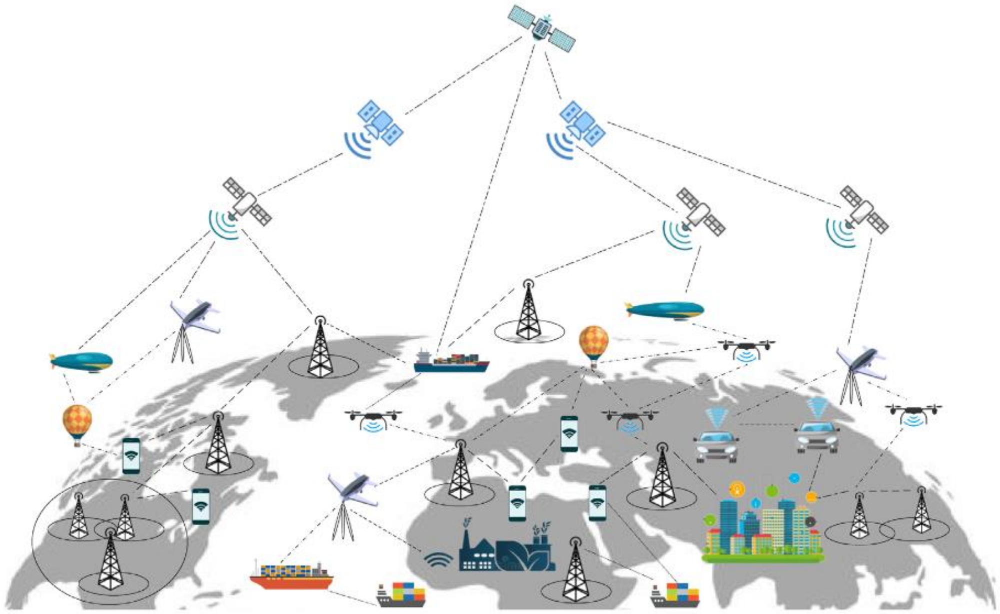
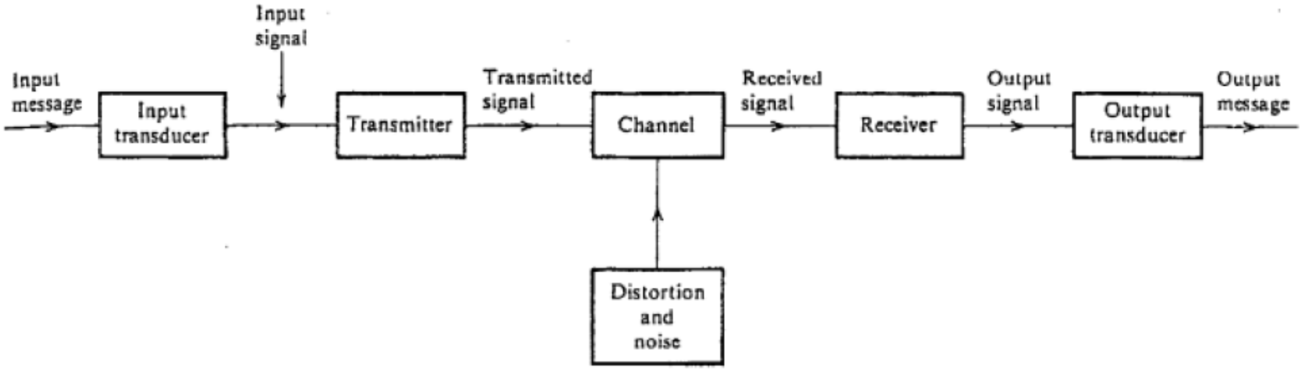
Lecture 1: Introduction to digital transmission systems

Communication Systems

Example of communication systems





Components of a communication system (1)

- *Input message is* source of data, e.g., human voices, pictures, videos, and messages
- *Input transducer* converts input message (non-electrical signals) to electrical signals

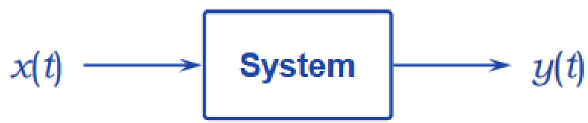
  Example: A voice message is converted to a current signal
- *Transmitter* includes several modules:
  - Modulator-amplifier enhances signal characteristics such as amplitude, frequency, and phase
  - Antenna(s) converts electrical signals into electromagnetic waves
- *Channel* reflects a propagation environment, e.g., air, underwater, space, copper cable, optical fibres

Components of a communication system (2)

- *Distortion and noise* reflect, for instance, imperfect hardware and thermal noise
- *Receiver* includes several modules
  - Demodulator obtains original message signals from received signals
  - Amplifier strengthen weak message signals
  - Filter removes noise and interference
- *Output transducer* converts electrical signals into non-electrical forms
  - Example: A loudspeaker converts electrical signals into voice signals

Signal & System

- Signal conveys information and data that may be voltage, current, or other measurements
- System manipulates signals



- Signal in time domain x(t) versus signal in frequency domain X(f)
  - Fourier transform $X(f) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$
  - Inverse Fourier transform $x(t) = \mathcal{F}^{-1}\{X(f)\} = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$
  - Spectrum of x(t) is X(f)
  - Amplitude of spectrum |X(f)|
  - Phase of spectrum arg{X(f)}
  - Parseval's theorem:

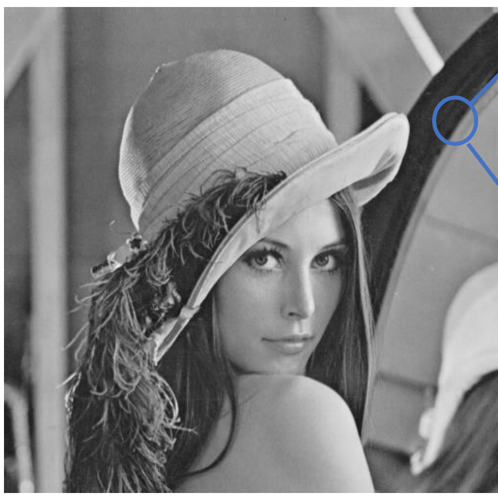
$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

Signal Classification

- Analog signals versus digital signals
- Continuous signals versus discrete signals (Time domain)
- Periodic signals versus non-periodic signals
- Energy signals versus power signals
- Deterministic signals versus non-deterministic signals

Digital signal versus analog signal (1)

- Signal can be digital or analog
  - Digital signal such as digital images, video
  - Analog signal such as human voice



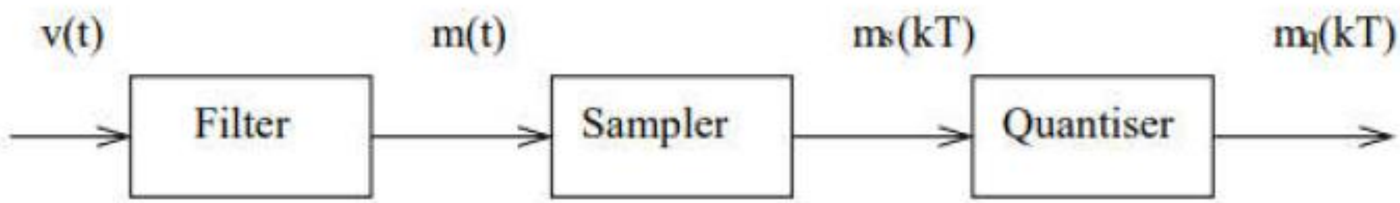
170	238	85	255	221	0
69	135	18	170	120	69
222	0	236	135	0	255
120	255	84	170	136	237
237	18	221	69	120	255
85	171	120	221	18	136

Natural image

Human speech

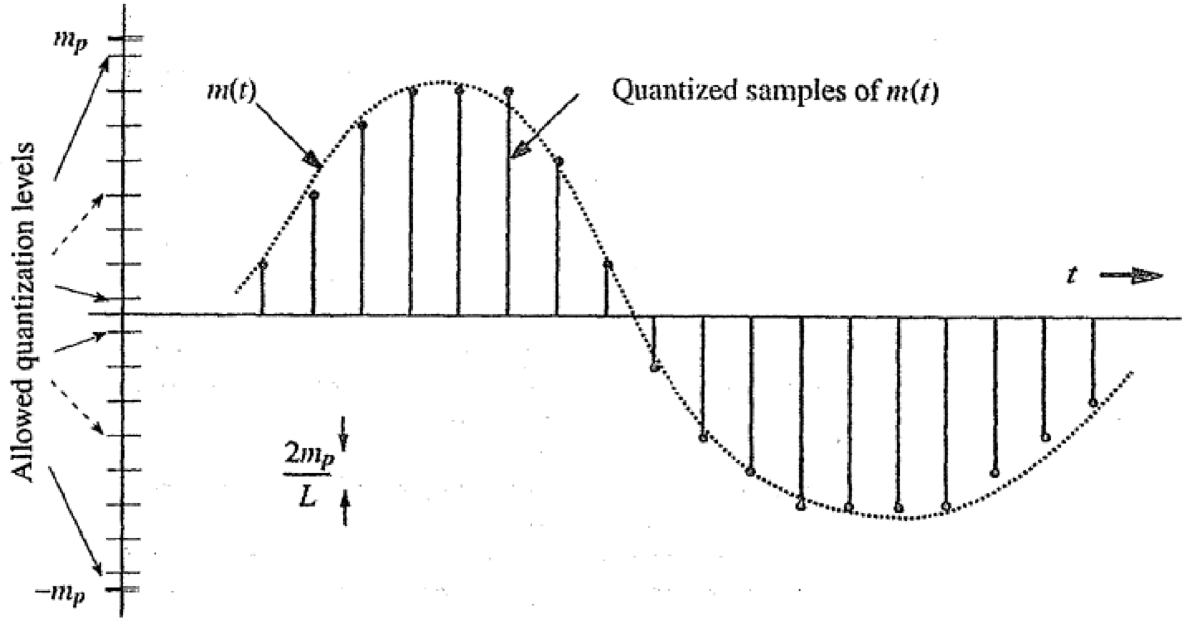
Digital signal versus analog signal (2)

- Digital signals conveys information with less noise, distortion, and interference than analog ones
- Digital signals can be obtained from analog signals by using an analog to digital converter (ADC)



Digital signal versus analog signal (3)

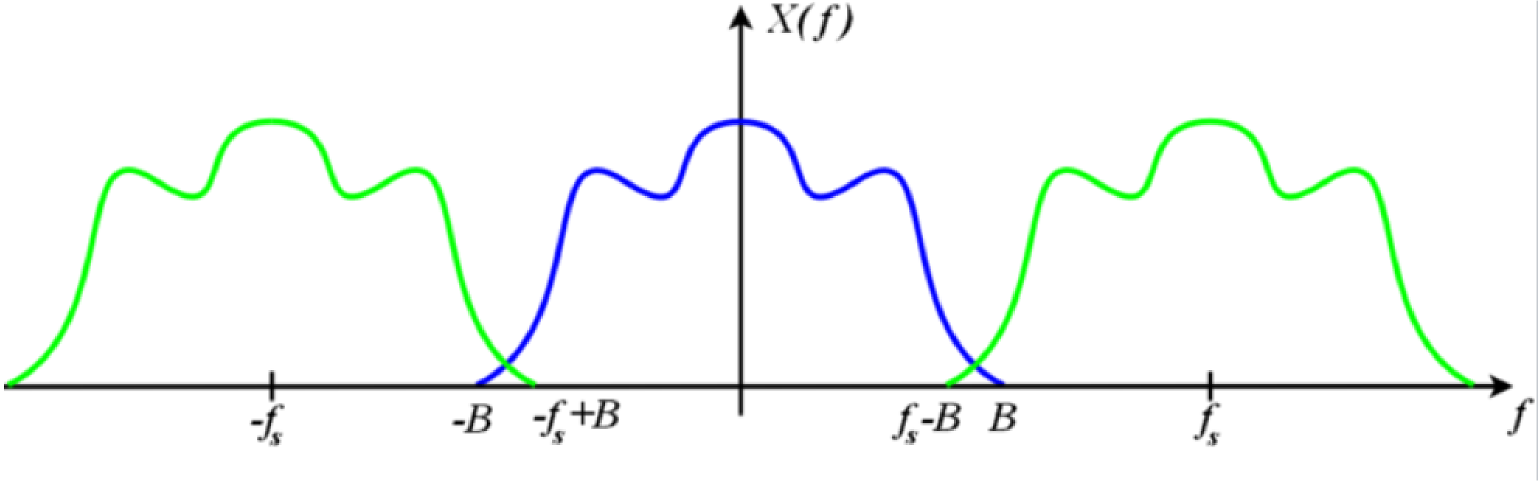
- Signal sampling : signal *m(t)* is sampled in time domain



- Magnitude of sampled signals *ms(kT)* are defined from a finite set of quantized levels

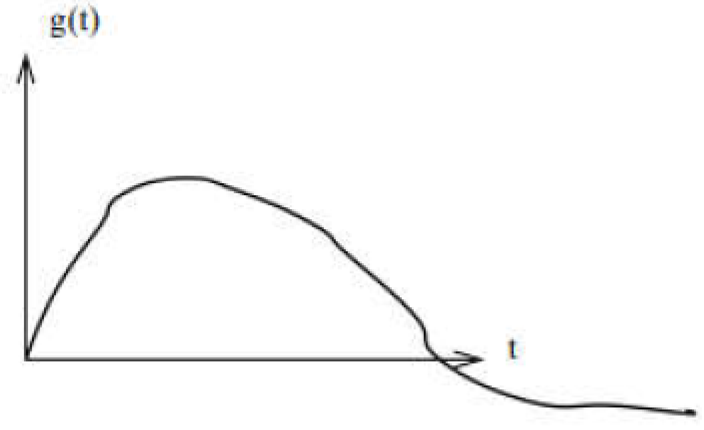
Digital signal versus analog signal (4)

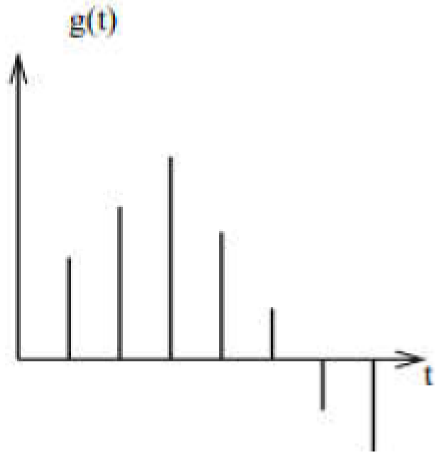
- Nyquist-Shannon sampling theorem:
  - Sampling rate should be at least twice the system bandwidth (fs $\geq 2\mathrm{B}$)
- If fs < 2B, then aliasing appears



Continuous SIGNAL vs Discrete signal

- Continuous signal has a value at each time instant
- Discrete signal is defined at some time instants



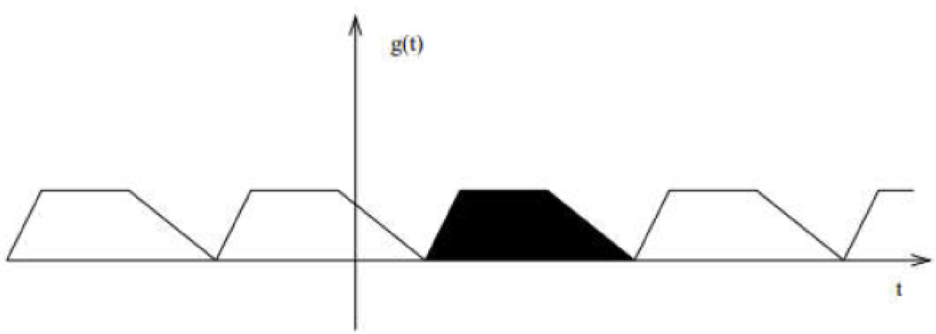


Periodic signals versus non-periodic signals

- Periodic signal: a signal g(t) is periodic if there exist a positive number To such that $g(t) = g(t + T_0), \forall t$
- Some periodic signals such as

$$\sin(\omega t), \cos(\omega t), e^{j\omega t}$$

  - where $\omega = 2\pi / T_0$
- Can extend g(t) from any time instant



Energy signal vs power signal

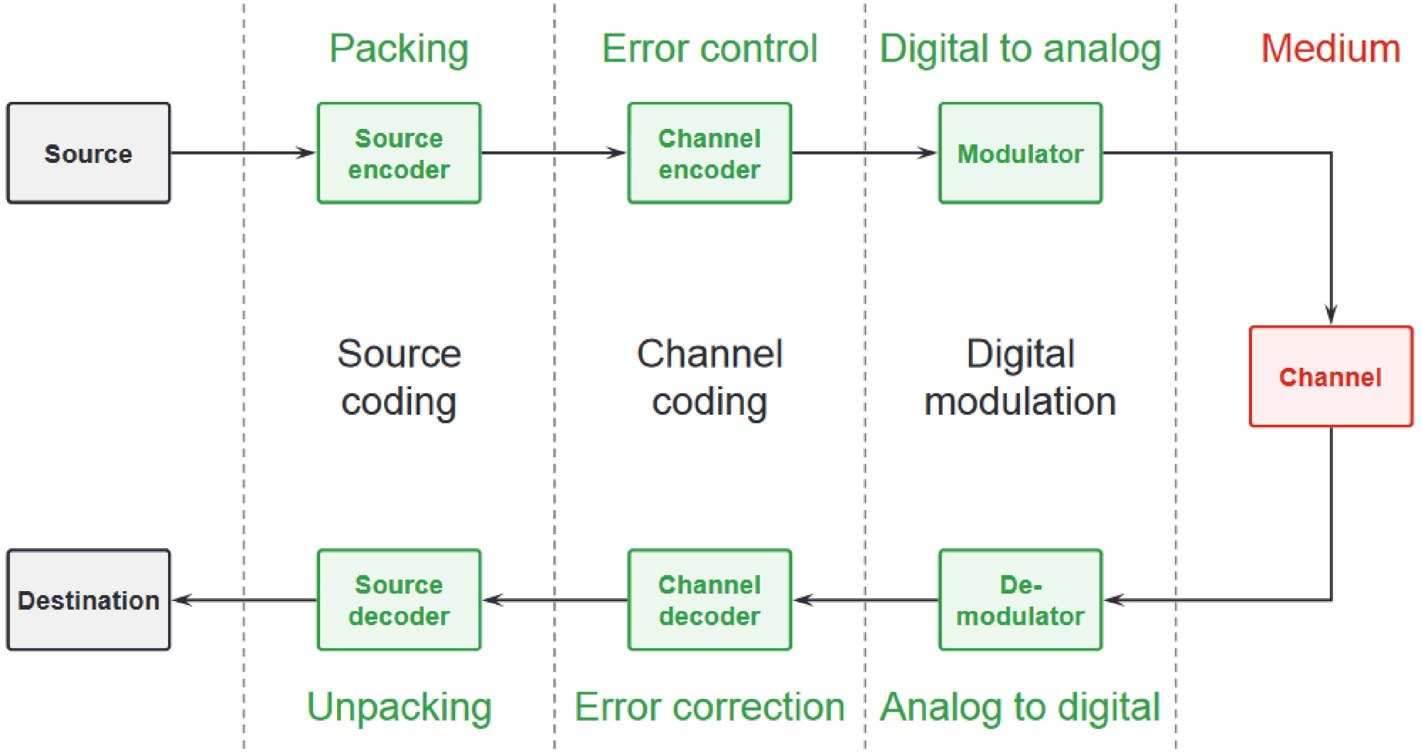
- Energy signal
  - Energy of a real signal g(t): $E_g = \int_{-\infty}^{\infty} g^2(t)dt$
  - Energy of a complex signal g(t): $E_g = \int_{-\infty}^{\infty} g^*(t)g(t)dt = \int_{-\infty}^{\infty} |g(t)|^2 dt$
  - g(t) is called an energy signal if $E_g < \infty$
    - Magnitude of g(t) should approach zero at an limiting regime of time
- Power signal
  - For a periodic signal, its power is a suitable metric:
  - Power signal is defined as $P_g = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |g|^2(t)dt$
- A signal cannot be both energy signal and power signal

$$0 < \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |g|^2(t)dt < \infty$$

Digital Communication

Digital system

Digital transmission systems: Transmits sequence of symbols belonging to a discrete alphabet



Example

- GSM(Global System for Mobile Communications)/UMTS (Universal Mobile Telecommunications System)/ 4G (Generation)/ 5G...
- Telephone Modem
- Optical Fibers
- Wired and Wireless LAN
- GPS/Galileo

Characteristics of a digital transmission system

Bit-Rate

A binary information sequence is characterized by its "speed"

- BIT-RATE Rb [bps] is the number of bits transmitted in a second

$$R_b = 1/T_b$$

- where Tb is time to transmit one bit

kilobit per second	**kbitps**	$\mathbf{10^3}$ **bps**
Megabit per second	**Mbitps**	$\mathbf{10^6}$ **bps**
Gigabit per second	**Gbitps**	$\mathbf{10^9}$ bps
Terabit per second	**Tbitps**	$\mathbf{10^{12}}$ bps

Bandwidth

- A binary sequence should be converted from digital domain to analog domain g(t) before sending over the medium
- Waveform g(t) is characterized by its power spectral density G(f) (see energy signal/ power signal)
- BANDWIDTH B [Hz] = frequency interval containing (a significant portion of) G(f)

Power

- Received signal power [W] ([dB]) depends on the transmitted signal and defined as the signal-to-noise ratio (SNR)
- Example: $y = \sqrt{p}hx + n$
  - Where h represents a channel coefficient; x is the transmitted signal with σ unit magnitude and the transmit power p [W]; and n is additive white Gaussian noise with standard variation
  - The received SNR is $p|h|^2/\sigma^2$
  - Channel capacity

$$R = B\log_2(1 + \mathrm{SNR}) = B\log_2\left(1 + \frac{p|h|^2}{\sigma^2}\right)\ \text{[bps]}$$

    - where B [Hz] is the system bandwidth

Transmission Error probability

- Transmitted binary information sequences $u_T = (u_T(i))$

$\Downarrow$

- Transmitted waveform s(t)

$\Downarrow$

- Received waveform r(t) ≠ s(t)

$\Downarrow$

- Transmitted binary information sequences $u_R = (u_R(i))$
- BIT ERROR PROBABILITY $P(u_T(i) \neq u_R(i))$

Complexity

COMPLEXITY implies engineering complexity of practical implementation

Delay

- Measured in seconds: Difference between time instants of the information bits
  - Enter transmitter → Exit receiver

Exercise

- HW1: Define the suitable metric for

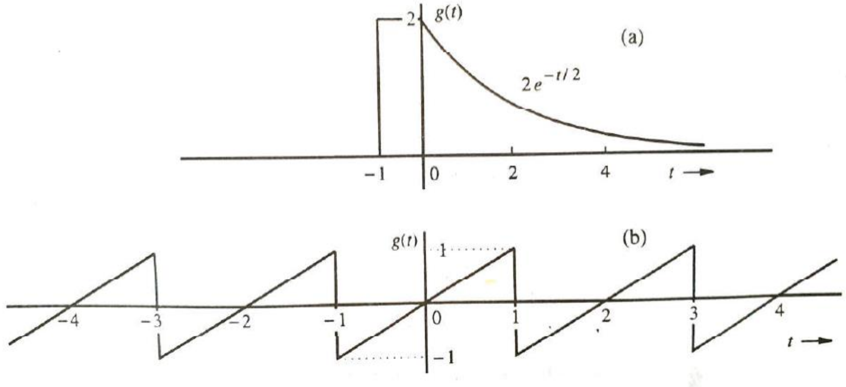
a) $x(t) = A\cos(2\pi f_0 t)$

b) $$x(t) = \begin{cases} A\cos(2\pi f_0 t) & \text{for } -T_0/2 \leq t \leq T_0/2, \\ 0 & \text{elsewhere.} \end{cases}$$

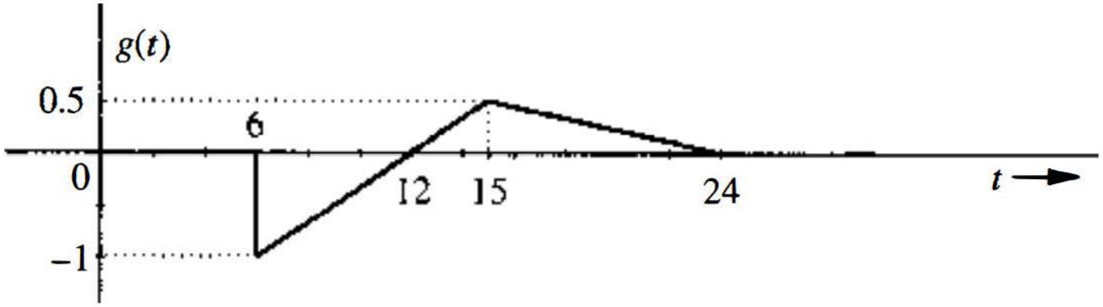
c) $$x(t) = \begin{cases} A\exp(-at) & \text{for } t > 0, a > 0 \\ 0 & \text{elsewhere.} \end{cases}$$

Exercise

- HW2: Define the suitable metric



- HW3: For given g(t) as follows



Define g(t-4), g(t+6), g(3t), and g(6-t)

Lecture 2: Signal constellation, labelling, and transmitted waveforms

Content

- Binary information sequences: definition and properties
  - Definition of binary information sequences
  - Properties of binary information sequences
  - Examples of binary information sequences
- Signal constellation
  - Definition of signal constellation
  - Examples of signal constellation
  - Hamming space
- Binary labelling
  - Principles of binary labelling
  - Examples of binary labelling
- Transmitted waveforms
  - Transmitted waveform design based on Hamming space
  - Example of binary transmitted waveform for a given bit rate

Binary information sequences: definition and properties

Definition (1)

- Binary alphabet defined as $Z_2 = \{0, 1\}$
- Binary information sequence

$$\mathbf{u}_T = (\mathbf{u}_T[0], \mathbf{u}_T[1], \ldots, \mathbf{u}_T[i], \ldots), i \in \mathbb{N}, \mathbf{u}_T[i] \in Z_2$$

- Example

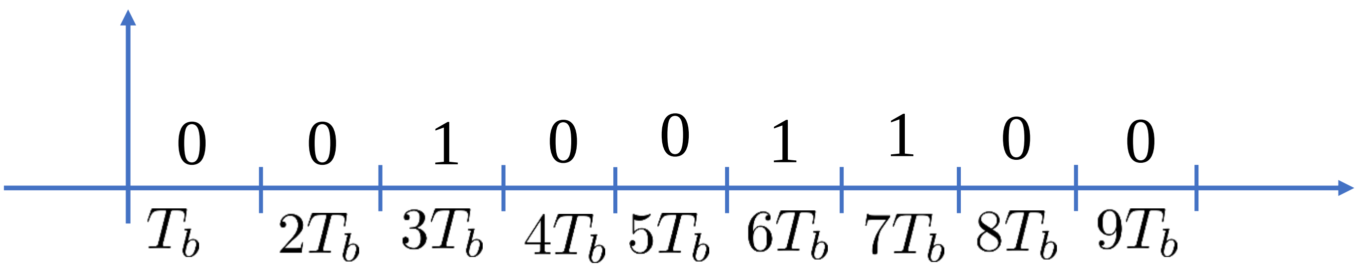
$$\mathbf{u}_T = (001001100\ldots)$$

Definition (2)

- Binary sequence $\mathbf{u}_T = (\mathbf{u}_T[0], \mathbf{u}_T[1], \ldots, \mathbf{u}_T[i], \ldots), i \in \mathbb{N}, \mathbf{u}_T[i] \in Z_2$
- For bit rate Rb, then each bit lasts $T_b = 1/R_b$

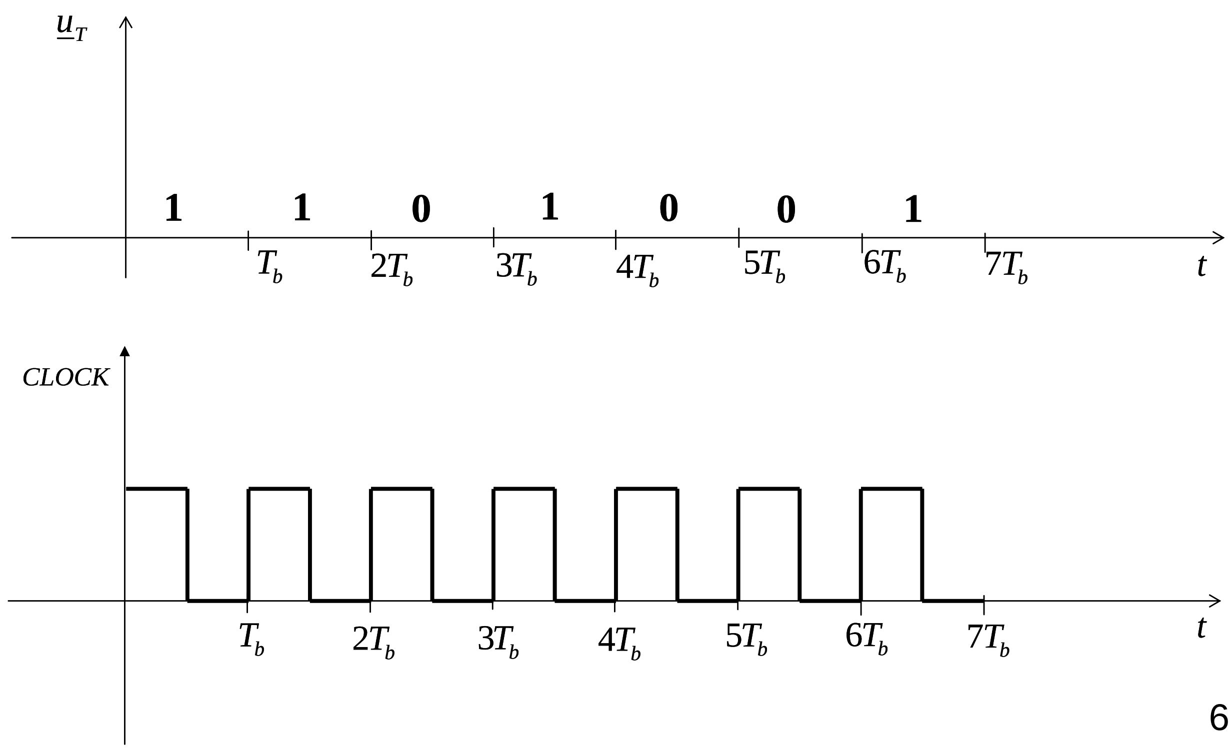
$$iT_b \leq t < (i+1)T_b$$

- Example: $\mathbf{u}_T = (001001100\ldots)$



Property (1)

- Each binary sequence $\mathbf{u}_T$ is characterized by
  - Bit information $\mathbf{u}_T[i]$
  - Transmission clock, with bit rate Rb



Property (2)

- Ideal binary sequences
  - Statistically independent bits

$$P(\mathbf{u}_T[i]|\mathbf{u}_T[j]) = P(\mathbf{u}_T[i])$$

  - Equiprobable bits

$$P(\mathbf{u}_T[i] = 0) = P(\mathbf{u}_T[i] = 1)$$

Signal Constellation

Definition

- Signal constellation *M*

$$M = \{s_1(t), s_2(t), \ldots, s_m(t)\}$$

  - The cardinality of the constellation set M:

$$|M| = m = 2^k \text{ signals}$$

<u>Hypothesis</u>: All the signals $s_i(t)$ are in finite time domain

$$0 \leq t \leq T = kT_b$$

Example

- Signal constellation with m =2

$$M = \{s_1(t) = P_T(t), s_2(t) = -P_T(t)\}$$

- Signal constellation with m = 4

$$M = \{s_1(t) = P_T(t)\cos(2\pi f_0 t), s_2(t) = P_T(t)\sin(2\pi f_0 t),$$
$$s_3(t) = -P_T(t)\cos(2\pi f_0 t), s_4(t) = -P_T(t)\sin(2\pi f_0 t)\}$$

Hamming space

- k-bit binary vector

$$\underline{v} = (u_0, \ldots, u_i, \ldots u_{k-1}) \qquad u_i \in Z_2$$

- Hamming space

$$H_k = \{\underline{v} = (u_0, \ldots, u_i, \ldots, u_{k-1}) \;\; u_i \in Z_2\}$$

  - Cardinality

$$H_k = 2^k \text{ vectors}$$

- Example

$$H_1 = \{(0)(1)\} = Z_2$$
$$H_2 = \{(00)(01)(10)(11)\}$$
$$H_3 = \{(000)(001)(010)(011)(100)(101)(110)(111)\}$$

Binary labelling

Principle

- Signal constellation M: cardinality 2^k
- Hamming space H_k: cardinality 2^k
- One-to-one mapping (Binary labeling)

$$\boxed{\begin{aligned} e: \quad & H_k \leftrightarrow M \\ & \underline{v} \in H_k \leftrightarrow s(t) = e(\underline{v}) \in M \end{aligned}}$$

Example (1)

- Signal constellation with m =2

$$M = \{s_1(t) = P_T(t), s_2(t) = -P_T(t)\}$$

$m=2 \rightarrow k=1$

$H_1 = \{\ (0)\ ,\ (1)\ \}$

$$\boxed{\begin{aligned} &e : H_1 \leftrightarrow M \\ &\quad (0) \leftrightarrow s_1(t) \\ &\quad (1) \leftrightarrow s_2(t) \end{aligned}}$$

Example (2)

- Signal constellation with m =4

$$M = \{s_1(t) = P_T(t)\cos(2\pi f_0 t), s_2(t) = P_T(t)\sin(2\pi f_0 t),$$
$$s_3(t) = -P_T(t)\cos(2\pi f_0 t), s_4(t) = -P_T(t)\sin(2\pi f_0 t)\}$$

$m=4 \rightarrow k=2$

$H_2 = \{ (00), (01), (11), (10) \}$

$$e : H_2 \leftrightarrow M$$

$$(00) \leftrightarrow s_1(t)$$
$$(01) \leftrightarrow s_2(t)$$
$$(10) \leftrightarrow s_3(t)$$
$$(11) \leftrightarrow s_4(t)$$

transmitted waveform

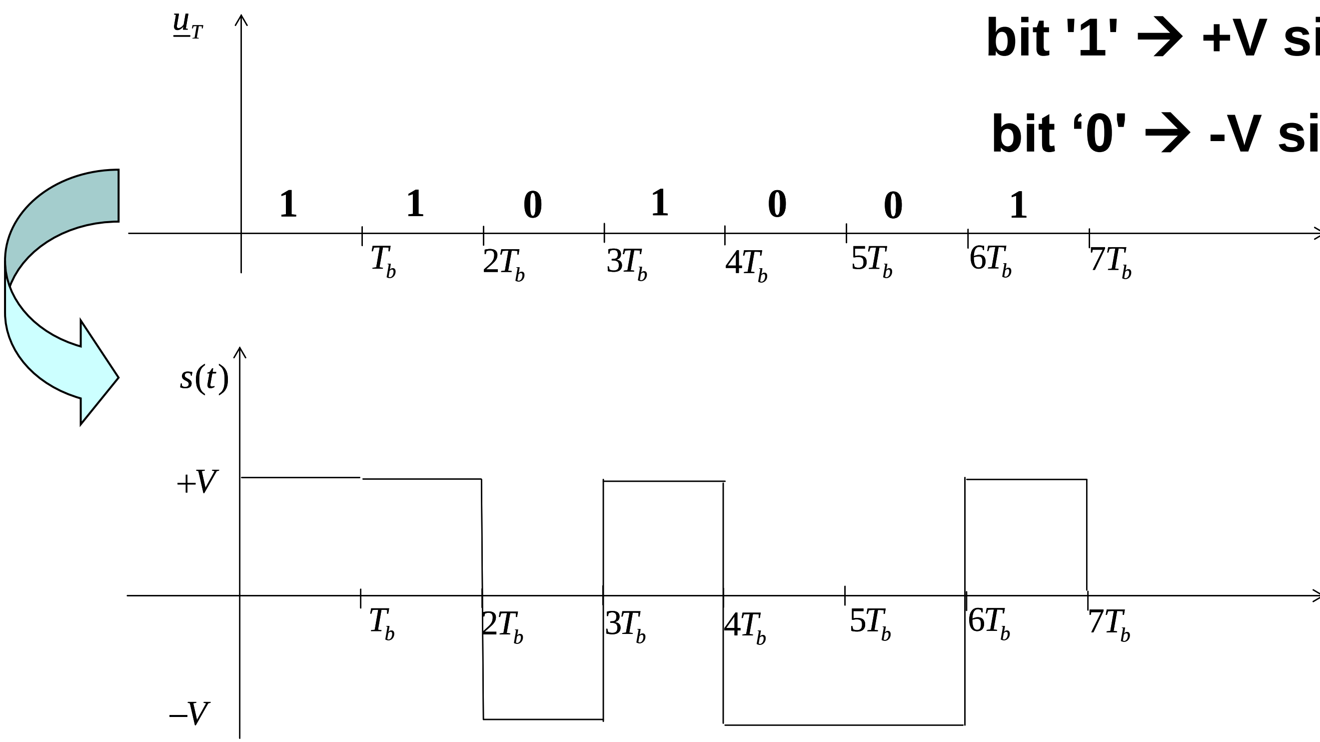
Definition (1)

- Transmitted waveform s(t) of a binary bit sequence is a real function of time
- Example $\underline{u}_T = (1101001...)$

Bipolar NRZ (Non Return Zero) representation

bit '1' → +V signal

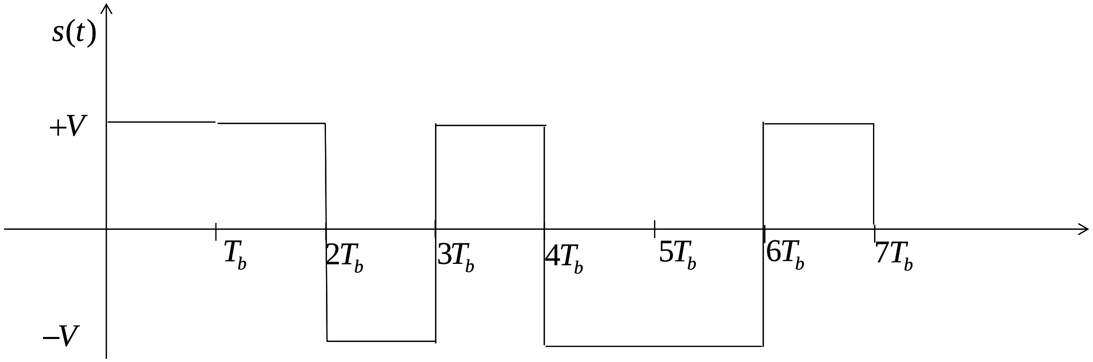
bit '0' → -V signal



Definition (2)

- Example

$\underline{u}_T = (1101001\ldots)$

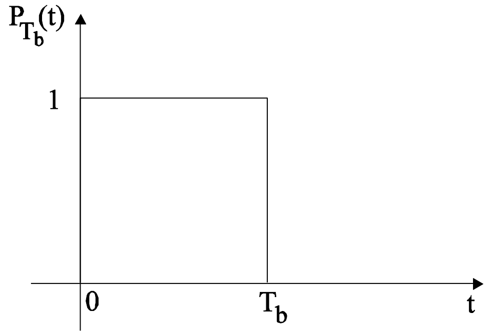


Two available signals

$$u_T[i] = 1 \rightarrow +VP_{T_b}(t - iT_b)$$

$$u_T[i] = 0 \rightarrow -VP_{T_b}(t - iT_b)$$

Rectangular Window of Duration T_b



Transmitted waveform based on Hamming space (1)

- For given
  - Binary sequence $\underline{u}_T$
  - Signal constellation M
  - Binary labeling e

➔ Construction of transmitted waveform s(t)

Transmitted waveform based on Hamming space (2)

- M has cardinality 2^k ➔ $e : H_k \leftrightarrow M$
- Divide $\underline{u}_T$ in k-bit vectors

$$\underline{\boldsymbol{u}}_T = (u_T[0], u_T[1], \ldots, u_T[i], \ldots)$$

$$\underline{\boldsymbol{u}}_T = \left(\underline{v_T[0]}, \underline{v_T[1]}, \ldots, \underline{v_T[n]}, \ldots\right)$$

- Vector [0]: $\underline{v}_T[0]=(u_T[0], \ldots, u_T[k\text{-}1])$

- Vector [n]: $\underline{v}_T[n]=(u_T[nk], \ldots, u_T[(n+1)k\text{-}1])$

Transmitted waveform based on Hamming space (3)

- Each bit lasts for Tb seconds
- Each k-bit vector lasts for $kT_b = T$ seconds

$$\underline{\boldsymbol{u}}_T = (\underbrace{\underline{v_T[0]}}_{T}, \underbrace{\underline{v_T[1]}}_{T}, \ldots, \underbrace{\underline{v_T[n]}}_{T}, \ldots)$$

- Each signal $s_i(t) \in M$ has finite domain T seconds

$$0 \leq t \leq T = kT_b$$

Transmitted waveform based on Hamming space (4)

- Binary labeling $\boxed{e : H_k \leftrightarrow M}$

$$\underline{\boldsymbol{u}}_T = \left(\underbrace{\underline{v_T[0]}}_{T},\ \underbrace{\underline{v_T[1]}}_{T},\ \ldots,\ \underbrace{\underline{v_T[n]}}_{T},\ \ldots \right)$$

$e \qquad\qquad e \qquad\qquad e$

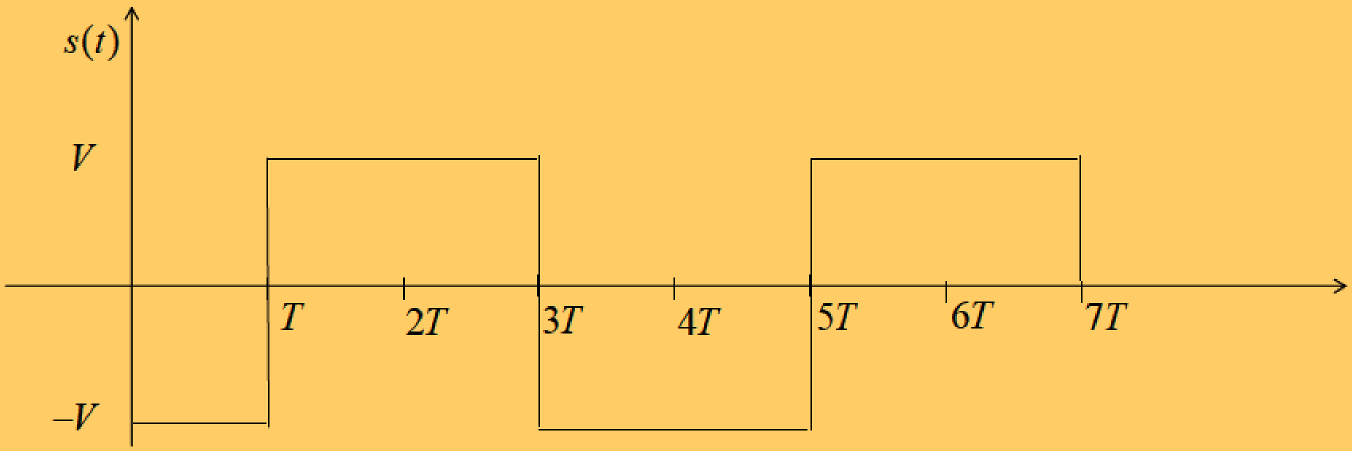
$$s(t) = \left(s[0](t),\ s[1](t),\ \ldots,\ s[n](t),\ \ldots \right)$$

$$\boxed{\text{Correct alignment} \quad s[n](t) = T_n(e(\underline{v}_T[n]))}$$

Example (1)

$\underline{\boldsymbol{u}}_{\mathrm{T}}$=(0110011...) R_b=1 Mbps

$$M = \{s_1(t) = -VP_T(t)\ ,\ s_2(t) = +VP_T(t)\}$$

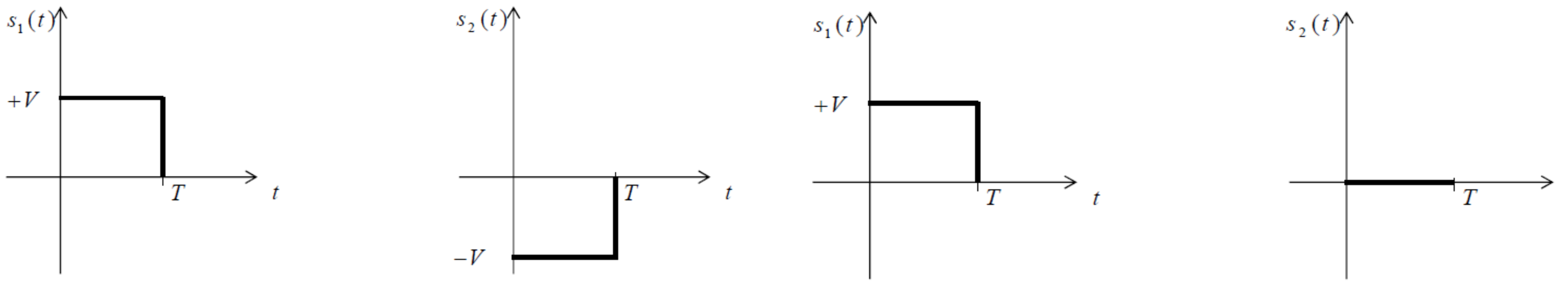


$\boxed{T = T_b = 1\,\mu s}$

Practical Signal Constellation (1)

$$M = \{s_1(t) = +VP_T(t), s_2(t) = -VP_T(t)\}$$

$$M = \{s_1(t) = +VP_T(t), s_2(t) = 0\}$$



$$m = 2 \rightarrow k = 1 \rightarrow T = T_b$$

$$m = 2 \rightarrow k = 1 \rightarrow T = T_b$$

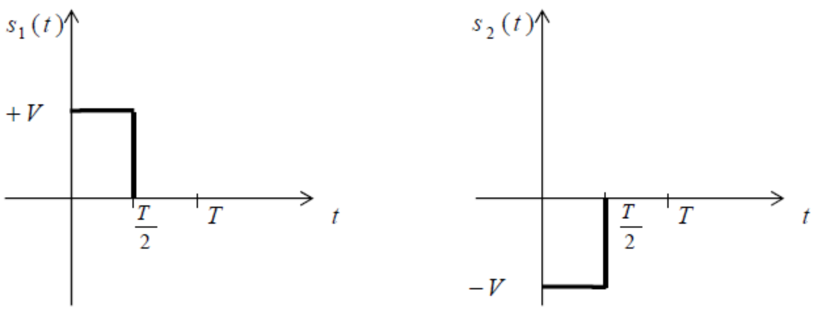
Polar - Non Return to Zero (NRZ)

Unipolar - Non Return to Zero (NRZ)

Practical Signal Constellation (2)

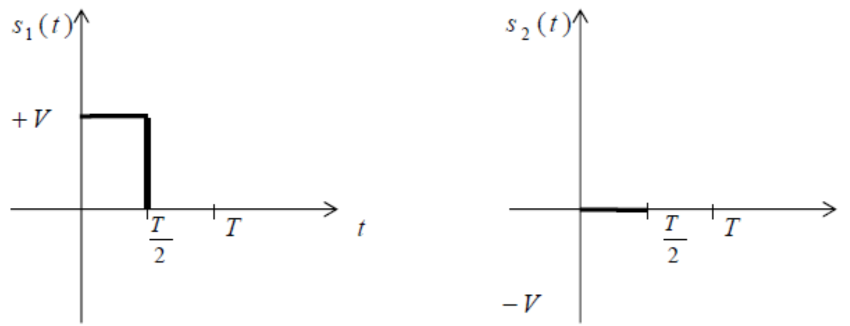
$$M = \{s_1(t) = +VP_{T/2}(t), s_2(t) = -VP_{T/2}(t)\}$$

$$M = \{s_1(t) = +VP_{T/2}(t), s_2(t) = 0\}$$



$$m = 2 \rightarrow k = 1 \rightarrow T = T_b$$

Polar - Return to Zero (RZ)



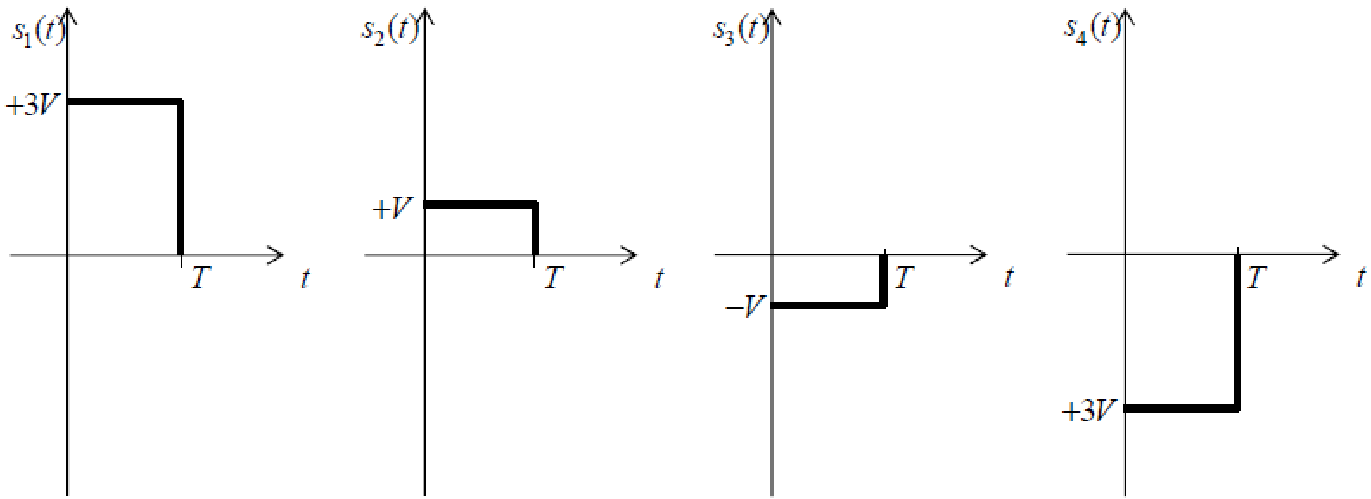
$$m = 2 \rightarrow k = 1 \rightarrow T = T_b$$

Unipolar - Return to Zero (RZ)

Practical Signal Constellation (3)

- **m-PAM (Pulse Amplitude Modulation)**

$$M = \{s_1(t) = +3VP_T(t), s_2(t) = +VP_T(t), s_3(t) = -VP_T(t), s_4(t) = -3VP_T(t)\}$$



$$m = 4 \rightarrow k = 2 \rightarrow T = 2T_b$$

4-PAM

Practical Signal Constellation (4)

- **m-ASK (Amplitude Shift Keying)**

$$M = \{s_1(t) = +3VP_T(t)\cos\left(2\pi f_0 t\right), s_2(t) = +VP_T(t)\cos\left(2\pi f_0 t\right),$$
$$s_3(t) = -VP_T(t)\cos\left(2\pi f_0 t\right), s_4(t) = -3VP_T(t)\cos\left(2\pi f_0 t\right)\}$$

$$m = 4 \rightarrow k = 2 \rightarrow T = 2T_b$$

4-ASK

Practical Signal Constellation (4)

- **m-ASK (Amplitude Shift Keying)**

$$M = \{s_1(t) = +3VP_T(t)\cos\left(2\pi f_0 t\right), s_2(t) = +VP_T(t)\cos\left(2\pi f_0 t\right),$$
$$s_3(t) = -VP_T(t)\cos\left(2\pi f_0 t\right), s_4(t) = -3VP_T(t)\cos\left(2\pi f_0 t\right)\}$$

$$m = 4 \rightarrow k = 2 \rightarrow T = 2T_b$$

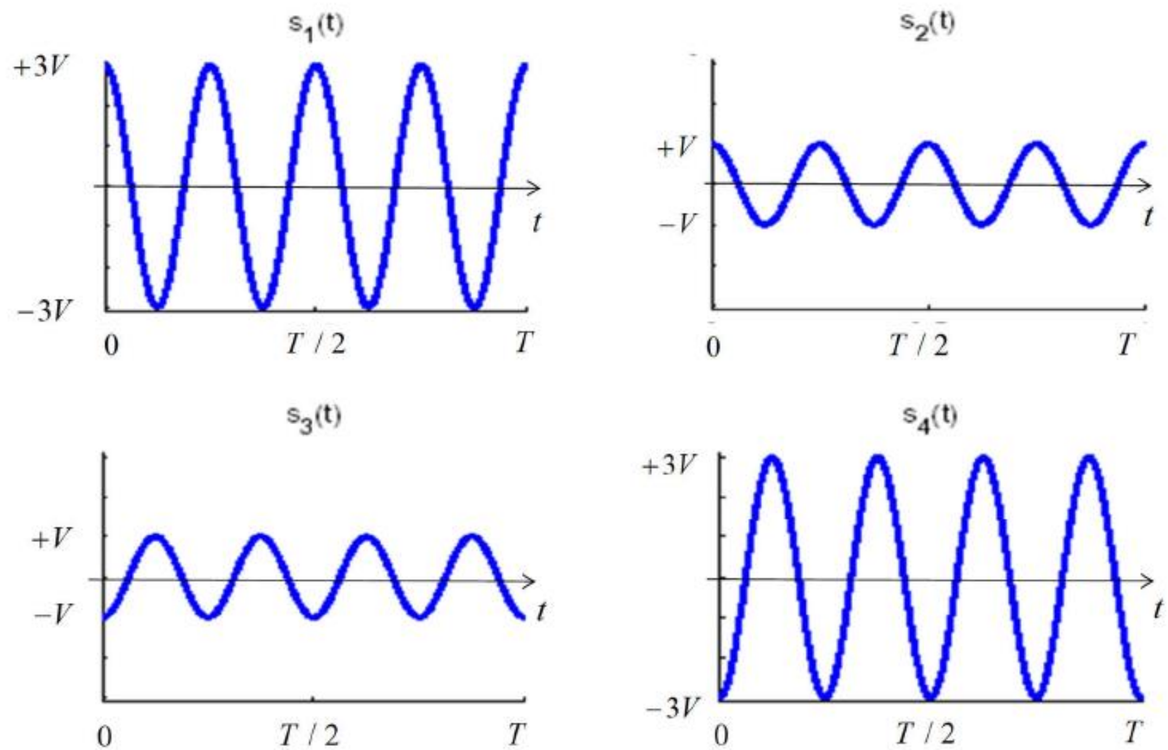
4-ASK

Practical Signal Constellation (5)

- **m-ASK (Amplitude Shift Keying)**

4-ASK

$$f_0 = 2R_b$$



Practical Signal Constellation (6)

- **m-PSK (Phase Shift Keying)**

Example: 2-PSK

$$M = \{s_1(t) = +VP_T(t)\cos\left(2\pi f_0 t\right), s_2(t) = -VP_T(t)\cos\left(2\pi f_0 t\right)\} =$$

$$= \{s_1(t) = +VP_T(t)\cos\left(2\pi f_0 t\right), s_2(t) = +VP_T(t)\cos\left(2\pi f_0 t - \pi\right)\}$$

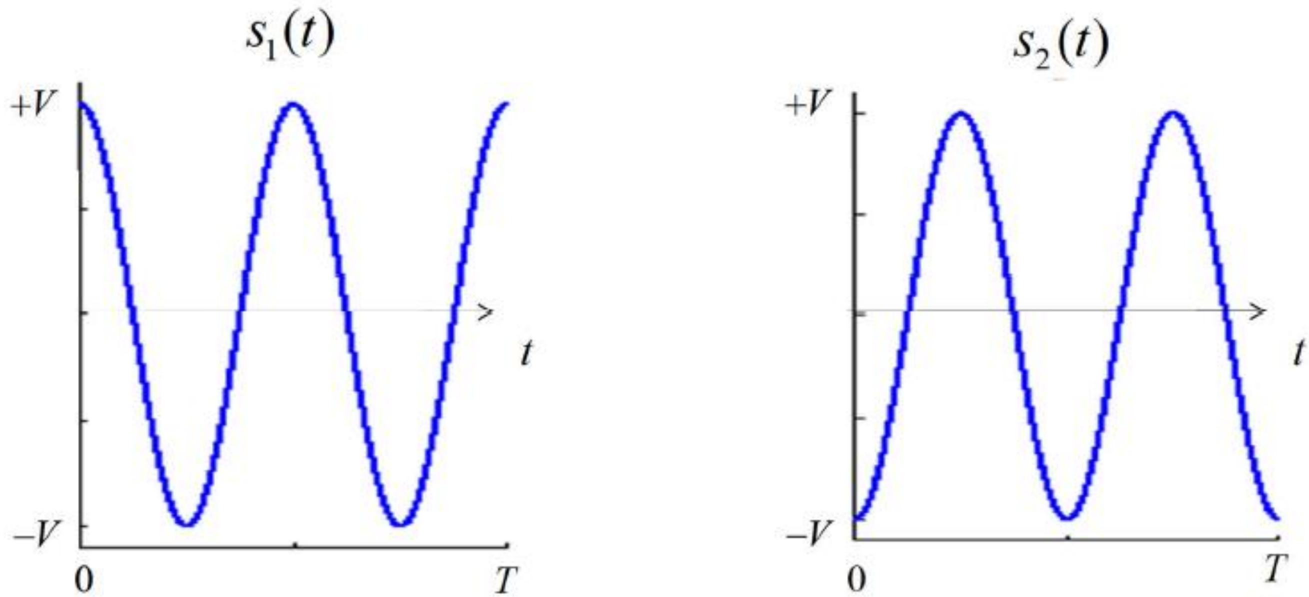
$$m = 2 \rightarrow k = 1 \rightarrow T = T_b$$

Practical Signal Constellation (7)

- **m-PSK (Phase Shift Keying)**

2-PSK

$$f_0 = 2R_b$$



Practical Signal Constellation (8)

- **m-PSK (Phase Shift Keying)**

Example: 4-PSK

$$M = \left\{ \begin{array}{l} s_1(t) = +VP_T(t)\cos\left(2\pi f_0 t\right), s_2(t) = +VP_T(t)\sin\left(2\pi f_0 t\right), \\ s_3(t) = -VP_T(t)\cos\left(2\pi f_0 t\right), s_4(t) = -VP_T(t)\sin\left(2\pi f_0 t\right) \end{array} \right\} =$$

$$= \left\{ \begin{array}{l} s_1(t) = +VP_T(t)\cos\left(2\pi f_0 t\right), s_2(t) = +VP_T(t)\cos\left(2\pi f_0 t - \dfrac{\pi}{2}\right), \\ s_3(t) = +VP_T(t)\cos\left(2\pi f_0 t - \pi\right), s_4(t) = VP_T(t)\cos\left(2\pi f_0 t - \dfrac{3\pi}{2}\right) \end{array} \right\}$$

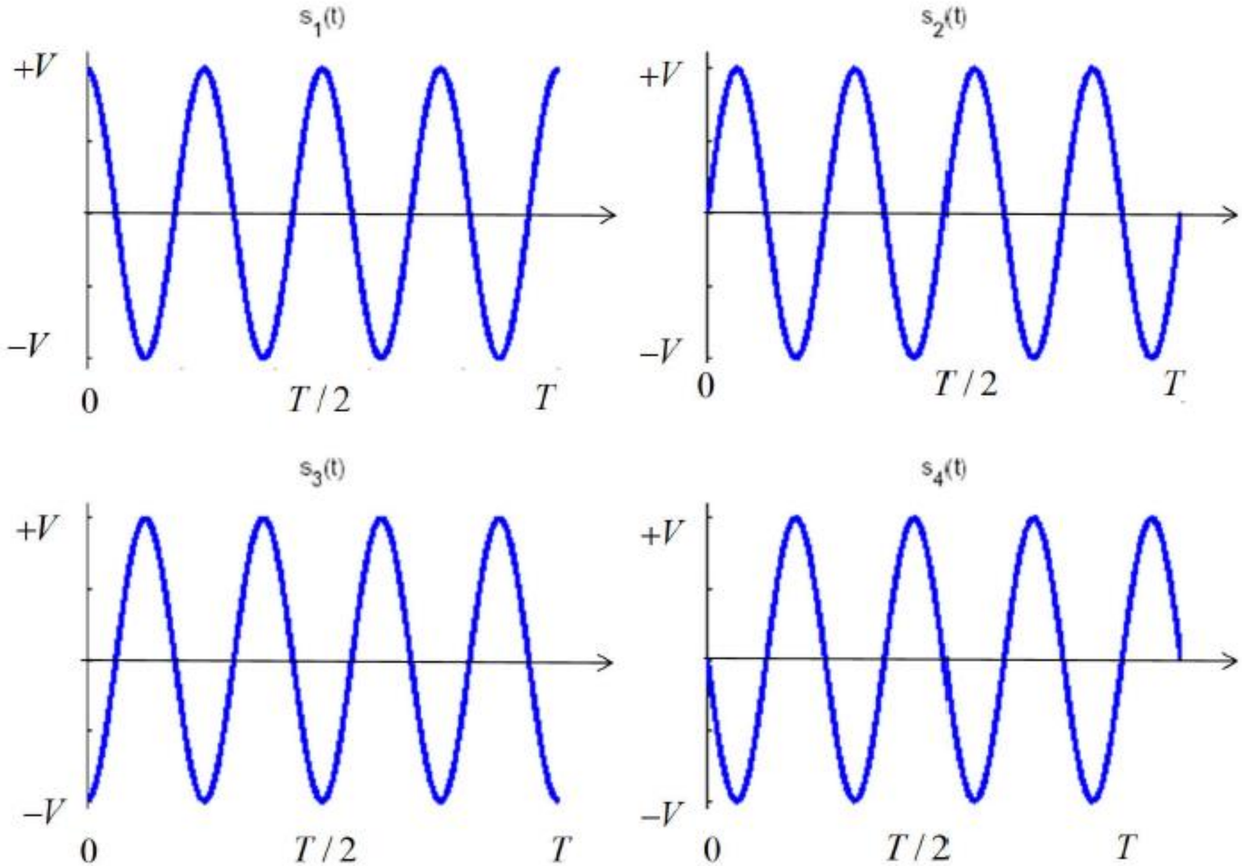
$$m = 4 \rightarrow k = 2 \rightarrow T = 2T_b$$

Practical Signal Constellation (9)

- **m-PSK (Phase Shift Keying)**

4-PSK

$$f_0 = 2R_b$$



Practical Signal Constellation (10)

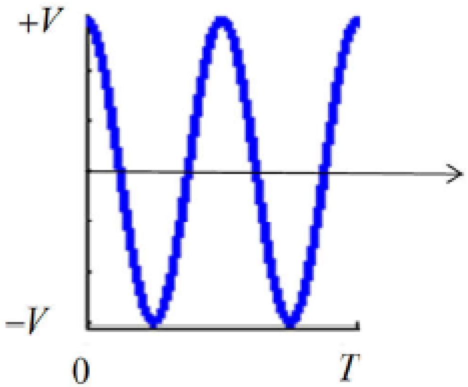
- **2-FSK (Frequency Shift Keying)**

$$M = \{s_1(t) = +VP_T(t)\cos\left(2\pi f_1 t\right), s_2(t) = +VP_T(t)\cos\left(2\pi f_2 t\right)\}$$

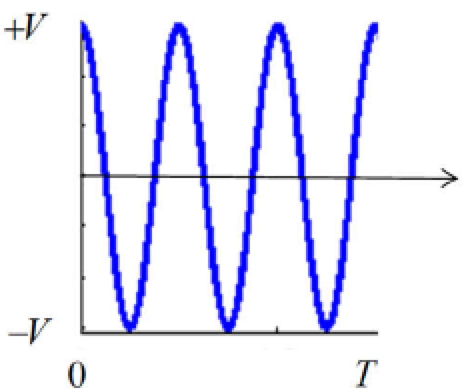
$$m = 2 \rightarrow k = 1 \rightarrow T = T_b$$

Practical Signal Constellation (10)

- **2-FSK**



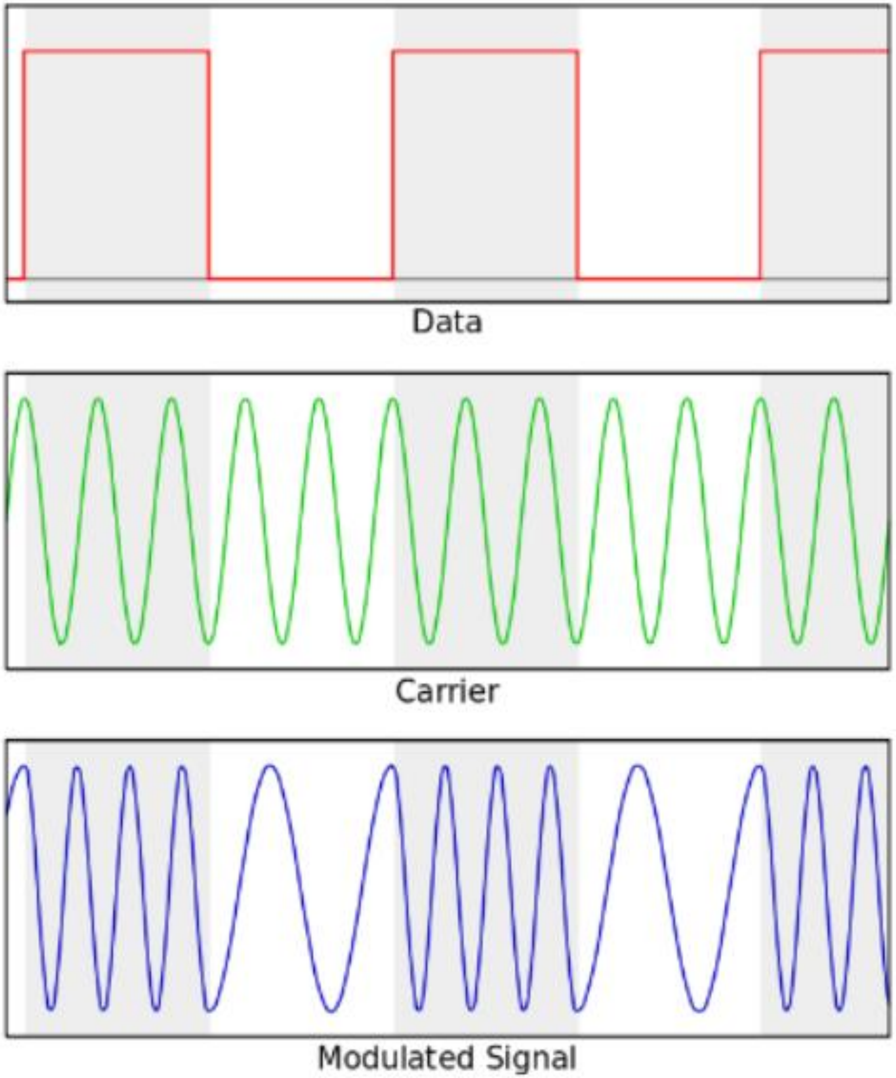
$$f_1 = 2R_b$$



$$f_2 = 3R_b$$

Practical Signal Constellation (11)

- **2-FSK**



Exercise (1)

$\underline{\boldsymbol{u}}_{\mathrm{T}}$=(10011100…) R_b=1 Mbps

$$\boxed{\begin{aligned}&M=\{s_1(t)=-3P_T(t)\ ,\ s_2(t)=-P_T(t),\\&s_3(t)=+P_T(t)\ ,\ s_4(t)=+3P_T(t)\}\end{aligned}}$$

Exercise (2)

$\underline{\boldsymbol{u}}_{\mathrm{T}}$=(10011100…) R_b=1 Mbps

$$M=\{s_1(t)=VP_T(t)\cos(2\pi f_0 t), s_2(t)=VP_T(t)\sin(2\pi f_0 t),$$
$$s_3(t)=-VP_T(t)\cos(2\pi f_0 t), s_4(t)=-VP_T(t)\sin(2\pi f_0 t)\}$$

(f_0=1MHz)

Lecture 3-1: Decision Theory

Content

- Signal space representation
  - AWGN channel
  - Receiver roles
  - Orthonormal basis formulation
  - Gram-Schmidt algorithm
  - Signal space based on orthonormal basis (vector presentation)
- ML and MAP criterions
  - Received signals and noise at the receiver side
  - Decision based on vector formulation
  - Detection with MAP criterion
  - Detection with ML criterion
  - Voronoi region

Signal space representation

AWGN channel (1)

Binary information sequence $\underline{u}_T$

↓

waveform $s(t)$

↓

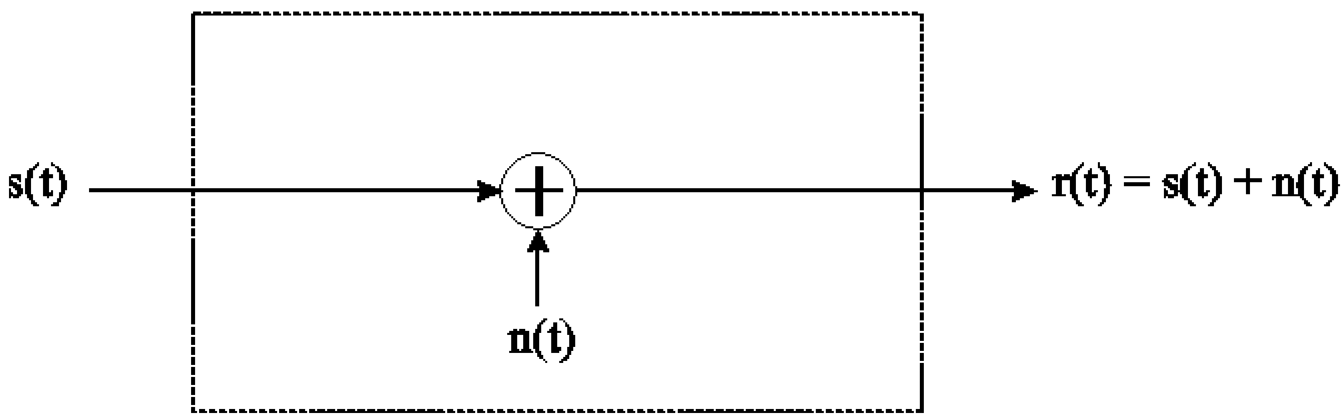
transmitted over the channel

Channel model:

Additive White Gaussian Noise (AWGN) channel

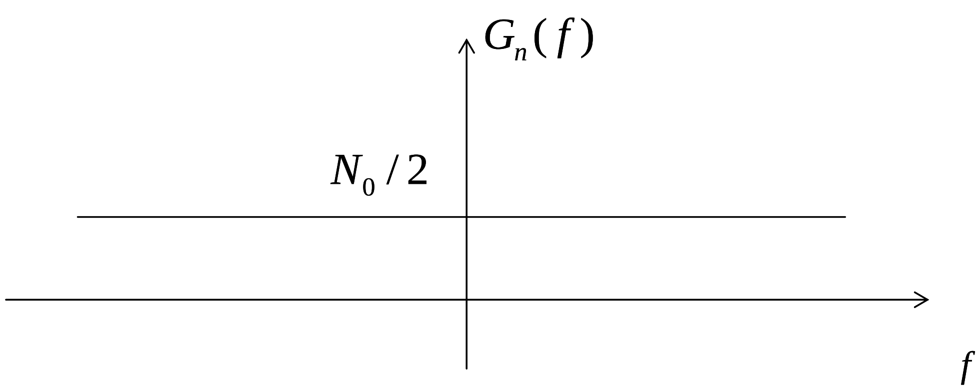
AWGN Channel (2)

- Linear and time-invariant
- Ideal frequency response $H(f)=1$
- Add white Gaussian noise n(t)



AWGN Channel (3)

- White Gaussian noise n(t)
  - Ergodic random process
  - Each random variable is a Gaussian random variable with zero average
  - Constant spectral density $G_n(f)=N_0/2$



Receiver (1)

- Transmission

$$\underline{\mathrm{u}}_{\mathrm{T}} \longrightarrow \boldsymbol{s}(t) \longrightarrow \boldsymbol{r}(t) = \boldsymbol{s}(t) + \boldsymbol{n}(t)$$

- PROBLEM

Given r(t) ➔ Recover u_T

- Divided in two steps:
  - Given *r(t)*, recover *s(t)*: ***(difficult problem)***
  - Given *s(t)*, recover $\underline{u}_T$: ***(easy problem:*** *labeling is a one-to-one mapping****)***

Receiver (2)

$$\underline{\mathrm{u}}_{\mathrm{T}} \longrightarrow s(t) \longrightarrow r(t) = s(t) + n(t)$$

- PROBLEM: Given *r(t)* → Recover *s(t)*

instead of working with real waveforms

easier to solve by working with **VECTORS**

Receiver (3)

1. Given M, build an orthonormal basis B

2. Work in the signal space S generated by B

3. Each signal of S can be expressed as a linear combination of the base elements → each signal of S corresponds to a real vector (= coefficients of the linear combination)

Basis (1)

- Given the signal constellation $M = \{ s_1(t) , \ldots , s_i(t), \ldots, s_m(t) \}$
- Build a basis $B = \{ b_1(t) , \ldots , b_j(t), \ldots, b_d(t) \} \quad (d \leq m)$
- B = set of signals

1. Orthogonal

$$\boxed{\int_0^T b_j(t) b_i(t)\, dt = 0 \quad \text{when} \quad j \neq i}$$

2. Unitary energy

$$\boxed{\int_0^T b^2{}_j(t)\, dt = 1}$$

3. A number d is minimal and sufficient for writing each signal of M as a linear combination

$$\boxed{s_i(t) = \sum_{j=1}^{d} s_{ij} b_j(t) \quad s_{ij} \in R}$$

Basis (2)

1. Given M, how to build B ?
2. For simple constellations, it is not difficult to write a basis B
3. Anyway, remember that there exists an algorithm which always provides a basis:

⟹ Gram-Schmidt algorithm

Gram-Schmidt algorithm (1)

$$M = \{ s_1(t) , \ldots , s_i(t), \ldots, s_m(t) \}$$

Given $s_1(t)$ → compute the first vector

1. define

$$\boxed{b_1^*(t) = s_1(t)}$$

2. compute

$$\boxed{b_1(t) = \frac{b_1^*(t)}{\sqrt{E(b_1^*)}}}$$

(If $b_1^*(t) = 0$ → $b_1(t) = 0$)

Gram-Schmidt algorithm (2)

- Given s2(t), look for the second versor

STEP 2

  - Compute the projection on the first versor

$$s_{21} = \int_0^T s_2(t) b_1(t) dt$$

Define

$$\boxed{b_2^*(t) = s_2(t) - s_{21} b_1(t)}$$

Compute

$$\boxed{b_2(t) = \frac{b_2^*(t)}{\sqrt{E(b_2^*)}}}$$

(If $b_2^*(t) = 0 \rightarrow$ $b_2(t) = 0$)

Gram-Schmidt algorithm (3)

$$s_{21} = \int_0^T s_2(t) b_1(t) dt \qquad b_2^*(t) = s_2(t) - s_{21} b_1(t)$$

Note:

- if $b_2^*(t) = 0$ ($s_2(t)$ is proportional to $b_1(t)$)
  → $b_2(t)=0$ and no new versor is found

- if $b_2^*(t) = 0$ ($s_2(t)$ is not proportional to $b_1(t)$)
  → $b_2(t) \neq 0$ and an new versor is found

Gram-Schmidt algorithm (4)

Given $s_i(t)$ $\qquad 3 \leq i \leq m$ $\qquad$ STEP i

Compute the projection on the previous versors

$$s_{ij} = \int_o^T s_i(t) b_j(t)\, dt \qquad 1 \leq j \leq i-1$$

define

$$\boxed{b_i^*(t) = s_i(t) - \sum_{j=1}^{i-1} s_{ij} b_j(t)}$$

compute

$$\boxed{b_i(t) = \frac{b_i^*(t)}{\sqrt{E(b_i^*)}}}$$

(If $b_1^*(t) = 0 \rightarrow b_i(t) = 0$)

Gram-Schmidt algorithm (5)

$$s_{ij} = \int_{o}^{T} s_i(t) b_j(t)\, dt \qquad b_i^*(t) = s_i(t) - \sum_{j=1}^{i-1} s_{ij} b_j(t)$$

Note:

- if $b_i^*(t) = 0$ ($s_i(t)$ is a linear combination of the current versors)
  → $b_i(t)=0$ and no new versor is found
- if $b_i^*(t) = 0$ ($s_i(t)$ is not a linear combination)
  → $b_i(t) \neq 0$ and a new versor is found

GRAM-SCHMIDT ALGORITHM (6)

FINAL STEP

- Delete all $b_i(t) = 0$
- Renumber the survived non-zero $b_i(t)$
- We have got the basis

$$\boxed{B = \{\, b_1(t)\ ,\ \dots\ ,\ b_j(t),\ \dots,\ b_d(t)\ \} \quad (d \le m)}$$

Exercise

Given the signal constellation

$$M = \{s_1(t) = +P_T(t), s_2(t) = -P_T(t)\}$$

Build an orthonormal basis B.

Base construction

Remember that, for simple constellations, it is often possible to find a basis B without applying Gram Schmidt.

It is sufficient to look for d signals satisfying the definition of orthonormal basis

1. orthogonal
2. with unitary energy
3. their number d is minimal and sufficient for writing each signal of M as a linear combination

Exercise

Given the signal constellation

$$M = \{s_1(t) = 0, s_2(t) = +P_T(t)\}$$

Build an orthonormal basis B.

Exercise

Given the signal constellation

$$M = \{s_1(t) = +P_T(t)\cos(2\pi f_0 t), s_2(t) = -P_T(t)\cos(2\pi f_0 t)\}$$

Build an orthonormal basis B.

Signal space

Given the basis B

$$B = \{\ b_1(t)\ ,\ \ldots\ ,\ b_j(t),\ \ldots,\ b_d(t)\ \}$$

The signal space S generated by B is

$$S = \left\{ a(t) = \sum_{j=1}^{d} a_j b_j(t) \quad a_j \in R \right\}$$

(set of all signals which can be expressed as linear combination of the basis signals)

Exercise

Given the basis B

$$B = \left\{ b_1(t) = +\frac{1}{\sqrt{T}} P_T(t) \right\}$$

What is the signal space S ?

Exercise

Given the basis B

$$B = \left\{ b_1(t) = +\sqrt{\frac{2}{T}} P_T(t) \cos\left(2\pi f_0 t\right) \right\}$$

What is the signal space S ?

Vector representation (1)

Fixed B, for each signal $a(t) \in S$ we have

$$a(t) = \sum_{j=1}^{d} a_j b_j(t)$$

The signal $a(t)$ corresponds to a real vector with d components (the coefficients a_j of the linear combination), and vice versa:

$$a(t) \equiv \underline{a} = (a_1 \ , \ldots, a_j , \ldots, a_d)$$

Vector representation (2)

1. From vector $\underline{a}$ to signal $a(t)$

$$\underline{a} = (a_1 \ ,\dots, a_j , \dots, a_d \,) \qquad\qquad a(t) = \sum_{j=1}^{d} a_j b_j(t)$$

2. From signal $a(t)$ to vector $\underline{a}$

Projection on $b_j(t)$

$$a(t) \;\Longrightarrow\; a_j = \int_0^T a(t) b_j(t)\, dt$$

$$\Downarrow$$

$$\underline{a} = (a_1 \ ,\dots, a_j , \dots, a_d \,)$$

Constellation vector representation (1)

We certainly have $M \subseteq S$

Each signal $s_i(t) \in S$ corresponds to a real vector with d components and viceversa:

$$s_i(t) \equiv \underline{s}_i = (s_{i1}, \ldots, s_{ij}, \ldots s_{id})$$

Constellation M as a signal set

$$M = \{ s_1(t), \ldots, s_i(t), \ldots, s_m(t) \}$$

Constellation M as a vector set

$$M = \{ \underline{s}_1, \ldots, \underline{s}_i, \ldots, \underline{s}_m \}$$

Constellation vector representation (2)

1. From vector $\underline{s_i}$ to signal $s_i(t)$

$$\underline{s_i} = (s_{i1} \;,\ldots, s_{ij}, \ldots, s_{id})$$

$$s_i(t) = \sum_{j=1}^{d} s_{ij} b_j(t)$$

2. From signal $s_i(t)$ to vector $\underline{s_i}$

Projection on the versor $b_j(t)$

$$s_i(t) \Rightarrow s_{ij} = \int_0^T s_i(t) b_j(t) dt$$

$$\Downarrow$$

$$\underline{s_i} = (s_{i1} \;,\ldots, s_{ij}, \ldots, s_{id})$$

Constellation vector representation (3)

Note that, as an alternative, the vector components can be computed without computing the projections.
We write

$$s_i(t) = s_{i1} b_1(t) + \ldots + s_{ij} b_j(t) + \ldots s_{id} b_d(t)$$

The basis signals $b_i(t)$ are known.
We look for a set of coefficients s_{ij} able to satisfy the equation.
The solution is unique.

Constellation vector representation (4)

The signal space S is isomorphic to the Euclidean space R^d (set of all vectors with d real components)

We can draw it as a Cartesian space

If d=1, $S \equiv R$ and can be drawn as a 1-D line

If d=2, $S \equiv R^2$ and can be drawn as the 2-D plane

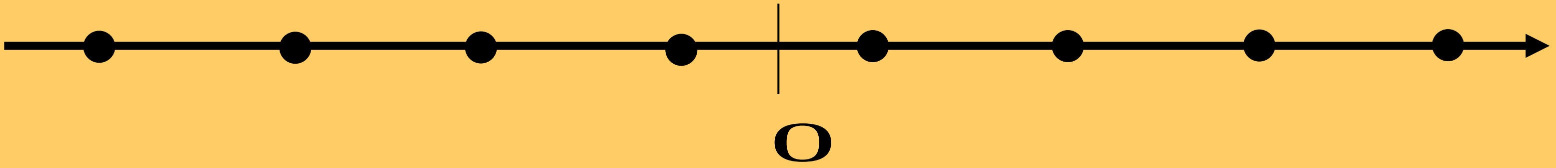
If d=3, $S \equiv R^3$ and can be drawn as the 3-D space

We will write $\boxed{M \subseteq R^d}$

(a constellation is a set of m points in the Euclidean space R^d)

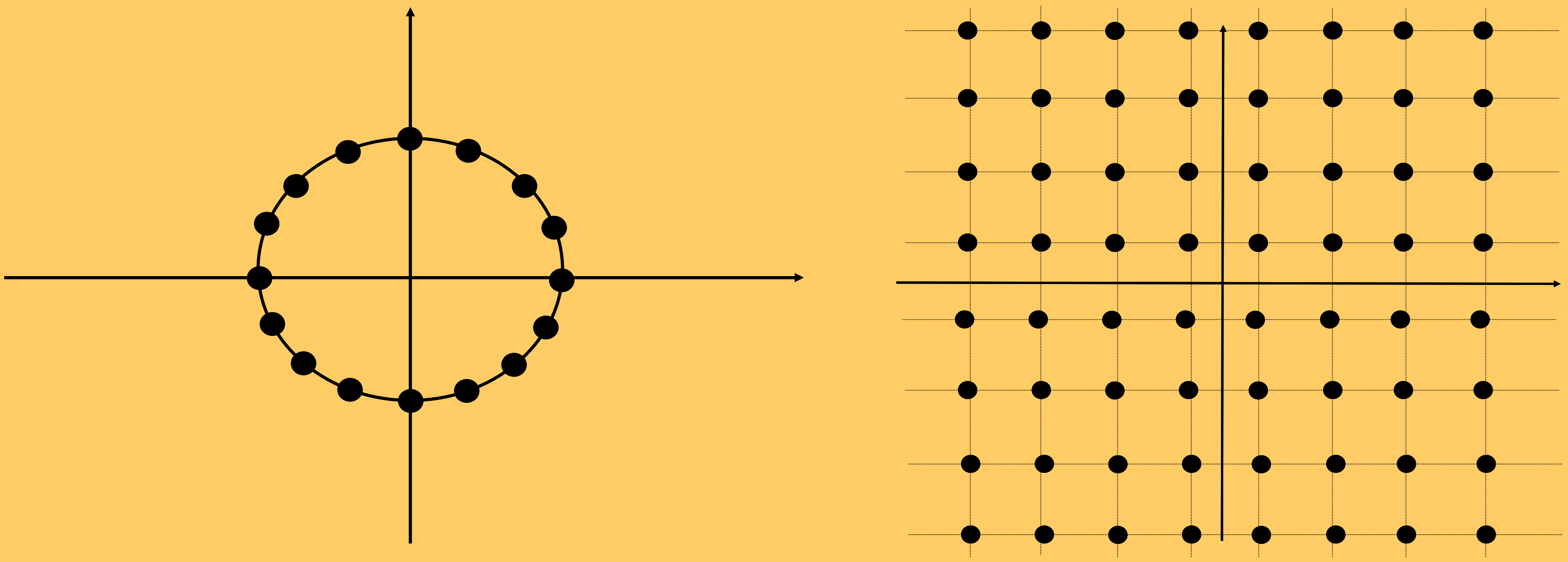
Example

Example of 1-D constellation



Example

Example of 2-D constellations



Signal energy (1)

Given a signal $a(t) \in S$

Its energy is given by $$E(a) = \int_0^T a^2(t)\,dt$$

Given its vector representation

$$a(t) \equiv (a_1, \ldots, a_j, \ldots a_d)$$

It is easy to show that $$E(a) = \sum_{j=1}^{d} a_j^2$$

Signal energy (2)

In fact, since $$a(t) = \sum_{j=1}^{d} a_j b_j(t)$$

$$E(a) = \int_0^T a^2(t)\, dt = \int_0^T [\sum_{j=0}^{d-1} a_j b_j(t)]^2\, dt = \sum_{j=0}^{d-1} a_j^2 \int_0^T b_j^2(t) dt = \sum_{j=0}^{d-1} a_j^2$$

Where we have used the orthogonality property

$$\int_0^T b_j(t)\, b_i(t) dt = 0 \quad se\;\; i \neq j$$

Constellation energy

Given a constellation $M = \left\{ \underline{s}_1, \ldots, \underline{s}_i, \ldots, \underline{s}_d \right\} \subseteq R^d$

with $\underline{s}_i = (s_{i1}\ , \ldots, s_{ij}, \ldots, s_{id})$

We have: $$E(s_i) = \sum_{j=1}^{d} s_{ij}^2$$

The (average) **constellation energy** is equal to:

$$E_s = \sum_{i=1}^{m} P(s_i) E(s_i)$$

where $P(s_i)$ is the probability of transmitting s_i

Constellation energy

Binary information sequences: ideal random

The binary vectors $\underline{v} \in H_k$ are equiprobable

The labeling is a one-to-one mapping $e : H_k \leftrightarrow M$

The constellation signals $\underline{s_i} \in M$ are equiprobable

$$P(s_i) = \frac{1}{m}$$

The signal constellation is simply:

$$\boxed{E_s = \frac{1}{m}\sum_{i=1}^{m} E(s_i)}$$

Energy per information bit

Average energy necessary to transmit an information bit via M

$$\boxed{E_b = \frac{E_S}{k}}$$

Exercise

Given the constellation

$$M = \{s_1(t) = +P_T(t), s_2(t) = -P_T(t)\}$$

- Build an orthonormal basis.
- Write the constellation as a vector set.
- Draw it.
- What is the signal space S ?
- Compute E_s and E_b.

Exercise

Given the constellation

$$M = \{s_1(t) = 0, s_2(t) = +P_T(t)\}$$

- Build an orthonormal basis.
- Write the constellation as a vector set.
- Draw it.
- What is the signal space S ?
- Compute E_s and E_b.

Exercise

Given the constellation

$$M = \{s_1(t) = +P_T(t)\cos(2\pi f_0 t), s_2(t) = -P_T(t)\cos(2\pi f_0 t)\}$$

- Build an orthonormal basis.
- Write the constellation as a vector set.
- Draw it.
- What is the signal space S ?
- Compute E_s and E_b.

Exercise

Given the constellation

$$M = \{s_1(t) = +P_T(t)\cos(2\pi f_0 t), s_2(t) = +P_T(t)\sin(2\pi f_0 t),$$
$$s_3(t) = -P_T(t)\cos(2\pi f_0 t), s_4(t) = -P_T(t)\sin(2\pi f_0 t)\}$$

- Build an orthonormal basis.
- Write the constellation as a vector set.
- Draw it.
- What is the signal space S ?
- Compute E_s and E_b.

Hint: $A\cos(2\pi f_0 t - \vartheta) = \left(A\cos\vartheta\right)\cos(2\pi f_0 t) + \left(A\sin\vartheta\right)\sin(2\pi f_0 t)$

LECTURE 3-2: DECISION THEORY

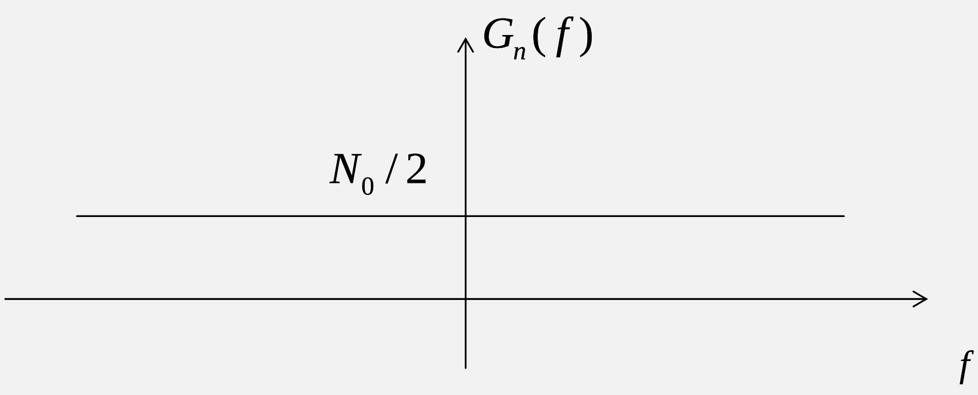
CONTENT

- Signal space representation
  - AWGN channel
  - Receiver roles
  - Orthonormal basis formulation
  - Gram-Schmidt algorithm
  - Signal space based on orthonormal basis (vector presentation)
- ML and MAP criterions
  - Received signals and noise at the receiver side
  - Decision based on vector formulation
  - Detection with MAP criterion
  - Detection with ML criterion
  - Voronoi region

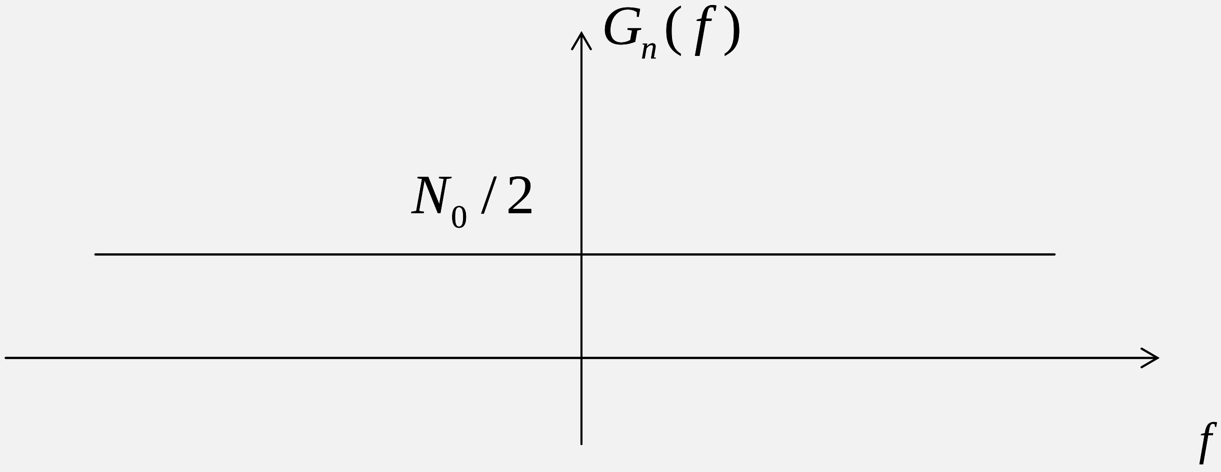
CHANNEL TRANSMISSION

white Gaussian noise $n(t)$

- **ergodic random process**
- **each random variable is a Gaussian random variable with zero average**
- **constant spectral density** $G_n(f)=N_0/2$

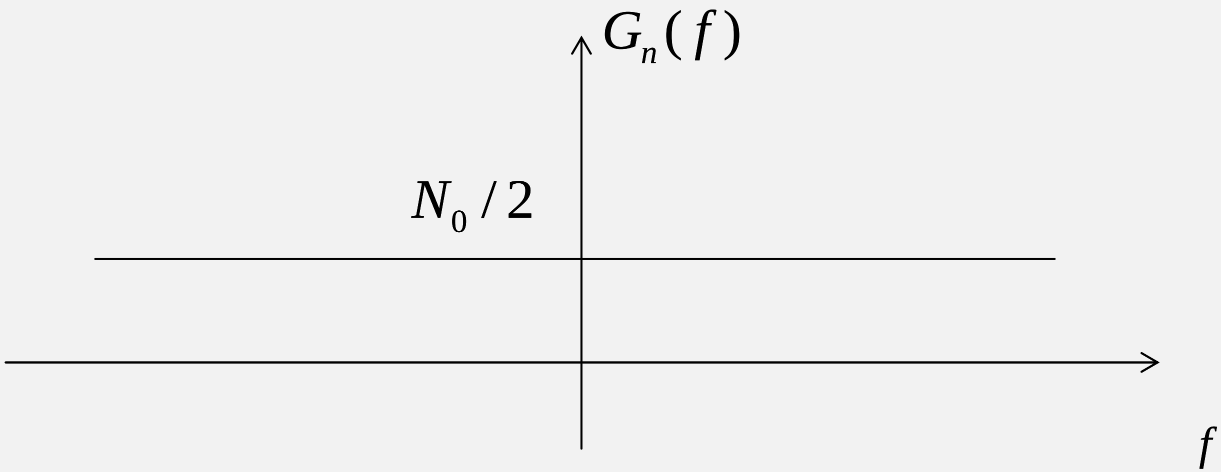


AWGN (1)



$$G_n(f) = N_0/2 \quad \Longleftrightarrow \quad R_n(\tau) = \frac{N_0}{2}\delta(\tau)$$

AWGN (2)



$$R_n(\tau) = \frac{N_0}{2}\delta(\tau) \quad \Longleftrightarrow \quad E\left[n(t_1)n(t_1+\tau)\right] = \frac{N_0}{2}\delta(\tau)$$

n(t) is an ergodic process

(temporal properties = statistical properties)

AWGN (3)

Fixed two different instant times t_1 and t_2
the random variables

$$t_1 \longrightarrow n(t_1)$$

$$t_2 \longrightarrow n(t_2)$$

are Gaussian random variables with

$$E[n(t_1)n(t_2)] = \frac{N_0}{2}\delta(t_1 - t_2)$$

Statistically independent

PROBLEM AT THE RECEIVER SIDE (1)

$$\underline{\mathrm{u}}_{\mathrm{T}} \longrightarrow s(t) \longrightarrow r(t) = s(t) + n(t)$$

PROBLEM: $\boxed{\text{given } r(t) \rightarrow \text{recover } s(t)}$

Divide $r(t)$ in segments of duration T

$$\Downarrow$$

$$r(t) = (\underbrace{r[0](t)}_{T} \mid \underbrace{r[1](t)}_{T} \mid \ldots \mid \underbrace{r[n](t)}_{T} \mid \ldots$$

PROBLEM AT THE RECEIVER SIDE (2)

Is it possible to independently analyze any single interval?

$$r(t) = (\underbrace{r[0](t)}_{T} \mid \underbrace{r[1](t)}_{T} \mid \ldots \mid \underbrace{r[n](t)}_{T} \mid \ldots$$

$$s(t) = (s[0](t) \mid s[1](t) \mid \ldots \mid s[n](t) \mid \ldots$$

$$\mathrm{n}(t) = (\mathrm{n}[0](t) \mid \mathrm{n}[1](t) \mid \ldots \mid \mathrm{n}[n](t) \mid \ldots$$

we have

$$r(t) = s(t) + n(t)$$

PROBLEM AT THE RECEIVER SIDE (3)

Consider the n-th interval $\boxed{nT \le t < (n+1)T}$

$$r[n](t) = s[n](t) + \mathrm{n}[n](t)$$

Each *r[n](t)* **certainly** depends on

- the corresponding transmitting signal *s[n](t)*
- the noise random variables extracted at $nT \le t < (n+1)T$

PROBLEM AT THE RECEIVER SIDE (4)

$$s(t) = (\underbrace{s[0](t)}_{T} \,|\, \underbrace{s[1](t)}_{T} \,|\, \ldots \,|\, \underbrace{s[m](t)}_{T} \,|\, \ldots\ldots \,|\, \underbrace{s[n](t)}_{T} \,|\, \ldots$$

Each transmitted signal $s[n](t)$

- has finite duration T
- is statistically independent with respect to any other transmitted signal $s[m](t)$, $m \neq n$

→ $r[n](t)$ is independent from $s[m](t)$, $m \neq n$

PROBLEM AT THE RECEIVER SIDE (5)

$$\mathrm{n}(t) = (\underbrace{\mathrm{n}[0](t)}_{T} \,|\, \underbrace{\mathrm{n}[1](t)}_{T} \,|\, \ldots \,|\, \underbrace{\mathrm{n}[m](t)}_{T} \,|\, \ldots\ldots \,|\, \underbrace{\mathrm{n}[n](t)}_{T} \,|\, \ldots$$

Each random variable $\mathrm{n}(t_i)$ is statistically independent

→$r[n](t)$ is independent from $n[m](t)$, $m \neq n$

PROBLEM AT THE RECEIVER SIDE (5)

Each $r[n](t)$ **only** depends on

- the corresponding transmitting signal $s[n](t)$
- the noise random variables extracted at

Each interval can be independently analyzed

NO INTERSYMBOL INTERFERENCE (ISI)

$$r(t) = (\underbrace{r[0](t)}_{T} \,|\, \underbrace{r[1](t)}_{T} \,|\, \ldots \,|\, \underbrace{r[n](t)}_{T} \,|\, \ldots$$

PROBLEM AT THE RECEIVER SIDE (6)

Each interval can be independently analyzed

Let us focus on the first one, for $0 \leq t < T$

$$r(t) = (\underbrace{r[0](t)}_{T} \,|\, r[1](t) \,|\, \ldots \,|\, r[n](t) \,|\, \ldots$$

PROBLEM AT THE RECEIVER SIDE (7)

Let us consider the first interval $0 \leq t < T$

$$s[0](t) \longrightarrow r[0](t) = s[0](t) + n[0](t)$$

<u>For simplicity, omit the index [0]</u>

$$s(t) \longrightarrow r(t) = s(t) + n(t)$$

PROBLEM: given $r(t)$ → recover $s(t)$

PROBLEM AT THE RECEIVER SIDE (8)

The transmitted signal $s(t)$ certainly belongs to the signal space S

Does the received signal $r(t)$ belong to S ?

$$r(t) = s(t) + n(t)$$

This depends on $n(t)$.

In general, $n(t)$ will be a generic signal not belonging to S: $n(t) \notin S$

In general

$$\boxed{r(t) \notin S}$$

RANDOM VARIABLES (1)

We know that $n(t) \notin S$

Let us try to project the noise on the basis signals.

$$B = \left(b_j(t)\right)_{j=1}^{d}$$

The *j*-th projection is:

$$n_j = \int_0^T n(t) b_j(t) dt$$

RANDOM VARIABLES (2)

$$n_j = \int_0^T n(t) b_j(t) dt$$

It is easy to show that these components n_j are

Gaussian random variables

- average $E[n_j]=0$
- variance $\sigma^2 = N_0/2$
- Statistically independent

RANDOM VARIABLES (3)

$$n_j = \int_0^T n(t) b_j(t) dt$$

- Gaussian random variables:

Obtained from a linear transformation of a Gaussian process

RANDOM VARIABLES (4)

$$n_j = \int_0^T n(t) b_j(t) dt$$

- Average value $E[n_j]=0$

$$E\left[n_j\right] = E\left[\int_0^T n(t) b_j(t) dt\right] = \int_0^T E\left[n(t)\right] b_j(t) dt = 0$$

RANDOM VARIABLES (5)

$$n_j = \int_0^T n(t) b_j(t) dt$$

- variance $\qquad \sigma^2 = N_0/2$
- statistically independent

$$E\left[n_j n_i\right] = E\left[\int_0^T n(t) b_j(t) dt \int_0^T n(x) b_i(x) dx\right] = E\left[\int_0^T \int_0^T n(t) n(x) b_j(t) b_i(x) dt dx\right] =$$

$$= \int_0^T \int_0^T E\left[n(t) n(x)\right] b_j(t) b_i(x) dt dx = \int_0^T \int_0^T \frac{N_0}{2} \delta(t-x) b_j(t) b_i(x) dt dx =$$

$$= \frac{N_0}{2} \int_0^T b_j(t) b_i(t) dt = \begin{cases} N_0/2 \ if \ \ j = i \\ 0 \ if \ \ j \neq i \end{cases}$$

RANDOM NOISE IN THE SIGNAL SPACE

Given $n(t)$ we have computed the projections on the basis signals:

$$n_j = \int_0^T n(t) b_j(t) dt$$

Let us introduce $$\boxed{n_S(t) = \sum_j n_j b_j(t)}$$

Clearly, $n(t) \in S$: it is the portion of $n(t)$ belonging to S

In general $n(t) \neq n_S(t)$

RANDOM NOISE EXTERNAL TO THE SIGNAL SPACE

We have $\boxed{n(t) = n_S(t) + e(t)}$

$e(t)$ = portion of $n(t)$ external to S

Fixed the time instant $t = t^*$

$\Downarrow$

$\boxed{n_S(t^*) \text{ and } e(t^*)}$
statistically independent

(Proof for exercise)

RANDOM NOISE EXTERNAL TO THE SIGNAL SPACE

Proof

$$E\left[n_S(t^*)e(t^*)\right] = 0 = E\left[n_S(t^*)\right]E\left[e(t^*)\right]$$

$$\Downarrow$$

$n_S(t^*)$ and $e(t^*)$
statistically independent

$$\Downarrow$$

The noise added outside the signal space is statistically independent

RECEIVED SIGNAL IN THE SIGNAL SPACE

We know that $r(t) \notin S$

Let us project *r(t)* on the basis signals.

$$B = \left(b_j(t)\right)_{j=1}^{d}$$

The *j*-th projection is:

$$r_j = \int_0^T r(t) b_j(t) dt$$

RECEIVED SIGNAL IN THE SIGNAL SPACE

Define $\boxed{r_S(t) = \sum_j r_j b_j(t)}$ Obviously $r_S(t) \in S$

In general $r(t) \neq r_S(t)$

But $r(t) = s(t) + n(t) = \underbrace{s(t) + n_S(t)}_{\in S} + \underbrace{e(t)}_{\notin S}$

Then $r(t) = r_S(t) + e(t)$ with $r_S(t) = s(t) + n_S(t)$

DECISION PROBLEM IN THE SIGNAL SPACE

$P1$ ORIGINAL PROBLEM:
given $r(t)=s(t)+n(t)$ → recover $s(t)$

$P2$ EQUIVALENT PROBLEM:
given $r_S(t) = s(t)+n_S(t)$ → recover $s(t)$

The only difference is $e(t)$:
noise (external to S) which is
statistically independent with respect to
both $s(t)$ and $n_S(t)$

DECISION PROBLEM IN THE SIGNAL SPACE

- $r_S(t)$ is a **sufficient statistics** for solving the problem
- It is sufficient to work in the signal space S
- All the other (infinite) dimensions do not carry useful information, but only noise

DECISION PROBLEM: VECTORIAL FORMULATION

P2

PROBLEM:

given $r_S(t) = s(t) + n_S(t) \rightarrow$ recover $s(t)$

The three signals belong to S

$\Downarrow$

Vector representation

DECISION PROBLEM: VECTORIAL FORMULATION

$$\boxed{r_S(t) = s(t) + n_S(t)} \Rightarrow \boxed{\underline{r} = \underline{s}_T + \underline{n}}$$

$$\underline{r} = (r_1, \ldots, r_j, \ldots, r_d) \qquad r_j = \int_0^T r(t) b_j(t)\, dt$$

$$\underline{s}_T = (s_1, \ldots, s_j, \ldots, s_d) \qquad s_j = \int_0^T s(t) b_j(t)\, dt$$

$$\underline{n} = (n_1, \ldots, n_j, \ldots, n_d) \qquad n_j = \int_0^T n(t) b_j(t)\, dt$$

RECEIVED VECTOR

The received vector $\underline{r}$ (in the signal space) is given by

$$\underline{r} = \underline{s}_T + \underline{n}$$

Where $\underline{s}_T = (s_1, \ldots, s_j, \ldots, s_d) \in M$ is the transmitted signal

and $\underline{n} = (n_1, \ldots, n_j, \ldots, n_d)$ is the noise vector (added in the signal space)

For each component we have $r_j = s_j + n_j$

RECEIVED VECTOR

$$r_j = s_j + n_j$$

The components r_j are

Gaussian random variables with

- Mean $E[r_j]=s_j$
- Variance $\sigma^2[r_j]=N_0/2$
- Statistically independent

$$\left(E[r_i r_j] = s_i s_j = E[r_i]E[r_j] \right)$$

DECISION PROBLEM: VECTORIAL FORMULATION

$P2$

PROBLEM:

given $r_S(t) = s(t) + n_S(t)$ → recover $s(t)$

$P3$

PROBLEM:

given $\underline{r} = \underline{s}_{\underline{T}} + \underline{n}$ → recover $\underline{s}_{\underline{T}}$

IMPORTANT:

given $r(t)$, the vector $\underline{r}$ is easy to compute
(the basis signals are known)

DECISION CRITERION

P3

PROBLEM:

given $\underline{r}=\underline{s}_T+\underline{n} \rightarrow$ recover $\underline{s}_T$

At the receiver side, given $\underline{r}$

we want to choose a **received signal** $\boxed{\underline{s}_R \in M}$

Goal: make the right choice: $\underline{s}_R = \underline{s}_T$

Unfortunately this in not always possible, due to noise.

DECISION CRITERION

Given $\underline{r}$ we have to establish a **<u>decision criterion</u>** which determines the choice of $\underline{s}_R$

Minimization of symbol (signal) error probability

$$\boxed{P_S(e) = P(\underline{s}_R \neq \underline{s}_T)}$$

DECISION CRITERION

P3

PROBLEM:

given $\underline{r}=\underline{s}_T+\underline{n} \rightarrow$ recover $\underline{s}_T$

Let us suppose to receive a given $\underline{r}=\underline{\rho}\in R^d$

→ we choose $\underline{s}_R \in M$ such that $P_S(e)$ is minimum

decision criterion

C1

$$\underline{s}_R = \arg\min_{\underline{s}_i \in M}\left[P(\underline{s}_R \neq \underline{s}_T \mid \underline{r} = \underline{\rho}) \right]$$

DETECTION

Problem of deciding which one, among a set of mutually exclusive alternatives, is correct.

- A random variable X with m possible sample values with known a priori probability $P(X=x)$

- We observe another random variable Y which is connected to X by known probabilities $P(Y=y|X=x)$ called **likelihoods**

When an experiment is performed, two samples $x \in X$ and $y \in Y$ are extracted.

The decision maker observes y but not x.

Let us suppose to receive a given $\underline{r}=\underline{\rho} \in R^d$

→ we choose $\underline{s}_R \in M$ such that $P_S(e)$ is minimum

DETECTION

Given y , the decision maker makes a **decision** $d(y)=x'$

The decision is correct if $x'=x$

Decision criterion adopted for choosing $d(y)$:

Maximization of correct decision $P(x' = x)$

=

Minimization of wrong decision $P(x' \neq x)$

MAP CRITERION

It is easy to show that the decision criterion must be **a MAXIMUM A POSTERIORI (MAP) criterion**

$$\boxed{d(y) = \arg\max_{x} \left[P(X = x \mid Y = y) \right]}$$

MAP CRITERION

Proof

$$P(X' \neq X) = \sum_x \sum_y P(X' \neq X, X = x, Y = y) =$$

$$= \sum_x \sum_y P(X' \neq X \mid X = x, Y = y) P(X = x, Y = y) =$$

$$= \sum_x \sum_y P(X'(y) \neq x \mid X = x, Y = y) P(X = x \mid Y = y) P(Y = y) =$$

$$= \sum_y \left[\sum_x (1 - \delta_{X'(y), x}) P(X = x \mid Y = y) \right] P(Y = y)$$

$$X'(y) = \arg\min_z \sum_x (1 - \delta_{z,x}) P(X = x \mid Y = y) = \arg\max_x P(X = x \mid Y = y)$$

ML CRITERION

Bayes theorem $$P(X=x \mid Y=y)=\frac{P(Y=y \mid X=x)P(X=x)}{P(Y=y)}$$

$$d(y)=\arg\max_{x}\left[P(X=x \mid Y=y)\right]$$

$$\Rightarrow \quad d(y)=\arg\max_{x}\left[P(Y=y \mid X=x)P(X=x)\right]$$

For equiprobable hypothesis $$P(X=x)=\frac{1}{m}$$

$$d(y)=\arg\max_{x}\left[P(Y=y \mid X=x)\right]$$

ML CRITERION

MAXIMUM LIKELIHOOD (ML) criterion

$$\boxed{d(y) = \arg\max_{x} \left[P(Y = y \mid X = x) \right]}$$

DETECTION PROBLEM AT THE RECEIVER SIDE

Random variable $X \equiv$ transmitted signal $\underline{s}_T \in M$

Observed variable $Y \equiv$ received signal $\underline{r} = \underline{s}_T + \underline{n} \in S$

DETECTION PROBLEM AT THE RECEIVER SIDE

$$\underline{r} = \underline{s}_T + \underline{n}$$

Connection between $\underline{r}$ and $\underline{s}_T$

$$f_{\underline{r}}(\underline{\rho} \mid \underline{s}_T = \underline{s}_i)$$

This is a Gaussian density function centered around $\underline{s}_i$ with variance $N_0/2$ in each dimension

GAUSSIAN DENSITY FUNCTION

Example: sinlgle Gaussian random variable r

- Mean μ
- Variance σ^2
- density function:

$$f_r(\rho) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(\rho-\mu)^2}{2\sigma^2}\right)$$

GAUSSIAN DENSITY FUNCTION

Example: pair of Gaussian random variables r_1 r_2

- Mean μ
- Variance σ^2
- statistically independent
- density function:

$$f_{r_1 r_2}(\rho_1\ \rho_2) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(\rho_1-\mu)^2}{2\sigma^2}\right) \bullet \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(\rho_2-\mu)^2}{2\sigma^2}\right)$$

$$f_{r_1 r_2}(\rho_1\ \rho_2) = \frac{1}{(\sqrt{2\pi}\sigma)^2}\exp\left(-\frac{(\rho_1-\mu)^2+(\rho_2-\mu)^2}{2\sigma^2}\right)$$

GAUSSIAN DENSITY FUNCTION

$$f_{\underline{r}}(\underline{\rho} \mid \underline{s}_T = \underline{s}_i)$$

$\underline{r}$ = Array of d Gaussian random variables

- Mean $\mu = s_{ij}$
- Variance $\sigma^2 = N_0/2$
- Statistically independent
- density function

$$f_{\underline{r}}(\underline{\rho} \mid \underline{s}_T = \underline{s}_i) = \frac{1}{(\sqrt{\pi N_0})^d} \exp\left(-\frac{\sum_{j=1}^{d} (\rho_j - s_{ij})^2}{N_0}\right)$$

ML CRITERION

$$d(y) = \arg\max_{x}\left[P(Y = y \mid X = x)\right]$$

For our problem becomes:

$$\boxed{\text{given } \underline{r} = \underline{\rho} \quad \text{choose} \quad \underline{s}_R = d\left(\underline{\rho}\right) = \arg\max_{\underline{s}_i \in M}\left[f_{\underline{r}}(\underline{\rho} \mid \underline{s}_T = \underline{s}_i)\right]}$$

$C2$

ML CRITERION

Using the expression of $f_{\underline{r}}(\underline{\rho} \mid \underline{s}_T = \underline{s}_i)$

$$\underline{s}_R = \arg\max_{\underline{s}_i \in M} \left[\frac{1}{(\sqrt{\pi N_0})^d} \exp\left(-\frac{\sum_{j=1}^{d} (\rho_j - s_{ij})^2}{N_0} \right) \right]$$

$$\underline{s}_R = \arg\min_{\underline{s}_i \in M} \sum_{j=1}^{d} (\rho_j - s_{ij})^2$$

MINIMUM DISTANCE CRITERION

$$\underline{s}_R = \arg\min_{\underline{s}_i \in M} \sum_{j=1}^{d} (\rho_j - s_{ij})^2$$

By introducing the Euclidean distance between vectors in R^d:

$$d_E^2(\underline{\rho} - \underline{s}_i) = \sum_{j=1}^{d} (\rho_j - s_{ij})^2$$

We have:

$$\underline{s}_R = \arg\min_{\underline{s}_i \in M} d_E^2(\underline{\rho} - \underline{s}_i)$$

MINIMUM DISTANCE CRITERION

The ML criterion is equivalent to a
minimum distance criterion

$$C3 \quad \boxed{\text{given } \underline{r} = \underline{\rho} \quad \text{choose} \quad \underline{s}_R = \arg\min_{\underline{s}_i \in M} d_E^2(\underline{\rho} - \underline{s}_i)}$$

VORONOI REGION

$$\text{given } \underline{r} = \underline{\rho} \quad \text{choose} \quad \underline{s}_R = \arg\min_{\underline{s}_i \in M} d_E^2(\underline{\rho} - \underline{s}_i)$$

This decision criterion associates to any vector $\underline{\rho} \in R^d$ a received signal $\underline{s}_R \in M$

We can introduce the **Voronoi (decision) region** $V(\underline{s}_i)$ =set of all received vectors which determine the choice $\underline{s}_R = \underline{s}_i$

$$V(\underline{s}_i) = \left\{ \underline{\rho} \in R^d : \underline{s}_R = \underline{s}_i \right\}$$

VORONOI REGION

Set of all received vectors

which determine the choice $\underline{s}_R = \underline{s}_i$

When do we have $\underline{s}_R = \underline{s}_i$?

When $\underline{\rho} \in R^d$ is nearest to $\underline{s}$ than to all other constellation signals

$$\boxed{V(\underline{s}_i) = \{\underline{\rho} \in R^d : \; d_E^2(\underline{\rho}, \underline{s}_i) \le d_E^2(\underline{\rho}, \underline{s}) \quad \forall \underline{s} \in M\}}$$

VORONOI REGION CRITERION

NOTE:

If we receive $\underline{\rho} \in V(\underline{s}_i)$ $\Rightarrow$ We certainly choose $\underline{s_R} = \underline{s}_i$

The minimum distance criterion

$$\boxed{\text{given } \underline{r} = \underline{\rho} \quad \text{choose} \quad \underline{s_R} = \arg\min_{\underline{s}_i \in M} d_E^2(\underline{\rho} - \underline{s}_i)}$$

can be expressed as a **Voronoi region criterion**

C4 $$\boxed{\text{given } \underline{r} = \underline{\rho} \quad \text{if} \quad \underline{\rho} \in V(\underline{s}) \text{ select } \underline{s_R} = \underline{s}}$$

Lecture 4: Receiver architectures

Signal space receiver

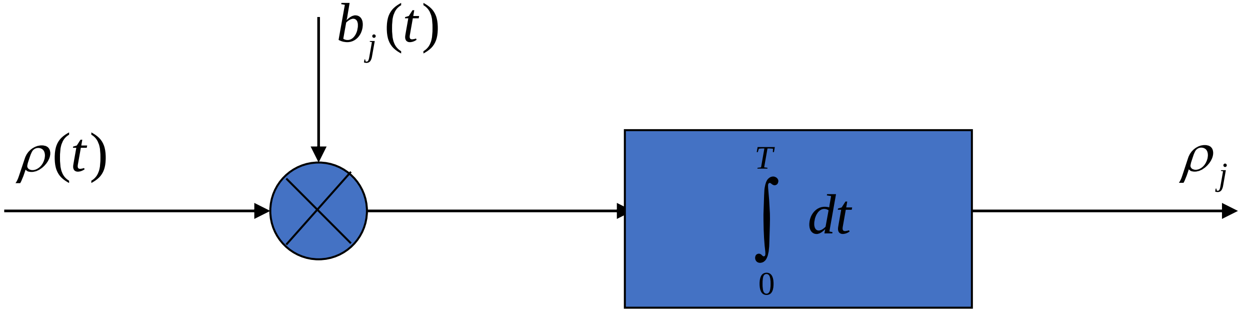
Given the received signal $\rho(t)$ for $0 \leq t < T$, the receiver must:

1. Compute the d projections $\rho_j = \int_0^T \rho(t) b_j(t) dt$
2. Given the received vector $\underline{\rho} = (\rho_1, \ldots, \rho_j, \ldots, \rho_d)$ choose $\underline{s_R} \in M$

   according to the ML criterion (minimum distance or Voronoi)
3. Given $\underline{s_R}$, recover the binary information vector $\underline{u_R}$ by inverting the binary labeling: $\underline{u_R} = e^{-1}(\underline{s_R})$

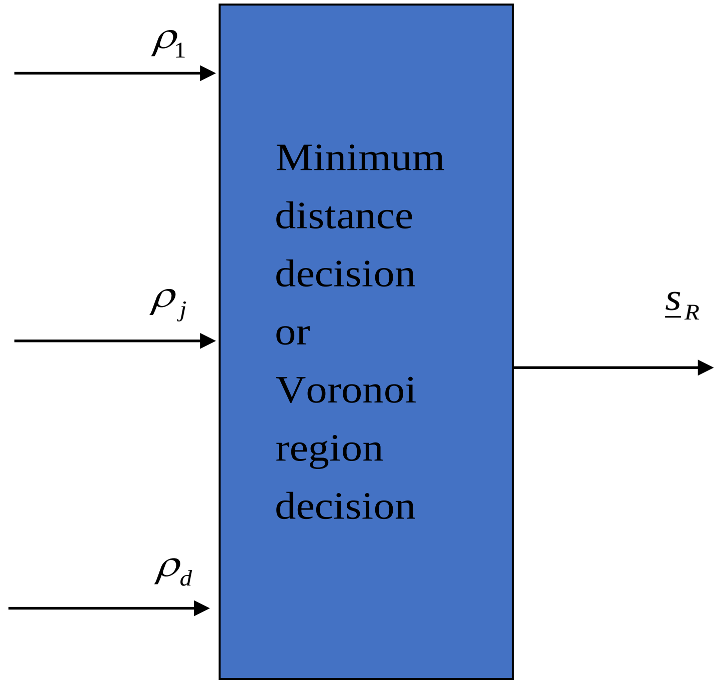
Signal space receiver (with integrator)

1. Given $\rho(t)$ compute the d projections $\rho_j = \int_0^T \rho(t) b_j(t)\, dt$



Signal space receiver (with integrator)

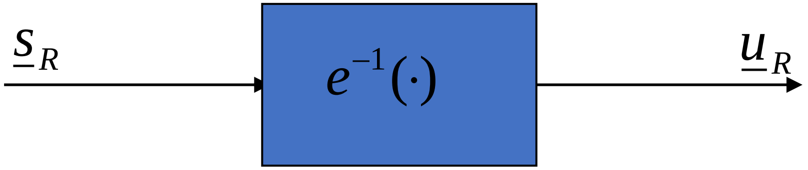
2. Given the vector $\underline{\rho}=(\rho_1,\ldots,\rho_j,\ldots\rho_d)$ apply the ML criterion for choosing $\underline{s}_R \in M$



Signal space receiver (with integrator)

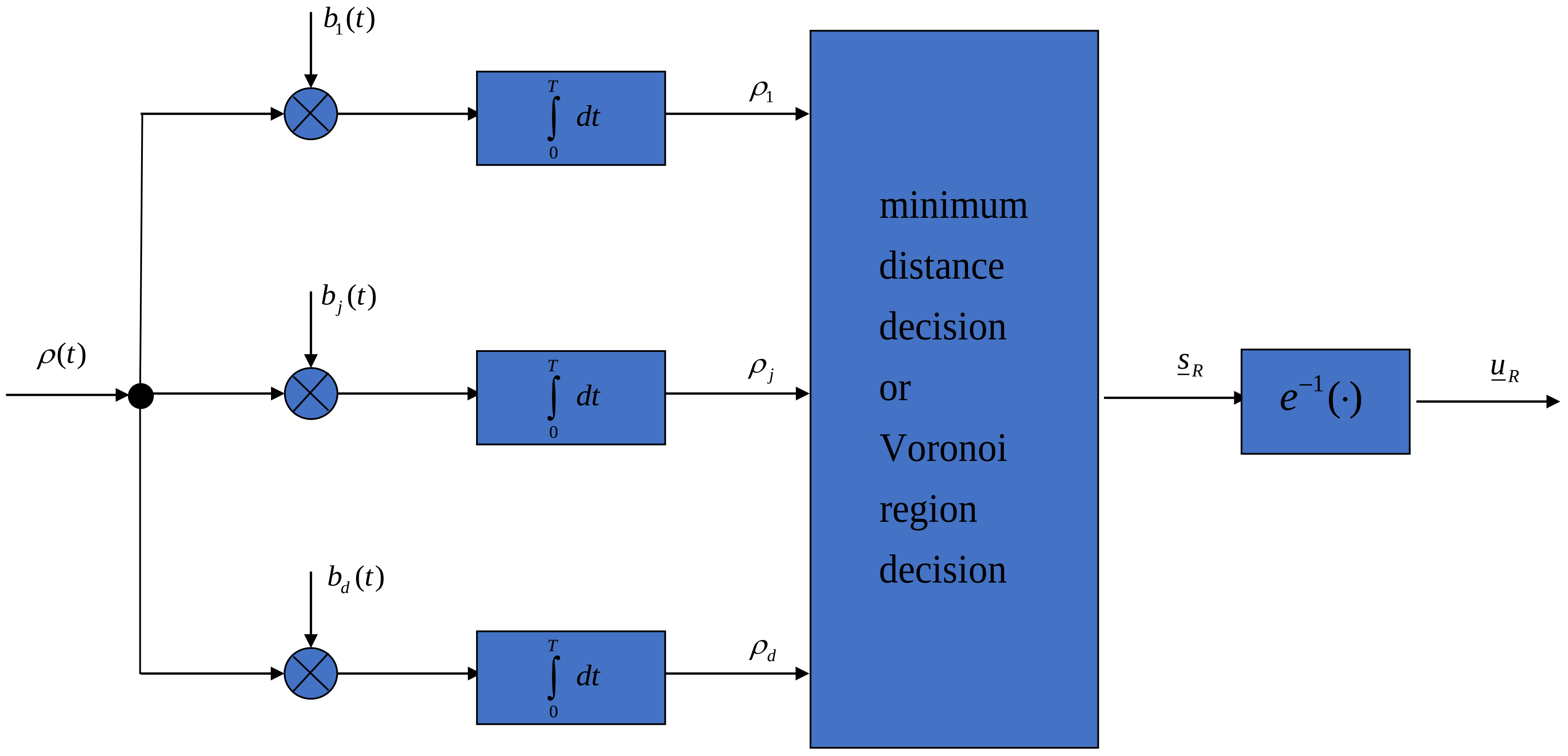
3. Given $\underline{s}_R$, recover the binary information vector $\underline{u}_R$ by inverting the binary labelling

$$\underline{u}_R = e^{-1}(\underline{s}_R)$$



Signal space receiver (with integrator)

Complete scheme



Matched filter

Given a filter with impulse response $h(t)$

the output waveform $y(t)$ is connected to the input waveform $x(t)$ by

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Matched filter

Let us suppose:

- The input waveform to be the received waveform $\rho(t)$
- The impulse response to be $h(t) = b_j(T - t)$

MATCHED FILTER

Matched filter

The output of the matched filter is

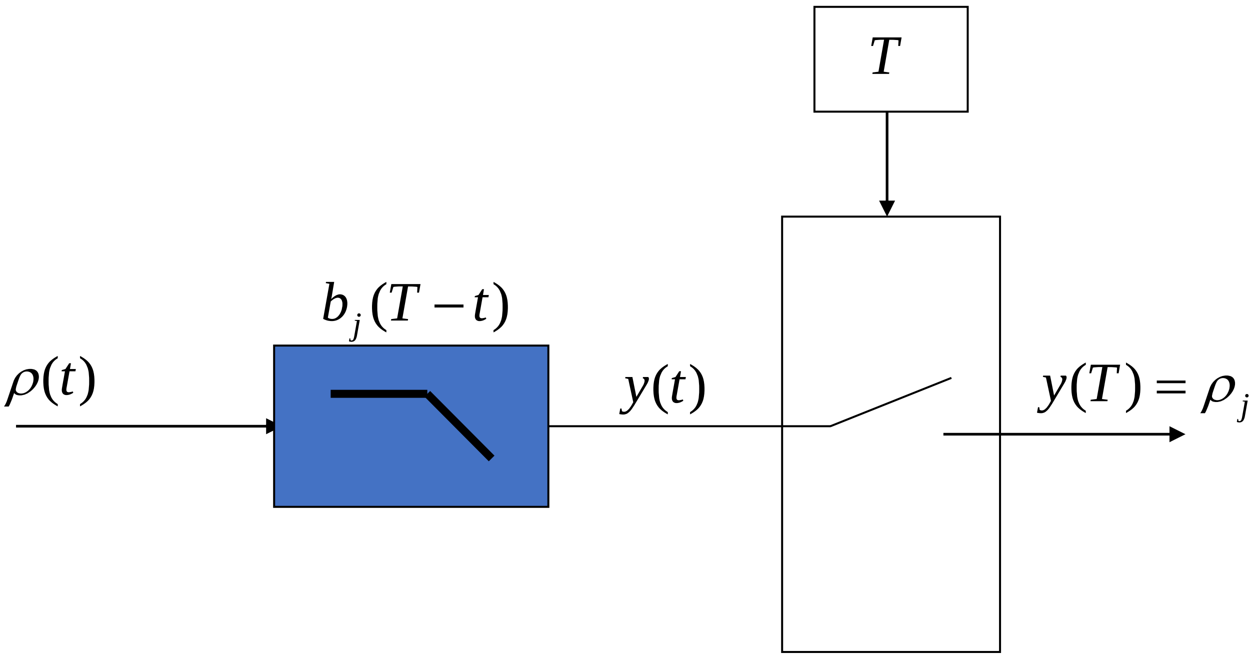
$$y(t) = \int_{-\infty}^{+\infty} \rho(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{+\infty} \rho(\tau)\, b_j(T-t+\tau)\, d\tau$$

Let us suppose to sample this output waveform at time $t=T$

$$y(t=T) = \int_{-\infty}^{+\infty} \rho(\tau)\, b_j(\tau)\, d\tau = \int_{0}^{T} \rho(\tau)\, b_j(\tau)\, d\tau = \rho_j$$

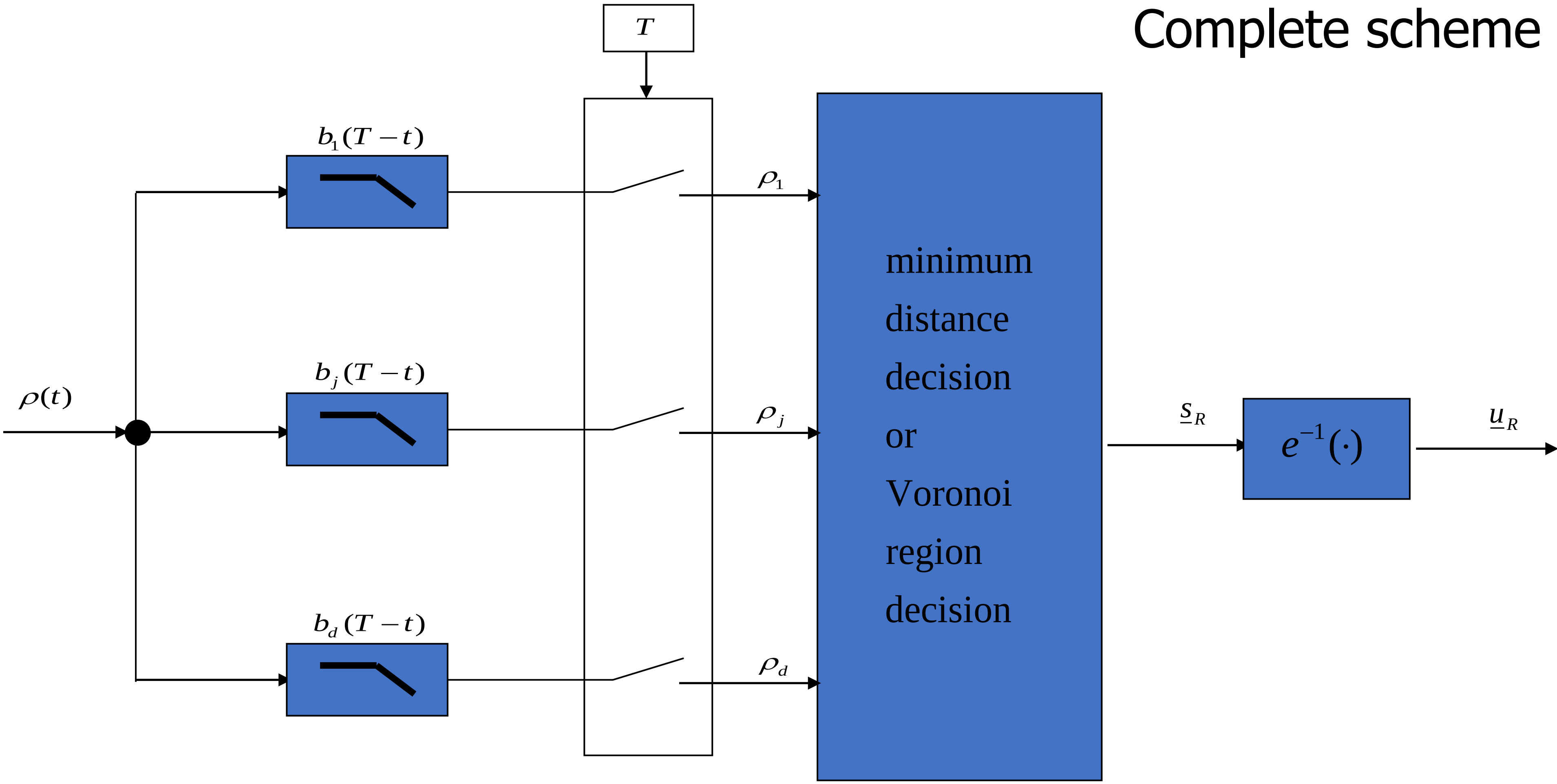
Matched filter

Alternative scheme for computing the projection $b_j(t)$ by using the matched filter instead of the integrator:



Signal space receiver (with matched filter)

Complete scheme



Complete receiver

Since now, we have focused on the first interval [0,T[

- The constellation $M = \{ s_1(t) , \ldots , s_i(t), \ldots, s_m(t) \}$ is composed by signals with domain $[0,T[$

- The basis $B = \{ b_1(t) , \ldots , b_j(t), \ldots, b_d(t) \}$ is composed by signals with domain $[0,T[$

- The projections are given by: $\rho_j = \rho_j[0] = \int_0^T \rho(t) b_j(t) dt$

Complete receiver

What about the other intervals, for example the second one $[T,2T[$?

The signals used for the second interval are the same of M, but translated of T

It is like to use a constellation $M' = \{ s'_1(t) , \ldots , s'_i(t), \ldots, s'_m(t) \}$

composed by signals with domain $[T,2T[$

obtained as $$s_i^{'}(t) = s_i(t - T)$$

Complete receiver

In the second interval $[T,2T[$

- The constellation is $M' = \{s'_1(t)\ ,\ \ldots\ ,s'_i(t),\ \ldots,\ s'_m(t)\ \}$ with
$$s_i^{'}(t) = s_i(t-T)$$
- The basis $B' = \{b'_1(t),\ \ldots\ ,b'_j(t),\ \ldots,b'_d(t)\}$ with
$$b_i^{'}(t) = b_i(t-T)$$
- The projections are given by: $\rho_j[1] = \int_T^{2T} \rho(t) b_j(t) dt$

Complete receiver

In the generic interval $[nT, (n+1)T[$

- The constellation is $M' = \{ s'_1(t) , \ldots , s'_i(t), \ldots, s'_m(t) \}$ with

$$s_i^{'}(t) = s_i(t - nT)$$

- The basis $B' = \{ b'_1(t) , \ldots , b'_j(t), \ldots, b'_d(t) \}$ with

$$b_i^{'}(t) = b_i(t - nT)$$

- The projections are given by: $\rho_j[n] = \displaystyle\int_{nT}^{(n+1)T} \rho(t) b_j(t) dt$

Complete receiver

Given the output of the matched filter

$$y(t)=\int_{-\infty}^{+\infty}\rho(\tau)\ f(t-\tau)\ d\tau=\int_{-\infty}^{+\infty}\rho(\tau)\,b_j(T-t+\tau)\ d\tau$$

Sample it at $t=(n+1)T$

$$y(t=(n+1)T)=\int_{-\infty}^{+\infty}\rho(\tau)\,b_j(\tau-nT)\ d\tau=\int_{nT}^{(n+1)T}\rho(\tau)\,b_j(\tau-nT)\ d\tau=\rho_j[n]$$

Complete receiver

The matched filter allows to compute the projections $\rho_j[n]$

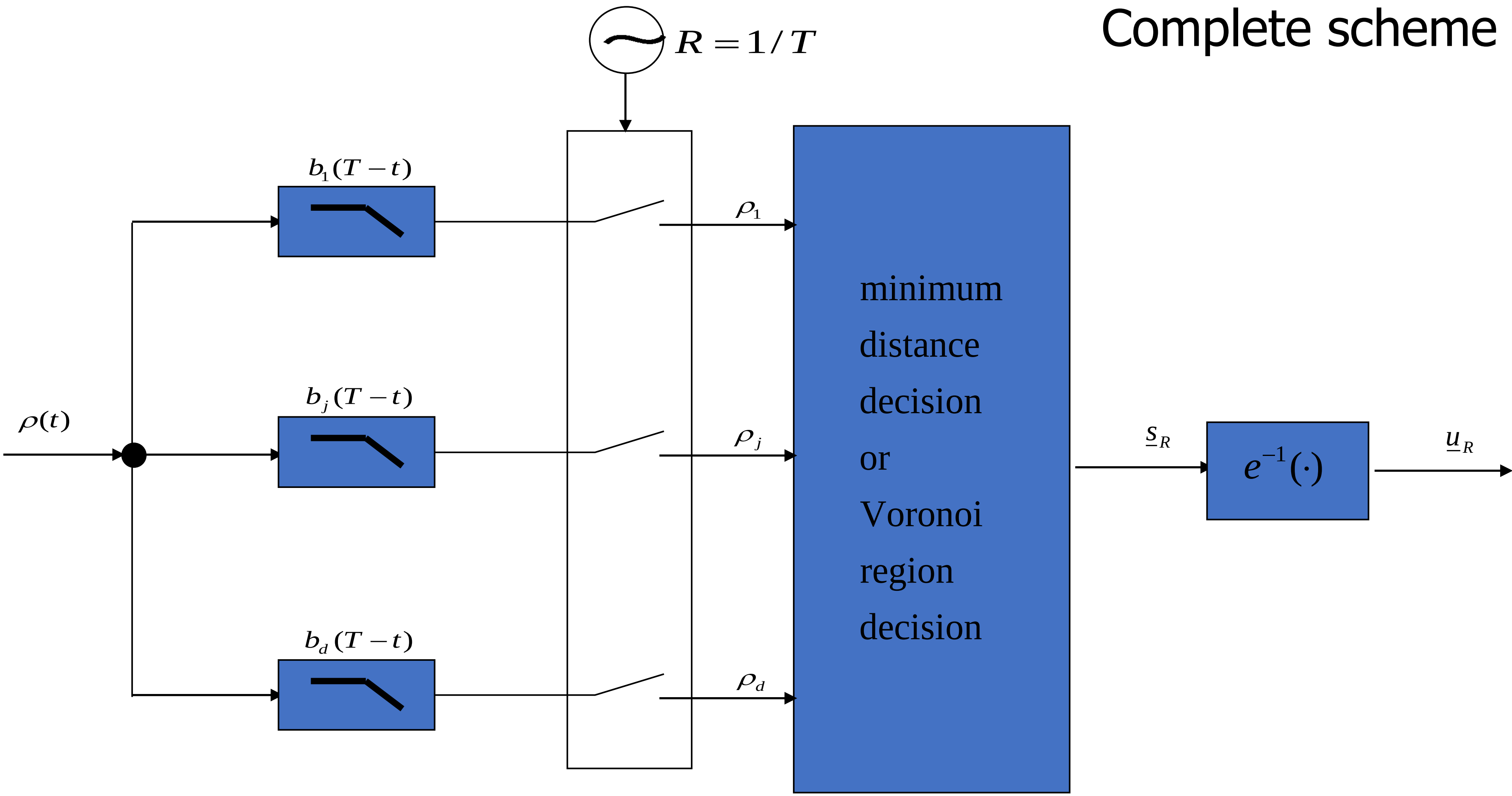
not only for the first interval, but for any other interval $[nT,(n+1)T[$

WE MUST SAMPLE THE MATCHED FILTER OUTPUT

- WITH FREQUENCY $R=1/T$
- AT $t=(n+1)T$

Signal space receiver (with matched filter)

Complete scheme



Symbol synchronization

A binary information sequence is characterized by a bit-rate R_b.

Each constellation signal corresponds to k bits and has duration $T=kT_b$.

The symbols are transmitted at frequency $R=1/T=R_b/k$ (symbol rate).

At the receiver side, the filter output must be sampled at frequency R.

The nominal value of R is known but its actual value is not!

Symbol synchronization

There are not two identical oscillators with frequency R:
the frequency must be recovered

The filter output must be sampled exactly at $t=(n+1)T$:
the phase (timing reference) must be recoverd

Symbol syncronization:
starting from the received signal, symbol rate frequency and phase must be exactly recovered.

Fundamental to exactly compute the projections and to detect the transmitted signal

Correlation receiver

Starting from the Euclidean distance criterion:

$$\underline{s_R} = \arg\min_{\underline{s}_i \in M} d_E^2(\underline{\rho}, \underline{s}_i)$$

We have

$$d_E^2(\underline{\rho}, \underline{s}_i) = \sum_{j=1}^{d} (\rho_j - s_{ij})^2 = \sum_{j=1}^{d} \rho_j^{\ 2} + \sum_{j=1}^{d} s_{ij}^{\ 2} - 2\sum_{j=1}^{d} \rho_j s_{ij}$$

It follows:

$$\underline{s_R} = \arg\min_{\underline{s}_i \in M} d_E^2(\underline{\rho} - \underline{s}_i) = \arg\min_{\underline{s} \in M} \left[\sum_{j=1}^{d} \rho_j^{\ 2} + \sum_{j=1}^{d} s_{ij}^{\ 2} - 2\sum_{j=1}^{d} \rho_j s_{ij} \right]$$

Correlation receiver

$$\underline{s_R} = \arg\min_{\underline{s}\in M} \; [\sum_{j=1}^{d} s_j{}^2 - 2\sum_{j=1}^{d}\rho_j s_{ij} \;] = \arg\max_{\underline{s}\in M} \; [\sum_{j=1}^{d}\rho_j s_{ij} - \frac{1}{2}\sum_{j=1}^{d} s_{ij}{}^2]$$

Since:

$$E(s_i) = \sum_{j=1}^{d} s_{ij}{}^2$$

We have:

$$\underline{s_R} = \arg\max_{\underline{s}_i\in M} \; [\sum_{j=1}^{d}\rho_j s_{ij} - \frac{1}{2}E(s_i)]$$

Correlation receiver

$$\underline{s}_R = \arg\max_{\underline{s}_i \in M} \left[\sum_{j=1}^{d} \rho_j s_{ij} - \frac{1}{2} E(s_i) \right]$$

Note that

$$\int_0^T \rho(t)\, s_i(t)\, dt = \int_0^T \rho(t) \left[\sum_{j=1}^{d} s_{ij} b_j(t) \right] dt = \sum_{j=1}^{d} s_{ij} \int_0^T \rho(t)\, b_j(t)\, dt = \sum_{j=1}^{d} s_{ij} \rho_j$$

This is the **correlation** between the received signal and the generic constellation signal $s_i(t)$:

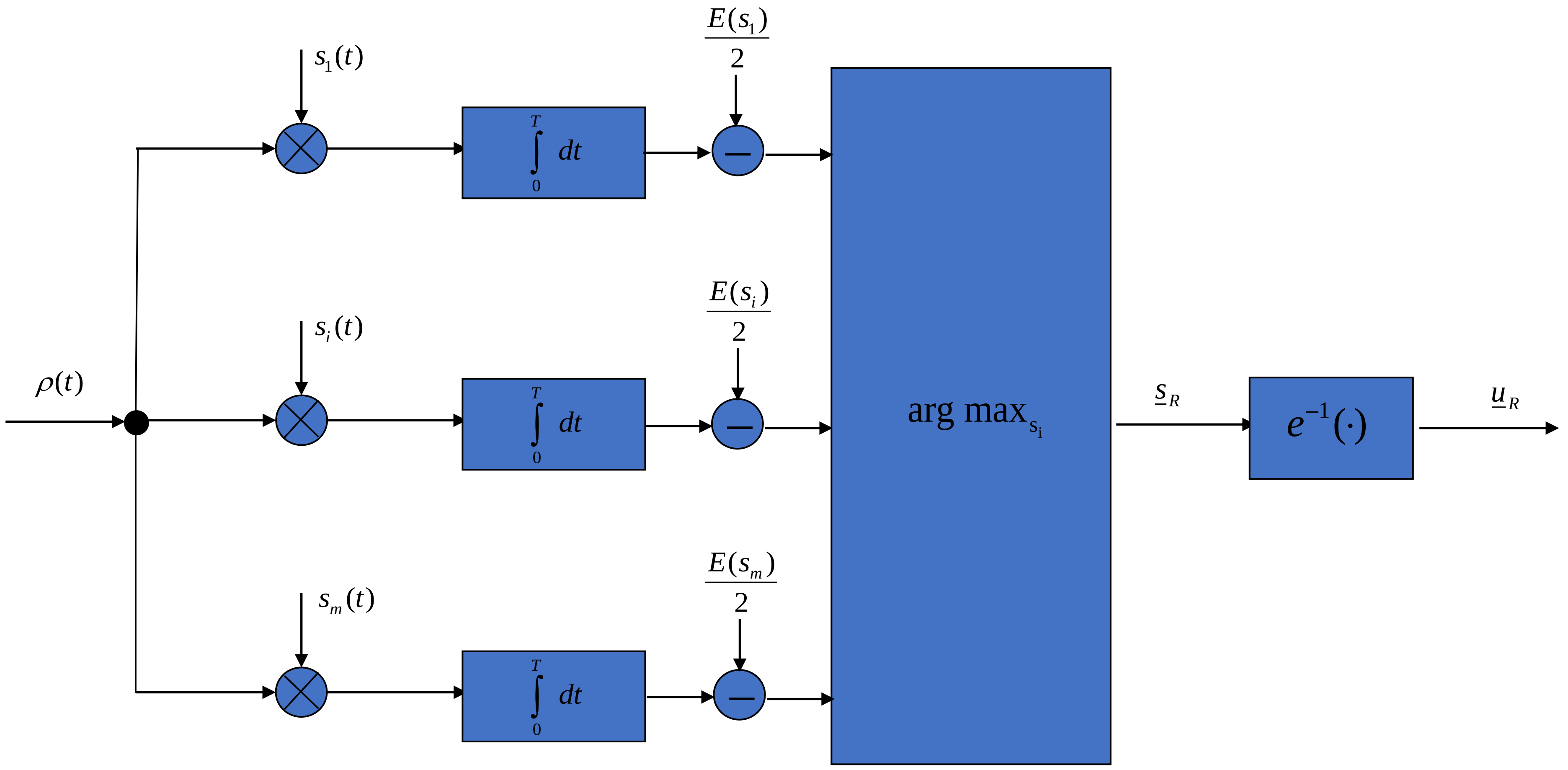
$$\int_0^T \rho(t)\, s_i(t)\, dt$$

Correlation receiver

ML criterion based on correlation

$$\underline{s_R} = \arg\max_{\underline{s}_i \in M} \left[\int_0^T \rho(t)\, s_i(t)\, dt - \frac{1}{2} E(s_i) \right]$$

Correlation receiver



Receiver comparison

The matched filter receiver has

- d filters
- a Euclidean distance (Voronoi) decisor

The correlation receiver has

- m integrators ($m>d$)
- a max decisor

Example

$$M = \{s_1(t) = P_T(t), s_2(t) = -P_T(t)\}$$

1. Draw the transmitted waveform corresponding to $\underline{u}_{\mathrm{T}}$=101010….

2. Determine the matched filter

3. Draw the matched filter output waveform (no noise)

4. Verify that the samples (at $t=(n+1)T$) coincide with the transmitted symbols

LECTURE 5: ERROR PROBABILITY COMPUTATION

INTRODUCTION TO ERROR PROBABILITY COMPUTATION

To measure the goodness of a digital radio link: error probability

SYMBOL ERROR RATE = SER = $P_s(e)$ =

$$P_S(e) = P(\underline{s}_R[n] \neq \underline{s}_T[n])$$

BIT ERROR RATE = BER = $P_b(e)$ =

$$P(u_R[i] \neq u_T[i])$$

INTRODUCTION TO ERROR PROBABILITY COMPUTATION

R_b
Bit Rate

$T_b = 1/ R_b$
Bit time

$T = k\, T_b$
Symbol time

$R = 1/T$
Symbol Rate

INTRODUCTION TO ERROR PROBABILITY COMPUTATION

E_b
Energy per information bit

E_S
Energy per transmitted signal

$S = E_b R_b = E_S R$
Signal power

INTRODUCTION TO ERROR PROBABILITY COMPUTATION

N_0
Noise spectral density

B
Signal Bandwidth

$N = N_0 B$
Noise power

INTRODUCTION TO ERROR PROBABILITY COMPUTATION

S/N

Signal to Noise ratio

E_b/N_0

Signal to Noise ratio referred to an information bit

Connection:

$$\frac{S}{N} = \frac{E_b}{N_0}\frac{R_b}{B} = \frac{E_b}{N_0}\eta$$

where $\eta = \frac{R_b}{B}$ **spectral efficiency**

INTRODUCTION TO ERROR PROBABILITY COMPUTATION

The system performance are expressed as a function of E_b/N_0

This ratio is proportional to the received power

$$S = \frac{S}{N} N = \frac{E_b}{N_0} \frac{R_b}{B} N_0 B = \frac{E_b}{N_0} R_b N_0$$

SER COMPUTATION

Definition: $P_S(e) = P(\underline{s}_R \neq \underline{s}_T)$

We have

$$P_S(e) = \sum_{i=1}^{m} P_S(e \mid \underline{s_T} = \underline{s_i}) P(\underline{s_T} = \underline{s_i}) = \frac{1}{m} \sum_{i=1}^{m} P_S(e \mid \underline{s_T} = \underline{s_i})$$

We have to compute

$$P_S(e \mid \underline{s_T} = \underline{s_i}) = P(\underline{s_R} \neq \underline{s_T} \mid \underline{s_T} = \underline{s_i})$$

SER COMPUTATION

First formulation:

$$P_S(e \mid \underline{s_T} = \underline{s_i}) = P(\underline{s_R} \neq \underline{s_T} \mid \underline{s_T} = \underline{s_i}) = 1 - P(\underline{s_R} = \underline{s_T} \mid \underline{s_T} = \underline{s_i}) =$$

$$= 1 - P(\underline{\rho} \in V(\underline{s_i}) \mid \underline{s_T} = \underline{s_i})$$

Second formulation:

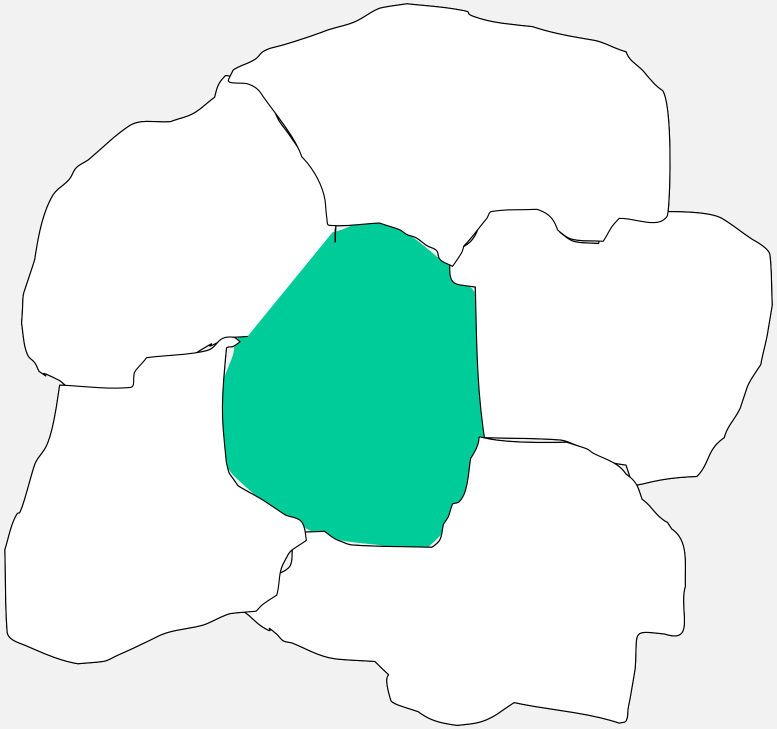
$$P_S(e \mid \underline{s_T} = \underline{s_i}) = P(\underline{s_R} \neq \underline{s_T} \mid \underline{s_T} = \underline{s_i}) = P(\underline{\rho} \notin V(\underline{s_i}) \mid \underline{s_T} = \underline{s_i}) =$$

$$= \sum_{j \neq i} P(\underline{s_R} = \underline{s_i} \mid \underline{s_T} = \underline{s_i}) = \sum_{j \neq i} P(\underline{\rho} \in V(\underline{s_j}) \mid \underline{s_T} = \underline{s_i})$$

SER COMPUTATION

First formulation

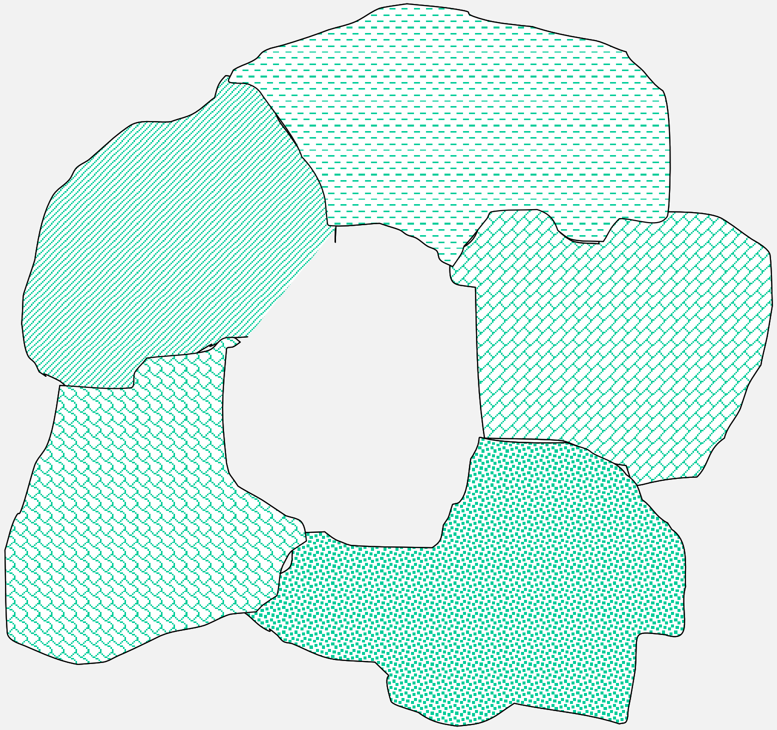
$$P_S(e \mid \underline{s_T} = \underline{s_i}) = 1 - P(\underline{\rho} \in V(\underline{s_i}) \mid \underline{s_T} = \underline{s_i})$$



SER COMPUTATION

Second formulation

$$P_S(e \mid \underline{s_T} = \underline{s_i}) = P(\underline{\rho} \notin V(\underline{s_i}) \mid \underline{s_T} = \underline{s_i}) = \sum_{j \neq i} P(\underline{\rho} \in V(\underline{s_j}) \mid \underline{s_T} = \underline{s_i})$$



BER COMPUTATION

When the received signal is correct ($\underline{s}_R = \underline{s}_T$), also the information vector is correct ($\underline{v}_R = \underline{v}_T$).

When the received signal is wrong ($\underline{s}_R \neq \underline{s}_T$), the binary information vector is certainly wrong ($\underline{v}_R \neq \underline{v}_T$), but the number of information bits depends on the labeling and is given by

$$\frac{d_H(\underline{v}_R, \underline{v}_T)}{k}$$

BER COMPUTATION

We have

$$P_b(e) = \frac{1}{m}\sum_{i=1}^{m} P_b(e \,|\, \underline{s_T} = \underline{s_i})$$

where

$$P_b(e \,|\, \underline{s_T} = \underline{s_i}) = \sum_{j \neq i} P_b(e, \underline{s_R} = \underline{s_j} \,|\, \underline{s_T} = \underline{s_i}) =$$

$$= \sum_{j \neq i} \frac{d_H(\underline{v_j}, \underline{v_i})}{k} P(\underline{s_R} = \underline{s_j} \,|\, \underline{s_T} = \underline{s_i}) =$$

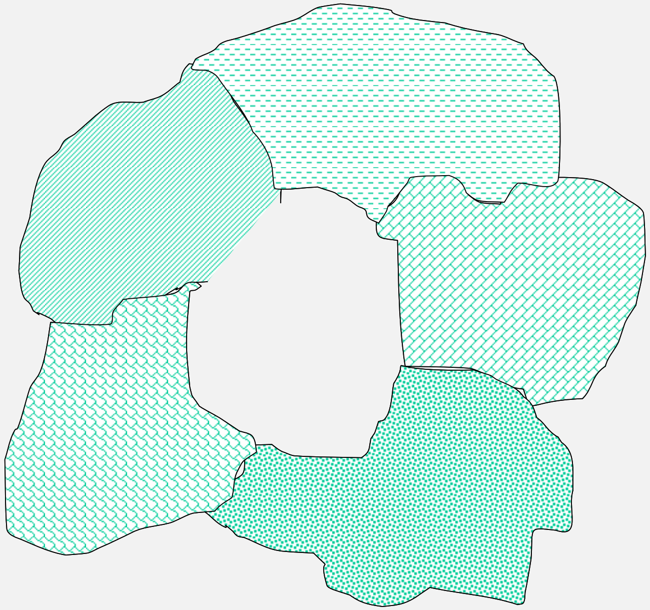
$$= \sum_{j \neq i} \frac{d_H(\underline{v_j}, \underline{v_i})}{k} P(\underline{\rho} \in V(\underline{s_j}) \,|\, \underline{s_T} = \underline{s_i})$$

$$\left[\text{where } \underline{v_i} = e^{-1}\left(\underline{s_i}\right) \text{ and } \underline{v_j} = e^{-1}\left(\underline{s_j}\right)\right]$$

BER COMPUTATION

$$P_b(e) = \frac{1}{m}\sum_{i=1}^{m} P_b(e \mid \underline{s_T} = \underline{s_i})$$

$$P_b(e \mid \underline{s_T} = \underline{s_i}) = \sum_{j \neq i} \frac{d_H(\underline{v_j}, \underline{v_i})}{k} \; P(\underline{\rho} \in V(\underline{s_j}) \mid \underline{s_T} = \underline{s_i})$$



ERFC

Given a Gaussian random variable n with

- mean μ
- variance σ^2
- density

$$f_n(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

We have

$$P(n > x) = \int_x^{+\infty} f_n(x)\, dx = \frac{1}{2} erfc\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)$$

ERFC

where

$$erfc(x) = \frac{2}{\sqrt{\pi}} \int_x^{+\infty} e^{-t^2} dt$$

In fact

$$P(n > x) = \int_x^{+\infty} f_n(x)\, dx = \int_x^{+\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx =$$

$$\frac{1}{\sqrt{\pi}} \int_{\frac{(x-\mu)}{\sqrt{2}\sigma}}^{+\infty} e^{-t^2} dt = \frac{1}{2} erfc\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)$$

In case of zero mean and variance $N_0/2$

$$P(n > x) = \frac{1}{2} erfc\left(\frac{x-\mu}{\sqrt{2}\sigma}\right) = \frac{1}{2} erfc\left(\frac{x}{\sqrt{N_0}}\right)$$

SER/BER COMPUTATION FOR BINARY ANTIPODAL SIGNALS

Let us consider a one-dimensional constellation (d=1) composed by two signals (m=2), symmetrical with respect to the origin:

$$M = \{\underline{s_1} = (+A) \;\; \underline{s_2} = (-A) \;\}$$

The Voronoi regions are:

$$V(\underline{s_1}) = \{\underline{\rho} = (\rho_1) \;\; , \;\; \rho_1 \geq 0\,\}$$

$$V(\underline{s_2}) = \{\underline{\rho} = (\rho_1) \;\; , \;\; \rho_1 \leq 0\,\}$$

SER/BER COMPUTATION FOR BINARY ANTIPODAL SIGNALS

We have:

$$P_S(e) = \frac{1}{m}\sum_{i=1}^{m} P_S(e \mid \underline{s_T} = \underline{s_i}) = \frac{1}{2}\left[P_S(e \mid \underline{s_T} = \underline{s_1}) + P_S(e \mid \underline{s_T} = \underline{s_2}) \right]$$

Let us compute

$$P_S(e \mid \underline{s_T} = \underline{s_1})$$

and

$$P_S(e \mid \underline{s_T} = \underline{s_2})$$

SER/BER COMPUTATION FOR BINARY ANTIPODAL SIGNALS

$$P_S(e \mid \underline{s_T} = \underline{s_1}) = P(\underline{\rho} \in V(\underline{s_2}) \mid \underline{s_T} = \underline{s_1}) = P(\rho_1 < 0 \mid \underline{s_T} = \underline{s_1})$$

We have:

$$\boxed{\underline{r} = \underline{s_T} + \underline{n} \qquad \underline{r} = \underline{\rho} \qquad \underline{s_T} = \underline{s_1}}$$

where $\underline{\rho} = (\rho_1)$ $\quad \underline{s_1} = (s_{11}) = (+A)$ $\quad \underline{n} = (n_1)$

It follows:

$$\rho_1 = A + n_1$$

SER/BER COMPUTATION FOR BINARY ANTIPODAL SIGNALS

$$P_S(e\,|\,\underline{s_T}=\underline{s_1}) = P(\rho_1 < 0\,|\,\underline{s_T}=\underline{s_1}) = P(A+n_1<0) = P(n_1<-A)$$

The random variable n_1 is Gaussian, with mean zero and variance $N_0/2$

$$P_S(e\,|\,\underline{s_T}=\underline{s_1}) = P(n_1<-A) = P(n_1>A) = \frac{1}{2}erfc\left(\frac{A}{\sqrt{N_0}}\right)$$

SER/BER COMPUTATION FOR BINARY ANTIPODAL SIGNALS

Compute now the error probbaility for $\underline{s}_T = \underline{s}_2$

$$P_S(e \mid \underline{s}_T = \underline{s}_2) = P(\underline{\rho} \in V(\underline{s}_1) \mid \underline{s}_T = \underline{s}_2) = P(\rho_1 > 0 \mid \underline{s}_T = \underline{s}_2)$$

We have:

$$\underline{r} = \underline{s}_T + \underline{n} \qquad \underline{r} = \underline{\rho} \qquad \underline{s}_T = \underline{s}_2$$

then

$$\underline{\rho} = (\rho_1) \qquad \underline{s}_2 = (s_{21}) = (-A) \qquad \underline{n} = (n_1)$$

$$\rho_1 = -A + n_1$$

SER/BER COMPUTATION FOR BINARY ANTIPODAL SIGNALS

$$P_S(e \mid \underline{s_T} = \underline{s_2}) = P(-A + n_1 > 0) = P(n_1 > A)$$

$$P_S(e \mid \underline{s_T} = \underline{s_2}) = \frac{1}{2} erfc\left(\frac{A}{\sqrt{N_0}}\right)$$

SER/BER COMPUTATION FOR BINARY ANTIPODAL SIGNALS

We have
$$P_S(e \mid \underline{s_T} = \underline{s_1}) = P_S(e \mid \underline{s_T} = \underline{s_2})$$

It follows:
$$P_S(e) = \frac{1}{2}\left[P_S(e \mid \underline{s_T} = \underline{s_1}) + P_S(e \mid \underline{s_T} = \underline{s_2})\right] = P_S(e \mid \underline{s_T} = \underline{s_1})$$

then
$$P_S(e) = P_S(e \mid \underline{s_T} = \underline{s_1}) = \frac{1}{2} erfc\left(\frac{A}{\sqrt{N_0}}\right)$$

[note that

$$P_S(e) = P_S(e \mid \underline{s_T} = \underline{s_1}) = \frac{1}{2} erfc\left(\frac{d}{2\sqrt{N_0}}\right)$$

SER/BER COMPUTATION FOR BINARY ANTIPODAL SIGNALS

We have:

$$P_S(e) = P_S(e \mid \underline{s_T} = \underline{s_1}) = \frac{1}{2} erfc\left(\frac{A}{\sqrt{N_0}} \right)$$

We want to write it as a function of E_b/N_0.

$$E(\underline{s_1}) = E(\underline{s_2}) = A^2$$

$$E_S = \frac{E(\underline{s_1}) + E(\underline{s_2})}{2} = A^2$$

$$E_b = \frac{E_S}{k} = E_S = A^2$$

SER/BER COMPUTATION FOR BINARY ANTIPODAL SIGNALS

Fundamental result

$$P_S(e) = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{N_0}}\right)$$

SER/BER COMPUTATION FOR BINARY ANTIPODAL SIGNALS

For this constellation we can establish this binary labeling:

$$e : H_1 \Leftrightarrow M$$

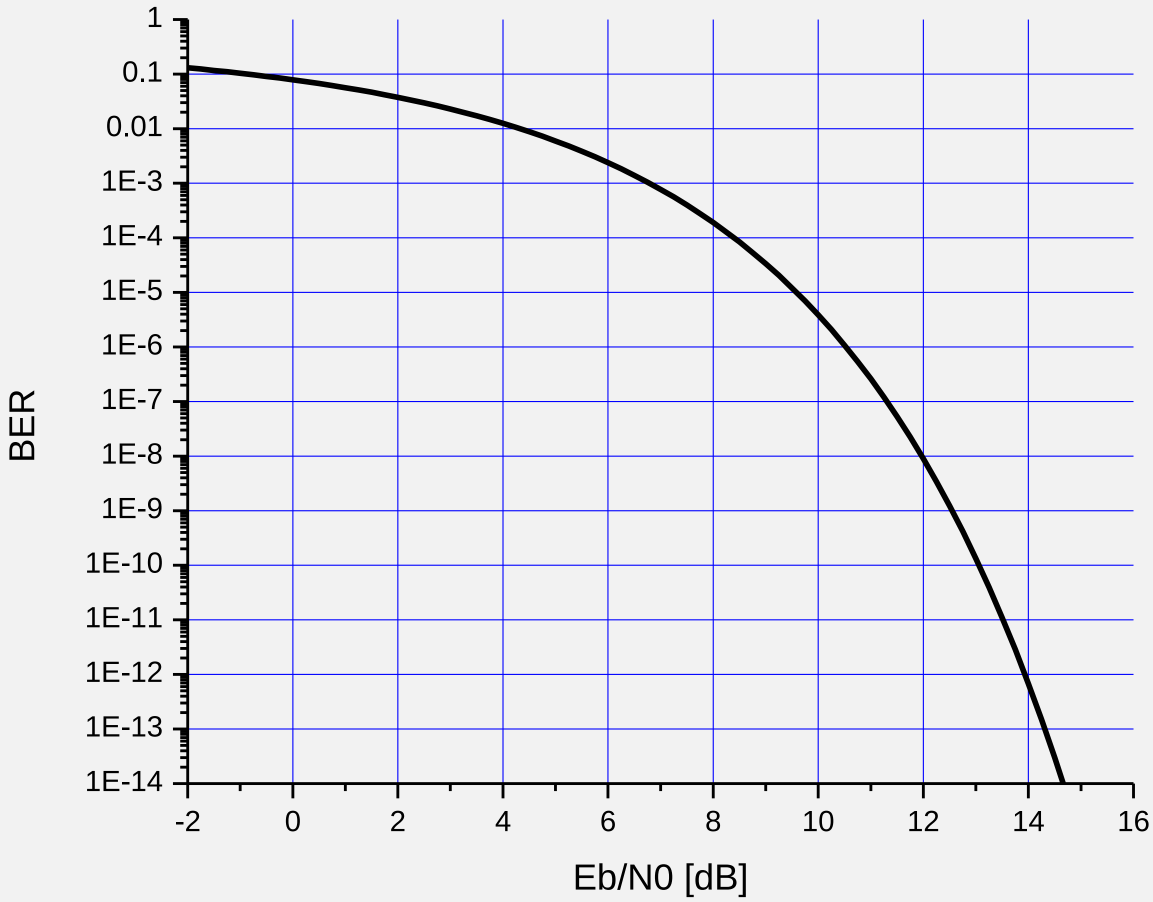
$$\underline{v_1} = (0) \Leftrightarrow \underline{s_1}$$

$$\underline{v_2} = (1) \Leftrightarrow \underline{s_2}$$

In this case, if the received signal is wrong, the information bit is wrong, too.

$$P_b(e) = P_S(e) = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{N_0}}\right)$$

SER/BER computation for binary antipodal signals



SER/BER COMPUTATION FOR BINARY ANTIPODAL SIGNALS

Different signal constellations (different transmitted waveforms) with the same vector constellations achieve equal BER performance!

As an example, the BER performance of a binary antipodal constellation does not depend on the versor

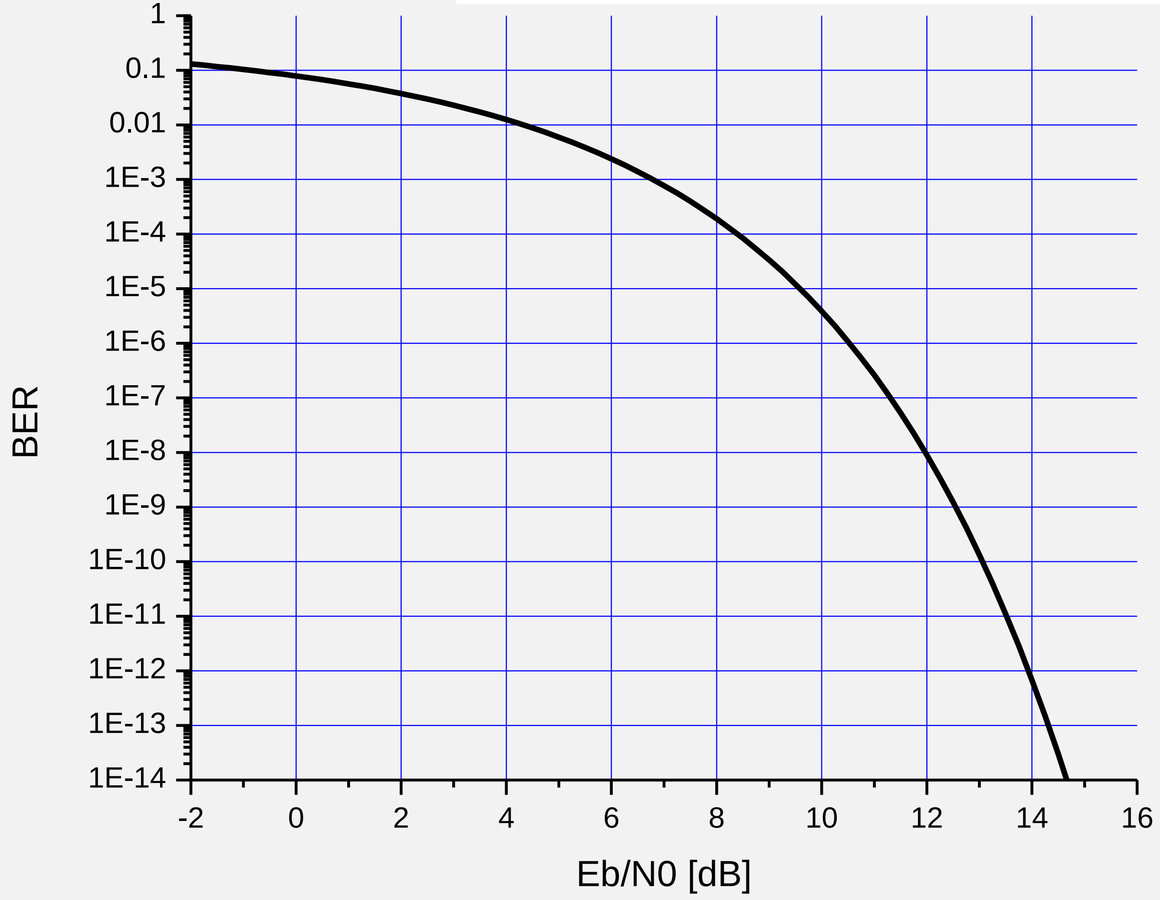
$$b_1(t) = \frac{1}{\sqrt{T}} P_T(t)$$

$$b_1(t) = \sqrt{\frac{2}{T}} P_T(t) \cos(2\pi f_0 t)$$

LECTURE 6: SPECTRAL ANALYSIS

SER/BER for (binary antipodal signal)

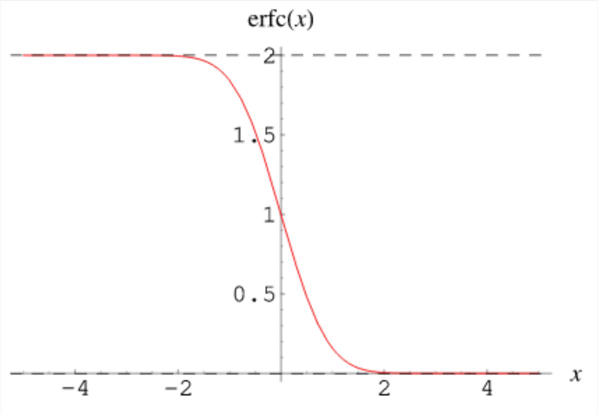
$$P_b(e) = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{N_0}}\right)$$



BER COMPARISON

Error probability of antipodal signals and orthogonal signals:

$$P_b(e)\,|_{antipodal} = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{N_0}}\right) \qquad P_b(e)\Big|_{orthogonal} = \frac{1}{2} erfc\left(\sqrt{\frac{1}{2}\frac{E_b}{N_0}}\right)$$

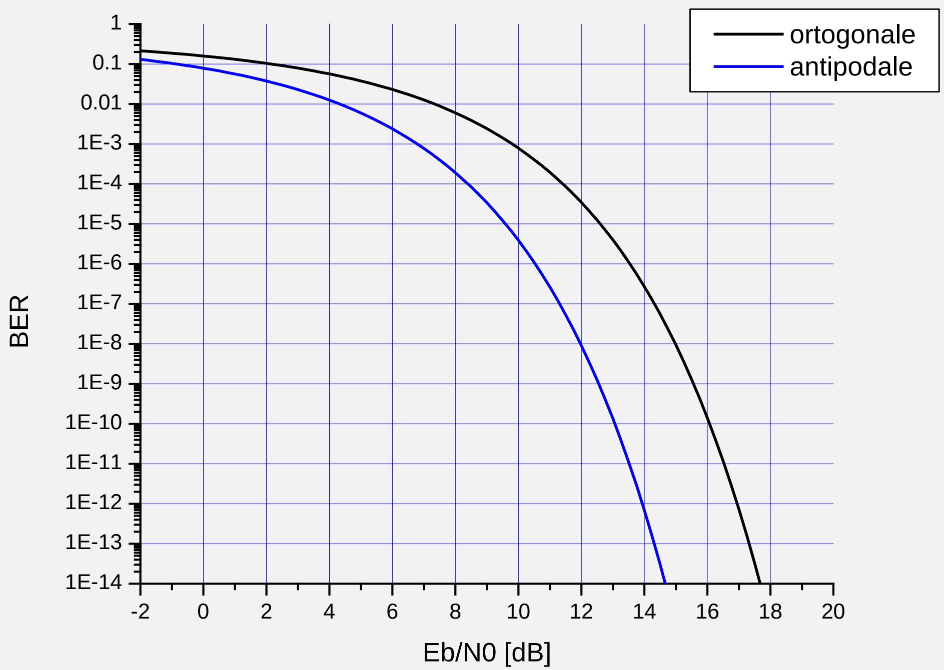


- Antipodal signals give lower error probability
- For a given BER, antipodal signals request E_b/N_0 lower than orthogonal signals
- For given E_b/N_0 , the system has lower BER

BER COMPARISON

$$P_b(e)\,|_{antipodal} = \frac{1}{2}\,erfc\left(\sqrt{\frac{E_b}{N_0}}\right)$$

$$P_b(e)\big|_{orthogonal} = \frac{1}{2}\,erfc\left(\sqrt{\frac{1}{2}\frac{E_b}{N_0}}\right)$$



BER COMPARISON

With E_b/N_0 =12 dB:

- Antipodal signals give $P_b(e)$ =1e-8
- Orthogonal signals give $P_b(e)$ =5e-5 (Higher error probability → Lower system performance)

To achieve $P_b(e)$=1e-6:

- Antipodal signals require: E_b/N_0 = 10.6 dB;
- Orthogonal signals require: E_b/N_0 =13.6 dB

(Antipodal signals gains 3 dB compared to orthogonal signals. Note that: Eb/No is related to the received power)

BER COMPARISON

Example: For a communication environment with

$$P_R = P_T \frac{G_T G_R}{\left(\frac{4\pi d}{\lambda}\right)^2}$$

The system with antipodal signals has $P_b(e)$=1e-6 ➔ The received signals are ½ that of orthogonal signals

With the same transmit power, the propagation distance corresponding the antipodal signals are $\sqrt{2}$ longer than that of the orthogonal signals

For a given propagation distance, the network can reduce ½ the transmit power if the network uses antipodal signals instead of orthogonal signals

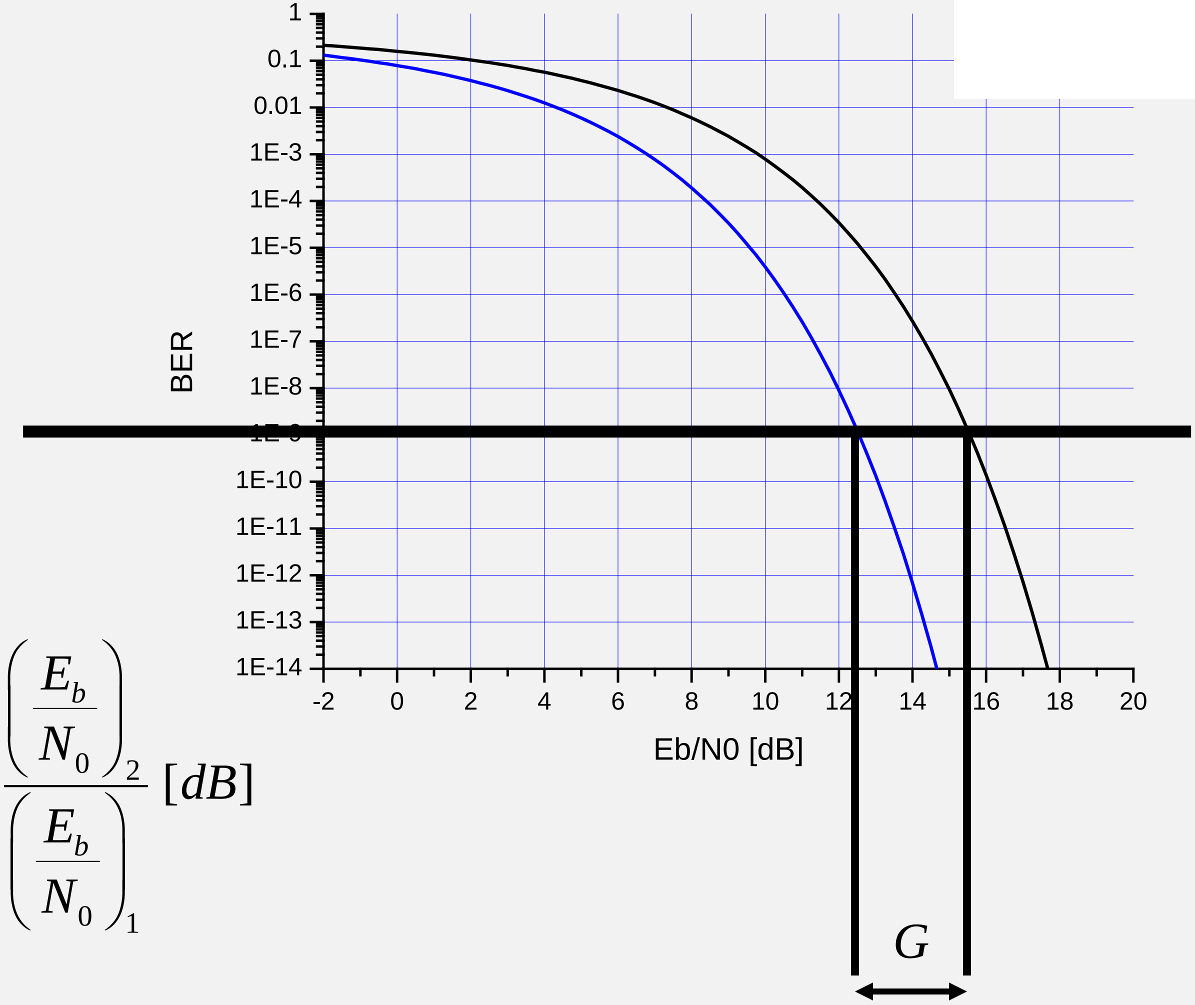
BER COMPARISON

For two constellation sets M_1 and M_2 with

$$P_b(e)|_1 \approx erfc\left(\sqrt{y_1 \frac{E_b}{N_0}}\right) \qquad P_b(e)|_2 \approx erfc\left(\sqrt{y_2 \frac{E_b}{N_0}}\right)$$

If $y_1 > y_2$, then M_1 performs better than M_2 (lower BER)

BER COMPARISON



$$G = 10\log_{10}\frac{\left(\dfrac{E_b}{N_0}\right)_2}{\left(\dfrac{E_b}{N_0}\right)_1}\ [dB]$$

ASYMPTOTIC PERFORMANCE

- As $E_b/N_o \rightarrow \infty$, the variance of noise is extremely small.
- Most of the received signal belongs to the correct Voronoi region.
- Once errors appears, it falls in the neighbour Voronoi regions with an overwhelming probability.
- The asymptotic performance is approximated as follows

$$\boxed{P_S(e) \approx \frac{1}{2} A_{\min} erfc\left(\sqrt{\frac{d_{\min}^2}{4N_0}}\right)}$$

$$\boxed{d_{\min} = \min_{\underline{s}_1\, \underline{s}_j \in M} d_E(\underline{s}_1, \underline{s}_j)}$$

$$\boxed{A_{\min} = \text{multiplicity=number of signals } \underline{s}_j \text{ with } d_{\mathrm{E}}(\underline{s}_j\ \underline{s}_1) = d_{\min}}$$

ASYMPTOTIC PERFORMANCE

BER performance:

$$\boxed{P_b(e) \approx \frac{1}{2}\frac{w_{\min}}{k}\, erfc\left(\sqrt{\frac{d_{\min}^2}{4N_0}}\right)}$$

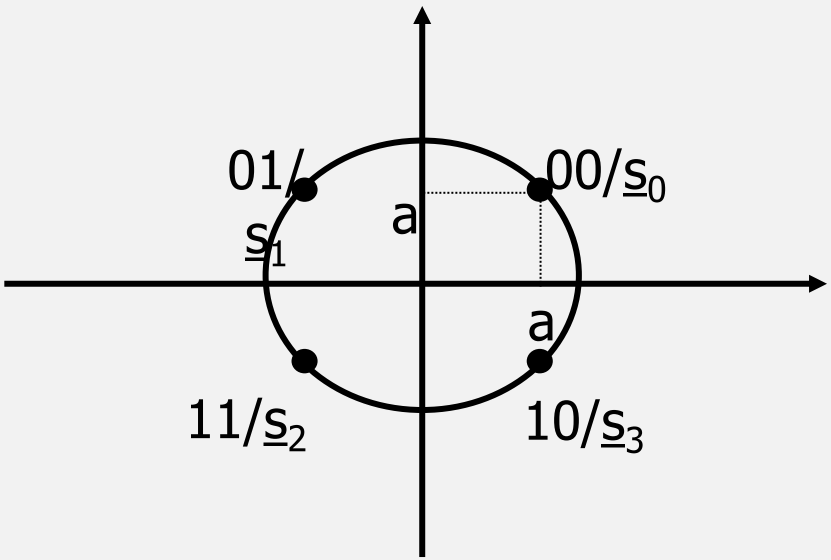
Where:

$$\boxed{w_{\min} = \text{input multiplicity} = \sum_{\underline{s_j}:d_E(\underline{s_1},\underline{s_j})=d_{\min}} d_H(\underline{v_1}, \underline{v_j})}$$

➔ All these formulations are upper bounds and the approximations for high SNR values

EXAMPLE

For 4-PSK



$$d_{\min} = 2a \qquad A_{\min} = 2 \qquad w_{\min} = 2$$

$$P_S(e) \approx erfc\left(\sqrt{\frac{d_{\min}^2}{4N_0}}\right) = erfc\left(\sqrt{\frac{E_b}{N_0}}\right)$$

$$P_b(e) \approx \frac{1}{2} erfc\left(\sqrt{\frac{d_{\min}^2}{4N_0}}\right) = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{N_0}}\right)$$

GRAY LABELLING

Asymptotic BER performance

$$P_b(e) \approx \frac{1}{2}\frac{w_{\min}}{k}\, erfc\left(\sqrt{\frac{d_{\min}^2}{4N_0}}\right)$$

One has $A_{\min} \le w_{\min}$

$$\boxed{A_{\min} = \text{multiplicity=number of signals } \underline{s}_j \text{ with } d_E(\underline{s}_j\ \underline{s}_1) = d_{\min}}$$

$$\boxed{w_{\min} = \text{input multiplicity=} \sum_{\underline{s}_j : d_E(\underline{s}_1, \underline{s}_j) = d_{\min}} d_H(\underline{v}_1, \underline{v}_j)}$$

For the optimality:

$$A_{\min} = w_{\min}$$

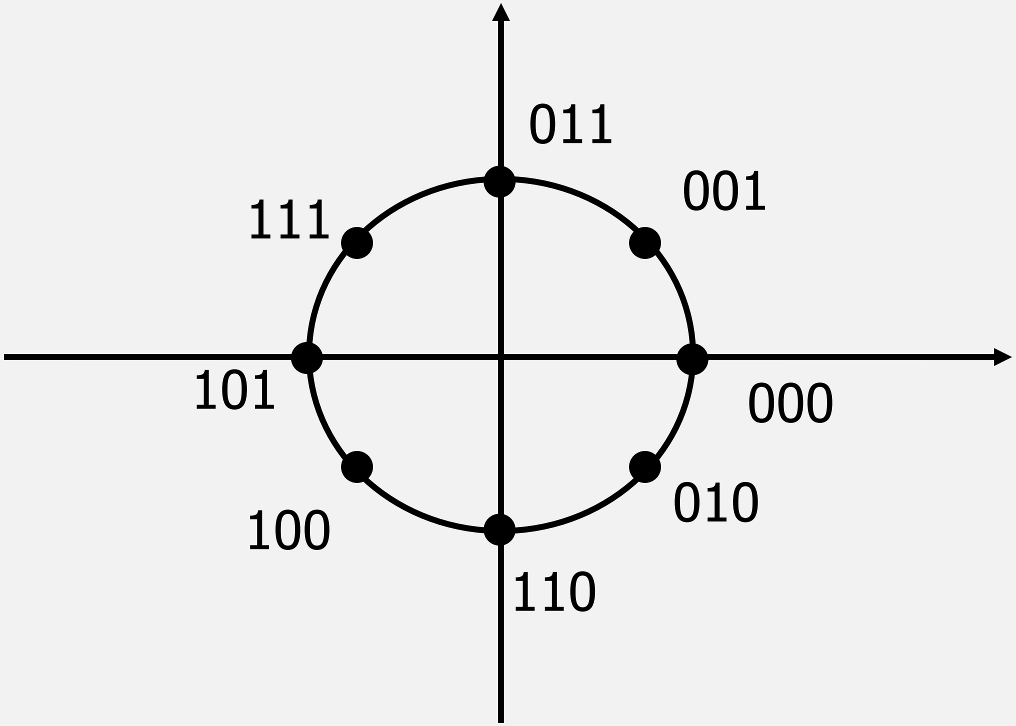
GRAY LABELLING

For given signal $\underline{s}_i$, its corresponding binary vector $\underline{v}_i = e^{-1}(\underline{s}_i)$.

"***adjacent***" $\underline{s}_i$ (minimum distance d_{min} to $\underline{s}_i$) has its binary vector with Hamming distance 1 compared wih $\underline{v}_i$ ..

➔ By this way, the asymptotic BER can be minimized

EXAMPLE



Signal Spectrum

PROPERTIES OF SIGNAL SPECTRUM

We want to study signal properties of $s(t)$ in the frequency domain by considering power spectral density $G_s(f)$, then define a suitable bandwidth (frequency band that contains almost information of $G_s(f)$)

ANTIPODAL SIGNAL

Consider the signal space:

$$M = \{ s_1(t) = +AP_T(t)\,,\, s_2(t) = -AP_T(t) \}$$

It is 1D signal space(d=1) with the basis:

$$B = \left\{ b_1(t) = \frac{1}{\sqrt{T}} P_T(t) \right\}$$

Hence, the signals in M can be represented as:

$$M = \{ \underline{s_1} = (+\alpha)\,,\, \underline{s_2} = (-\alpha) \} \qquad \alpha = A\sqrt{T}$$

ANTIPODAL SIGNAL

We consider the transmitted signal:

In the first interval *[0,T[*, the system sends the signal

$$s_1(t) = +\alpha b_1(t) \quad \text{or} \quad s_2(t) = -\alpha b_1(t)$$

In the (n+1)-th interval *[nT,(n+1)T[*, the system sends the signal

$$s_1(t-nT) = +\alpha b_1(t-nT) \quad \text{or} \quad s_2(t-nT) = -\alpha b_1(t-nT)$$

ANTIPODAL SIGNAL

The transmitted signals can be formulated as follows

$$\boxed{s(t) = \sum_{n=0}^{+\infty} a[n]p(t - nT)}$$

$$\boxed{a[n] \in \{+\alpha, -\alpha\}}$$

$$\boxed{p(t) = b_1(t)}$$

ANTIPODAL SIGNAL

For all the 1D signal space:

$$M = \{\underline{s_1} = (\alpha_1), \underline{s_2} = (\alpha_2), \ldots, \underline{s_m} = (\alpha_m)\} \subseteq R$$

with the basis vector: $b_1(t)$

The transmitted signal:

$$\boxed{s(t) = \sum_{n=0}^{+\infty} a[n]p(t - nT)}$$

$$\boxed{a[n] \in \{\alpha_1, \ldots, \alpha_i, \ldots, \alpha_m\}}$$

$$\boxed{p(t) = b_1(t)}$$

ANTIPODAL SIGNAL

a[n] has the stationary property of random variables, all symbols are independent with the probability:

$$P\left(a[n]=\alpha_i\right)=\frac{1}{m}$$

Mean
$$\mu_a=\frac{1}{m}\sum_{i=1}^{m}\alpha_i$$

Variance
$$\sigma_a^2=\frac{1}{m}\sum_{i=1}^{m}\left(\alpha_i-\mu_a\right)^2$$

WAVEFORM

We now compute the spectral density function for the signal in the form

$$s(t) = \sum_{n=-\infty}^{+\infty} a[n]p(t - nT)$$

$s(t)$ is a random process, the spectral density function is computed as

$$G_s(f) = \int_{-\infty}^{+\infty} R_s(\tau)\, e^{-j2\pi f \tau} d\tau$$

This value provides the information of the power distribution in the frequency domain:

$$P(s) = R_s(0) = \int_{-\infty}^{+\infty} G_s(f)\, df$$

THEORY OF SPECTRAL DENSITY

Consider a random process $$s(t) = \sum_{n=-\infty}^{+\infty} a[n]p(t-nT)$$

Where

- $a[n]$ is a stationary sequence of random variables with

  $$M_a(i)=E[a[n]a[n+i]]$$

- $p(t)$ is real signal with its Fourier transform version $P(f)$

Spectral density function: $$\boxed{G_s(f) = S_a(f)\frac{|P(f)|^2}{T}}$$

Where $$S_a(f) = \sum_i M_a(i)e^{-j2\pi fiT}$$

PROOF (FOR REFERENCE)

$$G_s(f) = \int_{-\infty}^{+\infty} R_s(\tau) e^{-j2\pi f \tau} d\tau$$

For a random process, one can compute

$$R_s(\tau) = \frac{1}{T} \int_0^T M_{SS}(t+\tau, t)\, dt$$

PROOF (FOR REFERENCE)

$$M_{SS}(t+\tau,t)=E[S(t+\tau)S(t)]=E\left[\sum_{m=-\infty}^{+\infty}a[m]p(t+\tau-mT)\sum_{n=-\infty}^{+\infty}a[n]p(t-nT)\right]=$$

$$=\sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty}E\big[a[m]a[n]\big]p(t+\tau-mT)\ p(t-nT)=$$

$$=\sum_{m=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty}M_a(m-n)p(t+\tau-mT)p(t-nT)=$$

For $i=(m-n)$ $[\ m = n+i\ ]$

$$\boxed{M_{SS}(t+\tau,t)=\sum_{i=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty}M_a(i)p(t+\tau-nT-iT)p(t-nT)}$$

PROOF (FOR REFERENCE)

$$M_{SS}(t+\tau,t)=\sum_{i=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} M_a(i)p(t+\tau-nT-iT)p(t-nT)$$

$$R_s(\tau)=\frac{1}{T}\int_0^T M_{SS}(t+\tau,t)\,dt=\frac{1}{T}\int_0^T\left[\sum_{i=-\infty}^{+\infty}\sum_{n=-\infty}^{+\infty} M_a(i)p(t+\tau-nT-iT)p(t-nT)\right]dt=$$

$$=\frac{1}{T}\sum_{i=-\infty}^{+\infty} M_a(i)\sum_{n=-\infty}^{+\infty}\int_0^T\left[p(t+\tau-nT-iT)p(t-nT)\right]dt=$$

For $t'=(t-nT)$

$$=\frac{1}{T}\sum_{i=-\infty}^{+\infty} M_a(i)\sum_{n=-\infty}^{+\infty}\int_{-nT}^{-(n-1)T}\left[p(t'+\tau-iT)p(t')\right]dt'$$

$$\boxed{R_s(\tau)=\frac{1}{T}\sum_{i=-\infty}^{+\infty} M_a(i)\int_{-\infty}^{+\infty}\left[p(t'+\tau-iT)p(t')\right]dt'}$$

PROOF (FOR REFERENCE)

$$R_s(\tau) = \frac{1}{T}\sum_{i=-\infty}^{+\infty} M_a(i) \int_{-\infty}^{+\infty} \left[p(t'+\tau - iT)p(t') \right] dt'$$

$$G_s(f) = \int_{-\infty}^{+\infty} R_s(\tau) e^{-j2\pi f\tau} d\tau = \int_{-\infty}^{+\infty} \left[\frac{1}{T}\sum_{i=-\infty}^{+\infty} M_a(i) \int_{-\infty}^{+\infty} \left[p(t'+\tau - iT)p(t') \right] dt' \right] e^{-j2\pi f\tau} d\tau =$$

$$= \frac{1}{T}\sum_{i=-\infty}^{+\infty} M_a(i) \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left[p(t'+\tau - iT)p(t')e^{-j2\pi f\tau} \right] dt'\, d\tau =$$

$$t'' = t' + \tau - iT \qquad\qquad \tau = t'' - t' + iT$$

$$e^{-j2\pi f\tau} = e^{-j2\pi ft''}\, e^{j2\pi ft'}\, e^{-j2\pi fiT}$$

$$G_s(f) = \frac{1}{T}\sum_{i=-\infty}^{+\infty} M_a(i) \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left[p(t'')p(t')e^{-j2\pi ft''}\, e^{j2\pi ft'}\, e^{-j2\pi fiT} \right] dt'\, dt'' =$$

PROOF (FOR REFERENCE)

$$G_s(f)=\frac{1}{T}\sum_{i=-\infty}^{+\infty} M_a(i)\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[p(t'')p(t')e^{-j2\pi ft''}e^{j2\pi ft'}e^{-j2\pi fiT}\right]dt'\,dt''=$$

$$=\frac{1}{T}\sum_{i=-\infty}^{+\infty} M_a(i)e^{-j2\pi fiT}\int_{-\infty}^{+\infty} p(t'')\,e^{-j2\pi ft''}dt''\int_{-\infty}^{+\infty} p(t')\;e^{j2\pi ft'}\,dt'=$$

$$=\frac{1}{T}S_a(f)P(f)P^*(f)$$

$$\boxed{G_v(f)=S_a(f)\,\frac{\left|P(f)\right|^2}{T}}$$

STATISTICALLY INDEPENDENT SYMBOLS WITH ZERO MEAN

<u>Fundamental case study</u>

Let us suppose the sequence $a[n]$ is characterized by:

- Statistically independent symbols
- Zero mean: $\mu_a = 0$

THIS CORRESPONDS TO A ONE-DIMENSIONAL CONSTELLATION WITH CENTRE OF MASS IN THE ORIGIN

STATISTICALLY INDEPENDENT SYMBOLS WITH ZERO MEAN

We have

$$\text{for } i \neq 0 \quad M_a(i) = E(a[n+1]a[n]) = E(a[n+1])E(a[n]) = 0$$

$$\text{for } i = 0 \quad M_a(i) = E\left(a[n]^2\right) = \sigma_a^2$$

STATISTICALLY INDEPENDENT SYMBOLS WITH ZERO MEAN

For:
$$S_a(f) = \sum_i M_a(i) e^{-j2\pi fiT} = \sigma_a^2$$

Spectral density: $G_s(f) = S_a(f) \dfrac{|P(f)|^2}{T}$

Another expression (for simplicity): $$\boxed{G_s(f) = \sigma_a^2 \frac{|P(f)|^2}{T}}$$

For an antipodal signal with centre-of-mass in the origin, the $|P(f)|^2$

EXAMPLE

Assume that:

$$M = \{s_1(t) = +AP_T(t)\,,\, s_2(t) = -AP_T(t)\,\}$$

It is 1D signal space (d=1) with the basis

$$B = \left\{ b_1(t) = \frac{1}{\sqrt{T}} P_T(t) \right\}$$

The signals are expressed as follows

$$M = \{\underline{s_1} = (+\alpha)\,,\, \underline{s_2} = (-\alpha)\} \qquad \alpha = A\sqrt{T}$$

EXAMPLE

The transmitted waveform: $$s(t) = \sum_{n=0}^{+\infty} a[n]p(t - nT)$$

with $$a[n] \in \{+\alpha, -\alpha\}$$ $$p(t) = b_1(t)$$

For the sequence $a[n]$, one obtains

Mean $$\mu_a = 0.5\,(-\alpha + \alpha) = 0$$

Variance $$\sigma_a^2 = 0.5\,(\alpha^2 + \alpha^2) = \alpha^2 = A^2 T$$

EXAMPLE

Sequence $a[n]$ includes independent random variables with zero mean

The spectral density function is

$$G_s(f) = \sigma_a^2 \frac{|P(f)|^2}{T}$$

EXAMPLE

We have: $$p(t) = b_1(t) = \frac{1}{\sqrt{T}} P_T(t)$$

Let us introduce: *sinc(x):*

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{(\pi x)}$$

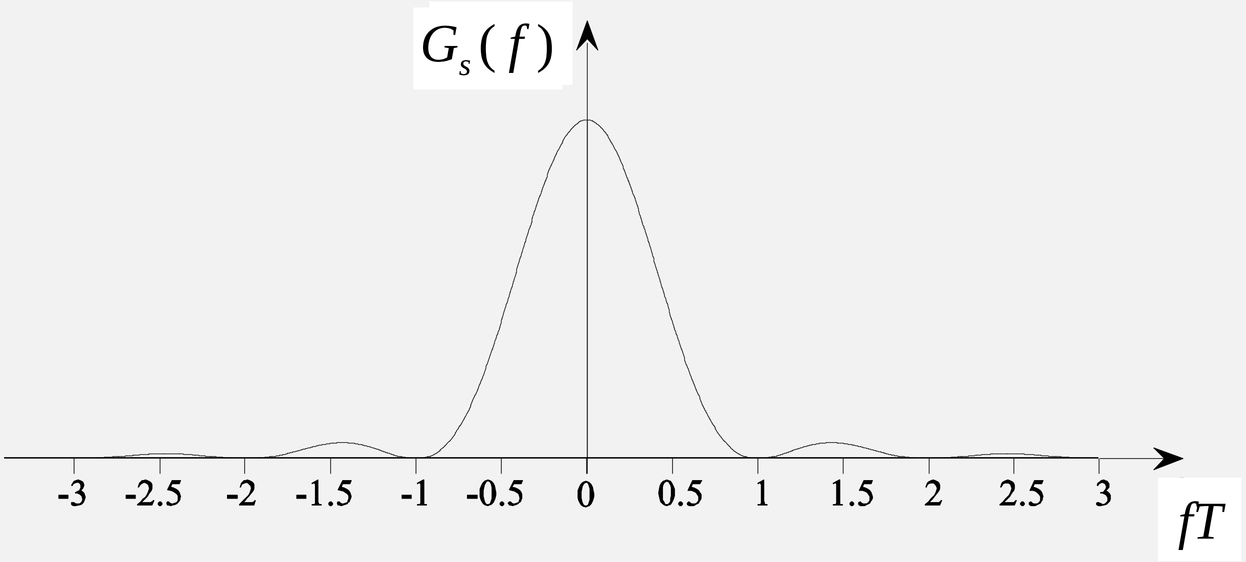
The Fourier transform of *s(t)* is:

$$P(f) = \sqrt{T}\,\mathrm{sinc}(fT)e^{-j\pi f\mathrm{T}} = \sqrt{T}\,\frac{\sin(\pi fT)}{(\pi fT)}e^{-j\pi f\mathrm{T}}$$

EXAMPLE

PSD of this signal space:

$$\boxed{G_s(f) = A^2 T \left(\frac{\sin(\pi fT)}{(\pi fT)} \right)^2}$$

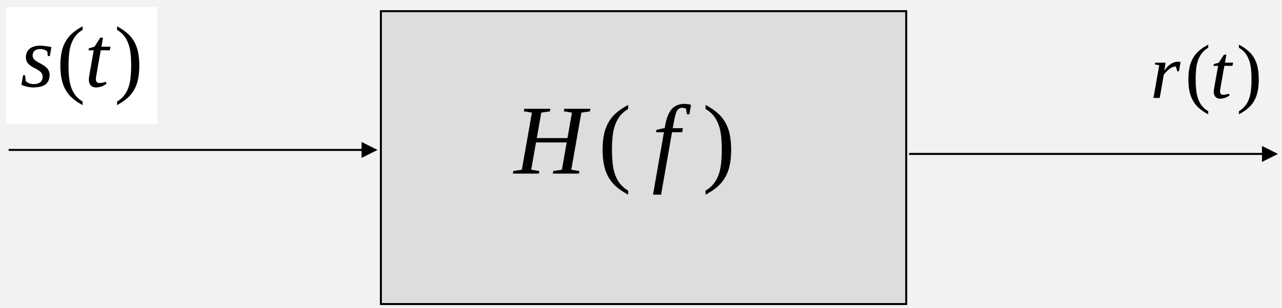


CONCLUSION

- Signal spectrum at the baseband(“centred around the origin = DC”)
- Spectrum is 0 as the frequency is a multiplication of $1/T$
- The “main lobe” occupies $2/T$, from $-1/T$ to $+1/T$
- The other lobes occupy $1/T$ and magnitude is reduced over time

CONSTELLATION I

The signal space is matched well with a “low-pass response”

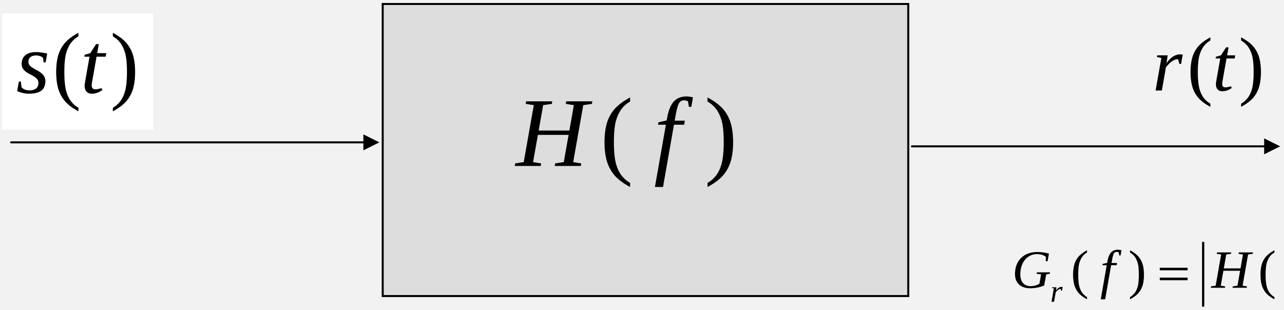


The received signal (with no interference) has the spectral density function:

$$G_r(f) = |H(f)|^2 G_s(f)$$

Since $G_s(f)$ spans the entire frequency axis, only $H(f)=1$ can keep the signal in its form

CONSTELLATION

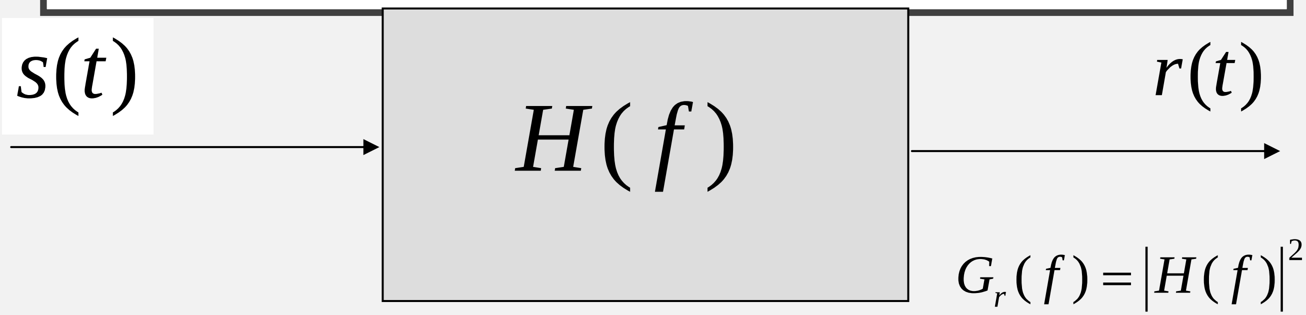


$$G_r(f) = |H(f)|^2 G_s(f)$$

If $H(f) \neq 1$, received signal is different from transmitted signal. However, if the signal bandwidth is sufficiently large, distortion is not quite significant (received signal is not quite different from transmitted signal.)

To reduce distortion→ Baseband channel should have a high "cut-off" frequency.

CONSTELLATION



$$G_r(f) = |H(f)|^2 G_s(f)$$

For a channel with the frequency impulse response $H(f)$

We must design transmitted signals such that $G_s(f)$ has its frequency around that of $H(f)$ ➔ "good" .

By this methodology, the received signal is approximate the transmitted signal

CONSTELLATION

Let us consider

$$M = \{ s_1(t) = +AP_T(t)\cos(2\pi f_0 t)\,,\, s_2(t) = -AP_T(t)\cos(2\pi f_0 t) \;\}$$

It is a one-dimensional constellation (d=1) , with basis

$$B = \left\{ b_1(t) = \sqrt{\frac{2}{T}} P_T(t)\cos\left(2\pi f_0 t\right) \right\}$$

The constellation corresponds to this vector set

$$M = \{ \underline{s_1} = (+\alpha)\,,\, \underline{s_2} = (-\alpha) \} \qquad \alpha = A\sqrt{\frac{T}{2}}$$

CONSTELLATION 2

The transmitted signal is given by $$s(t) = \sum_{n=0}^{+\infty} a[n] p(t - nT)$$

with $$a[n] \in \{+\alpha, -\alpha\}$$ $$p(t) = b_1(t)$$

Where the sequence $a[n]$ has

mean $$\mu_a = 0.5\,(-\alpha + \alpha) = 0$$

variance $$\sigma_a^2 = 0.5\,(\alpha^2 + \alpha^2) = \alpha^2 = A^2 \frac{T}{2}$$

CONSTELLATION 2

The power spectral density is given by

$$G_s(f) = \sigma_a^2 \frac{|P(f)|^2}{T}$$

Where $$p(t) = b_1(t) = \sqrt{\frac{2}{T}} P_T(t)\cos\left(2\pi f_0 t\right)$$

has Fourier transform $$P(f) = \left[\sqrt{2T}\,\frac{\sin(\pi fT)}{(\pi fT)} e^{-j\pi f \mathrm{T}}\right] * \left[\frac{1}{2}\left(\delta(f - f_0) + \delta(f + f_0)\right)\right] =$$

$$= \sqrt{\frac{T}{2}}\left[\left(\frac{\sin(\pi(f - f_0)T)}{\left(\pi(f - f_0)T\right)}\right) + \left(\frac{\sin(\pi(f + f_0)T)}{\left(\pi(f + f_0)T\right)}\right)\right] e^{-j\pi f \mathrm{T}}$$

CONSTELLATION 2

It follows

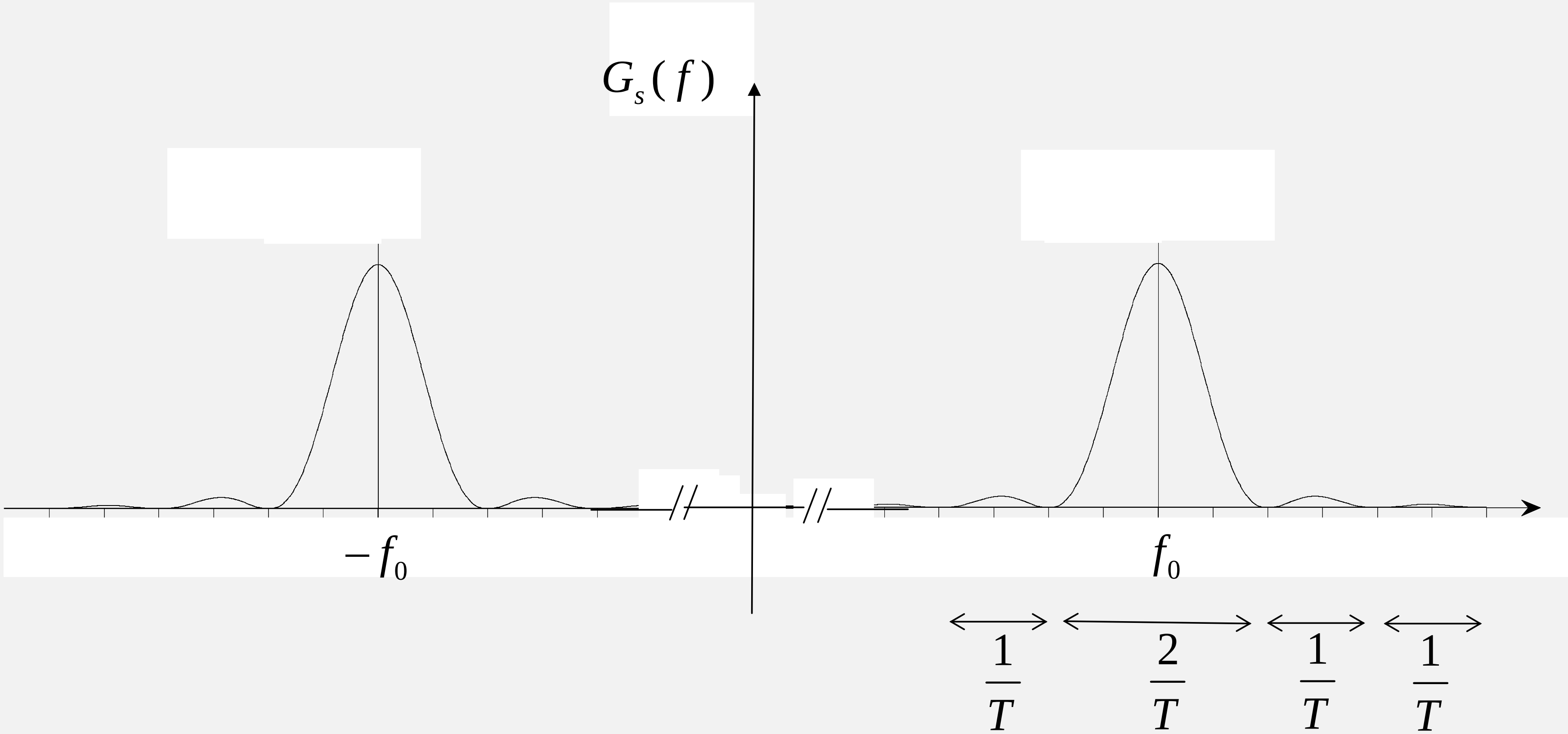
$$|P(f)|^2 = \frac{T}{2}\left[\left(\frac{\sin(\pi(f-f_0)T)}{\left(\pi(f-f_0)T\right)}\right)^2 + \left(\frac{\sin(\pi(f+f_0)T)}{\left(\pi(f+f_0)T\right)}\right)^2\right]$$

The power spectral density is

$$G_s(f) = \frac{1}{4}A^2T\left[\left(\frac{\sin(\pi(f-f_0)T)}{\left(\pi(f-f_0)T\right)}\right)^2 + \left(\frac{\sin(\pi(f+f_0)T)}{\left(\pi(f+f_0)T\right)}\right)^2\right]$$

CONSTELLATION 2

$$G_s(f) = \frac{1}{4}A^2T\left[\left(\frac{\sin(\pi(f-f_0)T)}{(\pi(f-f_0)T)}\right)^2 + \left(\frac{\sin(\pi(f+f_0)T)}{(\pi(f+f_0)T)}\right)^2\right]$$



CONSTELLATION 2

- It is a **bandpass (IF) spectrum** (centered around a non-zero frequency f_0)
- The spectrum is equal to zero for frequencies multiple of $1/T$
- The main lobe has width $2/T$ around f_0
- All the other lobes have width $1/T$ and decreasing amplitude

LINEAR MODULATION

In general, given $s(t)=\sum_n a[n]p(t-nT)$

with spectrum $G_s(f)=\sigma_a^2\dfrac{|P(f)|^2}{T}$

If we consider $s'(t)=\sum_n a_n p'(t-nT)$

with $p'(t)=p(t)\cos(2\pi f_0 t)$

Its spectrum is $G_{s'}(f)=\dfrac{1}{4}[G_s(f-f_0)+G_s(f+f_0)]$

CONSTELLATION 3

Let us consider

$$M = \left\{ s_1(t) = +A\frac{\sin(\pi t/T)}{(\pi t/T)}, s_2(t) = -A\frac{\sin(\pi t/T)}{(\pi t/T)} \right\}$$

It is a one-dimensional constellation (d=1) , with basis

$$B = \left\{ b_1(t) = \frac{1}{\sqrt{T}} \frac{\sin(\pi t/T)}{(\pi t/T)} \right\}$$

The constellation corresponds to this vector set

$$M = \{\underline{s_1} = (+\alpha), \underline{s_2} = (-\alpha)\} \qquad \alpha = A\sqrt{T}$$

CONSTELLATION 3

The transmitted signal is given by $$s(t) = \sum_{n=0}^{+\infty} a[n]p(t-nT)$$

with $$a[n] \in \{+\alpha, -\alpha\}$$ $$p(t) = b_1(t)$$

Where the sequence *a[n]* has

mean $$\mu_a = 0.5\,(-\alpha + \alpha) = 0$$

variance $$\sigma_a^2 = 0.5\,(\alpha^2 + \alpha^2) = \alpha^2 = A^2 T$$

CONSTELLATION 3

The power spectral density is given by

$$G_s(f) = \sigma_a^2 \frac{\left|P(f)\right|^2}{T}$$

Where
$$p(t) = \frac{1}{\sqrt{T}} \frac{\sin\left(\pi t / T\right)}{\left(\pi t / T\right)}$$

has Fourier transform
$$P(f) = \sqrt{T} rect_{\frac{1}{T}}\left(f\right)$$

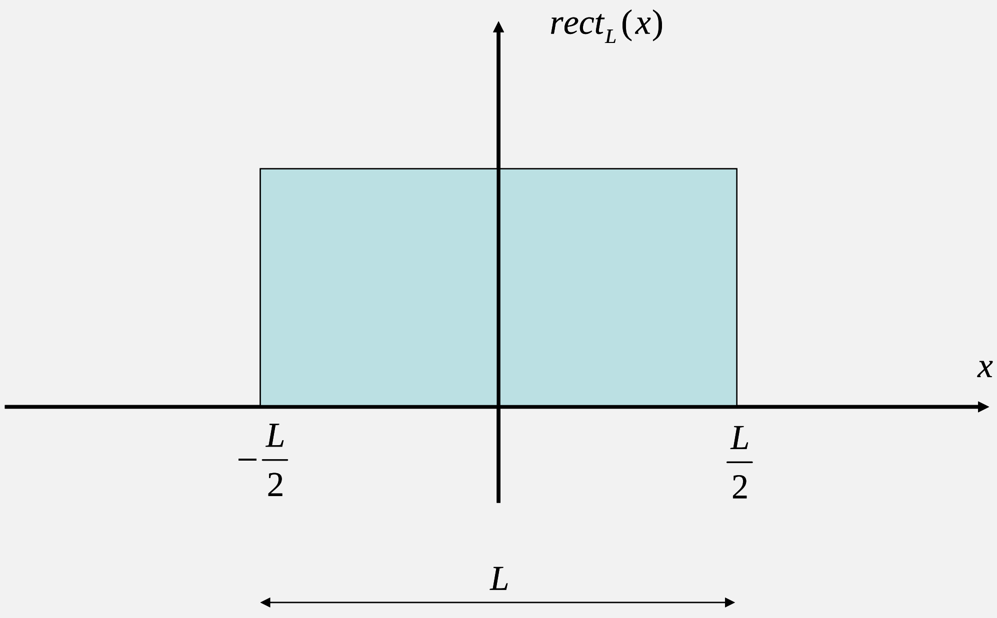
CONSTELLATION 3

$p(t)$: is an ideal low pass filter with Fourier transform constant between $-1/(2T)$ and $+1/(2T)$.

$$p(t)=\frac{1}{\sqrt{T}}\frac{\sin\left(\pi t/T\right)}{\left(\pi t/T\right)}$$

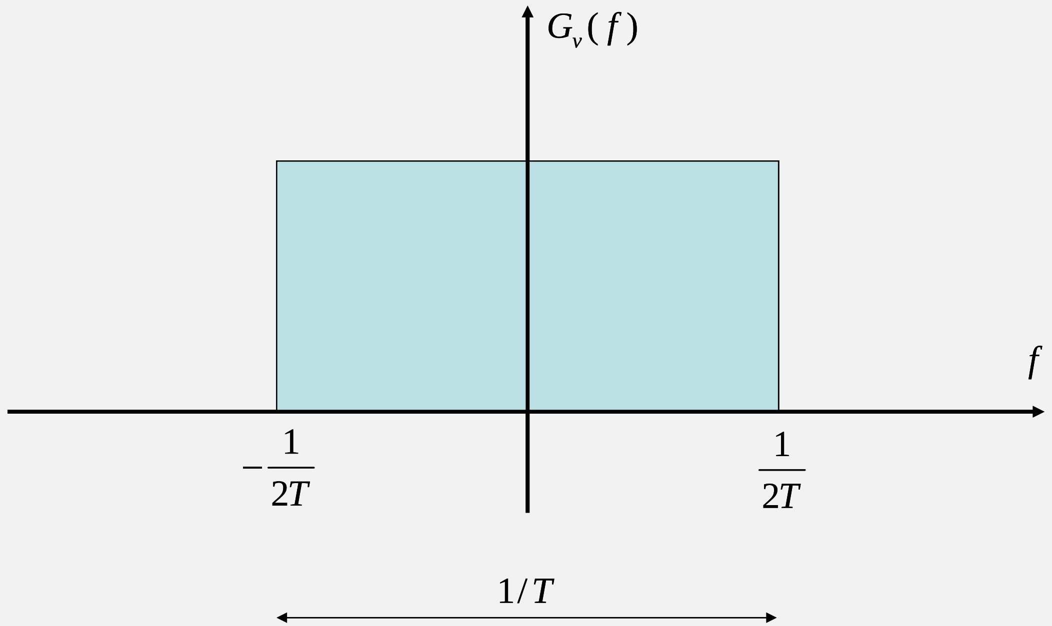
$$P(f)=\sqrt{T}\,rect_{\frac{1}{T}}\left(f\right)$$

$$\left|P(f)\right|^2=T\,rect_{\frac{1}{T}}\left(f\right)$$



CONSTELLATION 3

The power spectral density is $G_s(f) = A^2 T rect_{\frac{1}{T}}(f)$



It is a baseband spectrum

LECTURE 7: INTERSYMBOL INTERFERENCE

INFINITE TIME DOMAIN SIGNALS

Given a 1-D constellation with zero mean, its transmitted signal has form

$$s(t) = \sum_n a[n]\, p(t - nT)$$

and its power spectral density is given by

$$G_s(f) = \sigma_a^2 \frac{|P(f)|^2}{T}$$

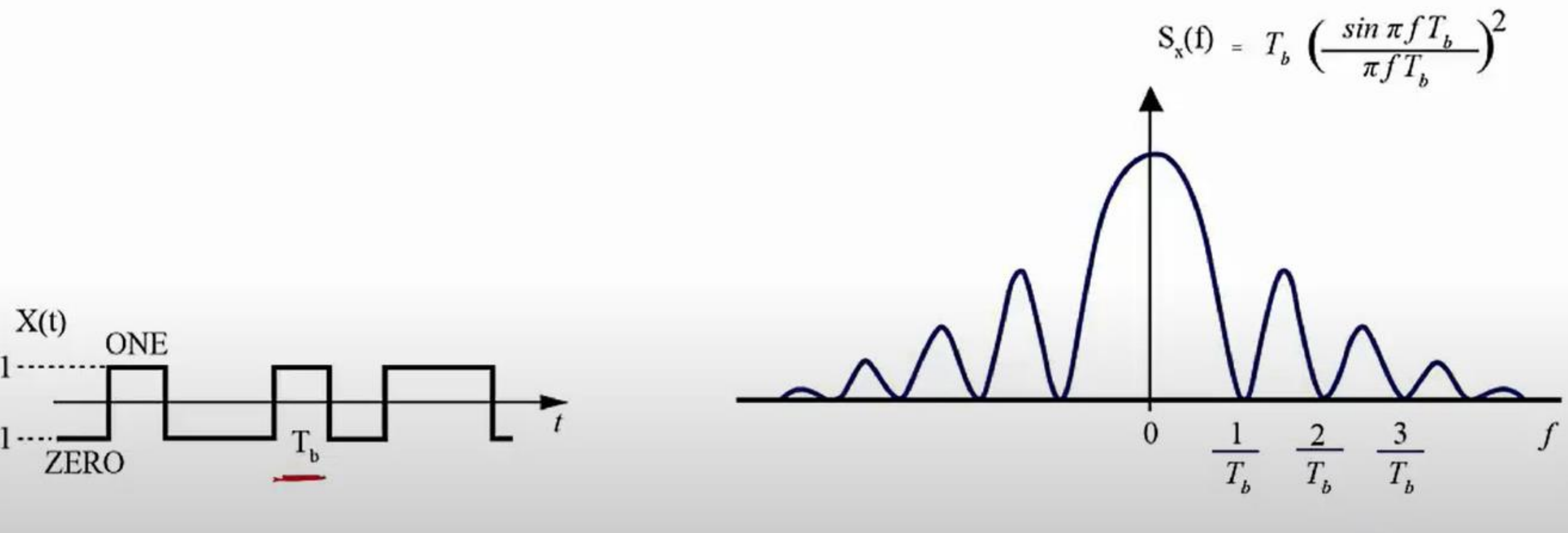
As a consequence, if $p(t) = b_1(t)$ has finite time domain, the transmitted signal $s(t)$ has infinite spectral domain.

INFINITE TIME DOMAIN SIGNALS

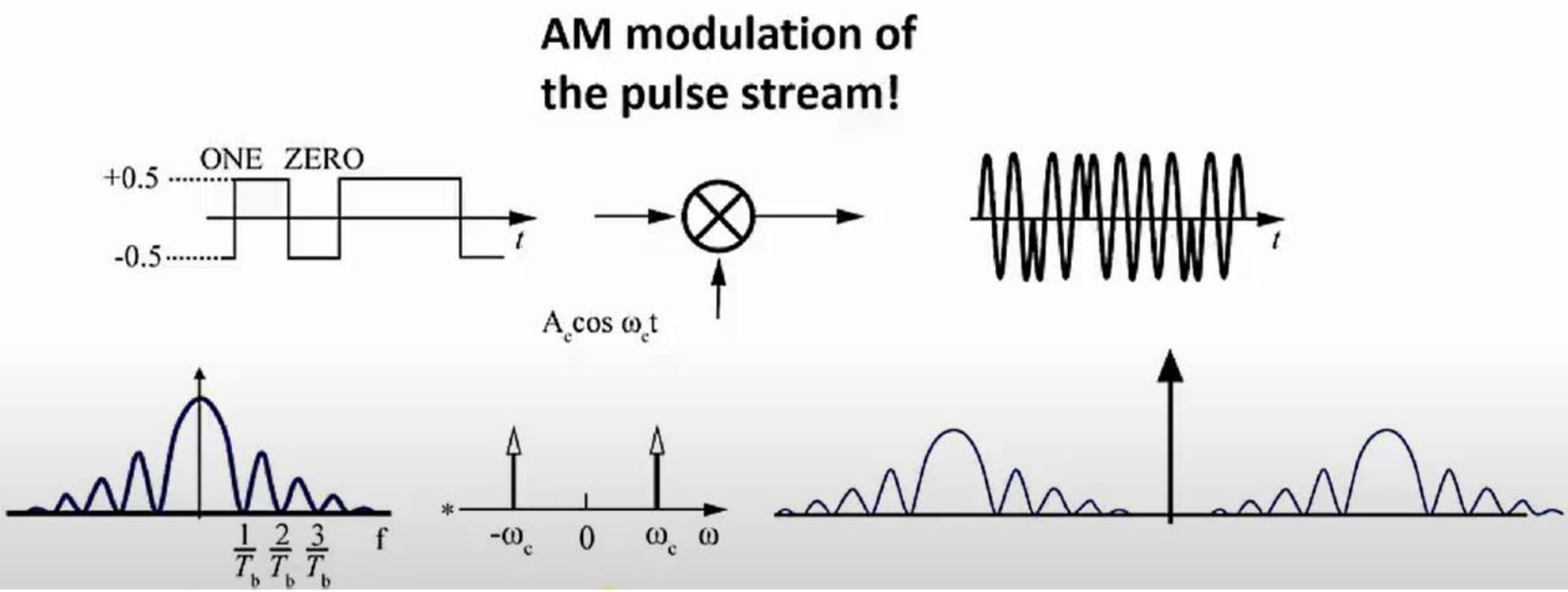
Fourier Transform Table

$x(t)$	$X(f)$	$X(\omega)$
$\delta(t)$	1	1
1	$\delta(f)$	$2\pi\delta(\omega)$
$\delta(t-t_0)$	$e^{-j2\pi f t_0}$	$e^{-j\omega t_0}$
$e^{j2\pi f_0 t}$	$\delta(f-f_0)$	$2\pi\delta(\omega-\omega_0)$
$\cos(2\pi f_0 t)$	$\frac{1}{2}\left[\delta(f-f_0)+\delta(f+f_0)\right]$	$\pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$
$\sin(2\pi f_0 t)$	$\frac{1}{2j}\left[\delta(f-f_0)-\delta(f+f_0)\right]$	$-j\pi\left[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\right]$
$rect(t)$	$\mathrm{sinc}(f)$	$\mathrm{sinc}\left(\frac{\omega}{2\pi}\right)$
$\mathrm{sinc}(t)$	$rect(f)$	$rect\left(\frac{\omega}{2\pi}\right)$
$\Lambda(t)$	$\mathrm{sinc}^2(f)$	$\mathrm{sinc}^2\left(\frac{\omega}{2\pi}\right)$
$\mathrm{sinc}^2(t)$	$\Lambda(f)$	$\Lambda\left(\frac{\omega}{2\pi}\right)$
$e^{-\alpha t}u(t),\quad \alpha>0$	$\frac{1}{\alpha+j2\pi f}$	$\frac{1}{\alpha+j\omega}$
$te^{-\alpha t}u(t),\quad \alpha>0$	$\frac{1}{(\alpha+j2\pi f)^2}$	$\frac{1}{(\alpha+j\omega)^2}$
$e^{-\alpha\|t\|},\quad \alpha>0$	$\frac{2\alpha}{(\alpha^2+(2\pi f)^2}$	$\frac{2\alpha}{(\alpha^2+(\omega)^2}$
$e^{-\pi t^2}$	$e^{-\alpha f^2}$	$e^{-\alpha f^2}$
$\mathrm{sgn}(t)$	$\frac{1}{j\pi f}$	$\frac{2}{j\omega}$
$u(t)$	$\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$	$\pi\delta(\omega)+\frac{1}{j\omega}$

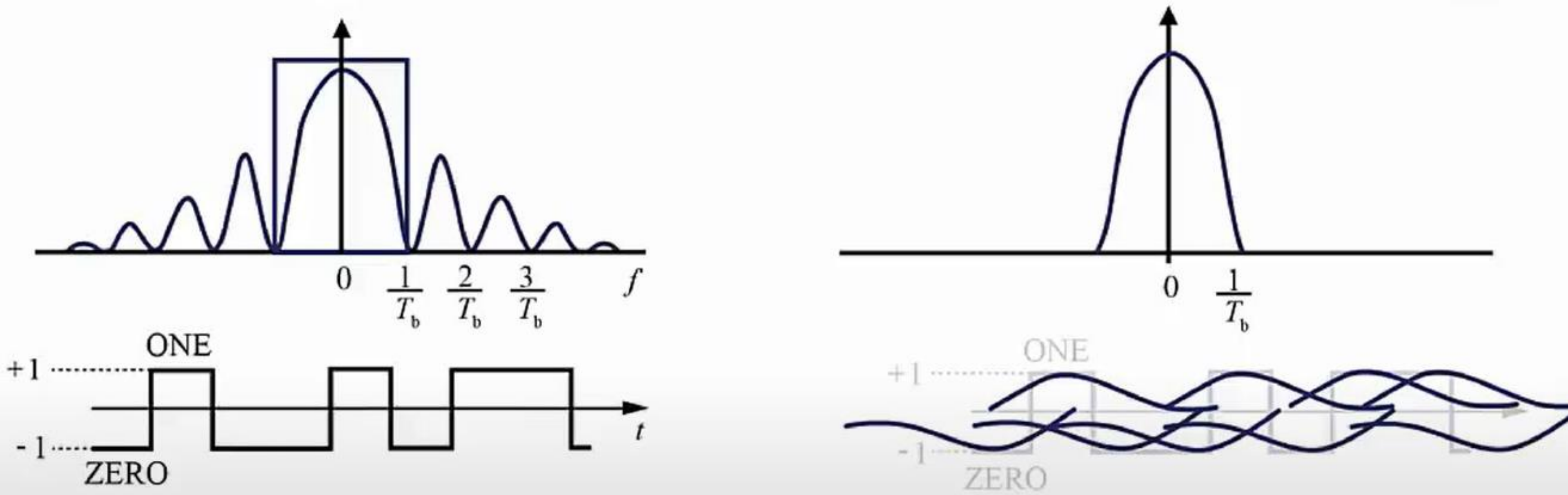
INFINITE TIME DOMAIN SIGNALS



INFINITE TIME DOMAIN SIGNALS



INFINITE TIME DOMAIN SIGNALS



INFINITE TIME DOMAIN SIGNALS

To overcome this problem, we should use constellation signals with infinite time domain (note this is a non causal situation, impossible to be implemented in practical but fundamental for analytical study).

Consider a constellation M composed by signals with unlimited time domain (but finite energy), and suppose M to be a one-dimensional constellation characterized by a basis

$$B = \{b_1(t)\}$$

INFINITE TIME DOMAIN SIGNALS: SINGLE SYMBOL TRANSMISSION

Let us suppose to transmit a **single symbol** $a[0]$.

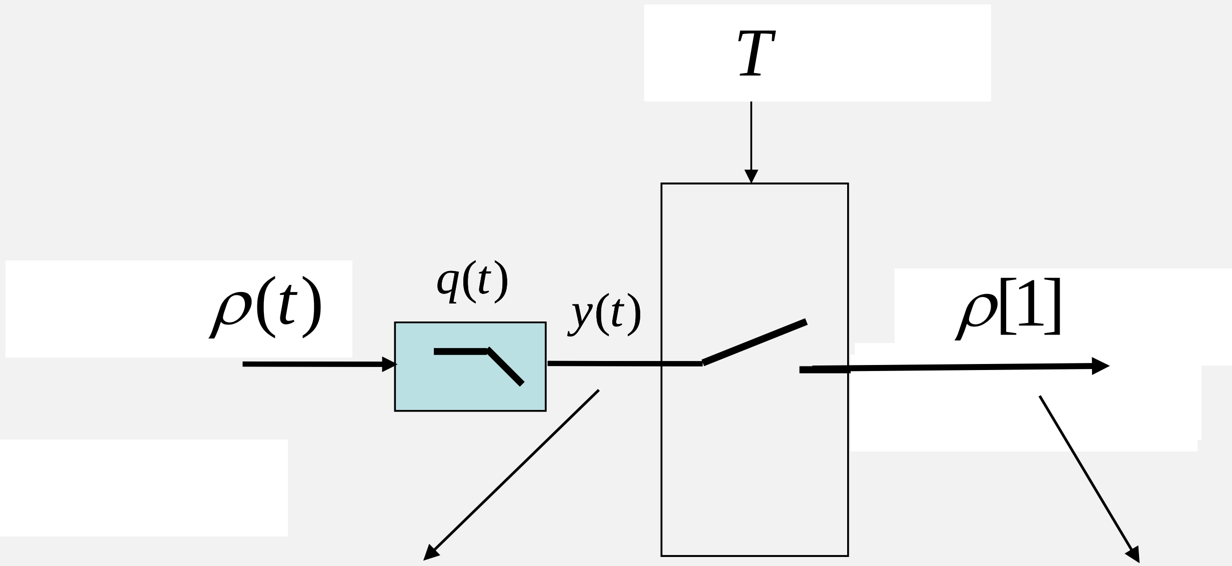
Given the received signal $r(t)=\rho(t)$, we must compute its projection on the basis signal $b_1(t)$:

$$\rho[1] = \int_{-\infty}^{+\infty} \rho(t) b_1(t) dt$$

(note that the integration domain is no more limited between 0 and T)

INFINITE TIME DOMAIN SIGNALS: SINGLE SYMBOL TRANSMISSION

This projection can be computed by using a matched filter $q(t) = b_1(T-t)$



$$y(t) = \int_{-\infty}^{+\infty} r(\tau) q(t-\tau) d\tau = \int_{-\infty}^{+\infty} r(\tau) b_1(T-t+\tau) d\tau$$

$$y(t=T) = \int_{-\infty}^{+\infty} \rho(t) b_1(t) dt = \rho[1]$$

INFINITE TIME DOMAIN SIGNALS: SEQUENCE TRANSMISSION

Consider now the transmission of an infinite symbol **sequence** $(a[n])$

$$\mathbf{s(t)} = \sum_{n=-\infty}^{+\infty} a[n]\,p(t - nT)$$

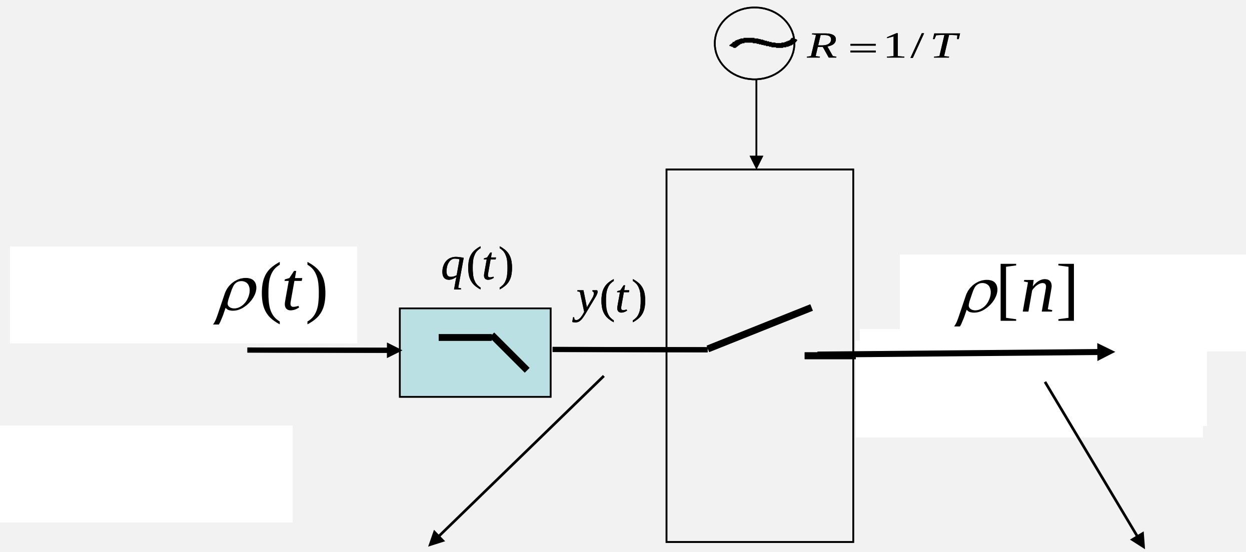
The channel is supposed to be completely ideal:

- $H(f)=1$
- $n(t)=0$

$$\Rightarrow \qquad \rho\mathbf{(t)} = \mathbf{s(t)}$$

INFINITE TIME DOMAIN SIGNALS: SINGLE SYMBOL TRANSMISSION

At the received side, the projections $\rho[n]$ are computed by using the matched filter and sampling it at $(n+1)T$



$$y(t) = \int_{-\infty}^{+\infty} r(\tau)q(t-\tau)d\tau = \int_{-\infty}^{+\infty} r(\tau)b_1(T-t+\tau)d\tau$$

$$y(t=(n+1)T) = \rho[n]$$

ON THE SAMPLING TIME

In ideal conditions we should have $\rho[n] = y((n+1)T) = y(T + nT)$

It is more useful to write $\rho[n] = y(t_0 + nT)$

In ideal conditions $t_0 = T$

In practical conditions $t_0 = T + D$

Where the delay D takes into account:

- Propagation delay
- Implementation delay
- ...

(at the receiver side, the symbol synchronizer will exactly acquire the sampling time)

INTERSYMBOL INTERFERENCE

The matched filter output waveform is equal to

$$y(t) = \rho(t) * q(t)$$

Since the channel is ideal $\rho(t) = s(t)$ and we have:

$$y(t) = s(t) * q(t) = \left(\sum_{n=-\infty}^{+\infty} a[n] p(t - nT) \right) * q(t) =$$

$$= \sum_{n=-\infty}^{+\infty} a[n] x(t - nT)$$

where

$$\boxed{x(t) = p(t) * q(t)}$$

INTERSYMBOL INTERFERENCE

The received symbol is then equal to

$$\rho[n] = y(t_0 + nT) = \sum_{m=-\infty}^{+\infty} a[m]x(t_0 + nT - mT) = \sum_{i=-\infty}^{+\infty} a[n-i]x(t_0 + iT) =$$

$$= \sum_{i=-\infty}^{+\infty} x[i]a[n-i]$$

where $x[i] = x(t_0 + iT)$

INTERSYMBOL INTERFERENCE

The generic received symbol $\rho[n]$ is then connected to the corresponding transmitted symbol $a[n]$ via:

$$\boxed{\rho[n] = \sum_{i=-\infty}^{+\infty} x[i]a[n-i]}$$

we can write

$$\rho[n] = x[0]a[n] + \sum_{i=1}^{+\infty} x[i]a[n-i] + \sum_{i=-\infty}^{-1} x[i]a[n-i]$$

INTERSYMBOL INTERFERENCE

Previous symbols contribution

Received symbol

$$\rho\,[n] = x[0]a[n] + \sum_{i=1}^{+\infty} x[i]a[n-i] + \sum_{i=-\infty}^{-1} x[i]a[n-i]$$

Transmitted symbol contribution

Future symbols contribution

In line of principle, the received symbol $\rho[n]$ does not depend only on the transmitted symbol $a[n]$, but also on all the other transmitted symbols

We may have **Intersymbol interference (ISI)**

INTERSYMBOL INTERFERENCE

Received symbol

$$\rho[n] = \sum_{i=-\infty}^{+\infty} x[i]a[n-i] =$$

$$= x[0]a[n] +$$

Transmitted symbol contribution

$$+ x[1]a[n-1] + x[2]a[n-2] + ...$$

Previous symbols contribution

$$+ x[-1]a[n+1] + x[-2]a[n+2] + ...$$

Future symbols contribution

INTERSYMBOL INTERFERENCE

We are transmitting over an ideal channel → we should have:

$$\rho[n] = a[n]$$

(received symbol = transmitted symbol)

This is verified if and only if

$$\boxed{\begin{array}{lll} x[i]=1 & \text{if} & i=0 \\ x[i]=0 & \text{if} & i\neq 0 \end{array}}$$

$$\boxed{\begin{array}{lll} x\left(t_0+iT\right)=1 & \text{if} & i=0 \\ x\left(t_0+iT\right)=0 & \text{if} & i\neq 0 \end{array}}$$

SIGNALS WITH FINITE DOMAIN [0,T[

For constellation signals with finite domain $[0,T[$ we already verified the absence of ISI: for ideal channel we always had

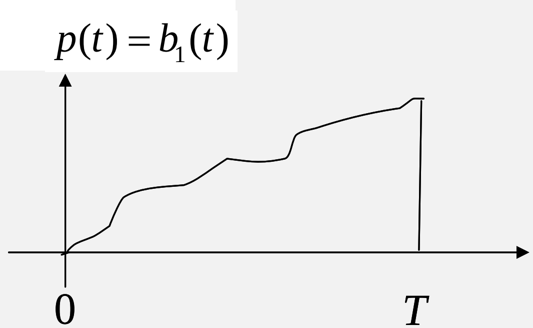
$$\rho[n] = a[n]$$

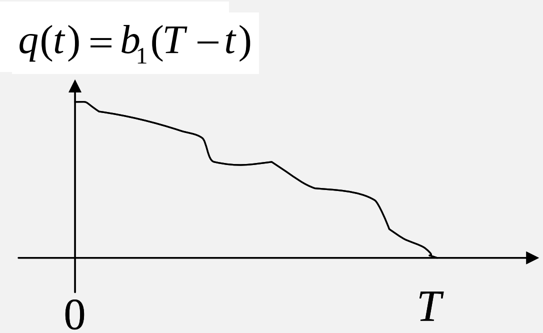
This means that in this case the function $x(t)$ automatically satisfies the previous condition.

INTERSYMBOL INTERFERENCE

In fact, let

- $b_1(t)$ be a versor with finite time domain $[0,T[$
- $p(t)= b_1(t)$
- $q(t)=p(T-t)$





Consider $$x(t)=p(t)*q(t)=\int_{-\infty}^{+\infty} p(\tau)q(t-\tau)d\tau$$

We have:

1. for $t \le 0 \quad x(t)=0$
2. $x(T)=\int_{-\infty}^{+\infty} p(\tau)q(T-\tau)d\tau=\int_{-\infty}^{+\infty} b_1(\tau)b_1(T-T+\tau)d\tau=1$
3. for $t \ge 0 \quad x(t)=0$

INTERSYMBOL INTERFERENCE

As a consequence, we have

$$
\begin{aligned}
x\left(t_0+iT\right)&=1 \quad \text{if} \quad i=0 \\
x\left(t_0+iT\right)&=0 \quad \text{if} \quad i\neq 0
\end{aligned}
\qquad \text{for } t_0=T
$$

$$
\boxed{
\begin{aligned}
x[i]&=1 \quad \text{if} \quad i=0 \\
x[i]&=0 \quad \text{if} \quad i\neq 0
\end{aligned}}
$$

The function x(t) automatically satisfies the NO ISI conditions

when the constellation signal have finite time domain $[0,T[$.

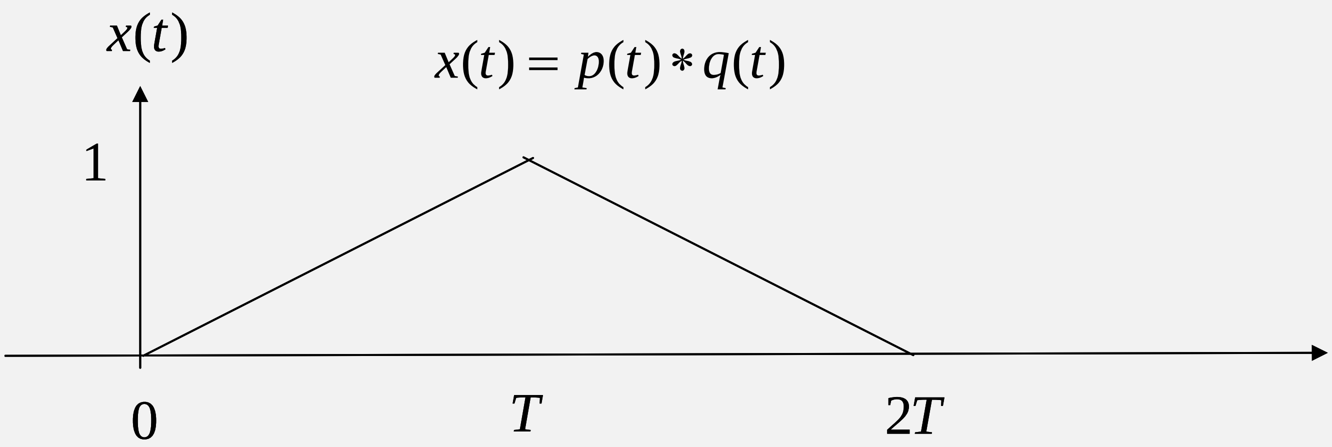
EXAMPLE I

1-D constellation with versor $b_1(t) = \frac{1}{\sqrt{T}} P_T(t)$

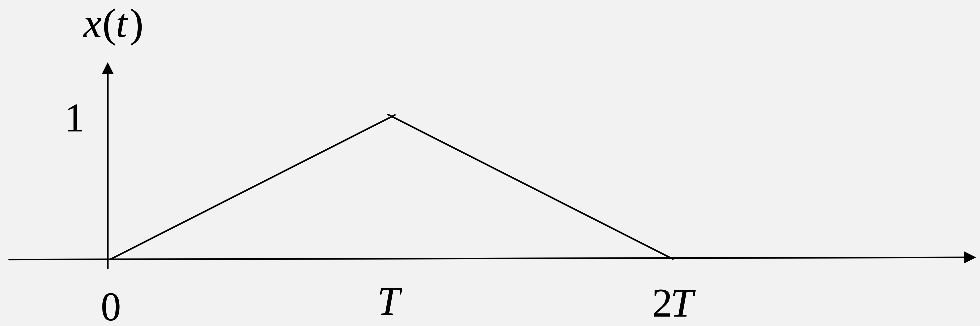
$$p(t) = b_1(t) = \frac{1}{\sqrt{T}} P_T(t)$$

$$q(t) = p(T-t) = \frac{1}{\sqrt{T}} P_T(t)$$

$$x(t) = p(t) * q(t)$$



EXAMPLE 1



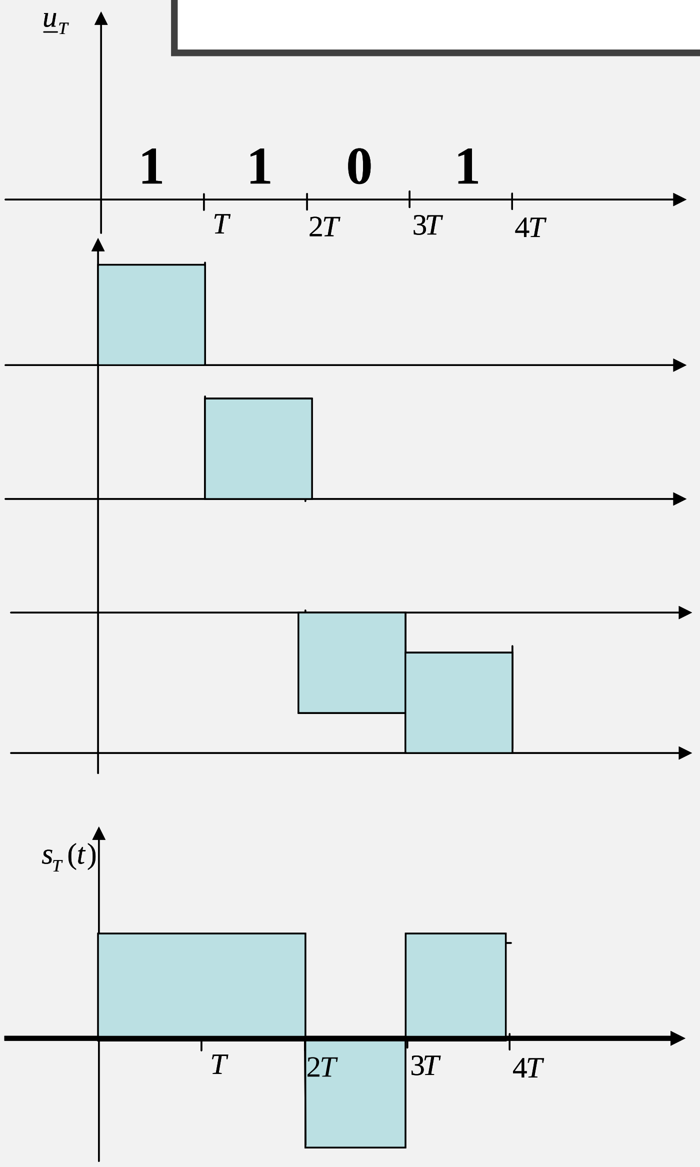
This function $x(t)$ satisfies the condition

$$x(t_0 + iT) = 1 \quad \text{if} \quad i = 0$$
$$x(t_0 + iT) = 0 \quad \text{if} \quad i \neq 0 \qquad \text{for } t_0 = T$$

$$\Rightarrow \quad \boxed{\begin{array}{lll} x[i] = 1 & \text{if} & i = 0 \\ x[i] = 0 & \text{if} & i \neq 0 \end{array}}$$

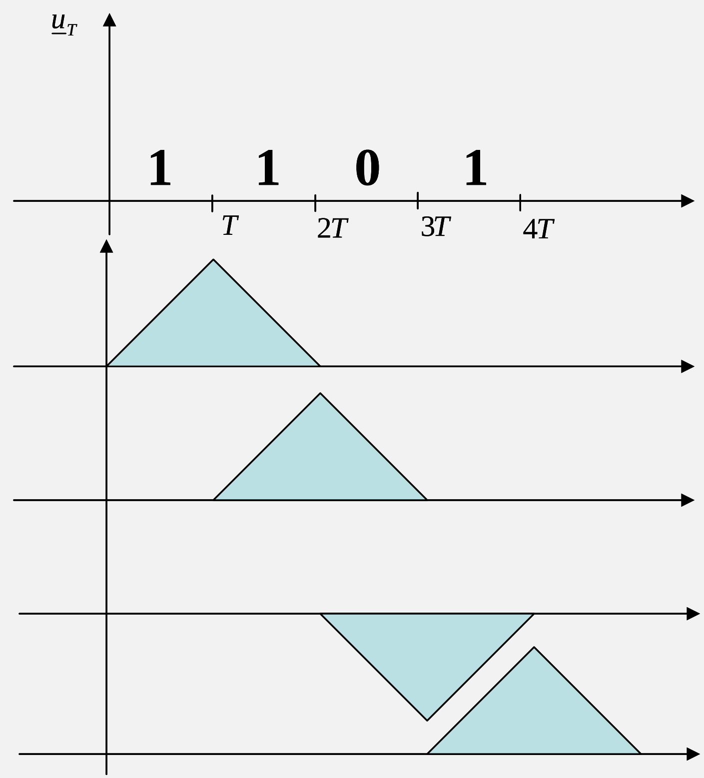
We certainly have NO ISI: $\rho[n] = y(T + nT) = a[n]$

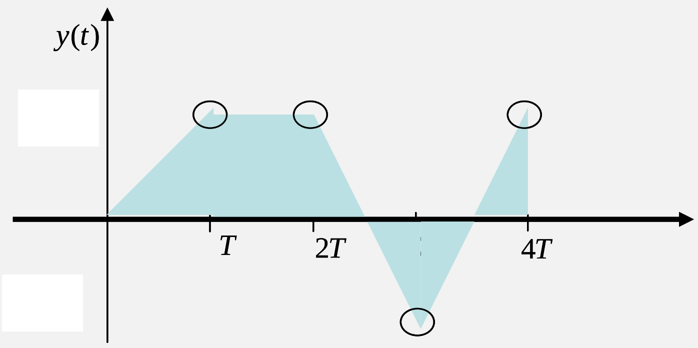
EXAMPLE 1



$$s(t) = \sum_n a[n]p(t - nT)$$

EXAMPLE 1





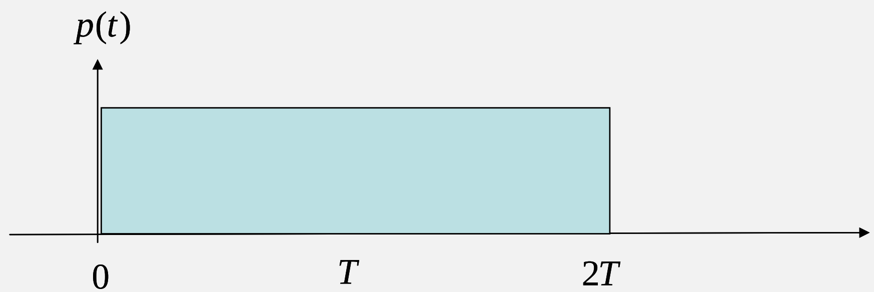
$$y(t) = \sum_n a[n]x(t - nT)$$

$$\rho[n] = y(T + nT) = a[n]$$

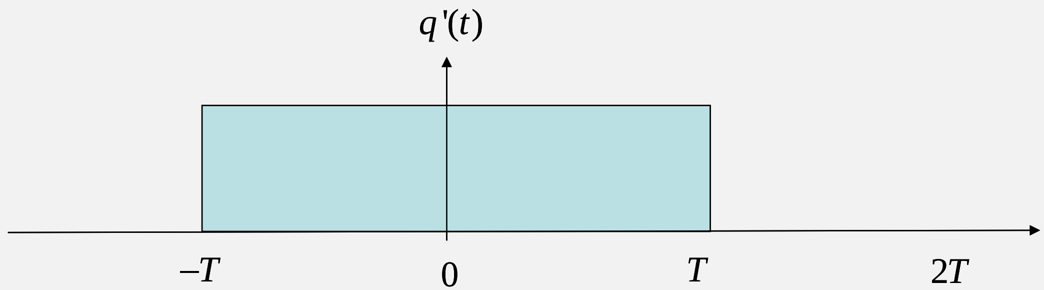
EXAMPLE 2

1-D constellation with versor $$b_1(t) = \frac{1}{\sqrt{2T}} P_{2T}(t)$$

$$p(t) = b_1(t)$$

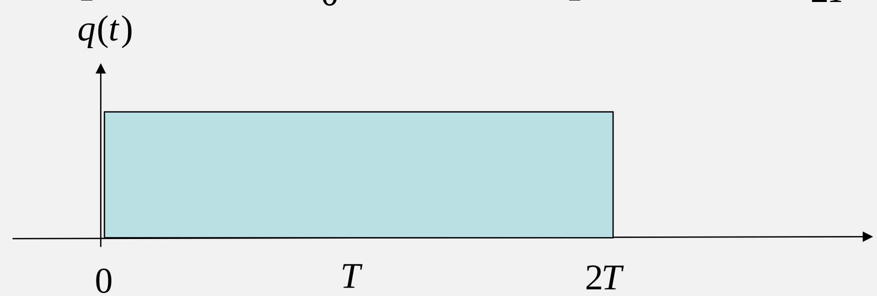


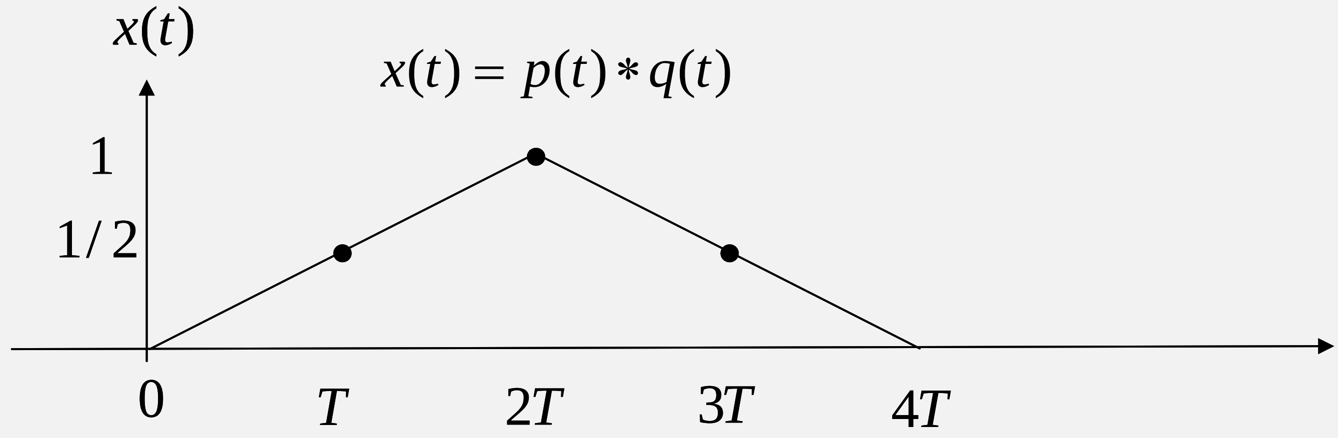
$$q'(t) = p(T-t)$$



Non causal: we have to delay it

of $D' = T$ $$q(t) = q'(t-T)$$





We should have

$$x(t_0 + iT) = 1 \quad \text{if} \quad i = 0$$
$$x(t_0 + iT) = 0 \quad \text{if} \quad i \neq 0$$

$$\text{for} \;\; t_0 = T + D' = 2T$$

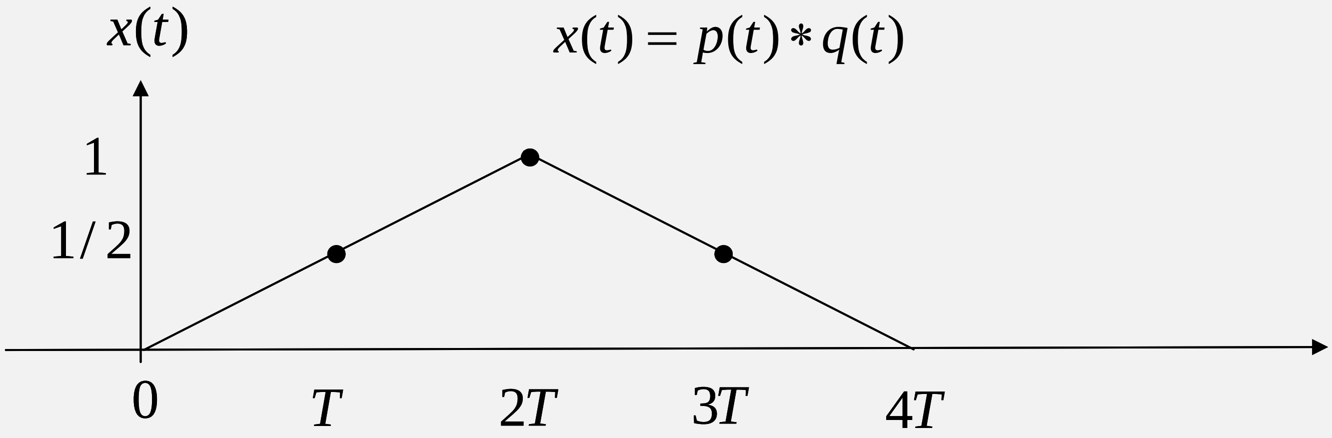
Instead we have:

$$x(t_0 + iT) = 1 \quad \text{if} \quad i = 0$$
$$x(t_0 + iT) = 1/2 \quad \text{if} \quad i = 1 \quad i = -1$$
$$x(t_0 + iT) = 0 \quad \text{if} \quad i \neq 0 \quad i \neq 1 \quad i \neq -1$$

$$\boxed{\begin{array}{lll} x[i] = 1 & \text{if} & i = 0 \\ x[i] = 1/2 & \text{if} & i = 1 \;\; i = -1 \\ x[i] = 0 & \text{if} & i \neq 0 \;\; i \neq 1 \;\; i \neq -1 \end{array}}$$

EXAMPLE 2

$$x(t) = p(t) * q(t)$$



We have intersymbol interference:

$$\rho[n] = y(t_0 + nT) = a[n] + \frac{1}{2}a[n-1] + \frac{1}{2}a[n+1] \neq a[n]$$

LECTURE 8: NYQUIST CRITERION FOR NO ISI

NYQUIST CRITERION

Given the function $x(t) = p(t) * q(t)$

The NO ISI condition is

$$x(t_0 + iT) = 1 \quad \text{if} \quad i = 0$$
$$x(t_0 + iT) = 0 \quad \text{if} \quad i \neq 0$$

For simplicity we pose here $t_0=0$ (discussed later).

The NO ISI condition becomes:

$$\boxed{\begin{array}{l} x(iT) = 1 \quad \text{if} \quad i = 0 \\ x(iT) = 0 \quad \text{if} \quad i \neq 0 \end{array}}$$

We call it the **<u>time domain Nyquist criterion</u>**

NYQUIST CRITERION

Second Nyquist theorem

If a function *x(t)* verifies the time domain Nyquist criterion

$$x(iT)=1 \quad \text{if} \quad i=0 \qquad\qquad x(iT)=0 \quad \text{if} \quad i\neq 0$$

It follows

$$x(t)\sum_i \delta(t-iT)=\delta(t)$$

Then:

$$X(f)*\frac{1}{T}\left[\sum_n \delta\left(f-\frac{n}{T}\right)\right]=1$$

which is the **frequency domain Nyquist criterion**

$$\boxed{\sum_n X\left(f-\frac{n}{T}\right)=T}$$

NYQUIST CRITERION

Given a function $x(t)$, to test the frequency domain Nyquist criterion we must:

- consider all the replica of $X(f)$ centred around frequencies multiple of $1/T$
- sum all of them

The result must be a constant on the frequency axis

$$\boxed{\sum_n X\left(f-\frac{n}{T}\right)=T}$$

NYQUIST CRITERION

$$\boxed{\begin{array}{l} x(iT) = 1 \quad \text{if} \quad i = 0 \\ x(iT) = 0 \quad \text{if} \quad i \neq 0 \end{array}}$$

$$\boxed{\sum_n X\left(f - \frac{n}{T}\right) = T}$$

Which are the functions $x(t)$ satisfying the Nyquist criterion?

Let us consider:

- Functions $x(t)$ characterized by a Fourier transform $X(f)$ with infinite frequency domain.
- Functions $x(t)$ characterized by a Fourier transform $X(f)$ with finite frequency domain.

NYQUIST CRITERION

Functions $x(t)$ characterized by a Fourier transform $X(f)$ with infinite frequency domain.

There are infinite solutions.

Among them, we already know that all the functions $x(t)=p(t)*q(t)$ where:

- $p(t)$ = versor with time domain $[0,T[$
- $q(t) = p(T-t)$

certainly satisfy the Nyquist criterion

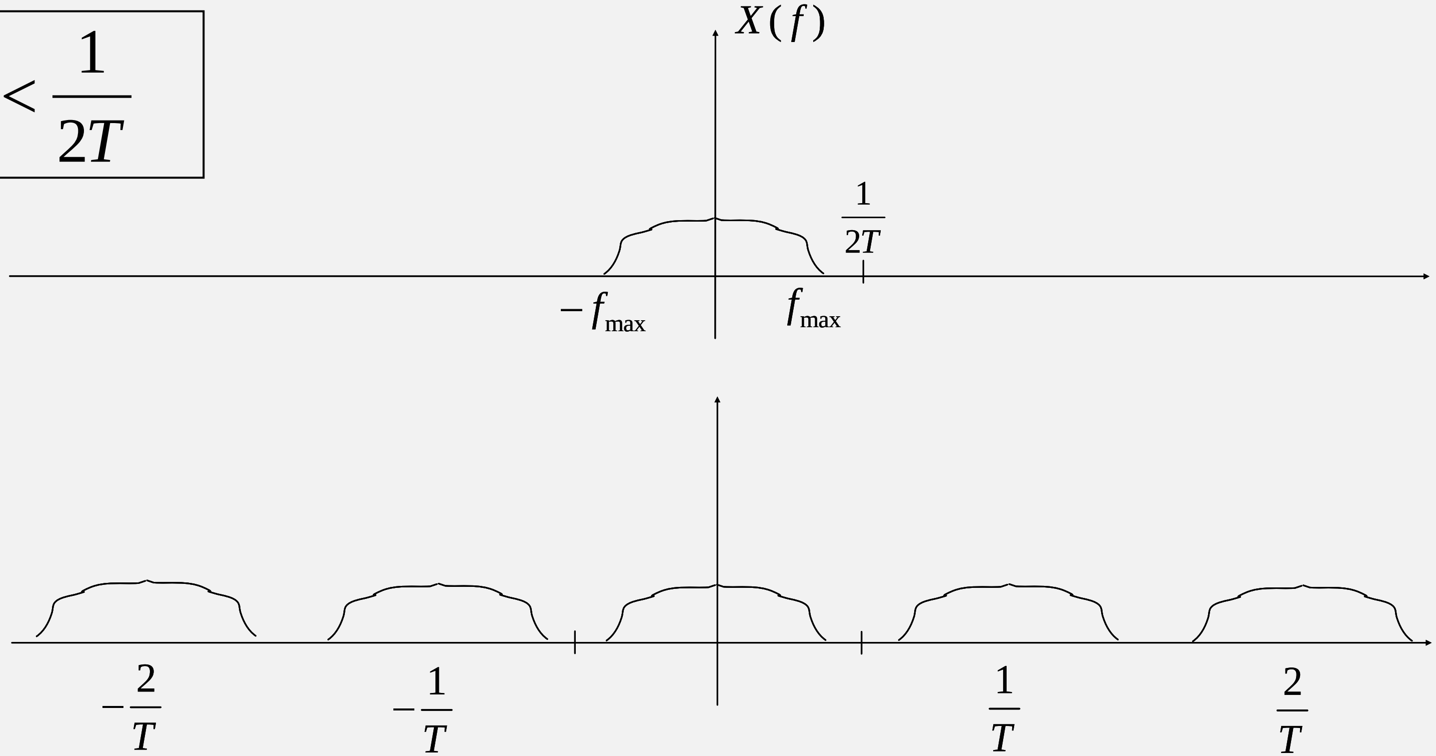
NYQUIST CRITERION

Functions $x(t)$ characterized by a Fourier transform $X(f)$ with finite frequency domain $[-f_{max} , f_{max}]$

Does exist any solution?

NYQUIST CRITERION, CASE 1

Case 1: $\boxed{f_{\max} < \frac{1}{2T}}$

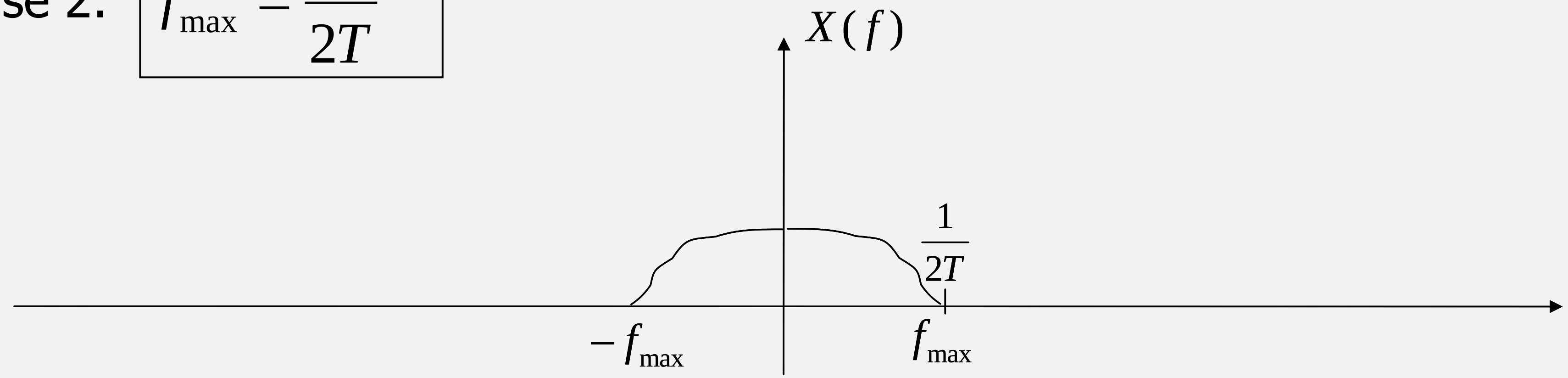


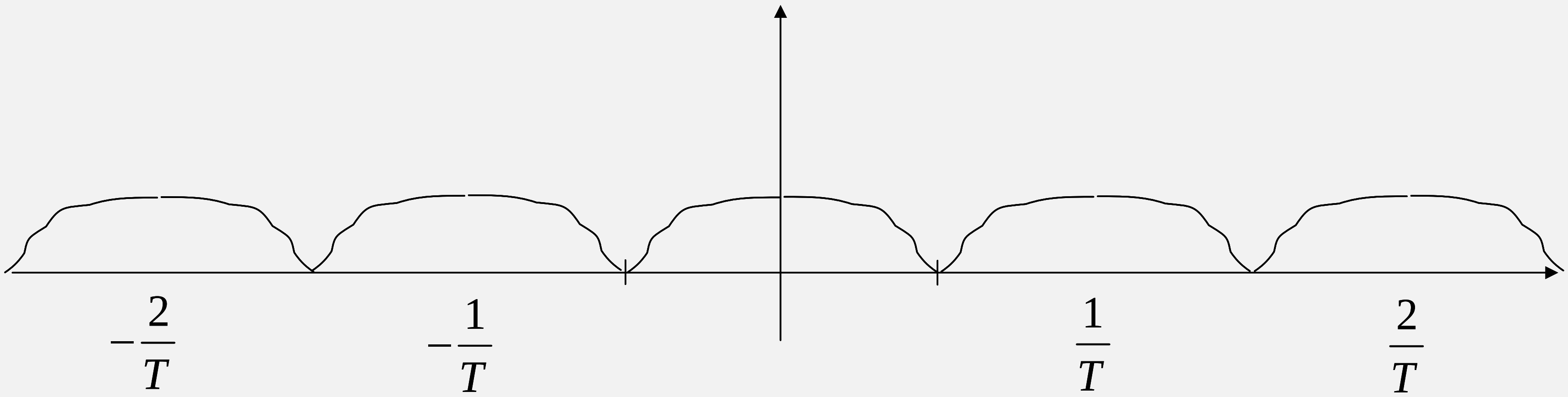
In this case it is impossible to satisfy the Nyquist criterion $\sum_n X\left(f - \frac{n}{T}\right) = T$

(holes at frequencies $n/2T$)

NYQUIST CRITERION, CASE 2

Case 2: $\boxed{f_{\max} = \dfrac{1}{2T}}$

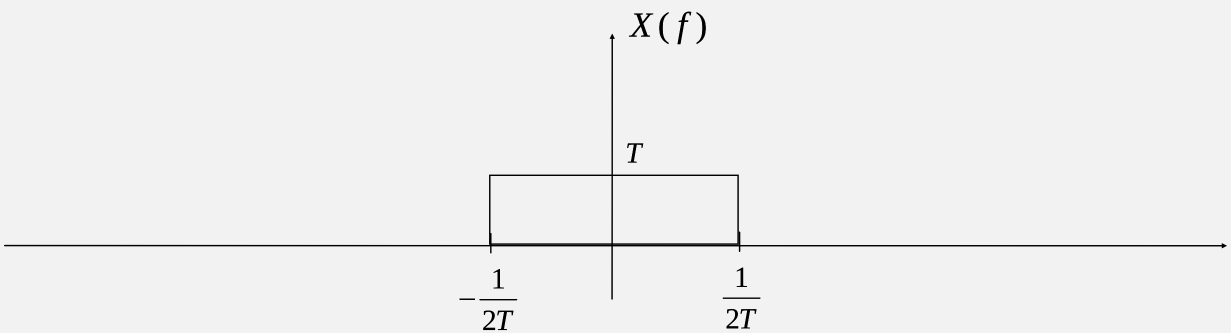


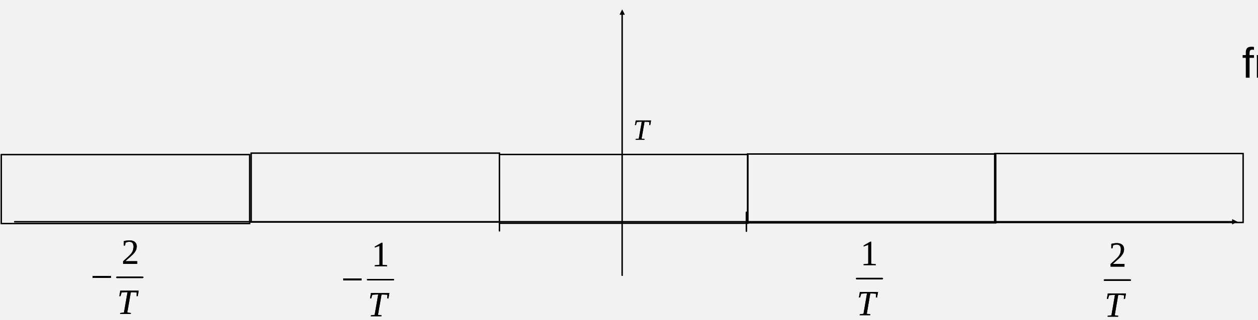


NYQUIST CRITERION, CASE 2

Case 2: $\boxed{f_{\max} = \dfrac{1}{2T}}$

A solution exists: the **ideal low pass filter**



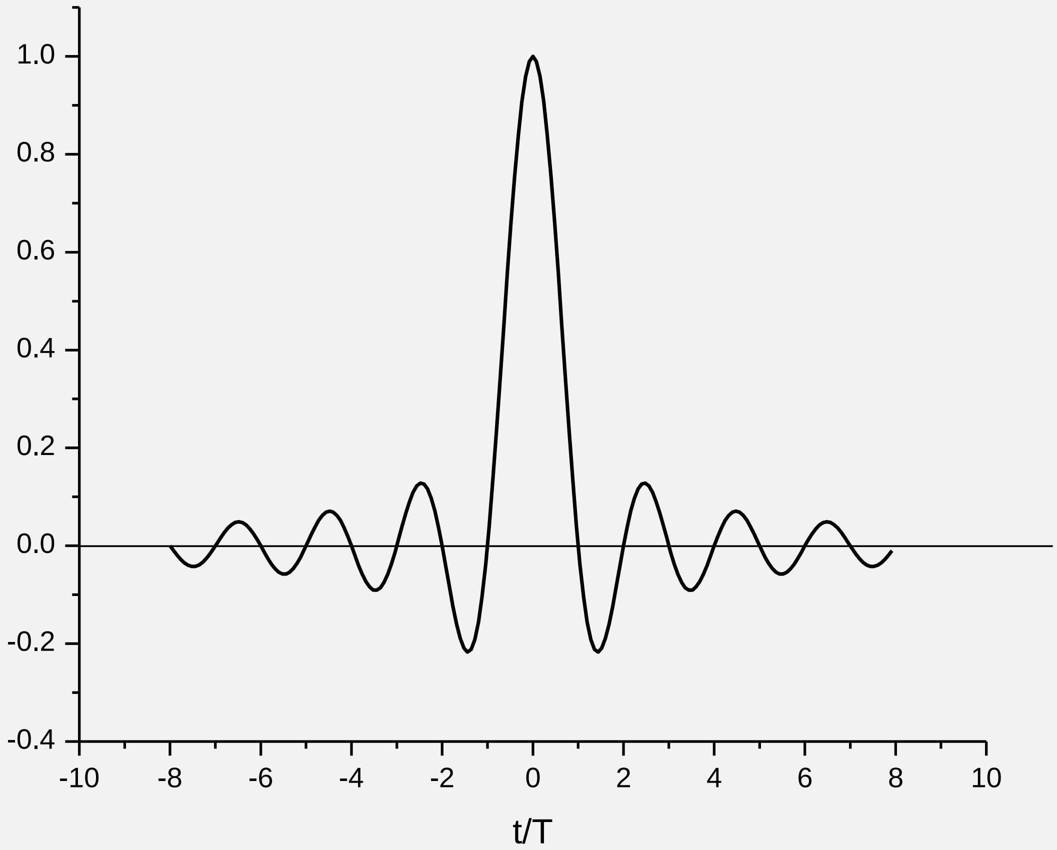


It satisfies the frequency-domain Nyquist criterion

$$\sum_n X\left(f-\frac{n}{T}\right)=T$$

IDEAL LOW PASS FILTER

The **ideal low pass filter** $$x(t) = \frac{\sin(\pi t / T)}{(\pi t / T)}$$



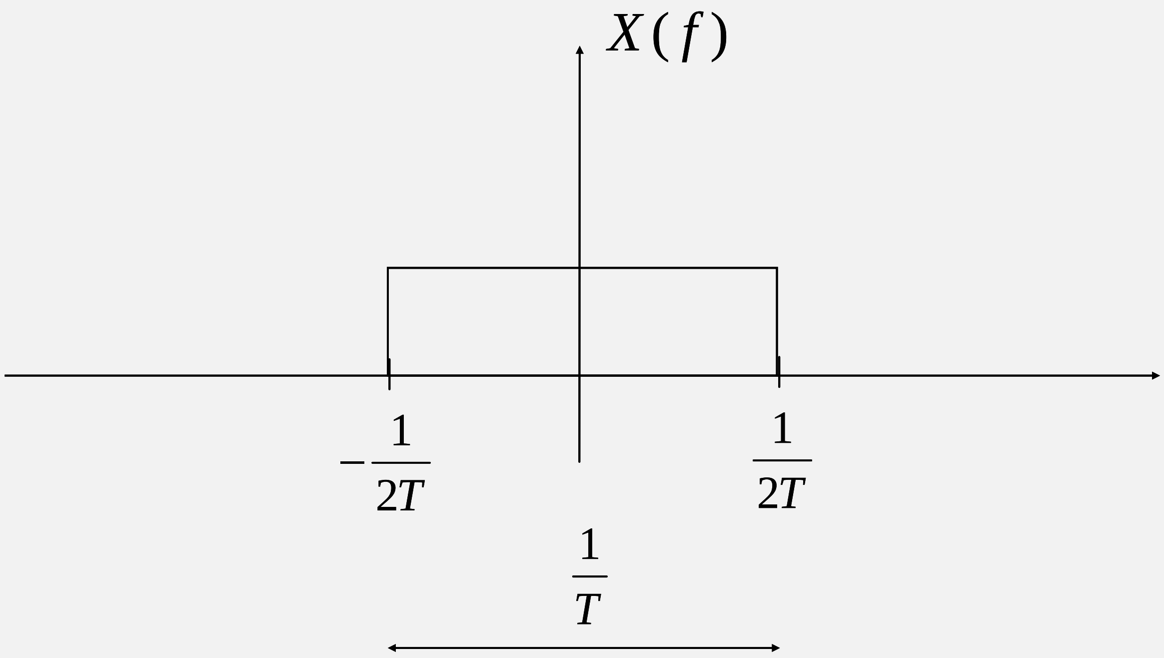
Note that it clearly satisfies the time-domain Nyquist criterion

$$x(iT) = 1 \quad \text{if} \quad i = 0$$

$$x(iT) = 0 \quad \text{if} \quad i \neq 0$$

IDEAL LOW PASS FILTER

The **ideal low pass filter**

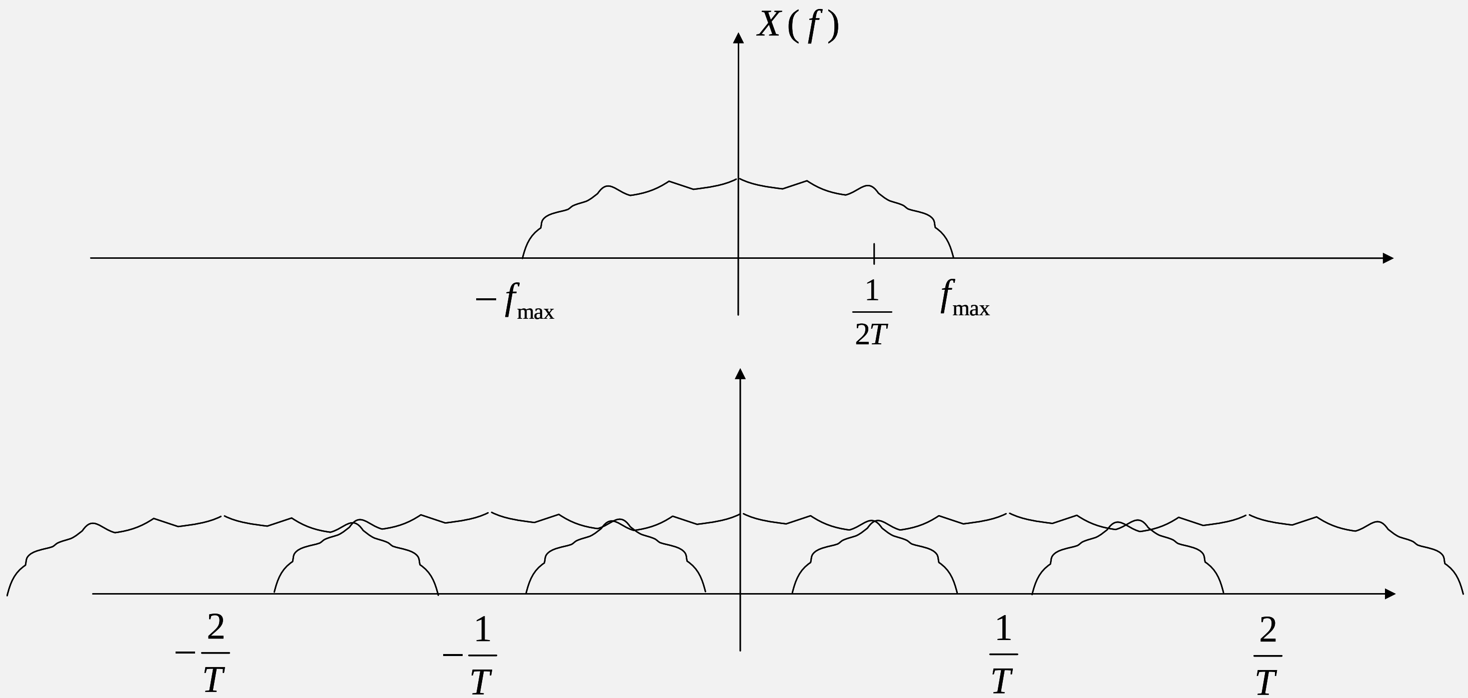


This is the waveform satisfying Nyquist

with the smaller bandwidth occupation

NYQUIST CRITERION, CASE 3

Case 3: $\boxed{f_{\max} > \dfrac{1}{2T}}$



There are a lot of solutions.

RAISED COSINE FILTERS

Example (very important for applications)

Raised cosine filters

$$\boxed{x(t)=\frac{\sin(\pi t/T)}{(\pi t/T)}\frac{\cos(\alpha\pi t/T)}{1-(2\alpha t/T)^2}}$$

roll-off coefficient α

$$0\le\alpha\le 1$$

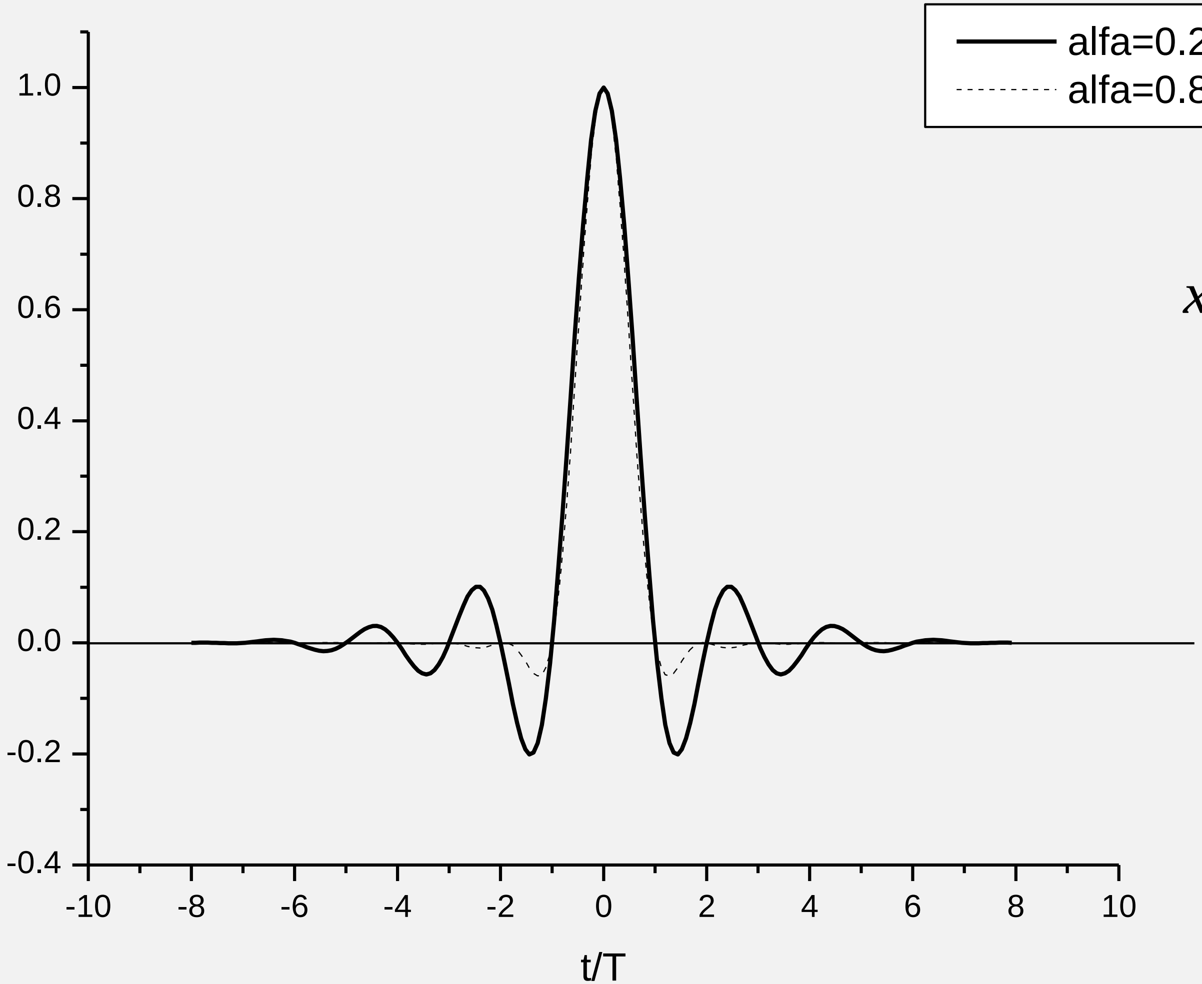
Note that

1. They clearly satisfy the time-domain Nyquist criterion

$$x(iT)=1 \quad \text{if} \quad i=0$$
$$x(iT)=0 \quad \text{if} \quad i\neq 0$$

2. For α=0 we obtain the ideal low pass filter

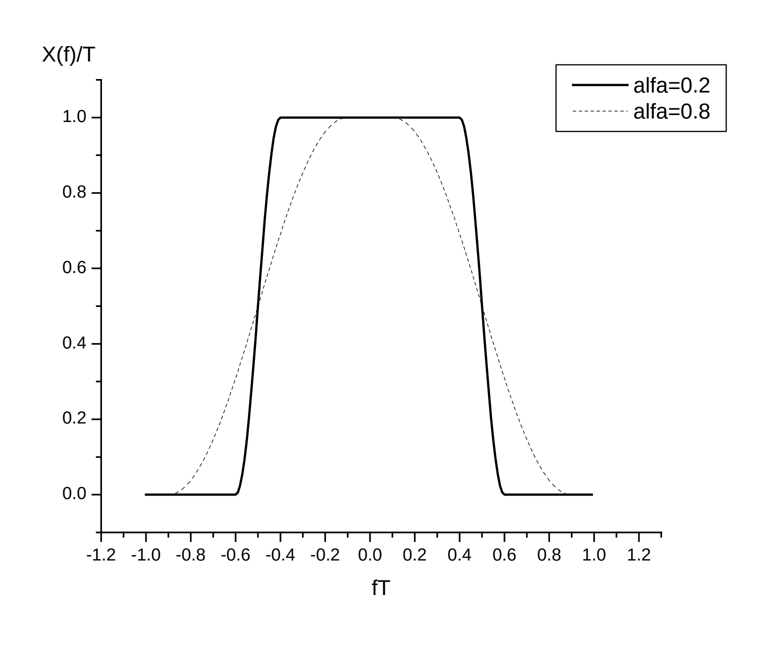
RAISED COSINE FILTERS



$$x(t) = \frac{\sin(\pi t / T)}{(\pi t / T)} \frac{\cos(\alpha \pi t / T)}{1 - (2\alpha t / T)^2}$$

RAISED COSINE FILTERS

Frequency response



$$X(f) = T \qquad \text{for} \qquad |f| \le \frac{(1-\alpha)}{2T}$$

$$X(f) = \frac{T}{2}\left[1 - \sin\left(\frac{\pi T}{\alpha}\left(|f| - \frac{1}{2T}\right)\right)\right] \qquad \text{for} \qquad \frac{(1-\alpha)}{2T} \le |f| \le \frac{(1+\alpha)}{2T}$$

$$X(f) = 0 \qquad \text{for} \qquad |f| \le \frac{(1+\alpha)}{2T}$$

RAISED COSINE FILTER

Raised cosine filters

$$x(t) = \frac{\sin(\pi t / T)}{(\pi t / T)} \frac{\cos(\alpha \pi t / T)}{1-(2\alpha t / T)^2}$$

They can be approximated very well

(Example, FIR digital filter, windowing design, 16 intervals = 64 taps with sampling frequency $4R$)

ON TIME DELAY

Until now, we have studied solutions for $t_0=0$

$$\rho[n] = y(t_0 + nT) \text{ with } t_0 = 0 \quad \Rightarrow \quad \begin{aligned} x(iT) &= 1 \quad \text{if} \quad i = 0 \\ x(iT) &= 0 \quad \text{if} \quad i \neq 0 \end{aligned}$$

Anyway, given a function $x(t)$ which satisfies the criterion for $t_0=0$, the function $x'(t) = x(t-t_0)$ satisfies the condition for any t_0

(note that, at the receiver side, the symbol synchronizer circuit is always able to exactly acquire the time sampling t_0)

TX AND RX FILTER

Remember that we have studied the properties of the function $x(t)$, where

$$x(t)=p(t)*q(t)$$

The matched filter $q(t)$ is given by

$$q(t)=p(T-t)$$

$$Q(f)=P(f)^{*}\,e^{-j2\pi fT}$$

TX AND RX FILTER

If $x(t)$=ideal low pass filter or raised cosine filter, what about $p(t)$ and $q(t)$?

If $p(t)$ is an even function $p(t)=p(-t)$

We have $q(t) = p(T-t)=p(t-T)$

We can realize $q(t) = p(t)$

The extra delay T is recovered by the symbol synchronization circuit.

TX AND RX FILTER

We have $X(f) = P(f)\; Q(f)$

If $q(t)=p(t)$ it follows $Q(f)=P(f)$ and

$$X(f) = P(f)^2 \quad \rightarrow \quad P(f) = Q(f) = \sqrt{X(f)}$$

We separate the function $x(t)$ in two identical functions, called **transmission filter** $p(t)$ and **receiver filter** $q(t)$

IDEAL LOW PASS TX FILTER

For the ideal low pass filter $$x(t) = \frac{\sin(\pi t / T)}{\left(\pi t / T\right)}$$

We obtain $$\boxed{p(t) = \frac{1}{\sqrt{T}} \frac{\sin(\pi t / T)}{\left(\pi t / T\right)}}$$

RRC TX FILTER

For the raised cosine filter

$$x(t) = \frac{\sin(\pi t/T)}{(\pi t/T)} \frac{\cos(\alpha \pi t/T)}{1-(2\alpha t/T)^2}$$

We obtain

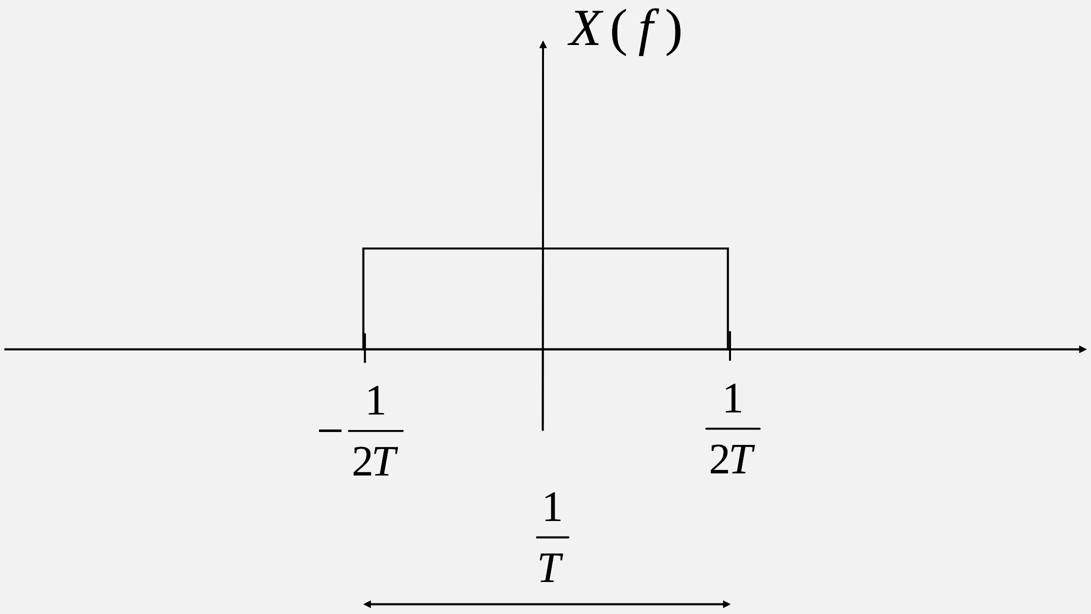
Root Raised Cosine (RRC)

$$\boxed{p(t) = \frac{1}{\sqrt{T}} \frac{\sin(\pi \frac{t}{T}(1-\alpha)) + 4\alpha \frac{t}{T}\cos(\pi \frac{t}{T}(1+\alpha))}{\pi \frac{t}{T}(1-(4\alpha \frac{t}{T})^2)}}$$

IDEAL LOW PASS TX FILTER

Transmission filter $p(t)$: ideal low pass filter

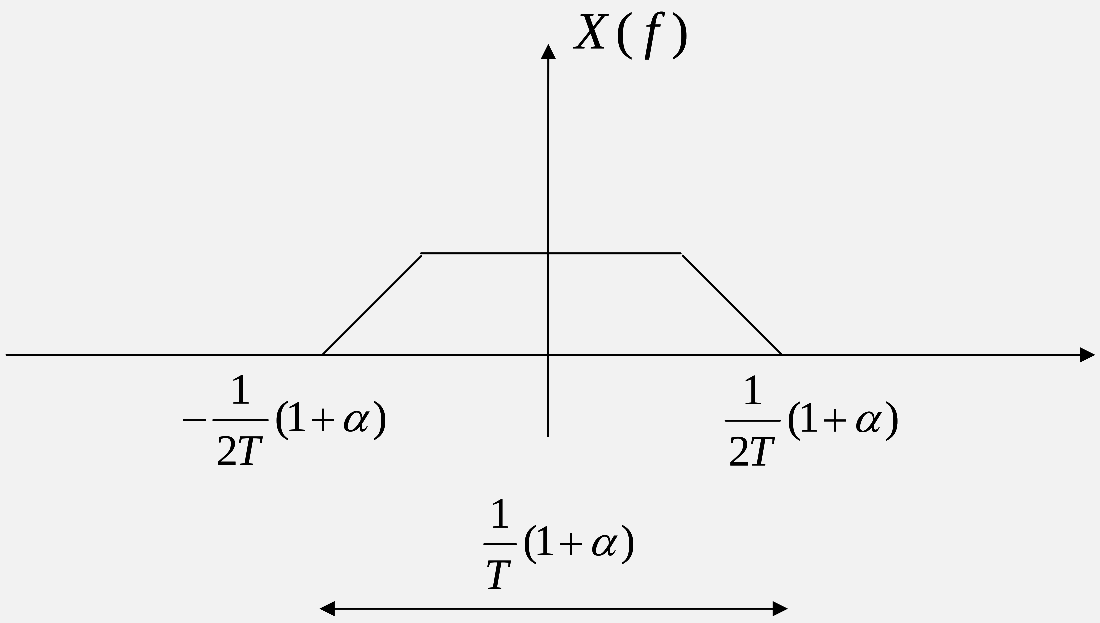
Minimal bandwidth occupancy equal to $\boxed{\frac{1}{2T}}$



RRC TX FILTER

Transmission filter $p(t)$: root raised cosine filter

Bandwidth occupancy equal to $\boxed{\frac{1}{2T}(1+\alpha)}$



LECTURE 9: PULSE-AMPLITUDE MODULATION

2-PAM CONSTELLATION: CHARACTERISTICS

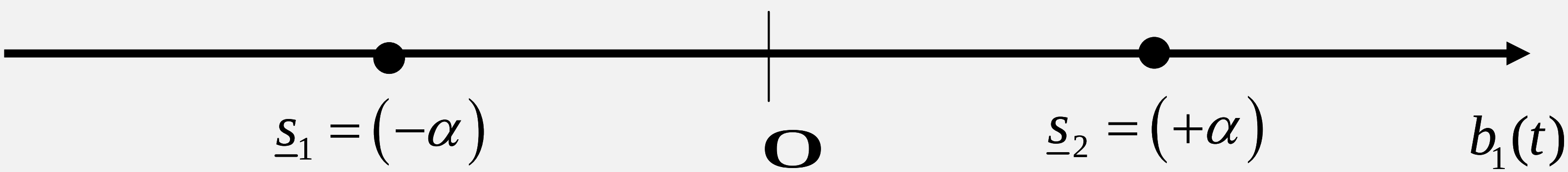
1. Base-band modulation
2. One-dimensional signal space
3. Antipodal binary constellation
4. Information associated to the impulse amplitude
   PAM=Pulse Amplitude Modulation

2-PAM CONSTELLATION: CONSTELLATION

SIGNAL SET $M=\{s_1(t)=-\alpha\, p(t)\,,\, s_2(t)=+\alpha\, p(t)\}$

Versor $b_1(t)=p(t) \quad (d=1)$

VECTOR SET $M=\{s_1=(-\alpha)\,,\, s_2=(+\alpha)\}\subseteq R$



$k=1$ $\boxed{T=T_b}$ $\boxed{R=R_b}$

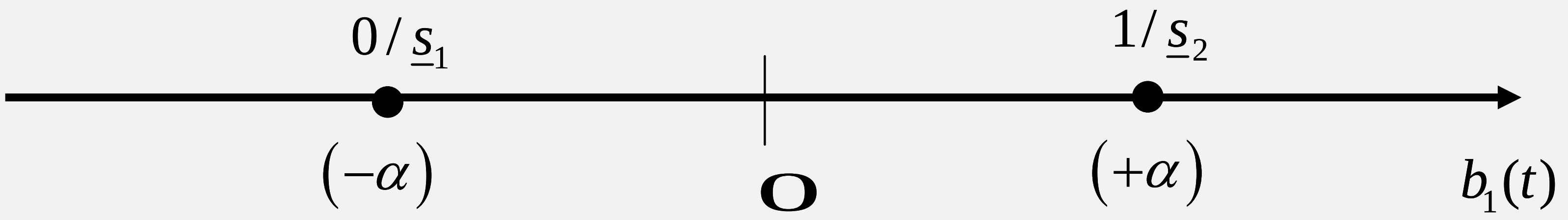
2-PAM CONSTELLATION: BINARY LABELING

(example)

$$e : H_1 \leftrightarrow M$$

$$e(0) = \underline{s}_1$$

$$e(1) = \underline{s}_2$$



2-PAM CONSTELLATION: TRANSMITTED WAVEFORM

$$s(t) = \sum_{n=-\infty}^{+\infty} a[n]\, p(t - nT)$$

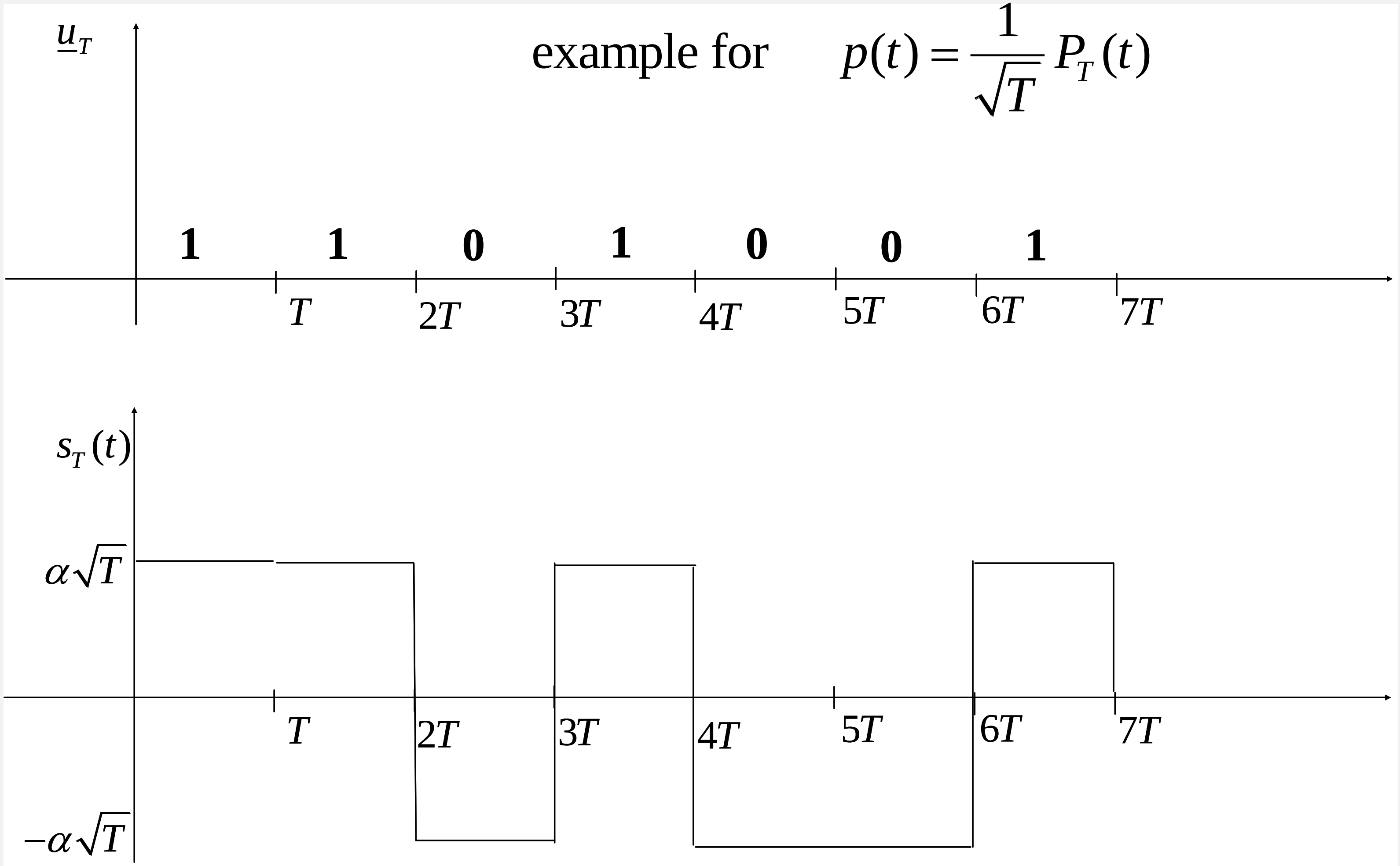
where

$$\boxed{T = T_b}$$

$$a[n] \in \{-\alpha, +\alpha\}$$

2-PAM CONSTELLATION: TRANSMITTED WAVEFORM

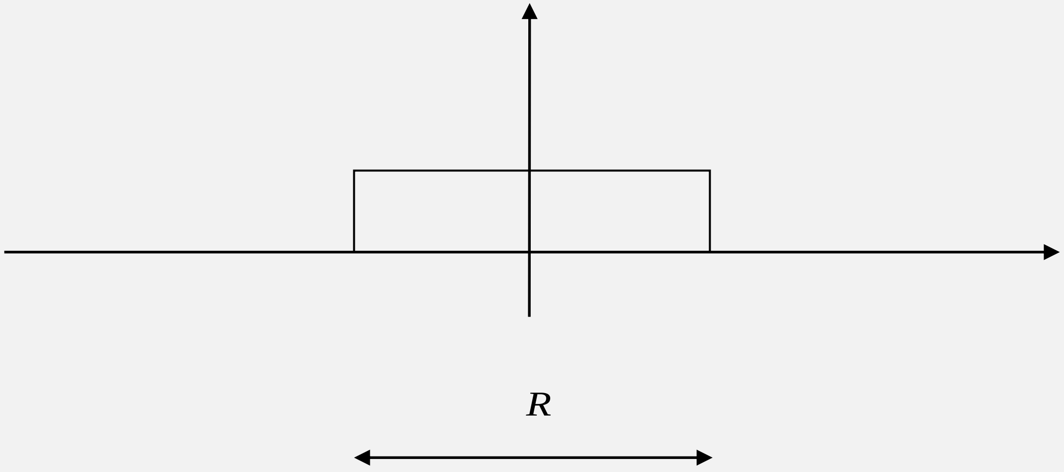
example for $p(t) = \frac{1}{\sqrt{T}} P_T(t)$



2-PAM CONSTELLATION: SIGNAL SPECTRUM

$$G_s(f) = \sigma_a^2 \frac{|P(f)|^2}{T} = x|P(f)|^2 \qquad\qquad x \in R$$

Case 1: $p(t)$ = ideal low pass filter



BANDWIDTH DEFINITION

BANDWIDTH B [Hz] = frequency interval containing (a significant portion of) $G_s(f)$

Alternative definitions:

1. TOTAL BANDWIDTH (contains all the spectrum)
2. Half power bandwidth (-3dB below the maximum)
3. Equivalent noise bandwidth (rectangular (with amplitude=maximum) containing the whole power)
4. Null to null bandwidth (main lobe width)

BANDWIDTH DEFINITION

5. 99% (99.9% etc.) bandwidth or fractional power containment bandwidth (contains the 99% of the power)

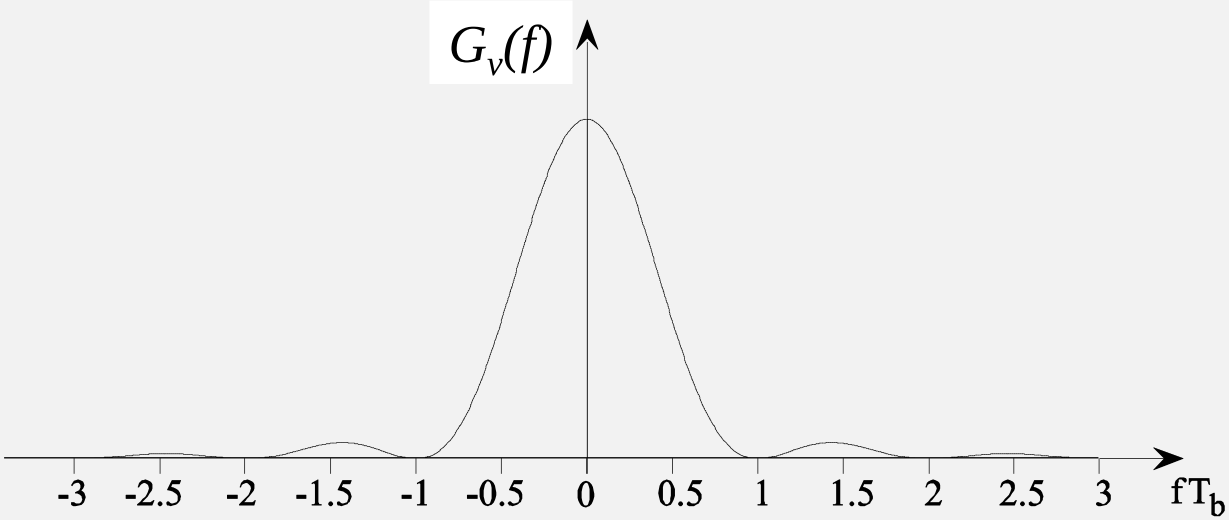
6. - 35 dB (-50 dB) or bounded power spectral density bandwidth (outside the interval, Gs(f) is -35 dB below its maximum)

EXAMPLE

Binary antipodal constellation with rectangular pulse

$$p(t) = b_1(t) = \frac{1}{\sqrt{T}} P_T(t)$$

$$\boxed{G_s(f) = A^2 T \left(\frac{\sin(\pi f T)}{(\pi f T)} \right)^2}$$



EXAMPLE

Alternative definitions:

1. TOTAL BANDWIDTH = ∞
2. Half power bandwidth $\approx 0.44/T_b$
3. Equivalent noise bandwidth = $0.5/T_b$
4. Null to null bandwidth = $1/T_b$
5. 99% bandwidth $\approx 10.29/T_b$
6. -35 dB bandwidth $\approx 17.57/T_b$

6. -50 dB bandwidth $\approx 100.52/T_b$

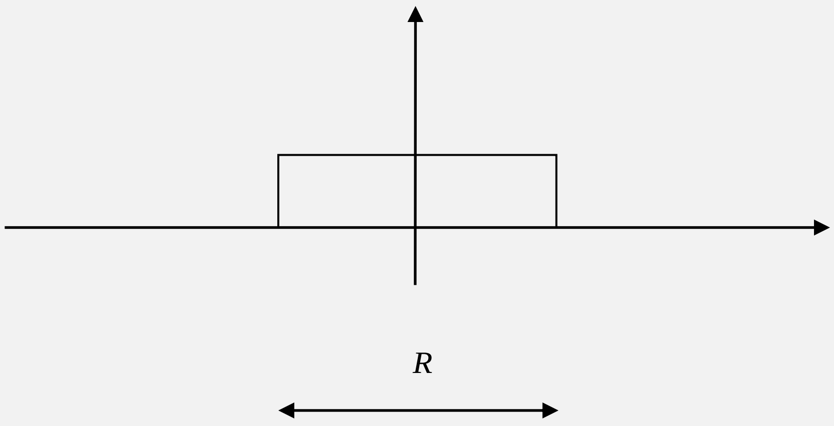
SPECTRAL EFFICIENCY

SPECTRAL EFFICIENCY [bps/Hz]

$$\boxed{\eta = \frac{R_b}{B}}$$

2-PAM CONSTELLATION: BANDWIDTH AND SPECTRAL EFFICIENCY

Case 1: $p(t)$ = ideal low pass filter



Total bandwidth (ideal case)

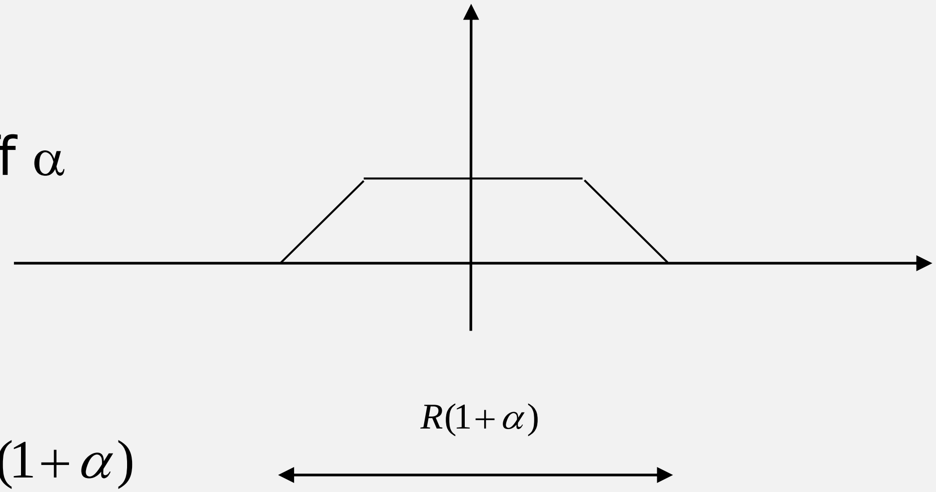
$$\boxed{B_{id} = \frac{R}{2} = \frac{R_b}{2}}$$

Spectral efficiency (ideal case)

$$\boxed{\eta_{id} = \frac{R_b}{B_{id}} = 2\ bps/Hz}$$

2-PAM CONSTELLATION: BANDWIDTH AND SPECTRAL EFFICIENCY

Case 2: $p(t)$ = RRC filter with roll off α



Total bandwidth $B = \frac{R}{2}(1+\alpha) = \frac{R_b}{2}(1+\alpha)$

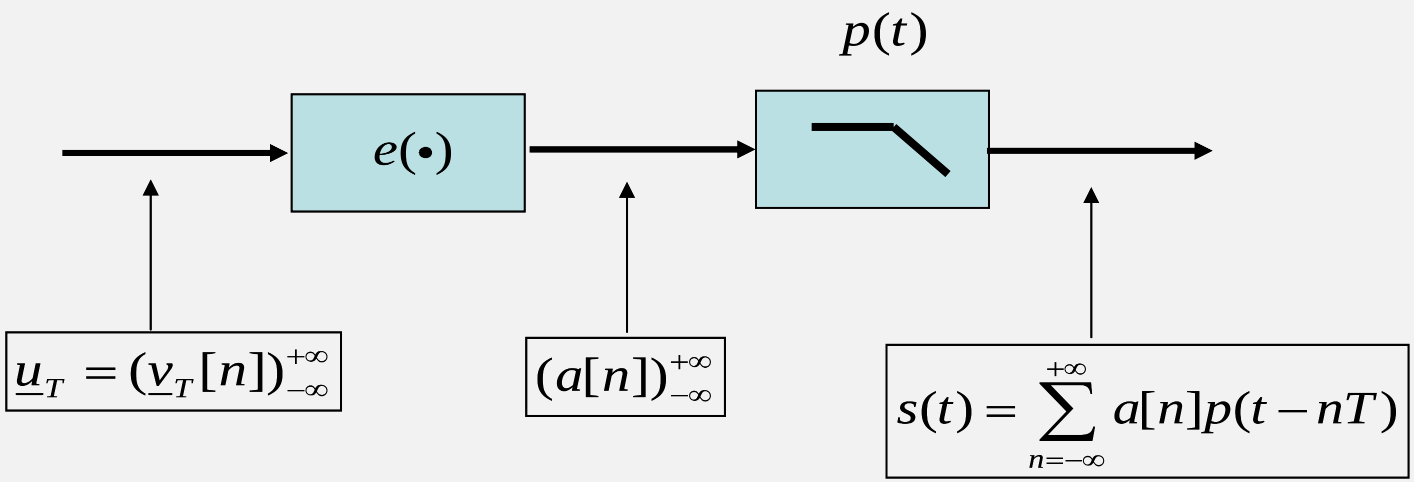
Spectral efficiency $\boxed{\eta = \frac{R_b}{B} = \frac{2}{(1+\alpha)}\, bps/Hz}$

EXERCIZE

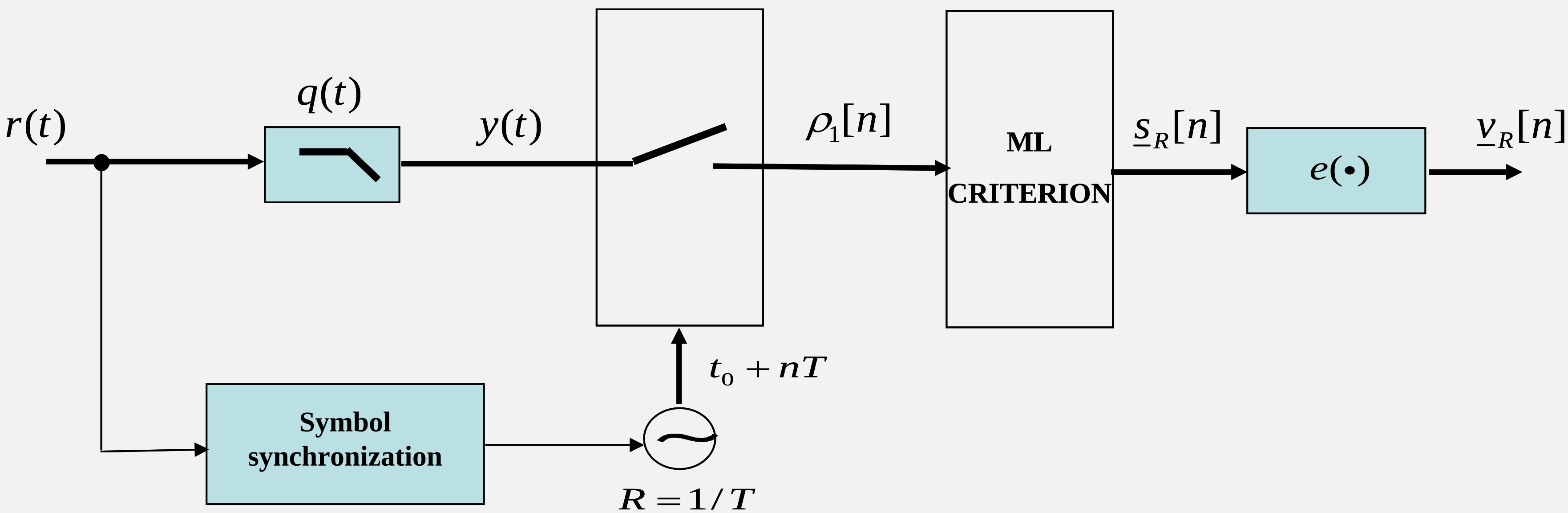
Given a baseband channel with bandwidth B up to 4000 Hz, compute the maximum bit rate R_b we can transmit over it with a 2-PAM constellation in the two cases:

- Ideal low pass filter
- RRC filter with α=0.25

2-PAM CONSTELLATION: MODULATOR



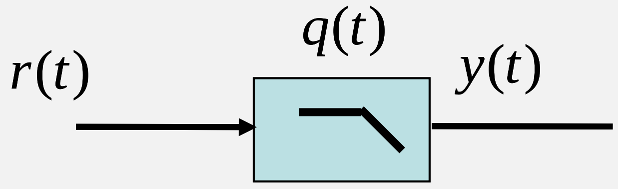
2-PAM CONSTELLATION: DEMODULATOR



EYE DIAGRAM

Given the matched filter output waveform

- Divide it in segments of duration $2T$
- Overlap them (oscilloscope)

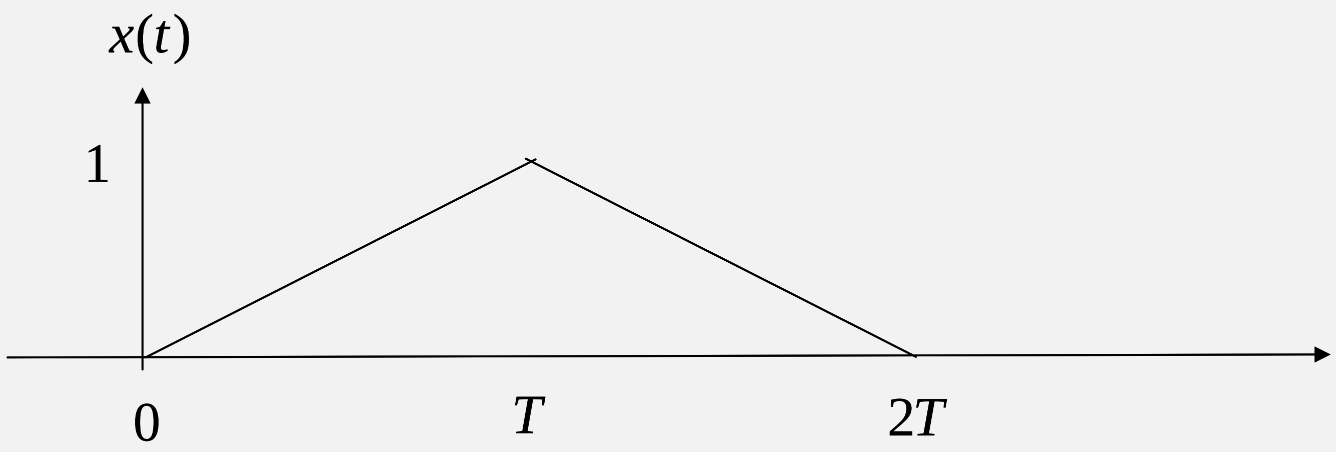


EXAMPLE

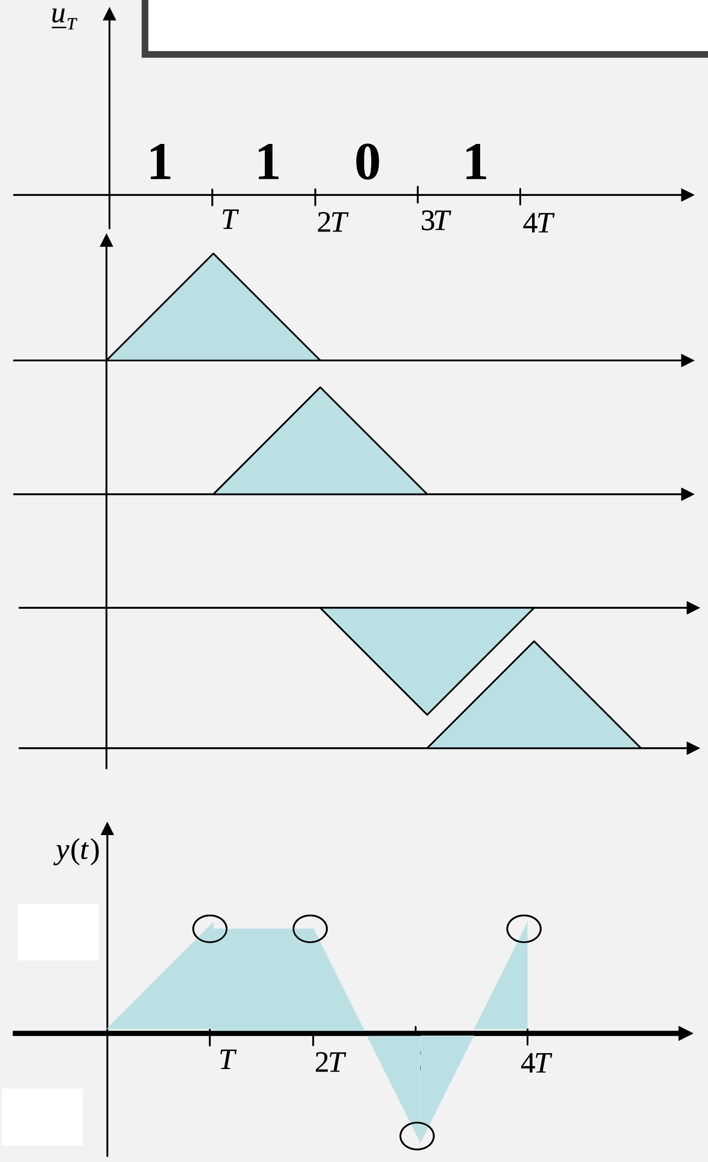
$$p(t) = b_1(t) = \frac{1}{\sqrt{T}} P_T(t)$$

$$x(t) = p(t) * q(t)$$

$$q(t) = p(T - t) = \frac{1}{\sqrt{T}} P_T(t)$$



EXAMPLE

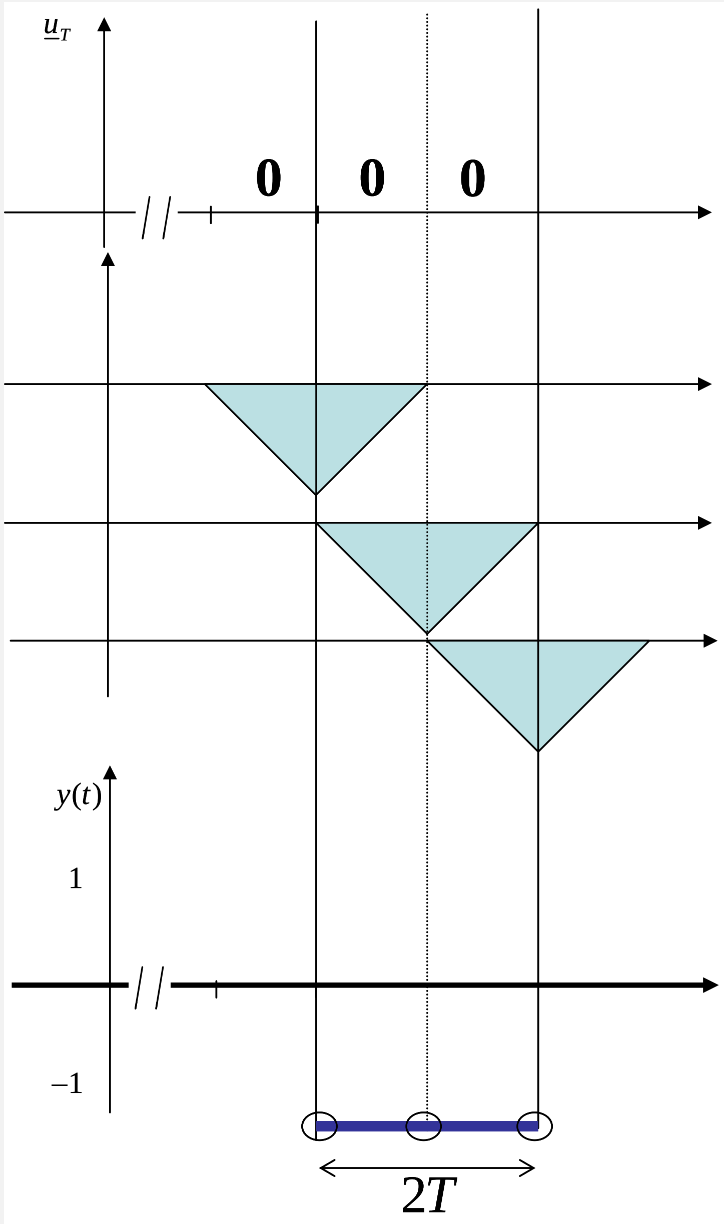


$$y(t) = \sum_n a[n]x(t - nT)$$

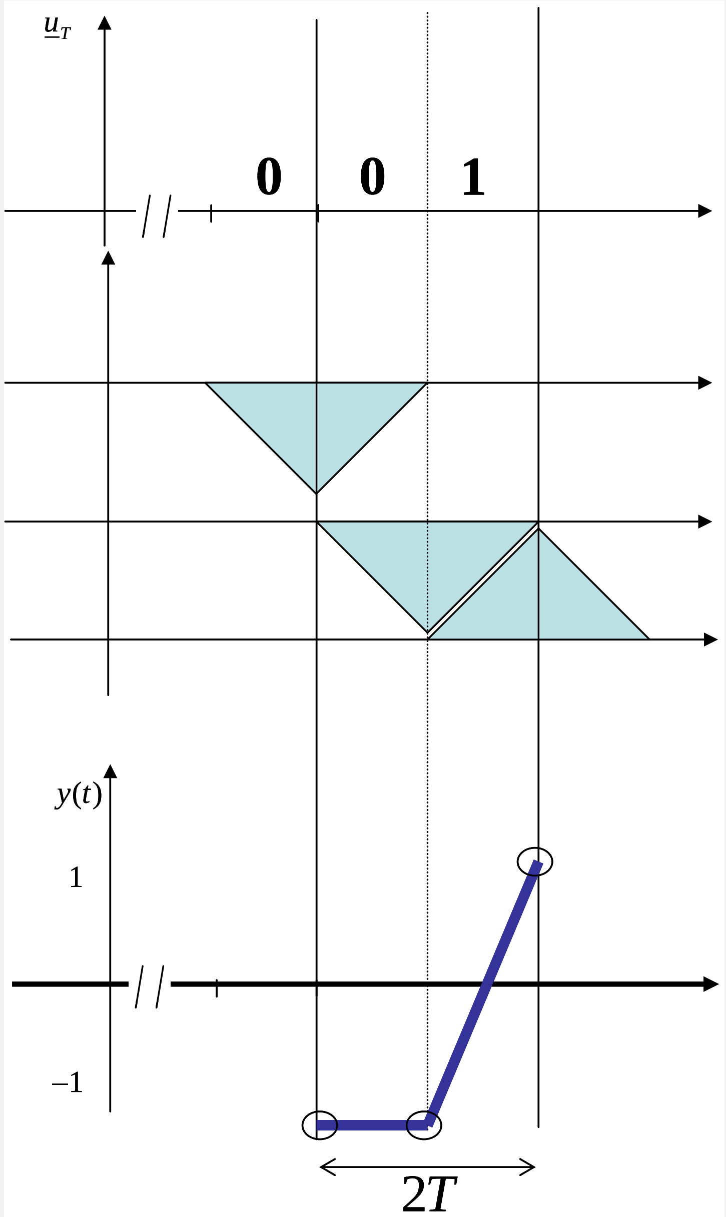
$$\rho[n] = y(T + nT) = a[n]$$

EXAMPLE

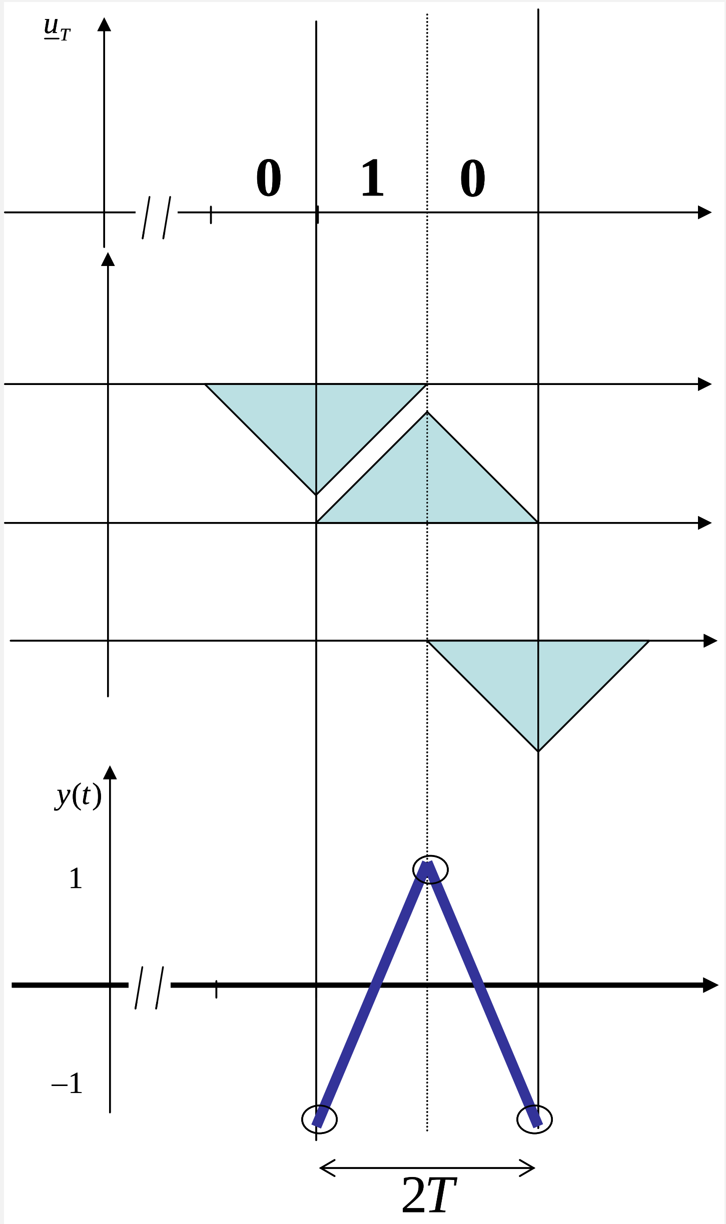
Consider all possible segments of duration $2T$



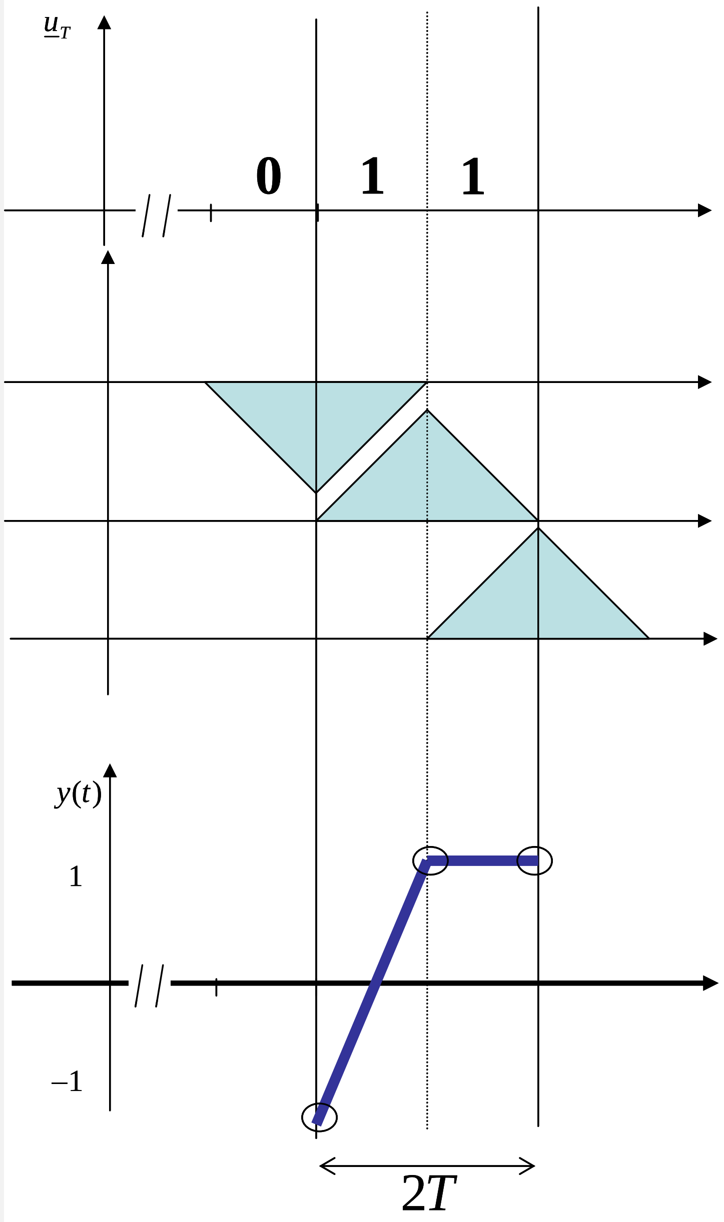
EXAMPLE



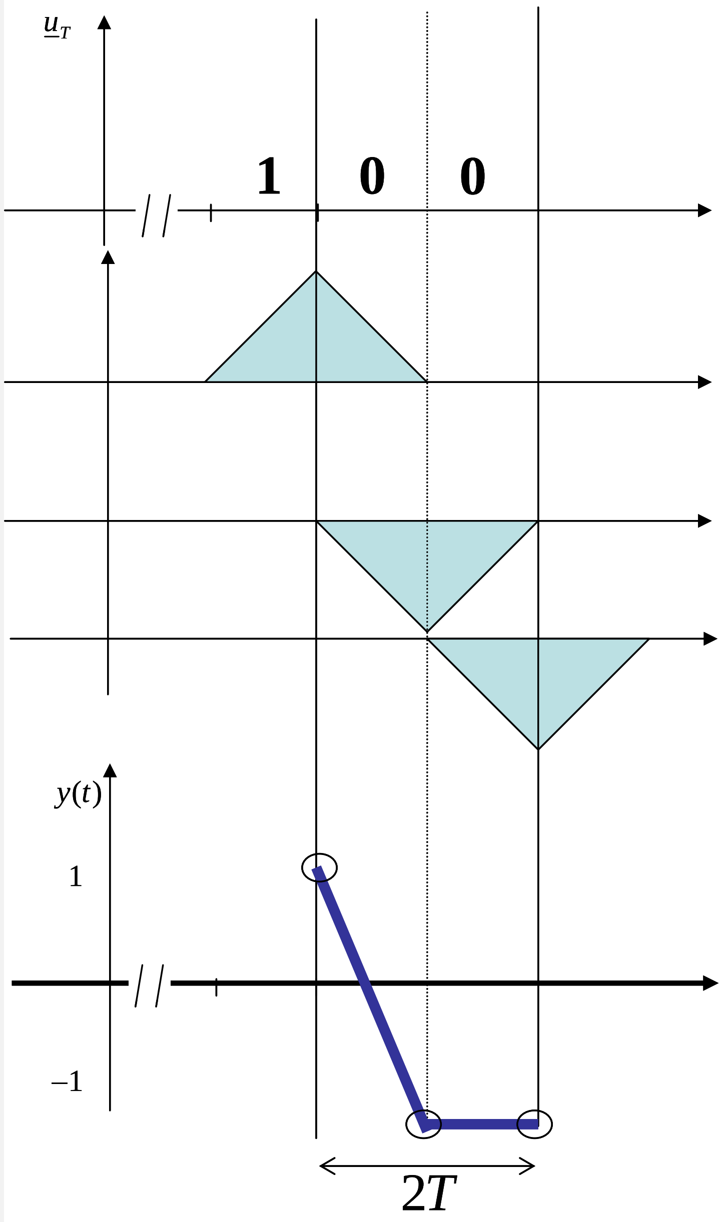
EXAMPLE



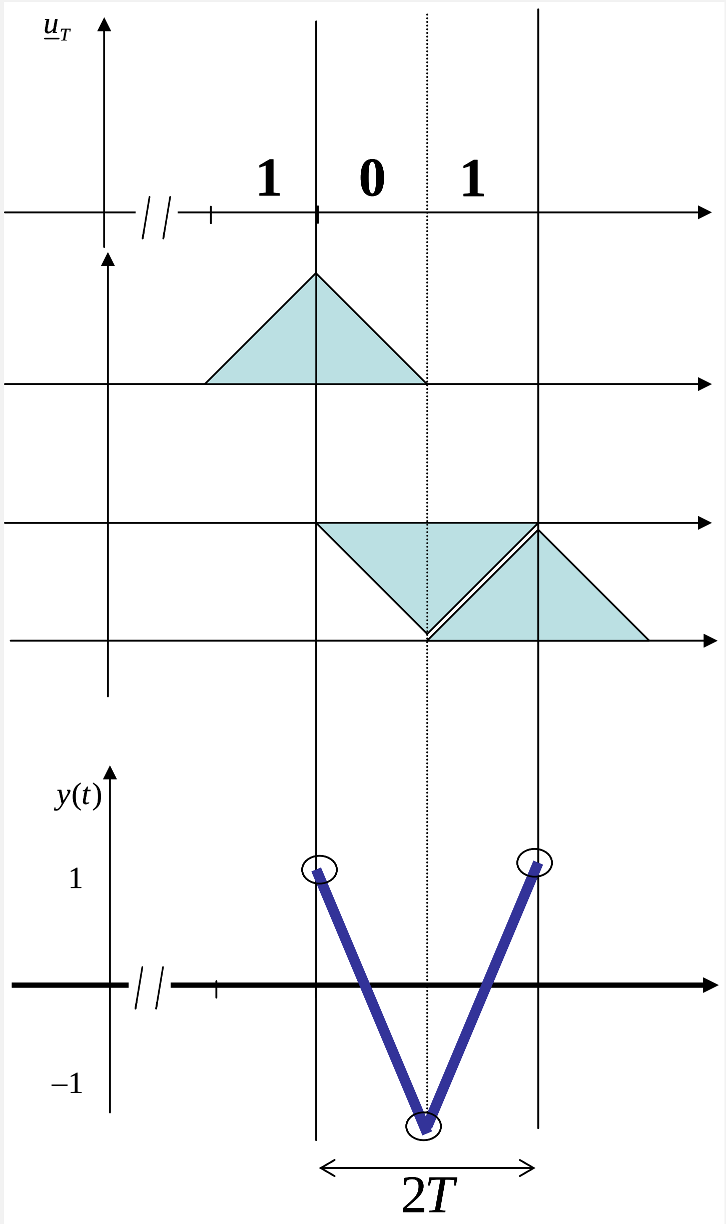
EXAMPLE



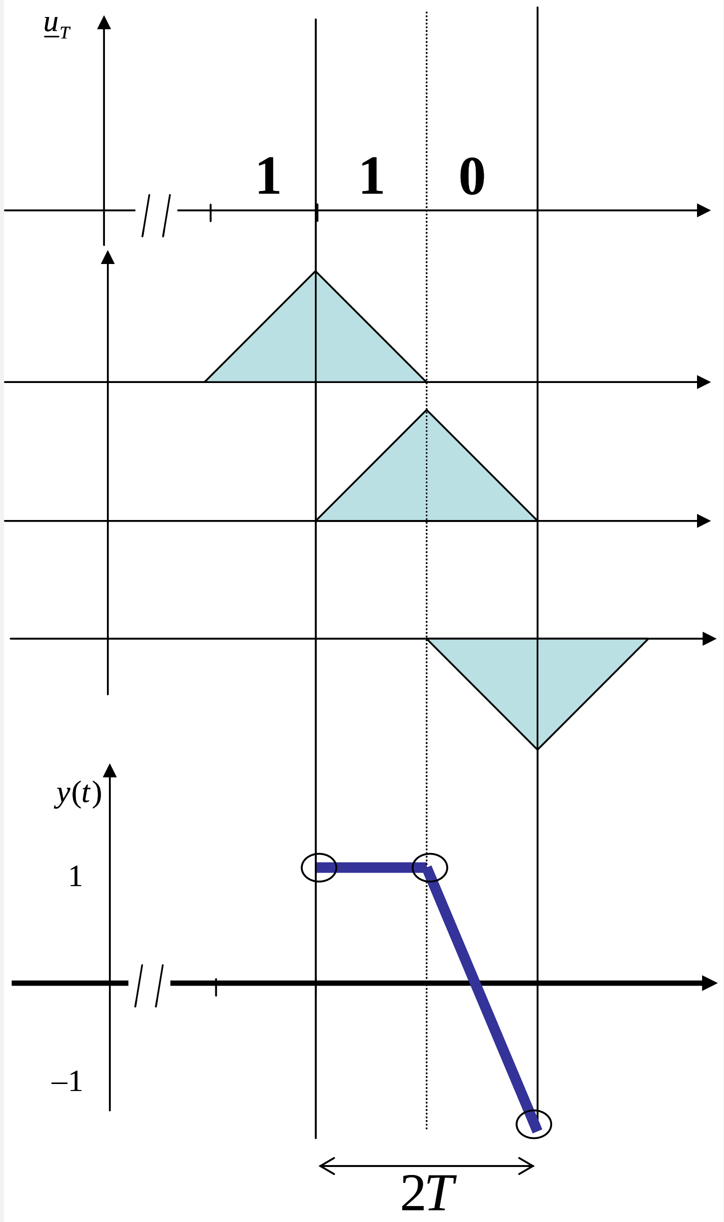
EXAMPLE



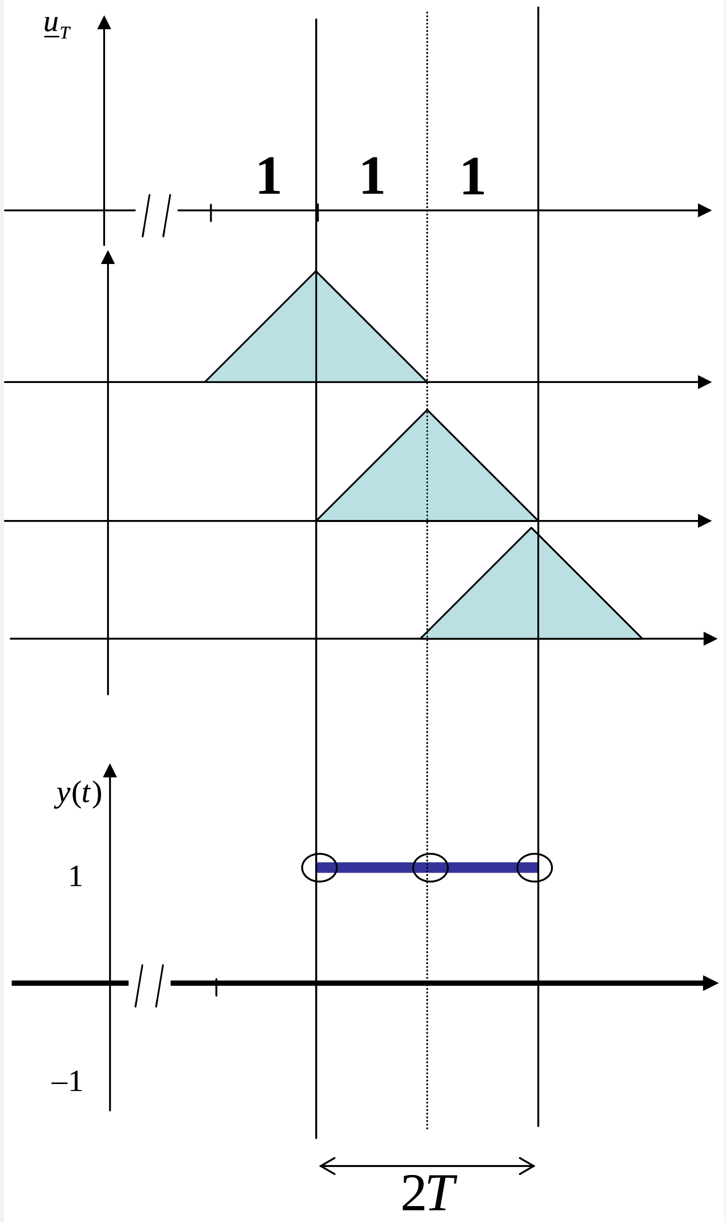
EXAMPLE



EXAMPLE

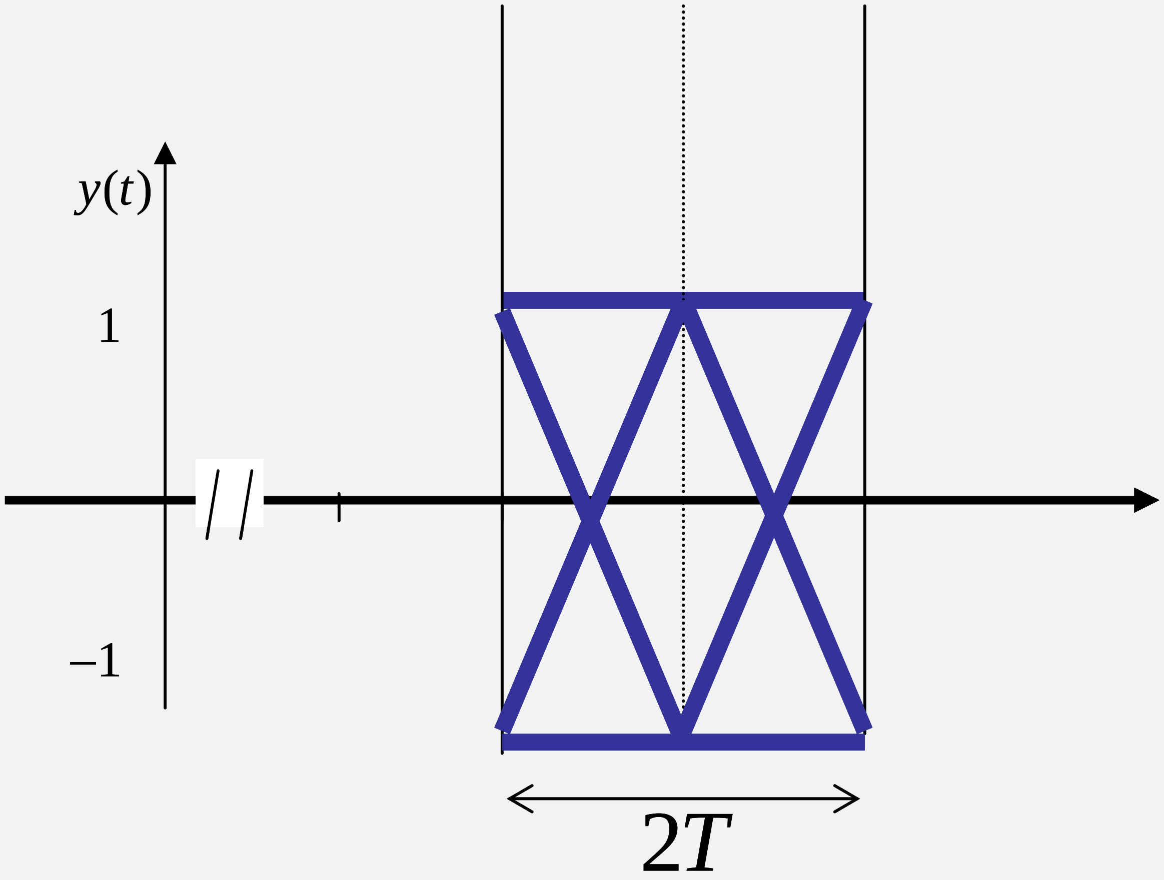


EXAMPLE



EXAMPLE

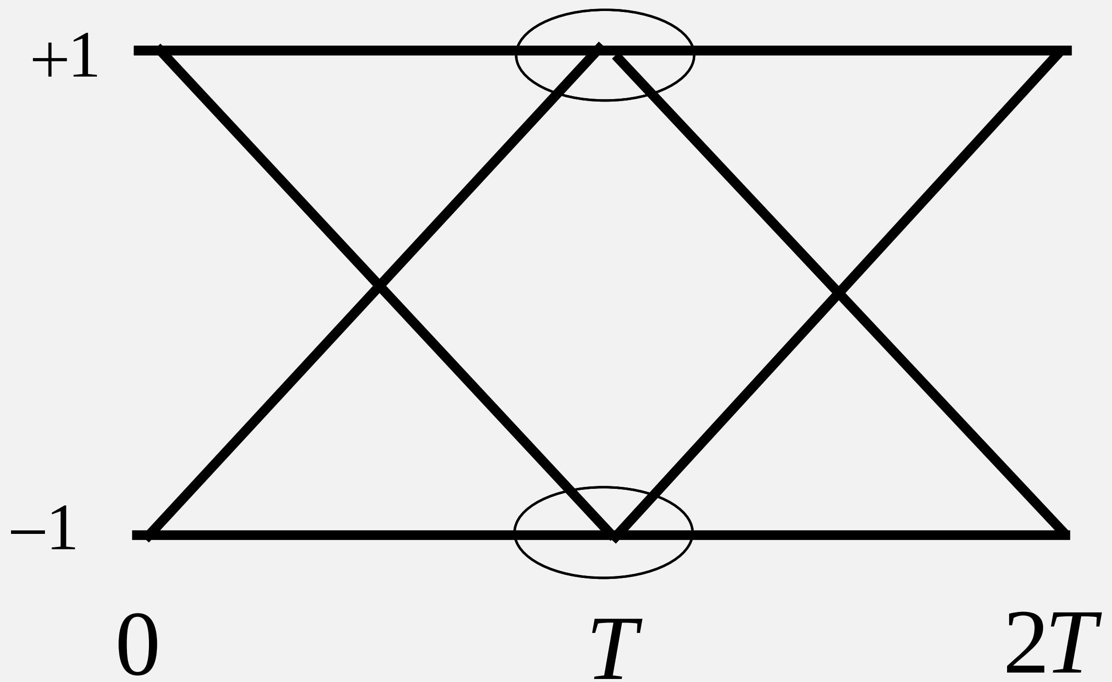
Overlap all the segments



EYE DIAGRAM

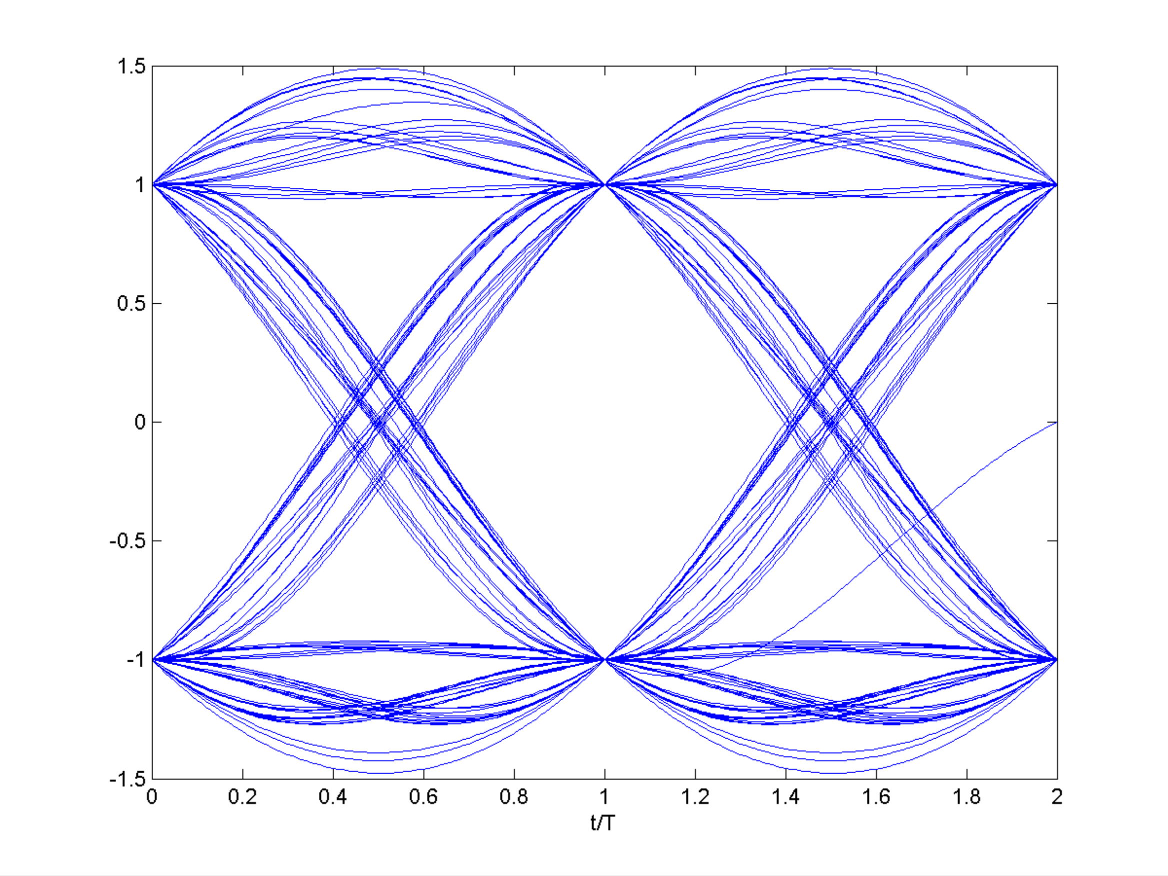
2-PAM constellation with rectangular window

$$p(t) = \frac{1}{\sqrt{T}} P_T(t)$$



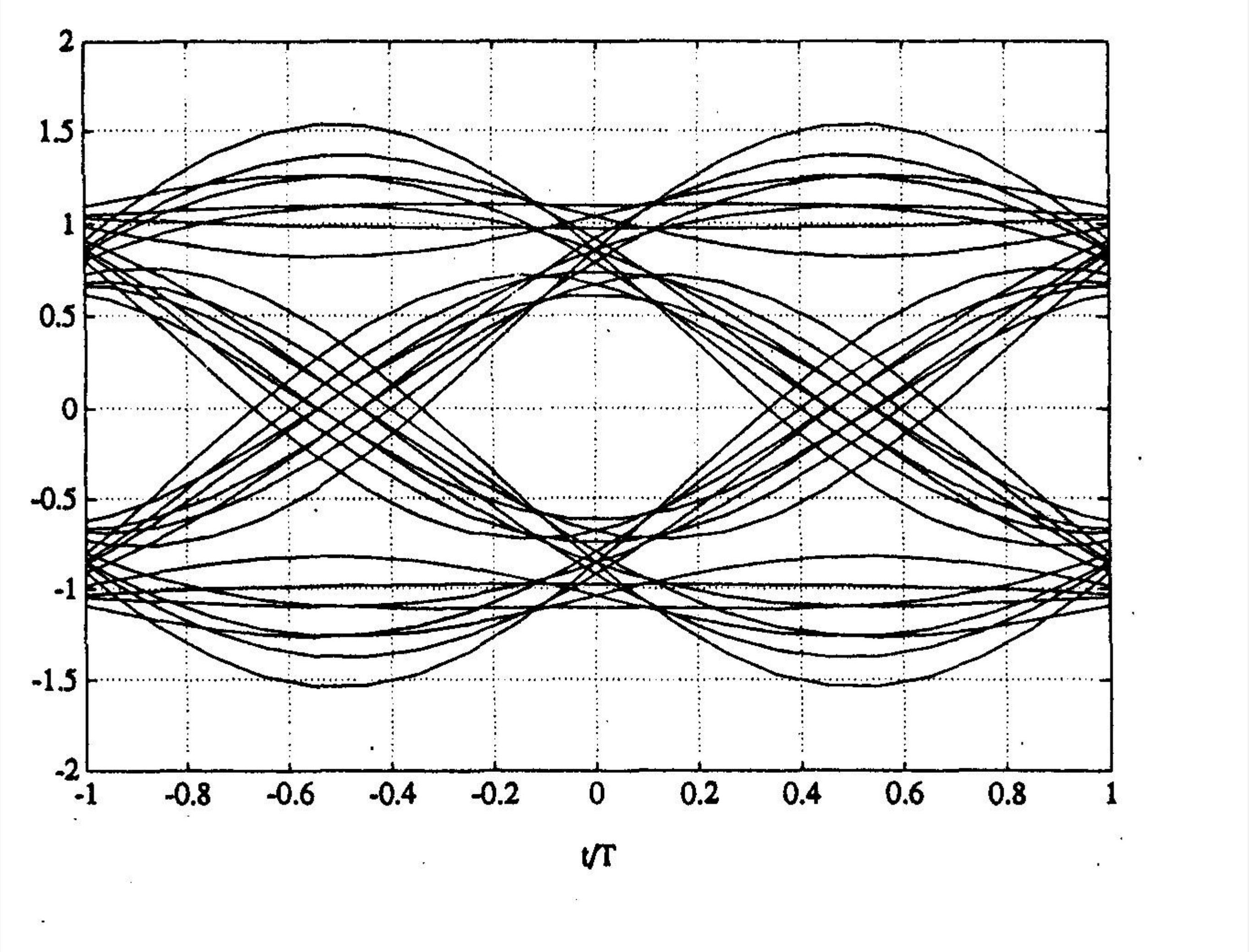
EYE DIAGRAM

2-PAM constellation with RRC filter (α=0.5)



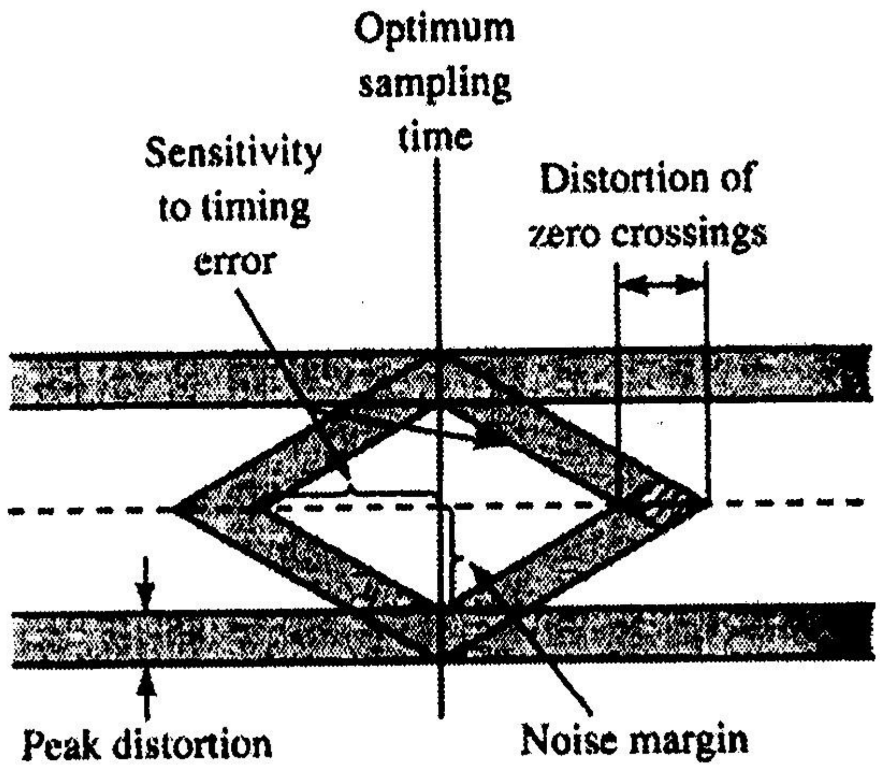
EYE DIAGRAM

2-PAM constellation in real conditions



EYE DIAGRAM

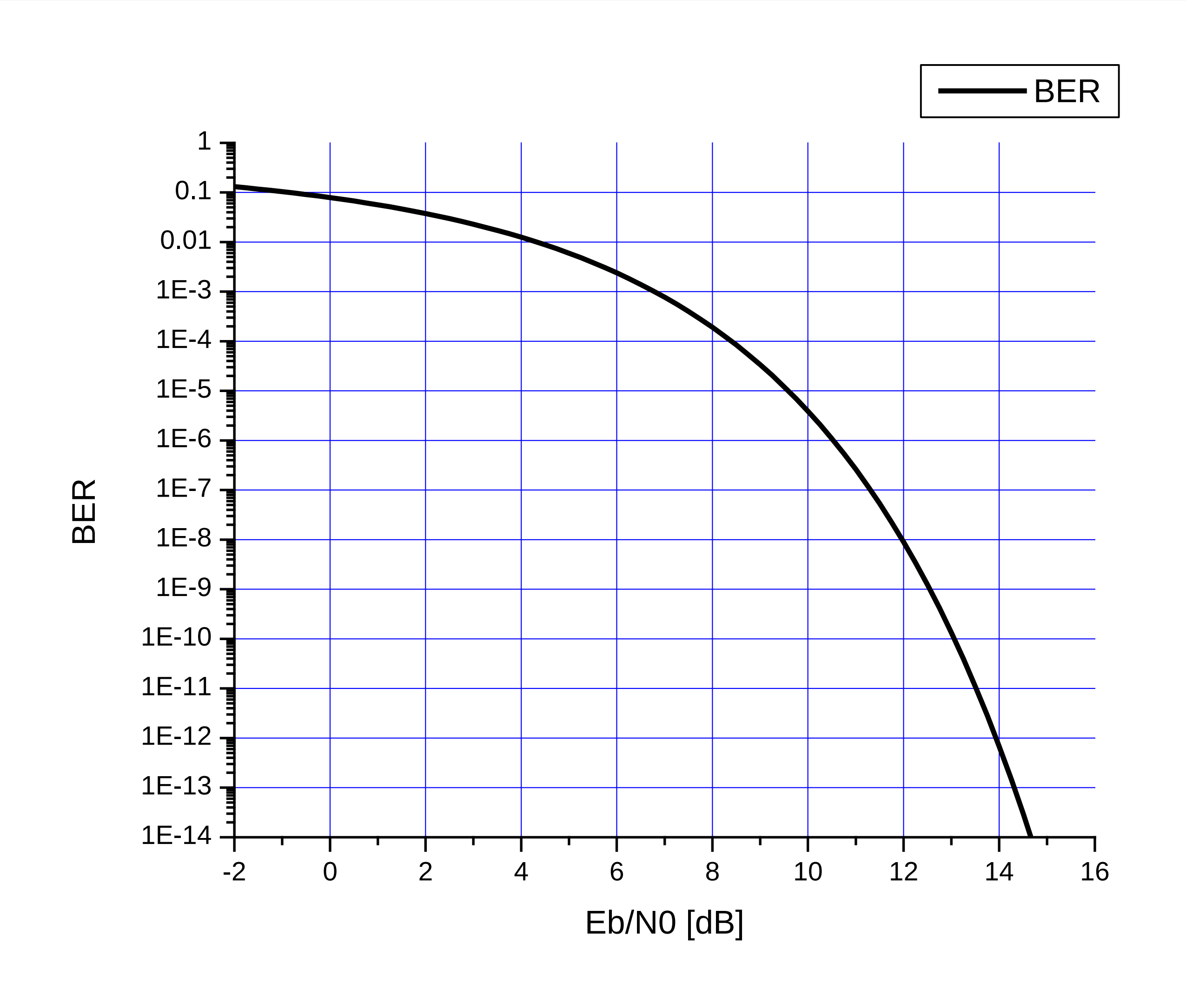
Fundamental quantities



2-PAM CONSTELLATION: ERROR PROBABILITY

$$BER = \frac{1}{2} erfc \sqrt{\frac{E_b}{N_0}}$$

ERROR PROBABILITY



LECTURE 9-2: PULSE-AMPLITUDE MODULATION

INTRODUCTION TO LINE CODING

Techniques used for improving baseband transmission.

Focus on a 2-PAM modulation. Let us suppose to modify the TX filter $p(t)$ for

- Spectral shaping: shaping of the transmitted waveform spectrum for matching the channel response characteristics

- Expedite the recovery operation at the receiver side.

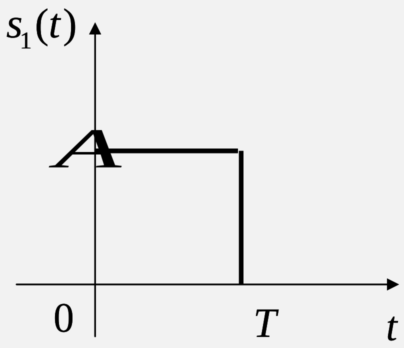
Typical examples are:

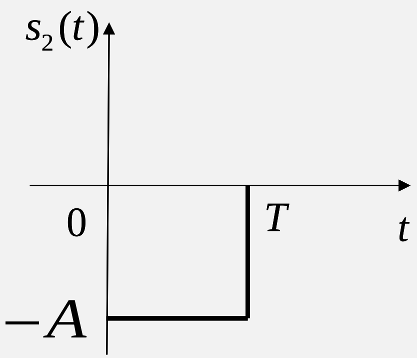
in the first case, spectral efficiency improvement and/or reduction of transmitted power around DC (where many wired connections show bad response)

in the second case, symbol synchronization

BIPOLAR NRZ (NON RETURN TO ZERO)

Signal set $\boxed{M = \{ s_1(t) = +AP_T(t)\, ,\, s_2(t) = -AP_T(t) \}}$





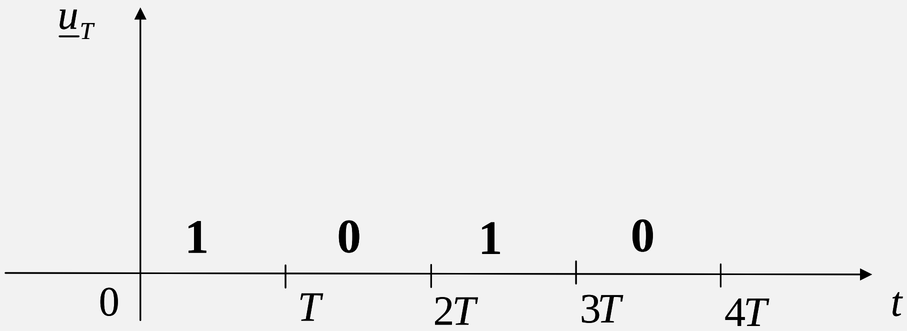
Versor $b_1(t) = \frac{1}{\sqrt{T}} P_T(t)$

Vector set $M = \{ \underline{s}_1 = (+\alpha)\, ,\, \underline{s}_2 = (-\alpha) \}$

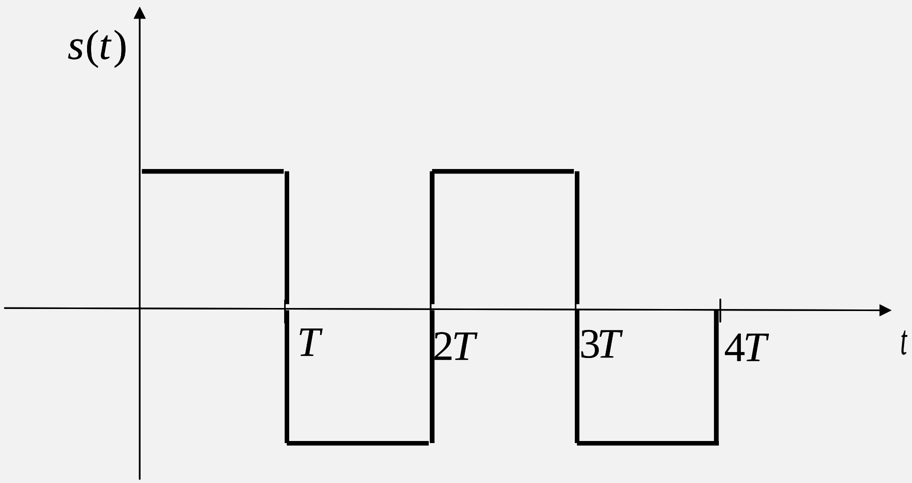
(it coincides with a <u>2-PAM with rectangular pulse)</u>

BIPOLAR NRZ

Transmitted waveform



$$s(t) = \sum_n a[n]p(t - nT)$$

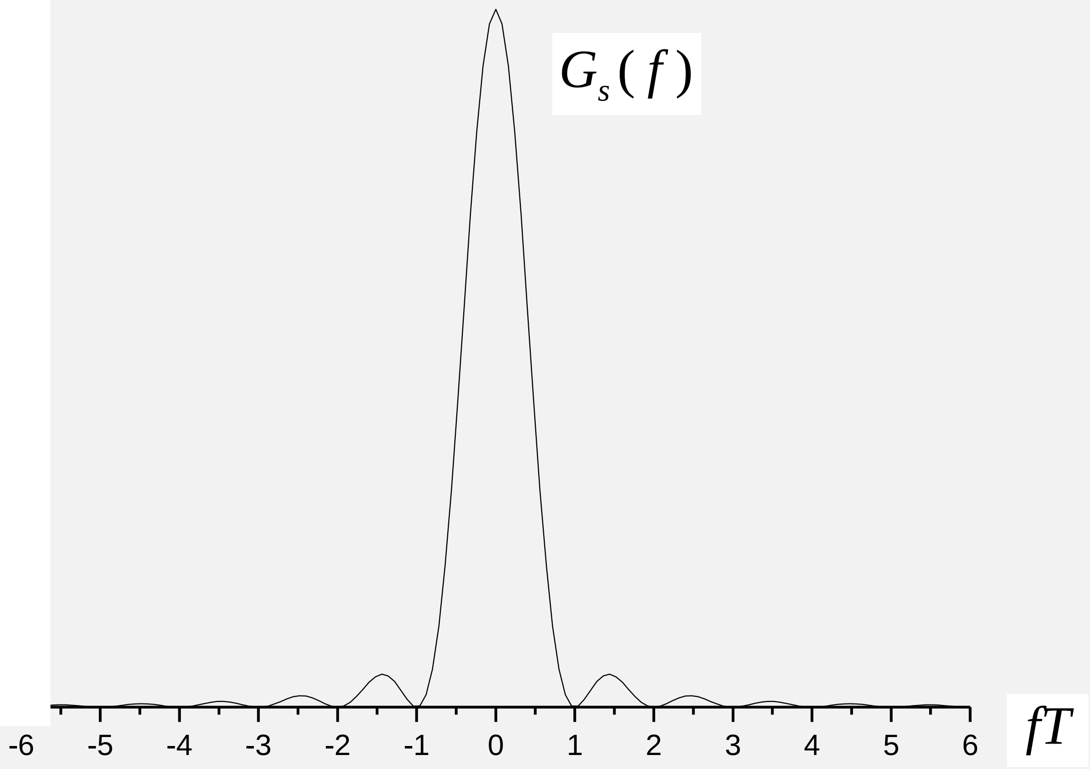


$$a[n] \in \{+\alpha, -\alpha\}$$

BIPOLAR NRZ

Signal spectrum

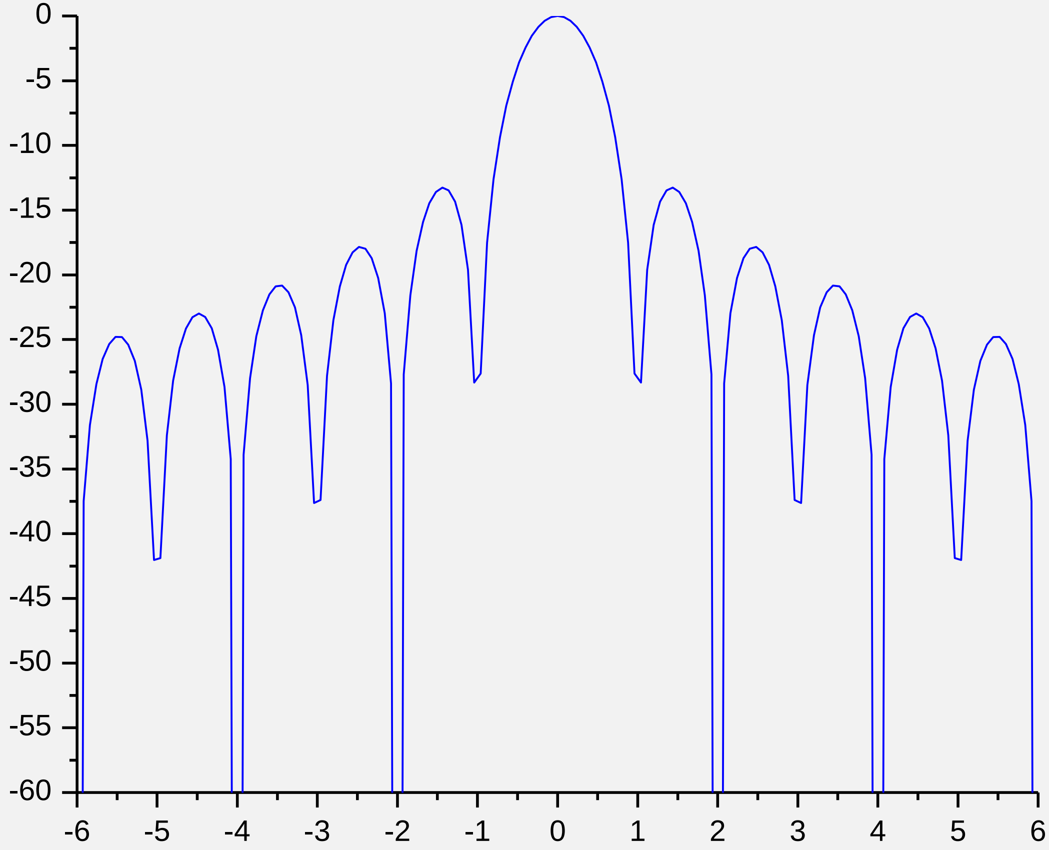
$$G_s(f) = \sigma_a^2 \frac{|P(f)|^2}{T} = A^2 T \mathrm{sinc}^2(fT)$$



BIPOLAR NRZ

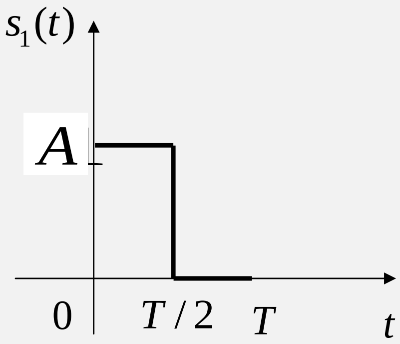
Signal spectrum

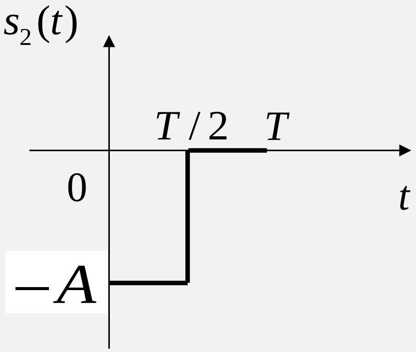
$$G_s(f) = \sigma_a^2 \frac{|P(f)|^2}{T} = A^2 T \mathrm{sinc}^2(fT)$$



BIPOLAR RZ (RETURN TO ZERO)

Signal set $\boxed{M = \{ s_1(t) = +AP_{T/2}(t)\,,\, s_2(t) = -AP_{T/2}(t) \}}$



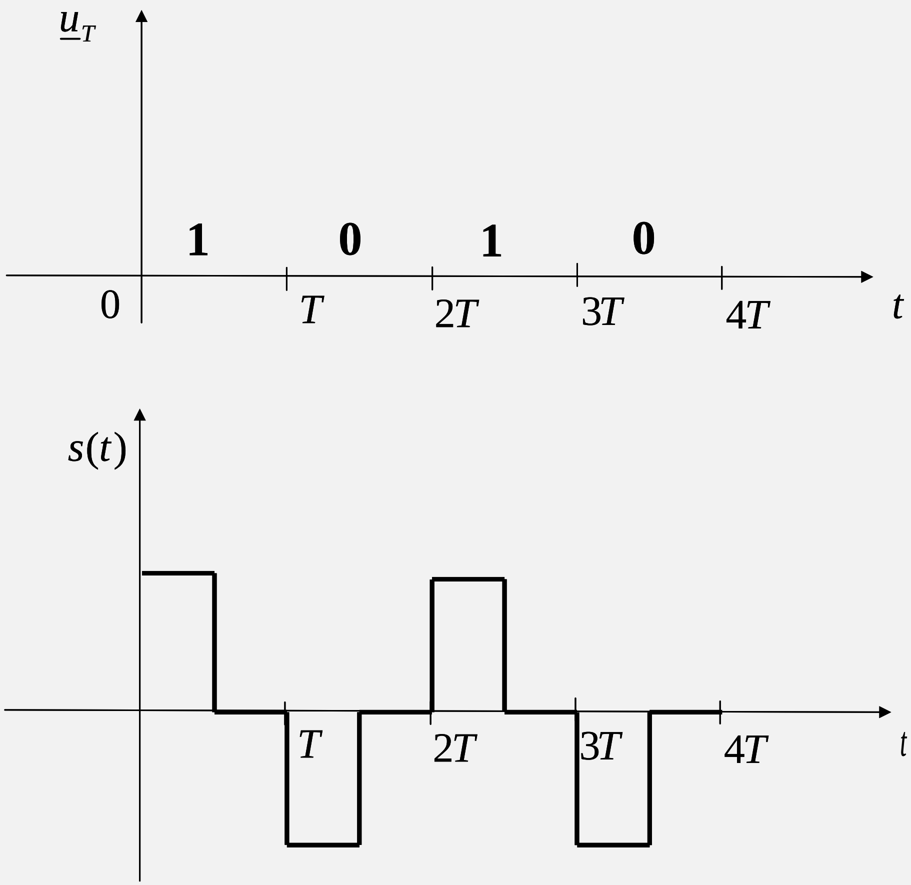


Versor $b_1(t) = \sqrt{\frac{2}{T}} P_{T/2}(t)$

Vector set $M = \{ \underline{s_1} = (+\alpha)\,,\, \underline{s_2} = (-\alpha) \}$

BIPOLAR RZ

Transmitted waveform



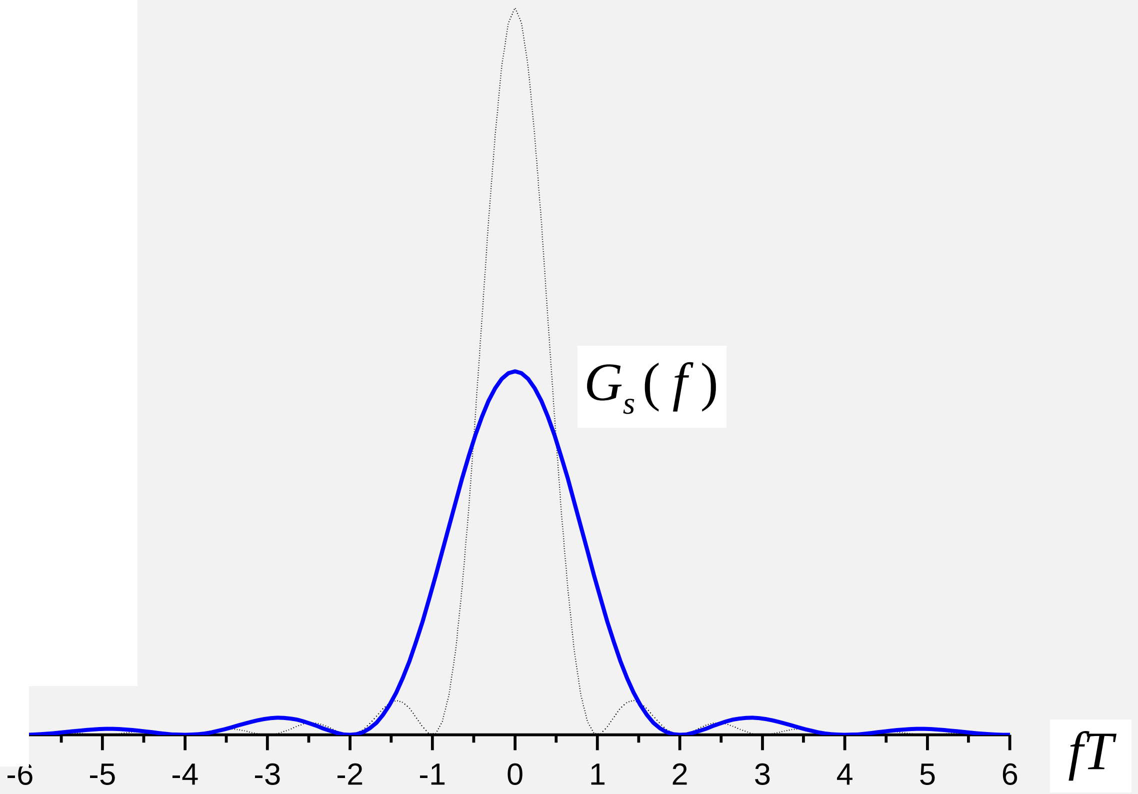
$$s(t)=\sum_n a[n]p(t-nT)$$

$$a[n]\in\{+\alpha,-\alpha\}$$

BIPOLAR RZ

Signal spectrum

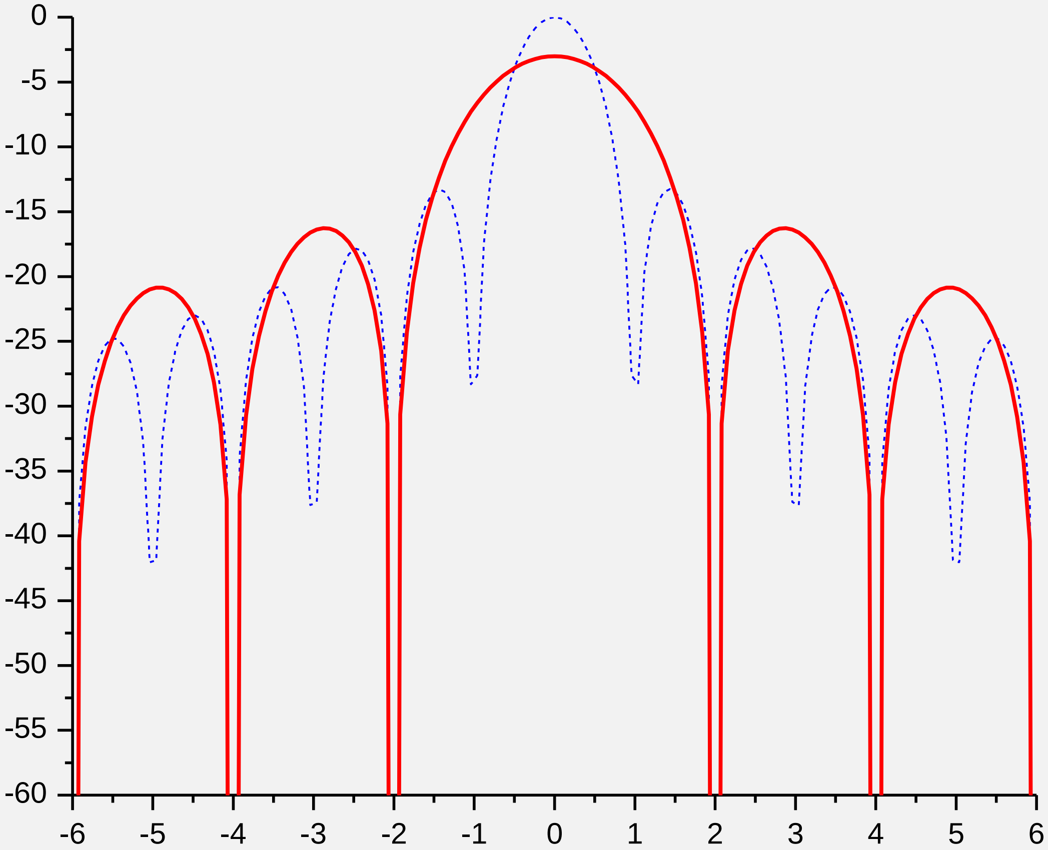
$$G_s(f) = \sigma_a^2 \frac{|P(f)|^2}{T} = \frac{A^2 T}{4} \mathrm{sinc}^2(fT/2)$$



BIPOLAR RZ

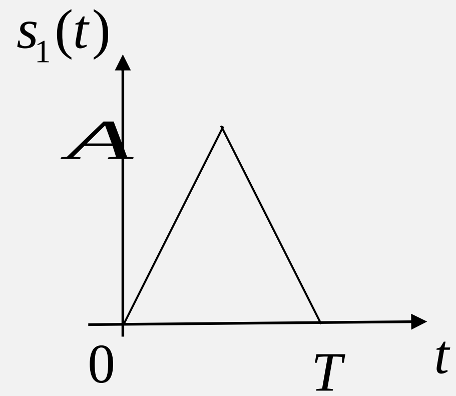
Signal spectrum

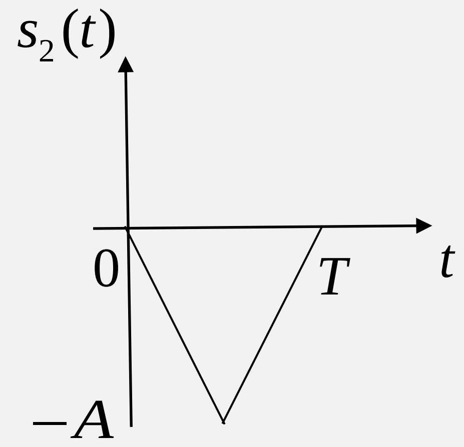
$$G_s(f) = \sigma_a^2 \frac{|P(f)|^2}{T} = \frac{A^2 T}{4} \mathrm{sinc}^2(fT/2)$$



EXAMPLE: BIPOLAR TRIANGULAR

Signal set $$\boxed{M = \{ s_1(t) = +A\Delta_T(t)\,,\, s_2(t) = -A\Delta_T(t) \}}$$



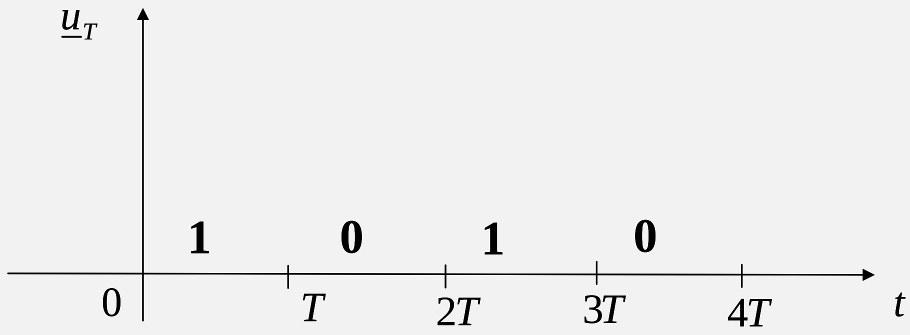


Versor $$b_1(t) = \sqrt{\frac{3}{T}}\Delta_T(t)$$

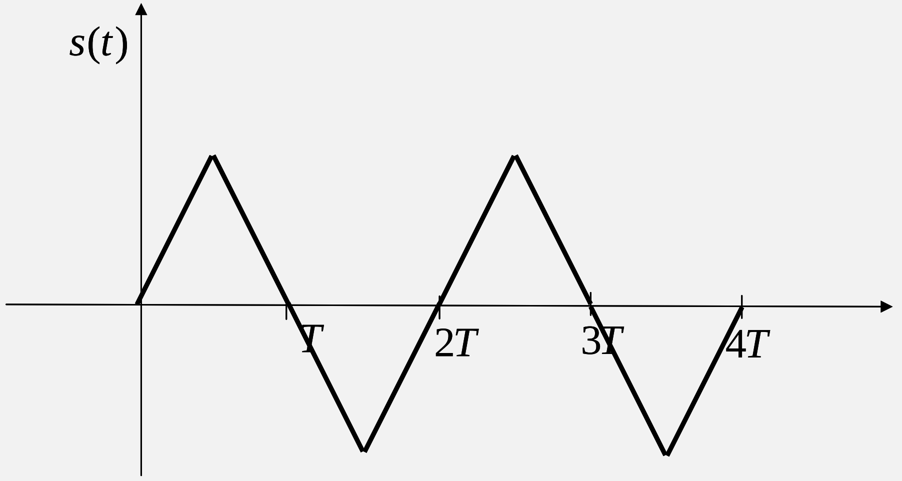
Vector set $$M = \{ \underline{s_1} = (+\alpha)\,,\, \underline{s_2} = (-\alpha) \}$$

EXAMPLE: BIPOLAR TRIANGULAR

Transmitted waveform



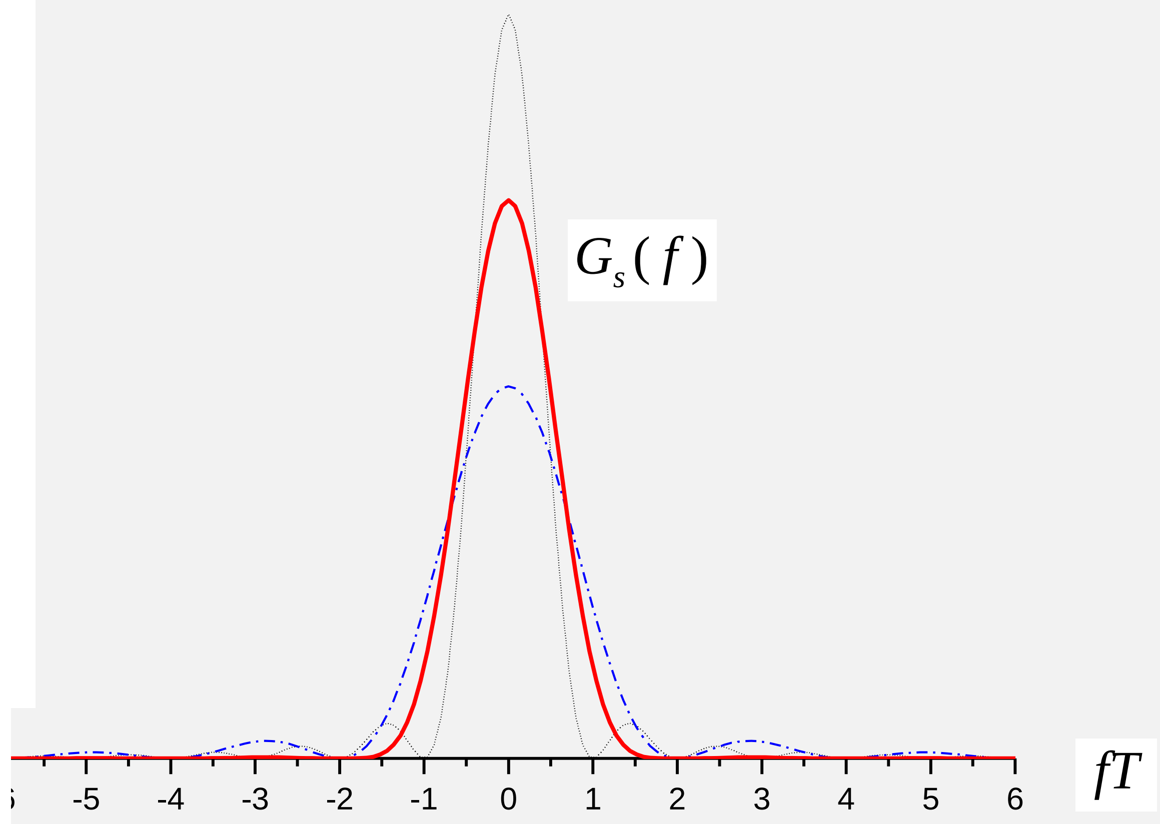
$$s(t) = \sum_n a[n]p(t - nT)$$



$$a[n] \in \{+\alpha, -\alpha\}$$

EXAMPLE: BIPOLAR TRIANGULAR

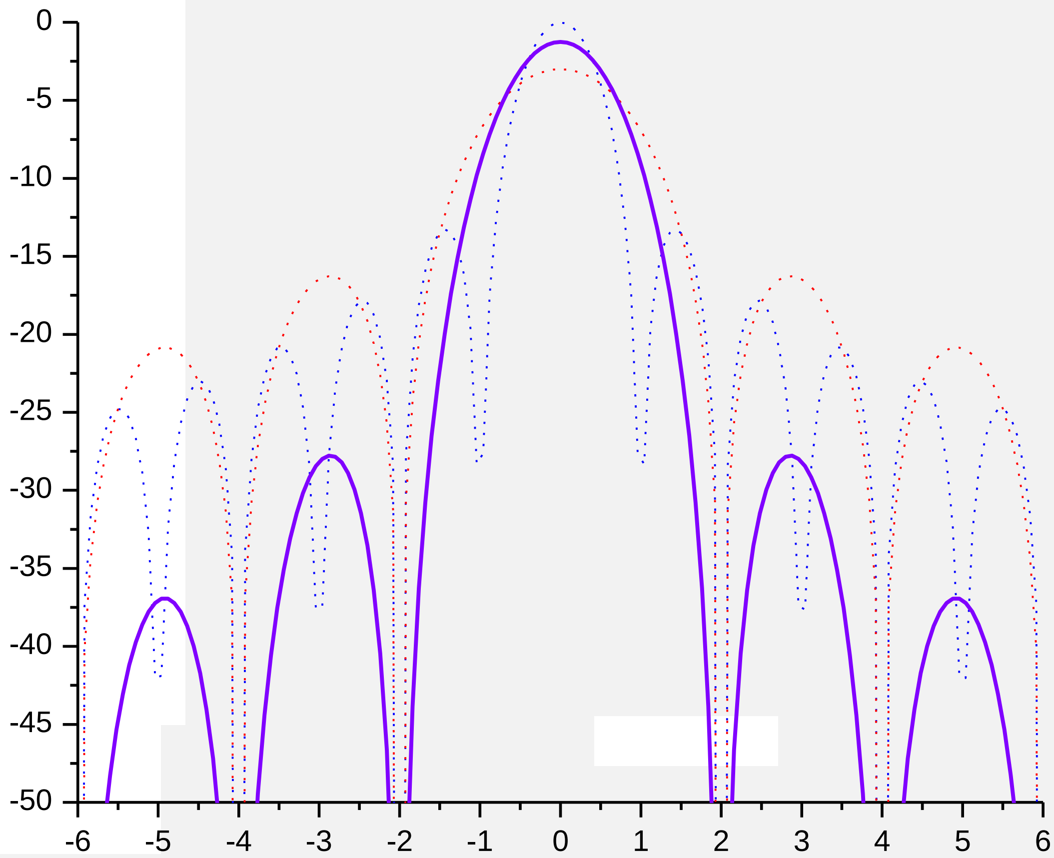
Signal spectrum $$G_s(f) = \sigma_a^2 \frac{|P(f)|^2}{T} = \frac{A^2 T}{4} \operatorname{sinc}^4(fT/2)$$



EXAMPLE: BIPOLAR TRIANGULAR

Signal spectrum

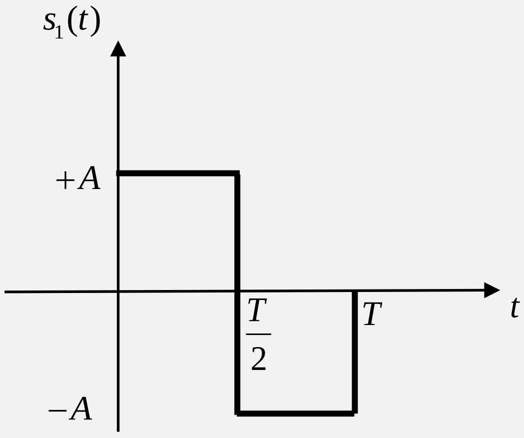
$$G_s(f) = \sigma_a^2 \frac{|P(f)|^2}{T} = \frac{A^2 T}{4} \mathrm{sinc}^4(fT/2)$$

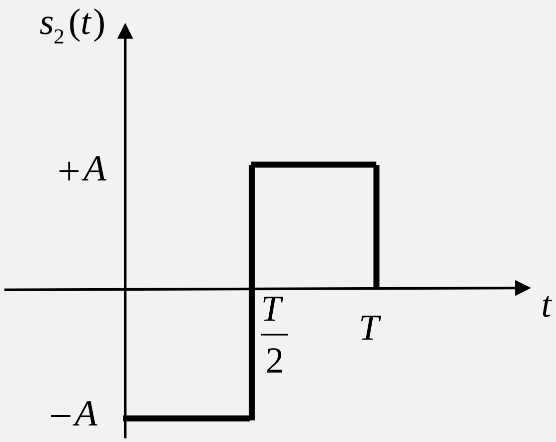


MANCHESTER (BIPHASE)

Signal set $$\boxed{M=\{s_1(t)=+Ax(t)\,,\,s_2(t)=-Ax(t)\,\}}$$

$$x(t)=\left[+P_{T/2}(t)-P_{T/2}(t-T/2)\right]$$



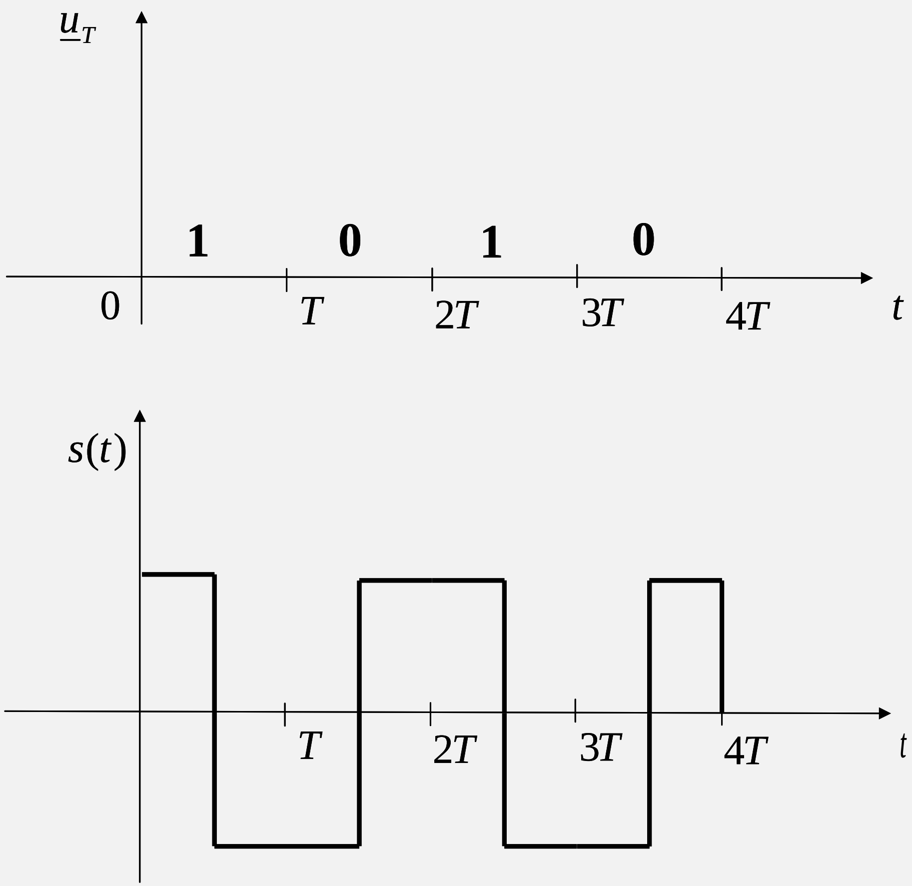


Versor $$b_1(t)=\frac{1}{\sqrt{T}}\left[+P_{T/2}(t)-P_{T/2}(t-T/2)\right]$$

Vector set $$M=\{\underline{s_1}=(+\alpha)\,,\,\underline{s_2}=\left(-\alpha\right)\}$$

MANCHESTER (BIPHASE)

Transmitted waveform



$$s(t) = \sum_n a[n] p(t - nT)$$

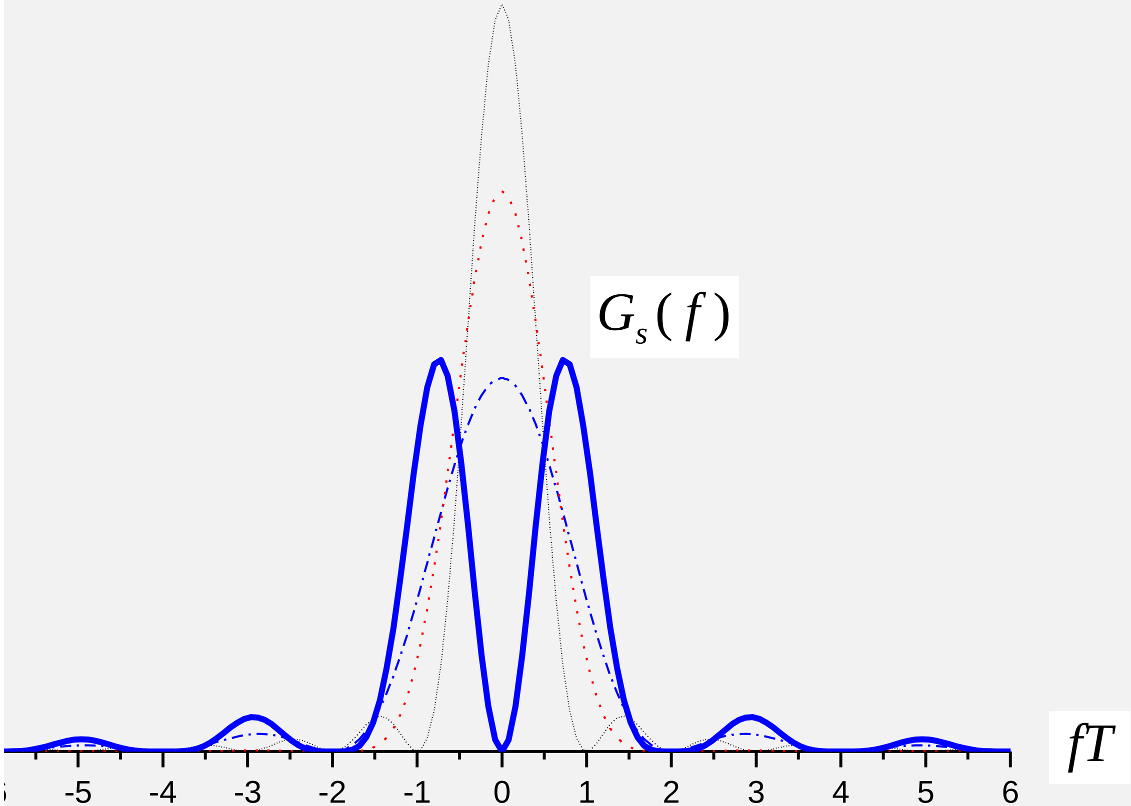
$$a[n] \in \{+\alpha, -\alpha\}$$

MANCHESTER (BIPHASE)

Signal spectrum

$$G_s(f) = \sigma_a^2 \frac{|P(f)|^2}{T} = A^2 T \frac{\sin^4(\pi fT/2)}{(\pi fT/2)^2}$$

(maximum at $f \approx 0.74/T$)

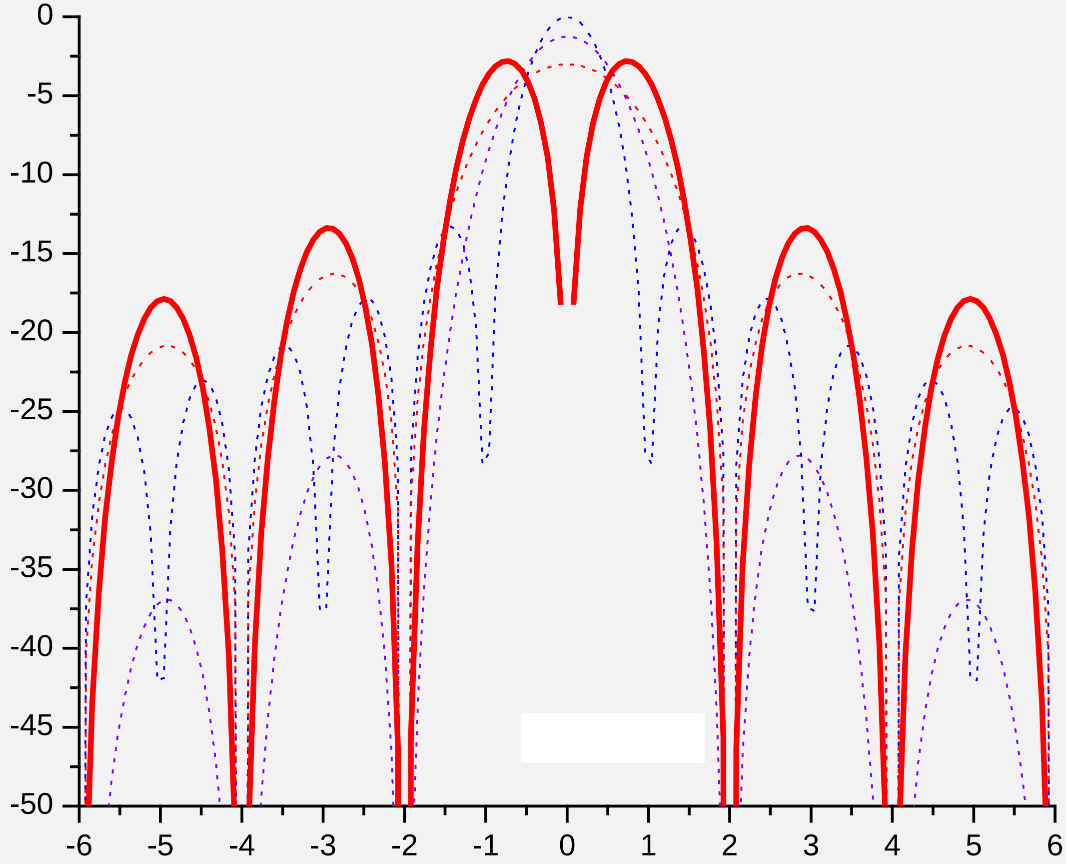


MANCHESTER (BIPHASE)

Signal spectrum

$$G_s(f) = \sigma_a^2 \frac{|P(f)|^2}{T} = A^2 T \frac{\sin^4(\pi f T/2)}{(\pi f T/2)^2}$$

(maximum at $f \approx 0.74/T$)



MANCHESTER (BIPHASE)

$$p(t)=b_1(t)=\frac{1}{\sqrt{T}}\left[+P_{T/2}(t)-P_{T/2}\left(t-\frac{T}{2}\right)\right]$$

$$P(f)=\frac{1}{\sqrt{T}}\left[+\frac{T}{2}\text{sinc}\left(f\frac{T}{2}\right)\exp\left(-j2\pi f\frac{T}{4}\right)-\frac{T}{2}\text{sinc}\left(f\frac{T}{2}\right)\exp\left(-j2\pi f\frac{3T}{4}\right)\right]=$$

$$=\left[+\frac{\sqrt{T}}{2}\text{sinc}\left(f\frac{T}{2}\right)\exp\left(-j2\pi f\frac{T}{4}\right)\right]\left[1-\exp\left(-j\pi fT\right)\right]$$

$$\left|P(f)\right|^2=\frac{T}{4}\text{sinc}^2\left(f\frac{T}{2}\right)\left|1-\cos\left(-\pi fT\right)-j\sin\left(-\pi fT\right)\right|^2=$$

$$=\frac{T}{4}\text{sinc}^2\left(f\frac{T}{2}\right)\left|1-\cos\left(\pi fT\right)+j\sin\left(\pi fT\right)\right|^2=$$

$$=\frac{T}{4}\text{sinc}^2\left(f\frac{T}{2}\right)\left[1+\cos^2\left(\pi fT\right)-2\cos\left(\pi fT\right)+\sin^2\left(\pi fT\right)\right]=$$

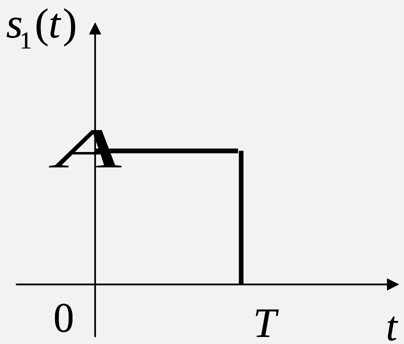
$$\boxed{\sin\left(\frac{A}{2}\right)=\sqrt{\frac{1-\cos A}{2}}}$$

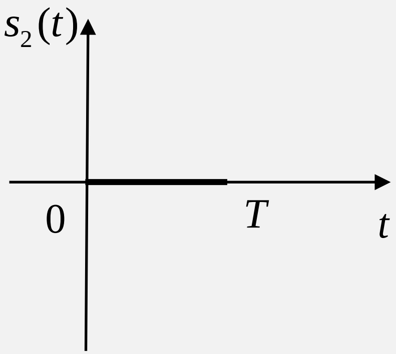
$$=\frac{T}{2}\text{sinc}^2\left(f\frac{T}{2}\right)\left[1-\cos\left(\pi fT\right)\right]=T\text{sinc}^2\left(f\frac{T}{2}\right)\sin^2\left(\pi f\frac{T}{2}\right)$$

UNIPOLAR NRZ

Signal set

$$\boxed{M = \{s_1(t) = +AP_T(t)\,,\, s_2(t) = 0\,\}}$$





Versor

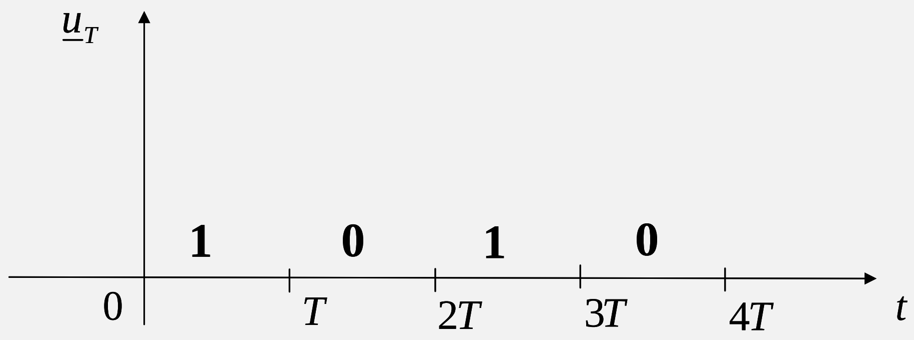
$$b_1(t) = \frac{1}{\sqrt{T}} P_T(t)$$

Vector set

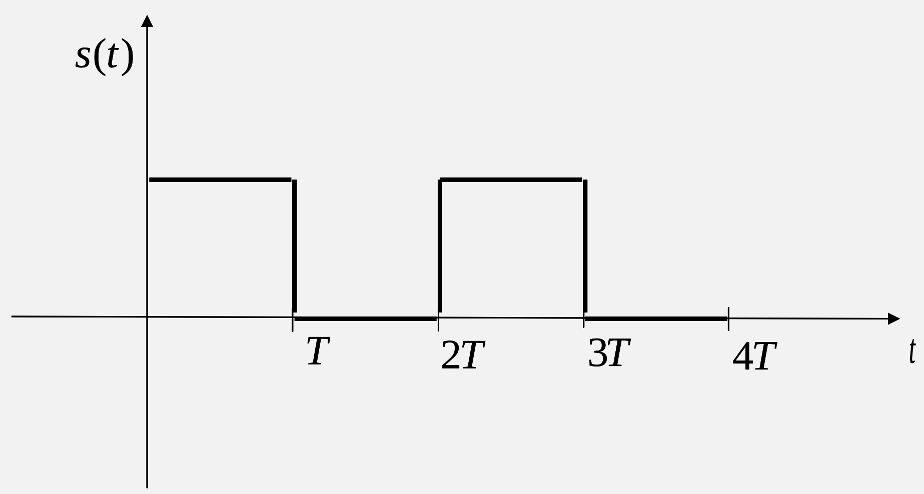
$$M = \{\underline{s_1} = (+\alpha)\,,\, \underline{s_2} = \left(0\right)\}$$

UNIPOLAR NRZ

Transmitted waveform



$$s(t) = \sum_n a[n]p(t - nT)$$



$$a[n] \in \{+\alpha, 0\}$$

UNIPOLAR NRZ

Signal spectrum

$$G_s(f) = \sigma_a^2 \frac{|P(f)|^2}{T} + \frac{\mu_a^2}{T^2} \sum_{n=-\infty}^{+\infty} \left| P\left(\frac{n}{T}\right) \right|^2 \delta\left(f - \frac{n}{T}\right)$$

$$|P(f)|^2 = x \,\mathrm{sinc}^2(\pi f T) \qquad\qquad x \in R$$

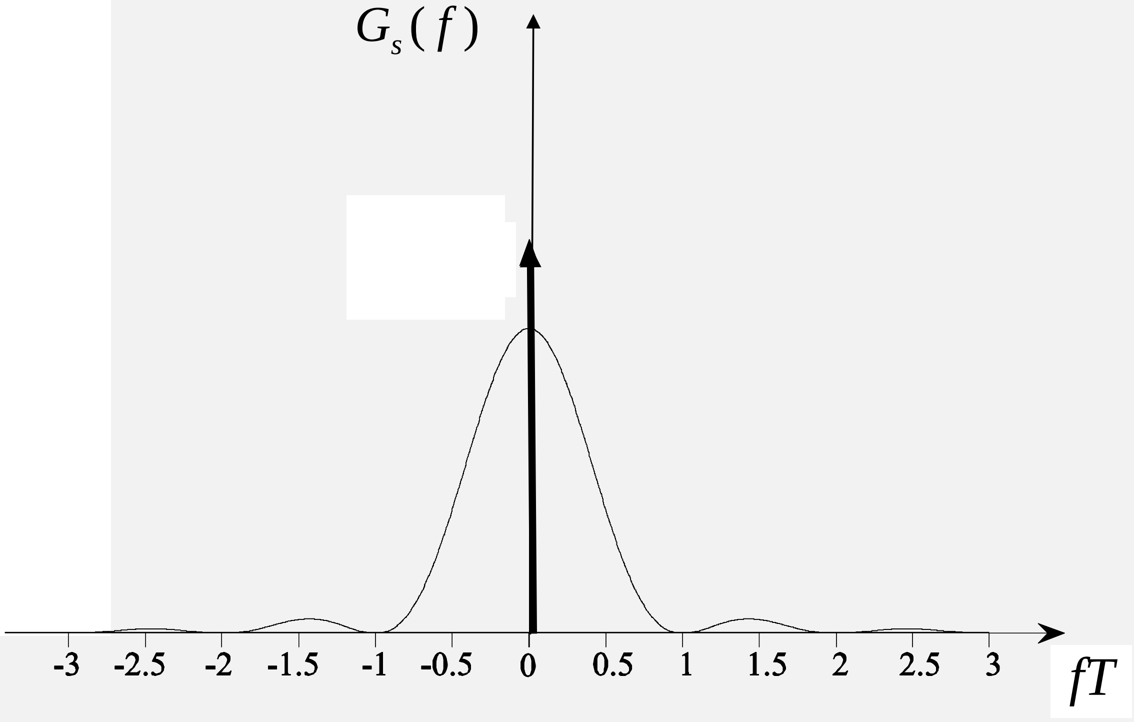
A Dirac delta at zero frequency

$$G_s(f) = \frac{A^2}{4} T \,\mathrm{sinc}^2(f\mathrm{T}) + \frac{A^2}{4} \delta(f)$$

UNIPOLAR NRZ

Signal spectrum

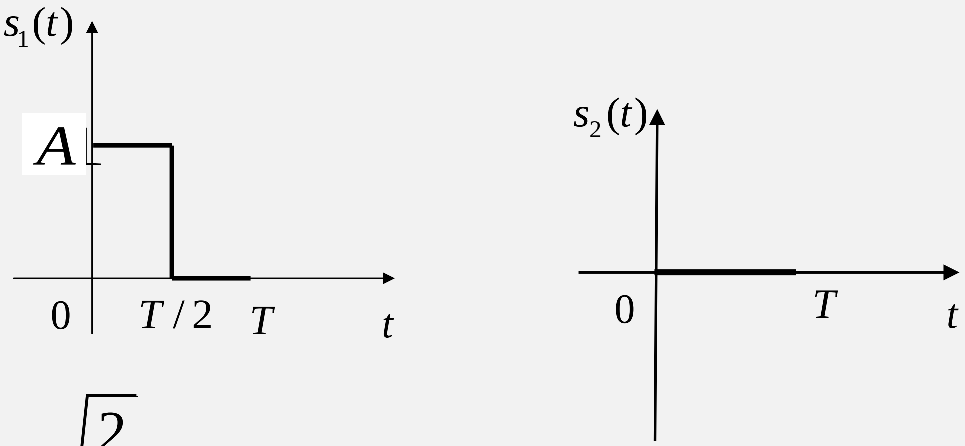
$$G_s(f) = \frac{A^2}{4} T \operatorname{sinc}^2(f\mathrm{T}) + \frac{A^2}{4}\delta(f)$$



UNIPOLAR RZ

Signal set

$$\boxed{M = \{s_1(t) = +AP_{T/2}(t)\,,\, s_2(t) = 0\,\}}$$



Versor

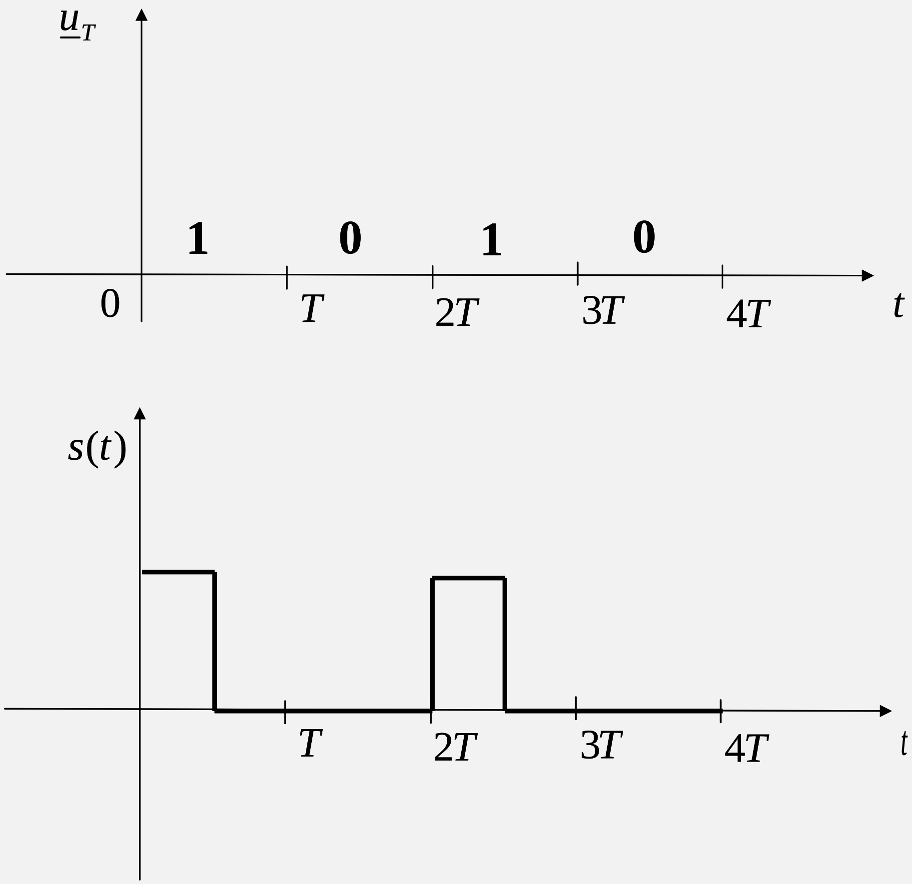
$$b_1(t) = \sqrt{\frac{2}{T}}P_{T/2}(t)$$

Vector set

$$M = \{\underline{s_1} = (+\alpha)\,,\, \underline{s_2} = \left(0\right)\}$$

UNIPOLAR RZ

Transmitted waveform



$$s(t) = \sum_n a[n] p(t - nT)$$

$$a[n] \in \{+\alpha, 0\}$$

UNIPOLAR RZ

Signal spectrum

$$G(f)=\sigma_a^2\frac{|P(f)|^2}{T}+\frac{\mu_a^2}{T^2}\sum_{n=-\infty}^{+\infty}\left|P\left(\frac{n}{T}\right)\right|^2\delta\left(f-\frac{n}{T}\right)$$

$$\left|P(f)\right|^2=z\left[\frac{\sin(\pi fT/2)}{(\pi fT/2)}\right]^2 \qquad (z\in R)$$

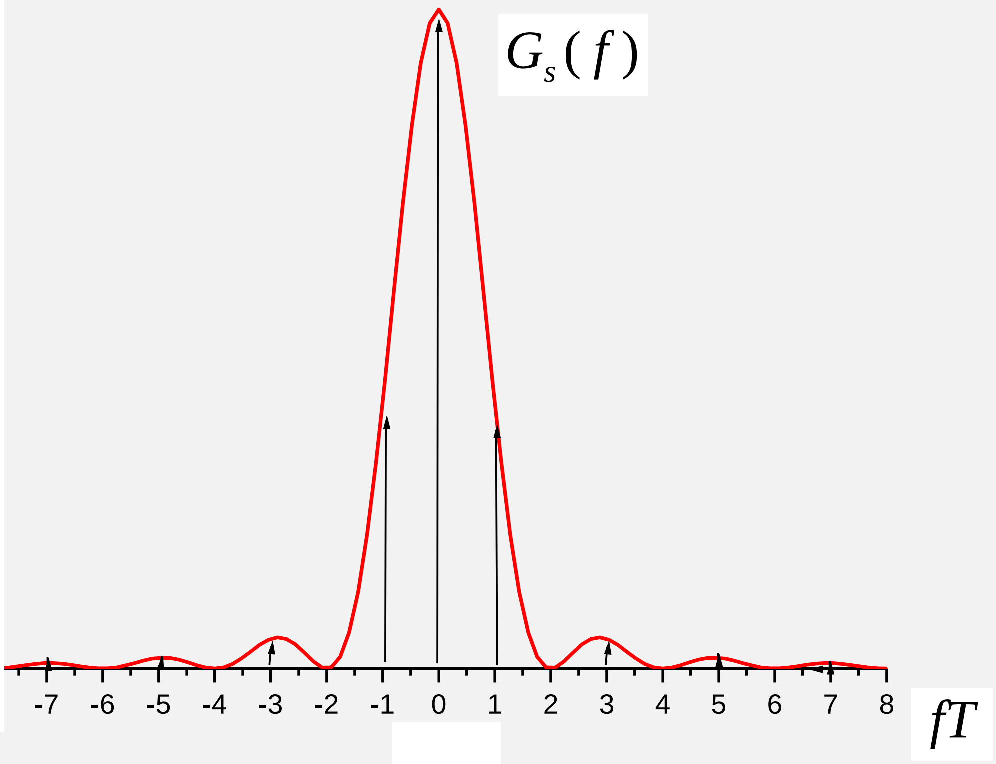
Dirac deltas at zero frequency and at odd multiples of $1/T$

$$G_s(f)=\frac{A^2}{16}T\mathrm{sinc}^2(fT/2)+\frac{A^2}{16}\sum_{i=-\infty}^{+\infty}\mathrm{sinc}^2\left(\frac{(2i+1)}{2}\right)\delta\left(f-\frac{(2i+1)}{T}\right)$$

UNIPOLAR RZ

Signal spectrum

$$G_s(f)=\frac{A^2}{16}T\operatorname{sinc}^2(fT/2)+\frac{A^2}{16}\sum_{i=-\infty}^{+\infty}\operatorname{sinc}^2\left(\frac{(2i+1)}{2}\right)\delta\left(f-\frac{(2i+1)}{T}\right)$$



M-PAM CONSTELLATION: CHARACTERISTICS

1. Base-band modulation
2. One-dimensional signal space
3. m signals, symmetrical with respect to the origin
4. Information associated to the impulse amplitude
   PAM=Pulse Amplitude Modulation

M-PAM CONSTELLATION: CONSTELLATION

SIGNAL SET $\boxed{M = \{s_i(t) = \alpha_i\, p(t)\}_{i=1}^{m}}$

Versor $b_1(t)=p(t) \quad (d=1)$

VECTOR SET

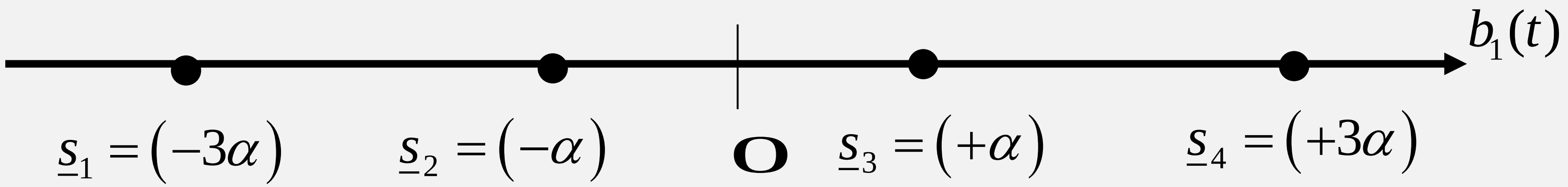
$$\boxed{M = \{\underline{s}_1 = (-(m-1)\alpha)\,,\ \underline{s}_2 = (-(m-3)\alpha), \ldots, \underline{s}_{m-1} = (+(m-3)\alpha)\,,\ \underline{s}_m = (+(m-1)\alpha)\} \subseteq R}$$

$k = \log_2(m)$ $\boxed{T = kT_b}$ $\boxed{\boxed{R = \frac{R_b}{k}}}$

M-PAM CONSTELLATION: CONSTELLATION

Example: 4-PAM constellation

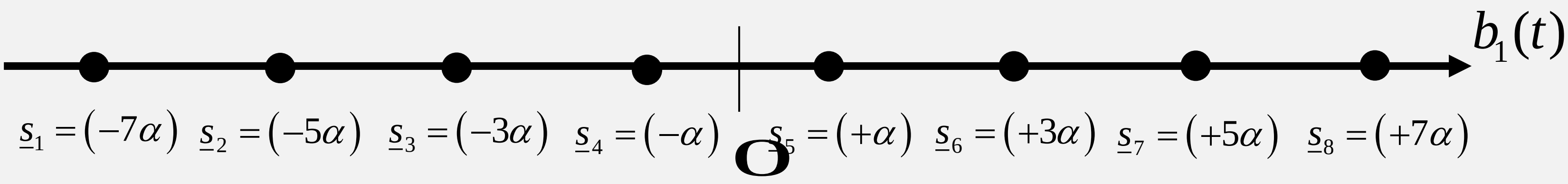
$$M = \{\underline{s}_1 = (\mathbf{-3\alpha}), \underline{s}_2 = (\mathbf{-\alpha}), \underline{s}_3 = (+\boldsymbol{\alpha}), \underline{s}_4 = (\mathbf{+3\alpha})\} \subseteq R$$



M-PAM CONSTELLATION: CONSTELLATION

Example: 8-PAM constellation

$$M = \{\underline{s}_1 = (\mathbf{-7\alpha}),\ \underline{s}_2 = (\mathbf{-5\alpha}), \underline{s}_3 = (\mathbf{-3\alpha}),\ \underline{s}_4 = (\mathbf{-\alpha}), \underline{s}_5 = (+\boldsymbol{\alpha}),\ \underline{s}_6 = (\mathbf{+3\alpha}), \underline{s}_7 = (\mathbf{+5\alpha}),\ \underline{s}_8 = (\mathbf{+7\alpha})\} \subseteq R$$

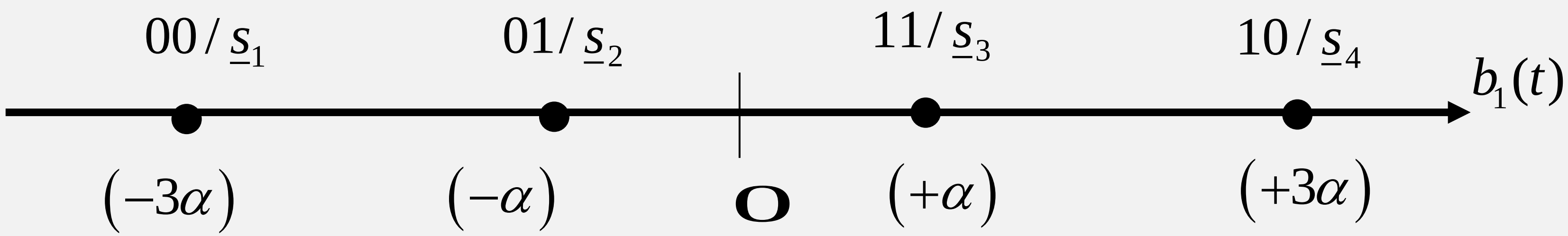


M-PAM CONSTELLATION: BINARY LABELLING

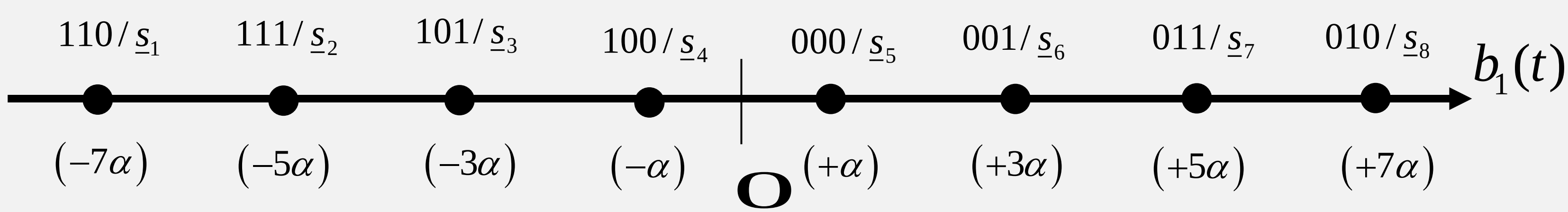
$$e : H_k \leftrightarrow M$$

It is always possible to build a Gray labeling

4-PAM:



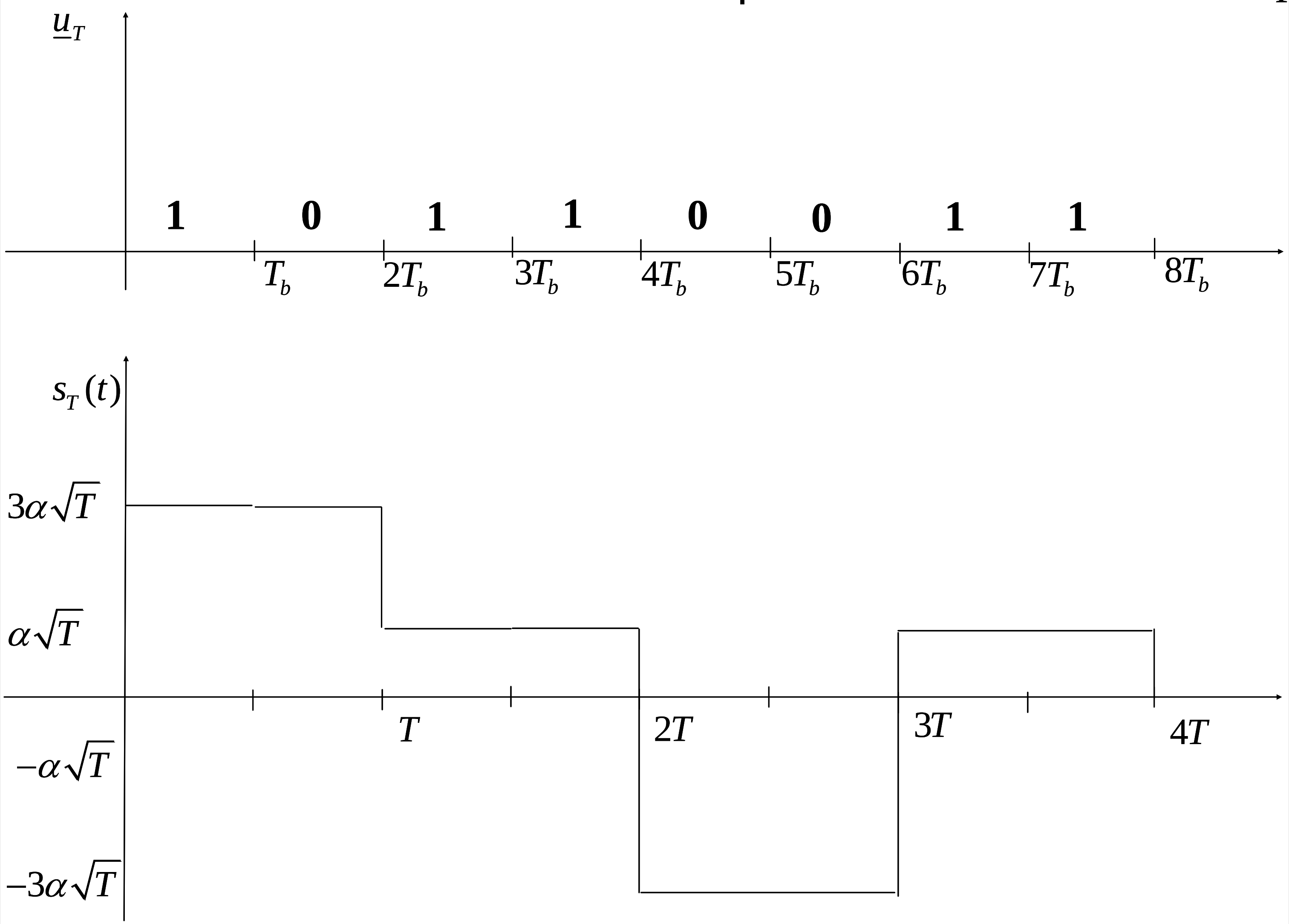
8-PAM:



M-PAM CONSTELLATION: TRANSMITTED WAVEFORM

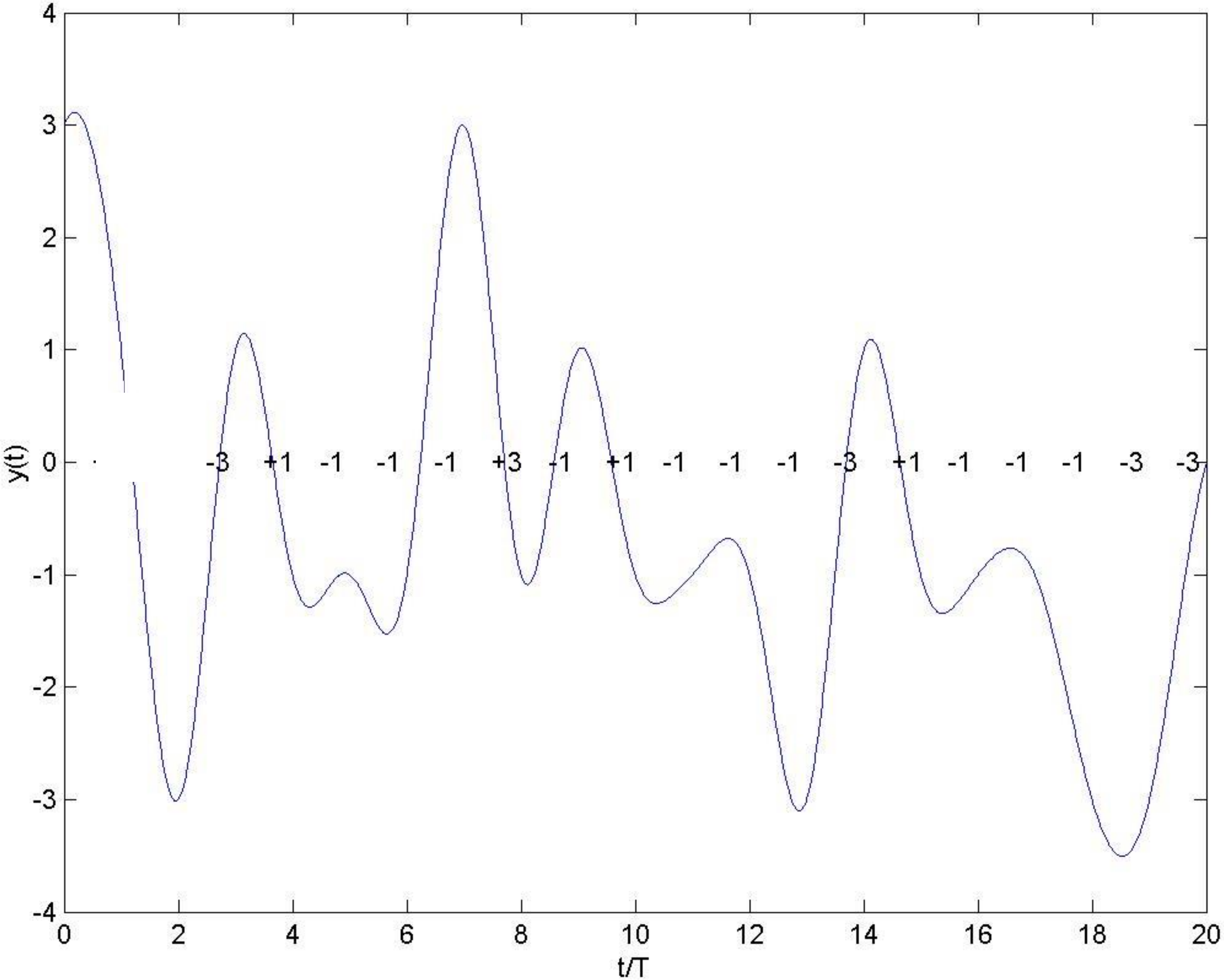
Example: 4-PAM

$$p(t) = \frac{1}{\sqrt{T}} P_T(t)$$



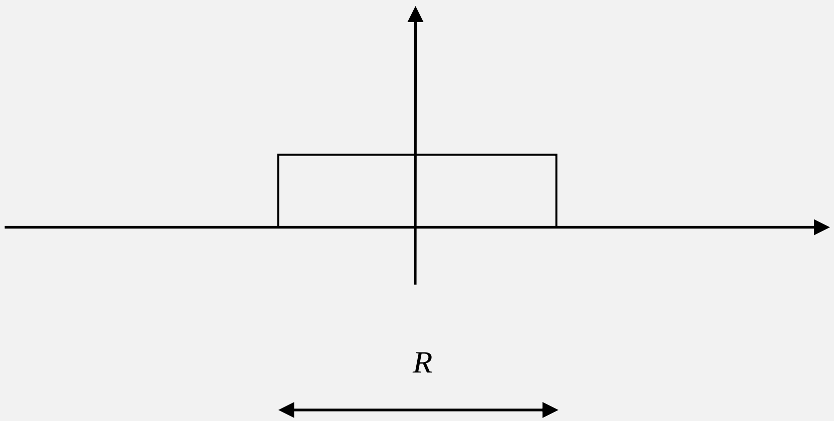
M-PAM CONSTELLATION: TRANSMITTED WAVEFORM

Example: 4-PAM $\quad p(t) = \text{RRC} \quad \alpha = 0.5$



M-PAM CONSTELLATION: BANDWIDTH AND SPECTRAL EFFICIENCY

Case 1: $p(t)$ = ideal low pass filter



Total bandwidth (ideal case)

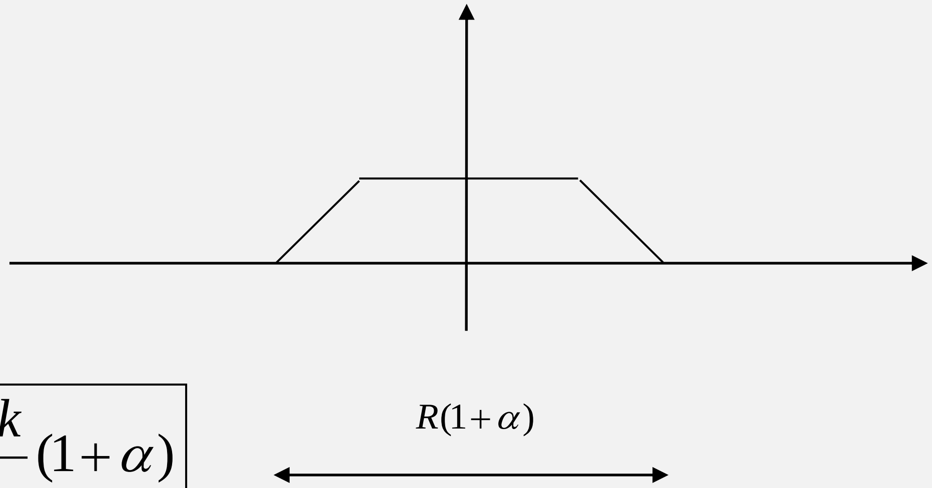
$$\boxed{B_{id} = \frac{R}{2} = \frac{R_b / k}{2}}$$

Spectral efficiency (ideal case)

$$\boxed{\eta_{id} = \frac{R_b}{B_{id}} = 2k \ bps / Hz}$$

M-PAM CONSTELLATION: BANDWIDTH AND SPECTRAL EFFICIENCY

Case 2: $p(t)$ = RRC filter roll off α



Total bandwidth $\boxed{B = \frac{R}{2}(1+\alpha) = \frac{R_b / k}{2}(1+\alpha)}$

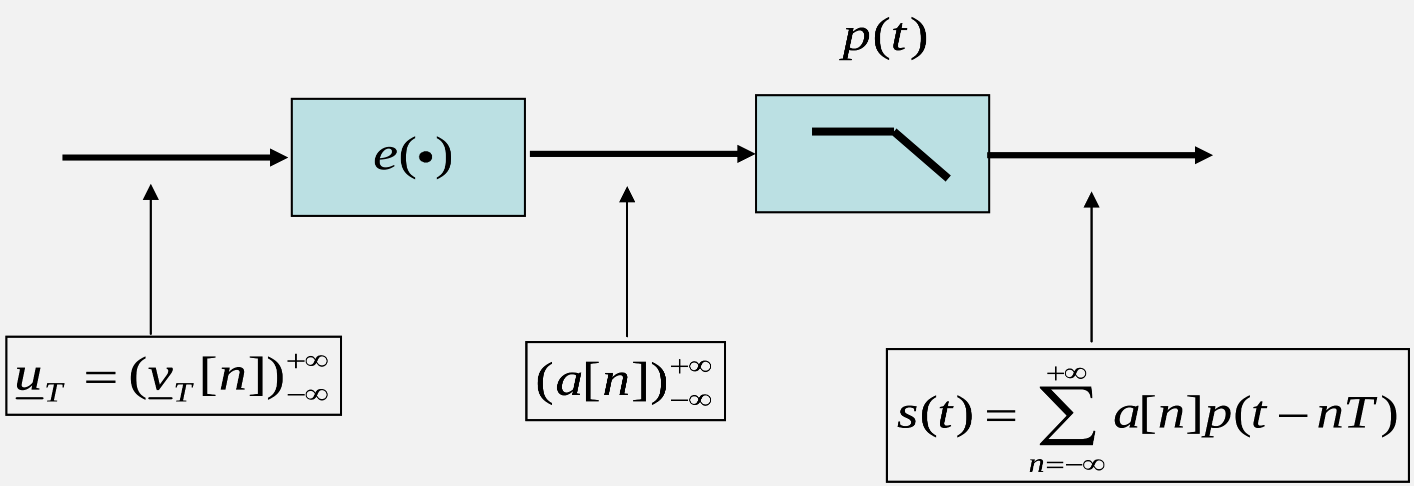
Spectral efficiency $\boxed{\eta = \frac{R_b}{B} = \frac{2k}{(1+\alpha)}\, bps / Hz}$

EXERCIZE

Given a baseband channel with bandwidth B up to 4000 Hz, compute the maximum bit rate R_b we can transmit over it with a 256-PAM constellation in the two cases:

- Ideal low pass filter
- RRC filter with α=0.25

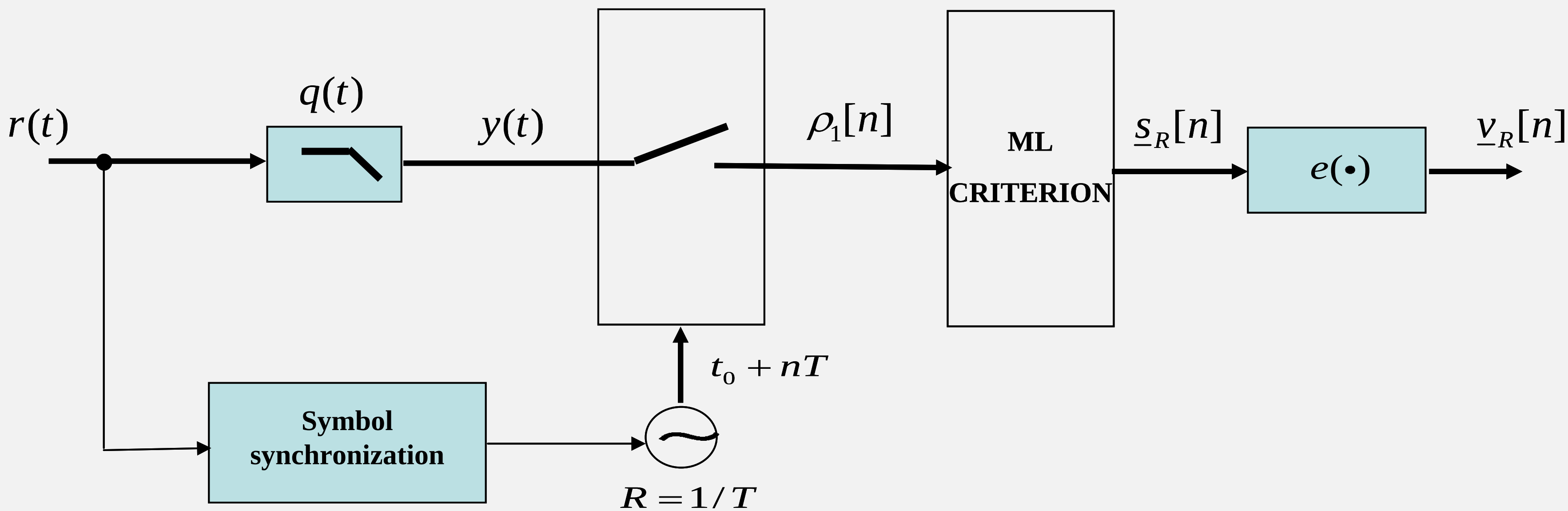
M-PAM CONSTELLATION: MODULATOR



Equal to 2-PAM, but we have *m* possible levels:

$$a[n] \in \{-(m-1)\alpha, -(m-3)\alpha, \ldots, +(m-3)\alpha, +(m-1)\alpha\}$$

M-PAM CONSTELLATION: DEMODULATOR

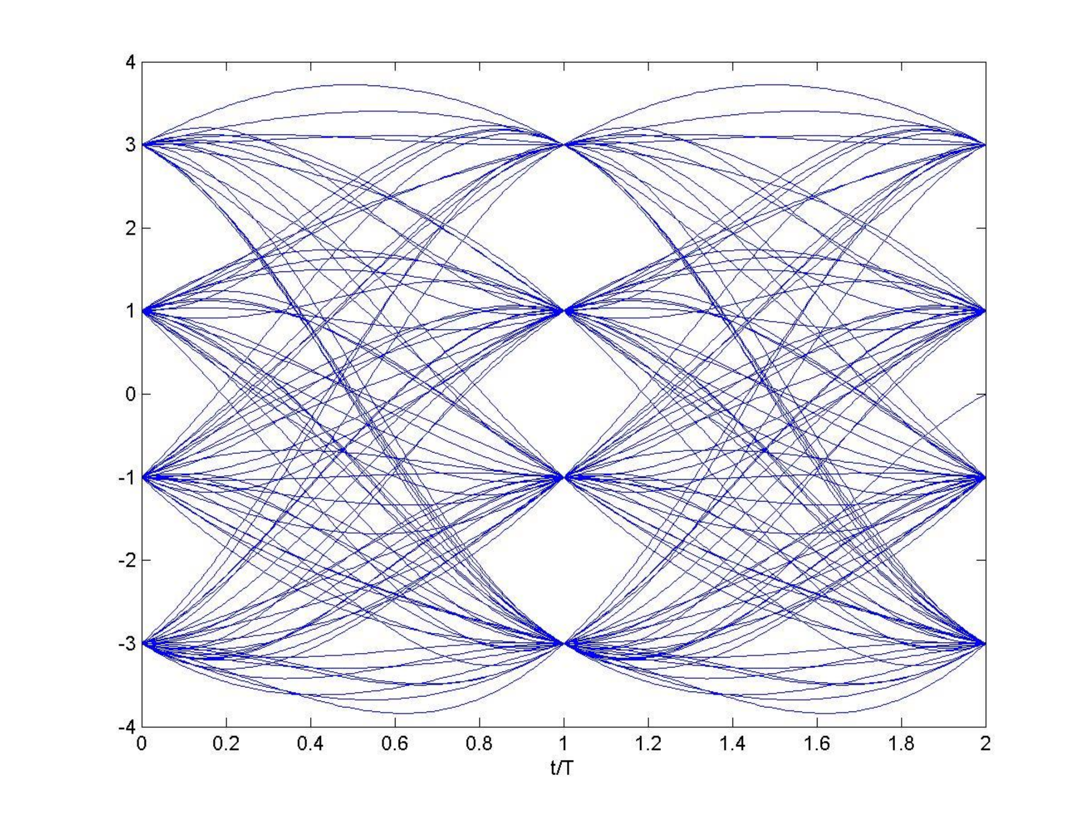


Equal to 2-PAM, but we have *m* possible levels:

$$a[n] \in \{-(m-1)\alpha, -(m-3)\alpha, \ldots, +(m-3)\alpha, +(m-1)\alpha\}$$

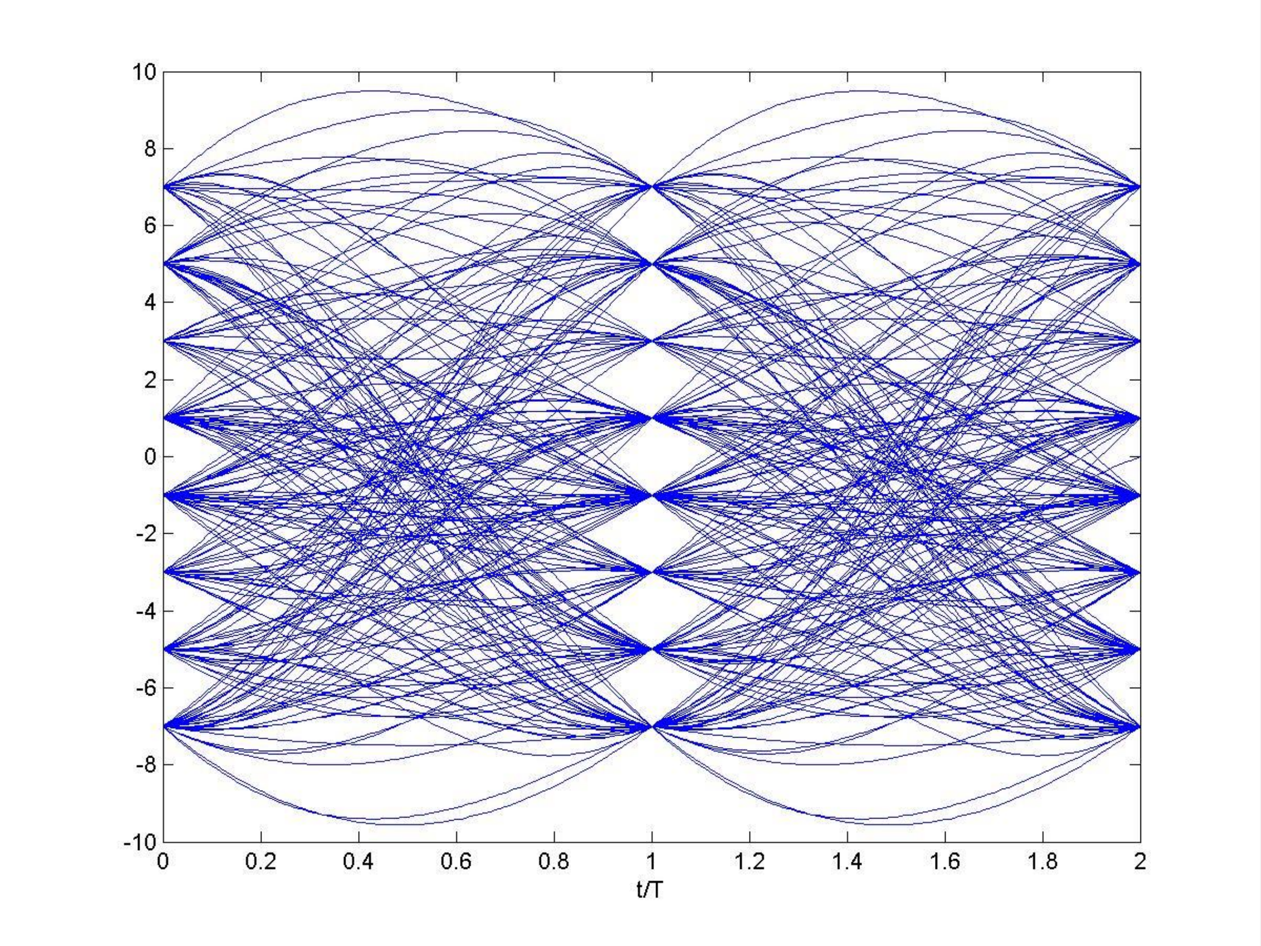
M-PAM CONSTELLATION: EYE DIAGRAM

4-PAM, $p(t)$ = RRC with α =0.5



M-PAM CONSTELLATION: EYE DIAGRAM

8-PAM, $p(t)$ = RRC with α =0.5



M-PAM CONSTELLATION: ERROR PROBABILITY

By applying the asymptotic approximation we can obtain

$$P_b(e) \approx \frac{m-1}{mk} erfc\left(\sqrt{\frac{3k}{m^2-1}\frac{E_b}{N_0}}\right)$$

M-PAM CONSTELLATION: ERROR PROBABILITY

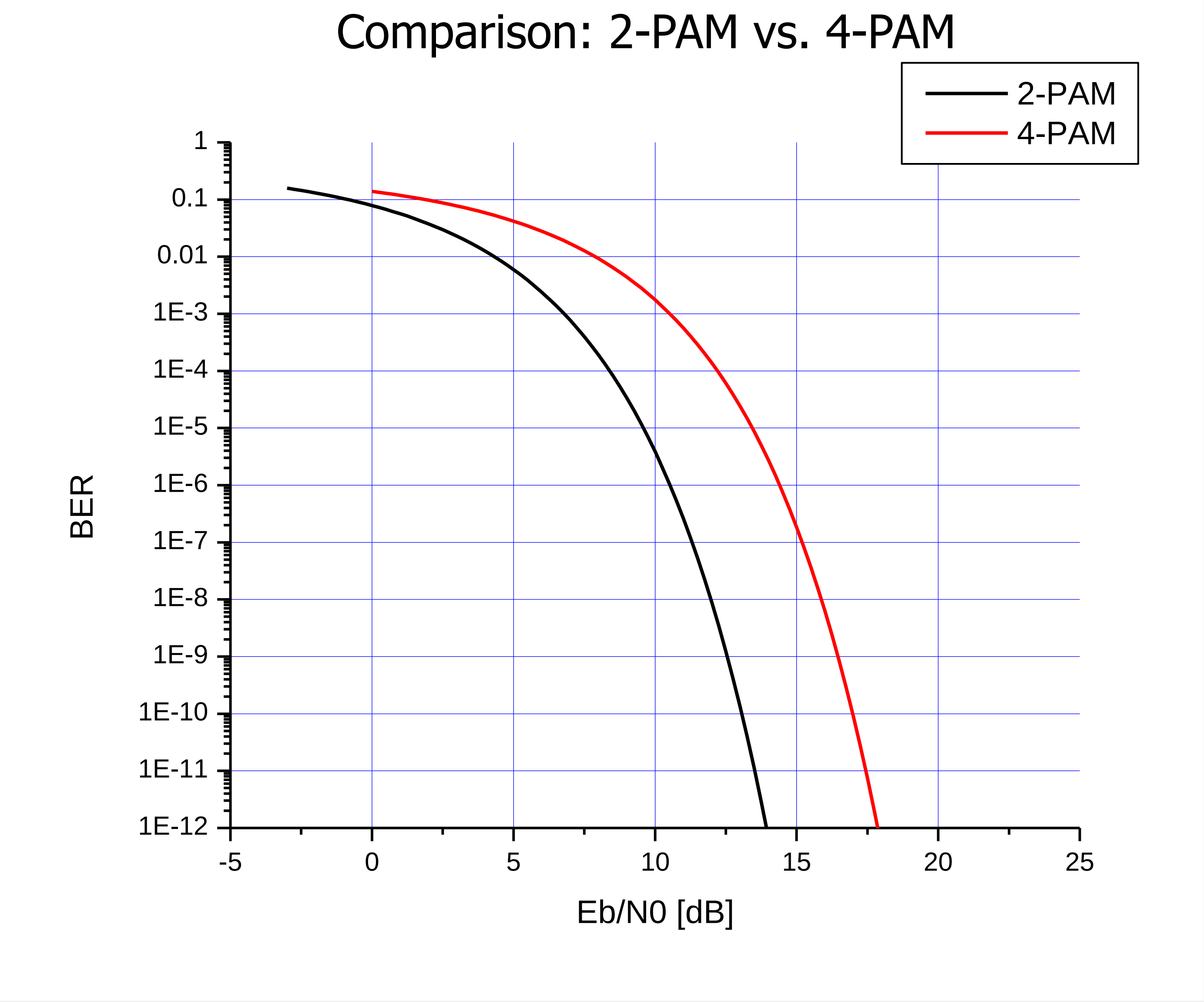
Comparison: 2-PAM vs. 4-PAM

$$2-\mathrm{PAM:} \qquad P_b(e) = \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{N_0}}\right)$$

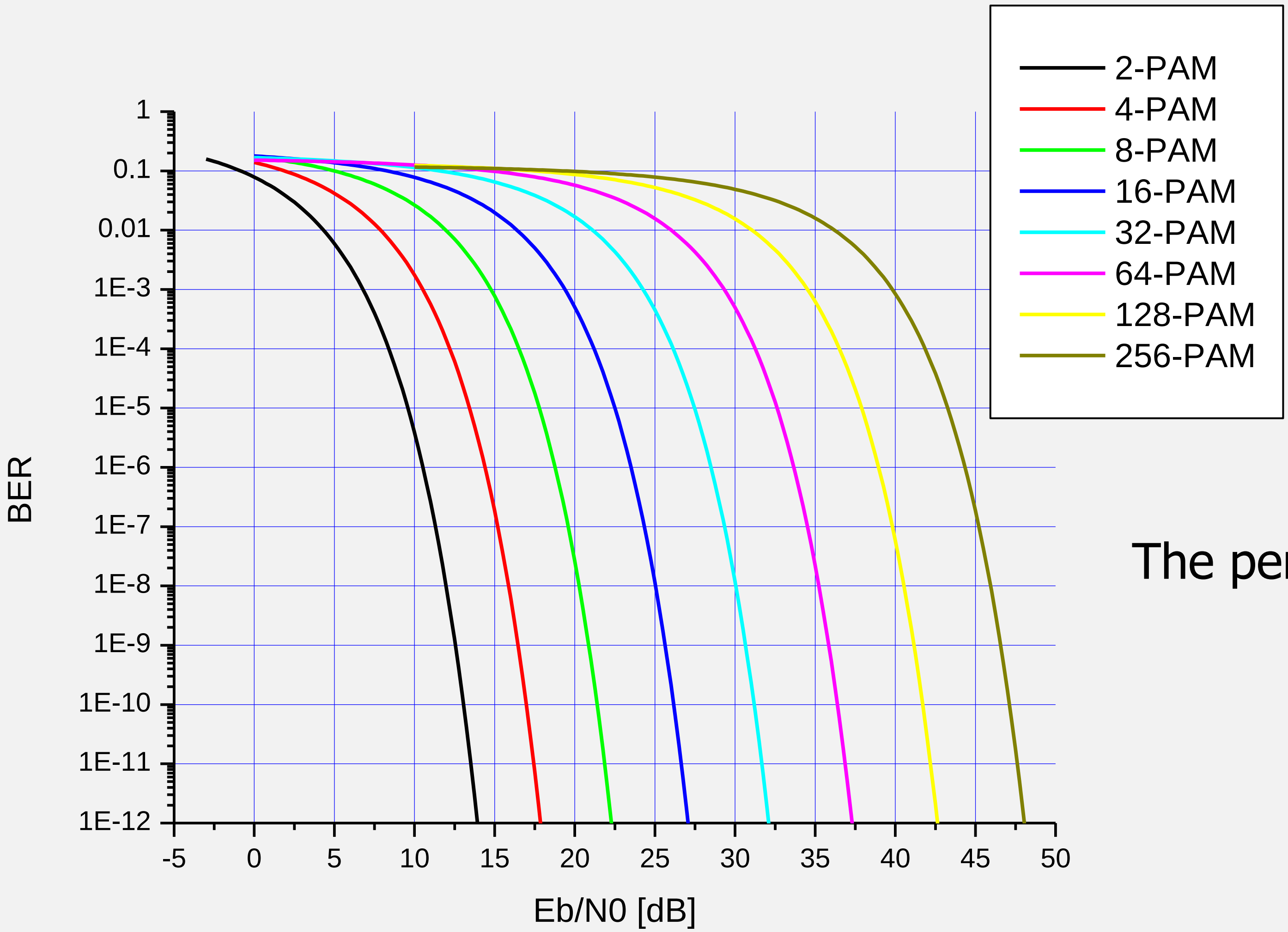
$$4-\mathrm{PAM:} \qquad P_b(e) \approx \frac{3}{8} erfc\left(\sqrt{\frac{2}{5}\frac{E_b}{N_0}}\right)$$

The 2-PAM constellation has better performance
The constellation gain is in the order of 10 log(5/2) = 4 dB

M-PAM CONSTELLATION: ERROR PROBABILITY



M-PAM CONSTELLATION: ERROR PROBABILITY



The performance decrease

for increasing m

M-PAM CONSTELLATION: PERFORMANCE/SPECTRAL EFFICIENCY TRADE-OFF

Given a baseband channel with bandwidth B and an m-PAM constellation, by increasing the number of signals $m=2^k$ we increase the spectral efficiency

$$\eta_{id} = R_b / B = 2k \quad bps / Hz$$

then we can transmit a higher bit rate R_b.

Unfortunately, the performance decrease:
fixed a BER value, the signal-to-noise ratio E_b/N_0 necessary to achieve it increases with m.

EXAMPLE

Suppose B=4kHz.

With a (ideal) 2-PAM we transmit R_b = 8 kbps

With a (ideal) 256-PAM we transmit R_b = 64 kbps

However, fixed a target BER (e.g. BER=1e-10), a 256-PAM requires a larger ratio E_b/N_0 (34 dB of difference!).

As an example, at the parity of transmitted power, the link distance is very lower (by a factor of 50!)

LINEAR MODULATION

An m-PAM constellation is a base-band modulation characterized by a low pass TX filter $p(t)$.

Let us suppose to change this TX filter from $p(t)$ to $p(t)\cos(2\pi f_0 t)$

- The constellation stays unchanged → **the BER performance are the same**
- **The signal spectrum changes**

LINEAR MODULATION

$$s(t) = \sum_n a[n]\, p(t - nT)$$

$$G(f) = \sigma_a^2 \frac{|P(f)|^2}{T}$$

$$\left.\begin{aligned} s'(t) &= \sum_n a[n]\, p'(t - nT) \\ p'(t) &= p(t)\cos(2\pi f_0 t) \end{aligned}\right\}$$

$$G'(f) = \frac{1}{4}[G(f - f_0) + G(f + f_0)]$$

The signal spectrum is translated around frequency f_0

LINEAR MODULATION

A linear modulation simply translates the spectrum around frequency f_0

(carrier frequency or Intermediate Frequency IF)

The modulation formats obtained by applying a linear modulation to

m-PAM modulations are called m-ASK (Amplitude Shift Keying).

The only one really important is 2-ASK, which is always called 2-PSK

(Phase Shift Keying).

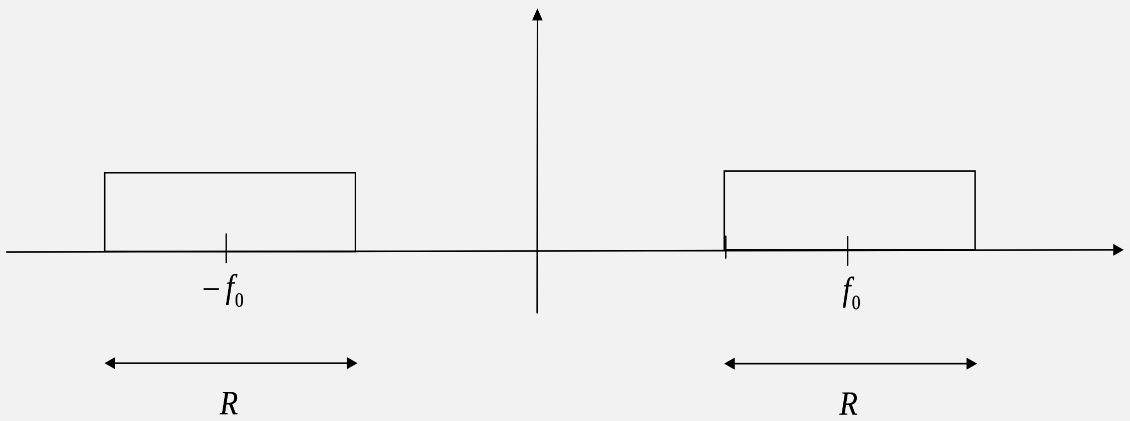
M-ASK CONSTELLATION: CHARACTERISTICS

1. One-dimensional constellation identical to m-PAM
2. Versor $b_1(t) = p'(t) = p(t)\cos(2\pi f_0 t)$
3. Signal spectrum centred around f_0 → bandpass modulations
4. ASK (Amplitude Shift Keying)

M-ASK CONSTELLATION: SIGNAL SPECTRUM

$$G_s(f) = x\left[\left|P(f - f_0)\right|^2 + \left|P(f + f_0)\right|^2\right] \qquad x \in R$$

Example: $p(t)$ = ideal low pass filter



$$B_{id} = R = \frac{R_b}{k}$$

$$\boxed{\eta_{id} = \frac{R_b}{B_{id}} = k\ bps/Hz}$$

M-ASK CONSTELLATION: PROPERTIES

Properties

- **Spectral efficiency halved with respect to m-PAM**
- BER performance identical to m-PAM
- No practical applications
  (only exception 2-ASK which is always called 2-PSK)

LECTURE 10: PHASE SHIFT KEYING MODULATION

2-PSK: CHARACTERISTICS

1. Bandpass modulation
2. One-dimensional signal space and antipodal binary constellation (equal to 2-PAM)
3. TX filter $p(t)cos(2\pi f_0 t)$
4. Information associated to the carrier phase = Phase Shift Keying

2-PSK: CONSTELLATION

SIGNAL SET $M = \{s_1(t) = +\alpha\, p(t)\cos(2\pi f_0 t)\ ,\ s_2(t) = -\alpha\, p(t)\cos(2\pi f_0 t)\}$

Information associated to the impulse amplitude
BUT
we can also write

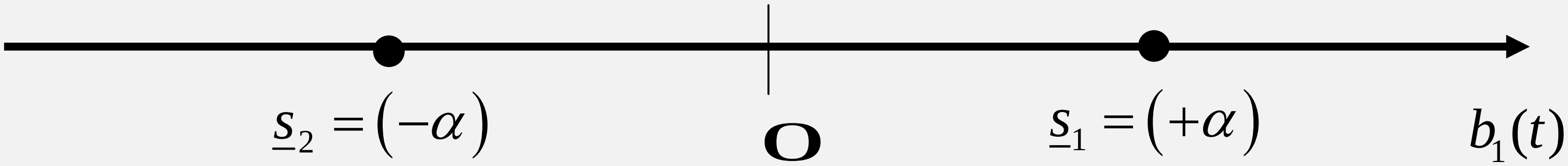
SIGNAL SET $\boxed{M = \{s_1(t) = +\alpha\, p(t)\cos(2\pi f_0 t)\ ,\ s_2(t) = +\alpha\, p(t)\cos(2\pi f_0 t - \pi)\}}$

Information associated to the carrier phase

2-PSK: CONSTELLATION

Versor $b_1(t)=p(t)\cos(2\pi f_0 t)$ $(d=1)$

VECTOR SET $M=\{\underline{s}_1=(+\alpha),\ \underline{s}_2=(-\alpha)\}\subseteq R$



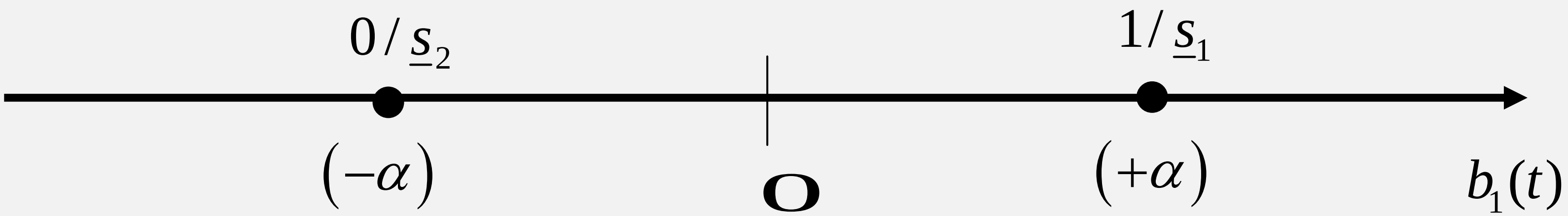
2-PSK: BINARY LABELING

(example)

$$e : H_1 \leftrightarrow M$$

$$e(1) = \underline{s}_1$$

$$e(0) = \underline{s}_2$$



2-PSK: TRANSMITTED WAVEFORM

$m = 2 \rightarrow k = 1$ $\boxed{R = R_b}$ $\boxed{T = T_b}$

Transmitted waveform $$s(t) = \sum_{n=-\infty}^{+\infty} a[n] b_1(t - nT)$$

where $$a[n] \in \{+\alpha, -\alpha\}$$

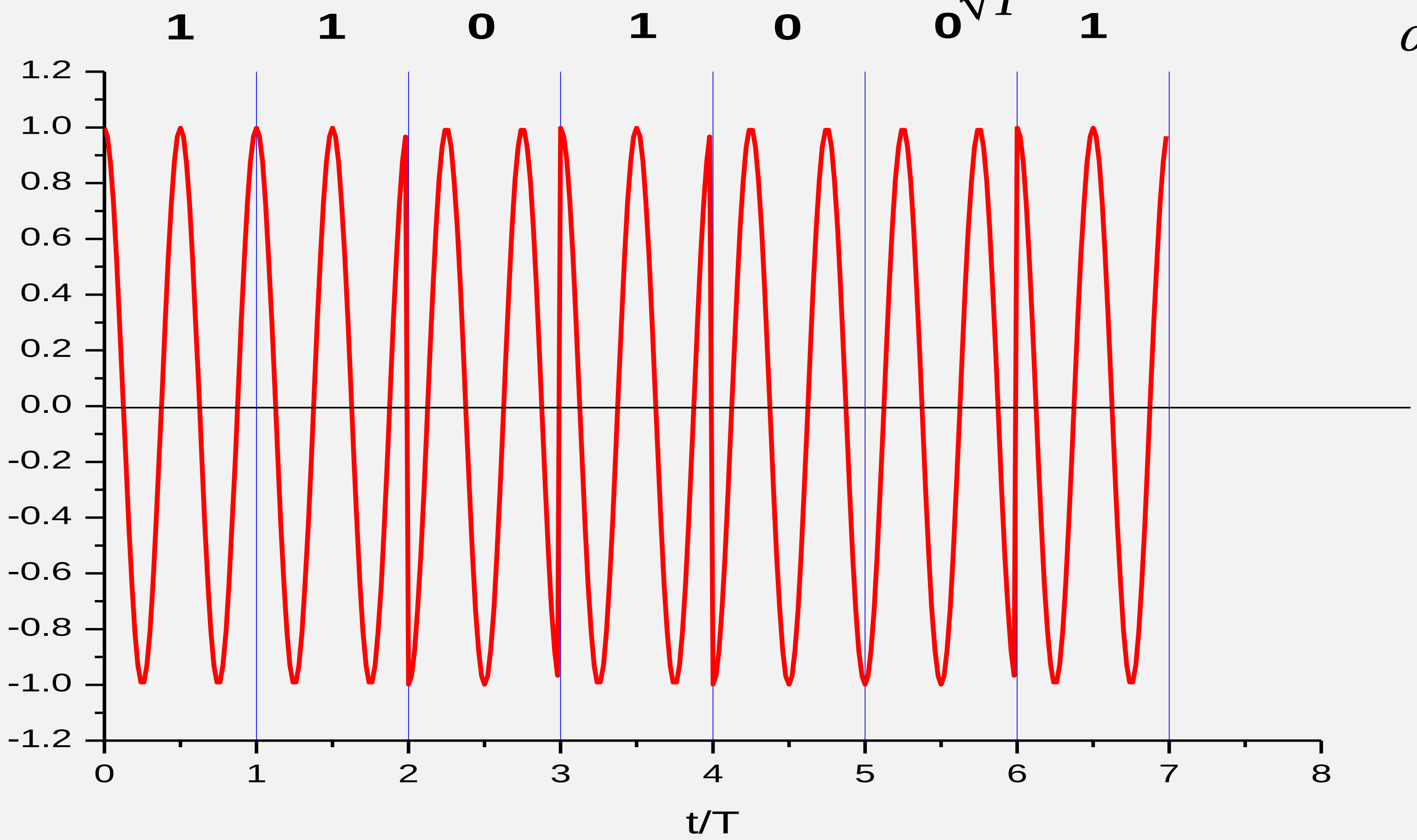
$$b_1(t) = p(t)\cos(2\pi f_0 t)$$

2-PSK: TRANSMITTED WAVEFORM

example for $p(t) = \frac{1}{\sqrt{T}} P_T(t)$

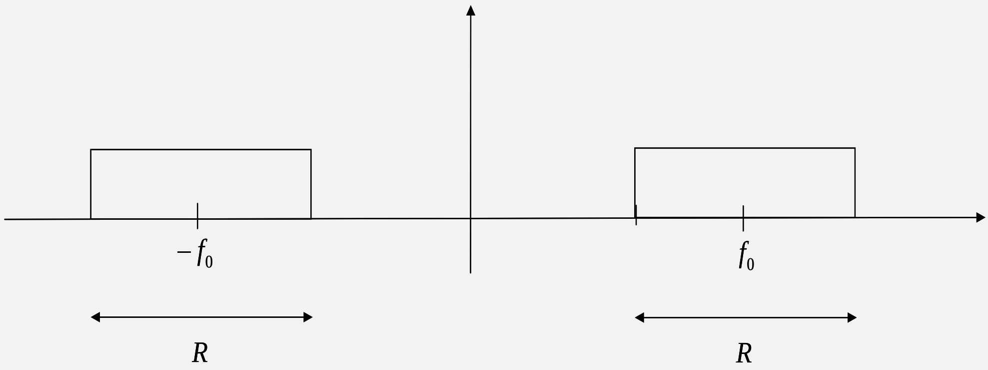
$f_0 = 2R_b$

$\alpha = \sqrt{T}$



2-PSK: BANDWIDTH AND SPECTRAL EFFICIENCY

Case 1: $p(t)$ = ideal low pass filter



Total bandwidth (ideal case)

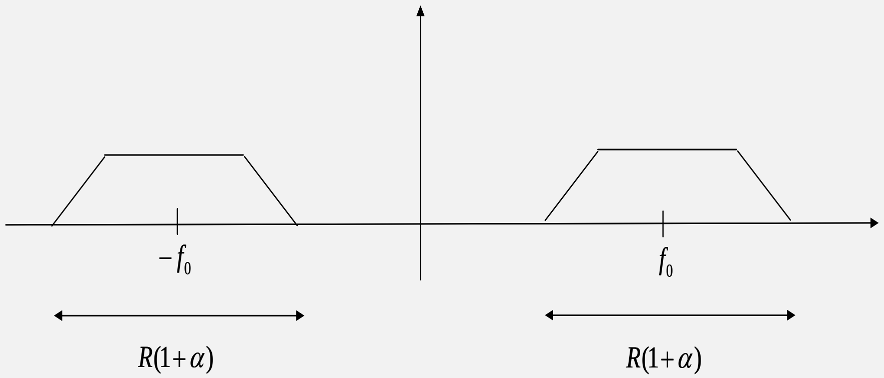
$$\boxed{B_{id} = R = R_b}$$

Spectral efficiency (ideal case)

$$\boxed{\eta_{id} = \frac{R_b}{B_{id}} = 1\,bps/Hz}$$

2-PSK: BANDWIDTH AND SPECTRAL EFFICIENCY

Case 2: $p(t)$ = RRC filter with roll off α



Total bandwidth $B = R(1+\alpha) = R_b(1+\alpha)$

Spectral efficiency $\boxed{\eta = \frac{R_b}{B} = \frac{1}{(1+\alpha)}\, bps/Hz}$

EXERCIZE

Given a bandpass channel with bandwidth B = 4000 Hz, centred around f_0=2 GHz, compute the maximum bit rate R_b we can transmit over it with a 2-PSK constellation in the two cases:

- Ideal low pass filter
- RRC filter with α=0.25

2-PSK: MODULATOR

The transmitted waveform is given by $s(t) = \sum_n a[n] b_1(t - nT)$

Where $b_1(t) = p(t)\cos(2\pi f_0 t)$

Then we must generate $s(t) = \sum_n a[n] p(t - nT)\cos(2\pi f_0 (t - nT))$

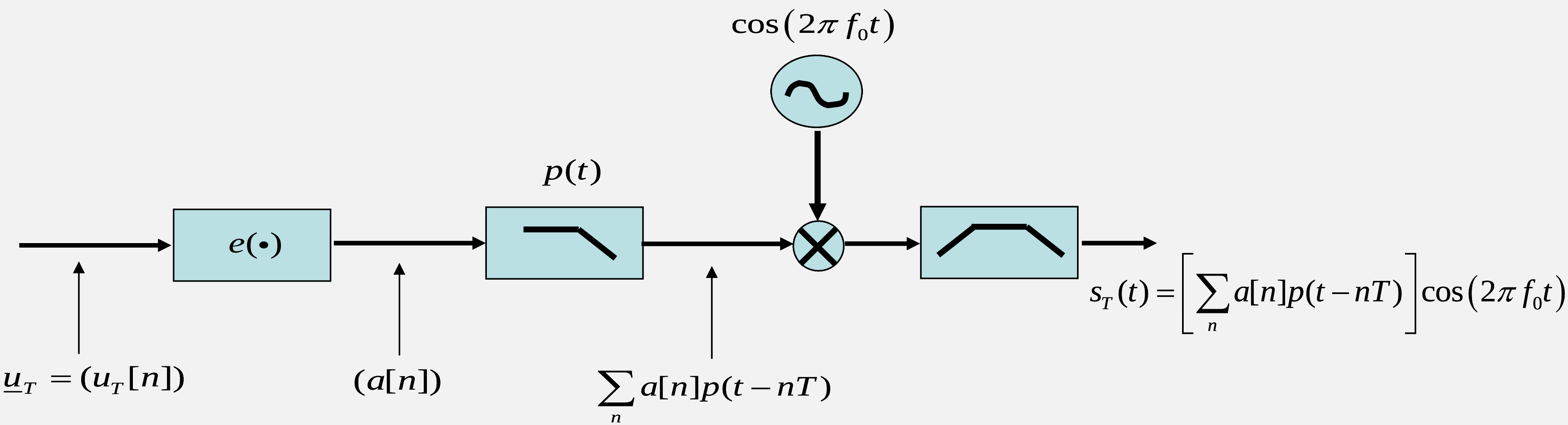
We choose f_0 multiple of $R=1/T$

It follows $\cos(2\pi f_0 (t - nT)) = \cos(2\pi f_0 t - 2\pi f_0 nT) = \cos(2\pi f_0 t)$

Then we can generate

$$\boxed{s(t) = \left[\sum_n a[n] p(t - nT)\right]\cos(2\pi f_0 t)}$$

2-PSK: MODULATOR



2-PSK: DEMODULATOR

Given the received signal $\rho(t)$

the received symbol is obtained by projecting it on the versor $b_1(t) = p(t)\cos(2\pi f_0 t)$

$$\rho[0] = \int_{-\infty}^{+\infty} \rho(t) b_1(t) dt = \boxed{\int_{-\infty}^{+\infty} \rho(t) p(t) \cos(2\pi f_0 t) dt}$$

This projection could be computed by using a matched filter

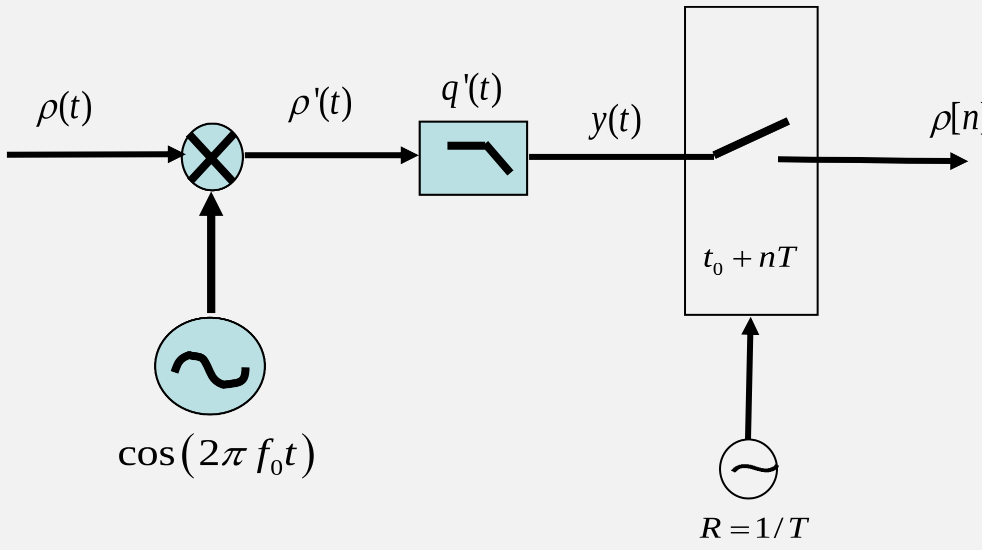
$$q(t) = b_1(T-t) = p(T-t)\cos(2\pi f_0 (T-t))$$

2-PSK: DEMODULATOR

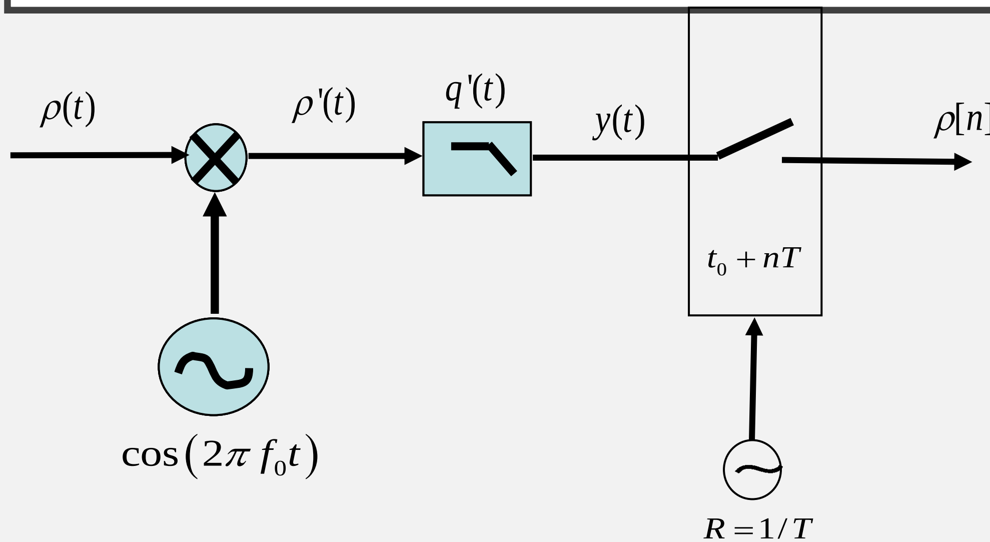
As an alternative, we can work as follows:

1. Given the received signal $\rho(t)$ multiply it by $\cos(2\pi f_0 t)$

2. Use a filter matched to $p(t)$: $q'(t) = p(T-t)$



2-PSK: DEMODULATOR

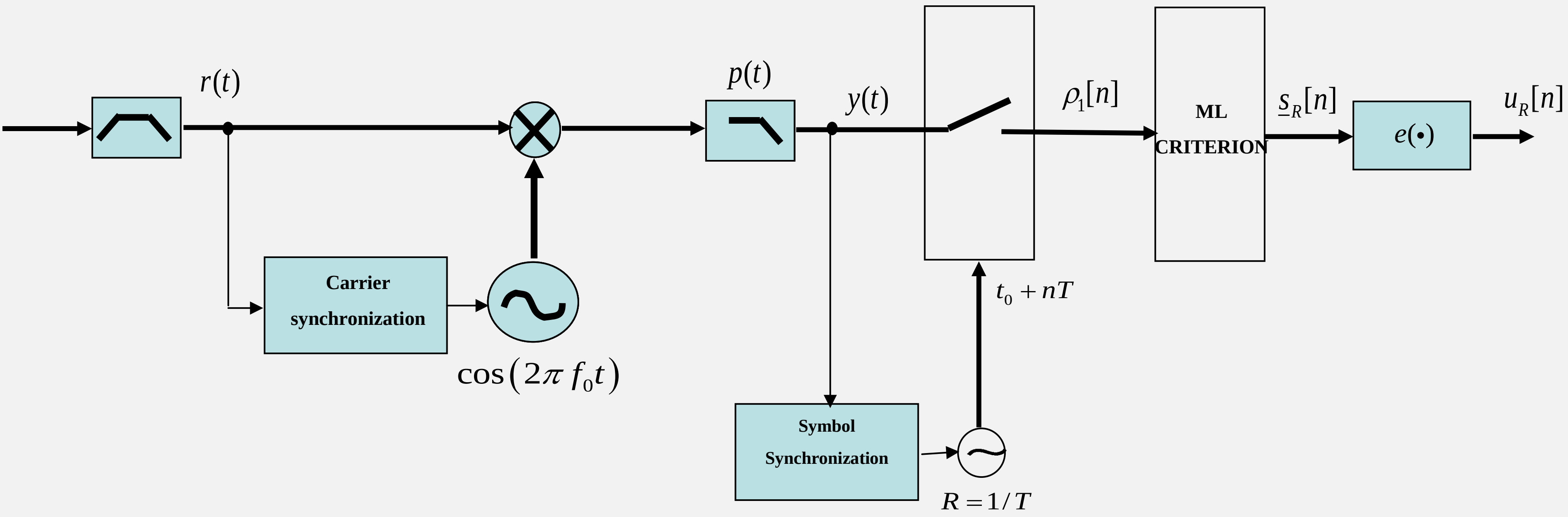


By sampling the matched filter output waveform we obtain

$$y(t) = \int_{-\infty}^{+\infty} \rho'(\tau) q'(t-\tau) d\tau = \int_{-\infty}^{+\infty} \rho(\tau) \cos(2\pi f_0 \tau) p(T - t + \tau) d\tau$$

$$y(t = T) = \boxed{\int_{-\infty}^{+\infty} \rho(\tau) \cos(2\pi f_0 \tau) p(\tau) d\tau} = \rho[0]$$

2-PSK: DEMODULATOR



2-PSK: INTERPRETATION

We generate a baseband signal

$$v(t) = \sum_n a[n]p(t - nT)$$

Multiplication by cosine shifts the spectrum around f_0

$$s(t) = v(t)\cos(2\pi f_0 t)$$

2-PSK: INTERPRETATION

At the receiver side, multiplication by cosine generates

$$s(t)\cos(2\pi f_0 t) = v(t)\cos(2\pi f_0 t)\cos(2\pi f_0 t) = v(t)\cos^2(2\pi f_0 t) = v(t)\left[\frac{1+\cos(2\pi(2f_0)t)}{2}\right]$$

This signal enters the matched filter $q(t)=p(T-t)$.

It is a low pass filter: the high frequency component around $2f_0$ is eliminated

Only the baseband component $v(t)=\sum_n a[n]p(t-nT)$ survives

The matched filter output is then equal to $a[n]$ when sampled at t_0+nT

2-PSK: ANALYTIC SIGNAL

The 2-PSK transmitted waveform

$$s(t) = \left[\sum_n a[n] p(t - nT) \right] \cos(2\pi f_0 t)$$

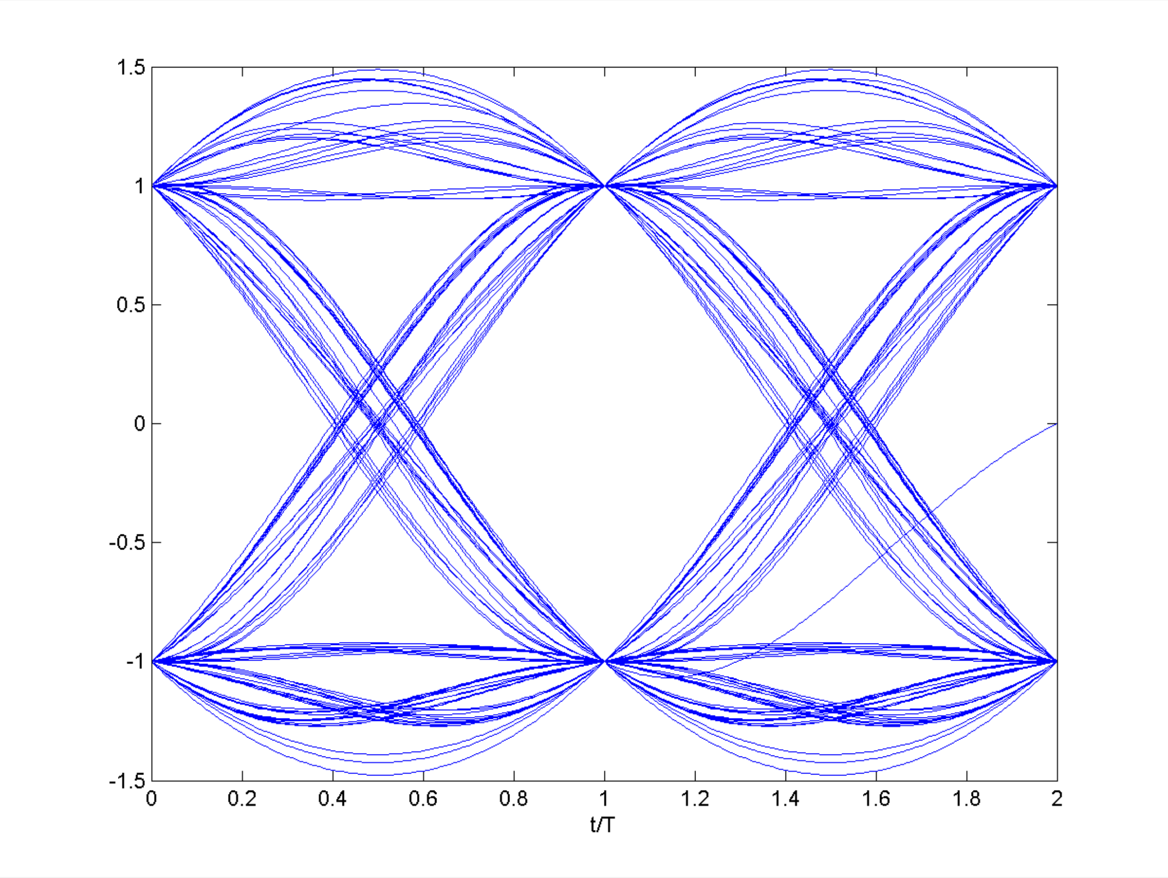
can be written as

$$s(t) = \mathrm{Re}\left[\dot{s}(t)\right] = \mathrm{Re}\left[\sum_n a[n] p(t - nT)\, \mathrm{e}^{j2\pi f_0 t} \right]$$

Where $\dot{s}(t)$ is called the **analytic signal** associated with $s(t)$

2-PSK: EYE DIAGRAM

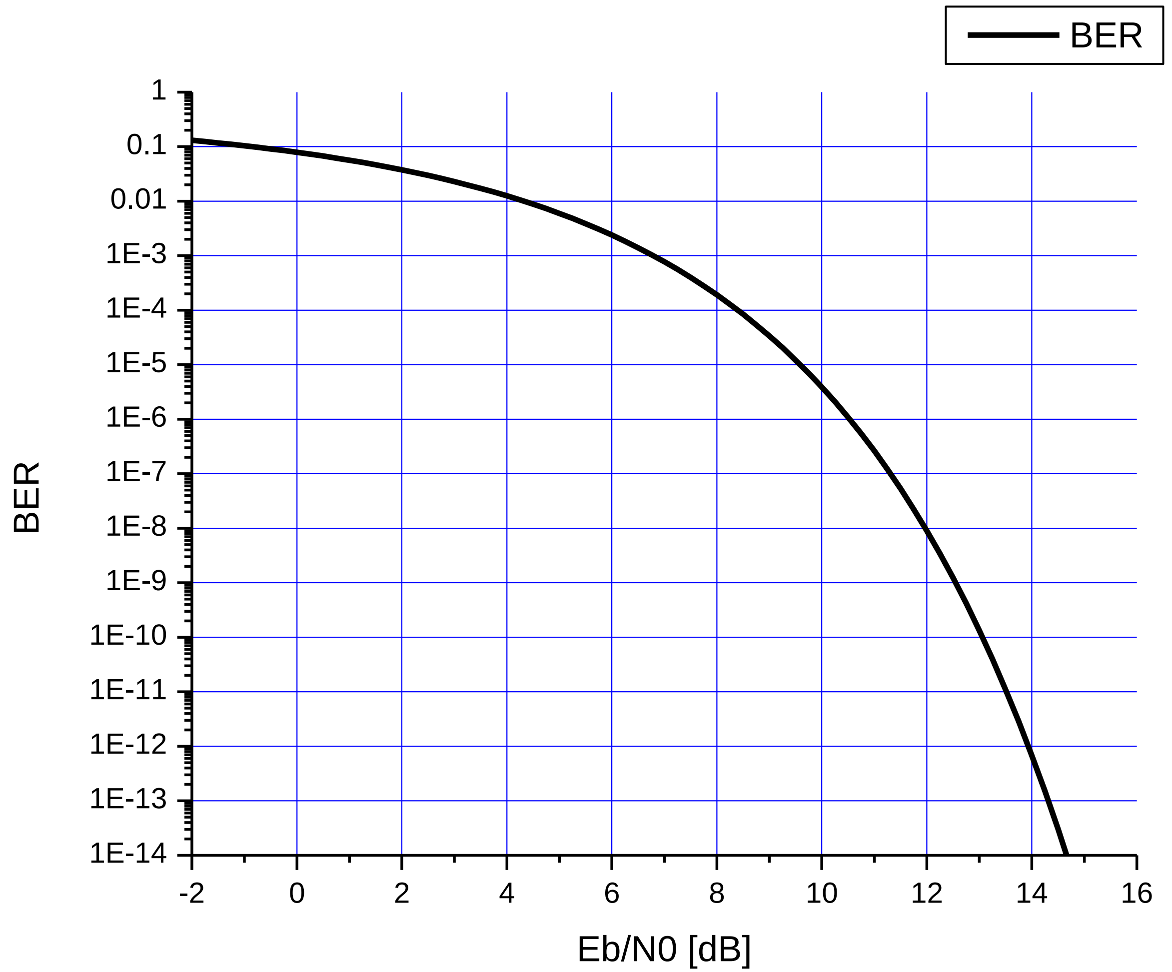
2-PSK constellation with RRC filter (α=0.5)



2-PSK: ERROR PROBABILITY

ERROR PROBABILITY

$$BER = \frac{1}{2}\,erfc\sqrt{\frac{E_b}{N_0}}$$



LECTURE 11:4-PSK AND M-PSK

QUADRATURE MODULATION

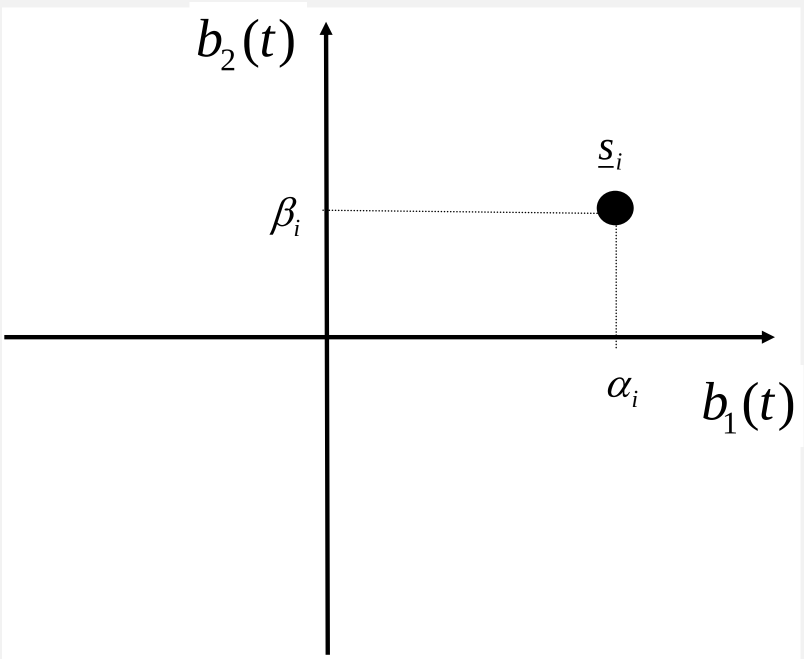
Consider a 2-D constellation, suppose that basis signals =cosine and sine

$$b_1(t) = p(t)\cos(2\pi f_0 t)$$

$$b_2(t) = p(t)\sin(2\pi f_0 t)$$

Each constellation symbol corresponds to a vector with two real components

$$M = \{\underline{s_i} = (\alpha_i, \beta_i)\}$$



QUADRATURE MODULATION

DURATION T

Binary information sequence $\underline{v}_T[n] \in H_k$

DURATION T

Symbol sequence $\underline{s}_T[n] \in M \subseteq R^2$

$\boxed{\alpha[n] \in R}$ $\boxed{\beta[n] \in R}$

DURATION T

Transmitted signal

$$s(t) = \sum_n \alpha[n] b_1(t - nT) + \sum_n \beta[n] b_2(t - nT) = a(t) + b(t)$$

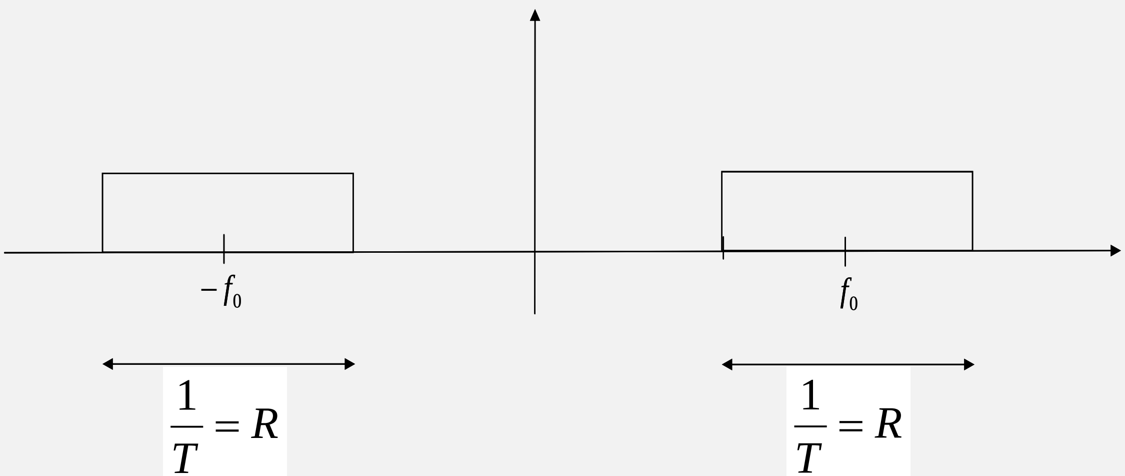
QUADRATURE MODULATION

Spectrum of $a(t)$:

$$a(t) = \sum_n \alpha[n] b_1(t - nT) = \left[\sum_n \alpha[n] p(t - nT) \right] \cos\left(2\pi f_0 t\right)$$

$$G_a = x\left[\left|P(f - f_0)\right|^2 + \left|P(f - f_0)\right|^2 \right] \qquad x \in R$$

when $p(t)$ = ideal low pass filter



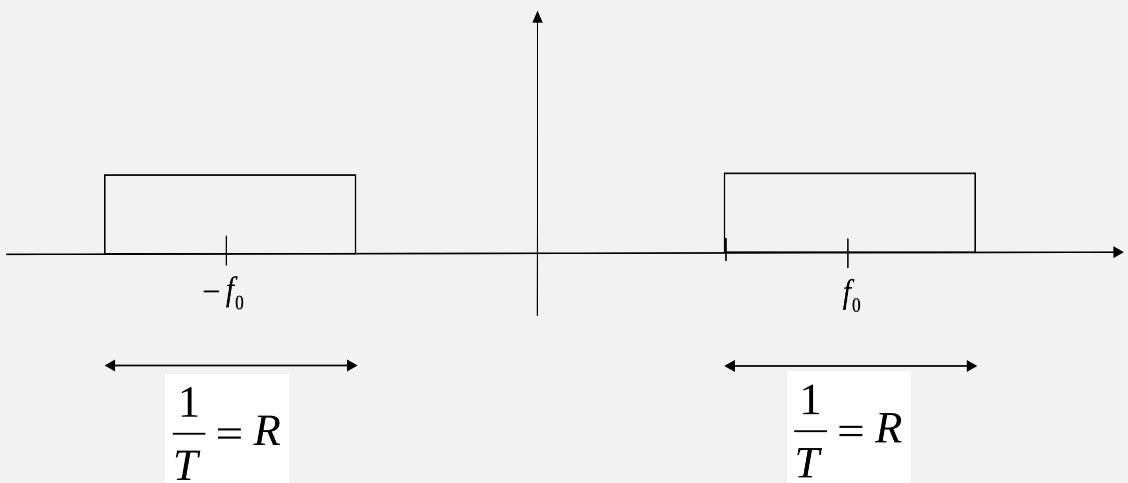
QUADRATURE MODULATION

Spectrum of $b(t)$:

$$b(t) = \sum_n \beta[n] b_1(t - nT) = \left[\sum_n \beta[n] p(t - nT) \right] \sin\left(2\pi f_0 t\right)$$

$$G_b = y\left[\left|P(f - f_0)\right|^2 + \left|P(f - f_0)\right|^2 \right] \qquad y \in R$$

when $p(t)$ = ideal low pass filter



QUADRATURE MODULATION

$$s(t) = a(t) + b(t)$$

It can be proved that

$$G_s(f) = G_a(f) + G_b(f)$$

QUADRATURE MODULATION

$$s(t) = a(t) + b(t)$$

$$G_s = G_a + G_b$$

$$G_a = x\left[\left|P(f-f_0)\right|^2 + \left|P(f-f_0)\right|^2\right] \qquad x \in R$$

$$G_b = y\left[\left|P(f-f_0)\right|^2 + \left|P(f-f_0)\right|^2\right] \qquad y \in R$$

$$G_s = z\left[\left|P(f-f_0)\right|^2 + \left|P(f-f_0)\right|^2\right] \qquad z \in R$$

G_a and G_b have the same shape and live on the same frequencies

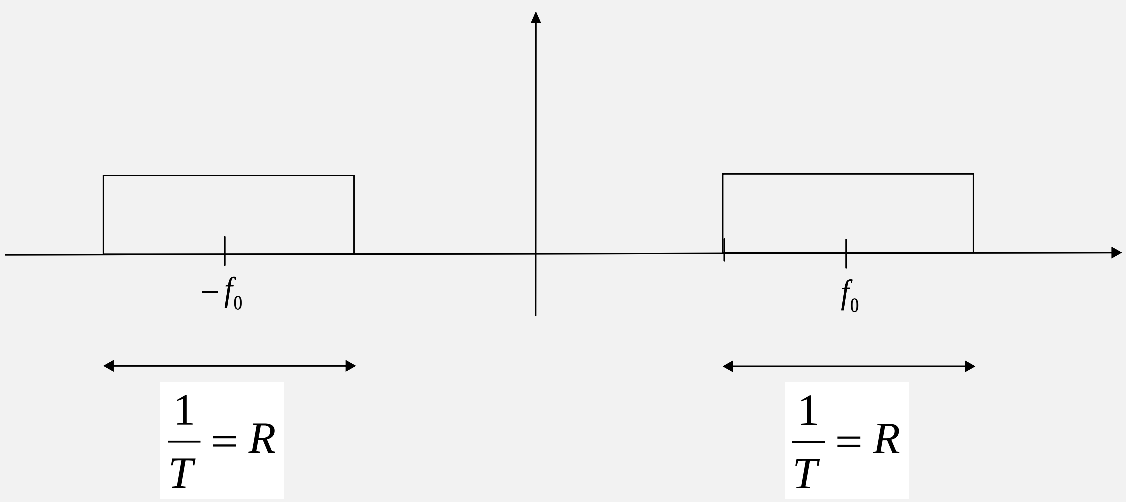
This is also the case for G_s

The spectrum of $s(t)$ only depends on $|P(f)|^2$

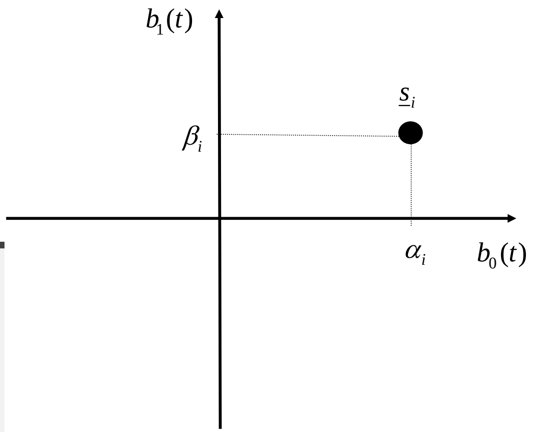
QUADRATURE MODULATION

Example when $p(t)$ = ideal low pass filter

$$G_s = z\left[\left|P(f-f_0)\right|^2 + \left|P(f-f_0)\right|^2\right] \qquad z \in R$$



I/Q COMPONENT



Given a quadrature modulation,
let us consider its transmitted waveform

$$s(t) = a(t) + b(t) =$$

$$= \underbrace{\left[\sum_n \alpha[n] p(t-nT)\right]}_{i(t)} \cos\left(2\pi f_0 t\right) + \underbrace{\left[\sum_n \beta[n] p(t-nT)\right]}_{q(t)} \sin\left(2\pi f_0 t\right)$$

I component (in phase)

Q component (in quadrature)

COMPLEX ENVELOPE

$$s(t) = \left[i(t)\right]\cos\left(2\pi f_0 t\right) + \left[q(t)\right]\sin\left(2\pi f_0 t\right)$$

Complex envelope $\boxed{\tilde{s}(t) = i(t) - jq(t)}$

$$i(t) = \sum_n \alpha[n]p(t - nT)$$

$$q(t) = \sum_n \beta[n]p(t - nT)$$

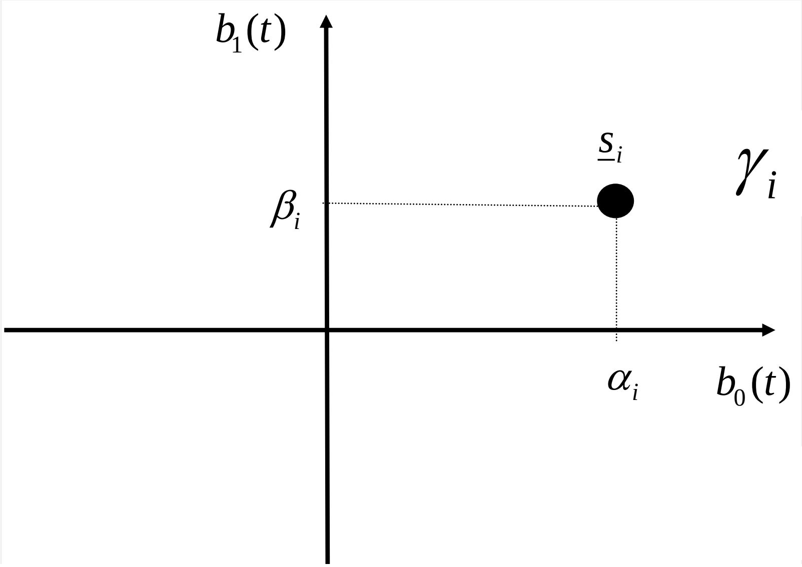
Complex symbol $\boxed{\gamma[n] = \alpha[n] - j\beta[n]}$

$$\tilde{s}(t) = \sum_n \gamma[n]p(t - nT)$$

COMPLEX ENVELOPE

$$\tilde{s}(t) = \sum_{n} \gamma[n] p(t - nT)$$

$$\gamma[n] = \alpha[n] - j\beta[n]$$



$$\gamma_i = \alpha_i - j\beta_i$$

Quadrature constellation as a set of complex numbers

$$M = \left\{ \gamma_i = \alpha_i - j\beta_i \right\}_{i=1}^{m}$$

ANALYTIC SIGNAL

$$s(t) = [i(t)]\cos(2\pi f_0 t) + [q(t)]\sin(2\pi f_0 t)$$

$$\tilde{s}(t) = i(t) - jq(t)$$

$$s(t) = \mathrm{Re}\left[\tilde{s}(t)e^{j2\pi f_0 t}\right] = \mathrm{Re}[\dot{s}(t)]$$

Analytic signal

$$\boxed{\dot{s}(t) = \tilde{s}(t)e^{j2\pi f_0 t}}$$

$$\dot{s}(t) = \tilde{s}(t)e^{j2\pi f_0 t} = \left[\sum_n \gamma[n]p(t - nT)\right]e^{j2\pi f_0 t}$$

4-PSK: CHARACTERISTICS

1. Band-pass modulation

2. 2D signal set

3. Basis signals $p(t)cos(2\pi f_0 t)$ & $p(t)sin(2\pi f_0 t)$

4. Constellation = 4 signals, equidistant on a circle

5. Information associated to the carrier phase

4-PSK: CONSTELLATION

SIGNAL SET

$$\boxed{\begin{aligned} M = \{ & s_1(t) = Ap(t)\cos(2\pi f_0 t)\,, s_2(t) = Ap(t)\sin(2\pi f_0 t) \\ & s_3(t) = -Ap(t)\cos(2\pi f_0 t)\,, s_4(t) = -Ap(t)\sin(2\pi f_0 t)\,\} \end{aligned}}$$

If we write

$$M = \left\{ \begin{aligned} & s_1(t) = Ap(t)\cos(2\pi f_0 t)\,, \\ & s_2(t) = Ap(t)\sin(2\pi f_0 t) = Ap(t)\cos\left(2\pi f_0 t - \frac{\pi}{2}\right), \\ & s_3(t) = -Ap(t)\cos(2\pi f_0 t) = Ap(t)\cos\left(2\pi f_0 t - \pi\right), \\ & s_4(t) = -Ap(t)\sin(2\pi f_0 t) = Ap(t)\cos\left(2\pi f_0 t - \frac{3\pi}{2}\right) \end{aligned} \right\}$$

Information associated to the carrier phase

4-PSK: CONSTELLATION

SIGNAL SET

$$\boxed{\begin{aligned} &M = \{s_i(t) = Ap(t)\cos(2\pi f_0 t - \varphi_i)\}_{i=1}^{4} \\ &\varphi_i = (i-1)\frac{\pi}{2} \end{aligned}}$$

Versors

$$b_1(t) = p(t)\cos(2\pi f_0 t)$$

$$b_2(t) = p(t)\sin(2\pi f_0 t)$$

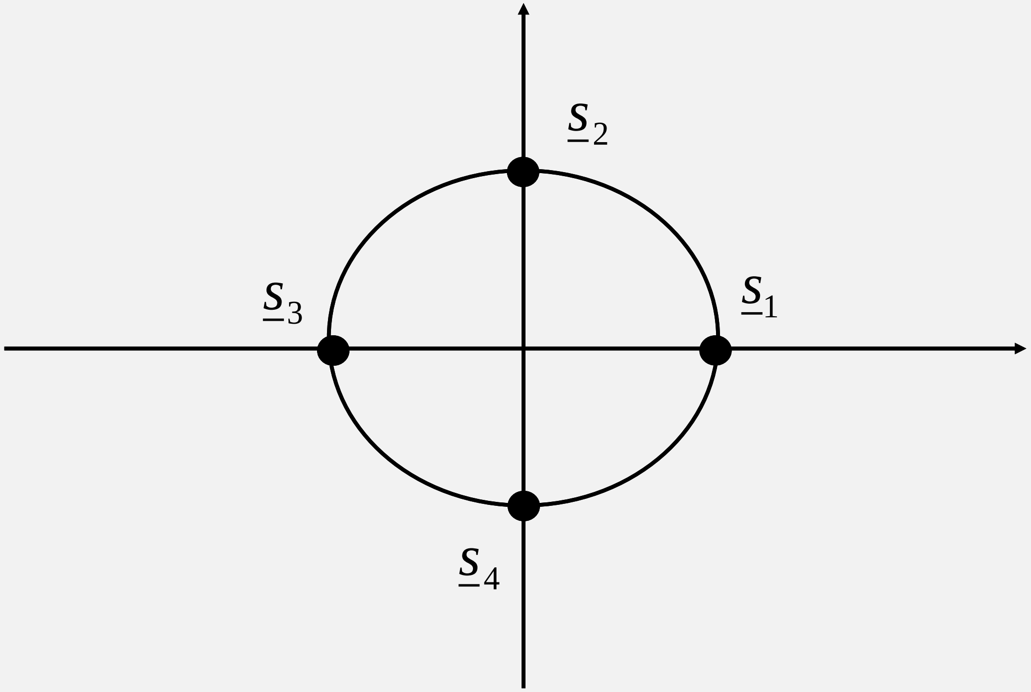
VECTOR SET

$$\boxed{M = \{\underline{s}_1 = (A,0), \underline{s}_2 = (0,A), \underline{s}_3 = (-A,0), \underline{s}_4 = (0,-A)\} \subseteq R^2}$$

4-PSK: CONSTELLATION

VECTOR SET

$$\boxed{M = \{\underline{s}_1 = (A,0), \underline{s}_2 = (0,A), \underline{s}_3 = (-A,0), \underline{s}_4 = (0,-A)\} \subseteq R^2}$$



4-PSK: CONSTELLATION

SIGNAL SET (with arbitary starting phase)

$$M = \{ s_i(t) = Ap(t)\cos(2\pi f_0 t - \varphi_i) \}_{i=1}^{4}$$

$$\varphi_i = \Phi + (i-1)\frac{\pi}{2}$$

$$s_i(t) = (A\cos\varphi_i)\, p(t)\cos(2\pi f_0 t) + (A\sin\varphi_i)\, p(t)\sin(2\pi f_0 t)$$

Versors

$$b_1(t) = p(t)\cos(2\pi f_0 t)$$

$$b_2(t) = p(t)\sin(2\pi f_0 t)$$

Vector set

$$M = \{ \underline{s}_i = (\alpha_i, \beta_i) \}_{i=1}^{4} \subseteq R^2$$

$$\alpha_i = A\cos\varphi_i$$

$$\beta_i = A\sin\varphi_i$$

$$\varphi_i = \Phi + (i-1)\frac{\pi}{2}$$

4-PSK: CONSTELLATION

Example: $\Phi = 0$

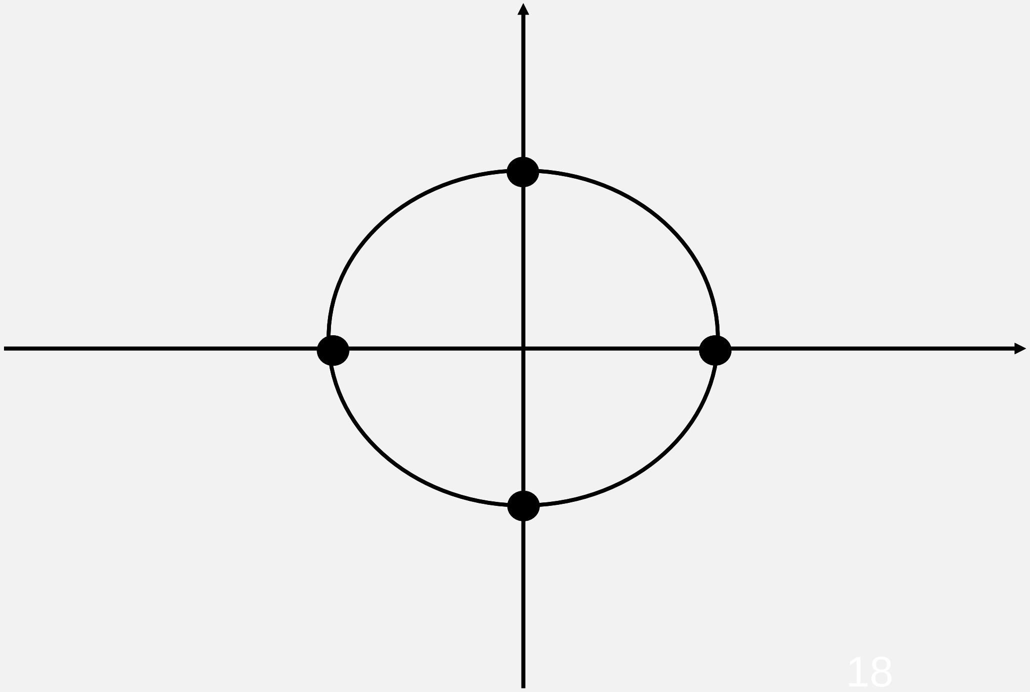
$$M = \{\underline{s}_1 = (A,0)\ ,\ \underline{s}_2 = (0,A)\ ,\ \underline{s}_3 = (-A,0)\ ,\ \underline{s}_4 = (0,-A)\} \subseteq R^2$$

$$M = \{\underline{s}_i = (\alpha_i, \beta_i)\ \}_{i=1}^{4} \subseteq R^2$$

$$\alpha_i = A\cos\varphi_i$$

$$\beta_i = A\sin\varphi_i$$

$$\varphi_i = (i-1)\frac{\pi}{2} \in \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}$$



4-PSK: CONSTELLATION

Example: $\Phi = \frac{\pi}{4}$

$$M = \{\underline{s}_1 = (-\alpha, -\alpha), \underline{s}_2 = (+\alpha, -\alpha), \underline{s}_3 = (+\alpha, +\alpha), \underline{s}_4 = (-\alpha, +\alpha)\} \subseteq R^2$$

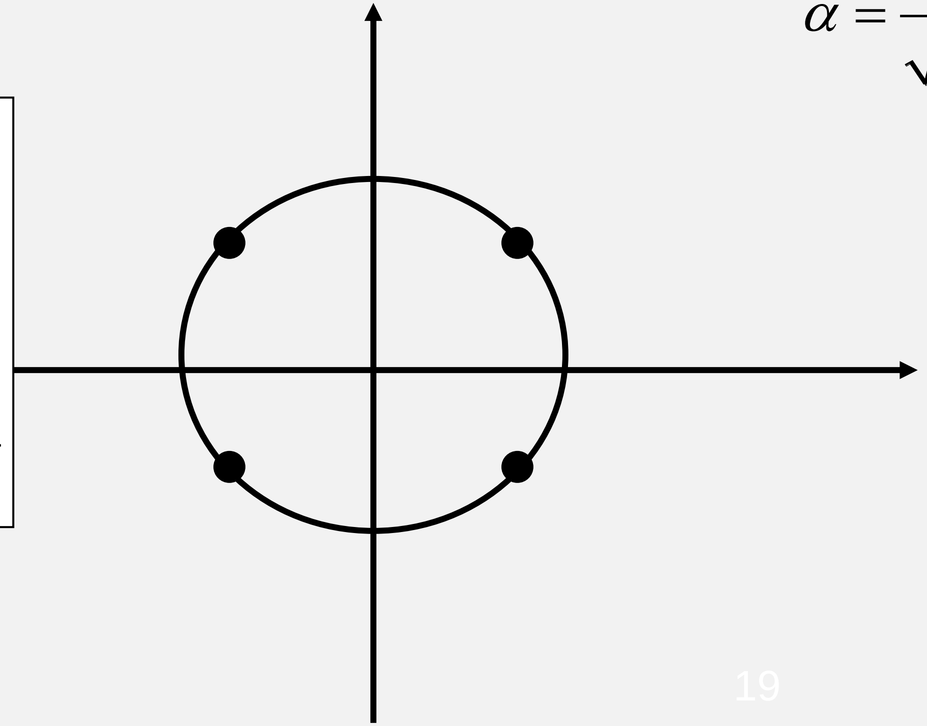
$$\alpha = \frac{A}{\sqrt{2}}$$

$$M = \{\underline{s}_i = (\alpha_i, \beta_i)\}_{i=1}^{4} \subseteq R^2$$

$$\alpha_i = A\cos\varphi_i$$

$$\beta_i = A\sin\varphi_i$$

$$\varphi_i = \frac{\pi}{4} + (i-1)\frac{\pi}{2} \in \left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$$

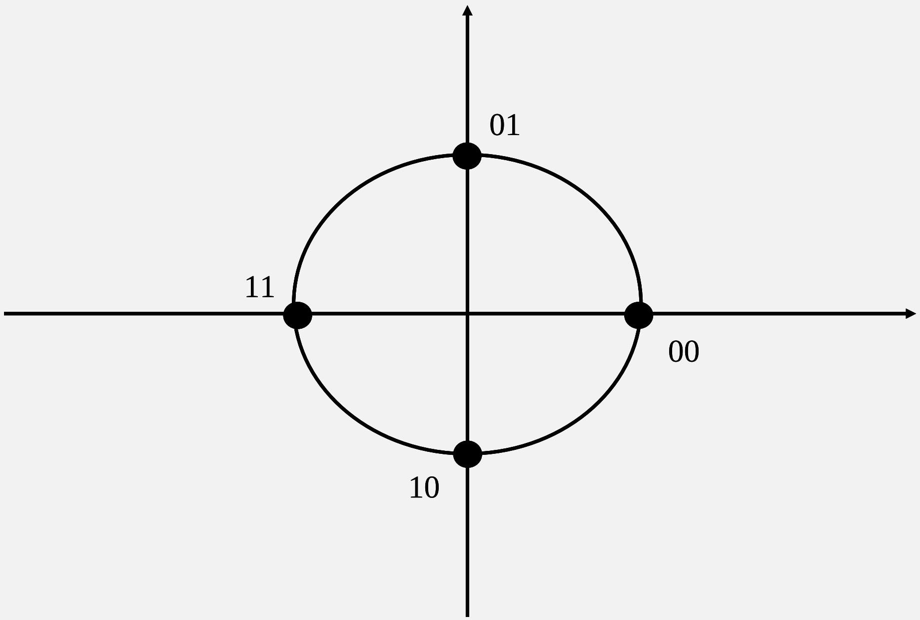


4-PSK: BINARY LABELING

Example of Gray labeling

$$e : H_2 \leftrightarrow M$$

$$e(00) = \underline{s}_0$$
$$e(01) = \underline{s}_1$$
$$e(11) = \underline{s}_2$$
$$e(10) = \underline{s}_3$$



4-PSK: TRANSMITTED WAVEFORM

$m = 2 \rightarrow k = 2$

$\boxed{T = 2T_b}$

$\boxed{R = \dfrac{R_b}{2}}$

Each symbol has duration T
Each symbol component (α and β) lasts for T second

Transmitted waveform

$$s(t) = \underbrace{\left[\sum_n \alpha[n] p(t - nT) \right]}_{i(t)} \cos(2\pi f_0 t) + \underbrace{\left[\sum_n \beta[n] p(t - nT) \right]}_{q(t)} \sin(2\pi f_0 t)$$

I component (in phase)

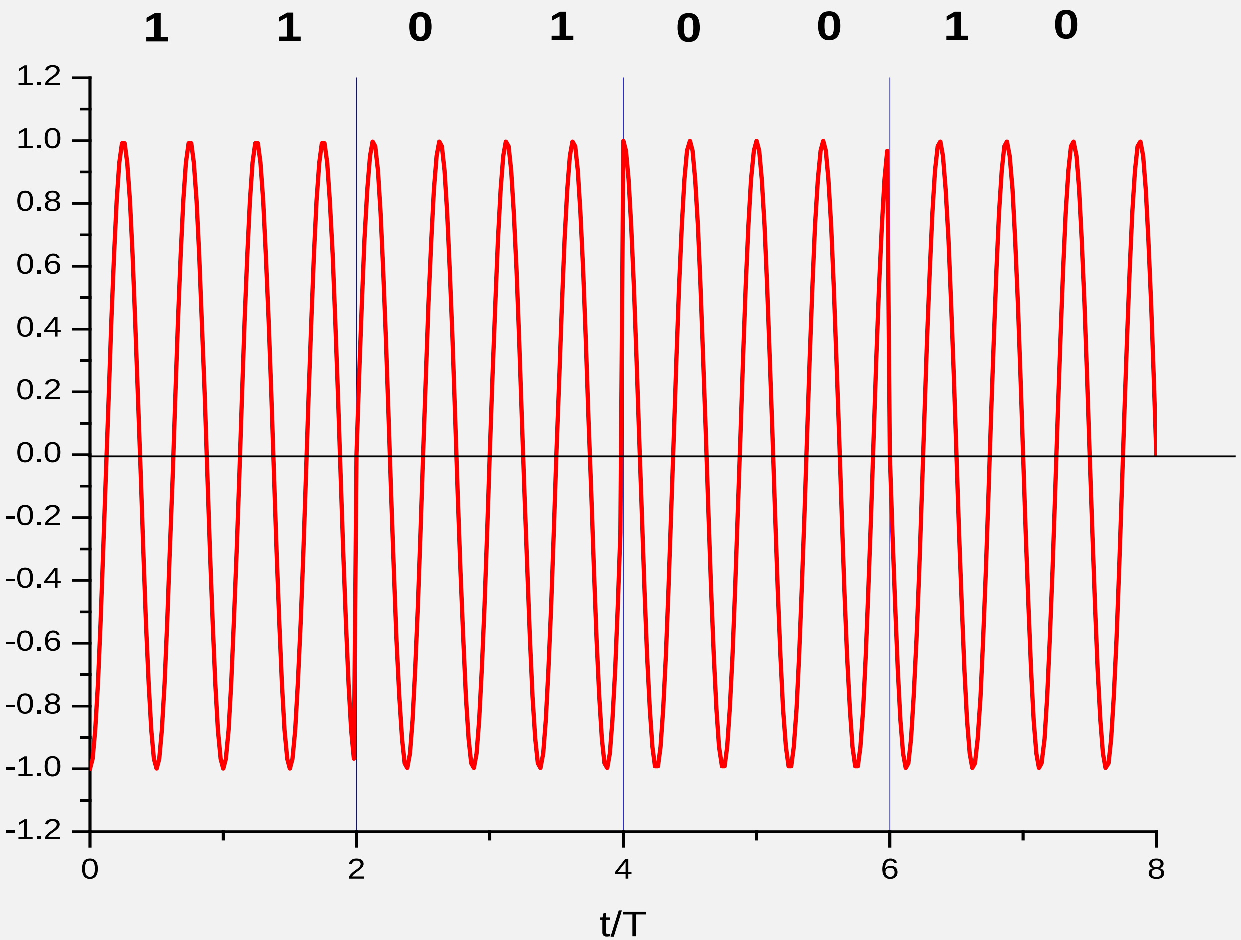
Q component (in quadrature)

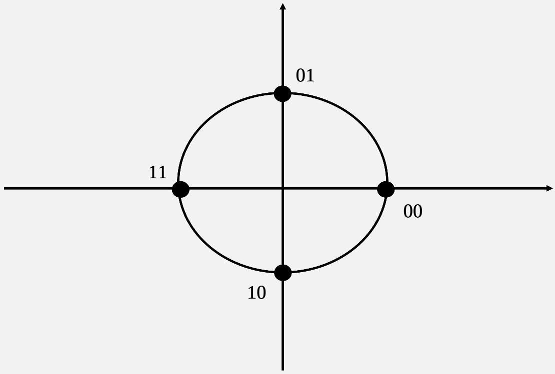
4-PSK: TRANSMITTED WAVEFORM

example for $$p(t) = \frac{1}{\sqrt{T}} P_T(t)$$

$$f_0 = 2R_b$$

$$\alpha = \sqrt{T}$$





4-PSK: ANALYTIC SIGNAL

$$s(t) = \underbrace{\left[\sum_n \alpha[n] p(t-nT) \right]}_{i(t)} \cos(2\pi f_0 t) + \underbrace{\left[\sum_n \beta[n] p(t-nT) \right]}_{q(t)} \sin(2\pi f_0 t)$$

$$s(t) = \mathrm{Re}\left[\dot{s}(t) \right] = \mathrm{Re}\left[\tilde{s}(t) e^{j 2\pi f_0 t} \right]$$

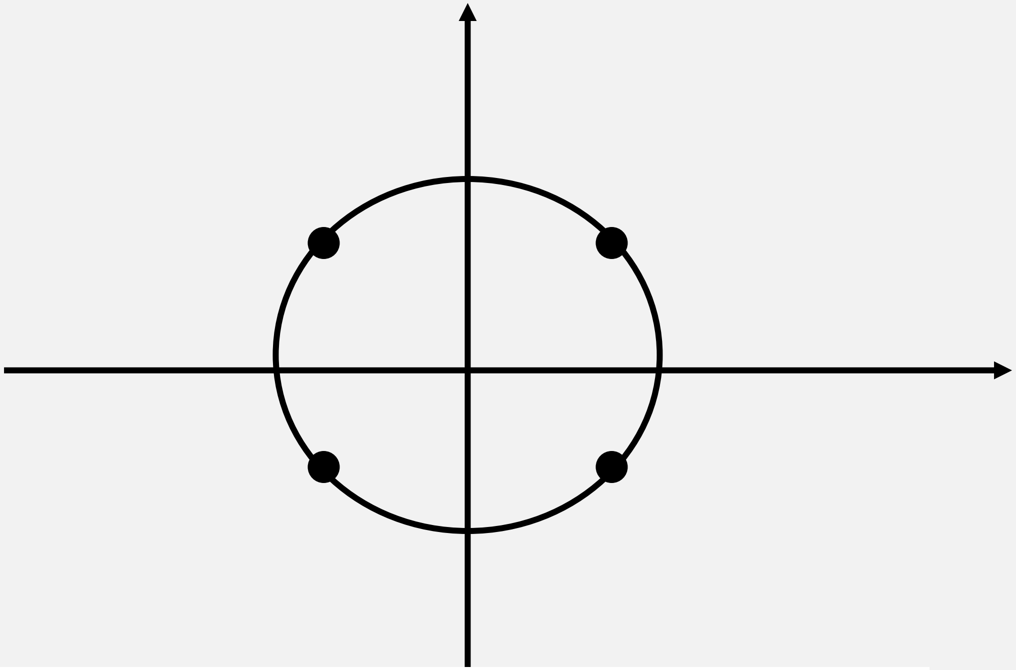
$$\tilde{s}(t) = i(t) - jq(t) = \sum_n \gamma[n] p(t-nT)$$

$$\gamma[n] = \alpha[n] - j\beta[n]$$

4-PSK: ANALYTIC SIGNAL

$$\tilde{s}(t) = \sum_n \gamma[n] p(t - nT)$$

$$\gamma[n] = \alpha[n] - j\beta[n]$$



$$M = \{ s_1 = (a - ja), s_2 = (-a - ja), s_3 = (-a + ja), s_4 = (a + ja), \}$$

4-PSK: BANDWIDTH AND SPECTRAL EFFICIENCY

Transmitted waveform

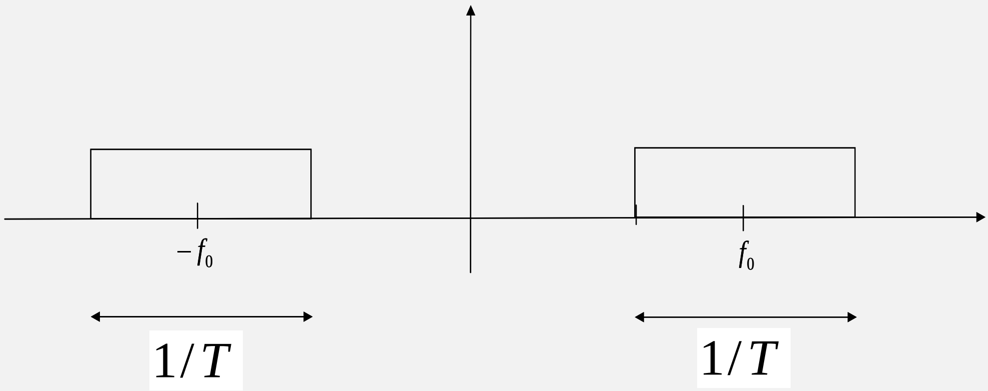
$$s(t) = \left[\sum_n \alpha[n] p(t - nT) \right] \cos(2\pi f_0 t) + \left[\sum_n \beta[n] p(t - nT) \right] \sin(2\pi f_0 t)$$

$$G_s(f) = z\left[\left|P(f - f_0)\right|^2 + \left|P(f + f_0)\right|^2 \right] \qquad z \in R$$

Each symbol $\alpha[n]$ and $\beta[n]$ has time duration $T = 2T_b$

4-PSK: BANDWIDTH AND SPECTRAL EFFICIENCY

Case 1: $p(t)$ = ideal low pass filter



Total bandwidth (ideal case)

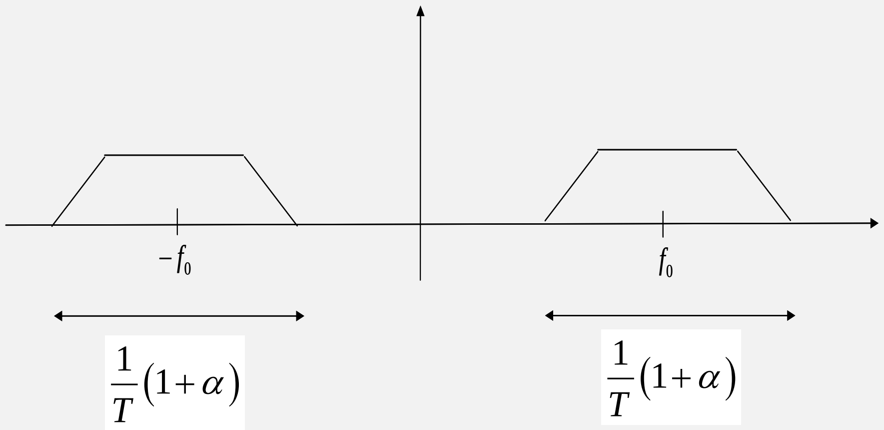
$$\boxed{B_{id} = R = \frac{R_b}{2}}$$

Spectral efficiency (ideal case)

$$\boxed{\eta_{id} = \frac{R_b}{B_{id}} = 2\ bps/Hz}$$

4-PSK: BANDWIDTH AND SPECTRAL EFFICIENCY

Case 2: $p(t)$ = RRC filter with roll off α



Total bandwidth

$$B = R(1+\alpha) = \frac{R_b}{2}(1+\alpha)$$

Spectral efficiency

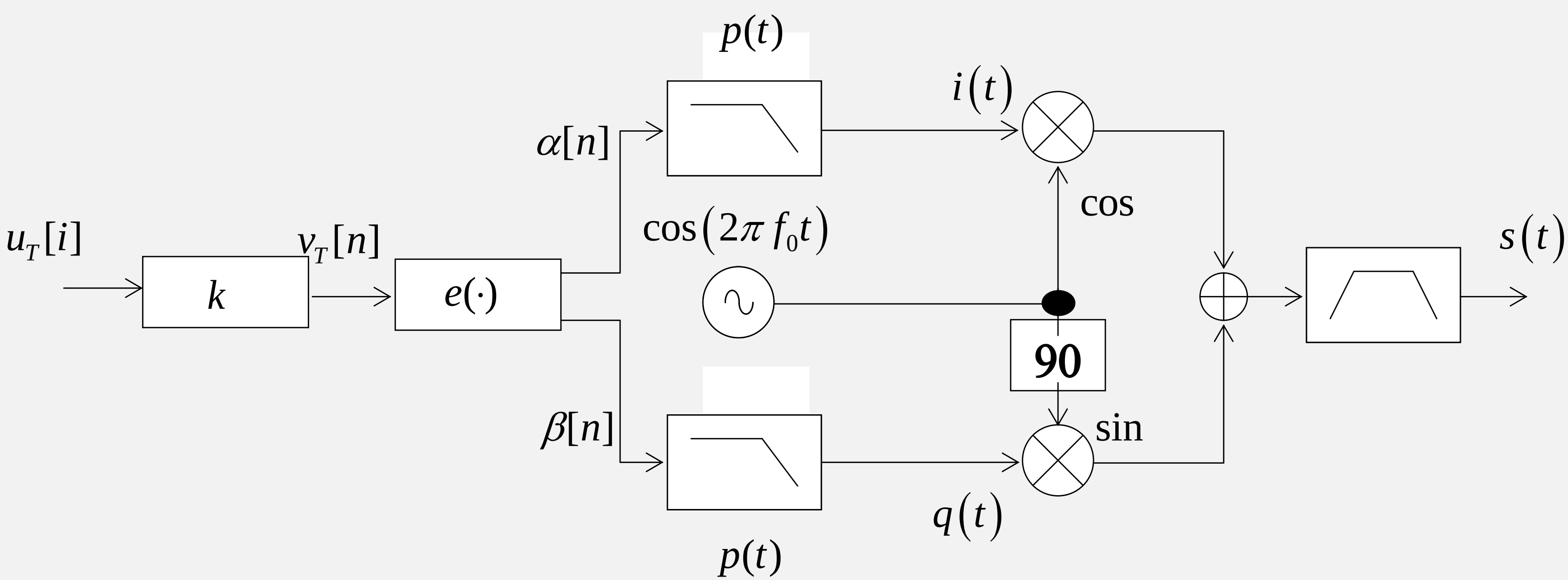
$$\boxed{\eta = \frac{R_b}{B} = \frac{2}{(1+\alpha)}\ bps/Hz}$$

EXERCIZE

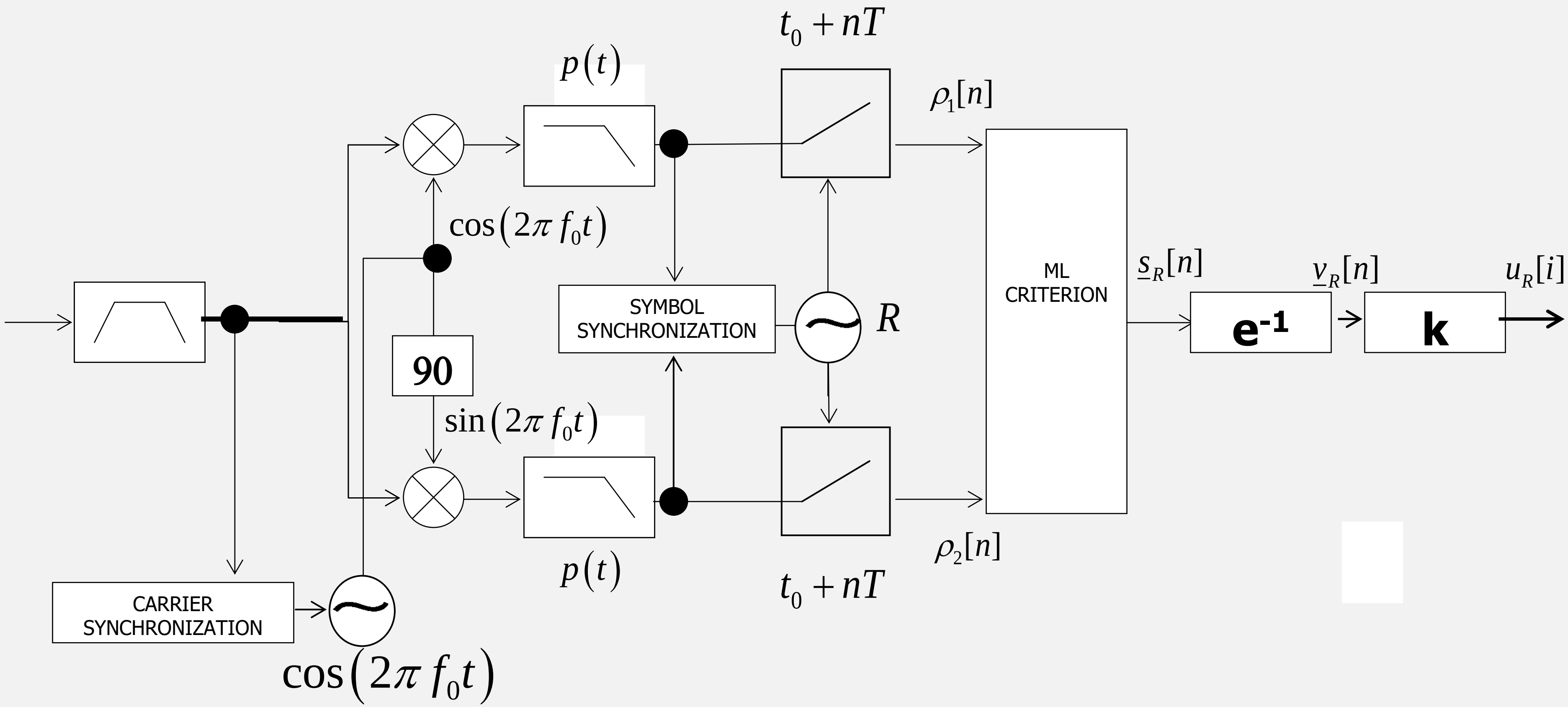
Given a bandpass channel with bandwidth B = 4000 Hz, centred around f_0=2 GHz, compute the maximum bit rate R_b we can transmit over it with a 4-PSK constellation in the two cases:

- Ideal low pass filter
- RRC filter with α=0.25

4-PSK: MODULATOR

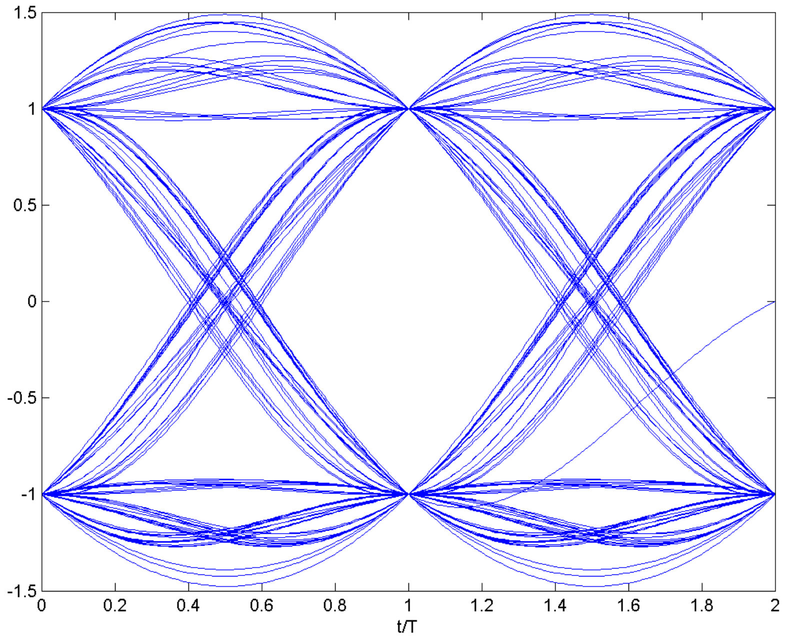


4-PSK: DEMODULATOR

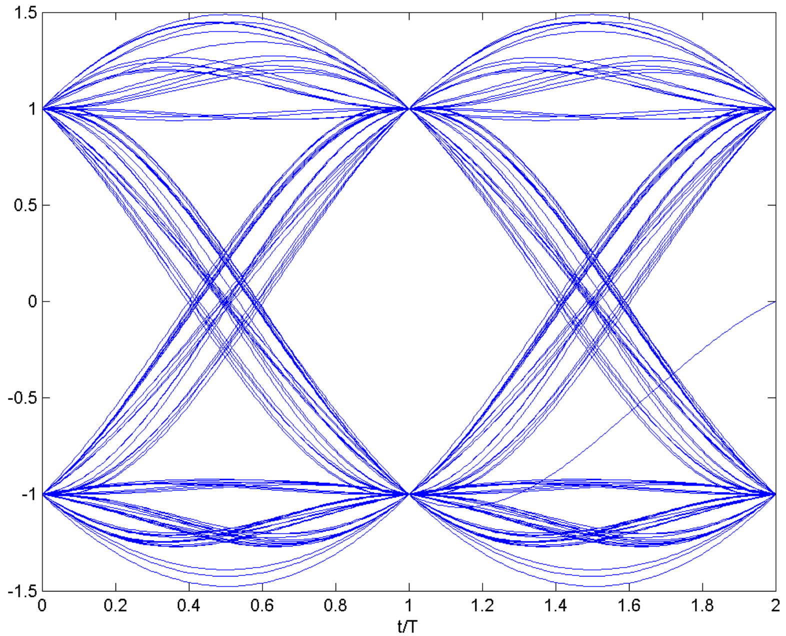


4-PSK: EYE DIAGRAM

4-PSK constellation with RRC filter (α=0.5)



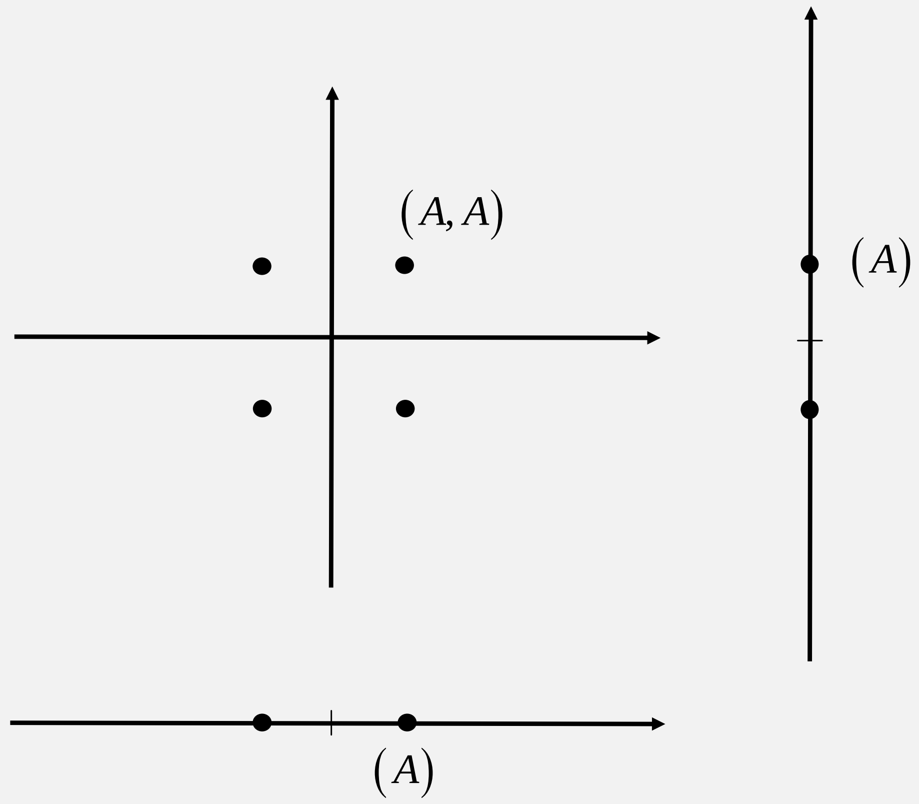
Canale I



Canale Q

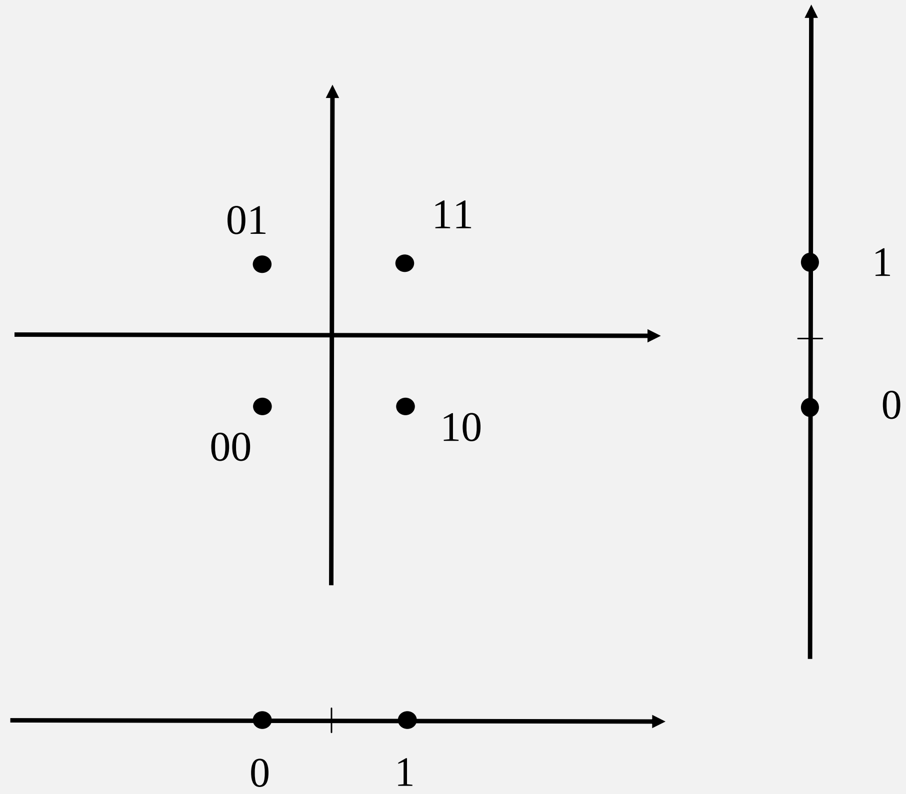
4-PSK: INTEPRETATION

The 4-PSK vector set can be viewed as
the Cartesian product of two 2-PSK constellations



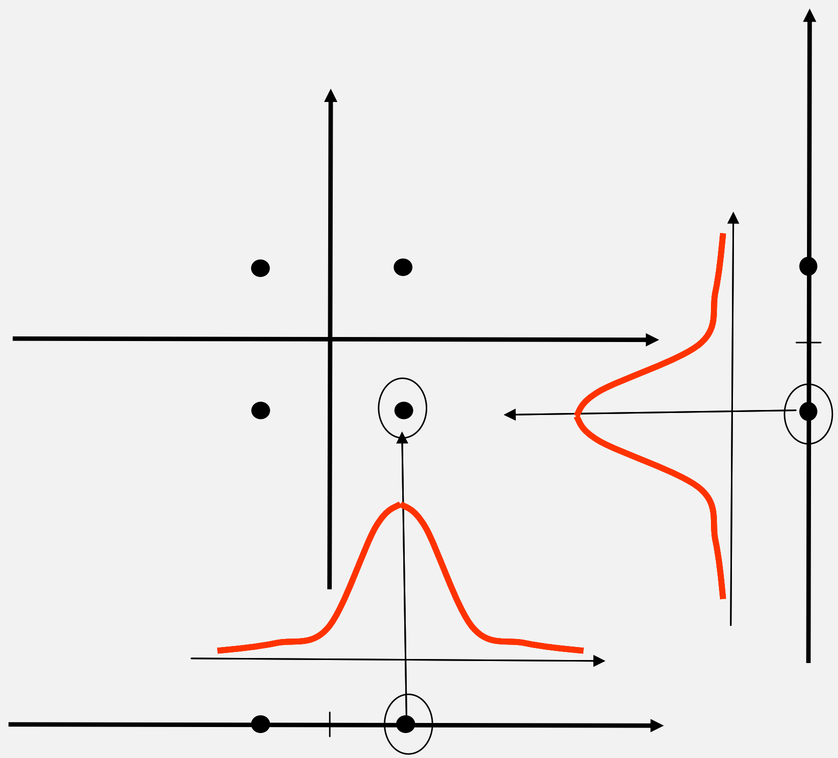
4-PSK: INTEPRETATION

This is also true for the binary Gray labeling
(first bit = I component, second bit = Q component)



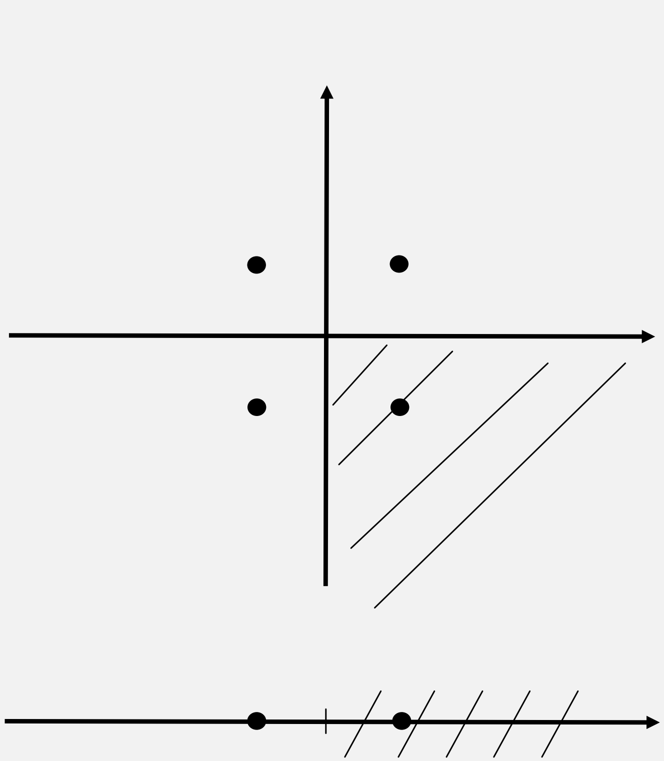
4-PSK: INTEPRETATION

The AWGN channel adds two Gaussian components which are statistically independent



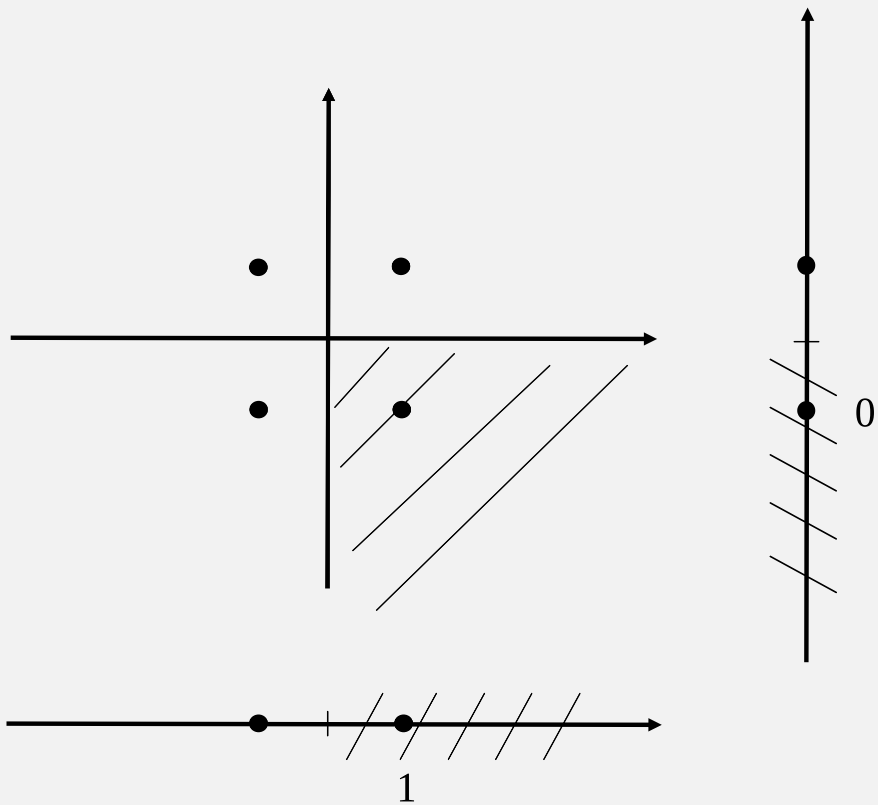
4-PSK: INTEPRETATION

The Voronoi regions of 4-PSK signals are the Cartesian product of the Voronoi regions of the constituent 2-PSK constellations



4-PSK: INTEPRETATION

The Voronoi regions of 4-PSK signals are the Cartesian product of the Voronoi regions of the constituent 2-PSK constellations



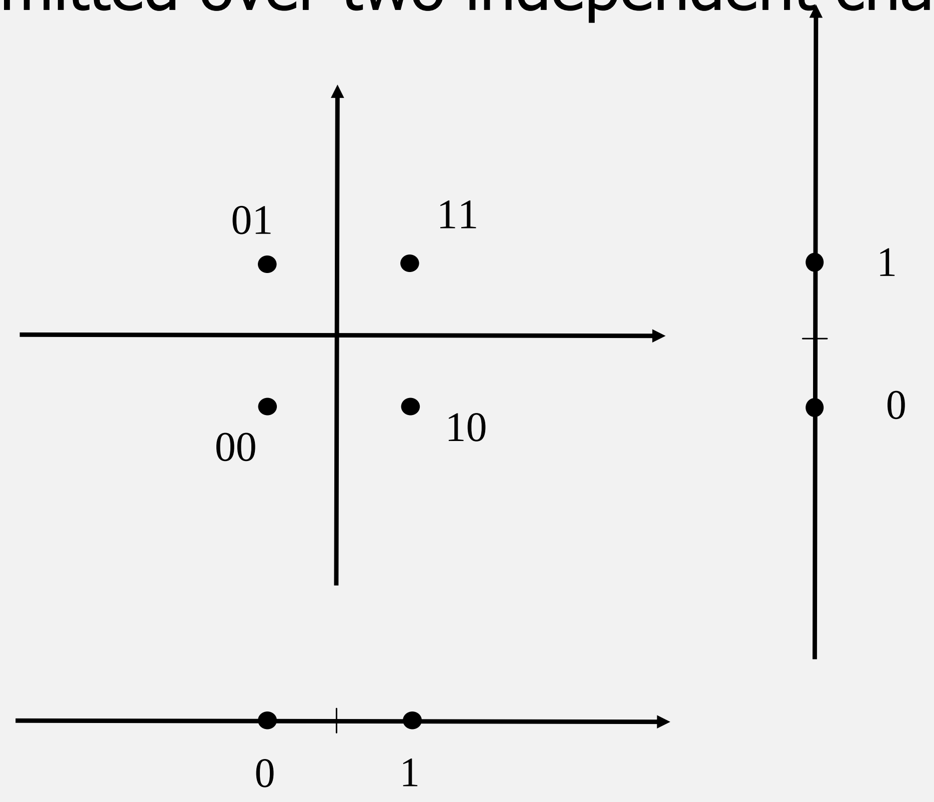
Given the received vector $(\rho_1[n], \rho_2[n])$

The sign of the first component $\rho_1[n]$ determines the first received bit

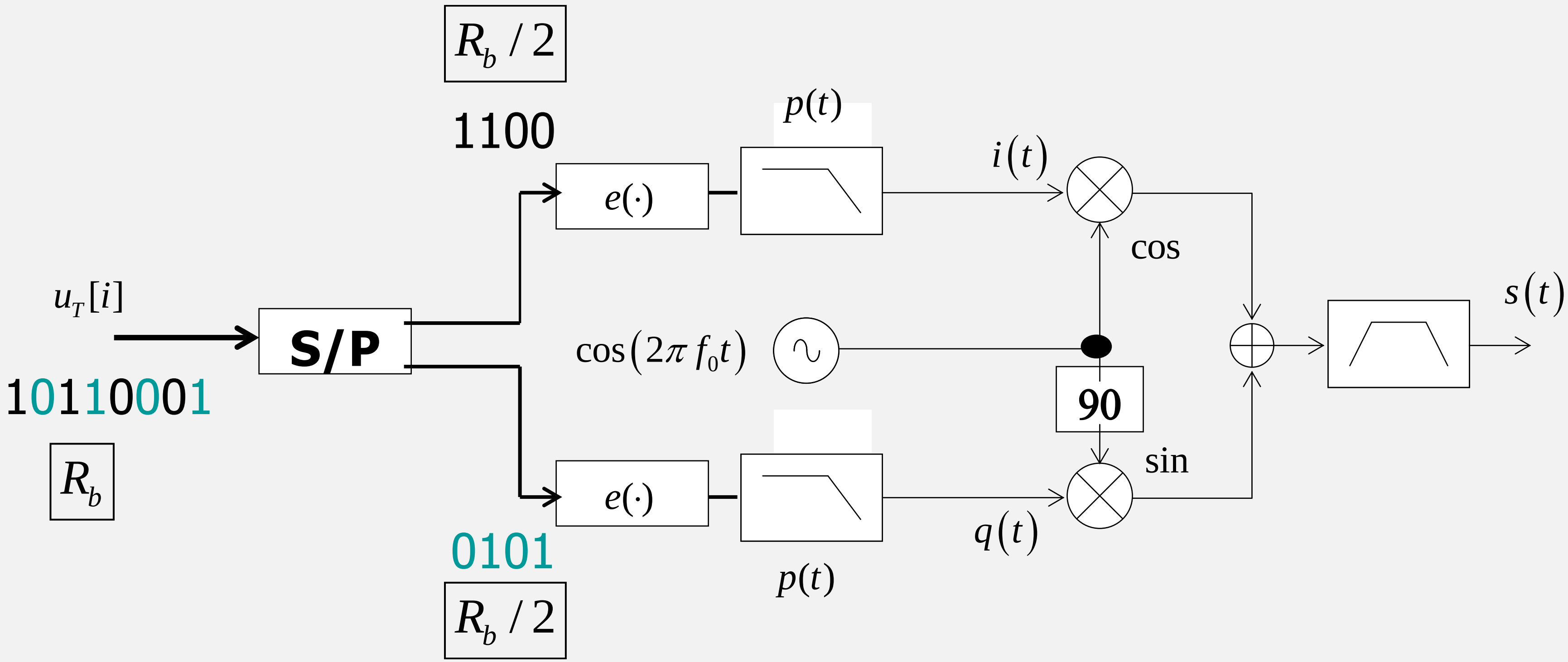
The sign of the second component $\rho_2[n]$ determines the second received bit

4-PSK: INTEPRETATION

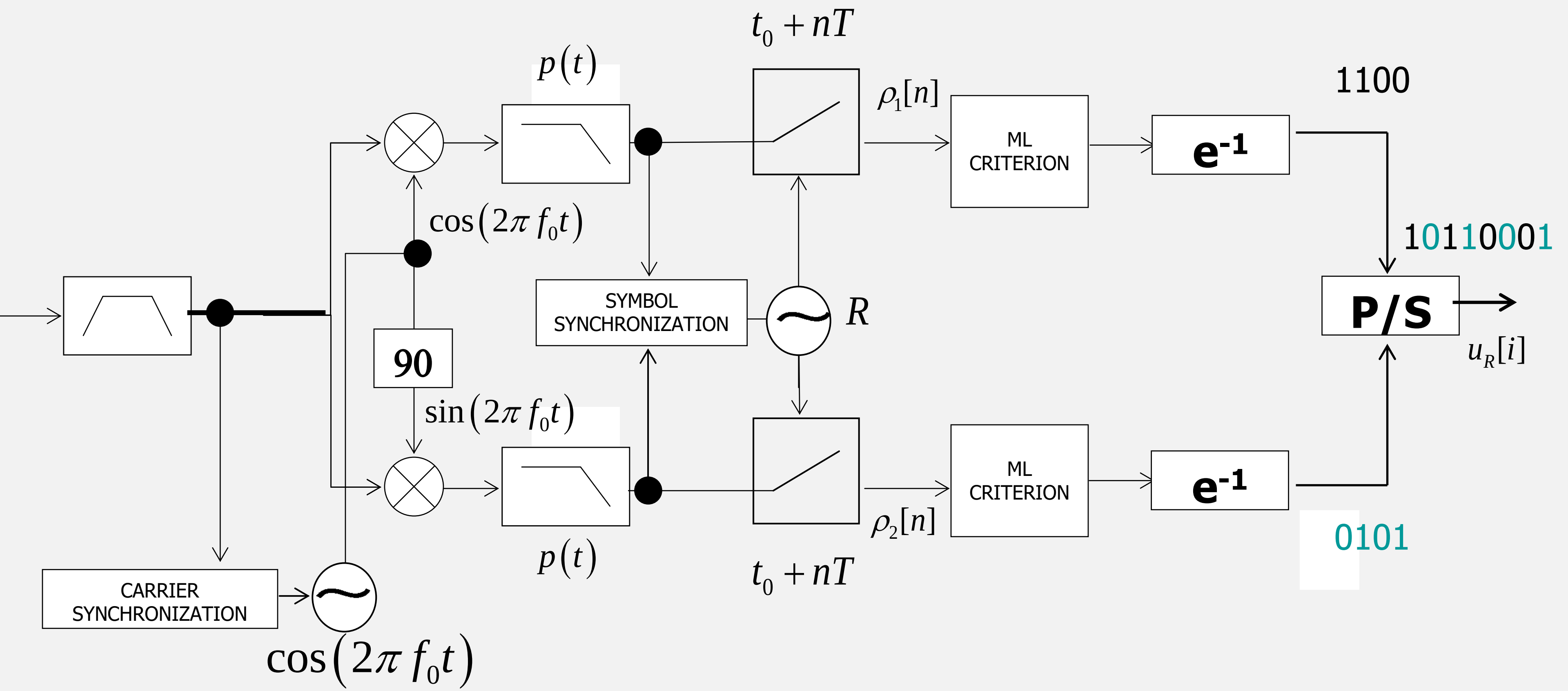
The 4-PSK modulation can be viewed as
the Cartesian product of two 2-PSK constellations
transmitted over two independent channels



4-PSK: MODULATOR



4-PSK: DEMODULATOR

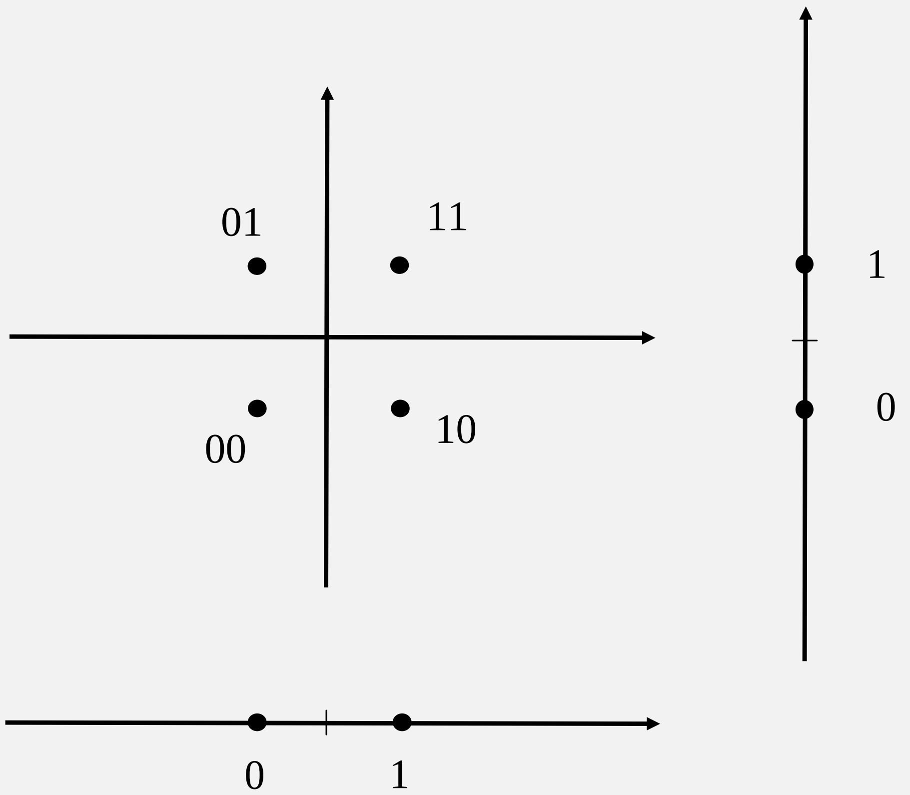


4-PSK: INTEPRETATION

The Cartesian product interpretation clarifies why a 4-PSK constellation

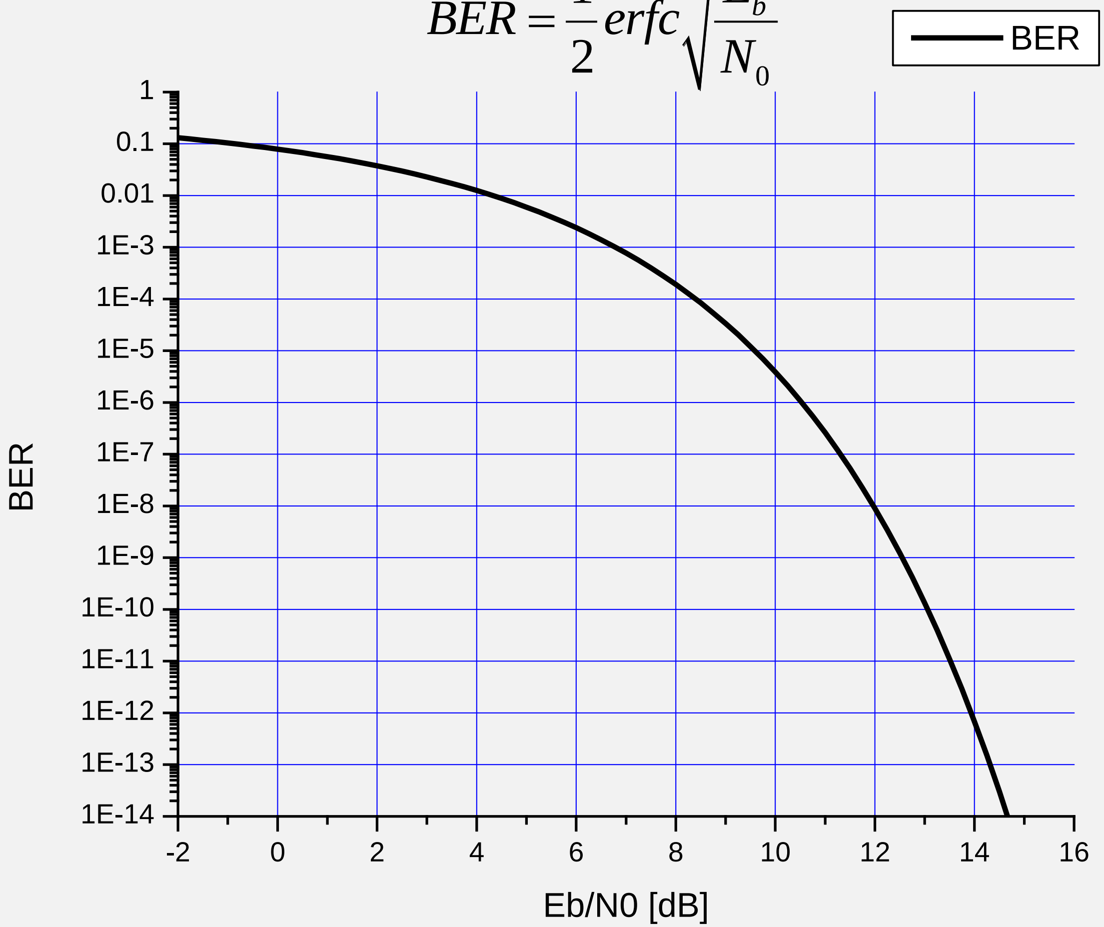
1. **Has the same BER performance of a 2-PSK**

2. **Has double spectral efficiency** (two sequences with half bit-rate transmitted on the same frequencies)



4-PSK: ERROR PROBABILITY

$$BER = \frac{1}{2} erfc \sqrt{\frac{E_b}{N_0}}$$



4-PSK: APPLICATIONS

Probably the most used digital modulation

- Satellite links
- Terrestrial radio links (with low spectral efficiency)
- GPS/Galileo
- UMTS
- ...

M-PSK: CHARACTERISTICS

1. Band-pass modulation
2. 2D signal set
3. Basis signals $p(t)cos(2\pi f_0 t)$ e $p(t)sin(2\pi f_0 t)$
4. Costellation = m signals, equidistant on a circle
5. Information associated to the carrier phase

M-PSK: CONSTELLATION

SIGNAL SET

$$M = \{ s_i(t) = Ap(t)\cos(2\pi f_0 t - \varphi_i) \}_{i=1}^{m}$$

$$\varphi_i = \Phi + (i-1)\frac{2\pi}{m}$$

Information associated to the carrier phase

M-PSK: CONSTELLATION

$$\boxed{s_i(t) = Ap(t)\cos(2\pi f_0 t - \varphi_i)}$$

$$\varphi_i = \Phi + (i-1)\frac{2\pi}{m}$$

We can write

$$\boxed{s_i(t) = (A\cos\varphi_i)\,p(t)\cos(2\pi f_0 t) + (A\sin\varphi_i)\,p(t)\sin(2\pi f_0 t)}$$

Clearly, we have two versors

$$\boxed{b_1(t) = p(t)\cos(2\pi f_0 t)}$$

$$\boxed{b_2(t) = p(t)\sin(2\pi f_0 t)}$$

M-PSK: CONSTELLATION

SIGNAL SET

$$M = \{ s_i(t) = Ap(t)\cos(2\pi f_0 t - \varphi_i) \}_{i=1}^{m} \qquad \varphi_i = \Phi + (i-1)\frac{2\pi}{m}$$

VERSORS

$$b_1(t) = p(t)\cos(2\pi f_0 t)$$

$$b_2(t) = p(t)\sin(2\pi f_0 t)$$

VECTOR SET

$$M = \{ \underline{s}_i = (\alpha_i, \beta_i) \}_{i=1}^{m} \subseteq R^2$$

$$\alpha_i = A\cos\varphi_i$$

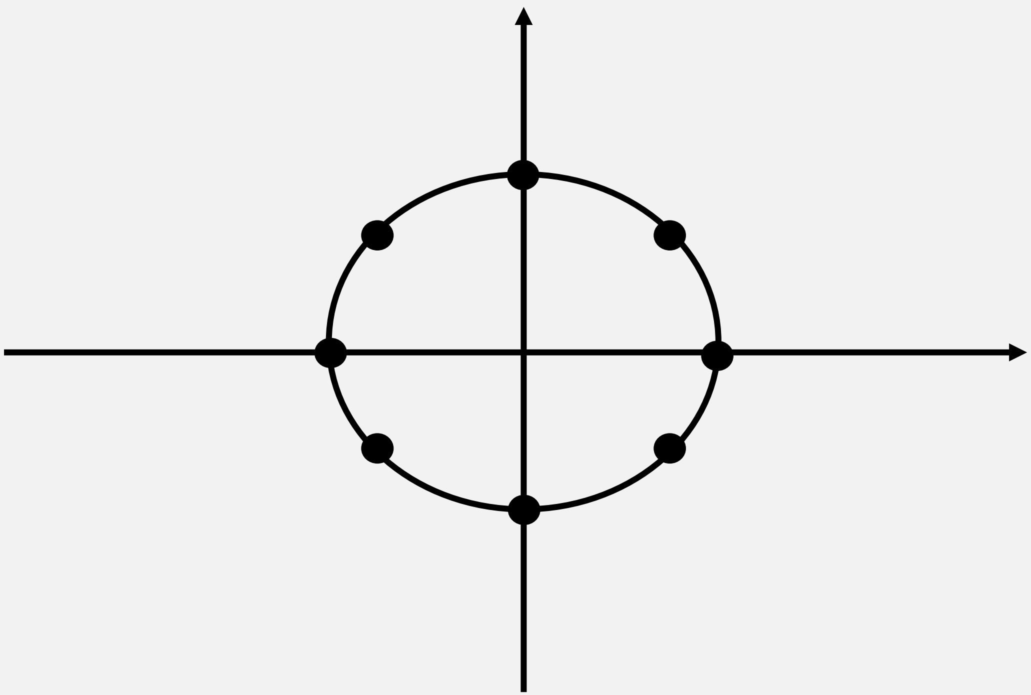
$$\beta_i = A\sin\varphi_i$$

$$\varphi_i = \Phi + (i-1)\frac{2\pi}{m}$$

EXAMPLE

8-PSK $\Phi = 0$

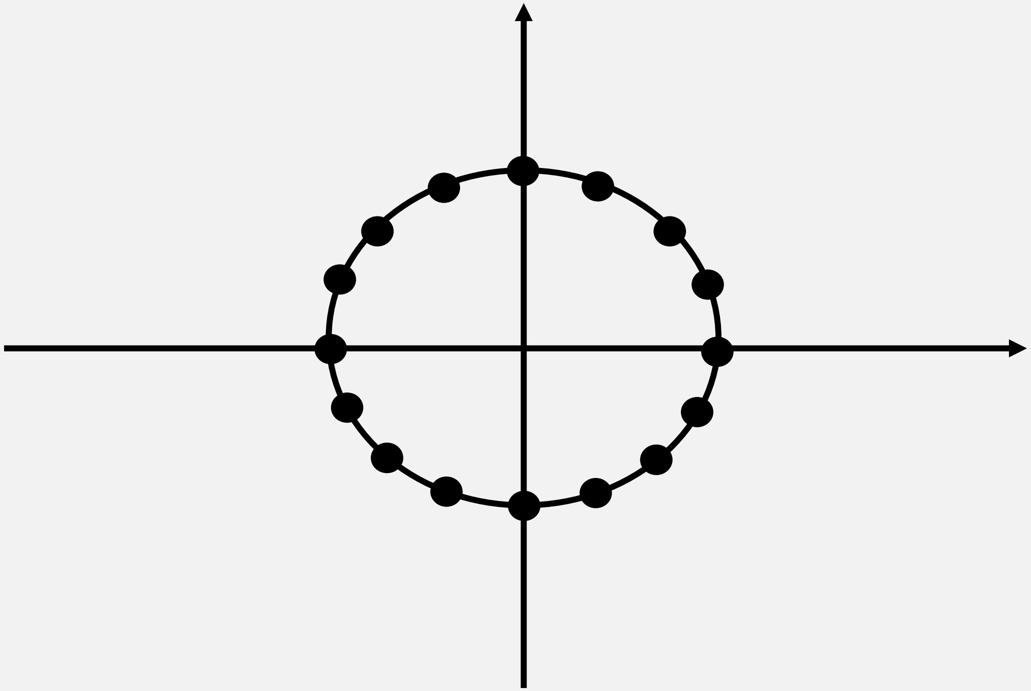
$$M = \{\underline{s}_1 = (A,0)\,,\ \underline{s}_2 = (A/\sqrt{2}, A/\sqrt{2})\,,\ \underline{s}_3 = (0,A)\,,\ \underline{s}_4 = (-A/\sqrt{2}, A/\sqrt{2}),$$
$$\underline{s}_5 = (-A,0)\,,\ \underline{s}_6 = (-A/\sqrt{2}, -A/\sqrt{2})\,,\ \underline{s}_7 = (0,-A)\,,\ \underline{s}_8 = (A/\sqrt{2}, -A/\sqrt{2})\} \subseteq R^2$$



EXAMPLE

16-PSK $\Phi = 0$

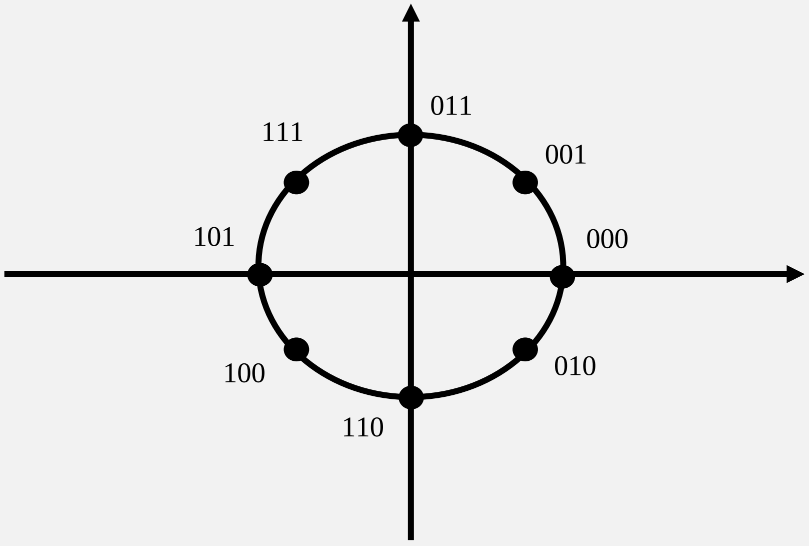
$$M = \{\underline{s}_1 = (A,0)\,,\ \underline{s}_2 = (0.924A, 0.383A)\,,\ \underline{s}_3 = (A/\sqrt{2}, A/\sqrt{2})\,,\ \underline{s}_4 = (0.383A, 0.924A),$$
$$\underline{s}_5 = (0, A)\,,\ \underline{s}_6 = (-0.383A, 0.924A,)\,,\ \underline{s}_7 = (-A/\sqrt{2}, A/\sqrt{2})\,,\ \underline{s}_8 = (-0.924A, 0.383A),$$
$$\underline{s}_9 = (-A,0)\,,\ \underline{s}_{10} = (-0.924A, -0.383A)\,,\ \underline{s}_{11} = (-A/\sqrt{2}, -A/\sqrt{2})\,,\ \underline{s}_{12} = (-0.383A, -0.924A),$$
$$\underline{s}_{13} = (0,-A)\,,\ \underline{s}_{14} = (0.383A, -0.924A,)\,,\ \underline{s}_{15} = (A/\sqrt{2}, -A/\sqrt{2})\,,\ \underline{s}_{16} = (0.924A, -0.383A)\} \subseteq R^2$$

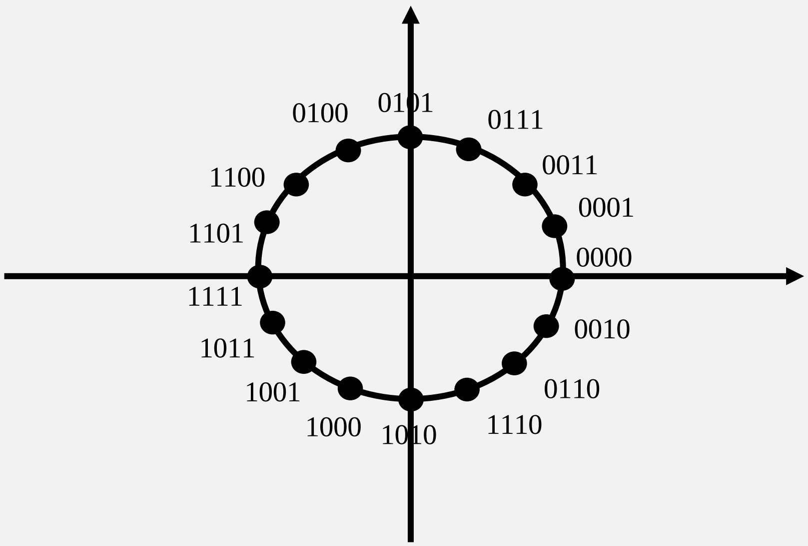


M-PSK: BINARY LABELING

$$e : H_k \leftrightarrow M$$

It is always possible to build Gray labelings





M-PSK: TRANSMITTED WAVEFORM

$k = \log_2 m$

$\boxed{T = kT_b}$

$\boxed{R = \frac{R_b}{k}}$

Each symbol has duration T
Each symbol component (α and β) lasts for T second

Transmitted waveform

$$s(t) = \underbrace{\left[\sum_n \alpha[n]p(t-nT)\right]}_{i(t)}\cos(2\pi f_0 t) + \underbrace{\left[\sum_n \beta[n]p(t-nT)\right]}_{q(t)}\sin(2\pi f_0 t)$$

I component (in phase)

Q component (in quadrature)

M-PSK: ANALYTIC SIGNAL

$$s(t) = \Bigg[\underbrace{\sum_n \alpha[n] p(t - nT)}_{i(t)} \Bigg] \cos(2\pi f_0 t) + \Bigg[\underbrace{\sum_n \beta[n] p(t - nT)}_{q(t)} \Bigg] \sin(2\pi f_0 t)$$

$$s(t) = \mathrm{Re}\left[\dot{s}(t) \right] = \mathrm{Re}\left[\tilde{s}(t) e^{j 2\pi f_0 t} \right]$$

$$\tilde{s}(t) = i(t) - jq(t) = \sum_n \gamma[n] p(t - nT)$$

$$\gamma[n] = \alpha[n] - j\beta[n]$$

M-PSK: BANDWIDTH AND SPECTRAL EFFICIENCY

Transmitted waveform

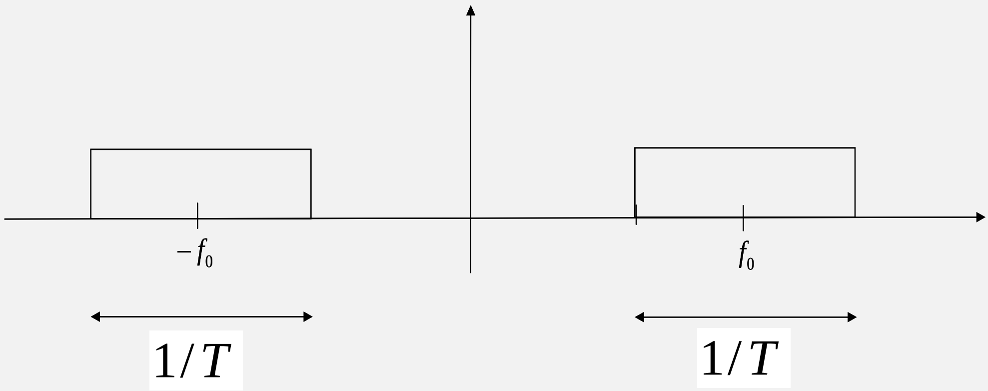
$$s(t) = \left[\sum_n \alpha[n] p(t - nT) \right] \cos(2\pi f_0 t) + \left[\sum_n \beta[n] p(t - nT) \right] \sin(2\pi f_0 t)$$

$$G_s(f) = z\left[\left|P(f - f_0)\right|^2 + \left|P(f + f_0)\right|^2 \right] \qquad z \in R$$

Each symbol $\alpha[n]$ and $\beta[n]$ has time duration $T = kT_b$

M-PSK: BANDWIDTH AND SPECTRAL EFFICIENCY

Case 1: $p(t)$ = ideal low pass filter



Total bandwidth (ideal case)

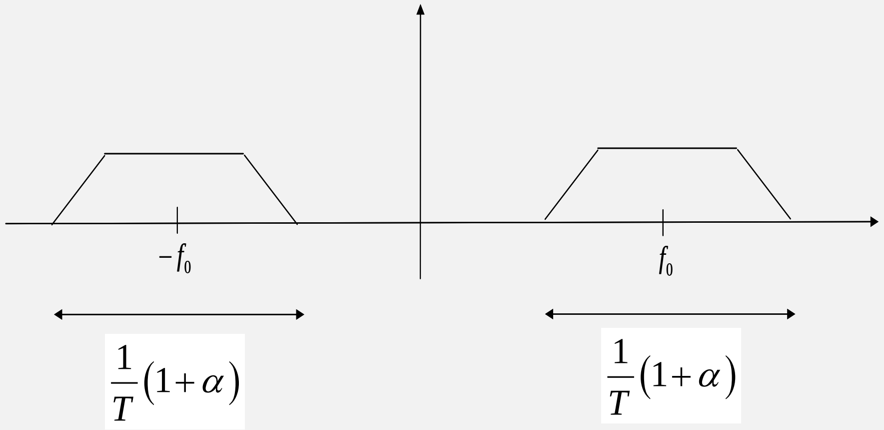
$$\boxed{B_{id} = R = \frac{R_b}{k}}$$

Spectral efficiency (ideal case)

$$\boxed{\eta_{id} = \frac{R_b}{B_{id}} = k\ bps\,/\,Hz}$$

M-PSK: BANDWIDTH AND SPECTRAL EFFICIENCY

Case 2: $p(t)$ = RRC filter with roll off α



Total bandwidth

$$B = R(1+\alpha) = \frac{R_b}{k}(1+\alpha)$$

Spectral efficiency

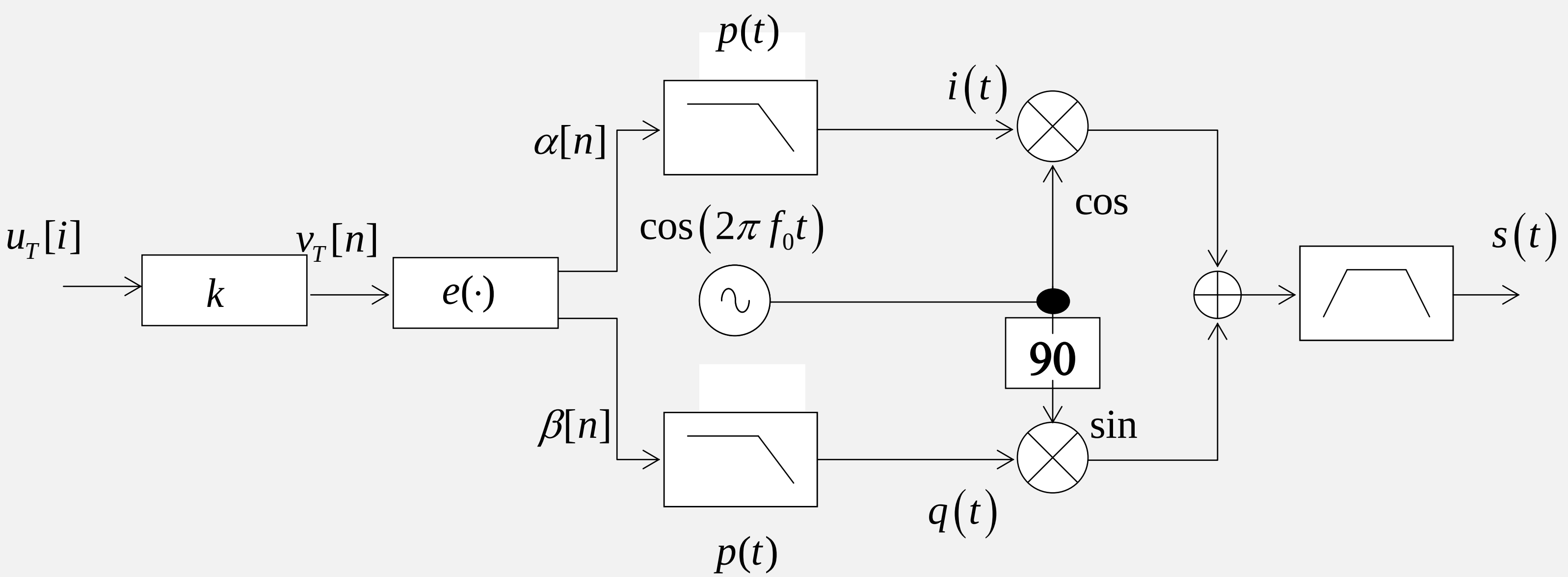
$$\boxed{\eta = \frac{R_b}{B} = \frac{k}{(1+\alpha)}\, bps/Hz}$$

EXERCIZE

Given a bandpass channel with bandwidth B = 4000 Hz, centred around f_0=2 GHz, compute the maximum bit rate R_b we can transmit over it with an 8-PSK constellation or a 16-PSK constellation in the two cases:

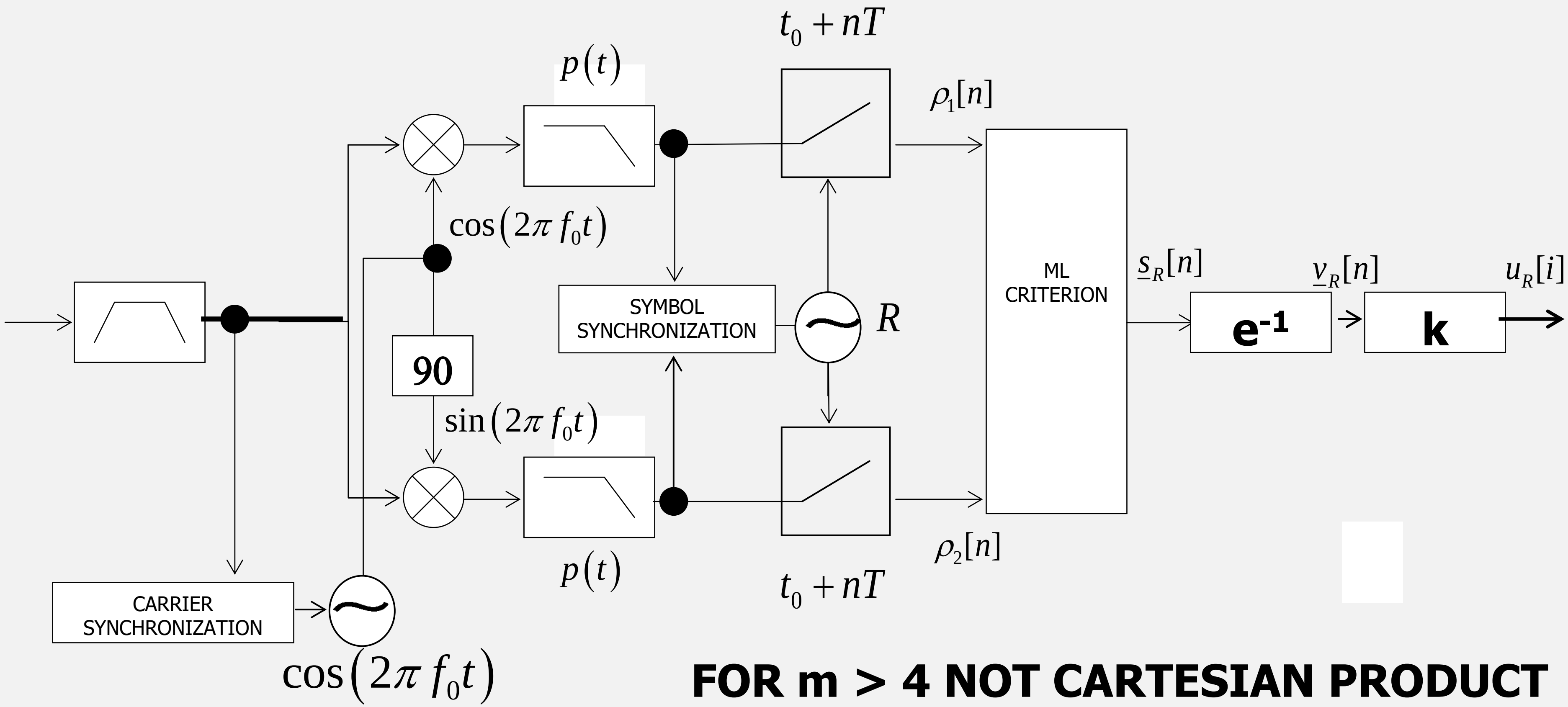
- Ideal low pass filter
- RRC filter with α=0.25

M-PSK: MODULATOR



FOR m > 4 NOT CARTESIAN PRODUCT

M-PSK: DEMODULATOR



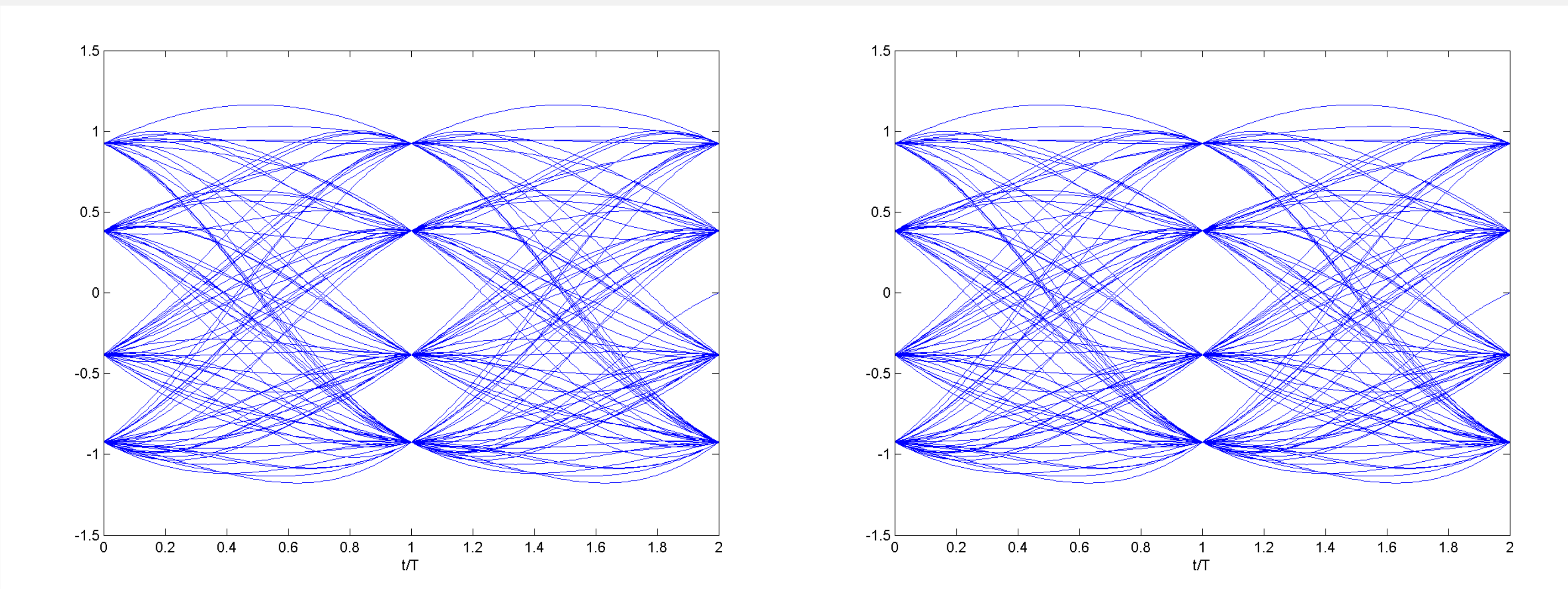
FOR m > 4 NOT CARTESIAN PRODUCT

Voronoi regions = plane sectors

M-PSK: EYE DIAGRAM

8-PSK constellation with RRC filter (α=0.5)

[α and β components = 0.924,0.383,-0.383,-0.924]



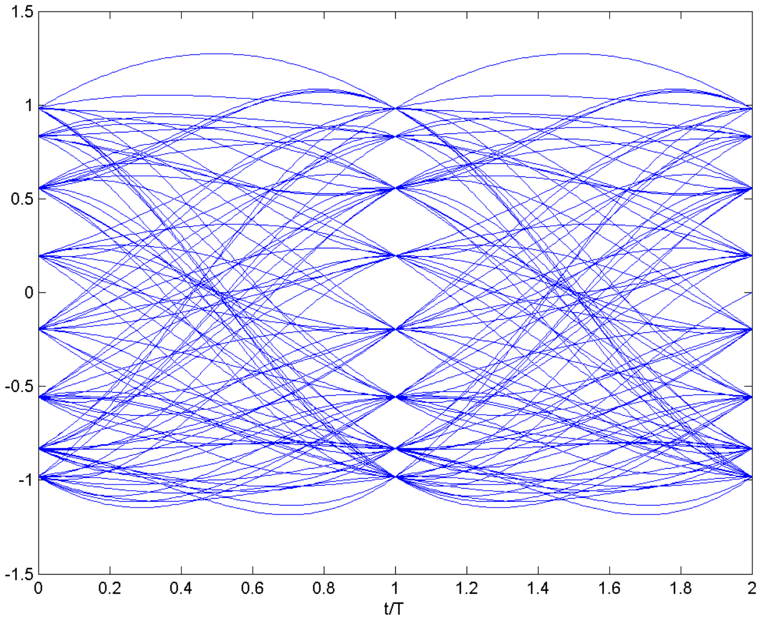
Channel I

Channel Q

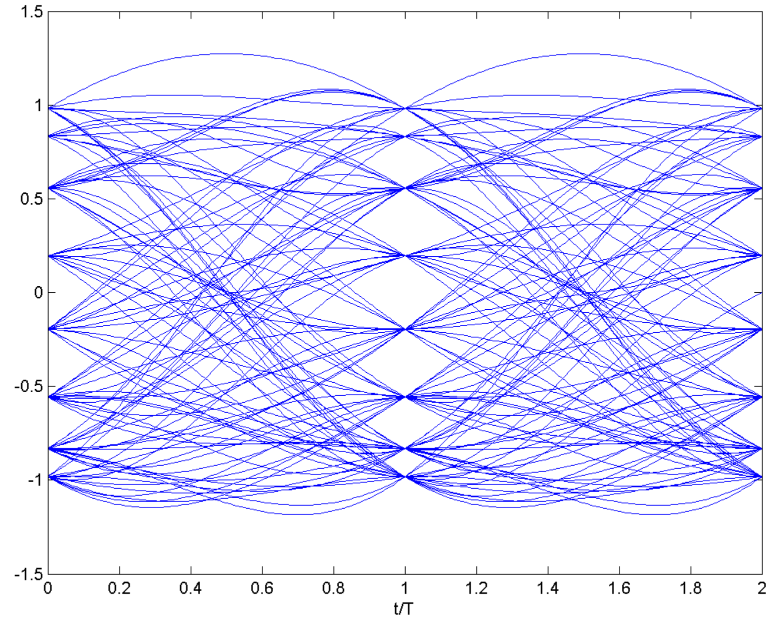
M-PSK: EYE DIAGRAM

16-PSK constellation with RRC filter (α=0.5)

[α and β components = 0.981,0.832,0.556,0.195,-0.195,-0.556,-0.832,-0.981]



Channel I



Channel Q

M-PSK CONSTELLATION: ERROR PROBABILITY

By applying the asymptotic approximation we can obtain

$$\boxed{P_b(e) \approx \frac{1}{k} erfc\left(\sqrt{k \frac{E_b}{N_0} \sin^2\left(\frac{\pi}{m}\right)} \right)}$$

The performance decrease for increasing *m*

(minimum distance decreases)

M-PSK CONSTELLATION: ERROR PROBABILITY

4-PSK: $P_b(e) \approx \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{N_0}}\right)$

8-PSK: $P_b(e) \approx \frac{1}{3} erfc\left(\sqrt{0.439\frac{E_b}{N_0}}\right)$ - 3.6 dB with respect to 4-PSK

16-PSK: $P_b(e) \approx \frac{1}{4} erfc\left(\sqrt{0.152\frac{E_b}{N_0}}\right)$ - 4.6 dB with respect to 8-PSK

No one uses m-PSK for m > 16: very poor BER performance

M-PSK CONSTELLATION: ERROR PROBABILITY