Autonomous and Mobile Robotics Project

Leonardo Pio Lo Porto, Gioele Migno, Filippo Ficarola

Safe Navigation for a three wheels mobile robot through Hamilton-Jacobi Reachability Analysis

DIPARTIMENTO DI INGEGNERIA INFORMATICA
AUTOMATICA E GESTIONALE ANTONIO RUBERTI

SAPIENZA
UNIVERSITÀ DI ROMA

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outline

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introduction

What is Hamilton-Jacobi Reachability Analysis (HJ-RA)?

- HJ-RA is a verification method for guaranteeing performance and safety properties of systems based on reachability analysis and differential games theory

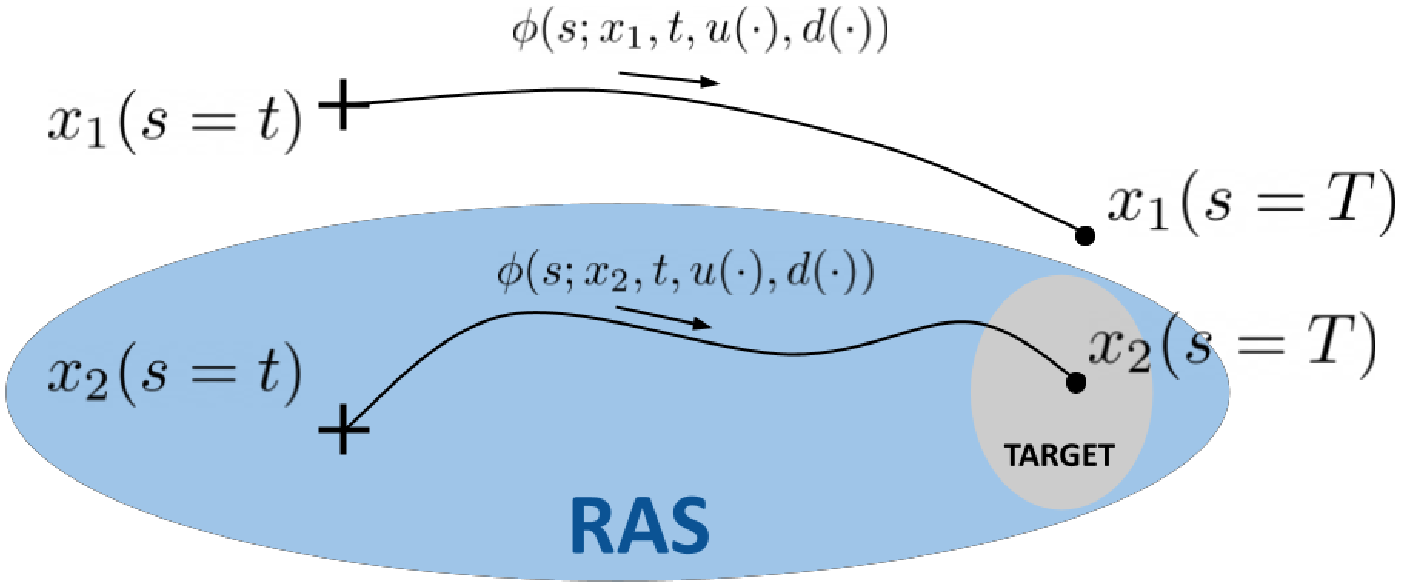
HJ-RA	=	Reachability Analysis	+	Differential Game

- HJ-RA is applicable to general nonlinear systems

theoretical foundations: reachability analysis

What is the goal of reachability analysis?

- The goal is to compute the **reach-avoid set (RAS)** that is defined as:
  «*The* ***set of initial states*** *from which the system, using an* ***optimal input****, can be driven to a* ***target set*** *within a finite horizon and satisfying time-varying state constraints at all times*»



theoretical foundations: reachability analysis

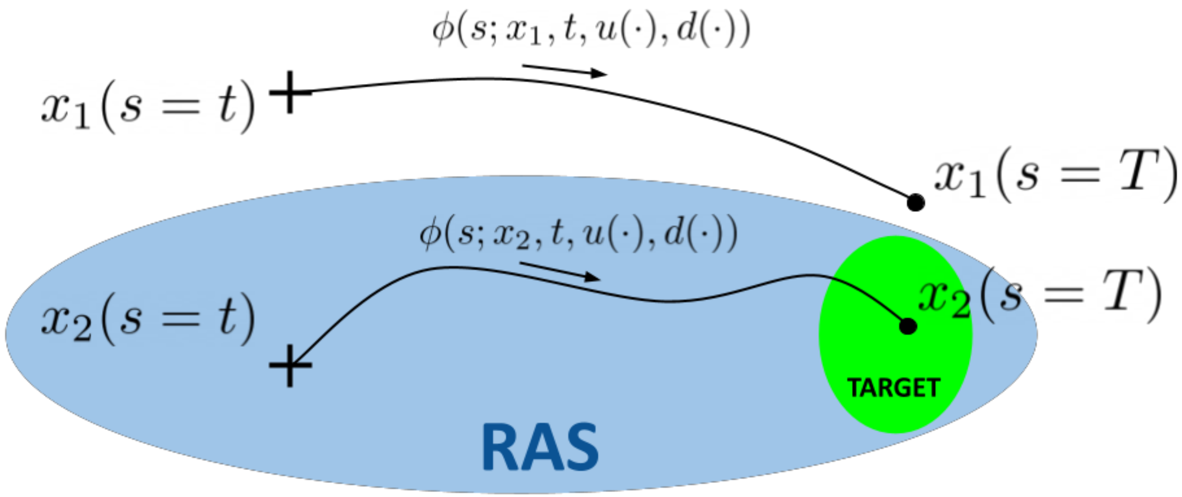
What does the target set represent?

- **Case 1**: it represents a set of **undesired states** (unsafe) -> RAS must be avoided since there exists an optimal input *d* (**disturbance**) that leads the state into an unsafe region

- **Case 2**: it represents a set of **desired states** (safe) -> State must start in RAS since applying an optimal input (**control law**) the system can reach a desired state

We assume Case 2:

TARGET SET = Desired States



theoretical foundations: differential games

Why two-players differential game?

- Systems in real world are subjected to **two types of inputs**, a controllable one *u* (**control**) and another uncontrollable *d* (**disturbance**)

$$\dot{x} = f(x(\cdot), u(\cdot), d(\cdot))$$

- The two inputs are seen as two players:
    - ***u*** tries **to reach** the target set (Player 1)
    - ***d*** tries **to steer away** the system state from target set (Player 2)

theoretical foundations: differential games

Definition of the differential game

- **System dynamics**

$$\begin{cases} \dot{x}(s) = f(x(s), u(s), d(s)) \\ x(t) = x \end{cases} \quad s \in [t, T] \quad t \in [0, T] \quad u(\cdot) \in U \subseteq \mathbb{R}^m$$

$$x(\cdot) \in \mathbb{R}^n \qquad d(\cdot) \in D \subseteq \mathbb{R}^p$$

- *u* and *d* are the two players
- Game starts at time $s = t$ with state $x(t) = x$ and ends at time $s = T$ with state $x(T)$

- We denote the trajectory of the system as:

$$\phi(s, x, t, u(\cdot), d(\cdot)) \colon [t, T] \to \mathbb{R}^n$$

theoretical foundations: differential games

Definition of the differential game (cont'd)

- **Payoff function** of the game starting at time $s = t$ from initial state x

$$J(x, t, u(\cdot), d(\cdot)) = \underbrace{\int_t^T c(x(s), d(s), u(s), s)\, ds}_{\text{RUNNING COST}} + \underbrace{q(x(T))}_{\text{FINAL COST}}$$

$$c : \mathbb{R}^n \to \mathbb{R}$$
$$q : \mathbb{R}^n \to \mathbb{R}$$

- $c(\cdot)$ indicates the running cost which represents the reward gained during the game while $q(\cdot)$ is used to evaluate the final state $x(T)$ reached

- The payoff function represents the reward/cost obtained by the two players at the end of the differential game

theoretical foundations: differential games

Definition of the differential game (cont'd)

- **Payoff function** of the game starting at time $s = t$ from initial state x

$$J(x, t, u(\cdot), d(\cdot)) = \underbrace{\int_t^T c(x(s), d(s), u(s), s)\, ds}_{\text{RUNNING COST}} + \underbrace{q(x(T))}_{\text{FINAL COST}}$$

$$c : \mathbb{R}^n \to \mathbb{R}$$
$$q : \mathbb{R}^n \to \mathbb{R}$$

Without loss of generality, we assume:

- u wants minimize $J(\cdot)$
- d wants to maximize $J(\cdot)$

theoretical foundations: differential games

Outcome of the game

- In the case of simple games, there exists a **dominant strategy** for each player, namely an optimal strategy that is <u>better than others independently of any opponent's strategy</u>

- In the case of differential games in which no longer exist dominant strategies, we cannot predict the **outcome** (= value of payoff function at the end of the game)

→ Solution: define two quantities:
  - *Lower value* $V^{-}(\cdot)$: indicates the lowest possible outcome of the game
  - *Upper value* $V^{+}(\cdot)$: indicates the highest possible outcome of the game

theoretical foundations: differential games

Information Pattern

- To define mathematically the outcome of the game we give a strategic advantage to a player with respect to the other by using a *non-anticipative strategy*
- *Non-anticipative strategy*: A player knows the current input chosen by its opponent
- Lower and Upper value definition

$$V^{-}(x,t) \triangleq \inf_{\gamma(\cdot)\in\Gamma(\cdot)} \sup_{d(\cdot)\in\mathcal{D}(\cdot)} J(x,t,\gamma[d](\cdot),d) \qquad V^{+}(x,t) \triangleq \sup_{\delta(\cdot)\in\Delta(\cdot)} \inf_{u(\cdot)\in\mathcal{U}(\cdot)} J(x,t,u,\delta[u](\cdot))$$

- Where $\gamma(\cdot)$ represents the non-anticipative strategy when we want to give the advantage to the player that wants to minimize ($u(\cdot)$) and $\delta(\cdot)$ to the player that want to maximize ($d(\cdot)$)

theoretical foundations: differential games

How is differential games theory related to a reachability problem?

- Consider the case in which we have a set to reach (target set)
- Player 1 tries to bring the system towards the target with its control input, in the meanwhile the disturbance (Player 2) tries to steer the system away from the target
- In order to consider the worst possible scenario, the strategic advantage is usually given to the disturbance $d(\cdot)$ → the outcome of the game is defined by $V^+(x,t)$
- We will see later that the RAS is related to the outcome of the game $V^+(x,t)$ allowing us to solve the reachability problem using the Upper value of the game

theoretical foundations: level set method

How to model a reachability problem using differential game theory

GAME OF KIND
(Binary outcome)

Using level set method

- Implicit game definition of a reachability problem:
  - System reaches target set
  - System does not reach target set

GAME OF DEGREE
(Real-value outcome)

- Definition compatible to differential game theory. The players want to optimize a cost function $J(\cdot)$ with opposite goals:
  - one tries to maximize it
  - the other to minimize it

theoretical foundations: level set method

How to use level set method (reachability problem)

- Target set is a reach set R (desired set of state)
- **Basic idea:** represent the reach set using a real-value function
- Define a Lipschitz function $g\big(x(s)\big)$, where $x(s)$ represents the current system state, such that:

$$R = \{x \in \mathbb{R}^n \,|\, g(x) \leq 0\}$$

theoretical foundations: level set method

- **Payoff function** $J(\cdot)$ considering only the value of $g(\cdot)$ at the end of the game $s = T$ (not interested in the running cost):

$$J(x, t, u(\cdot), d(\cdot)) = g(x(T))$$

- Giving the strategic advantage to $d(\cdot)$ (use the non-anticipative strategy $\delta[d](\cdot)$)

- Outcome of the game starting from time $s = t$ with initial state x

$$V^+(x,t) = \sup_{\delta(\cdot)\in\Delta(\cdot)} \inf_{u(\cdot)\in\mathcal{U}(\cdot)} J(x,t,u,\delta[u](\cdot)) = \sup_{\delta(\cdot)\in\Delta(\cdot)} \inf_{u(\cdot)\in\mathcal{U}(\cdot)} g(x(T))$$

theoretical foundations: level set method

How to use level set method (constrained reachability problem)

- Our reachability problem consists in reaching a goal and simultanously avoid obstacles along the trajectory, therefore we have two kind of sets:
  - Reach-set R (target set)
  - **Constrained set** (free space) → $K = A^C$ complementary set of the avoid set (obstacles to avoid)

- K is characterized similarly to R by a Lipschitz function $h(\cdot)$:

$$K = \{x \in \mathbb{R}^n \,|\, h(x) \leq 0\}$$

theoretical foundations: level set method

Moving sets

- In real-world scenarios, both target and avoid set can change position over time
- We add time dependency to the two functions $g(\cdot), h(\cdot)$:

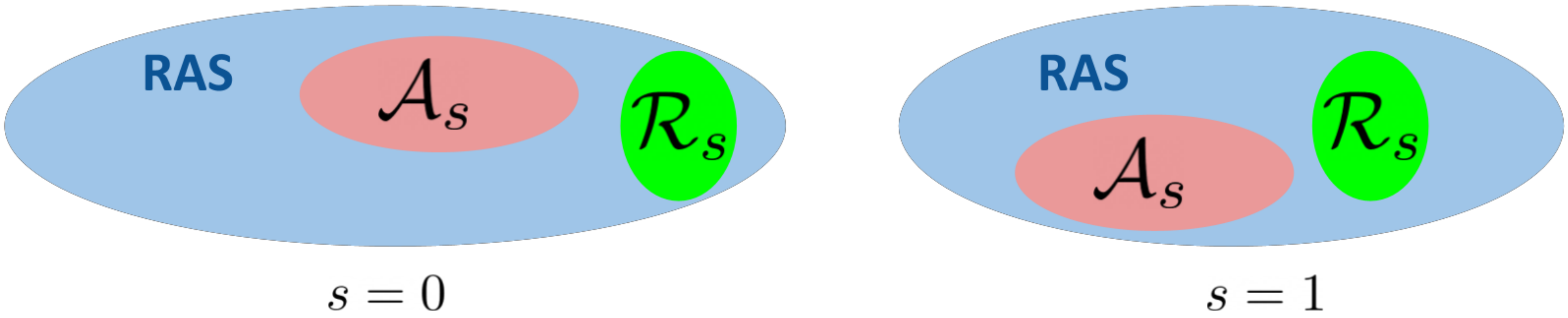
$$\mathcal{R}_s \triangleq R = \{(x, s) \in \mathbb{R}^n \times [t, T] \,|\, g(x, s) \leq 0\}$$

$$\mathcal{K}_s \triangleq K = \{(x, s) \in \mathbb{R}^n \times [t, T] \,|\, h(x, s) \leq 0\}$$

theoretical foundations: level set method

Moving sets

$$\mathcal{K}_s = \mathcal{A}_s^C$$



theoretical foundations: level set method

- **Payoff function** considering the set to reach and the set to avoid both time-varying:

$$J(x,t,u(\cdot),d(\cdot)) = \min_{\tau\in[t,T]} \{\mathcal{J}(\tau,x,t,u(\cdot),d(\cdot))\}$$

where

$$\mathcal{J}(\tau,x,t,u(\cdot),d(\cdot)) = \max\left\{g(\mathcal{X}(\tau),\tau),\ \max_{r\in[t,\tau]} h(\mathcal{X}(\tau),r)\right\}$$

$$\mathcal{X}(\tau) = \phi(\tau;x,t,u(\cdot),d(\cdot))$$

If $\mathcal{J} \leq 0$ → the system trajectory reaches the reach set at time τ without ever having collided with an obstacle on $[t,\tau]$

- **Explanation:** Take the maximum between two quantities; the first one uses $g(\cdot)$ to determine if the current system state is inside the reach set *R;* the second uses $h(\cdot)$ to check whether or not the state had ever left the constrained set *K* in the time interval $[t,\tau]$

reach-avoid set: definition

Reach-Avoid Set (RAS) definition

- *Definition*: it is the set of initial point (x, t) for which the system trajectory $\phi(\cdot)$, starting at time t with initial state x and both players acting optimally, reaches the target set R at some instant $\tau \in [t, T]$ while remaining inside the constrained set K for all $s \in [t, \tau]$:

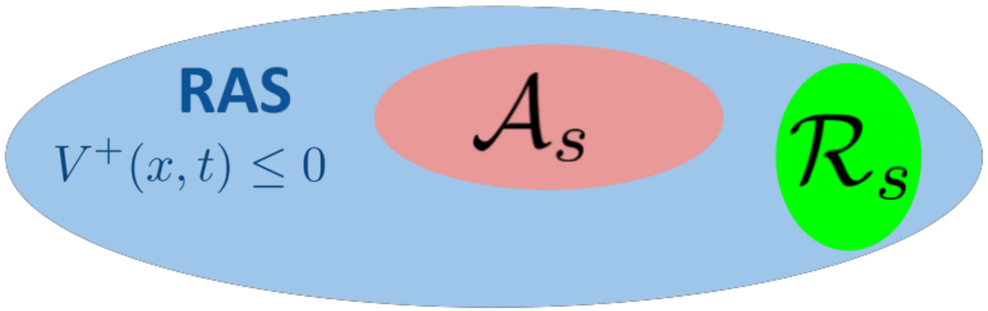
$$RAS = \{(x,t) \in \mathbb{R}^n \times [0,T] \,|\, \exists u^*(\cdot) \in \mathcal{U}_{[0,T]},$$
$$\forall \delta(\cdot) \in \Delta_{[0,T]}, \exists \tau \in [t,T], \phi(\tau; x, t, u^*(\cdot), \delta(\cdot)) \in \mathcal{R}_\tau \quad \wedge$$
$$\forall s \in [t,\tau], \phi(s; x, t, u^*(\cdot), \delta(\cdot)) \in \mathcal{K}_s\}$$

reach-avoid set: relation with outcome game

- **Proposition.** The reach-avoid set RAS is given by the sub-zero level set of the game's outcome

$$RAS = \{(x,t) \in \mathbb{R}^n \times [0,T] \,|\, V^+(x,t) \leq 0\}$$

$\mathcal{K}_s = \mathcal{A}_s^C$



$s = 0$

reach-avoid set: compute outcome game

- The game's outcome is the viscosity solution of the following Hamilton-Jacobi-Isaacs equation:

$$\max\left\{\mathcal{H}, \Delta h\right\} = 0 \qquad t \in [0, T] \quad x \in \mathbb{R}^n$$

$$\mathcal{H} \triangleq \min\left\{\frac{\partial V^+(x,t)}{\partial t} + H\left(x, \frac{\partial V^+(x,t)}{\partial x}, t\right), \Delta g\right\}$$

$$\Delta h \triangleq h(x,t) - V^+(x,t)$$

$$\Delta g \triangleq g(x,t) - V^+(x,t)$$

With terminal condition $V^+(x,T) = \max\left\{g(x,T), h(x,T)\right\}$

reach-avoid set: compute outcome game

$$\mathcal{H} \triangleq \min\left\{\frac{\partial V^+(x,t)}{\partial t} + H\left(x, \frac{\partial V^+(x,t)}{\partial x}, t\right), \Delta g\right\}$$

- The Hamiltonian H is given by:

$$H(x,p,t) = \min_{u\in U}\max_{d\in D} p^T f(x,u,d)$$

$$p = \frac{\partial V^+(x,t)}{\partial x}$$

System dynamics

case of study: use HJ-Reachability Analysis

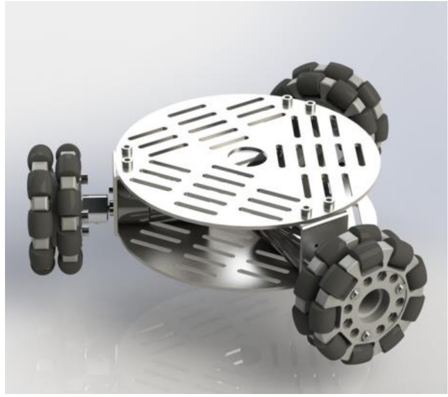
- In order to use HJ-RA to solve a reachability problem in the case of a robot, we have to follow several steps:

    - Definition of the game dynamics (kinematic model of the robot)
    - Definition and procedural solution of min-max problem in the Hamiltonian
    - Environment definition (definition of $g(x, s)$ and $h(x, s)$ that represent respectively the reach set R_S and the constraint set K_S at time s)
    - Numerical computation of $V^+(x, t)$

- Once computed $V^+(x, t)$, in a limited and discretized state-time space, we can use this value function to define a safe control law from each feasible initial states x to the reach set.

case of study: three-wheel omni-directional mobile robot presentation

- It is an holonomic robot with three omnidirectional wheels
- It has the ability to move simultaneously and independently in translation and rotation



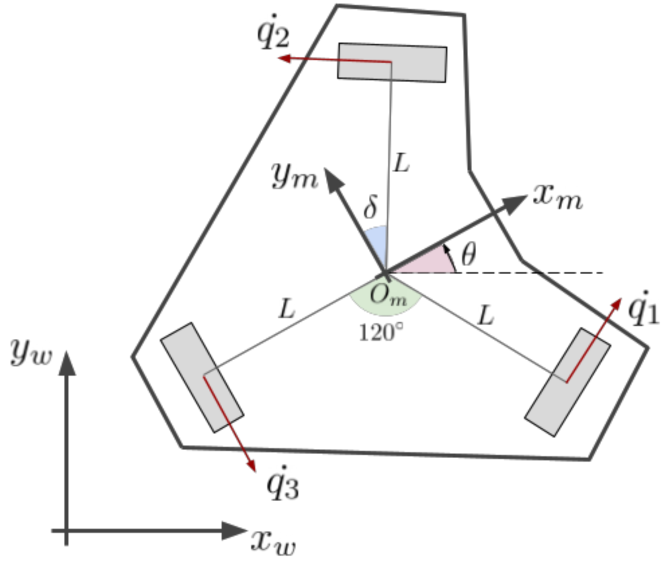
- Omnidirectional wheels allow to increase mobility
- The wheels are mounted symmetrically with 120 degrees from each other

case of study: nomenclature

- Three omni directional wheels disposition: 120 °

Nomenclature

- $[x_m, y_m, O_m]$: mobile robot reference frame (RF_m)
- $[x_w, y_w, O_w]$: fixed world reference frame (RF_w)
- $[\dot{q}_1, \dot{q}_2, \dot{q}_3]$: velocities
- L: distance of the wheels from the center of mass $r = O_m$
- θ: orientation of the robot w.r.t. RF_w
- δ: wheel orientation wrt RF_m (30 °)



case of study: kinematic model

- The kinematic model w.r.t. the world reference frame is:

$$\dot{x} = \begin{bmatrix} \frac{2}{3}\cos(\theta+\delta) & -\frac{2}{3}\cos(\theta-\delta) & \frac{2}{3}\sin\theta \\ \frac{2}{3}\sin(\theta+\delta) & -\frac{2}{3}\sin(\theta-\delta) & -\frac{2}{3}\cos\theta \\ \frac{1}{3L} & \frac{1}{3L} & \frac{1}{3L} \end{bmatrix} \dot{q} + \boxed{\begin{bmatrix} -sin\theta \\ cos\theta \\ 0 \end{bmatrix} d}$$

$$x = [{}^{w}x_r, {}^{w}y_r, \theta]^T$$

$$u = [u_1, u_2, u_3]^T = \dot{q} = [\dot{q}_1, \dot{q}_2, \dot{q}_3]^T$$

External disturbance given by a velocity with module d and perpendicular to robot orientation θ (i.e. it lies on Y_m).

case of study: Hamiltonian for three-wheel mobile robot

- Given the game dynamics $f(x(\cdot), u(\cdot), d(\cdot))$, the Hamiltonian can be computed as follows:

$$H(x,p,t) = \min_{u \in U} \max_{d \in D} [p_x, p_y, p_\theta] \; f(x,u,d) \qquad \text{with } u \in U[u_{min}, u_{max}],\, d \in D = [d_{min}, d_{max}]$$

- Developing and collecting terms w.r.t. controls and disturbance:

$$H(\cdot) = \min_{u \in U} \max_{d \in D} \overbrace{\left(\frac{2}{3}c(\theta+\delta)p_x + \frac{2}{3}s(\theta+\delta)p_y + \frac{1}{3L}p_\theta\right)}^{\sigma_1} u_1 + \overbrace{\left(-\frac{2}{3}c(\theta-\delta)p_x - \frac{2}{3}s(\theta-\delta)p_y + \frac{1}{3L}p_\theta\right)}^{\sigma_2} u_2$$

$$+ \underbrace{\left(-\frac{2}{3}s(\theta)p_x - \frac{2}{3}c(\theta)p_y + \frac{1}{3L}p_\theta\right)}_{\sigma_3} u_3 + \underbrace{\left(c(\theta)p_y - s(\theta)p_x\right)}_{\sigma_d} d$$

Where $c(\cdot) = \cos(\cdot)$, $s(\cdot) = \sin(\cdot)$

REMINDER:

$$p = \frac{\partial V^+(x,t)}{\partial x}$$

case of study: Hamiltonian for three-wheel mobile robot

- Since the inputs appear linearly in the Hamiltonian, we can easily provide a procedure to solve the min-max problem

$$H(\cdot) = \sigma_{u1} u_1^* + \sigma_{u2} u_2^* + \sigma_{u3} u_3^* + \sigma_d d^*$$

- Where $[u_1^*, u_2^*, u_3^*]^T$ and d^* represent the optimal moves for the two players and are given by:

$$u_i^* = \begin{cases} u_{min} & \text{if } \sigma_{ui} \geq 0 \\ u_{max} & \text{Otherwise} \end{cases}$$

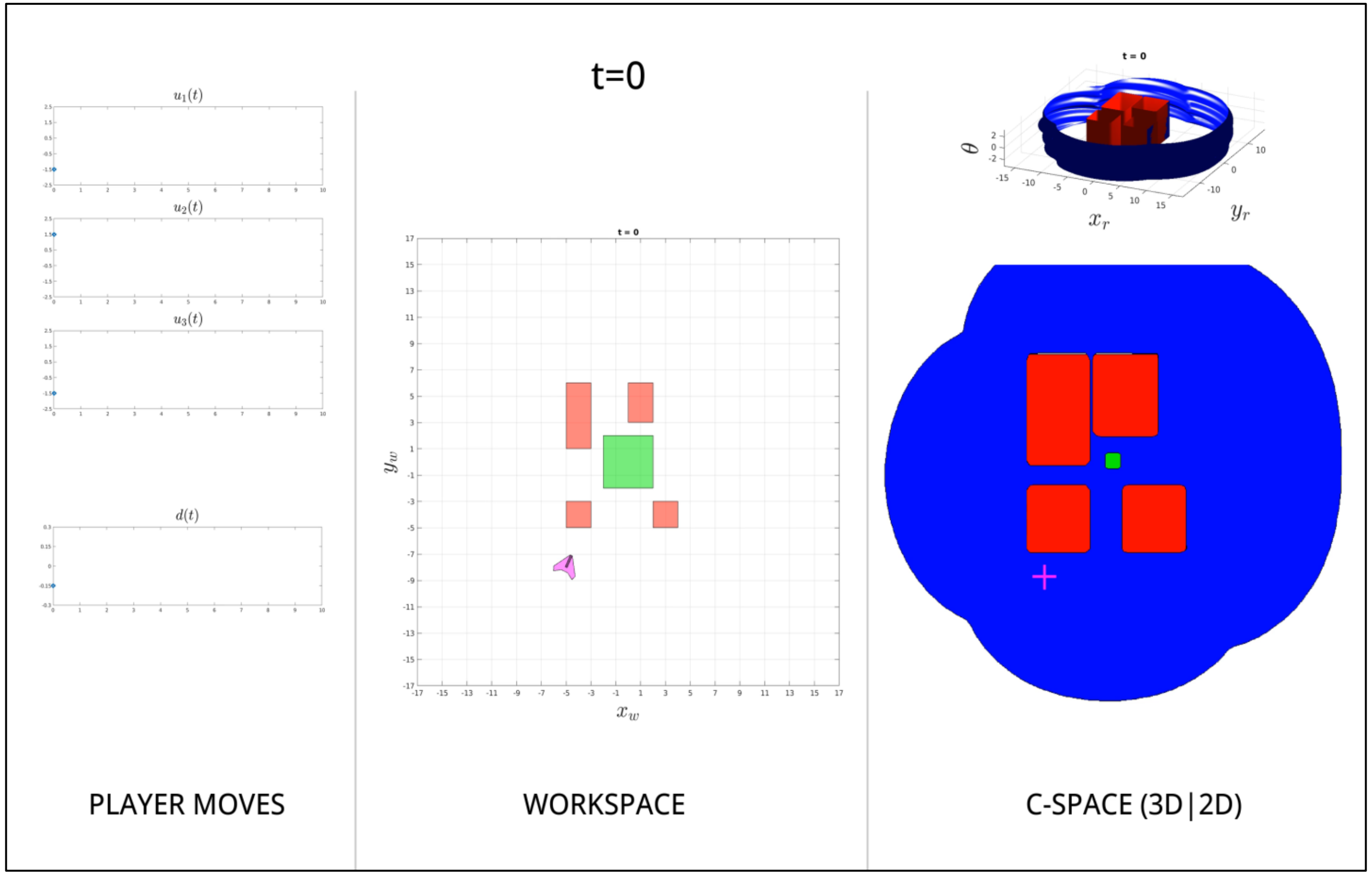
$$d^* = \begin{cases} d_{min} & \text{if } \sigma_d \leq 0 \\ d_{max} & \text{Otherwise} \end{cases}$$

- Once defined the Hamiltonian we can compute the outcome game $V^+(x, t)$.

REMARK:
We can compute optimal inputs **only** after computing the outcome of the game

simulation: static environment

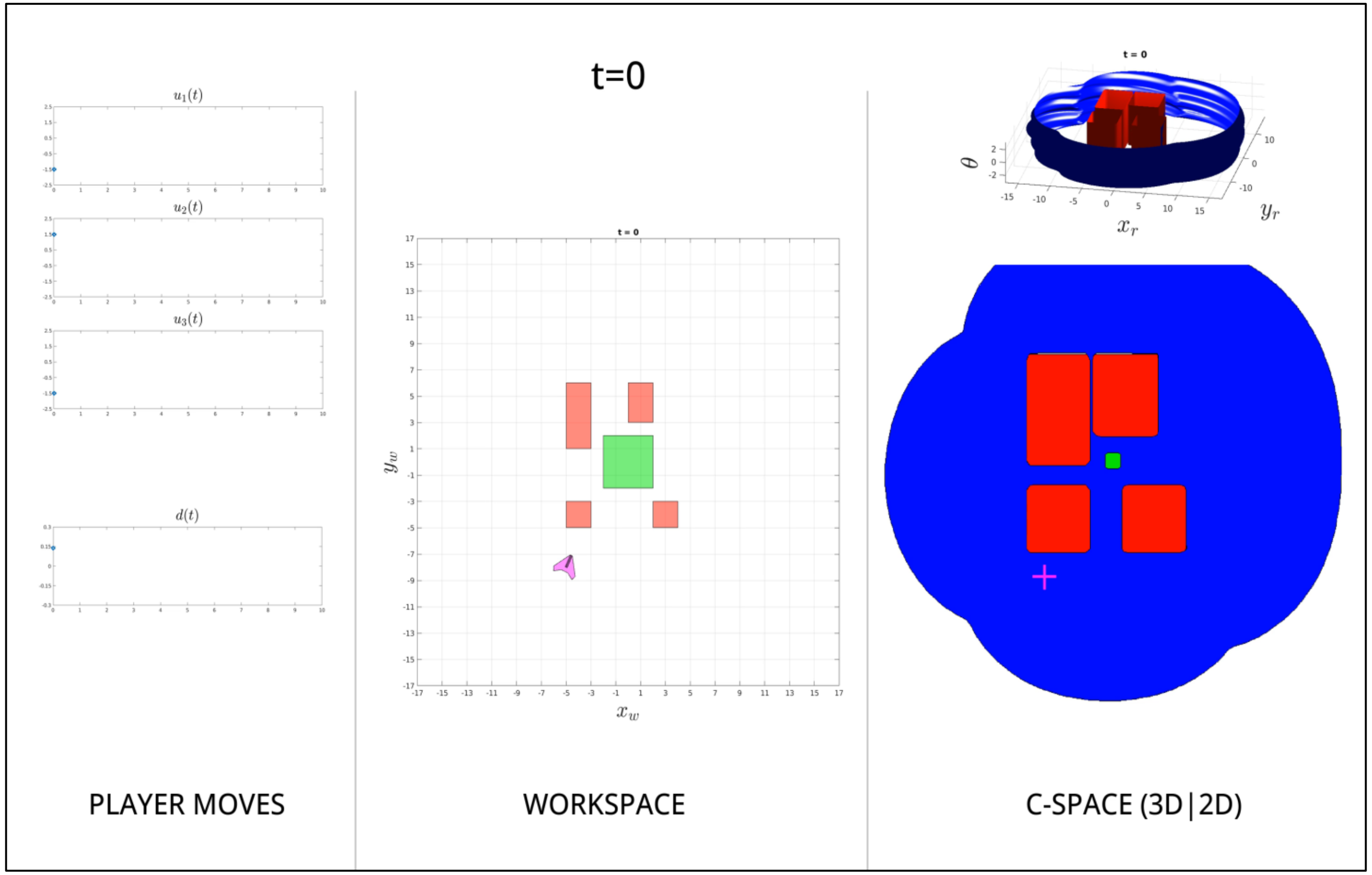
Experiment 1a: static environmet (optimal control and optimal disturbance)



- Initial position: [-5, -8] m
- Initial orientation: -2 rad
- Final desired orientation: [-0.1, -0.1] rad
- Time horizon: 10s
- Control: optimal
- Disturbance: optimal

simulation: static environment

Experiment 1b: static environmet (optimal control and random disturbance)



- Initial position: [-5. -8] m
- Initial orientation: -2 rad
- Final desired orientation: [-0.1, -0,1] rad
- Time horizon: 10s
- Control: optimal
- Disturbance: random (Fig.1)

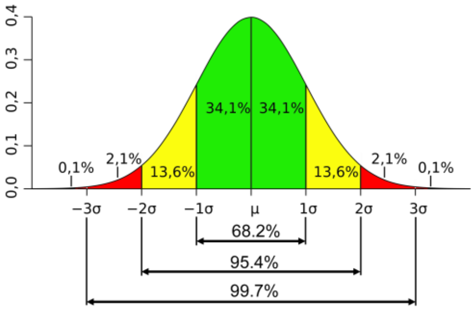
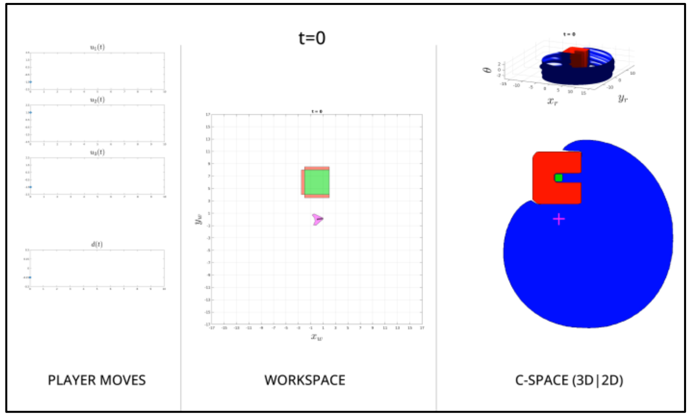


Fig.1 - Gaussian Disturbance
- Mean μ: 0
- Variance σ: $\frac{|d_{max}|}{3}$

simulation: dynamic environment

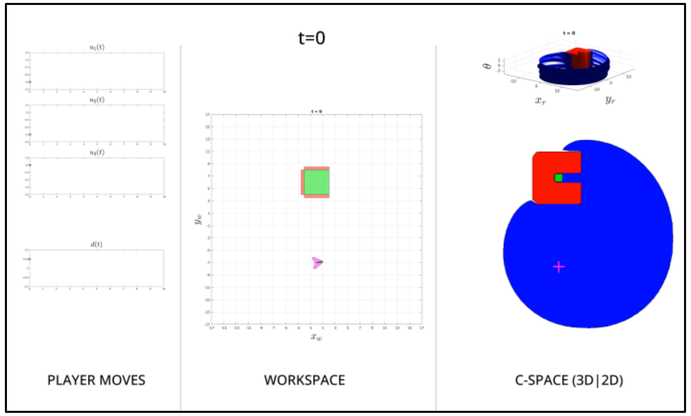
Experiment 2: dynamic environmet (optimal control and optimal disturbance)

Initial state inside RAS near the obstacle



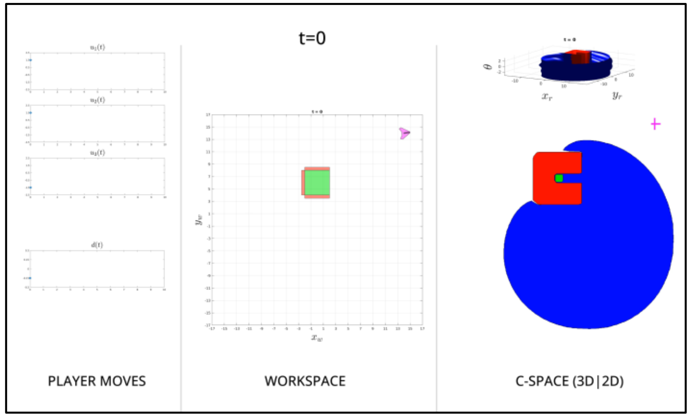
- Initial state: [0 m, 0 m, -3 rad]

Initial state inside RAS faraway the obstacle



- Initial state: [0 m, -7 m, -3 rad]

Initial state outside the RAS



- Initial state: [14 m, 14 m, -3 rad]

conclusions

- We first introduced all the basic concepts of differential games necessary to Hamilton-Jacobi reachability
- We showed how to model a reachability problem in the more complex case, namely with a reach set and an avoid set, both time-varying
- We concluded with the case of study and simulations of a three-wheeled robot in two different types of environments (static and dynamic).

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