Title

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***Abstract*—// TO DO**

I. Introduction

During system design, a crucial step is to determine if the system works according to the desired specifications such as performance and safety. This verification step is challenging for many reasons, the first one is the need to take into account all possible system behaviors, this makes necessary the use of a formal verification method since most simulation-based approaches are insufficient. The other main ones are the presence of unpredictable disturbance in practical systems, and the complexity of system dynamics that in general is nonlinear, evolves in continuous time, and has high dimensional state space.

Hamilton-Jacobi Reachability Analysis (HJ-RA) is a verification method for guaranteeing performance and safety properties of systems based on reachability analysis, which computes the set of states from which the system can be driven to a target set, while satisfying time-varying state constraints at all times. HJRA overcomes some of the above challenges, it is applicable to general nonlinear systems and easily handles control and disturbance variables.

// TO DO: explain structure of this report

II. Theoretical Fundamentals

In this section, we will introduce all the theoretical fundamentals necessary to understand the theory behind the Hamilton-Jacobi Reachability Analysis.

A. Reachability Analysis

The goal of reachability analysis is to compute the reach-avoid set (RAS) defined as the set of initial states from which the system, using an optimal input, can be driven to a target set within a finite time horizon and satisfying time-varying state constraints at all times. In a reachability problem formulation, the target set can represent a set of undesired states (unsafe), or a set of desired states, in the first case, the RAS contains states to avoid since there exists an optimal input (disturbance) that leads the state into an unsafe region. In the second case instead, the RAS represents safe states from which, applying an optimal input (control law), the system can reach a desired state. In the literature, the just described reach-avoid set, is also called backward reachable set (BRS), in alternative to it, in some cases one might be interested in computing a forward reachable set (FRS), defined as the set of all states that the system can reach from a given initial set of states after a finite time horizon. In order to understand the difference between BRS and FRS, consider a reachability problem in which the target set contains unsafe states, the FRS can be used to check whether the set of possible future states of the system includes undesired states, the BRS instead, can be used to compute, by starting from known unsafe conditions, those states that must be avoided. In this paper HJ-RA is used to compute the BRS of the system, however, it can be used also to compute the FRS.

All systems in the real world are subject to a disturbance, hence they have two different kinds of inputs, a controllable one u (control) and another uncontrollable d (disturbance); for that reason the computation of the BRS can be formulated in terms of a two-player game. For example, consider an aircraft that has to follow a trajectory to complete a task, the system has two inputs: a control input (Player 1) and a disturbance (Player 2), in this scenario, the disturbance could be the wind. Suppose now that there is a goal position to reach (target set) along the trajectory, therefore the control input tries to bring the state at the target and the disturbance to steer it away, in this case, the BRS contains all the initial states for which exists an optimal control command that despite the worst disturbance, brings the system at the goal position. Suppose instead of a goal, there is an obstacle along the trajectory, now the target set is defined by all states of the system that correspond to a collision with the obstacle, therefore, the BRS contains those states which could lead to a collision despite the best possible control actions. In both cases, the BRS can be computed by studying the outcome of the game between the two players. Due to the way they are formulated, those two games are "games of kind", namely games in which the outcome is binary: system reaches or not the target set. In order to solve this type of game it is necessary to translate it into a "games of degree" in which players want to optimize a cost function J, and have opposite goals, one tries to maximize and the other to minimize it. An approach we will see later, called Level Set Method, can perform this kind of transformation by translating the problem into a standard differential game.

B. Differential Games

In this section we will introduce the basic concepts of the game theory related to a two person, zero-sum differential game.

1) Definition of the differential game: Consider the system $\dot{x} = f(x, u, d)$ with $x \in \mathbb{R}^n$, $u \in U \subseteq \mathbb{R}^m$, $d \in D \subseteq \mathbb{R}^p$, $f : \mathbb{R}^n \times U \times D \to \mathbb{R}^n$ and $t \in [\tau_i, \tau_f]$. The inputs $u(\cdot)$, $d(\cdot)$ represent Player 1 (control) and Player 2 (disturbance) respectively, we assume that they are drawn from the set of measurable functions [6][1]:

$$u(\cdot) \in \mathcal{U}_{[\tau_i,\tau_f]} \triangleq \{\sigma : [\tau_i, \tau_f] \to U | \sigma(\cdot) \text{ is measurable}\}$$

$$d(\cdot) \in \mathcal{D}_{[\tau_i,\tau_f]} \triangleq \{\sigma : [\tau_i, \tau_f] \to D | \sigma(\cdot) \text{ is measurable}\}$$

Assume U, D are compact, $f(\cdot)$ is bounded and Lipschitz continuous in x and continuous in u and d, therefore, the system dynamics admits a unique trajectory $x(t)$ that represents the response of the system to the controls $u(\cdot)$, $d(\cdot)$ starting from a initial state $x_i = x(\tau_i)$. The differential equation associated to the game is then the following one:

$$\begin{cases} \dot{x}(t) = f(x(t), u(t), d(t)) & t \in [\tau_i, \tau_f] \\ x(\tau_i) = x_i \end{cases} \tag{1}$$

In order to complete the game definition is necessary to introduce the payoff function of the game $J(\cdot)$ and to do that we use two Lipschitz continuous functions $c : \mathbb{R}^n \to \mathbb{R}$ and $q : \mathbb{R}^n \to \mathbb{R}$:

$$J(x, t, u(\cdot), d(\cdot)) = \int_t^{\tau_f} c(x(s), d(s), u(s), s)\, ds + q(x(\tau_f)) \tag{2}$$

The payoff function represents the reward/cost obtained by the two players at the end of the differential game (1) started at time t with an initial state x, and subject to the commands $u(\cdot)$, $d(\cdot)$ chosen by Player 1 and Player 2 respectively. The function $c(\cdot)$ indicates a running cost, therefore the first term of (2) is the reward gained during the game (trajectory). The second term instead, uses $q(\cdot)$ to evaluate the final state $x(\tau_f)$ reached.

2) Value of the game: In order to solve the game, namely computes its outcome, it's necessary to define the goal and the information pattern of each player. Since we are referring to a zero-sum game, the aims of the players must be opposite, without loss of generality we assume Player 1 ($u(\cdot)$) wants to minimize the cost function $J(\cdot)$ and Player 2 ($d(\cdot)$) tries to maximize it. The information pattern of a game indicates which information a player has respect to its opponent. In the case of simple games, usually there exists a dominant strategy for each player, namely an optimal strategy that is better than other ones independently to the opponent's strategy. In these cases it is easy to solve the game since, assuming rational players, we know a priori the strategies that will be chosen and therefore we can compute the game outcome in advantage. In more complex games like the differential game (1) in which no longer exists dominant strategies, we cannot predict the outcome and therefore we have to find another way to solve game. For this reason in game theory two quantities called lower value $V(\cdot)$ and upper value $U(\cdot)$ are defined. The lower value $V(\cdot)$ indicates the lowest possible outcome of the game (lowest value of $J(\cdot)$, the upper value $U(\cdot)$ is instead the highest one. These values can be defined by giving to a player an advantage respect to the other one, as done in [6] [3] [1] to give a strategic advantage to a player we impose to him the use of a non-anticipative strategy

C. Hamilton-Jacobi Equation

In physics, the Hamilton-Jacobi equation is an alternative formulation of classical mechanics. It is a first-order nonlinear partial differential equation of the form $H(x, u_x(x, \alpha, t), t) + u_t(x, \alpha, t) = K(\alpha, t)$ with independent variables $(x, t) \in \mathbb{R}^n \times \mathbb{R}$ and parameters $\alpha \in \mathbb{R}^n$. It has wide applications in many scientific fields like mechanics. Its solutions determine infinite families of solutions of Hamilton's ordinary differential equations, which are the equations of motion of a mechanical system. In our case, the family of solutions that come from the computation of the HJI PDE and the variational inequality (see Theorems 1 and 2 in Section III) is the viscosity solution.

D. Viscosity Solution

Viscosity solution is a concept introduced by M.Crandall and P. Lions in 1983; it is a notion of solution that allows it to be, for example, nowhere differentiable but for which strong uniqueness theorems, stability theorems and general existence theorems are all valid. It is a generalization of the solution to a partial differential equation (PDE). It has been found that the viscosity solution is that kind of solution to use in many applications of PDE's, including for example first order equations arising in dynamic programming (the Hamilton–Jacobi–Bellman equation) or differential games (the Hamilton–Jacobi–Isaacs equation). This means that, given the classical concept of PDE

$$F(y, u(y), Du(y)) = 0 \tag{3}$$

under the viscosity solution concept, a certain variable u does not need to be everywhere differentiable. There may be points where Du does not exist and yet u satisfies the equation in an appropriate generalized sense.

Taking the main concepts of viscosity solutions from M. Crandall, L.C Evans and P.L Lions, let u be a function from O into $\mathbb{R}$ and let y_0 belong to $\mathcal{O}$. The superdifferential and subdifferential of u at y_0, denoted, respectively, by $D^+u(y_0)$ and $D^-u(y_0)$ are the set of points for which, respectively, the following two inequalities hold:

$$\limsup_{y \to y_0} (u(y) - u(y_0) - p_0 \cdot (y - y_0))|y - y_0|^{-1} \leq 0 \tag{4}$$

and

$$\liminf_{y \to y_0} (u(y) - u(y_0) - p_0 \cdot (y - y_0))|y - y_0|^{-1} \geq 0 \tag{5}$$

where $p_0 \cdot (y - y_0)$ is the Euclidean sclar product of p_0 and $(y - y_0)$. Given these concepts it is possible to define what is a viscosity solution mathematically. If $D^+u(y_0)$ and $D^-u(y_0)$ are nonempty at some x and u is differentiate at x, a viscosity solution of (3) is a function u belonging to $C(\mathcal{O})$ satisfying the two following conditions:

$$F(y, u(), p) \leq 0 \forall y \in \mathcal{O}, \forall p \in D^+u(y) \tag{6}$$

$$F(y, u(), p) \geq 0 \forall y \in \mathcal{O}, \forall p \in D^-u(y) \tag{7}$$

E. Level Set Method

As mentioned before, in order to compute the BRS is necessary to solve a game of kind where the outcome is boolean: the system state either reaches the target set or not. Level Set Method can be used to translate this game into a game of degree, where players share an objective function to optimize. The basic idea of this approach is to encode the boolean outcome through a quantitative function and compare its value at the end of the game to a threshold value, usually zero, to determine whether or not the system reached the target set. The first step is to define a Lipschitz function $g(x)$, where x represents the system state, such that the target set R corresponds to the zero sublevel set of $g(x)$, that is, $x \in R \Leftrightarrow g(x) \leq 0$. We indicated the target set with R (reach) since from now on we suppose that the set contains goal states, namely states to reach. Now we can define the cost function of the game $J(\cdot)$, we are not interested in any kind of running cost, therefore we consider only the value of $g(x)$ at the end of a game in which $t \in [\tau_i, \tau_f]$:

$$J(x, t, u(\cdot), d(\cdot)) = g(x(\tau_f)) \tag{8}$$

The lower value of the game is given by the following value function $V(x,t)$ in which the control u tries to minimize and the disturbance d to maximize the cost $J(\cdot)$. We assume that the player that wants to reach the target set R, namely the control input u (Player 1), is restricted to use a non-anticipative strategies $\gamma[d](t)$ and we indicates the class of strategies admissible in a time interval $[\tau_i, \tau_f]$ as $\Gamma_{[\tau_i,\tau_f]}$.

$$\begin{aligned} V(x,t) &= \inf_{\gamma(\cdot)\in\Gamma(\cdot)} \sup_{d(\cdot)} J(x, t, \gamma(\cdot), d(\cdot)) \\ &= \inf_{\gamma(\cdot)\in\Gamma(\cdot)} \sup_{d(\cdot)} g(x(\tau_f)) \end{aligned} \tag{9}$$

In practical scenarios, along the trajectory of a dynamical system there may be both goals to reach and obstacles to avoid. The goals to reach can be represented by the target set R as previously done, the set of states to avoid instead, can be defined with another set A (avoid) that contains all the system state x that corresponds to an object collision, this new kind of set can be defined using a function $h(x)$ similar to $g(x)$. Formally: consider the sets R, A related respectively to the level sets of two Lipschitz continuous and bounded functions $g : \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R}^n \to \mathbb{R}$, then the two sets can be characterized as:

$$R = \{x \in \mathbb{R}^n | g(x) \leq 0\} \text{ and } A = \{x \in \mathbb{R}^n | h(x) > 0\} \tag{10}$$

The most common choice for the function $g(x)$ and $h(x)$ is to use the distance between the state x and the set of interested, namely:

$$g(x) = \begin{cases} -d(x, R^c) & \text{if } x \in R \\ d(x, R) & \text{if } x \in R^c \end{cases} \tag{11}$$

$$h(x) = \begin{cases} d(x, A^c) & \text{if } x \in A \\ -d(x, A) & \text{if } x \in A^c \end{cases} \tag{12}$$

Since $g(x)$, $h(x)$ must be bounded, we will see later why, we can introduce two constants C_g C_h to impose a saturation to the distance functions, or alternatively, we can use the arctangent of the signed distance, in this way the resulting functions are bounded and also globally Lipschtz [2]. In the next section we will see how the value function $V(\cdot)$ is formulated when we have both a reach R and an avoid A set, and most importantly how it can be solved in order to calculate the BRS. In the following sections we will refer to the BRS as a reach-avoid set (RAS) to highlight the fact that there is both a set to reach and one to avoid.

III. Problem Formulation

Consider the system $\dot{x} = f(x, u, d)$ with $x \in \mathbb{R}^n$, $u \in U \subseteq \mathbb{R}^m$, $d \in D \subseteq \mathbb{R}^p$, $f : \mathbb{R}^n \times U \times D \to \mathbb{R}^n$ and $t \in [\tau_i, \tau_f]$. The inputs $u(\cdot)$, $d(\cdot)$ represent Player 1 (control) and Player 2 (disturbance) respectively, we assume that they are drawn from the set of measurable functions: (WHY??)

$$u(\cdot) \in \mathcal{U}_{[\tau_i,\tau_f]} \triangleq \{\sigma : [\tau_i, \tau_f] \to U | \sigma(\cdot) \text{ is measurable}\}$$

$$d(\cdot) \in \mathcal{D}_{[\tau_i,\tau_f]} \triangleq \{\sigma : [\tau_i, \tau_f] \to D | \sigma(\cdot) \text{ is measurable}\}$$

Consider also two functions $g : \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R}^n \to \mathbb{R}$ used to represent the target set R and the avoid set A respectively (11) (12). Assume U, D are compact, $f(\cdot)$, $g(\cdot)$, $h(\cdot)$ are bounded and Lipschitz continuous in x and continuous in u and d, therefore, the system dynamics admits a unique trajectory $x(t)$ from an initial state x_i at time τ_i under input $u(\cdot)$ and $d(\cdot)$. We denote this solutions as:

$$\phi(t; x_i, \tau_i, u(\cdot), d(\cdot)) : [\tau_i, \tau_f] \to \mathbb{R}^n$$

In order to define the information patterns of the differential game is necessary to compute the RAS, we assume that the control input u (Player 1), is restricted to use a non-anticipative strategies $\gamma(\cdot)$ as before. Given the previous sets R and A (10), whereby for technical reasons we assume R closed and A open, we can define two different kinds of reachability problems. In the first one we are interested in reaching safely R exactly at the end of the game $(t = \tau_f)$, in the second instead, the system can reach safely R at any t inside the time horizon $[\tau_i, \tau_f]$. In the following we will formulate and solve both of them.

A. *Reach-avoid at the terminal time*

In this first type of reachability problem, we are interested in characterizing the RAS as the set of initial states from which the system trajectory $\sigma(\cdot)$ can start and reach the target set R at the terminal time τ_f, without passing through the avoid set A over the time interval $[\tau_i, \tau_f]$. Formally the RAS contains all the initial states x_i for which there exists an optimal strategy $\gamma[d](t) \in \Gamma[\tau_i, \tau_f]$ such that for all $d(\cdot) \in \mathcal{D}_{[\tau_i,\tau_f]}$, the system trajectory satisfies $x(\tau_f) \in R$ and $x(t) \in A^c$ for all $t \in [\tau_i, \tau_f]$:

$$\begin{aligned} RAS_{\tau_f}(t) = \{ & x_i \in \mathbb{R}^n | \exists\gamma(\cdot) \in \Gamma_{[t,\tau_f]}, \forall d(\cdot) \in \mathcal{D}_{[t,\tau_f]} \\ & (\phi(\tau_f; x_i, t, \gamma(\cdot), d(\cdot)) \in R) \quad \wedge \\ & (\forall\tau \in [t, \tau_f], \phi(\tau; x_i, t, \gamma(\cdot), d(\cdot)) \notin A) \ \} \end{aligned} \tag{13}$$

The cost function $J(\cdot)$ of the game now must take into account also the presence of obstacles along the trajectory, therefore we define it as:

$$J(x,t,u(\cdot),d(\cdot)) = \max\left\{g(\mathcal{X}(\tau_f)), \max_{\tau\in[\tau_i,\tau_f]} h(\mathcal{X}(\tau))\right\} \tag{14}$$

Where:

$$\mathcal{X}(\tau) = \phi(\tau; x, t, u(\cdot), d(\cdot))$$

Then the value function $V : \mathbb{R}^n \times [\tau_i, \tau_f] \to R$ is given by:

$$V(x,t) = \inf_{\gamma(\cdot)\in\Gamma_{[t,\tau_f]}} \sup_{d(\cdot)\in\mathcal{D}_{[t,\tau_f]}} J(x,t,\gamma(\cdot),d(\cdot))$$

The RAS is linked to the level set of the value function $V(\cdot)$ through the following proposition proved in [3] :

Proposition 1. $RAS_{\tau_f}(t) = \{x \in \mathbb{R}^n | V(x,t) \le 0\}$

We are finally ready to introduce the theorem that allows us to compute $V(\cdot)$ and then thanks to Prop.1, to calculate RAS_{term}. The proof can be found in [3].

Theorem 1. *$V(\cdot)$ is the unique viscosity solution over $(x,t) \in \mathbb{R}^n \times [\tau_i, \tau_f]$ of the variational inequality:*

$$\max\left\{h(x) - V(x,t), \frac{\partial V}{\partial t}(x,t) + H(x,t)\right\} = 0$$

$$H(x,t) = \sup_{d\in D} \inf_{u\in U} \frac{\partial V}{\partial t}(x,t) f(x,u,d)$$

with terminal condition:

$$V(x,\tau_f) = \max\{g(x), h(x)\}$$

B. Reach-avoid at any time

The second type of reachability problem is similar to the previous one, however, in this case we are not interested to reach the target set exactly at $t = \tau_f$ but at any time $t \in [\tau_i, \tau_f]$, therefore, the RAS now contains the set of initial states x_i from which the system trajectory can start and, using an optimal control input u, reaches the target set R at some time t without passing through the set A until it hits R.

$$\begin{aligned}\widetilde{RAS}_t(t) = \big\{&x_i \in \mathbb{R}^n | \exists\gamma(\cdot) \in \Gamma_{[t,\tau_f]}, \forall d(\cdot) \in \mathcal{D}_{[t,\tau_f]} \\ &\exists\tau_1 \in [t,\tau_f], (\phi(\tau_1; x_i, t, \gamma(\cdot), d(\cdot)) \in R) \quad \wedge \\ &\quad (\forall\tau_2 \in [t,\tau_f], \phi(\tau_2; x_i, t, \gamma(\cdot), d(\cdot)) \notin A) \;\big\}\end{aligned} \tag{15}$$

For technical reasons related to the current differential game [3] [5], it is necessary to define an augmented system dynamics in which Player 1 uses an augmented input $\widetilde{u} = [u, \overline{u}] \in U \times [0,1]$

$$\widetilde{f}(x,\widetilde{u},d) = \overline{u} f(x,u,d)$$

Assume $\widetilde{U}, \widetilde{\mathcal{U}}, \widetilde{\Gamma}$ defined similarly to the previous case, we denote the augmented system trajectory as $\widetilde{\phi}(\tau; x_i, t, \widetilde{u}(\cdot), d(\cdot))$. The value function is then similar to the previous case:

$$\widetilde{V}(x,t) = \inf_{\widetilde{\gamma}(\cdot)\in\widetilde{\Gamma}_{[t,\tau_f]}} \sup_{d(\cdot)\in\mathcal{D}_{[t,\tau_f]}} \widetilde{J}(x,t,\widetilde{\gamma}(\cdot),d(\cdot)) \tag{16}$$

Where

$$\widetilde{J}(x,t,\widetilde{u}(\cdot),d(\cdot)) = \max\left\{g(\widetilde{\mathcal{X}}(\tau_f)), \max_{\tau\in[\tau_i,\tau_f]} h(\widetilde{\mathcal{X}}(\tau))\right\}$$

$$\widetilde{\mathcal{X}}(\tau) = \widetilde{\phi}(\tau; x, t, \widetilde{u}(\cdot), d(\cdot))$$

Also in this case the RAS is linked to the value function $\widetilde{V}(x,t)$, proof in [3].

Proposition 2. *For any $t \in [\tau_i, \tau_f]$:*

$$\widetilde{RAS}_t(t) = \left\{x \in \mathbb{R}^n | \widetilde{V}(x,t) \le 0\right\}$$

Finally, the following theorem allows us to compute the RAS also in this case. Proof in [3].

Theorem 2. *$\widetilde{V}(\cdot)$ is the unique viscosity solution over $(x,t) \in \mathbb{R}^n \times [\tau_i, \tau_f]$ of the variational inequality:*

$$\max\left\{h(x) - V(x,t), \frac{\partial\widetilde{V}}{\partial t}(x,t) + \min\left\{0, \widetilde{H}(x,t)\right\}\right\} = 0$$

$$\widetilde{H}(x,t) = \sup_{d\in D} \inf_{u\in U} \frac{\partial\widetilde{V}}{\partial t}(x,t) f(x,u,d)$$

with terminal condition:

$$\widetilde{V}(x,\tau_f) = \max\{g(x), h(x)\}$$

IV. Case of Study

case of study test

V. Simulation

simulation

VI. Conclusion

conclusion

References

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