Introduction to Graph Theory

Giovani Tavares
giovanitavares@usp.br

University of Sao Paulo — 2025.1

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- Bondy & Murty
- Diestel
- Combinatoria (Botler)
- SageMath

1.1 Basic Concepts

Definition 1.1 (Grid). *The grid $G_{m,n}$ is the graph (V,E) such that:*

- $V = [m] \times [n] = (x, y)|x \in [m], y \in [n]$
- $E = \{(u, v), (x, y) : (u = x, \textit{ and } |v - y| = 1) \textit{ or } (v = y \textit{ and } |u - x| = 1)\}$

Definition 1.2 (Cycle). *The cycle $G_{m,n}$ is the graph (V,E) such that:*

- $V = [n]$
- $E = \{(x, (x \mod n) + 1\}) : x \in [n]\}$

Definition 1.3 (Order). *The order of a graph G is its number of vertices.*

- $v(G) = |V|$

Definition 1.4 (Size). *The size of a graph G is its number of edges.*

- $e(G) = |E|$

1.2 Adjascency and Incidency

If $e = uv$, we say that e links u and v. We also say:

- u and v are adjascent
- u and v are neighbors
- e is incicent to u and v
- We also say that two egdes that share the same vertex are adjascent.
- Two edges that share both vertices are parallel or multiple
- An edge that links a vertex to itself is called a lasso
- A pair of vertices or edges that are not adjascent are called independent
- A set of vertices such that every pair or two vertices are independent is called an independent set or stable

1.3 Degrees of a Vertex

The degree of a vertex, denoted by $d_G(v)$ is the number of edges (of G) incident to v.

- The minimum degree of G is $\delta(G) = min\{d(v) : v \in V(G)\}$
- The maximum degree of G is $\Delta(G) = max\{d(v) : v \in V(G)\}$
- The average degree of G is $\bar{d}(G) = \frac{1}{v(G)} \sum_{v \in V} d(v)$

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- Entregar lista no overleaf
- Monitoria as quintas (13h)

- $E \subseteq \binom{V}{2}$

Proposition 1 (Lema do Aperto de Mao)**.** *For every graph G, the sum of the degrees of its vertices is equal to twice its number of vertices:*

$$\sum_{v \in V(G)} d_G(v) = 2e(G) = 2|E(G)| \tag{1}$$

Proof: Contagem Dupla

$$S = \{(e, u) \in E \times V : u \in e\} \tag{2}$$

As every edge is at precisely two elements of S, then $|S| = 2e(G)$. Furthermore, if $u \in V$, then u is at $d(v)$ elements of S . Then $S =$

Corolary 1. *Every graph has an even number of odd degreed vertices*
Proof
Using the handshake lemma, we can do induction in $e(G)$

Definition 2.1. *If $d_G(u) = 0$, we say that u is an isolated vertex*

Proposition 2. *If G does not have any isolated vertex and $e(G) < v(G)$, then G has at least two vertices with degree 1.*
Proof *We know $e(G) \leq v(G) - 1$. Using the handshake lemma, we have:*

$$\begin{aligned}
2(v(G) - 1) \geq 2e(G) &= \sum_{u \in V(G)} d(u) && (3)\\
&= x + \sum_{u \in V(G), d(u) \geq 2} d(u) && (4)\\
&\geq x + \sum_{u \in V(G)} 2 && (5)\\
&= x + 2(v(G) - x) && (6)\\
&= && (7)
\end{aligned}$$

Proposition 3. *If $e(G) \geq v(G) + 1$, then G has at least one vertex with degree ≥ 3.*
Proof *Suppose, by contradiction, that $d(u) \leq 2$ for every $u \in V(G)$.*

$$\begin{aligned}
e(G) = \frac{1}{2} \sum_V d(u) \leq \frac{1}{2} \sum_V 2 = \frac{1}{2} 2V(G) = V(G) && (8)\\
\implies e(G) \leq V(G), \text{ which is a contradiction.} && (9)\\
&& (10)
\end{aligned}$$

3 Special Types of Graphs

- A graph is **simple** if is does not contain "arestas multiplas nem lacos"
- A graph is **complete** is a simple graph such that $G = (V, E) : E = \binom{V}{2}$
- A complete graph with n vertices is denoted by K_n
- G is **empty** if $V(G) = E(G) = \emptyset$
- G is **trivial** if $E(G) = \emptyset$
- We say that G is $K - regular$ if $d(u) = k$ for all and every $u \in V(G)$
- G is regular if it is $K - regular$ for some $k \in N$
- Every complete graph is regular
- K_n is $(n-1) - Regular$
- G is **bipartite** if $V(G)$ can be partitioned into two distinct sets X and Y such that every edge has a vertex in X and another one in Y. Such partition is the bipartition of G.
- The bipartite complete graph with a bipartition (X, Y) is the graph G such that $V(G) = XUY$ and $E(G) = X \times Y$. There is no way for a graph to be complete bipartite, just the other way around. The notation for a bipartite complete graph is $K_{n,m}$, with $|X| = n$ and $|Y| = m$.
- The complement of G, denoted by $\bar{G}$, is the graph such that $V(\bar{G}) = V(G)$ and $E(\bar{G}) = \{xy \in \binom{V}{2} : xy \notin E(G)\}$. $\bar{G}$ is basically the graph that has all the edges that G does not have.

Proposition 4. *Every graph G such that $v(G) \geq 2$ has two vertices with the same degree.*
Proof by induction
Induction hypothesis: every graph with $n-1$ vertices has two vertices with the same degree.
Suppose $v(G) \geq 3$, we have two cases:

- *i. G has a vertex u such that $d(u) = 0$. Hence, by the induction hypothesis, $G' = G - u$ also has two vertices with the same degree.*
- *$d(u) \geq 1$ for all $u \in V$*

It is easy to computationally test whether a graph is bipartite. (Exercise)