \nabla \times \mathbf{B} - \dfrac{\partial \mathbf{E}}{\partial t} = 0, \end{cases} \quad \begin{cases} \nabla \cdot \mathbf{B} = 0, \\ \nabla \times \mathbf{E} + \dfrac{\partial \mathbf{B}}{\partial t} = 0, \end{cases}$$

where the first two equations come from $\partial_\mu F^{\mu\nu} = 0$ and need further terms when sources are present, while the last two come from the Bianchi identity, after recognizing the definitions of the electric and magnetic fields, and are true even in the presence of sources.

If we now expand the lagrangian density in terms of the potentials, we can exploit the fact that not every component of the vector potential A^μ is physical, since the EM field is known to have only two physical degrees of freedom (the two polarization states of the photon). Thus the lagrangian density can be rewritten as

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2} F_{0i} F^{0i} - \frac{1}{4} F_{ij} F^{ij}.$$

Thus we understand that the lagrangian density depends only on the spatial components of the vector potential A^i and their time derivatives, while the time component A^0 does not have a kinetic term (no dependence on $\frac{1}{2}(\partial_0 A^0)^2$):

$$\mathcal{L} \supset \sum_i \frac{1}{2} (\dot{A}^i)^2, \quad \text{but} \quad \mathcal{L} \not\supset \frac{1}{2} (\dot{A}^0)^2.$$

This indicates that A^0 is not a dynamical propagating degree of freedom, but rather acts as a Lagrange multiplier enforcing a constraint on the physical states of the theory.

Thus if we were to solve the equations of motion directly for A_μ, we would find that only A_i and $\dot{A}_i$ are necessary as initial conditions ($A_i(t_0, \mathbf{x})$, $\dot{A}_i(t_0, \mathbf{x})$) to determine the evolution of the field, while A_0 is determined by the equation $\nabla \cdot \mathbf{E} = 0$ and does not represent an independent degree of freedom. We have indeed

$$-\nabla \cdot \mathbf{E} = \nabla \cdot \nabla A_0 + \nabla \cdot \dot{\mathbf{A}} = 0 \implies \nabla^2 A_0 = -\nabla \cdot \dot{\mathbf{A}},$$

so that A_0 is fixed by the spatial components A_i and their time derivatives. We could compute a solution for A_0 using Green's functions of the Laplacian operator:

$$\begin{cases} \nabla^2 A_0(t_0, \mathbf{x}) = -\nabla \cdot \dot{\mathbf{A}}(t_0, \mathbf{x}), \\ \nabla^2 G(\mathbf{x}, \mathbf{y}) = \delta^{(3)}(\mathbf{x} - \mathbf{y}) \iff G(\mathbf{x}, \mathbf{y}) = \dfrac{-1}{4\pi|\mathbf{x} - \mathbf{y}|}, \end{cases}$$

so that

$$A_0(t_0, \mathbf{x}) = \int \mathrm{d}^3\mathbf{y}\, G(\mathbf{x}, \mathbf{y}) \left(-\nabla \cdot \dot{\mathbf{A}}(t_0, \mathbf{y})\right) = -\int \mathrm{d}^3\mathbf{y} \frac{\nabla \cdot \dot{\mathbf{A}}(t_0, \mathbf{y})}{4\pi|\mathbf{x} - \mathbf{y}|}.$$

In the end , we have only three independent degrees of freedom in the vector potential A^μ (the three spatial components A^i), but we know that the photon has only two physical polarization states. This discrepancy arises because the electromagnetic field is invariant under **gauge transformations** of the form

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu \alpha(x),$$

where $\alpha(x)$ is an arbitrary scalar function of the spacetime whose derivative has to vanish at infinity. This gauge invariance implies that not all configurations of A_μ correspond to physically distinct states: different choices of $\alpha(x)$ can lead to the same physical electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$. The field strength tensor $F_{\mu\nu}$ is invariant under these gauge transformations:

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = \partial_\mu(A_\nu + \partial_\nu \alpha) - \partial_\nu(A_\mu + \partial_\mu \alpha) = F_{\mu\nu}.$$

A_μ and A'_μ describe the same physical situation, they need to be identified, since this is a physical equivalence class. This is very different from global transformations: $\psi(x) \to \psi'(x) = e^{i\theta}\psi(x)$ is a transformation that changes the field configuration, but does not identify different configurations as physically equivalent, since the phase factor $e^{i\theta}$ is constant and does not depend on spacetime. Here we have just a redundancy in the description of the electromagnetic field.

We can do another example of gauge transformation: consider a solution for the vector potential $A_\mu(x)$. After performing a gauge transformation with a function $\alpha(x)$:

TODO: insert picture of gauge orbit

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu \alpha(x),$$

we could repeat the gauge transformation with a different function $\beta(x)$:

$$A'_\mu(x) \to A''_\mu(x) = A'_\mu(x) + \partial_\mu \beta(x) = A_\mu(x) + \partial_\mu(\alpha(x) + \beta(x)).$$

Each of these configurations describes the same physical electromagnetic field, since the field strength tensor remains unchanged. This shows that there is an infinite number of gauge equivalent configurations of A_μ that correspond to the same physical situation.

6.1.1 | Popular Gauge Fixing Choices

We need to *choose a representative* for each equivalence class of gauge equivalent configurations in order to uniquely describe the physical electromagnetic field. This process is called **gauge fixing**: we fix some condition on the vector potential A_μ to eliminate the redundancy and select a unique element on the **gauge orbit** for each physical configuration. Some physiscal properties, such as the number of physical degrees of freedom, may become clearer after gauge fixing.

There are several popular choices for gauge fixing in electromagnetism, each with its own advantages and disadvantages. Two of the most common gauges are:

1. **Lorentz Gauge**: The Lorentz gauge condition is given by

$$\partial_\mu A^\mu = 0 = \partial_0 A^0 + \nabla \cdot \mathbf{A}.$$

This gauge is Lorentz invariant, making it convenient for relativistic calculations. In this gauge, the independent degrees of freedom are reduced down to two, corresponding to the two physical polarization states of the photon. But A^μ does not necessarily satisfy the wave equation:

$$\partial_\mu A^\mu(x) = f(x) \neq 0,$$

but we can always perform a gauge transformation to ensure that it does: if there exists a function $\alpha(x)$ such that $\Box\alpha(x) = -f(x)$, then the gauge transformed field

$$A'^\mu(x) = A^\mu(x) + \partial^\mu \alpha(x)$$

will satisfy the Gauge condition; but note that this does not completely fix the gauge, since we can still perform gauge transformations and obtain new fields that satisfy the Lorentz gauge condition: if $\Box\beta(x) = 0$, then

$$A''^\mu(x) = A'^\mu(x) + \partial^\mu \beta(x)$$

will also satisfy the Lorentz gauge condition $\partial_\mu A''^\mu = 0$.

But in this gauge, we have still some residual gauge freedom, since we can perform gauge transformations with functions $\beta(x)$ (called harmonic functions) that satisfy the homogeneous wave equation $\Box\beta(x) = 0$. Furthermore, this gauge fixing has the advantage of being **manifestly Lorentz invariant**, making it suitable for relativistic calculations.

2. **Coulomb Gauge**: The Coulomb gauge condition is given by

$$\nabla \cdot \mathbf{A} = 0,$$

which focuses on the spatial components of the vector potential. In this gauge, the scalar potential ϕ is determined by the charge distribution, while the vector potential $\mathbf{A}$ describes the transverse electromagnetic waves. This gauge is particularly useful in non-relativistic quantum mechanics and in problems involving static charges, since it is not manifestly Lorentz invariant.

One can use the residual gauge freedom in the Lorentz gauge to reach the Coulomb gauge by choosing a suitable gauge function $\alpha(x)$ that satisfies the appropriate conditions:

$$\nabla \cdot \mathbf{A} = \partial_i A^i = 0 \implies \partial_\mu A^\mu = \partial_0 A^0 + \nabla \cdot \mathbf{A} = \partial_0 A^0,$$

so that A_0 is constant in time, since $\partial_\mu A^\mu = \partial_0 A^0 = 0$ from the Lorentz gauge. However, the temporal component A^0

$$A_0(t_0, \mathbf{x}) = \int \mathrm{d}^3\mathbf{y} \frac{\partial_t \left(\nabla \cdot \mathbf{A}(t_0, \mathbf{y})\right)}{4\pi|\mathbf{x} - \mathbf{y}|} = 0,$$

since $\nabla \cdot \mathbf{A} = 0$. Thus, in the Coulomb gauge, the scalar potential is fixed to zero

$$A_0(\mathbf{x}) = 0,$$

and the vector potential $\mathbf{A}$ describes the two transverse polarization states of the photon: since we have imposed $\nabla \cdot \mathbf{A} = 0$, only the components of $\mathbf{A}$ perpendicular to the direction of propagation remain as physical degrees of freedom.

Note that in both gauges, we have successfully reduced the number of independent degrees of freedom in the vector potential A_μ from four (actually three, since we already know from Gauss' law that the temporal component is constrained) to two, corresponding to the two physical polarization states of the photon. The residual gauge freedom in the Lorentz gauge (which we used to go in the Coulomb gauge directly from Lorentz') was not a physical degree of freedom, but rather a redundancy in the description of the electromagnetic field: in Lorentz gauge A_0 is a function of A_i, while in Coulomb gauge A_0 is fixed to zero.

6.2 | Gauge Fixing of the Electromagnetic Field

Before starting the quantization procedure, we need to find the conjugate momenta associated with the fields A^μ: we will use it to find the expression for the Hamiltonian (via Legendre transform) and impose the canonical commutation relations.

From the lagrangian density

$$\mathcal{L} = -\frac{1}{2}\left(F_{0i}F^{0i} + \frac{1}{2}F_{ij}F^{ij}\right),$$

the **conjugate momenta** are defined as

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} = \begin{cases} \pi^0 = \dfrac{\partial \mathcal{L}}{\partial(\partial_0 A_0)} = 0, \text{ since } A_0 \text{ is not a dynamical field,} \\ \pi^i = \dfrac{\partial \mathcal{L}}{\partial(\partial_0 A_i)} = \dfrac{\partial}{\partial(\dot{A}_i)}\left[-\dfrac{1}{2}\left(\dot{A}_j - \partial_j A_0\right)\left(\dot{A}^j - \partial^j A^0\right)\right] = E^i, \end{cases}$$

since, for the spatial component, we can compute

$$\frac{\partial}{\partial(\dot{A}_i)}\left[-\frac{1}{2}\left(\dot{A}_j - \partial_j A_0\right)\left(\dot{A}^j - \partial^j A^0\right)\right] = -\frac{1}{2}F^{0i} - \frac{1}{2}F_{0j}\eta^{ji} = -F^{0i} = E^i.$$

Thus we have $\pi = (0, \mathbf{E})$ and the Hamiltonian density of the system can be computed via Legendre transform as

$$\mathcal{H} = \pi^\mu \partial_0 A_\mu - \mathcal{L} = \pi^i \partial_0 A_i - \mathcal{L},$$

since $\pi^0 = 0$. Recalling the expressions for π^i and $\mathcal{L}$, we get

$$\pi^i \partial_0 A_i = E^i \partial_0 A_i = E^i\left(F_{0i} + \partial_i A_0\right) = E^i F_{0i} + E^i \partial_i A_0,$$
$$\mathcal{L} = -\frac{1}{2}\left(F_{0i}F^{0i} + \frac{1}{2}F_{ij}F^{ij}\right) = -\frac{1}{2}\left(-|\mathbf{E}|^2 + |\mathbf{B}|^2\right) = \frac{1}{2}\left(|\mathbf{E}|^2 - |\mathbf{B}|^2\right),$$

we get to an hamiltonian density of

$$\mathcal{H} = E^i F_{0i} + E^i \partial_i A_0 - \frac{1}{2}\left(|\mathbf{E}|^2 - |\mathbf{B}|^2\right) = \frac{1}{2}\left(|\mathbf{E}|^2 + |\mathbf{B}|^2\right) + \mathbf{E}\cdot(\nabla A_0).$$

So that the Hamiltonian takes the form

$$H = \int \mathrm{d}^3\mathbf{x}\left(\frac{1}{2}\left(|\mathbf{E}|^2 + |\mathbf{B}|^2\right) - A_0(\nabla\cdot\mathbf{E})\right).$$

In this expression, the term involving A_0 (found after an integration by parts) does not act as a physical variable, but as a *Lagrange multiplier* enforcing Gauss's law $\nabla \cdot \mathbf{E} = 0$ in the absence of charges:

$$\frac{\partial \mathcal{H}}{\partial A_0} = -\nabla\cdot\mathbf{E} = 0 \implies \nabla\cdot\mathbf{E} = 0.$$

We have still to fix a gauge in order to proceed with the quantization, so the field has still some redundant degrees of freedom, and this is a constraint on the physical states of the theory (on the elements of A^μ).

Lorentz gauge fixing. Lastly, before starting the quantization procedure, we have to impose the Lorentz gauge condition $\partial_\mu A^\mu = 0$, since its Lorentz invariance makes it suitable for relativistic quantum field theory. From this choice, we get the equations of motion

$$\partial_\mu F^{\mu\nu} = 0 = \partial_\mu \partial^\mu A^\nu - \partial^\nu(\partial_\mu A^\mu) = \partial_\mu \partial^\mu A^\nu = \Box A^\nu.$$

Since $\Box A^\nu = 0$, each component of the vector potential A^μ satisfies the wave equation, particularly the massless KG scalar equation: at the quantum level, the field A^μ will describe massless particles, the **photons**, with energy $E_\mathbf{p} = \sqrt{|\mathbf{p}|^2} = |\mathbf{p}|$.

Feynman gauge fixing. Instead of imposing the gauge condition $\partial_\mu A^\mu = 0$ by hand, we can modify the lagrangian density slightly by adding a gauge-fixing term:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2,$$

where ξ let us choose different gauges: $\xi = 1$ corresponds to the **Feynman gauge**, while $\xi \to 0$ (computed after quantization) corresponds to the Landau gauge. Here we will choose $\xi = 1$ for simplicity, so

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2.$$

This modification does not change the equations of motion for physical fields (since we have redundancy in the description, we are just fixing a gauge implicitly from the Lagrangian), but it allows us to derive the Lorentz gauge condition from the equations of motion themselves. The new equations of motion become

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = \frac{\partial\mathcal{L}}{\partial A_\nu} = 0 \implies \partial_\mu F^{\mu\nu} + \partial_\mu \eta^{\mu\nu}(\partial_\sigma A^\sigma) = 0,$$

which simplifies to

$$\partial_\mu\partial^\mu A^\nu - \partial_\mu\partial^\nu A^\mu + \partial^\nu(\partial_\mu A^\mu) = \Box A^\nu = 0.$$

We reached the same equations as before, but now the gauge condition $\partial_\mu A^\mu = 0$ is automatically satisfied by the solutions of the equations of motion.

Proceeding to find the conjugate momenta, we get

$$\begin{cases} \pi^0 = \dfrac{\partial\mathcal{L}}{\partial(\partial_0 A_0)} = -\partial_\mu A^\mu; \\ \pi^i = \dfrac{\partial\mathcal{L}}{\partial(\partial_0 A_i)} = \partial^i A^0 - \partial^0 A^i = F^{i0}; \end{cases} \iff \quad \pi^\mu = F^{\mu 0} - \eta^{\mu 0}(\partial_\nu A^\nu),$$

different from the previous case where $\pi^0 = 0$, since now the Lagrangian has a dependence on $\partial_0 A_0$ in the new gauge-fixing term.

6.3 | Quantization of the Electromagnetic Field

Sticking to the Feynman gauge, we can finally proceed to the quantization for the electromagnetic field. The classical fields $A_\mu(x)$ and $\pi^\mu(x)$ are going to be promoted to operators acting on a Hilbert space. We impose the **canonical commutation relations** at equal times:[1]

$$\begin{aligned}\left[\hat{A}_\mu(\mathbf{x},t),\hat{\pi}^\nu(\mathbf{y},t)\right] &= i\delta_\mu{}^\nu\delta^{(3)}(\mathbf{x}-\mathbf{y}),\\ \left[\hat{A}_\mu(\mathbf{x},t),\hat{A}_\nu(\mathbf{y},t)\right] &= 0,\\ \left[\hat{\pi}^\mu(\mathbf{x},t),\hat{\pi}^\nu(\mathbf{y},t)\right] &= 0.\end{aligned} \tag{6.3.1}$$

This relations will ensure the correct quantization of the electromagnetic field, with symmetric multiparticle states corresponding to the bosonic nature of photons, while the use of anticommutation relations would lead to inconsistencies with the spin-statistics theorem.

The general solution to the equations of motion $\Box A^\mu = 0$ can be expressed as a Fourier expansion in terms of plane waves whose coefficients should operate as creation and annihilation operators for photons. We can write:

$$\hat{A}_\mu(x) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\left(\hat{\xi}_\mu(\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{x}} + \hat{\xi}^\dagger_\mu(\mathbf{p})e^{-i\mathbf{p}\cdot\mathbf{x}}\right),$$

where $p^\mu = (E_\mathbf{p},\mathbf{p})$ with $E_\mathbf{p} = |\mathbf{p}|$ for massless particles.

The operators $\hat{\xi}_\mu(\mathbf{p})$ and $\hat{\xi}^\dagger_\mu(\mathbf{p})$ will be identified as annihilation and creation operators for photons with momentum $\mathbf{p}$ and polarization indexed by μ. To explicitly see this, we can decompose these operators in terms of a **polarization basis**: we introduce polarization vectors $\epsilon^{(\lambda)}_\mu(\mathbf{p})$ with $\lambda = 0,1,2,3$ labeling the four possible polarization states of the photon

$$\epsilon^{(\lambda)}_\mu(\mathbf{p}),\quad \lambda = 0,1,2,3,\quad \begin{cases}\epsilon^{(\lambda)}_\mu(\mathbf{p})\epsilon^{(\lambda')\mu}(\mathbf{p}) &= \eta^{\lambda\lambda'},\\ \epsilon^{(\lambda)}_\mu(\mathbf{p})\epsilon^{(\lambda')}_\nu(\mathbf{p})\eta_{\lambda\lambda'} &= \eta_{\mu\nu}.\end{cases} \tag{6.3.2}$$

which satisfy the orthonormality with respect to the Minkowski metric $\eta_{\mu\nu}$. Lambda $\lambda = 0$ corresponds to the timelike polarization, $\lambda = 1,2$ correspond to the two transverse polarizations, while $\lambda = 3$ corresponds to the longitudinal polarization: it labels the only non zero component of the polarization vector.

We can then express the operators $\hat{\xi}_\mu(\mathbf{p})$ and $\hat{\xi}^\dagger_\mu(\mathbf{p})$ in terms of creation and annihilation operators $\hat{a}^{(\lambda)}_\mathbf{p}$ and $\hat{a}^{(\lambda)\,\dagger}_\mathbf{p}$ for photons with definite polarization:

$$\hat{\xi}_\mu(\mathbf{p}) = \sum_{\lambda=0}^{3}\epsilon^{(\lambda)}_\mu(\mathbf{p})\hat{a}^{(\lambda)}_\mathbf{p},\quad \hat{\xi}^\dagger_\mu(\mathbf{p}) = \sum_{\lambda=0}^{3}\epsilon^{(\lambda)}_\mu(\mathbf{p})\hat{a}^{(\lambda)\,\dagger}_\mathbf{p}.$$

We can now write the final expression for the quantized electromagnetic field and its conjugate momenta:[2]

$$\begin{cases}\hat{A}_\mu(\mathbf{x}) &= \displaystyle\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\sum_{\lambda=0}^{3}\epsilon^{(\lambda)}_\mu(\mathbf{p})\left(\hat{a}^{(\lambda)}_\mathbf{p}e^{i\mathbf{p}\cdot\mathbf{x}} + \hat{a}^{(\lambda)\,\dagger}_\mathbf{p}e^{-i\mathbf{p}\cdot\mathbf{x}}\right),\\ \hat{\pi}^\mu(\mathbf{x}) &= \displaystyle\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}(+i)\sqrt{\frac{E_\mathbf{p}}{2}}\sum_{\lambda=0}^{3}\epsilon^{\mu\,(\lambda)}(\mathbf{p})\left(\hat{a}^{(\lambda)}_\mathbf{p}e^{i\mathbf{p}\cdot\mathbf{x}} - \hat{a}^{(\lambda)\,\dagger}_\mathbf{p}e^{-i\mathbf{p}\cdot\mathbf{x}}\right).\end{cases} \tag{6.3.3}$$

[1] Since we know that integer spin particles are bosons and the correct spin-statistics relations are imposed via commutation.

[2] In the expression for the conjugate momenta, we have a term $(+i)$ changing sign with respect to the scalar field case, due to the different definition of π^μ in terms of the fields A^μ: after using equation (3.1.4), for KG we had $\pi = \partial_0\phi$, while here we have $\pi^\mu = -\dot{A}^\mu + \ldots$, so that in the end this sign changes.

We have the four vector polarization depending on the four momentum $p^\mu = (|\mathbf{p}|, \mathbf{p})$, but we know that the photon has only two physical polarization states. This discrepancy arises because we have not yet fully fixed the gauge: the presence of unphysical polarization states (longitudinal and timelike) is a consequence of the gauge redundancy in the electromagnetic field. To make contact with the two physical polarization states of the photon, we choose $\epsilon_\mu^{(1)}$ and $\epsilon_\mu^{(2)}$ as the two transverse polarization vectors

$$\epsilon_\mu^{(1)} p^\mu = \epsilon_\mu^{(2)} p^\mu = 0, \qquad \text{Lorentz invariant condition,}$$

thus they are perpendicular to the direction of the momentum, while $\epsilon_\mu^{(0)}$ and $\epsilon_\mu^{(3)}$ correspond to the unphysical timelike and longitudinal polarizations, respectively.

Example (Momentum along the z-axis). Consider a photon with momentum along the z-axis:

$$p^\mu = (E, 0, 0, E).$$

A possible choice for the polarization vectors is:

$$\begin{aligned}
\epsilon_\mu^{(0)} &= (1, 0, 0, 0) \quad \text{(timelike)},\\
\epsilon_\mu^{(1)} &= (0, 1, 0, 0) \quad \text{(transverse)},\\
\epsilon_\mu^{(2)} &= (0, 0, 1, 0) \quad \text{(transverse)},\\
\epsilon_\mu^{(3)} &= (0, 0, 0, 1) \quad \text{(longitudinal)}.
\end{aligned}$$

This choice satisfies the orthonormality conditions, since clearly

$$\epsilon_\mu^{(1)} p^\mu = E\left(\epsilon_0^{(1)} + \epsilon_3^{(1)}\right) = 0, \quad \epsilon_\mu^{(2)} p^\mu = E\left(\epsilon_0^{(2)} + \epsilon_3^{(2)}\right) = 0,$$

we have then

$$\epsilon_0^{(1)} = \epsilon_3^{(1)} = 0, \quad \epsilon_0^{(2)} = \epsilon_3^{(2)} = 0,$$

since the transverse polarization vectors have zero time and longitudinal components. We need to choose the transverse polarization vectors to be orthogonal to each other and normalized:

$$\epsilon_\mu^{(1)} \epsilon^{(2)\,\mu} = 0 = \epsilon_1^{(1)} \epsilon^{(2)\,1} + \epsilon_2^{(1)} \epsilon^{(2)\,2} = 0,$$

so now it is justified the initial choice of $\epsilon_\mu^{(1)} = (0, 1, 0, 0)$ and $\epsilon_\mu^{(2)} = (0, 0, 1, 0)$,[3] which clearly satisfies the orthonormality conditions with respect to the Minkowski metric. To clarify, we can write

$$\epsilon^{(1)\,\mu} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \epsilon_\mu^{(1)} = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}.$$

since we had written only row vectors before. The same applies to $\epsilon_\mu^{(2)}$.[4]

The creation and annihilation operators $a_\mathbf{p}^{(\lambda)}$ and $a_\mathbf{p}^{(\lambda)\,\dagger}$ satisfy the commutation relations

TODO: expliciit derivation of commutation relations.

$$\begin{aligned}
\left[\hat{a}_\mathbf{p}^{(\lambda)}, \hat{a}_\mathbf{q}^{(\lambda')\,\dagger}\right] &= -\eta^{\lambda\lambda'} (2\pi)^3 \delta^{(3)}(\mathbf{p} - \mathbf{q}),\\
\left[\hat{a}_\mathbf{p}^{(\lambda)}, \hat{a}_\mathbf{q}^{(\lambda')}\right] &= \left[\hat{a}_\mathbf{p}^{(\lambda)\,\dagger}, \hat{a}_\mathbf{q}^{(\lambda')\,\dagger}\right] = 0,
\end{aligned}$$

[3] We could have chosen the opposite identification, it would have been the same.

[4] Practically we are treating $\epsilon^{\mu\,(\lambda)}$ as a four vector in the Minkowski space, while $\epsilon_\mu^{(\lambda)}$ is its covariant counterpart, obtained by lowering the index with the Minkowski metric: thus the sign changes only in the spatial components.

which can be derived from the canonical commutation relations for the fields $\hat{A}_\mu$ and $\hat{\pi}^\mu$. Note the minus sign in the first commutation relation, which arises from the Minkowski metric $\eta^{\lambda\lambda'}$ and reflects the presence of unphysical polarization states.

If we indeed check the norm of the states created by the creation operators acting on the vacuum $|0\rangle$, defined by $\hat{a}_{\mathbf{p}}^{(\lambda)}|0\rangle = 0$ for all λ and $\mathbf{p}$, we find that for spacelike polarizations $\lambda = 1, 2, 3$ we have positive norm states:

$$
\begin{aligned}
\langle \mathbf{p}, \lambda | \mathbf{q}, \lambda' \rangle = \langle 0 | \hat{a}_{\mathbf{p}}^{(\lambda)} \hat{a}_{\mathbf{q}}^{(\lambda')\dagger} |0\rangle &= -\eta^{\lambda\lambda'} (2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{q}), \\
\implies \langle \mathbf{p}, \lambda | \mathbf{p}, \lambda \rangle &= +(2\pi)^3 \delta^{(3)}(0) > 0, \quad \text{for } \lambda = 1, 2, 3,
\end{aligned}
$$

while for the timelike polarization $\lambda = 0$ we have negative norm states:

$$
\langle \mathbf{p}, 0 | \mathbf{q}, 0 \rangle = \langle 0 | \hat{a}_{\mathbf{p}}^{(0)} \hat{a}_{\mathbf{q}}^{(0)\dagger} |0\rangle = -\eta^{00} (2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{q}) = -(2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{q}) < 0.
$$

This comes from the commutator relations

$$
\begin{aligned}
\left[\hat{a}_{\mathbf{p}}^{(0)}, \hat{a}_{\mathbf{q}}^{(0)\dagger}\right] &= -\eta^{00} (2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{q}) = -(2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{q}), \\
\left[\hat{a}_{\mathbf{p}}^{(i)}, \hat{a}_{\mathbf{q}}^{(i)\dagger}\right] &= -\eta^{ii} (2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{q}) = +(2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{q}), \quad i = 1, 2, 3,
\end{aligned}
$$

which in turn arise from the switching of the operators in the calculation of the norm.

This is the same problem we encountered when trying to quantize the Dirac field with commutator instead of anticommutators:[5] the presence of **negative norm states** indicates that our Hilbert space has an indefinite metric, which is unphysical since it usually leads to **negative probabilities** and **negative energies**. This states ended up being named **ghost states**.

6.3.1 | Imposing the Gauge Condition on the Fock Space

One can trace back the origin of these negative norm states to the gauge-fixing term in the lagrangian:

$$
\begin{aligned}
\mathcal{L} &= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\partial_\mu A^\mu)^2, \\
\mathcal{L} &\supset \frac{1}{2} \left((\dot{A}^1)^2 + (\dot{A}^2)^2 + (\dot{A}^3)^2 - (\dot{A}^0)^2 \right),
\end{aligned}
$$

where the positive time derivatives are contained in the original lagrangian $F_{\mu\nu}F^{\mu\nu}$ while the minus sign in front of the gauge-fixing term leads to negative time derivatives $\dot{A}_0^2$, thus to negative norm states for the unphysical polarizations (timelike). This is a manifestation of the indefinite metric in the space of states, which is a common feature in gauge theories before imposing physical state conditions. The solution is to *impose the gauge condition at the quantum level*, directly on the Fock space of states.[6]

(?) do not quite understand, ask

If we want to make the spin 1 particle massive, we need an additional degree of freedom, since a massive spin 1 particle has three polarization states, while a massless spin 1 particle has only two physical polarization states. We have seen that from the representations of the little group,

[5] The only difference being that we cannot try to use $\hat{a}_{\mathbf{p}}^\dagger$ as an annihilation operator, since timelike polarized states would still have a negative norm: this time the problem is of gauge nature, but instead of redundancies in the description of the system this time it may create bigger problems in the physical interpretation.

[6] We did not already impose the gauge conditions indeed. The term in the Lagrangian was added to impose implicitly the Lorentz gauge condition via the equations of motion, but when quantizing we have to take care of the unphysical states, which classically would be eliminated, but quantum mechanically they still appear in the Hilbert space.

a massive particle has the little group $SO(3)$ with three generators corresponding to the three spatial rotations, while a massless particle has the little group $ISO(2)$ with only one generator corresponding to rotations around the direction of motion, leading to only two physical polarization states (one rotation which implies two transverse polarizations). We are dealing with a massless spin 1 particle, the photon, so in the end we aim to describe its behavior with only two physical polarization states.

Moving now to Heisemberg picture, we will need to understand how to impose the gauge condition at the quantum level, and what does that condition imply for the physical states of the theory. Thus we promote the fields to operators

$$\hat{A}^\mu = \hat{A}^\mu(x) = \hat{A}^\mu(\mathbf{x}, t),$$

which has a time dependence in Heisemberg picture. We chose the gauge condition

$$\partial_\mu \hat{A}^\mu = 0,$$

since it should lead us towards Lorentz invariant solution while removing the negative time derivative in the Lagrangian. This condition makes sense only as an operator equation acting on the Hilbert space of states. However, we have at least three ways of imposing this condition:

- $\partial_\mu \hat{A}^\mu = 0$ as a strong operator equation, imposing that the matrix has to vanish identically. This approach is too restrictive and leads to inconsistencies in our theory. If we go back to the expression of the conjugate momenta

$$\hat{\pi}^\mu = \hat{F}^{\mu 0} - \eta^{\mu 0}(\partial_\nu \hat{A}^\nu),$$

  so that if we compute the time component

$$\hat{\pi}^0 = \hat{F}^{00} - \eta^{00}(\partial_\nu \hat{A}^\nu) = -(\partial_\nu \hat{A}^\nu) = 0,$$

  which would imply that $\hat{\pi}^0 = 0$ as an operator equation. But this is in contradiction with the canonical commutation relations imposed before:

$$\left[\hat{A}_\mu(\mathbf{x}, t), \hat{\pi}_\nu(\mathbf{y}, t)\right] = i\delta^{(3)}(\mathbf{x} - \mathbf{y})\eta_{\mu\nu}, \implies \left[\hat{A}_0(\mathbf{x}, t), \hat{\pi}_0(\mathbf{y}, t)\right] = i\delta^{(3)}(\mathbf{x} - \mathbf{y})\eta_{00} \neq 0,$$

  so we have a contradiction requesting that all the entries of the operator $\partial_\mu \hat{A}^\mu$ vanish identically.

- An alternative is to impose the gauge condition in a weak sense, meaning that we require that the expectation value of the operator $\partial_\mu \hat{A}^\mu$ vanishes when applied on physical states $|\psi\rangle$: We are *applying the gauge condition directly on the Fock space*, with the idea to consider unphysical all the states which do not respect

$$(\partial_\mu \hat{A}^\mu)\,|\psi\rangle = 0.$$

  This approach is less restrictive, but it still does not fully work in the recognition of physical and unphysical polarization states, since the vacuum state $|0\rangle$ would be recognized as unphysical: in Heisemberg picture (as done in (4.3.1)), we have indeed

$$\hat{A}_\mu(x) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} \sum_{\lambda=0}^{3} \epsilon_\mu^{(\lambda)}(\mathbf{p}) \left(a_\mathbf{p}^{(\lambda)} e^{ip_\mu x^\mu} + a_\mathbf{p}^{(\lambda)\,\dagger} e^{-ip_\mu x^\mu}\right) = \hat{A}_\mu^+(x) + \hat{A}_\mu^-(x),$$

where we have separated the positive and negative frequency parts of the field operator. The gauge condition operator $\partial_\mu \hat{A}^\mu$ then can be applied to the vacuum state:

$$\partial_\mu \hat{A}^\mu |0\rangle = \partial_\mu \hat{A}^{\mu+} |0\rangle + \partial_\mu \hat{A}^{\mu-} |0\rangle = \partial_\mu \hat{A}^{\mu+} |0\rangle \neq 0,$$

since when we act on the vacuum with the negative frequency part $\hat{A}^{\mu-}$, we get zero, but the positive frequency part $\hat{A}^{\mu+}$ acting on the vacuum does not vanish, leading to a non-zero result. Thus the vacuum state is not physical in this approach, it does not satisfy the gauge condition imposed on the Fock space $\partial_\mu \hat{A}^\mu |0\rangle = 0$.

- The most common and effective approach is the **Gupta-Bleuler formalism**, where we impose the gauge condition only on the positive frequency part of the field operator: in this way the vacuum remains a physical state, and we can consistently define the subspace of physical states in the Hilbert space. The Gupta-Bleuler condition is given by

$$\partial_\mu \hat{A}^{\mu+} |\psi\rangle = 0, \iff \langle\psi| \partial_\mu \hat{A}^{\mu-} = 0, \tag{6.3.4}$$

for physical states $|\psi\rangle$. This condition effectively eliminates the unphysical polarization states from the theory, while still allowing for a consistent quantization of the electromagnetic field. The physical states are then defined as those that satisfy this condition, leading to a well-defined physical subspace of the full Hilbert space.

We could note that the negative frequency part $\partial_\mu \hat{A}^{\mu-}$ functioning on physical states only when applied to the left, so that if we look at the matrix element of the operator $\partial_\mu \hat{A}^\mu$ between two physical states $|\psi\rangle$ and $|\phi\rangle$, we have:

$$\langle\phi| \partial_\mu \hat{A}^\mu |\psi\rangle = \langle\phi| \partial_\mu \hat{A}^{\mu+} |\psi\rangle + \langle\phi| \partial_\mu \hat{A}^{\mu-} |\psi\rangle = 0 + 0 = 0.$$

6.3.2 | Physical Implications of the Gupta-Bleuler Condition

Now that we found a consistent way to impose the gauge condition at the quantum level, we can analyze the implications of this condition on the physical states of the theory. The Gupta-Bleuler condition effectively removes the unphysical polarization states from the theory, leaving only the two transverse polarization states as physical degrees of freedom.

We can check that starting from

$$\partial^\mu \hat{A}^+_\mu |\psi\rangle = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_\mathbf{p}}} \sum_{\lambda=0}^{3} \left(-i\epsilon^{(\lambda)}_\mu(\mathbf{p})p^\mu\right) a^{(\lambda)}_\mathbf{p} e^{-ip_\mu x^\mu} |\psi\rangle = 0,$$

since one can notice how the contraction $\epsilon^{(\lambda)}_\mu(\mathbf{p})p^\mu$ vanishes for the transverse polarizations $\lambda = 1, 2$, orthogonal to the momentum p^μ: choosing the momentum along the z-axis as before $p^\mu = (E,\, 0,\, 0,\, E)$, we have

$$\epsilon^{(1)}_\mu p^\mu = 0, \quad \epsilon^{(2)}_\mu p^\mu = 0,$$

so that the only condition becomes

$$\partial^\mu \hat{A}^+_\mu |\psi\rangle = 0 \iff \left(\epsilon^{(0)}_\mu(\mathbf{p})p^\mu \hat{a}^{(0)}_\mathbf{p} + \epsilon^{(3)}_\mu(\mathbf{p})p^\mu \hat{a}^{(3)}_\mathbf{p}\right) |\psi\rangle = 0.$$

This implies a relation between the creation and annihilation operators for the unphysical polarization states

$$E\left(\epsilon^{(0)}_0(\mathbf{p})\hat{a}^{(0)}_\mathbf{p} + \epsilon^{(3)}_3(\mathbf{p})\hat{a}^{(3)}_\mathbf{p}\right) |\psi\rangle = 0,$$

which leads to a series of implications useful to understand the nature of physical states:[7]

$$\begin{cases} \left(\hat{a}^{(0)}_{\mathbf{p}} - \hat{a}^{(3)}_{\mathbf{p}}\right) |\psi\rangle = 0, \\ \hat{a}^{(0)}_{\mathbf{p}} |\psi\rangle = \hat{a}^{(3)}_{\mathbf{p}} |\psi\rangle \,, \\ \langle\psi| \, \hat{a}^{(0)\,\dagger}_{\mathbf{p}} = \langle\psi| \, \hat{a}^{(3)\,\dagger}_{\mathbf{p}}, \\ \langle\psi| \, \hat{a}^{(0)\,\dagger}_{\mathbf{p}} \hat{a}^{(0)}_{\mathbf{p}} |\psi\rangle = \langle\psi| \, \hat{a}^{(3)\,\dagger}_{\mathbf{p}} \hat{a}^{(3)}_{\mathbf{p}} |\psi\rangle \,. \end{cases} \tag{6.3.5}$$

This last relation shows that the number of timelike photons is equal to the number of longitudinal photons in physical states (since we are computing the expectation value of the number operators of the two unphysical polarizations photons on both sides), leading to a cancellation of their contributions to physical observables. Thus, only the two transverse polarization states remain as physical degrees of freedom, consistent with the known properties of photons (negative norm timelike photons are not physical).

To check this, we can consider $|\mathbf{p},\, \lambda = 0\rangle = \hat{a}^{(0)\,\dagger}_{\mathbf{p}} |0\rangle$ the state with timelike polarization, which if our condition is correct should be unphysical:

$$\implies \left(\hat{a}^{(0)}_{\mathbf{p}} - \hat{a}^{(3)}_{\mathbf{p}}\right) |\mathbf{p},\, \lambda = 0\rangle \neq 0,$$

but let us compute that explicitly:

$$\begin{aligned} \left(\hat{a}^{(0)}_{\mathbf{p}} - \hat{a}^{(3)}_{\mathbf{p}}\right) |\mathbf{q},\, 0\rangle &= \hat{a}^{(0)}_{\mathbf{p}} \hat{a}^{(0)\,\dagger}_{\mathbf{q}} |0\rangle - \hat{a}^{(3)}_{\mathbf{p}} \hat{a}^{(0)\,\dagger}_{\mathbf{p}} |0\rangle = \hat{a}^{(0)}_{\mathbf{p}} \hat{a}^{(0)\,\dagger}_{\mathbf{q}} |0\rangle \\ &= \left[\hat{a}^{(0)}_{\mathbf{p}}, \hat{a}^{(0)\,\dagger}_{\mathbf{q}}\right] |0\rangle + \hat{a}^{(0)\,\dagger}_{\mathbf{p}} \hat{a}^{(0)}_{\mathbf{q}} |0\rangle \\ &= \left[\hat{a}^{(0)}_{\mathbf{p}}, \hat{a}^{(0)\,\dagger}_{\mathbf{q}}\right] |0\rangle \\ &= -(2\pi)^3 \delta^{(3)}(0) \, |0\rangle \neq 0. \end{aligned}$$

This is good news: a state of only a timelike photon (which has negative norm) is not physical; but the Hilbert space defined by $\left(\hat{a}^{(0)}_{\mathbf{p}} - \hat{a}^{(3)}_{\mathbf{p}}\right) |\psi\rangle = 0$ contains also **zero norm states**. This comes as good news, since zero norm states do not contribute to physical observables, so they can be present in the physical Hilbert space without causing any issues: we need only two polarization states to describe the photon, so the presence of $|S\rangle - |L\rangle$ polarization (scalar minus longitudinal, the states created by $\hat{a}^{(0)\,\dagger}_{\mathbf{p}} - \hat{a}^{(3)\,\dagger}_{\mathbf{p}}$) could be a problem, but if they are zero norm states they do not contribute to physical observables, so they can be safely included in the physical Hilbert space.

To see this clearly, we have to perform a change of basis on the Fock space:

$$\hat{a}^{(\lambda)}_{\mathbf{p}} \quad \longrightarrow \quad \hat{a}^{(1)}_{\mathbf{p}}, \hat{a}^{(2)}_{\mathbf{p}}, \hat{b}^{(\pm)}_{\mathbf{p}} = \frac{1}{\sqrt{2}} \left(\hat{a}^{(0)}_{\mathbf{p}} \pm \hat{a}^{(3)}_{\mathbf{p}}\right).$$

In this new basis, we should be able to create states with definite numbers of transverse photons and definite combinations of timelike and longitudinal photons:

$$\begin{aligned} \hat{a}^{(1)\,\dagger}_{\mathbf{p}} |0\rangle &= |\mathbf{p},\, 1\rangle \,, \\ \hat{a}^{(2)\,\dagger}_{\mathbf{p}} |0\rangle &= |\mathbf{p},\, 2\rangle \,, \\ \hat{b}^{(+)\,\dagger}_{\mathbf{p}} |0\rangle &= \frac{1}{\sqrt{2}} \left(|\mathbf{p},\, 0\rangle + |\mathbf{p},\, 3\rangle\right), \\ \hat{b}^{(-)\,\dagger}_{\mathbf{p}} |0\rangle &= \frac{1}{\sqrt{2}} \left(|\mathbf{p},\, 0\rangle - |\mathbf{p},\, 3\rangle\right). \end{aligned}$$

[7] Remember that $\epsilon^{(\lambda)}_i$, with $i = 1,\, 2,\, 3$, are spacelike components, so they have a minus sign when the index is low (or equivalently we can have high index in the polarization and low index in the momentum, but the minus sign is always present).

so that we can create states with only transverse photons, or states with definite combinations of timelike and longitudinal photons (and combinations of these). Thus the Gupta-Bleuler condition $\left(\hat{a}_{\mathbf{p}}^{(0)} - \hat{a}_{\mathbf{p}}^{(3)}\right)|\psi\rangle = 0$ can be rewritten as

$$\hat{b}_{\mathbf{p}}^{(-)}|\psi\rangle = 0,$$

which means that physical states $|\psi\rangle$ cannot contain any $\hat{b}_{\mathbf{p}}^{(-)\,\dagger}$ excitations.

We can now check explicitly that states with only transverse photons are physical states applying the Gupta-Bleuler condition through the operator $\hat{b}_{\mathbf{p}}^{(-)}$. To do so we will need the commutation relations for the new operators $\hat{b}_{\mathbf{p}}^{(\pm)}$:

$$\begin{aligned}
\left[\hat{b}_{\mathbf{p}}^{(\pm)},\, \hat{b}_{\mathbf{q}}^{(\pm)\,\dagger}\right] &= \left[\hat{a}_{\mathbf{p}}^{(0)},\, \hat{a}_{\mathbf{q}}^{(0)\,\dagger}\right] + \left[\hat{a}_{\mathbf{p}}^{(3)},\, \hat{a}_{\mathbf{q}}^{(3)\,\dagger}\right] \\
&= -(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) + (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) = 0, \\
\left[\hat{b}_{\mathbf{p}}^{(\pm)},\, \hat{b}_{\mathbf{q}}^{(\mp)\,\dagger}\right] &= \left[\hat{a}_{\mathbf{p}}^{(0)},\, \hat{a}_{\mathbf{q}}^{(0)\,\dagger}\right] - \left[\hat{a}_{\mathbf{p}}^{(3)},\, \hat{a}_{\mathbf{q}}^{(3)\,\dagger}\right] \\
&= -2(2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{q}) \neq 0,
\end{aligned}$$

with all other commutators null. Now we can compute the action of $\hat{b}_{\mathbf{p}}^{(-)}$ on states with:

1. **only transverse photons**, created by $\hat{a}_{\mathbf{q}}^{(1/2)\,\dagger}$:

   $$\hat{b}_{\mathbf{p}}^{(-)}\left(\hat{a}_{\mathbf{q}}^{(1/2)\,\dagger}|0\rangle\right) = 0,$$

   since the commutators $\left[\hat{b}_{\mathbf{p}}^{(-)}, \hat{a}_{\mathbf{q}}^{(1/2)\,\dagger}\right]$ are proportional to the null commutators among $\hat{a}_{\mathbf{q}}^{(1/2)}$ and $\hat{a}_{\mathbf{p}}^{(0/3)}$. So the states with only transverse photons **are physical**.

2. **only timelike photons**, created by $\hat{a}_{\mathbf{q}}^{(0)\,\dagger} = \frac{1}{2}\left(\hat{b}_{\mathbf{q}}^{(+)\,\dagger} + \hat{b}_{\mathbf{q}}^{(-)\,\dagger}\right)$:

   $$\begin{aligned}
   \hat{b}_{\mathbf{p}}^{(-)\,\dagger}\left(\hat{a}_{\mathbf{q}}^{(0)\,\dagger}|0\rangle\right) &= \frac{1}{2}\hat{b}_{\mathbf{p}}^{(-)\,\dagger}\left(\hat{b}_{\mathbf{q}}^{(+)\,\dagger}|0\rangle + \hat{b}_{\mathbf{q}}^{(-)\,\dagger}|0\rangle\right) \\
   &= \frac{1}{2}\left[\hat{b}_{\mathbf{p}}^{(-)\,\dagger},\, \hat{b}_{\mathbf{p}}^{(+)\,\dagger}\right] + \frac{1}{2}\left[\hat{b}_{\mathbf{p}}^{(-)\,\dagger},\, \hat{b}_{\mathbf{p}}^{(-)\,\dagger}\right] \\
   &= \frac{1}{2}\left[\hat{b}_{\mathbf{p}}^{(-)\,\dagger},\, \hat{b}_{\mathbf{p}}^{(+)\,\dagger}\right] \neq 0.
   \end{aligned}$$

   This shows that timelike photons alone **form unphysical states**, since they do not satisfy the Gupta-Bleuler condition. This is expected, since timelike photons have negative norm.

3. **only longitudinal photons**, created by $\hat{a}_{\mathbf{q}}^{(3)\,\dagger} = \frac{1}{2}\left(\hat{b}_{\mathbf{q}}^{(+)\,\dagger} - \hat{b}_{\mathbf{q}}^{(-)\,\dagger}\right)$:

   $$\begin{aligned}
   \hat{b}_{\mathbf{p}}^{(-)\,\dagger}\left(\hat{a}_{\mathbf{q}}^{(0)\,\dagger}|0\rangle\right) &= \frac{1}{2}\hat{b}_{\mathbf{p}}^{(-)\,\dagger}\left(\hat{b}_{\mathbf{q}}^{(+)\,\dagger}|0\rangle - \hat{b}_{\mathbf{q}}^{(-)\,\dagger}|0\rangle\right) \\
   &= \frac{1}{2}\left[\hat{b}_{\mathbf{p}}^{(-)\,\dagger},\, \hat{b}_{\mathbf{p}}^{(+)\,\dagger}\right] - \frac{1}{2}\left[\hat{b}_{\mathbf{p}}^{(-)\,\dagger},\, \hat{b}_{\mathbf{p}}^{(-)\,\dagger}\right] \\
   &= \frac{1}{2}\left[\hat{b}_{\mathbf{p}}^{(-)\,\dagger},\, \hat{b}_{\mathbf{p}}^{(+)\,\dagger}\right] \neq 0.
   \end{aligned}$$

   This shows that longitudinal photons alone **form unphysical states**, since they do not satisfy the Gupta-Bleuler condition. This is expected, since longitudinal photons have negative norm.

4. **combination of timelike and longitudinal photons**, created by $\hat{b}_{\mathbf{q}}^{(+)\,\dagger}$ and $\hat{b}_{\mathbf{q}}^{(-)\,\dagger}$:

- For the combination created by $\hat{b}_{\mathbf{q}}^{(+)\,\dagger}$:

$$\hat{b}_{\mathbf{p}}^{(-)}\left(\hat{b}_{\mathbf{q}}^{(+)\,\dagger}|0\rangle\right)=\left[\hat{b}_{\mathbf{p}}^{(-)},\hat{b}_{\mathbf{q}}^{(+)\,\dagger}\right]|0\rangle\neq 0,$$

  so that states with one timelike and one longitudinal photon in the combination $|S\rangle+|L\rangle$[8] **are unphysical**, since they do not satisfy the Gupta-Bleuler condition (even if they have zero norm).

- For the combination created by $\hat{b}_{\mathbf{q}}^{(-)\,\dagger}$:

$$\hat{b}_{\mathbf{p}}^{(-)}\left(\hat{b}_{\mathbf{q}}^{(-)\,\dagger}|0\rangle\right)=\left[\hat{b}_{\mathbf{p}}^{(-)},\hat{b}_{\mathbf{q}}^{(-)\,\dagger}\right]|0\rangle= 0,$$

  so that states with one timelike and one longitudinal photon in the combination $|S\rangle-|L\rangle$ **are physical**, since they satisfy the Gupta-Bleuler condition.

It seems that we have found a physical state which is a combination of timelike and longitudinal photons, which is unexpected since both these polarization states have negative norm. However, if we compute the norm of the state created by $\hat{b}_{\mathbf{p}}^{(-)\,\dagger}$:

$$\begin{aligned}\langle 0|\,\hat{b}_{\mathbf{p}}^{(-)}\hat{b}_{\mathbf{p}}^{(-)\,\dagger}\,|0\rangle &= \langle 0|\left[\hat{b}_{\mathbf{p}}^{(-)},\hat{b}_{\mathbf{p}}^{(-)\,\dagger}\right]|0\rangle+\langle 0|\,\hat{b}_{\mathbf{p}}^{(-)\,\dagger}\hat{b}_{\mathbf{p}}^{(-)}\,|0\rangle\\ &= \langle 0|\left[\hat{b}_{\mathbf{p}}^{(-)},\hat{b}_{\mathbf{p}}^{(-)\,\dagger}\right]|0\rangle+0=0,\end{aligned}$$

we find that this linear combination (and its multiples) of timelike and longitudinal photons is the only physical one, but its norm is zero. This property indicates that it is a null state in the physical subspace, hence it does not contribute to physical observables: this linear combination is physical but cannot be measured (observed).

Computing the norm of the transverse polarization states, we find that they have positive norm:

$$\begin{aligned}\langle \mathbf{p},\lambda=(1/2)|\mathbf{p},\lambda=(1/2)\rangle &= \langle 0|\,\hat{a}_{\mathbf{p}}^{(1/2)}\hat{a}_{\mathbf{p}}^{(1/2)\,\dagger}\,|0\rangle\\ &= \langle 0|\left[\hat{a}_{\mathbf{p}}^{(1/2)},\hat{a}_{\mathbf{p}}^{(1/2)\,\dagger}\right]|0\rangle+\langle 0|\,\hat{a}_{\mathbf{p}}^{(1/2)\,\dagger}\hat{a}_{\mathbf{p}}^{(1/2)}\,|0\rangle\\ &= (2\pi)^3\delta^{(3)}(\mathbf{0})>0.\end{aligned}$$

Thus, the Gupta-Bleuler condition effectively removes the unphysical polarization states from the theory, leaving only the two transverse polarization states as physical degrees of freedom, while the timelike and longitudinal combinations which survived still lead to zero norm states that do not contribute to physical observables.

Now if we consider a state with two photons, one with transverse polarization $|T\rangle$ and one with $|S\rangle-|L\rangle$ polarization (scalar minus longitudinal, the zero norm states):

$$\hat{b}_{\mathbf{q}}^{(-)\,\dagger}\,|\mathbf{p},\,T\rangle=|\mathbf{p},\,T;\;\mathbf{q},\,S-L\rangle\,,$$

and if we compute its norm:

$$\begin{aligned}\langle \mathbf{p}',\,T;\;\mathbf{q}',\,S-L|\mathbf{p},\,T;\;\mathbf{q},\,S-L\rangle &= \langle 0|\,\hat{a}_{\mathbf{p}'}^{(T)}\hat{b}_{\mathbf{q}'}^{(-)}\hat{b}_{\mathbf{q}}^{(-)\,\dagger}\hat{a}_{\mathbf{p}}^{(T)\,\dagger}\,|0\rangle\\ &= \langle 0|\,\hat{b}_{\mathbf{q}'}^{(-)}\hat{b}_{\mathbf{q}}^{(-)\,\dagger}\left[\hat{a}_{\mathbf{p}'}^{(T)},\,\hat{a}_{\mathbf{p}}^{(T)\,\dagger}\right]|0\rangle+0\\ &= (2\pi)^3\delta^{(3)}(\mathbf{p}-\mathbf{p}')\,\langle 0|\left[\hat{b}_{\mathbf{q}'}^{(-)},\,\hat{b}_{\mathbf{q}}^{(-)\,\dagger}\right]|0\rangle+0=0.\end{aligned}$$

[8] We call it $|S\rangle+|L\rangle$ since for timelike states we would have $|T\rangle$, but then it creates confusion with the transversal polarizations: transversal polarizations remain $|T\rangle$, while timelike ones will be addressed as *scalar*, or $|S\rangle$.

So in the end we can generalize this result to any state with any number of transverse photons and zero norm states (scalar minus longitudinal), leads to a total zero norm for the entire state (one photon in $|S\rangle - |L\rangle$ state is sufficient to zero out the norm of the state). Thus, the physical subspace of the Hilbert space is spanned only by states with transverse photons, while any state containing timelike or longitudinal photons leads to zero norm states that do not contribute to physical observables.

We are interested in th system's observables indeed, which are represented by operators acting on the Hilbert space. Physical observables do not mix physical and unphysical states, since due to the zero norm, any matrix element involving unphysical states vanishes. Thus, physical observables are effectively restricted to the physical subspace spanned by transverse photon states. Furthermore, if we were to compute the energy (or any other observable) of a state containing any number of transverse photons and any number of zero norm states, we would find that the contribution from the zero norm states vanishes always, leaving only the contribution from the transverse photons. This ensures that physical observables are well-defined and consistent with the known properties of photons.

TODO: insert figure

Intuitive Picture We can look at the total Fock space $\mathcal{F}$, which will contain all the possible states created by the creation operators on the vacuum: there will be states with transverse photons, timelike photons, longitudinal photons, and combinations thereof (it contains physical and unphysical states). Inside this total Fock space, we can identify the physical subspace $\mathcal{F}_{\text{phys}}$, which is spanned only by states with transverse photons (the physical degrees of freedom, respecting Gupta-Bleuler condition). The unphysical states (timelike and longitudinal photons) form a subspace $\mathcal{F}_{\text{unphys}}$ that is orthogonal to the physical subspace, and any state in this unphysical subspace has zero norm when projected onto the physical subspace.

Inside $\mathcal{F}_{\text{phys}}$ we will have both positive norm states (transverse photons) and zero norm states (combinations of timelike and longitudinal photons satisfying Gupta-Bleuler condition): transverse states and mixtures of the same number of trannsverse photons plus scalar-longitudinal combinations are gauge equivalent, since they differ by zero norm states. This gauge orbits are equivalence classes of states (states with the same physical observables) in $\mathcal{F}_{\text{phys}}$ that differ only by zero norm states, representing the same physical configuration.

Thus, the physical content of the theory is captured by the equivalence classes of states in $\mathcal{F}_{\text{phys}}$, while the unphysical states in $\mathcal{F}_{\text{unphys}}$ do not contribute to physical observables.

6.3.3 | Energy States

Finally, we can check that the energy operator (hamiltonian) acting on physical states gives positive energy eigenvalues, ensuring the stability of the vacuum and the consistency of the theory. The hamiltonian for the quantized electromagnetic field can be expressed in terms of the creation and annihilation operators as

$$\hat{H} = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} E_{\mathbf{p}} \left(\sum_{\lambda=1}^{3} \hat{a}_{\mathbf{p}}^{(\lambda)\,\dagger} \hat{a}_{\mathbf{p}}^{(\lambda)} - \hat{a}_{\mathbf{p}}^{(0)\,\dagger} \hat{a}_{\mathbf{p}}^{(0)} \right), \tag{6.3.6}$$

after normal ordering, where $E_{\mathbf{p}} = |\mathbf{p}|$ for massless photons. In order to derive this expression, we have used the Legendre transformation of the Lagrangian density for the electromagnetic field, so we need to compute the conjugate momenta and the lagrangian density. Starting from the latter:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2 = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu = -\frac{1}{2}\left(\partial_0 A_\mu \partial^0 A^\mu - \partial_i A_\mu \partial^i A^\mu\right),$$

since the mixed terms in the first part cancel out with the gauge fixing term. We can then compute the conjugate momenta as:

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} = -\frac{1}{2}\frac{\partial}{\partial \dot{A}_\mu}\eta^{\mu\nu}\dot{A}_\mu \dot{A}_\nu = -\partial^0 A^\mu = -\dot{A}^\mu,$$

leading to the hamiltonian:

$$H = \int \mathrm{d}^3\mathbf{x}\,(\pi^\mu \partial_0 A_\mu - \mathcal{L}) = \frac{1}{2}\int \mathrm{d}^3\mathbf{x}\left(\partial_i A_\mu \partial^i A^\mu - \pi_\mu \pi^\mu\right),$$

where we used $\pi^\mu \dot{A}_\mu = \pi_\mu \dot{A}^\mu = -\pi_\mu \pi^\mu$. Let us compute terms separately: starting from the spatial derivative of the fields and substituting the field expansion of (6.3.3), we have

$$\partial_i \hat{A}_\mu(\mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{-ip_i}{\sqrt{2|\mathbf{p}|}}\sum_{\lambda=0}^{3}\epsilon_\mu^{(\lambda)}(\mathbf{p})\left(\hat{a}_{\mathbf{p}}^{(\lambda)}e^{-i\omega t}e^{ip^j x^j} - \hat{a}_{\mathbf{p}}^{(\lambda)\,\dagger}e^{i\omega t}e^{-ip^j x^j}\right),$$

$$\partial^i \hat{A}^\mu(\mathbf{x}) = \int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\frac{-iq^i}{\sqrt{2|\mathbf{q}|}}\sum_{\lambda'=0}^{3}\epsilon^{\mu\,(\lambda')}(\mathbf{q})\left(\hat{a}_{\mathbf{q}}^{(\lambda')}e^{-i\omega t}e^{iq^j x^j} - \hat{a}_{\mathbf{q}}^{(\lambda')\,\dagger}e^{i\omega t}e^{-iq^j x^j}\right),$$

since $\partial_i e^{ip^j x^j} = \frac{\partial}{\partial x^i}e^{ip^j x^j} = ip^i e^{ip^j x^j} = -ip_i e^{ip^j x^j}$.[9] Thus, after integrating in $\mathrm{d}^3\mathbf{x}$ (which generates $\delta(\mathbf{p}+\mathbf{q})$ and $\delta(\mathbf{p}-\mathbf{q})$) and $\mathrm{d}^3\mathbf{q}$ (which is responsible for the minus signs in the first two terms of the next equation) in order to resolve the delta functions, we find

$$\frac{1}{2}\int \mathrm{d}^3\mathbf{x}\,\partial_i A_\mu \partial^i A^\mu = \frac{1}{4}\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}|\mathbf{p}|\sum_{\lambda\lambda'}\eta_{\lambda\lambda'}\left[-\hat{a}_{\mathbf{p}}^{(\lambda)}\hat{a}_{-\mathbf{p}}^{(\lambda')} - \hat{a}_{\mathbf{p}}^{(\lambda)\,\dagger}\hat{a}_{-\mathbf{p}}^{(\lambda')\,\dagger} - \left(\hat{a}_{\mathbf{p}}^{(\lambda)}\hat{a}_{\mathbf{p}}^{(\lambda')\,\dagger} + \hat{a}_{\mathbf{p}}^{(\lambda)\,\dagger}\hat{a}_{\mathbf{p}}^{(\lambda')}\right)\right],$$

$$-\frac{1}{2}\int \mathrm{d}^3\mathbf{x}\,\pi_\mu \pi^\mu = \frac{1}{4}\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}|\mathbf{p}|\sum_{\lambda\lambda'}\eta_{\lambda\lambda'}\left[\hat{a}_{\mathbf{p}}^{(\lambda)}\hat{a}_{-\mathbf{p}}^{(\lambda')} + \hat{a}_{\mathbf{p}}^{(\lambda)\,\dagger}\hat{a}_{-\mathbf{p}}^{(\lambda')\,\dagger} - \left(\hat{a}_{\mathbf{p}}^{(\lambda)}\hat{a}_{\mathbf{p}}^{(\lambda')\,\dagger} + \hat{a}_{\mathbf{p}}^{(\lambda)\,\dagger}\hat{a}_{\mathbf{p}}^{(\lambda')}\right)\right],$$

where we used the orthogonality relation for the polarization vectors:

$$\epsilon_\mu^{(\lambda)}(\mathbf{p})\epsilon^{(\lambda')\,\mu}(\mathbf{p}) = \eta_{\lambda\lambda'}.$$

Summing these two contributions, we find that the terms with two creation or two annihilation operators cancel out, leaving only the mixed terms. Thus, after normal ordering the expression (we will not use the explicit notation for brevity), we find

$$\begin{aligned} H &= -\frac{1}{2}\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}E_{\mathbf{p}}\sum_{\lambda\lambda'}\eta_{\lambda\lambda'}\left(\hat{a}_{\mathbf{p}}^{(\lambda')\,\dagger}\hat{a}_{\mathbf{p}}^{(\lambda)} + \hat{a}_{\mathbf{p}}^{(\lambda)\,\dagger}\hat{a}_{\mathbf{p}}^{(\lambda')}\right) \\ &= \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}E_{\mathbf{p}}\left(\sum_{\lambda=1}^{3}\hat{a}_{\mathbf{p}}^{(\lambda)\,\dagger}\hat{a}_{\mathbf{p}}^{(\lambda)} - \hat{a}_{\mathbf{p}}^{(0)\,\dagger}\hat{a}_{\mathbf{p}}^{(0)}\right), \end{aligned}$$

where in the last step we used the metric signature $-\eta_{\lambda\lambda'} = \mathrm{diag}(-1,1,1,1)$ to separate the contributions from the different polarizations.

When acting on physical states $|\psi\rangle \in \mathcal{F}_{\mathrm{phys}}$, which contain only transverse photons, the expected energy eigenvalues are positive, since each transverse photon contributes positively to the total energy:

$$\langle\psi|\,\hat{H}\,|\psi\rangle = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}E_{\mathbf{p}}\left[\langle\psi|\left(\hat{a}_{\mathbf{p}}^{(1)\,\dagger}\hat{a}_{\mathbf{p}}^{(1)} + \hat{a}_{\mathbf{p}}^{(2)\,\dagger}\hat{a}_{\mathbf{p}}^{(2)}\right)|\psi\rangle + \langle\psi|\,\hat{a}_{\mathbf{p}}^{(3)\,\dagger}\hat{a}_{\mathbf{p}}^{(3)}\,|\psi\rangle - \langle\psi|\,\hat{a}_{\mathbf{p}}^{(0)\,\dagger}\hat{a}_{\mathbf{p}}^{(0)}\,|\psi\rangle\right],$$

[9] When computing the contraction $\partial_i \hat{A}_\mu \partial^i \hat{A}^\mu$, we have to pay attention to the other derivative, since $\partial^i e^{ip^j x^j} = \frac{\partial}{\partial x_i}e^{ip^j x^j} = ip^i \eta_{ij} e^{ip^j x^j} = ip_i e^{ip^j x^j} = -ip^i$; then when contracting $(-ip_i)(-ip^i)$ we need to change sign since $|\mathbf{p}|^2 = p^i p^i = -p_i p^i$.

but since the Gupta-Bleuler condition ensures that the contributions from the timelike and longitudinal photons cancel out in physical states (as shown in eq. (6.3.5)), we have

$$\langle\psi|\,\hat{H}\,|\psi\rangle = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3} E_{\mathbf{p}} \langle\psi| \left(\hat{a}_{\mathbf{p}}^{(1)\,\dagger}\hat{a}_{\mathbf{p}}^{(1)} + \hat{a}_{\mathbf{p}}^{(2)\,\dagger}\hat{a}_{\mathbf{p}}^{(2)}\right)|\psi\rangle \geq 0,$$

confirming that the energy eigenvalues for physical states are indeed positive. This result is crucial for the stability of the vacuum and the overall consistency of the quantum theory of the electromagnetic field.

Notice that the vacuum state $|0\rangle$ has zero energy, as expected, but so do all the zero norm states created by combinations of timelike and longitudinal photons satisfying the Gupta-Bleuler condition. Thus, the vacuum and all zero norm states have the same eigenvalues when acting with operators on the Fock space, while physical states with transverse photons carry the information about the actual physical excitations of the electromagnetic field. We could say that the vacuum is degenerate, since there are multiple states (the vacuum and all zero norm states) that share the same energy eigenvalue of zero. However, only the vacuum state itself is physically relevant, while the zero norm states do not contribute to physical observables; they could be treated as gauge artifacts or unphysical configurations that do not affect measurable quantities in the theory.

Appendices

A | Notation and Conventions

In this Appendix, we summarize the conventions, notation, and definitions used throughout these notes. Unless stated otherwise, we adhere to the conventions commonly used in standard Quantum Field Theory texts (e.g., Peskin & Schroeder).

TODO: Check for consistency with other chapters and modify

Units and Dimensions

We employ **natural units**, defined by setting the reduced Planck constant and the speed of light in vacuum to unity:

$$\hbar = c = 1\,. \tag{A.1}$$

Consequently, quantities that are usually measured in different units become dimensionally related. All physical quantities can be expressed in terms of powers of mass (or energy). Denoting the mass dimension of a quantity X as $[X]$, we have:

- $[L] = [T] = -1$ (Length and Time have inverse mass dimension $[M]^{-1}$).
- $[E] = [p] = [m] = 1$ (Energy, momentum, and mass have mass dimension 1 $[M]^{1}$).

The action S is dimensionless ($[S] = 0$) in natural units (since $\hbar = 1$). This fact is used to determine the canonical dimensions of field operators. For example, in $d = 4$ spacetime dimensions:

- Scalar field $[\phi] = 1$.
- Spinor field $[\psi] = 3/2$.
- Vector field $[A_\mu] = 1$.

Spacetime and Metric

We work in 4-dimensional Minkowski spacetime. The coordinates are denoted by a four-vector x^μ:

$$x^\mu = (x^0, x^1, x^2, x^3) = (t, \mathbf{x})\,, \tag{A.2}$$

where Greek indices $\mu, \nu, \ldots$ run over $\{0, 1, 2, 3\}$. Latin indices $i, j, \ldots$ run over spatial dimensions $\{1, 2, 3\}$.

We utilize the **"mostly minus"** metric signature $(+,-,-,-)$. The metric tensor is:

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \text{diag}(1,-1,-1,-1)\,. \tag{A.3}$$

Scalar products of four-vectors are defined as:

$$a\cdot b = \eta_{\mu\nu}a^\mu b^\nu = a^\mu b_\mu = a^0 b^0 - \mathbf{a}\cdot\mathbf{b}\,, \tag{A.4}$$

where boldface letters denote 3-vectors. Note that $p\cdot x = Et - \mathbf{p}\cdot\mathbf{x}$.

Tensor Indices and Contraction

- **Contravariant vectors** (upper indices) usually represent coordinate differentials dx^μ or momenta $p^\mu = (E,\mathbf{p})$.
- **Covariant vectors** (lower indices) are obtained by contracting with the metric: $x_\mu = \eta_{\mu\nu}x^\nu = (t,-\mathbf{x})$. Note the sign flip in the spatial components.
- **Einstein Summation Convention:** Repeated indices (one upper and one lower) are implicitly summed over. For example, $\partial_\mu A^\mu \equiv \sum_{\mu=0}^{3}\partial_\mu A^\mu$.
- Differential operators are defined as $\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = (\partial_t,\nabla)$ and $\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = (\partial_t,-\nabla)$. Thus, the d'Alembertian is $\Box = \partial_\mu\partial^\mu = \partial_t^2 - \nabla^2$.

The Levi-Civita Symbol

We treat the totally antisymmetric Levi-Civita object $\epsilon^{\mu\nu\rho\sigma}$ as a symbol (tensor density) rather than a strictly geometrical tensor, defined by the convention:

$$\epsilon^{0123} = +1\,. \tag{A.5}$$

Even permutations of (0123) are $+1$, odd permutations are -1, and any repeated index results in 0. Due to the metric signature with three minus signs, the version with lower indices has the opposite sign:

$$\epsilon_{0123} = \eta_{0\alpha}\eta_{1\beta}\eta_{2\gamma}\eta_{3\delta}\epsilon^{\alpha\beta\gamma\delta} = (1)(-1)(-1)(-1)\epsilon^{0123} = -1\,. \tag{A.6}$$

Dirac Algebra and Gamma Matrices

The Dirac matrices γ^μ satisfy the Clifford algebra anticommutation relations:

$$\{\gamma^\mu,\gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbf{1}_{4\times4}\,. \tag{A.7}$$

Hermiticity properties:

$$(\gamma^0)^\dagger = \gamma^0\,, \quad (\gamma^i)^\dagger = -\gamma^i\,. \tag{A.8}$$

These can be summarized compactly as $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$.

Chirality matrix: We define γ^5 (or γ_5) as:

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3\,. \tag{A.9}$$

Properties: $\{\gamma^5,\gamma^\mu\} = 0$, $(\gamma^5)^\dagger = \gamma^5$, and $(\gamma^5)^2 = \mathbf{1}$.

Feynman Slash Notation: Any contraction of a four-vector with the gamma matrices is denoted by a slash:

$$\not{a} \equiv \gamma^\mu a_\mu = \gamma^0 a_0 - \boldsymbol{\gamma} \cdot \mathbf{a}\,. \tag{A.10}$$

Common identities include $\not{p}\not{p} = p^2$ and $\mathrm{Tr}[\not{a}\not{b}] = 4a \cdot b$.

Operators and Fields

We adopt the interaction picture or Heisenberg picture depending on the context. For free fields, the expansions are given in terms of ladder operators.

- **Ladder Operators:**
  - $a^\dagger_{\mathbf{p}}, b^\dagger_{\mathbf{p}}, d^\dagger_{\mathbf{p}}$: Creation operators. They create a particle with momentum $\mathbf{p}$ when acting on the vacuum $|0\rangle$.
  - $a_{\mathbf{p}}, b_{\mathbf{p}}, d_{\mathbf{p}}$: Annihilation operators. $a_{\mathbf{p}}|0\rangle = 0$.
- **Commutation Relations:** For bosons, $[a_{\mathbf{p}}, a^\dagger_{\mathbf{q}}] = (2\pi)^3 2E_{\mathbf{p}} \delta^{(3)}(\mathbf{p} - \mathbf{q})$. For fermions, $\{b_{\mathbf{p}}, b^\dagger_{\mathbf{q}}\} = (2\pi)^3 2E_{\mathbf{p}} \delta^{(3)}(\mathbf{p} - \mathbf{q})$. Note the relativistic normalization factor $2E_{\mathbf{p}}$.
- **Field Interpretation:** A complex scalar field $\phi(x)$ contains annihilation operators for particles (a) and creation operators for antiparticles ($b^\dagger$):

$$\phi(x) \sim \int d\tilde{p}\, (a_{\mathbf{p}} e^{-ipx} + b^\dagger_{\mathbf{p}} e^{ipx})\,. \tag{A.11}$$

  Thus, $\phi(x)$ annihilates a particle or creates an antiparticle at x. Conversely, $\phi^\dagger(x)$ creates a particle or annihilates an antiparticle.

B | Computations and further developments

Sign of complex radical

Nam dui ligula, fringilla a, euismod sodales, sollicitudin vel, wisi. Morbi auctor lorem non justo. Nam lacus libero, pretium at, lobortis vitae, ultricies et, tellus. Donec aliquet, tortor sed accumsan bibendum, erat ligula aliquet magna, vitae ornare odio metus a mi. Morbi ac orci et nisl hendrerit mollis. Suspendisse ut massa. Cras nec ante. Pellentesque a nulla. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Aliquam tincidunt urna. Nulla ullamcorper vestibulum turpis. Pellentesque cursus luctus mauris.

Unitary operators are generated by Hermitian operators

Let U be a unitary operator, that is,

$$U^\dagger U = UU^\dagger = I.$$

We want to show that every unitary operator can be written as the exponential of an Hermitian operator.

Consider an operator A such that

$$U = e^{iA}.$$

We can check the unitarity of U:

$$U^\dagger = (e^{iA})^\dagger = e^{-iA^\dagger}.$$

Then

$$U^\dagger U = e^{-iA^\dagger} e^{iA} = e^{i(A-A^\dagger)}.$$

For U to be unitary we must have $U^\dagger U = I$, hence

$$A^\dagger = A.$$

Therefore, A must be Hermitian. Conversely, given any Hermitian operator A, the exponential e^{iA} is unitary because:

$$(e^{iA})^\dagger = e^{-iA} \quad \Rightarrow \quad (e^{iA})^\dagger e^{iA} = e^{-iA} e^{iA} = I.$$

In summary, unitary operators are generated by Hermitian operators via the exponential map:

$$U = e^{iA}, \qquad A = A^\dagger.$$

In quantum mechanics, this result implies that any continuous unitary transformation can be written as the exponential of an Hermitian generator, which corresponds physically to an observable. For instance, the time evolution operator

$$U(t) = e^{-\frac{i}{\hbar} Ht}$$

is generated by the Hamiltonian H, an Hermitian operator representing the energy of the system.

Free EM field Lagrangian, field strenght and energy-momentum tensors

As anticipated in section 3.4.2, the lagrangian of a free EM field in absence of external sources reads:

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}(\partial_\mu A^\mu)^2.$$

We want to show how it is possible to rewrite it in the following form:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}.$$

Let us develop further this last form, then we will find the same expression from the original lagrangian; then:

$$\begin{aligned} -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} &= -\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) \\ &= -\frac{1}{4}\left(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu - \partial_\nu A_\mu \partial^\mu A^\nu + \partial_\nu A_\mu \partial^\nu A^\mu\right), \end{aligned}$$

but since the last two terms are identical to the first two if we rename the indices $\mu \leftrightarrow \nu$ (on the latter couple), then

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{2}\left(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu\right) = \frac{1}{2}(\partial_\mu A_\nu)(\partial^\nu A^\mu - \partial^\mu A^\nu).$$

Now, developing from the first presented expression for the lagrangian, we implicitly perform some integrations by part with the idea to simplify the boundary terms: we integrate because the dynamics is governed by the action (through its estremization) and we neglect the boundary terms because we assume that *no interesting physics is taking place on* $\partial\mathbb{R}^4$ (at infinite distances in an infinitely further future). Instead of explicit integrals we substitute terms with the inverse product rule for derivatives and neglect the term associated to the total derivative:

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}(\partial_\mu A^\mu)(\partial_\nu A^\nu) \\ &= -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{2}\left[\underbrace{\partial_\mu(A^\mu \partial_\nu A^\nu)}_{\text{on } \partial\mathbb{R}^4 \to 0} - A^\mu \partial_\mu(\partial_\nu A^\nu)\right] \\ &= -\frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) - \frac{1}{2}\left[\underbrace{\partial_\nu(A^\mu \partial_\mu A^\nu)}_{\text{on } \partial\mathbb{R}^4 \to 0} - (\partial_\nu A^\mu)(\partial_\mu A^\nu)\right] \\ &= \frac{1}{2}\left[\eta^{\nu\sigma}\eta_{\nu\rho}(\partial^\rho A^\mu)(\partial_\mu A_\sigma) - (\partial_\mu A_\nu)(\partial^\mu A^\nu)\right], \end{aligned}$$

and finally, after the two "integrations by part" and since $\eta^{\nu\sigma}\eta_{\nu\rho} = \delta^\sigma_\rho$, we rename the indices from the first term as σ, $\rho \to \nu$ and we obtain

$$\mathcal{L} = \frac{1}{2}(\partial_\mu A_\nu)(\partial^\nu A^\mu - \partial^\mu A^\nu),$$

obtaining in the end the same expression from the previous development and so the desired result: the Lagrangian in terms of **field strenght tensor**

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}.$$

From this last form of the Lagrangian it's easy to derive a practical expression for the **energy-momentum tensor**: from the expression in equation (3.3.2) (using A^μ as the dynamical field) we have:

$$\begin{aligned} T^{\mu\nu} &= \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\rho)} \partial^\nu A_\rho - \eta^{\mu\nu}\mathcal{L} \\ &= \frac{\partial}{\partial(\partial_\mu A_\rho)} \left(-\frac{1}{4}F_{\sigma\lambda}F^{\sigma\lambda}\right) \partial^\nu A_\rho - \eta^{\mu\nu}\left(-\frac{1}{4}F_{\sigma\lambda}F^{\sigma\lambda}\right). \end{aligned}$$

Let us ignore the last term, we develop the former:

$$\begin{aligned} \frac{\partial}{\partial(\partial_\mu A_\rho)} \left(-\frac{1}{4}F_{\sigma\lambda}F^{\sigma\lambda}\right) \partial^\nu A_\rho &= -\frac{1}{4}\frac{\partial}{\partial(\partial_\mu A_\rho)} \left(\eta^{\sigma\alpha}\eta^{\lambda\beta}F_{\sigma\lambda}F_{\alpha\beta}\right)\partial^\nu A_\rho \\ &= -\frac{1}{4}\eta^{\sigma\alpha}\eta^{\lambda\beta}\left[F_{\alpha\beta}\left(\delta^\mu{}_\sigma\delta^\rho{}_\lambda - \delta^\mu{}_\lambda\delta^\rho{}_\sigma\right) + F_{\sigma\lambda}\left(\delta^\mu{}_\alpha\delta^\rho{}_\beta - \delta^\mu{}_\beta\delta^\rho{}_\alpha\right)\right]\partial^\nu A_\rho \\ &= -\frac{1}{4}\left[F_{\alpha\beta}\left(\eta^{\mu\alpha}\eta^{\rho\beta} - \eta^{\rho\alpha}\eta^{\mu\beta}\right) + F_{\sigma\lambda}\left(\eta^{\sigma\mu}\eta^{\lambda\rho} - \eta^{\sigma\rho}\eta^{\lambda\mu}\right)\right]\partial^\nu A_\rho \\ &= -\frac{1}{4}(F^{\mu\rho} - F^{\rho\mu} + F^{\mu\rho} - F^{\rho\mu})\partial^\nu A_\rho = -\frac{1}{2}(F^{\mu\rho} - F^{\rho\mu})\partial^\nu A_\rho \\ &= -F^{\mu\rho}\partial^\nu A_\rho. \end{aligned}$$

So, restoring the latter term we obtain the wnated expression for the energy-momentum tensor:

$$T^{\mu\nu} = -F^{\mu\rho}\partial^\nu A_\rho + \frac{1}{4}\eta^{\mu\nu}\left(F_{\sigma\lambda}F^{\sigma\lambda}\right).$$

Hamiltonian operator for KG as harmonic oscillators

Following the approach initiated in section 4.1.2, we substitute the mode expansions for the field and conjugate momenta into the Hamiltonian expression:

$$\hat{H} = \frac{1}{2}\int \mathrm{d}^3\mathbf{x}\left(\hat{\pi}^2 + |\nabla\hat{\psi}|^2 + m^2\hat{\psi}^2\right).$$

Recall the Fourier transform expressions in terms of creation and annihilation operators:

$$\begin{aligned} \hat{\pi}^2 &= -\frac{\sqrt{\omega_\mathbf{p}\omega_\mathbf{q}}}{2}\left[\hat{a}_\mathbf{p}e^{i\mathbf{p}\cdot\mathbf{x}} - \hat{a}^\dagger_\mathbf{p}e^{-i\mathbf{p}\cdot\mathbf{x}}\right]\left[\hat{a}_\mathbf{q}e^{i\mathbf{q}\cdot\mathbf{x}} - \hat{a}^\dagger_\mathbf{q}e^{-i\mathbf{q}\cdot\mathbf{x}}\right], \\ |\nabla\hat{\psi}|^2 &= \frac{1}{2\sqrt{\omega_\mathbf{p}\omega_\mathbf{q}}} i\mathbf{p}\left[\hat{a}_\mathbf{p}e^{i\mathbf{p}\cdot\mathbf{x}} - \hat{a}^\dagger_\mathbf{p}e^{-i\mathbf{p}\cdot\mathbf{x}}\right] i\mathbf{q}\left[\hat{a}_\mathbf{q}e^{i\mathbf{q}\cdot\mathbf{x}} - \hat{a}^\dagger_\mathbf{q}e^{-i\mathbf{q}\cdot\mathbf{x}}\right], \\ m^2\hat{\psi}^2 &= \frac{m^2}{2\sqrt{\omega_\mathbf{p}\omega_\mathbf{q}}}\left[\hat{a}_\mathbf{p}e^{i\mathbf{p}\cdot\mathbf{x}} + \hat{a}^\dagger_\mathbf{p}e^{-i\mathbf{p}\cdot\mathbf{x}}\right]\left[\hat{a}_\mathbf{q}e^{i\mathbf{q}\cdot\mathbf{x}} + \hat{a}^\dagger_\mathbf{q}e^{-i\mathbf{q}\cdot\mathbf{x}}\right]. \end{aligned}$$

By substituting these terms into the Hamiltonian and regrouping, we obtain:

$$\begin{aligned} \hat{H} = \frac{1}{2}\int \mathrm{d}^3\mathbf{x}\frac{\mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{q}}{(2\pi)^6}\Bigg[& e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}\left(-\frac{\sqrt{\omega_\mathbf{p}\omega_\mathbf{q}}}{2} - \frac{\mathbf{p}\cdot\mathbf{q}}{2\sqrt{\omega_\mathbf{p}\omega_\mathbf{q}}} + \frac{m^2}{2\sqrt{\omega_\mathbf{p}\omega_\mathbf{q}}}\right)\hat{a}_\mathbf{p}\hat{a}_\mathbf{q} \\ & + e^{-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}\left(-\frac{\sqrt{\omega_\mathbf{p}\omega_\mathbf{q}}}{2} - \frac{\mathbf{p}\cdot\mathbf{q}}{2\sqrt{\omega_\mathbf{p}\omega_\mathbf{q}}} + \frac{m^2}{2\sqrt{\omega_\mathbf{p}\omega_\mathbf{q}}}\right)\hat{a}^\dagger_\mathbf{p}\hat{a}^\dagger_\mathbf{q} \\ & + e^{i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}\left(+\frac{\sqrt{\omega_\mathbf{p}\omega_\mathbf{q}}}{2} + \frac{\mathbf{p}\cdot\mathbf{q}}{2\sqrt{\omega_\mathbf{p}\omega_\mathbf{q}}} + \frac{m^2}{2\sqrt{\omega_\mathbf{p}\omega_\mathbf{q}}}\right)\hat{a}_\mathbf{p}\hat{a}^\dagger_\mathbf{q} \\ & + e^{-i(\mathbf{p}-\mathbf{q})\cdot\mathbf{x}}\left(+\frac{\sqrt{\omega_\mathbf{p}\omega_\mathbf{q}}}{2} + \frac{\mathbf{p}\cdot\mathbf{q}}{2\sqrt{\omega_\mathbf{p}\omega_\mathbf{q}}} + \frac{m^2}{2\sqrt{\omega_\mathbf{p}\omega_\mathbf{q}}}\right)\hat{a}^\dagger_\mathbf{p}\hat{a}_\mathbf{q}\Bigg]. \end{aligned}$$

We proceed by integrating over $\mathrm{d}^3\mathbf{x}$. Using the integral representation of the Dirac delta, $\int \frac{\mathrm{d}^3\mathbf{x}}{(2\pi)^3} e^{\pm i(\mathbf{p}\pm\mathbf{q})\cdot\mathbf{x}} = \delta(\mathbf{p}\pm\mathbf{q})$, we isolate the terms proportional to $\delta(\mathbf{p}+\mathbf{q})$ and $\delta(\mathbf{p}-\mathbf{q})$:

$$\begin{aligned}\hat{H} &= \frac{1}{2}\int \frac{\mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\,\delta(\mathbf{p}+\mathbf{q})\left(\hat{a}_{\mathbf{p}}\hat{a}_{\mathbf{q}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}}\right)\left(-\frac{\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}}{2} - \frac{\mathbf{p}\cdot\mathbf{q}}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}} + \frac{m^2}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}}\right)\\ &+ \frac{1}{2}\int \frac{\mathrm{d}^3\mathbf{p}\,\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\,\delta(\mathbf{p}-\mathbf{q})\left(\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{q}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{q}}\right)\left(+\frac{\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}}{2} + \frac{\mathbf{p}\cdot\mathbf{q}}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}} + \frac{m^2}{2\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}}\right).\end{aligned}$$

Evaluating the delta functions, which means integrating in $\mathrm{d}^3\mathbf{q}$, and noting the symmetry $\omega_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2+m^2} = \omega_{-\mathbf{p}}$, the expression simplifies to:

$$\begin{aligned}\hat{H} &= \frac{1}{2}\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{p}}}\left(\hat{a}_{\mathbf{p}}\hat{a}_{(-\mathbf{p})} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{(-\mathbf{p})}\right)\left(-\omega^2_{\mathbf{p}} + |\mathbf{p}|^2 + m^2\right)\\ &+ \frac{1}{2}\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{p}}}\left(\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}}\right)\left(\omega^2_{\mathbf{p}} + |\mathbf{p}|^2 + m^2\right),\end{aligned}$$

The first term vanishes identically because the factor $-\omega^2_{\mathbf{p}} + |\mathbf{p}|^2 + m^2$ is zero by definition of the dispersion relation. We are thus left with the final Hamiltonian in terms of ladder operators:

$$\hat{H} = \frac{1}{2}\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\,\omega_{\mathbf{p}}\left(\hat{a}_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}} + \hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}}\right).$$

Commutators between ladder and charge operators in complex KG

As anticipated in section 4.2, we want to compute the commutators between the ladder operators and the charge operator for the complex Klein-Gordon field: let us recall the expressions for the charge operator and the ladder operators

$$\hat{Q} = -i\int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^3}\left(\hat{a}^\dagger_{1,\mathbf{p}}\hat{a}_{2,\mathbf{p}} - \hat{a}^\dagger_{2,\mathbf{p}}\hat{a}_{1,\mathbf{p}}\right),$$

$$\hat{a}_{\pm,\mathbf{p}} = \frac{1}{\sqrt{2}}\left(\hat{a}_{1,\mathbf{p}} \pm i\hat{a}_{2,\mathbf{p}}\right), \quad \hat{a}^\dagger_{\pm,\mathbf{p}} = \frac{1}{\sqrt{2}}\left(\hat{a}^\dagger_{1,\mathbf{p}} \mp i\hat{a}^\dagger_{2,\mathbf{p}}\right).$$

Then we can start the computation, remembering that commutators among ladder operators of different fields vanish:

$$\begin{aligned}\left[\hat{Q},\,\hat{a}_{\pm,\mathbf{p}}\right] &= \frac{1}{\sqrt{2}}\int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\left[-i(\hat{a}^\dagger_{1,\mathbf{q}}\hat{a}_{2,\mathbf{q}} - \hat{a}^\dagger_{2,\mathbf{q}}\hat{a}_{1,\mathbf{q}}),\,\hat{a}_{1,\mathbf{p}} \pm i\hat{a}_{2,\mathbf{p}}\right]\\ &= \frac{1}{\sqrt{2}}\int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\left(-i\hat{a}_{2,\mathbf{q}}\left[\hat{a}^\dagger_{1,\mathbf{q}},\,\hat{a}_{1,\mathbf{p}}\right] \pm \hat{a}_{1,\mathbf{q}}\left[\hat{a}^\dagger_{2,\mathbf{q}},\,\hat{a}_{2,\mathbf{p}}\right]\right)\\ &= \frac{1}{\sqrt{2}}\int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\left(-i\hat{a}_{2,\mathbf{q}} \pm \hat{a}_{1,\mathbf{q}}\right)(-1)\delta^{(3)}(\mathbf{q}-\mathbf{p})\\ &= \pm\hat{a}_{1,\mathbf{p}} + i\hat{a}_{2,\mathbf{p}} = \pm\hat{a}_{\pm,\mathbf{p}}.\end{aligned}$$

Similarly, for the creation operators we have:

$$\begin{aligned}\left[\hat{Q},\,\hat{a}^\dagger_{\pm,\mathbf{p}}\right] &= \frac{1}{\sqrt{2}}\int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\left[-i(\hat{a}^\dagger_{1,\mathbf{q}}\hat{a}_{2,\mathbf{q}} - \hat{a}^\dagger_{2,\mathbf{q}}\hat{a}_{1,\mathbf{q}}),\,\hat{a}^\dagger_{1,\mathbf{p}} \mp i\hat{a}^\dagger_{2,\mathbf{p}}\right]\\ &= \frac{1}{\sqrt{2}}\int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\left(+i\hat{a}^\dagger_{2,\mathbf{q}}\left[\hat{a}_{1,\mathbf{q}},\,\hat{a}^\dagger_{1,\mathbf{p}}\right] \mp \hat{a}^\dagger_{1,\mathbf{q}}\left[\hat{a}_{2,\mathbf{q}},\,\hat{a}^\dagger_{2,\mathbf{p}}\right]\right)\\ &= \frac{1}{\sqrt{2}}\int \frac{\mathrm{d}^3\mathbf{q}}{(2\pi)^3}\left(+i\hat{a}^\dagger_{2,\mathbf{q}} \mp \hat{a}^\dagger_{1,\mathbf{q}}\right)\delta^{(3)}(\mathbf{q}-\mathbf{p})\\ &= \mp\hat{a}^\dagger_{1,\mathbf{p}} + i\hat{a}^\dagger_{2,\mathbf{p}} = \mp\hat{a}^\dagger_{\pm,\mathbf{p}}.\end{aligned}$$