Title

Filippo Ficarola, Gioele Migno, Leonardo Pio Lo Porto

***Abstract*—abstract**

I. Introduction

intro

II. Theoretical Fundamentals

A. Level Set Method

As mentioned before, in order to compute the BRS is necessary to solve a game of kind where the outcome is boolean: the system state either reaches the target set or not. Level Set Method can be used to translate this game into a game of degree, where players share an objective function to optimize. The basic idea of this approach is to encode the boolean outcome through a quantitative function and compare its value at the end of the game to a threshold value, usually zero, to determine whether or not the system reached the target set. The first step is to define a Lipschitz function $g(x)$, where x represents the system state, such that the target set R corresponds to the zero sublevel set of $g(x)$, that is, $x \in R \Leftrightarrow g(x) \leq 0$. We indicated the target set with R (reach) since from now on we suppose that the set contains goal states, namely states to reach. Now we can define the cost function of the game $J(\cdot)$, we are not interested in any kind of running cost, therefore we consider only the value of $g(x)$ at the end of a game in which $t \in [\tau_i, \tau_f]$:

$$J(x, t, u(\cdot), d(\cdot)) = g(x(\tau_f)) \tag{1}$$

The lower value of the game is given by the following value function $V(x,t)$ in which the control u tries to minimize and the disturbance d to maximize the cost $J(\cdot)$. We assume that the player that wants to reach the target set R, namely the control input u (Player 1), is restricted to use a non-anticipative strategies $\gamma[d](t)$ and we indicates the class of strategies admissible in a time interval $[\tau_i, \tau_f]$ as $\Gamma_{[\tau_i,\tau_f]}$.

$$\begin{aligned} V(x,t) &= \inf_{\gamma(\cdot)\in\Gamma(\cdot)} \sup_{d(\cdot)} J(x,t,\gamma(\cdot),d(\cdot)) \\ &= \inf_{\gamma(\cdot)\in\Gamma(\cdot)} \sup_{d(\cdot)} g(x(\tau_f)) \end{aligned} \tag{2}$$

In practical scenarios, along the trajectory of a dynamical system there may be both goals to reach and obstacles to avoid. The goals to reach can be represented by the target set R as previously done, the set of states to avoid instead, can be defined with another set A (avoid) that contains all the system state x that corresponds to an object collision, this new kind of set can be defined using a function $h(x)$ similar to $g(x)$. Formally: consider the sets R, A related respectively to the level sets of two Lipschitz continuous and bounded functions $g : \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R}^n \to \mathbb{R}$, then the two sets can be characterized as:

$$R = \{x \in \mathbb{R}^n | g(x) \leq 0\} \text{ and } A = \{x \in \mathbb{R}^n | h(x) > 0\} \tag{3}$$

The most common choice for the function $g(x)$ and $h(x)$ is to use the distance between the state x and the set of interested, namely:

$$g(x) = \begin{cases} -d(x, R^c) & \text{if } x \in R \\ d(x, R) & \text{if } x \in R^c \end{cases} \tag{4}$$

$$h(x) = \begin{cases} d(x, A^c) & \text{if } x \in A \\ -d(x, A) & \text{if } x \in A^c \end{cases} \tag{5}$$

Since $g(x)$, $h(x)$ must be bounded, we will see later why, we can introduce two constants C_g C_h to impose a saturation to the distance functions, or alternatively, we can use the arctangent of the signed distance, in this way the resulting functions are bounded and also globally Lipschtz [2]. In the next section we will see how the value function $V(\cdot)$ is formulated when we have both a reach R and an avoid A set, and most importantly how it can be solved in order to calculate the BRS. In the following sections we will refer to the BRS as a reach-avoid set (RAS) to highlight the fact that there is both a set to reach and one to avoid.

III. Problem Formulation

Consider the system $\dot{x} = f(x, u, d)$ with $x \in \mathbb{R}^n$, $u \in U \subseteq \mathbb{R}^m$, $d \in D \subseteq \mathbb{R}^p$, $f : \mathbb{R}^n \times U \times D \to \mathbb{R}^n$ and $t \in [\tau_i, \tau_f]$. The inputs $u(\cdot)$, $d(\cdot)$ represent Player 1 (control) and Player 2 (disturbance) respectively, we assume that they are drawn from the set of measurable functions: (WHY??)

$$u(\cdot) \in \mathcal{U}_{[\tau_i,\tau_f]} \triangleq \{\sigma : [\tau_i, \tau_f] \to U | \sigma(\cdot) \text{ is measurable}\}$$

$$d(\cdot) \in \mathcal{D}_{[\tau_i,\tau_f]} \triangleq \{\sigma : [\tau_i, \tau_f] \to D | \sigma(\cdot) \text{ is measurable}\}$$

Consider also two functions $g : \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R}^n \to \mathbb{R}$ used to represent the target set R and the avoid set A respectively (4) (5). Assume U, D are compact, $f(\cdot)$, $g(\cdot)$, $h(\cdot)$ are bounded and Lipschitz continuous in x and continuous in u and d, therefore, the system dynamics admits a unique trajectory $x(t)$ from an initial state x_i at time τ_i under input $u(\cdot)$ and $d(\cdot)$. We denote this solutions as:

$$\phi(t; x_i, \tau_i, u(\cdot), d(\cdot)) : [\tau_i, \tau_f] \to \mathbb{R}^n$$

In order to define the information patterns of the differential game is necessary to compute the RAS, we assume that the control input u (Player 1), is restricted to use a non-anticipative strategies $\gamma(\cdot)$ as before. Given the previous sets R and A (3), whereby for technical reasons we assume R closed and

A open, we can define two different kinds of reachability problems. In the first one we are interested in reaching safely R exactly at the end of the game ($t = \tau_f$), in the second instead, the system can reach safely R at any t inside the time horizon $[\tau_i, \tau_f]$. In the following we will formulate and solve both of them.

A. Reach-avoid at the terminal time

In this first type of reachability problem, we are interested in characterizing the RAS as the set of initial states from which the system trajectory $\sigma(\cdot)$ can start and reach the target set R at the terminal time τ_f, without passing through the avoid set A over the time interval $[\tau_i, \tau_f]$. Formally the RAS contains all the initial states x_i for which there exists an optimal strategy $\gamma[d](t) \in \Gamma[\tau_i, \tau_f]$ such that for all $d(\cdot) \in \mathcal{D}_{[\tau_i,\tau_f]}$, the system trajectory satisfies $x(\tau_f) \in R$ and $x(t) \in A^c$ for all $t \in [\tau_i, \tau_f]$:

$$\begin{aligned} RAS_{\tau_f}(t) = \big\{ & x_i \in \mathbb{R}^n | \exists \gamma(\cdot) \in \Gamma_{[t,\tau_f]}, \forall d(\cdot) \in \mathcal{D}_{[t,\tau_f]} \\ & (\phi(\tau_f; x_i, t, \gamma(\cdot), d(\cdot)) \in R) \quad \wedge \\ & (\forall \tau \in [t, \tau_f], \phi(\tau; x_i, t, \gamma(\cdot), d(\cdot)) \notin A) \ \} \end{aligned} \tag{6}$$

The cost function $J(\cdot)$ of the game now must take into account also the presence of obstacles along the trajectory, therefore we define it as:

$$J(x, t, u(\cdot), d(\cdot)) = \max \left\{ g(\mathcal{X}(\tau_f)), \max_{\tau \in [\tau_i, \tau_f]} h(\mathcal{X}(\tau)) \right\} \tag{7}$$

Where:

$$\mathcal{X}(\tau) = \phi(\tau; x, t, u(\cdot), d(\cdot))$$

Then the value function $V : \mathbb{R}^n \times [\tau_i, \tau_f] \to R$ is given by:

$$V(x, t) = \inf_{\gamma(\cdot) \in \Gamma_{[t,\tau_f]}} \sup_{d(\cdot) \in \mathcal{D}_{[t,\tau_f]}} J(x, t, \gamma(\cdot), d(\cdot))$$

The RAS is linked to the level set of the value function $V(\cdot)$ through the following proposition proved in [3] :

Proposition 1. $RAS_{\tau_f}(t) = \{x \in \mathbb{R}^n | V(x, t) \leq 0\}$

We are finally ready to introduce the theorem that allows us to compute $V(\cdot)$ and then thanks to Prop.1, to calculate RAS_{term}. The proof can be found in [3].

Theorem 1. *$V(\cdot)$ is the unique viscosity solution over $(x, t) \in \mathbb{R}^n \times [\tau_i, \tau_f]$ of the variational inequality:*

$$\max \left\{ h(x) - V(x, t), \frac{\partial V}{\partial t}(x, t) + H(x, t) \right\} = 0$$

$$H(x, t) = \sup_{d \in D} \inf_{u \in U} \frac{\partial V}{\partial t}(x, t) f(x, u, d)$$

with terminal condition:

$$V(x, \tau_f) = \max \{g(x), h(x)\}$$

B. Reach-avoid at any time

The second type of reachability problem is similar to the previous one, however, in this case we are not interested to reach the target set exactly at $t = \tau_f$ but at any time $t \in [\tau_i, \tau_f]$, therefore, the RAS now contains the set of initial states x_i from which the system trajectory can start and, using an optimal control input u, reaches the target set R at some time t without passing through the set A until it hits R.

$$\begin{aligned} \widetilde{RAS}_t(t) = \big\{ & x_i \in \mathbb{R}^n | \exists \gamma(\cdot) \in \Gamma_{[t,\tau_f]}, \forall d(\cdot) \in \mathcal{D}_{[t,\tau_f]} \\ & \exists \tau_1 \in [t, \tau_f], (\phi(\tau_1; x_i, t, \gamma(\cdot), d(\cdot)) \in R) \quad \wedge \\ & (\forall \tau_2 \in [t, \tau_f], \phi(\tau_2; x_i, t, \gamma(\cdot), d(\cdot)) \notin A) \ \} \end{aligned} \tag{8}$$

For technical reasons related to the current differential game [3] [5], it is necessary to define an augmented system dynamics in which Player 1 uses an augmented input $\widetilde{u} = [u, \overline{u}] \in U \times [0, 1]$

$$\widetilde{f}(x, \widetilde{u}, d) = \overline{u} f(x, u, d)$$

Assume $\widetilde{U}, \widetilde{\mathcal{U}}, \widetilde{\Gamma}$ defined similarly to the previous case, we denote the augmented system trajectory as $\widetilde{\phi}(\tau; x_i, t, \widetilde{u}(\cdot), d(\cdot))$. The value function is then similar to the previous case:

$$\widetilde{V}(x, t) = \inf_{\widetilde{\gamma}(\cdot) \in \widetilde{\Gamma}_{[t,\tau_f]}} \sup_{d(\cdot) \in \mathcal{D}_{[t,\tau_f]}} \widetilde{J}(x, t, \widetilde{\gamma}(\cdot), d(\cdot)) \tag{9}$$

Where

$$\widetilde{J}(x, t, \widetilde{u}(\cdot), d(\cdot)) = \max \left\{ g(\widetilde{\mathcal{X}}(\tau_f)), \max_{\tau \in [\tau_i, \tau_f]} h(\widetilde{\mathcal{X}}(\tau)) \right\}$$

$$\widetilde{\mathcal{X}}(\tau) = \widetilde{\phi}(\tau; x, t, \widetilde{u}(\cdot), d(\cdot))$$

Also in this case the RAS is linked to the value function $\widetilde{V}(x, t)$, proof in [3].

Proposition 2. *For any $t \in [\tau_i, \tau_f]$:*

$$\widetilde{RAS}_t(t) = \left\{ x \in \mathbb{R}^n | \widetilde{V}(x, t) \leq 0 \right\}$$

Finally, the following theorem allows us to compute the RAS also in this case. Proof in [3].

Theorem 2. *$\widetilde{V}(\cdot)$ is the unique viscosity solution over $(x, t) \in \mathbb{R}^n \times [\tau_i, \tau_f]$ of the variational inequality:*

$$\max \left\{ h(x) - V(x, t), \frac{\partial \widetilde{V}}{\partial t}(x, t) + \min \left\{ 0, \widetilde{H}(x, t) \right\} \right\} = 0$$

$$\widetilde{H}(x, t) = \sup_{d \in D} \inf_{u \in U} \frac{\partial \widetilde{V}}{\partial t}(x, t) f(x, u, d)$$

with terminal condition:

$$\widetilde{V}(x, \tau_f) = \max \{g(x), h(x)\}$$

IV. Case of Study

case of study

V. Simulation

simulation

VI. Conclusion

conclusion

References

[1] S. Bansal, Mo Chen, S. Herbert and C. J. Tomlin, "Hamilton-Jacobi Reachability: A Brief Overview and Recent Advances", *arxiv.org/abs/1709.07523*.

[2] K. Margellos and J. Lygeros, "Hamilton-Jacobi formulation for reach-avoid problems with an application to air traffic management", *Proceedings of the 2010 American Control Conference*, 2010, pp. 3045-3050, doi: 10.1109/ACC.2010.5530514.

[3] K. Margellos and J. Lygeros, "Hamilton–Jacobi Formulation for Reach–Avoid Differential Games," *in IEEE Transactions on Automatic Control*, vol. 56, no. 8, pp. 1849-1861, Aug. 2011, doi: 10.1109/TAC.2011.2105730.

[4] Datar, Mandar & Ketkar, Vishwanath & Mejari, Manas & Gupta, Ankit & Singh, N.M. & Kazi, Faruk, "Motion Planning for the Three Wheel Mobile Robot using the Reachable Set Computation under Constraints, (2014)" *IFAC Proceedings Volumes (IFAC-PapersOnline)*, 3. 10.3182/20140313-3-IN-3024.00178.

[5] I. M. Mitchell, A. M. Bayen and C. J. Tomlin, "A time-dependent Hamilton-Jacobi formulation of reachable sets for continuous dynamic games," *in IEEE Transactions on Automatic Control*, vol. 50, no. 7, pp. 947-957, July 2005, doi: 10.1109/TAC.2005.851439.