Nhập môn Kỹ thuật Truyền thông

Phần 2: Các kỹ thuật điều chế số
(Digital Modulations)

Bài 11: Không gian tín hiệu 4-PSK và m-PSK

Quadrature modulation

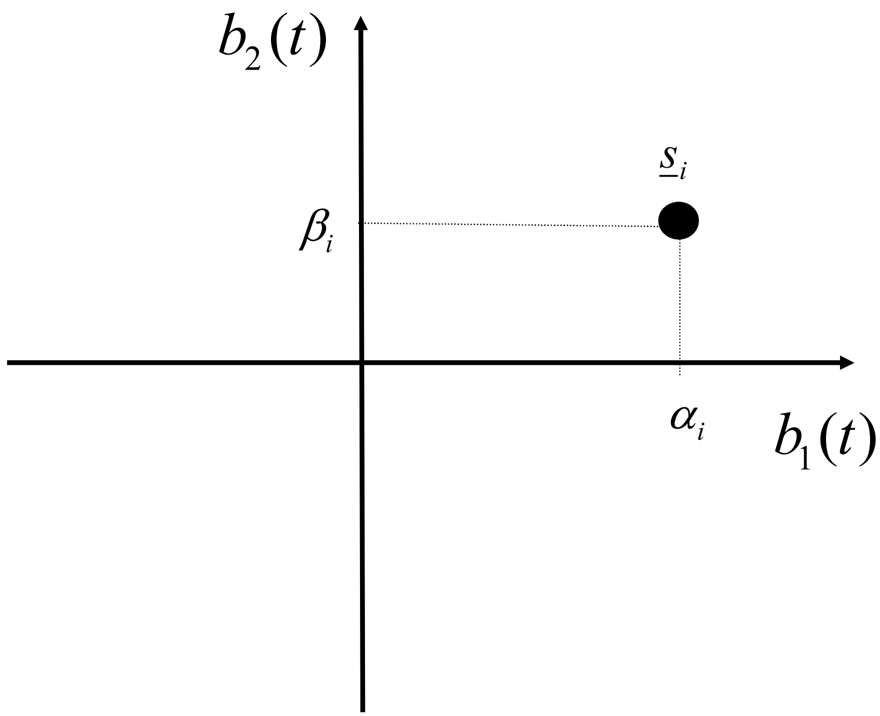
Consider a 2-D constellation, suppose that basis signals =cosine and sine

$$b_1(t) = p(t)\cos(2\pi f_0 t)$$

$$b_2(t) = p(t)\sin(2\pi f_0 t)$$

Each constellation symbol corresponds to a vector with two real components

$$M = \{\underline{s_i} = (\alpha_i, \beta_i)\}$$



Quadrature modulation

Binary information sequence $\underline{v}_T[n] \in H_k$ **DURATION T**

Symbol sequence $\underline{s}_T[n] \in M \subseteq R^2$ **DURATION T**

$\boxed{\alpha[n] \in R}$ $\boxed{\beta[n] \in R}$

DURATION T

Transmitted signal

$$s(t) = \sum_n \alpha[n] b_1(t - nT) + \sum_n \beta[n] b_2(t - nT) = a(t) + b(t)$$

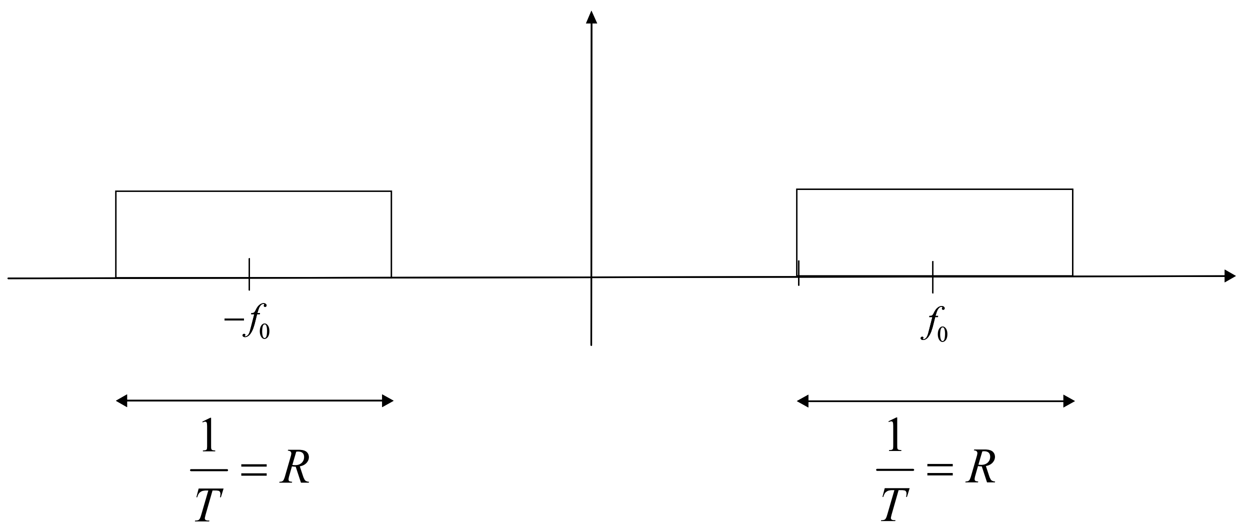
Quadrature modulation

Spectrum of $a(t)$:

$$a(t) = \sum_n \alpha[n] b_1(t - nT) = \left[\sum_n \alpha[n] p(t - nT) \right] \cos\left(2\pi f_0 t\right)$$

$$G_a = x\left[\left|P(f - f_0)\right|^2 + \left|P(f - f_0)\right|^2 \right] \qquad x \in R$$

when $p(t)$ = ideal low pass filter



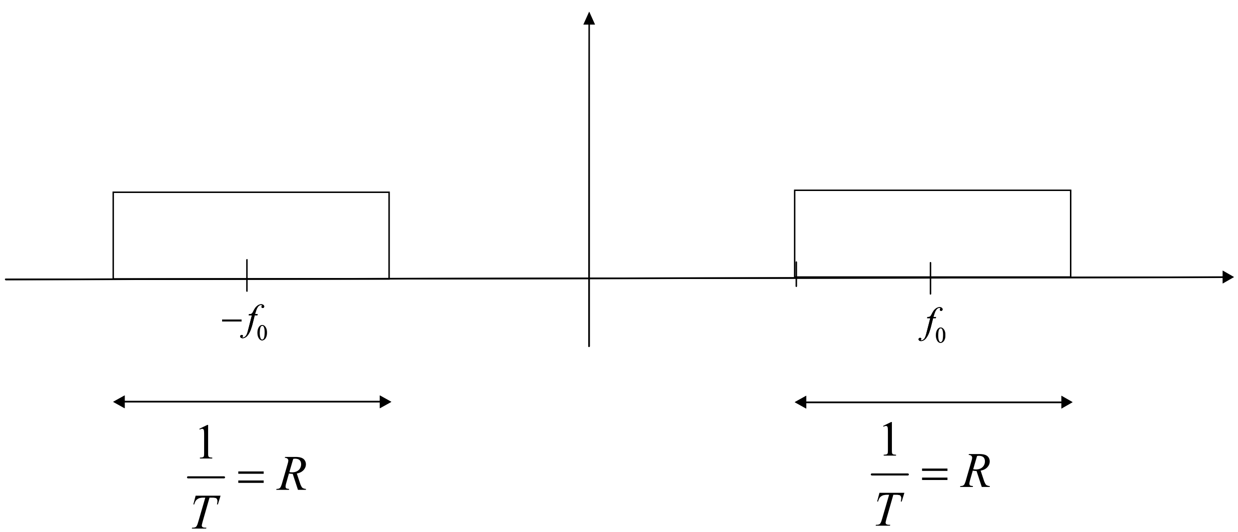
Quadrature modulation

Spectrum of $b(t)$:

$$b(t) = \sum_n \beta[n] b_1(t - nT) = \left[\sum_n \beta[n] p(t - nT) \right] \sin\left(2\pi f_0 t\right)$$

$$G_b = y\left[\left|P(f - f_0)\right|^2 + \left|P(f - f_0)\right|^2 \right] \qquad y \in R$$

when $p(t)$ = ideal low pass filter



Quadrature modulation

$$s(t) = a(t) + b(t)$$

It can be proved that

$$G_s(f) = G_a(f) + G_b(f)$$

Quadrature modulation

$$s(t) = a(t) + b(t)$$

$$G_s = G_a + G_b$$

$$G_a = x\left[\left|P(f - f_0)\right|^2 + \left|P(f - f_0)\right|^2\right] \qquad x \in R$$

$$G_b = y\left[\left|P(f - f_0)\right|^2 + \left|P(f - f_0)\right|^2\right] \qquad y \in R$$

$$G_s = z\left[\left|P(f - f_0)\right|^2 + \left|P(f - f_0)\right|^2\right] \qquad z \in R$$

G_a and G_b have the same shape and live on the same frequencies

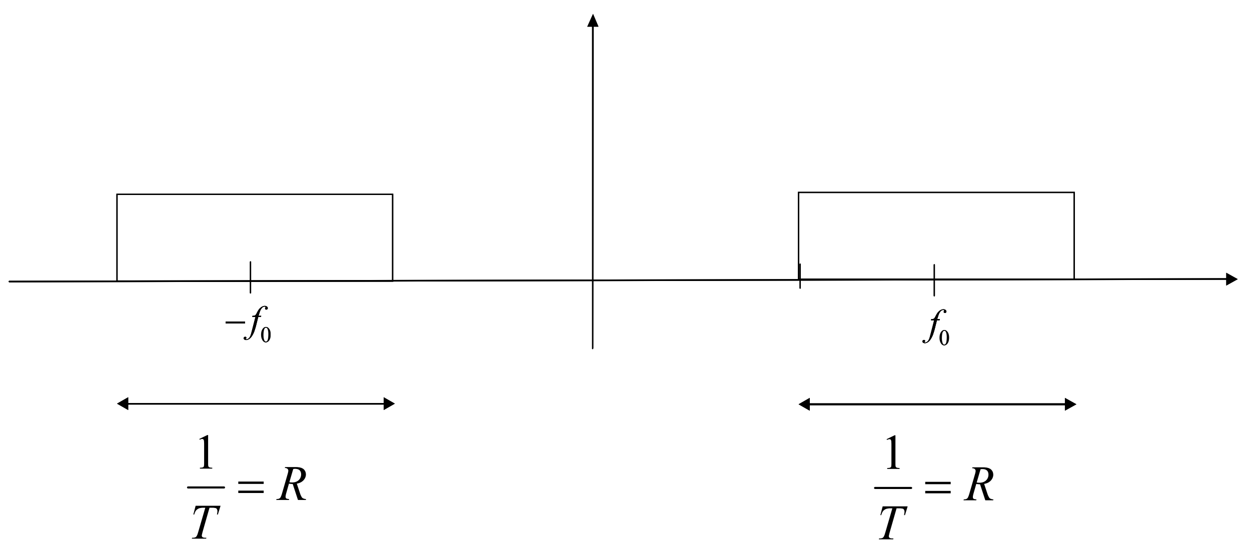
This is also the case for G_s

The spectrum of $s(t)$ only depends on $|P(f)|^2$

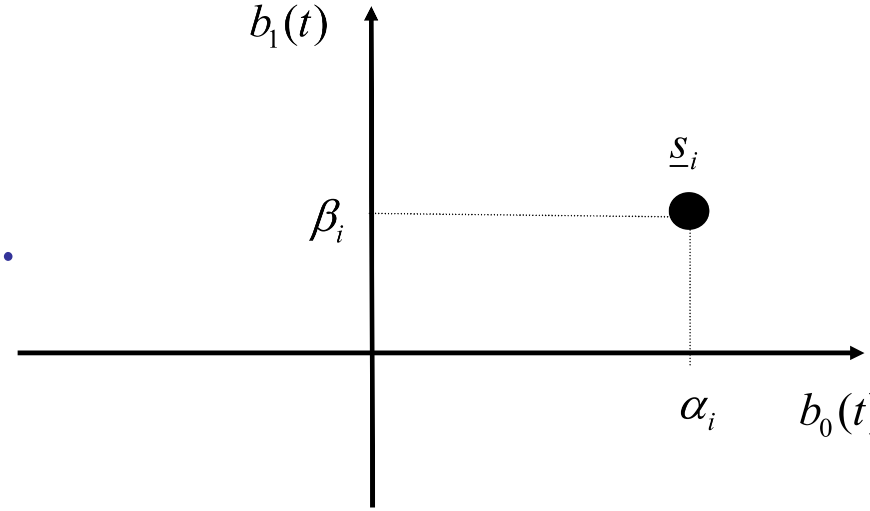
Quadrature modulation

Example when $p(t)$ = ideal low pass filter

$$G_s = z\left[\left|P(f-f_0)\right|^2 + \left|P(f-f_0)\right|^2\right] \qquad z \in R$$



I/Q component



Given a quadrature modulation,
let us consider its transmitted waveform

$$s(t) = a(t) + b(t) =$$

$$= \underbrace{\left[\sum_n \alpha[n] p(t - nT)\right]}_{i(t)} \cos\left(2\pi f_0 t\right) + \underbrace{\left[\sum_n \beta[n] p(t - nT)\right]}_{q(t)} \sin\left(2\pi f_0 t\right)$$

I component (in phase)

Q component (in quadrature)

Complex envelope

$$s(t) = [i(t)]\cos(2\pi f_0 t) + [q(t)]\sin(2\pi f_0 t)$$

Complex envelope $\boxed{\tilde{s}(t) = i(t) - jq(t)}$

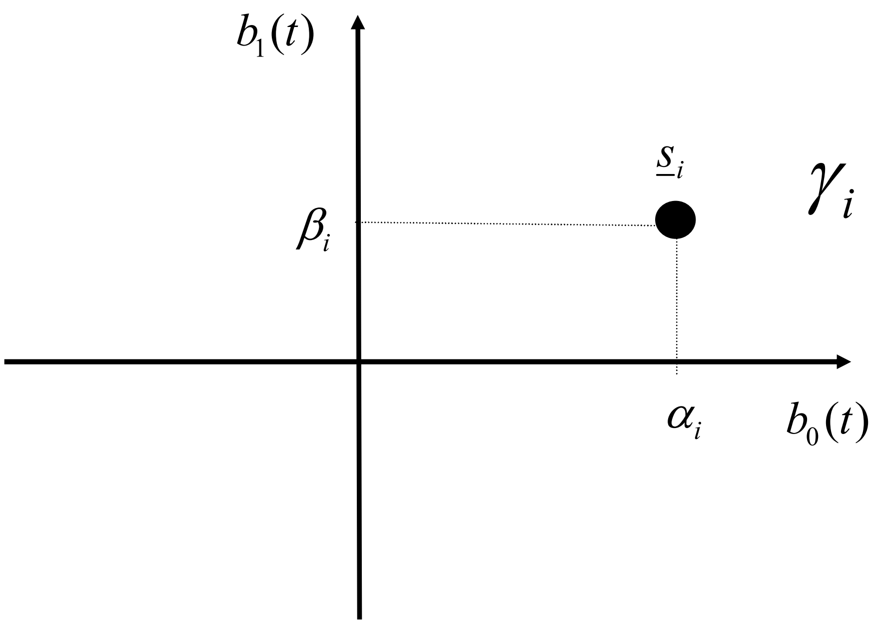
$$i(t) = \sum_n \alpha[n] p(t - nT) \qquad q(t) = \sum_n \beta[n] p(t - nT)$$

Complex symbol $\boxed{\gamma[n] = \alpha[n] - j\beta[n]}$

$$\tilde{s}(t) = \sum_n \gamma[n] p(t - nT)$$

Complex envelope

$$\tilde{s}(t) = \sum_n \gamma[n] p(t - nT) \qquad \gamma[n] = \alpha[n] - j\beta[n]$$



$$\gamma_i = \alpha_i - j\beta_i$$

Quadrature constellation as a set of complex numbers

$$M = \{\gamma_i = \alpha_i - j\beta_i\}_{i=1}^{m}$$

Analytic signal

$$s(t) = \left[i(t)\right]\cos\left(2\pi f_0 t\right) + \left[q(t)\right]\sin\left(2\pi f_0 t\right)$$

$$\tilde{s}(t) = i(t) - jq(t)$$

$$s(t) = \mathrm{Re}\left[\tilde{s}(t)e^{j2\pi f_0 t}\right] = \mathrm{Re}\left[\dot{s}(t)\right]$$

Analytic signal

$$\boxed{\dot{s}(t) = \tilde{s}(t)e^{j2\pi f_0 t}}$$

$$\dot{s}(t) = \tilde{s}(t)e^{j2\pi f_0 t} = \left[\sum_n \gamma[n]p(t-nT)\right]e^{j2\pi f_0 t}$$

***4-PSK**: characteristics*

1. Band-pass modulation
2. 2D signal set
3. Basis signals $p(t)cos(2\pi f_0 t)$ and $p(t)sin(2\pi f_0 t)$
4. Costellation = 4 signals, equidistant on a circle
5. Information associated to the carrier phase

***4-PSK**: constellation*

SIGNAL SET

$$\boxed{\begin{aligned} M = \{ & s_1(t) = Ap(t)\cos(2\pi f_0 t)\,, s_2(t) = Ap(t)\sin(2\pi f_0 t) \\ & s_3(t) = -Ap(t)\cos(2\pi f_0 t)\,, s_4(t) = -Ap(t)\sin(2\pi f_0 t)\,\} \end{aligned}}$$

If we write

$$M = \left\{ \begin{aligned} & s_1(t) = Ap(t)\cos(2\pi f_0 t)\,, \\ & s_2(t) = Ap(t)\sin(2\pi f_0 t) = Ap(t)\cos\left(2\pi f_0 t - \frac{\pi}{2}\right), \\ & s_3(t) = -Ap(t)\cos(2\pi f_0 t) = Ap(t)\cos\left(2\pi f_0 t - \pi\right), \\ & s_4(t) = -Ap(t)\sin(2\pi f_0 t) = Ap(t)\cos\left(2\pi f_0 t - \frac{3\pi}{2}\right) \end{aligned} \right\}$$

Information associated to the carrier phase

4-PSK: *constellation*

SIGNAL SET

$$\boxed{\begin{array}{l} M = \{s_i(t) = Ap(t)\cos(2\pi f_0 t - \varphi_i)\}_{i=1}^{4} \\ \varphi_i = (i-1)\dfrac{\pi}{2} \end{array}}$$

Versors

$$b_1(t) = p(t)\cos(2\pi f_0 t)$$

$$b_2(t) = p(t)\sin(2\pi f_0 t)$$

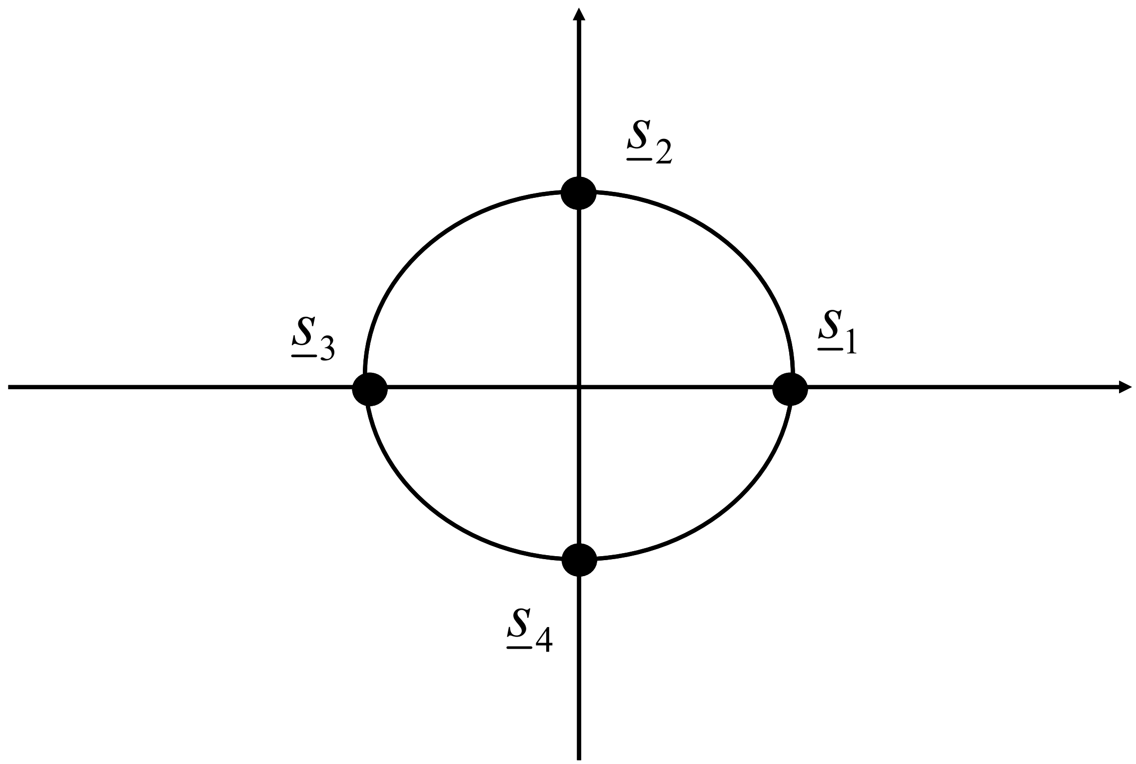
VECTOR SET

$$\boxed{M = \{\underline{s}_1 = (A,0), \underline{s}_2 = (0,A), \underline{s}_3 = (-A,0), \underline{s}_4 = (0,-A)\} \subseteq R^2}$$

***4-PSK**: constellation*

VECTOR SET

$$M = \{\underline{s}_1 = (A,0), \underline{s}_2 = (0,A), \underline{s}_3 = (-A,0), \underline{s}_4 = (0,-A)\} \subseteq R^2$$



4-PSK: constellation

SIGNAL SET (with arbitary starting phase)

$$\boxed{\begin{aligned} &M = \{ s_i(t) = Ap(t)\cos(2\pi f_0 t - \varphi_i) \}_{i=1}^{4} \\ &\varphi_i = \Phi + (i-1)\frac{\pi}{2} \end{aligned}}$$

$$\boxed{s_i(t) = (A\cos\varphi_i)\, p(t)\cos(2\pi f_0 t) + (A\sin\varphi_i)\, p(t)\sin(2\pi f_0 t)}$$

Versors

$$\boxed{b_1(t) = p(t)\cos(2\pi f_0 t)}$$

$$\boxed{b_2(t) = p(t)\sin(2\pi f_0 t)}$$

Vector set

$$\boxed{\begin{aligned} &M = \{ \underline{s}_i = (\alpha_i, \beta_i) \}_{i=1}^{4} \subseteq R^2 \\ &\alpha_i = A\cos\varphi_i \\ &\beta_i = A\sin\varphi_i \\ &\varphi_i = \Phi + (i-1)\frac{\pi}{2} \end{aligned}}$$

***4-PSK**: constellation*

Example: $\Phi = 0$

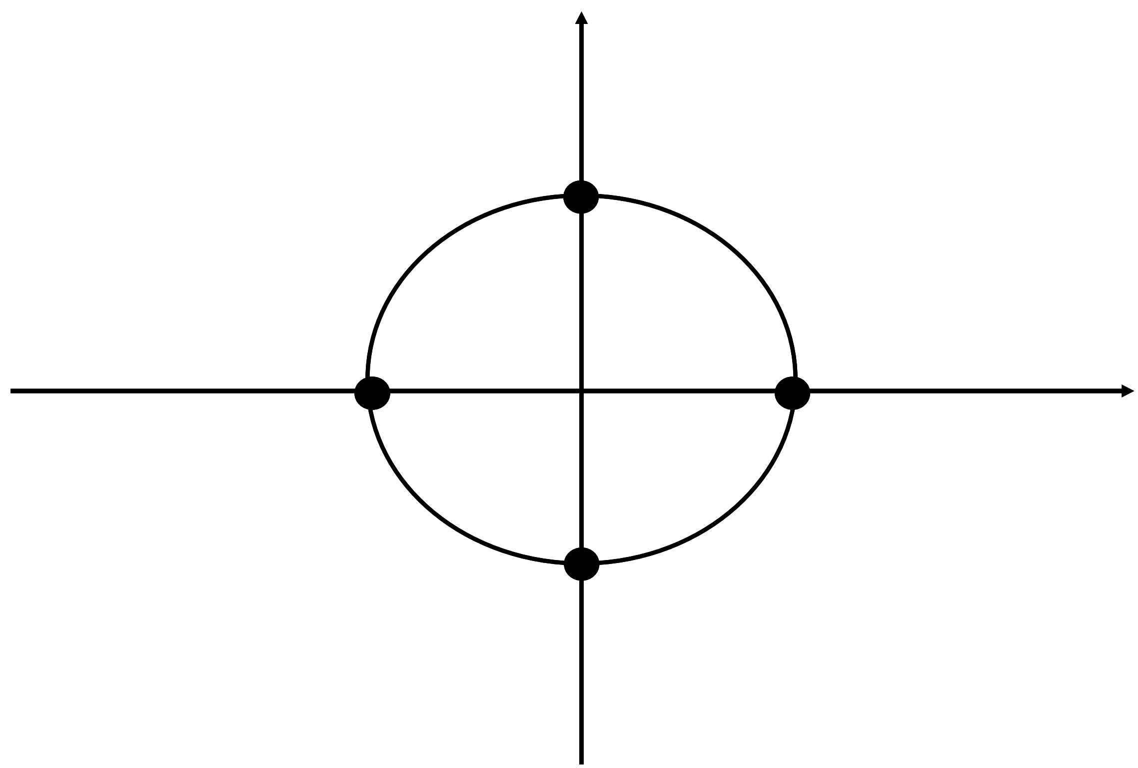
$$\boxed{M = \{\underline{s}_1 = (A,0)\,,\, \underline{s}_2 = (0,A)\,,\, \underline{s}_3 = (-A,0)\,,\, \underline{s}_4 = (0,-A)\} \subseteq R^2}$$

$$M = \{\underline{s}_i = (\alpha_i, \beta_i)\}_{i=1}^{4} \subseteq R^2$$

$$\alpha_i = A\cos\varphi_i$$

$$\beta_i = A\sin\varphi_i$$

$$\varphi_i = (i-1)\frac{\pi}{2} \in \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}$$



4-PSK: constellation

Example: $\Phi = \dfrac{\pi}{4}$

$$\boxed{M = \{\underline{s}_1 = (-\alpha, -\alpha)\,,\, \underline{s}_2 = (+\alpha, -\alpha)\,,\, \underline{s}_3 = (+\alpha, +\alpha)\,,\, \underline{s}_4 = (-\alpha, +\alpha)\} \subseteq R^2}$$

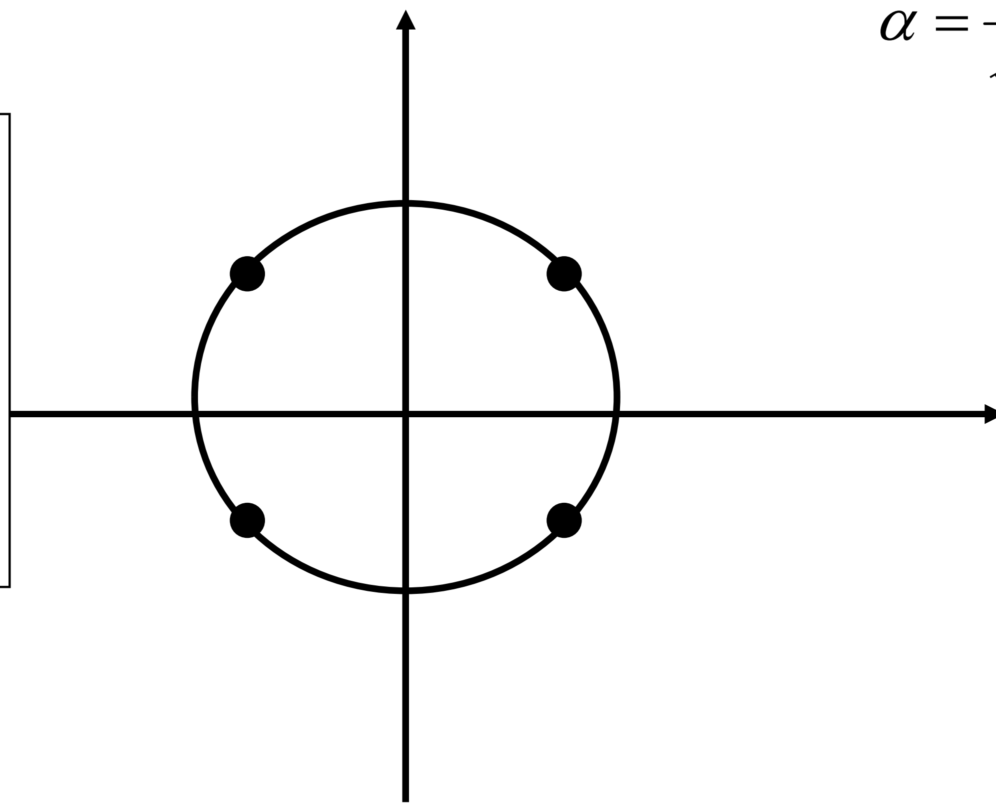
$$\alpha = \frac{A}{\sqrt{2}}$$

$$M = \{\underline{s}_i = (\alpha_i, \beta_i)\}_{i=1}^{4} \subseteq R^2$$

$$\alpha_i = A\cos\varphi_i$$

$$\beta_i = A\sin\varphi_i$$

$$\varphi_i = \frac{\pi}{4} + (i-1)\frac{\pi}{2} \in \left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$$



***4-PSK**: binary labeling*

Example of Gray labeling

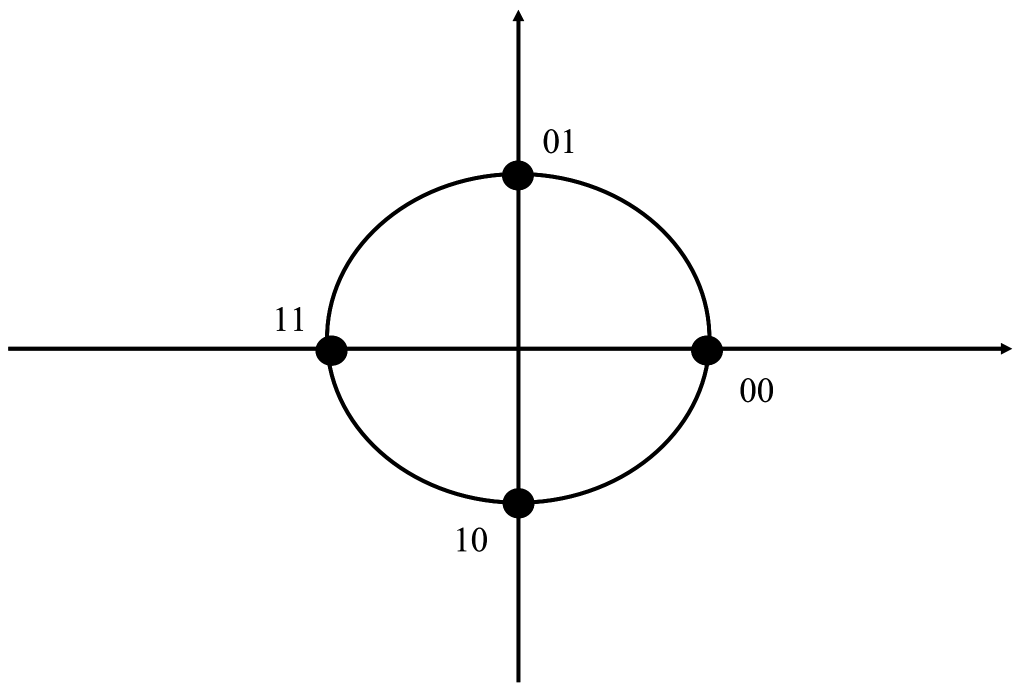
$$e : H_2 \leftrightarrow M$$

$$e(00) = \underline{s}_0$$

$$e(01) = \underline{s}_1$$

$$e(11) = \underline{s}_2$$

$$e(10) = \underline{s}_3$$



4-PSK: *transmitted waveform*

$m = 2 \rightarrow k = 2$

$\boxed{T = 2T_b}$

$\boxed{R = \frac{R_b}{2}}$

Each symbol has duration T
Each symbol component (α and β) lasts for T second

Transmitted waveform

$$s(t) = \underbrace{\left[\sum_n \alpha[n] p(t - nT) \right]}_{i(t)} \cos(2\pi f_0 t) + \underbrace{\left[\sum_n \beta[n] p(t - nT) \right]}_{q(t)} \sin(2\pi f_0 t)$$

I component (in phase)

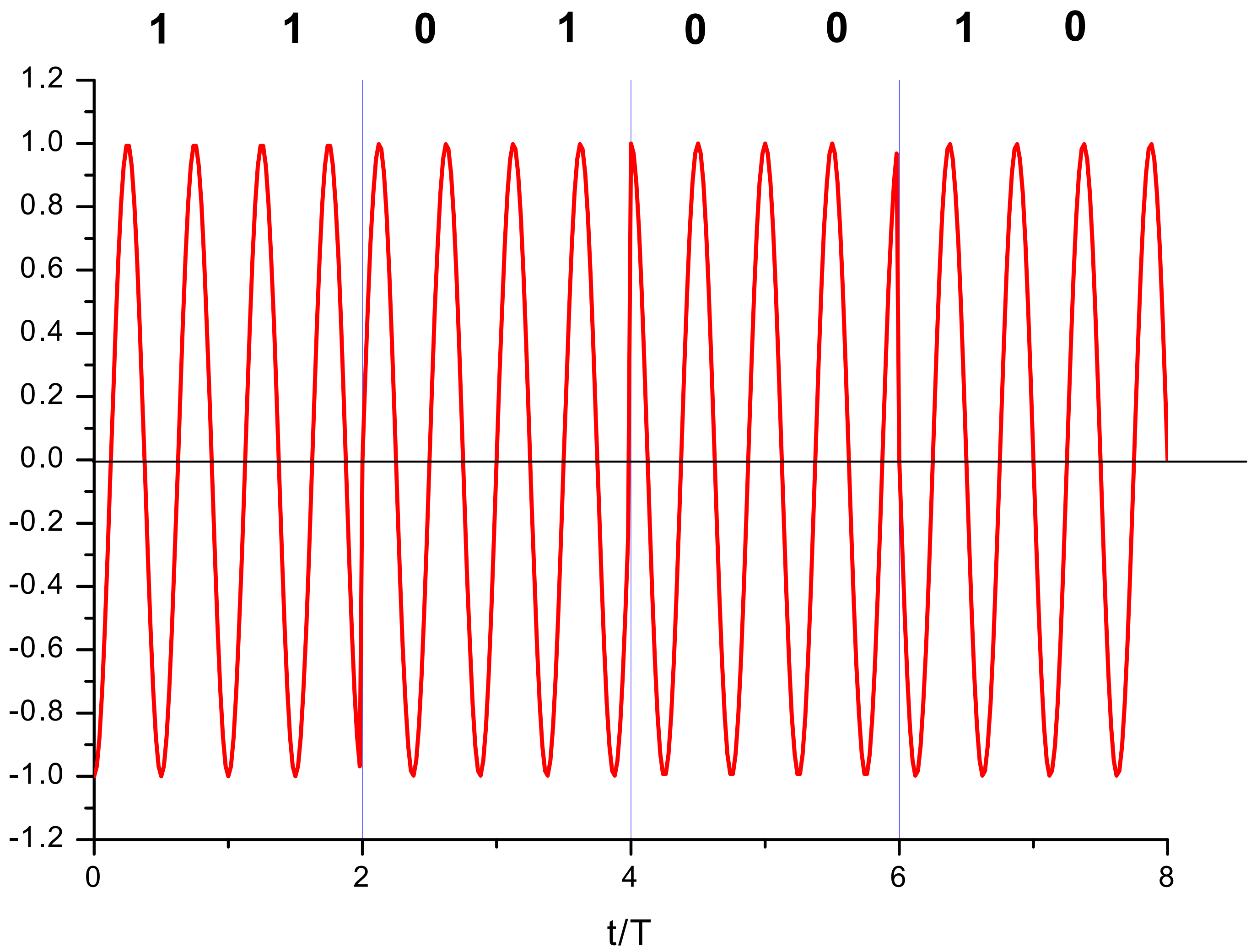
Q component (in quadrature)

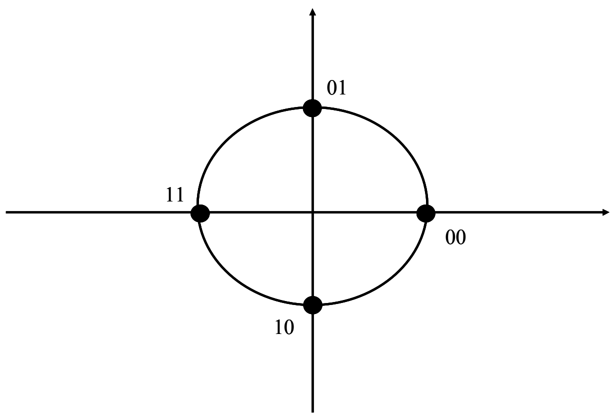
4-PSK*: transmitted waveform*

example for $p(t) = \frac{1}{\sqrt{T}} P_T(t)$

$f_0 = 2R_b$

$\alpha = \sqrt{T}$





4-PSK: *analytic signal*

$$s(t) = \underbrace{\left[\sum_n \alpha[n]p(t-nT)\right]}_{i(t)}\cos(2\pi f_0 t) + \underbrace{\left[\sum_n \beta[n]p(t-nT)\right]}_{q(t)}\sin(2\pi f_0 t)$$

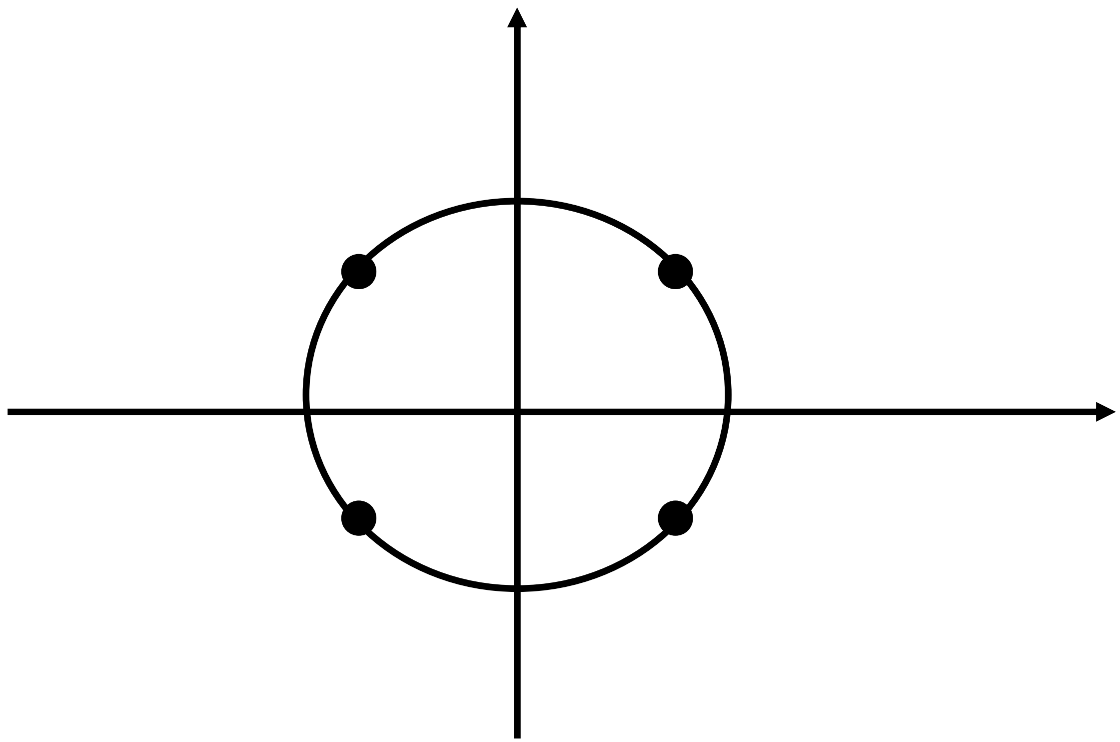
$$s(t) = \mathrm{Re}\left[\dot{s}(t)\right] = \mathrm{Re}\left[\tilde{s}(t)e^{j2\pi f_0 t}\right]$$

$$\tilde{s}(t) = i(t) - jq(t) = \sum_n \gamma[n]p(t-nT) \qquad \gamma[n] = \alpha[n] - j\beta[n]$$

4-PSK: analytic signal

$$\tilde{s}(t) = \sum_n \gamma[n] p(t - nT)$$

$$\gamma[n] = \alpha[n] - j\beta[n]$$



$$M = \{s_1 = (a - ja), s_2 = (-a - ja), s_3 = (-a + ja), s_4 = (a + ja),\}$$

***4-PSK**: bandwidth and spectral efficiency*

Transmitted waveform

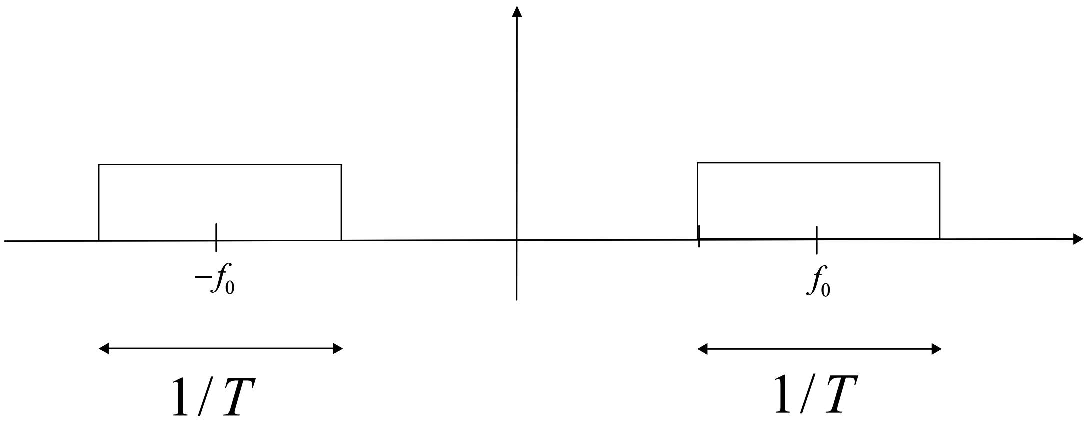
$$s(t) = \left[\sum_n \alpha[n] p(t - nT) \right] \cos(2\pi f_0 t) + \left[\sum_n \beta[n] p(t - nT) \right] \sin(2\pi f_0 t)$$

$$G_s(f) = z\left[\left| P(f - f_0) \right|^2 + \left| P(f + f_0) \right|^2 \right] \qquad z \in R$$

Each symbol $\alpha[n]$ and $\beta[n]$ has time duration $T = 2T_b$

***4-PSK**: bandwidth and spectral efficiency*

Case 1: $p(t)$ = ideal low pass filter



Total bandwidth
(ideal case)

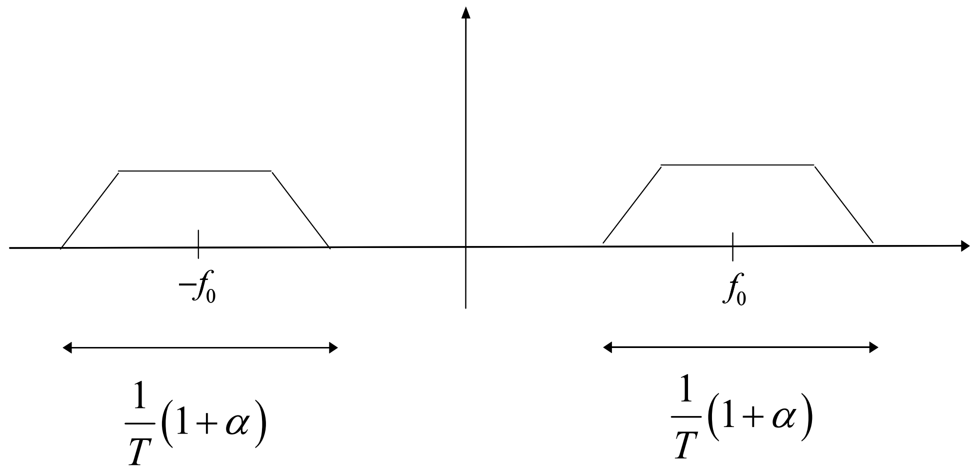
$$\boxed{B_{id} = R = \frac{R_b}{2}}$$

Spectral efficiency
(ideal case)

$$\boxed{\eta_{id} = \frac{R_b}{B_{id}} = 2\ bps/Hz}$$

***4-PSK**: bandwidth and spectral efficiency*

Case 2: $p(t)$ = RRC filter with roll off α



Total bandwidth $$B = R(1+\alpha) = \frac{R_b}{2}(1+\alpha)$$

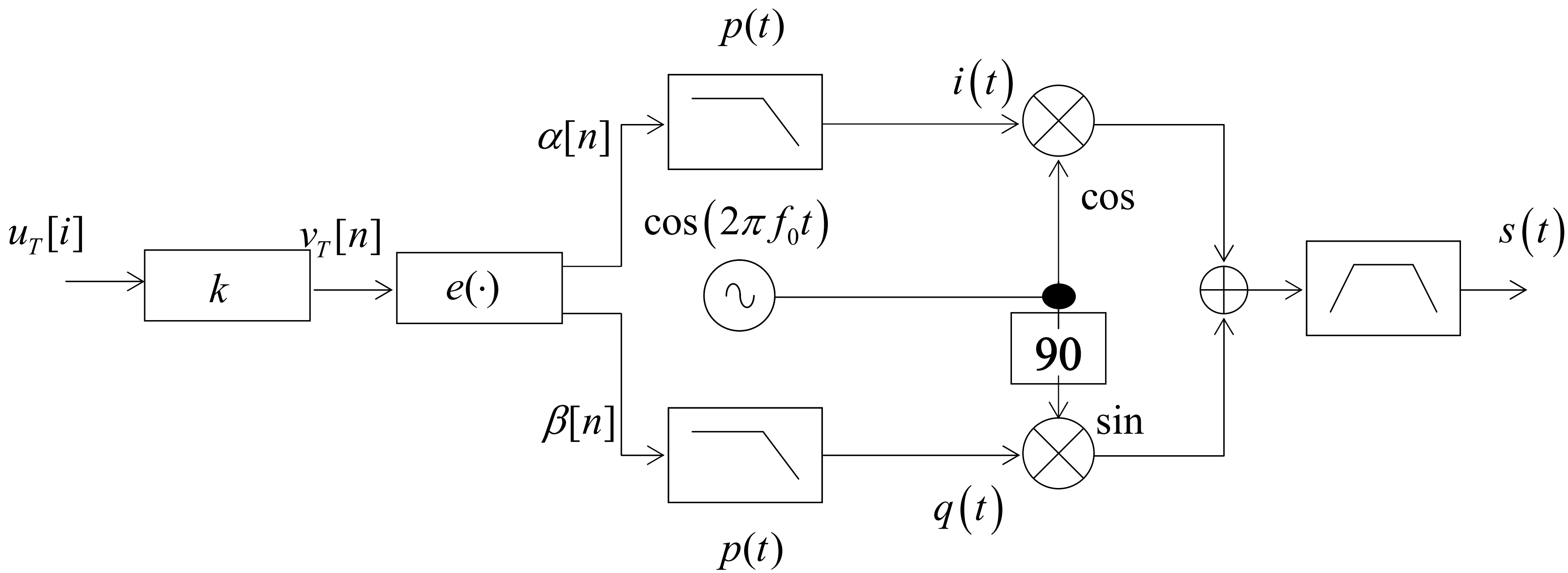
Spectral efficiency $$\boxed{\eta = \frac{R_b}{B} = \frac{2}{(1+\alpha)} \, bps/Hz}$$

Exercize

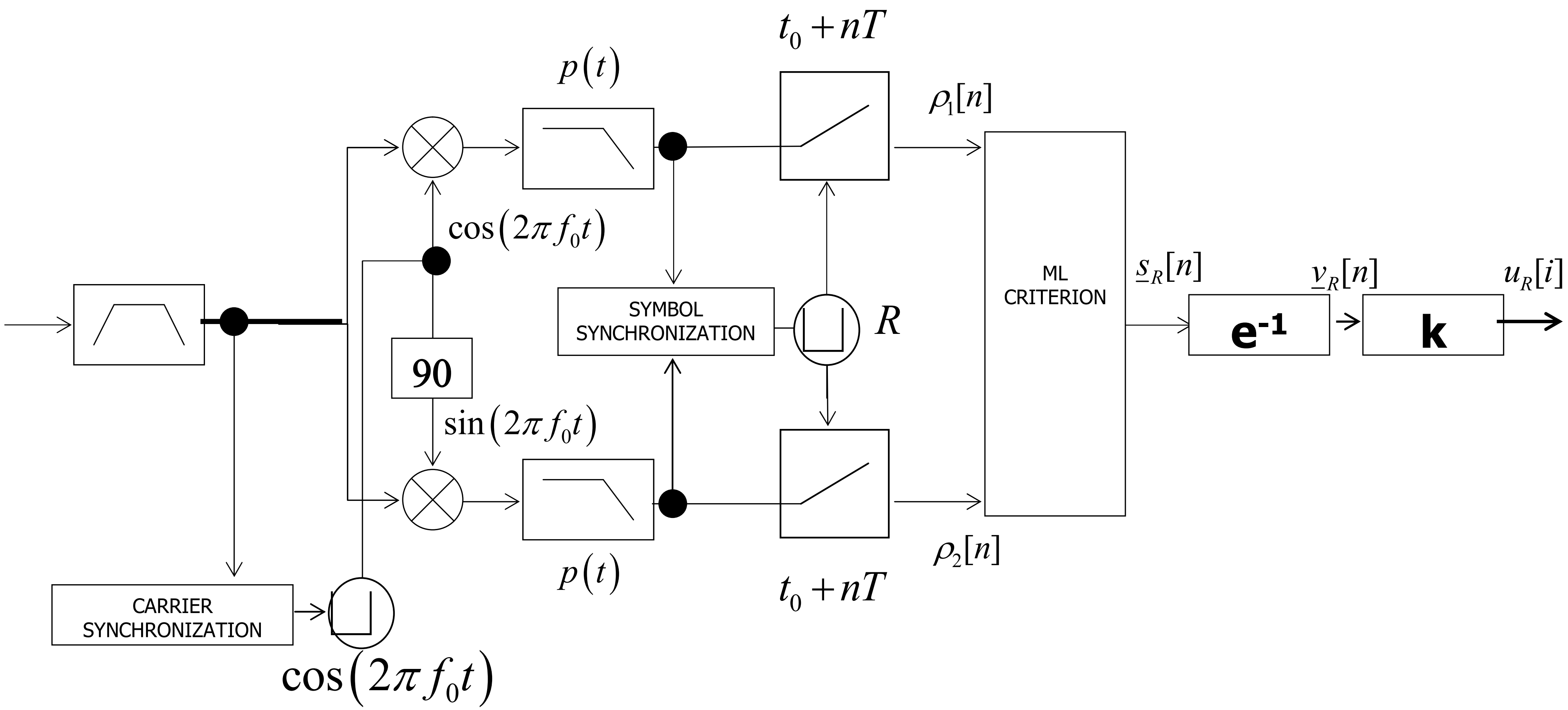
Given a bandpass channel with bandwidth B = 4000 Hz, centred around f_0=2 GHz, compute the maximum bit rate R_b we can transmit over it with a 4-PSK constellation in the two cases:

- Ideal low pass filter
- RRC filter with α=0.25

4-PSK: *modulator*

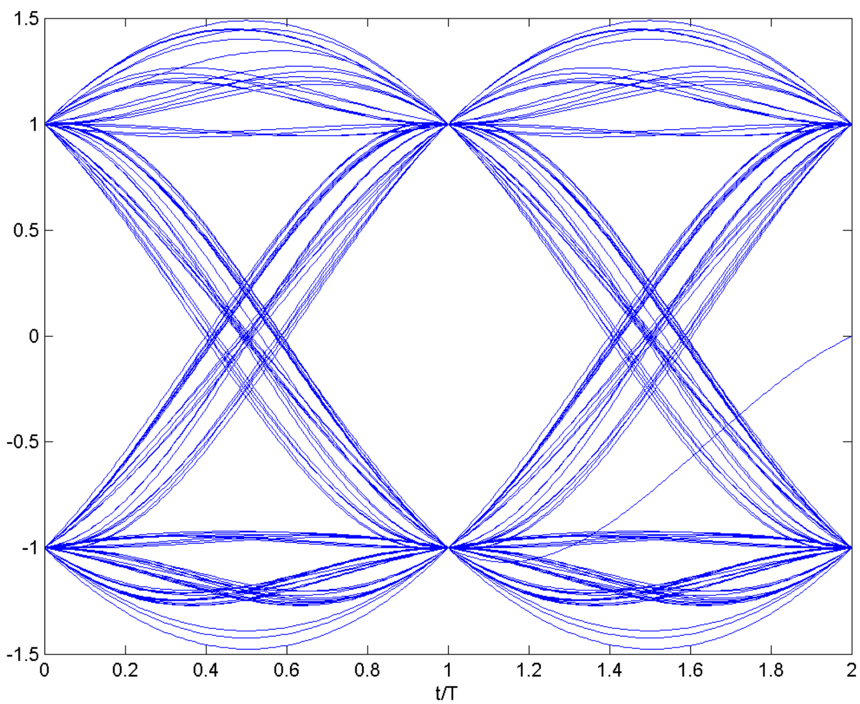


***4-PSK**: demodulator*

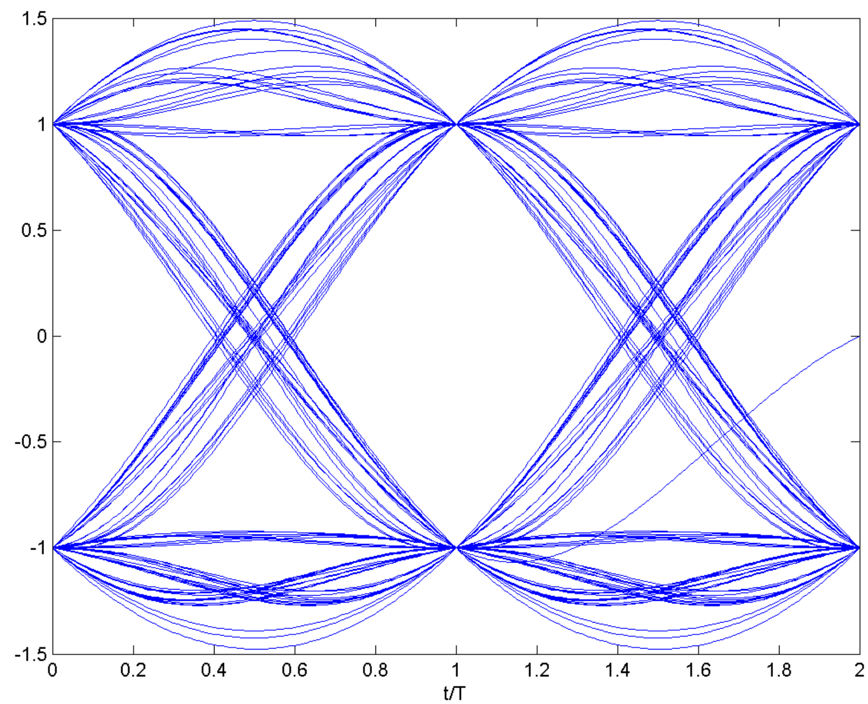


***4-PSK**: Eye diagram*

4-PSK constellation with RRC filter (α=0.5)



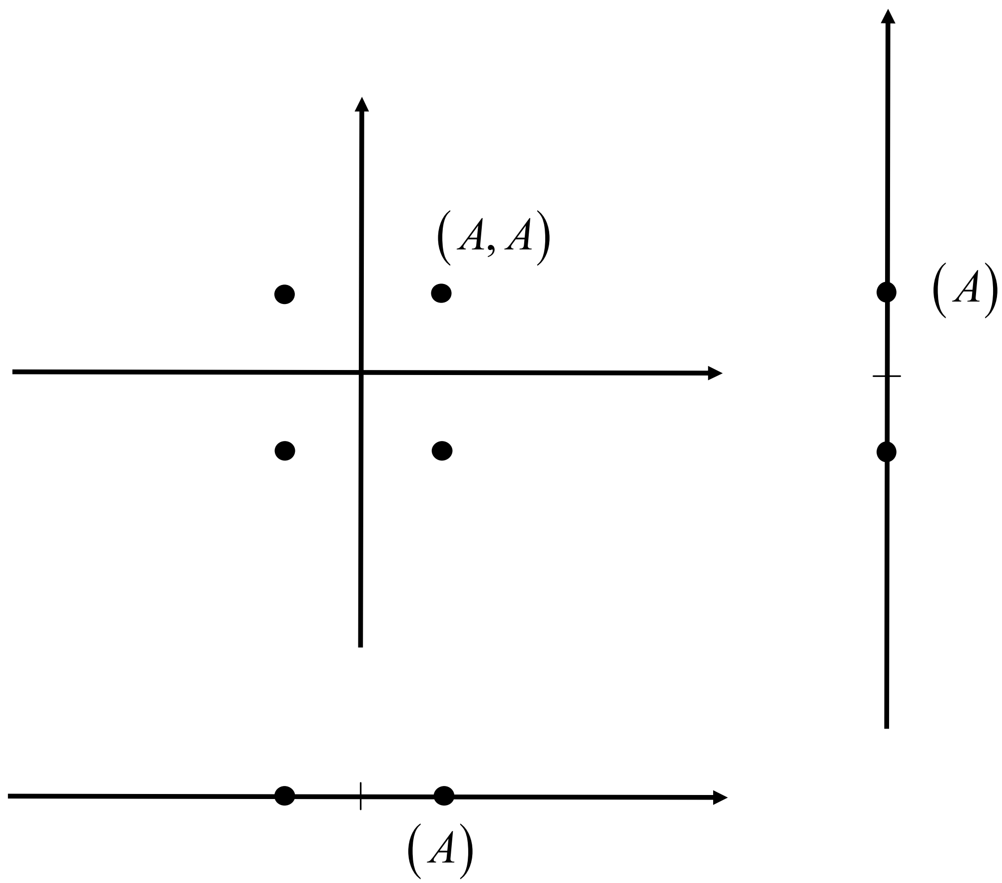
Canale I



Canale Q

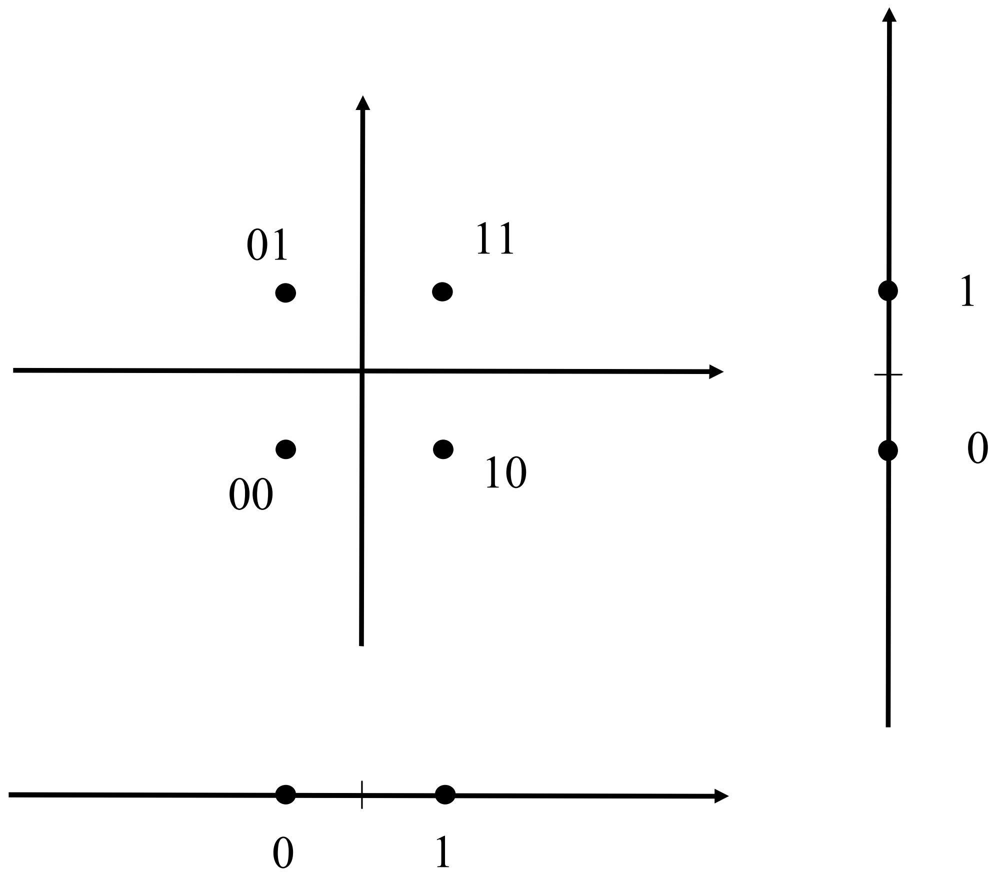
4-PSK: *intepretation*

The 4-PSK vector set can be viewed as
the Cartesian product of two 2-PSK constellations



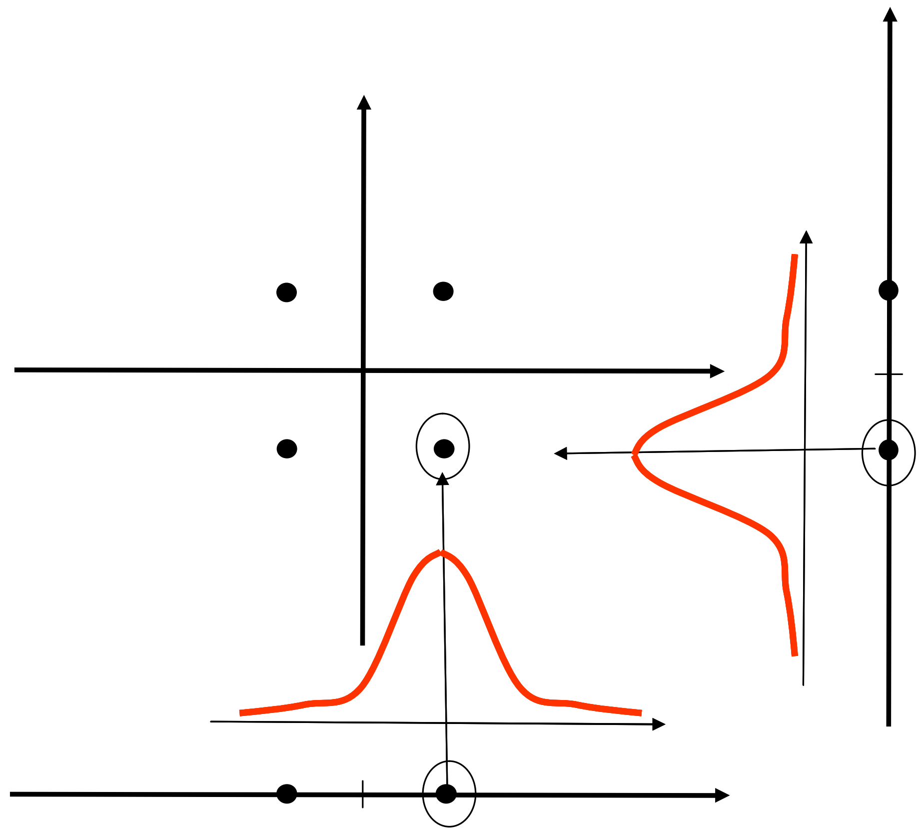
***4-PSK**: intepretation*

This is also true for the binary Gray labeling
(first bit = I component, second bit = Q component)



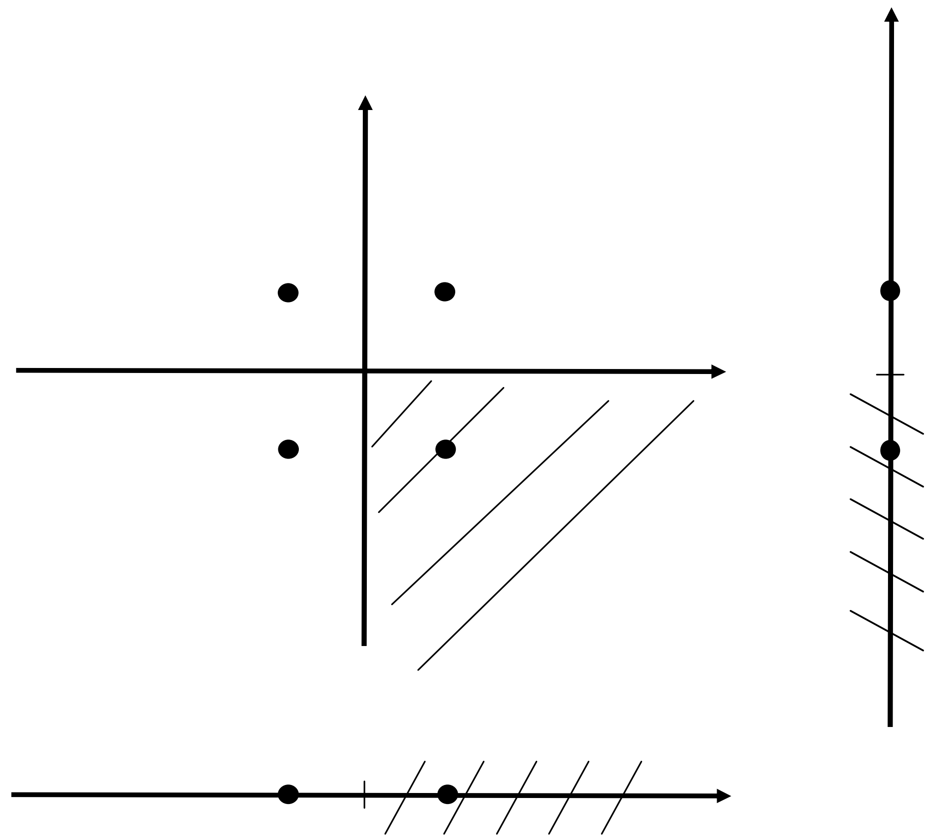
***4-PSK**: intepretation*

The AWGN channel adds two Gaussian components which are statistically independent



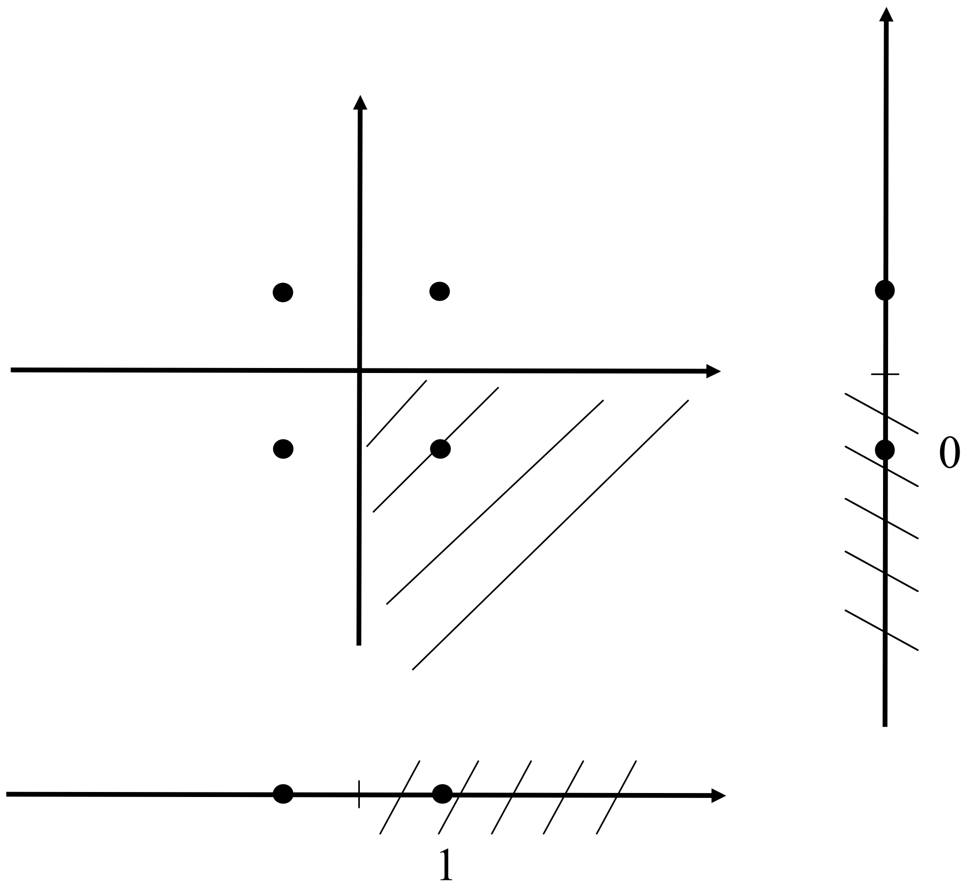
4-PSK: intepretation

The Voronoi regions of 4-PSK signals are the Cartesian product of the Voronoi regions of the constituent 2-PSK constellations



***4-PSK**: intepretation*

The Voronoi regions of 4-PSK signals are the Cartesian product of the Voronoi regions of the constituent 2-PSK constellations



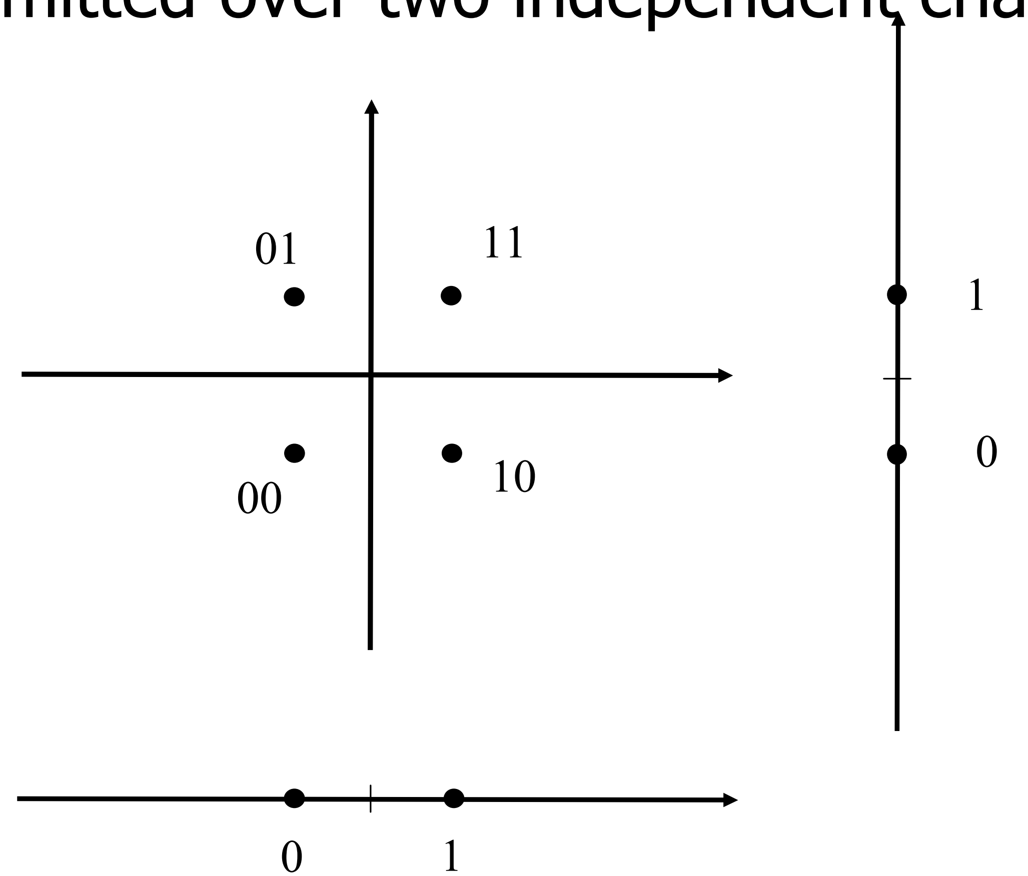
Given the received vector $(\rho_1[n]\ ,\ \rho_2[n])$

The sign of the first component $\rho_1[n]$ determines the first received bit

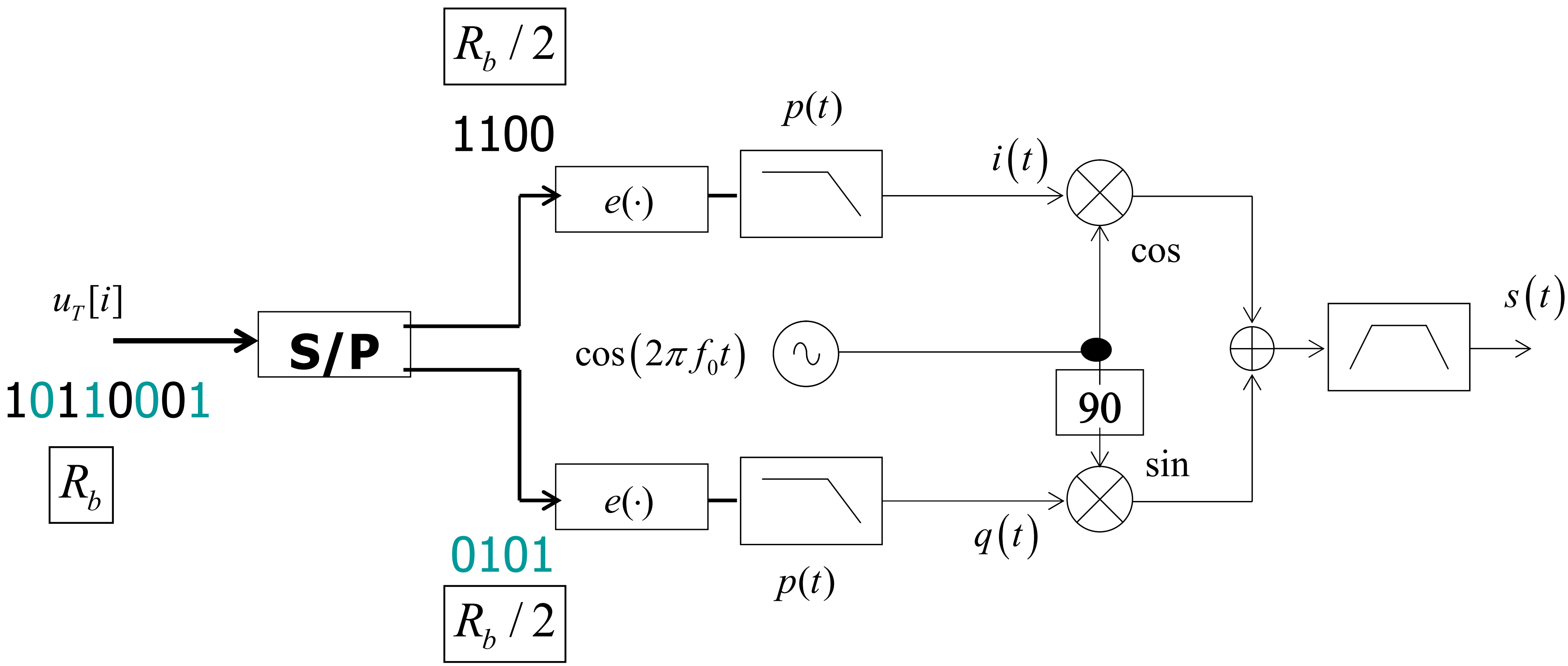
The sign of the second component $\rho_2[n]$ determines the second received bit

4-PSK: *intepretation*

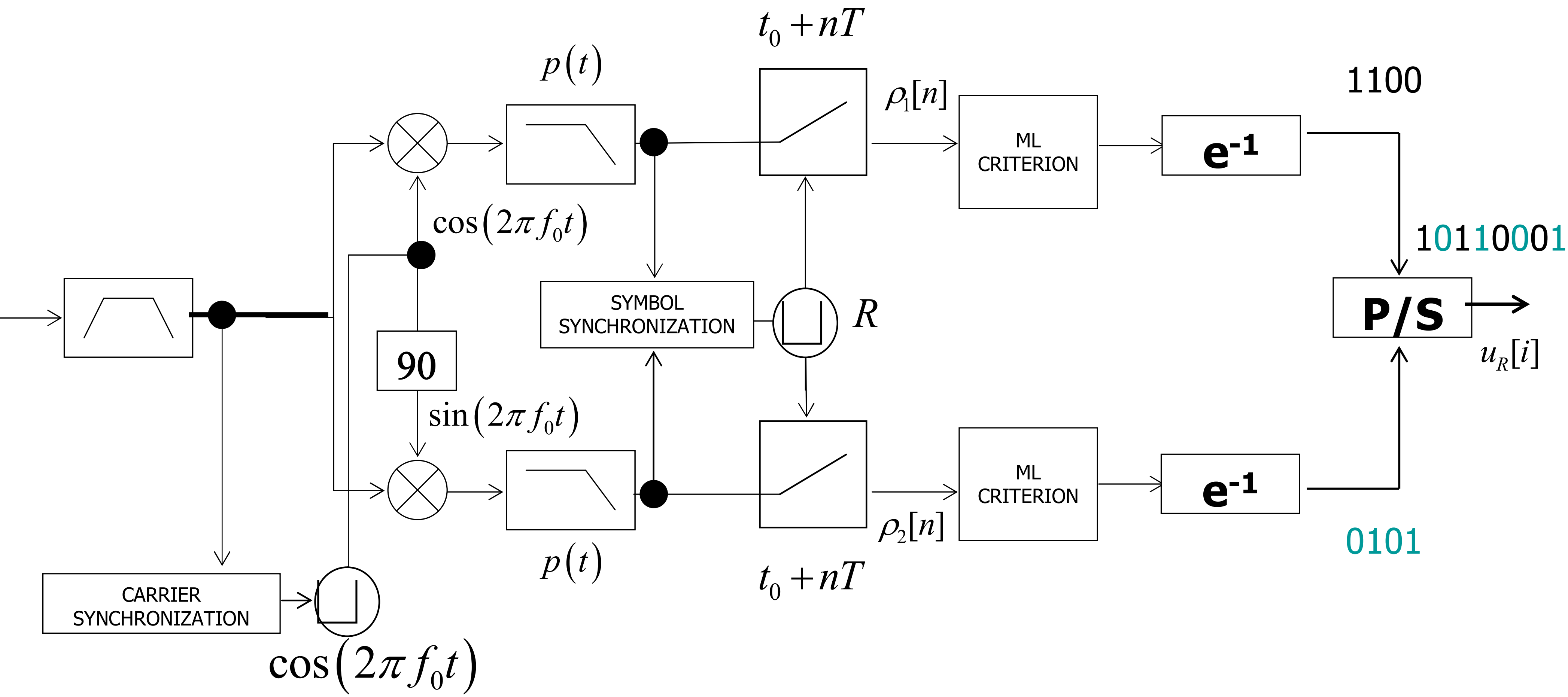
The 4-PSK modulation can be viewed as
the Cartesian product of two 2-PSK constellations
transmitted over two independent channels



***4-PSK**: modulator*



4-PSK: demodulator

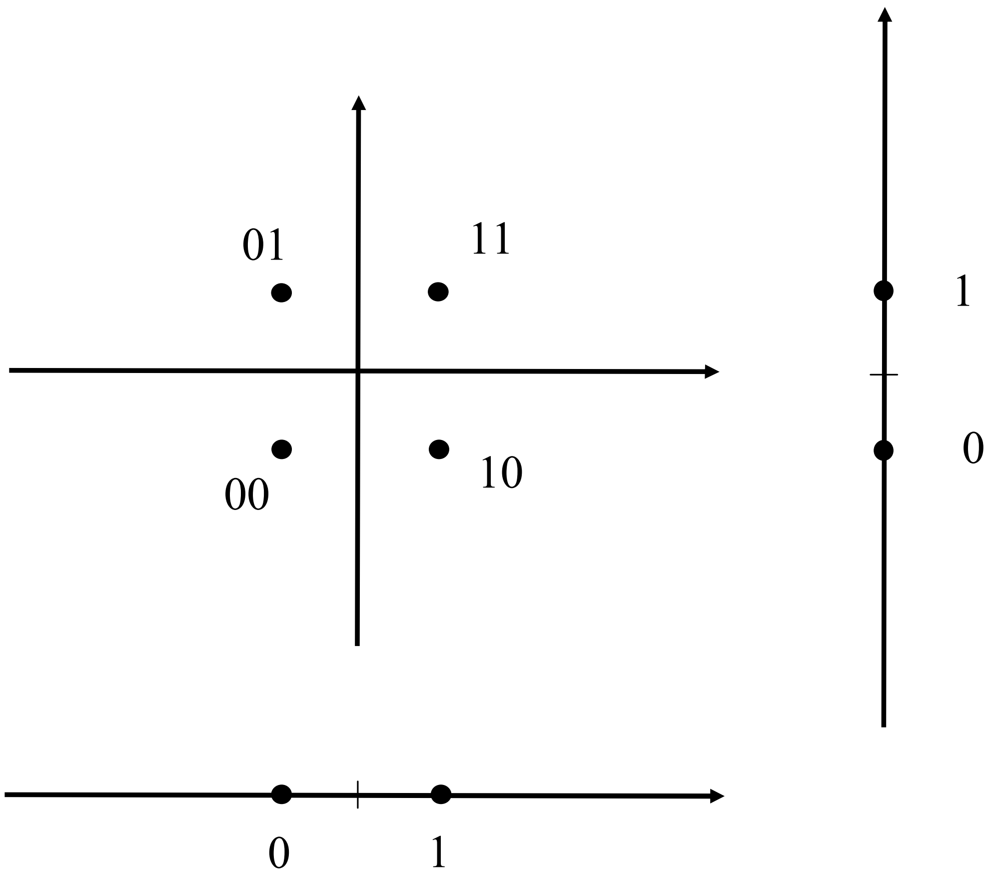


4-PSK: *intepretation*

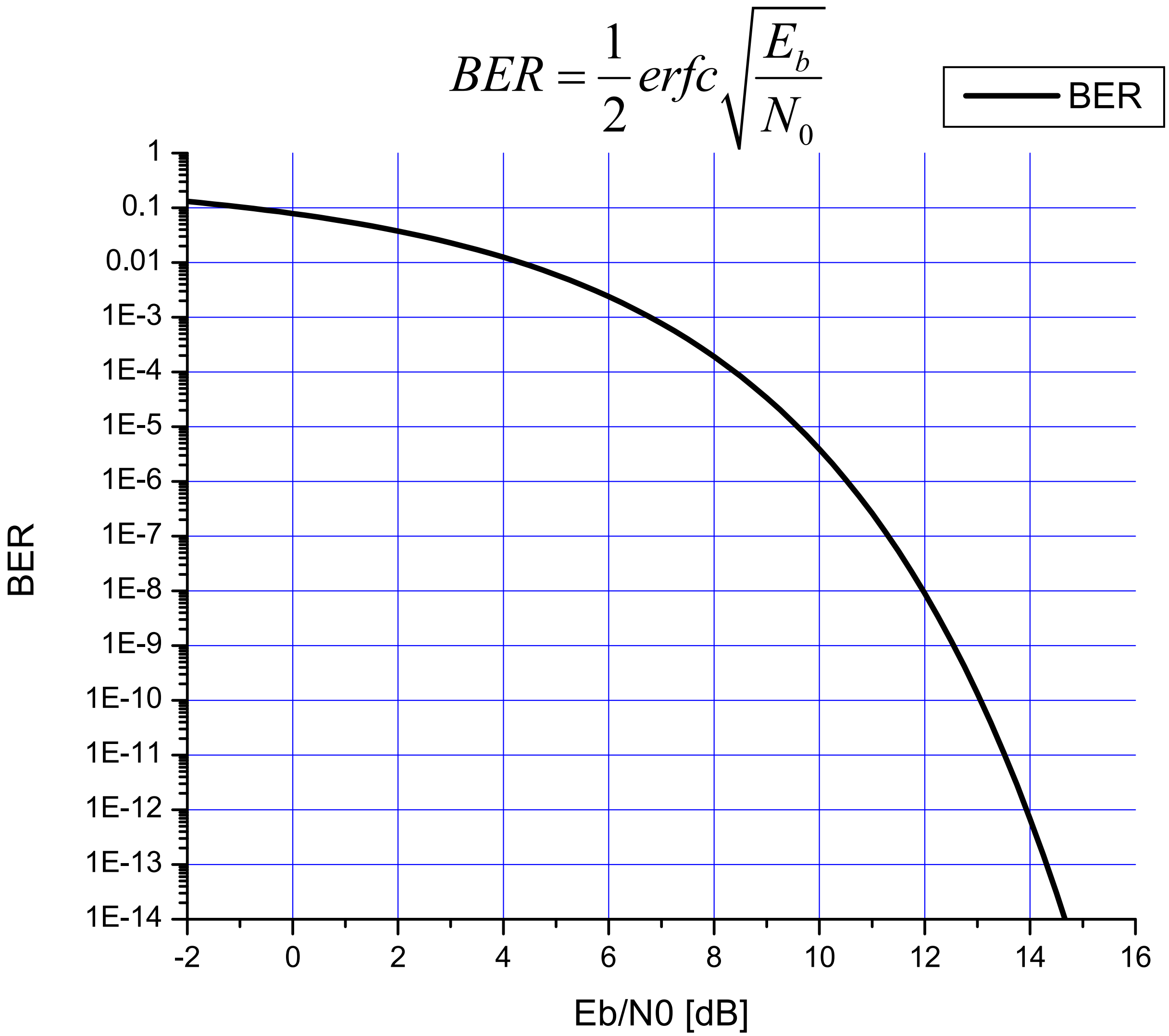
The Cartesian product interpretation clarifies why a 4-PSK constellation

1. **Has the same BER performance of a 2-PSK**

2. **Has double spectral efficiency** (two sequences with half bit-rate transmitted on the same frequencies)



4-PSK: *error probability*



***4-PSK**: applications*

Probably the most used digital modulation

- Satellite links
- Terrestrial radio links (with low spectral efficiency)
- GPS/Galileo
- UMTS
- ...

m-PSK: *characteristics*

1. Band-pass modulation
2. 2D signal set
3. Basis signals $p(t)cos(2\pi f_0 t)$ e $p(t)sin(2\pi f_0 t)$
4. Costellation = m signals, equidistant on a circle
5. Information associated to the carrier phase

m-PSK: *constellation*

SIGNAL SET

$$M = \{ s_i(t) = Ap(t)\cos(2\pi f_0 t - \varphi_i) \}_{i=1}^{m}$$

$$\varphi_i = \Phi + (i-1)\frac{2\pi}{m}$$

Information associated to the carrier phase

m-PSK: constellation

$$\boxed{s_i(t) = Ap(t)\cos(2\pi f_0 t - \varphi_i)}$$

$$\varphi_i = \Phi + (i-1)\frac{2\pi}{m}$$

We can write

$$\boxed{s_i(t) = (A\cos\varphi_i)\,p(t)\cos(2\pi f_0 t) + (A\sin\varphi_i)\,p(t)\sin(2\pi f_0 t)}$$

Clearly, we have two versors

$$\boxed{b_1(t) = p(t)\cos(2\pi f_0 t)}$$

$$\boxed{b_2(t) = p(t)\sin(2\pi f_0 t)}$$

m-PSK: constellation

SIGNAL SET

$$M = \{ s_i(t) = Ap(t)\cos(2\pi f_0 t - \varphi_i) \}_{i=1}^{m} \qquad \varphi_i = \Phi + (i-1)\frac{2\pi}{m}$$

VERSORS

$$b_1(t) = p(t)\cos(2\pi f_0 t)$$

$$b_2(t) = p(t)\sin(2\pi f_0 t)$$

VECTOR SET

$$M = \{ \underline{s}_i = (\alpha_i, \beta_i) \}_{i=1}^{m} \subseteq R^2$$

$$\alpha_i = A\cos\varphi_i$$

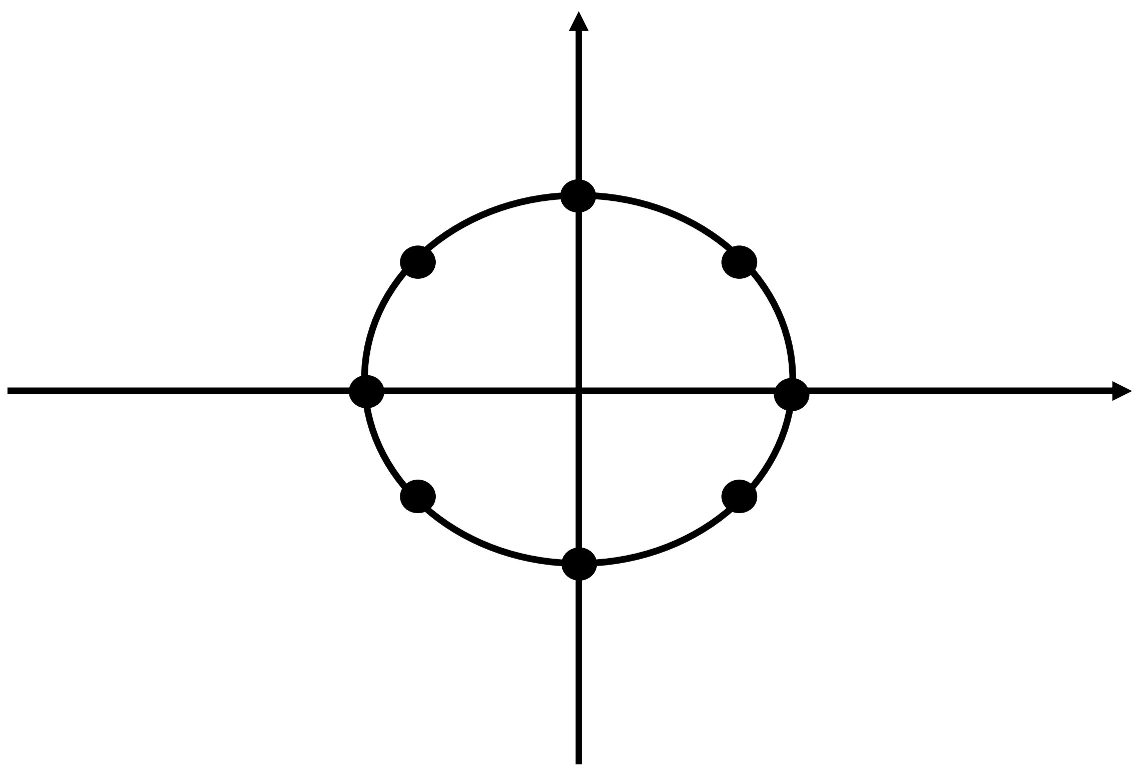
$$\beta_i = A\sin\varphi_i$$

$$\varphi_i = \Phi + (i-1)\frac{2\pi}{m}$$

Example

8-PSK $\Phi = 0$

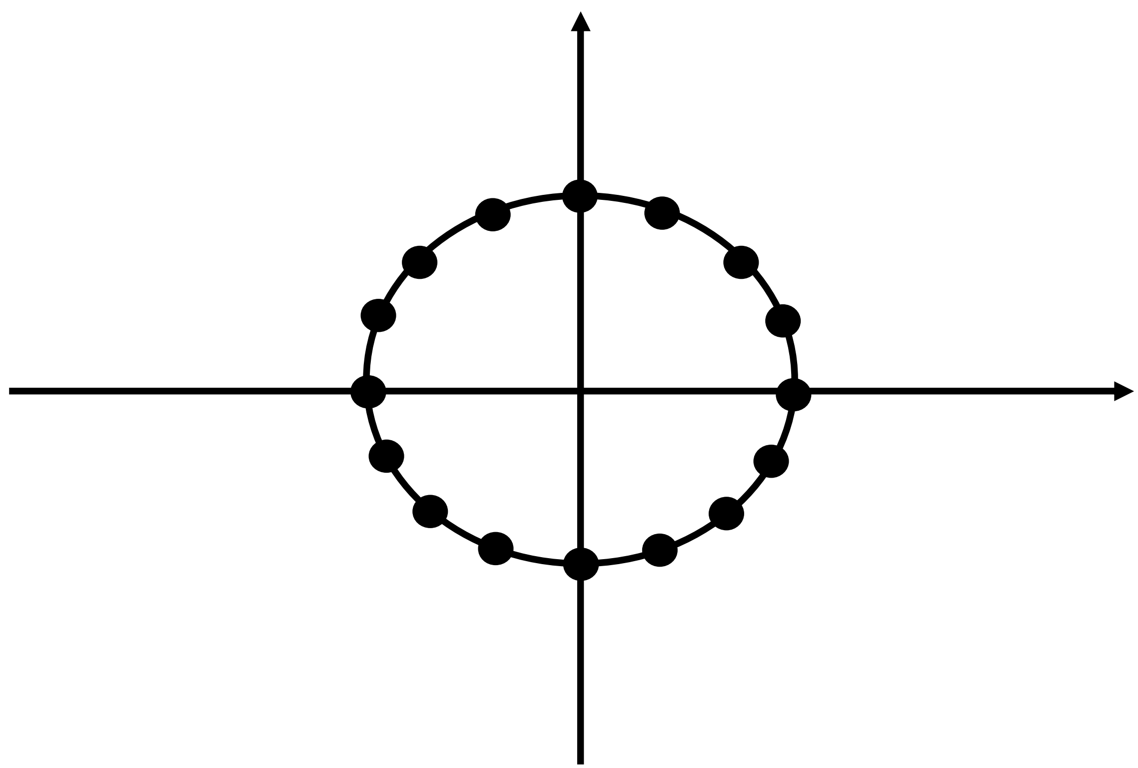
$$M = \{\underline{s}_1 = (A,0)\ ,\ \underline{s}_2 = (A/\sqrt{2}, A/\sqrt{2})\ ,\ \underline{s}_3 = (0, A)\ ,\ \underline{s}_4 = (-A/\sqrt{2}, A/\sqrt{2}),$$
$$\underline{s}_5 = (-A,0)\ ,\ \underline{s}_6 = (-A/\sqrt{2}, -A/\sqrt{2})\ ,\ \underline{s}_7 = (0,-A)\ ,\ \underline{s}_8 = (A/\sqrt{2}, -A/\sqrt{2})\} \subseteq R^2$$



Example

16-PSK $\quad \Phi = 0$

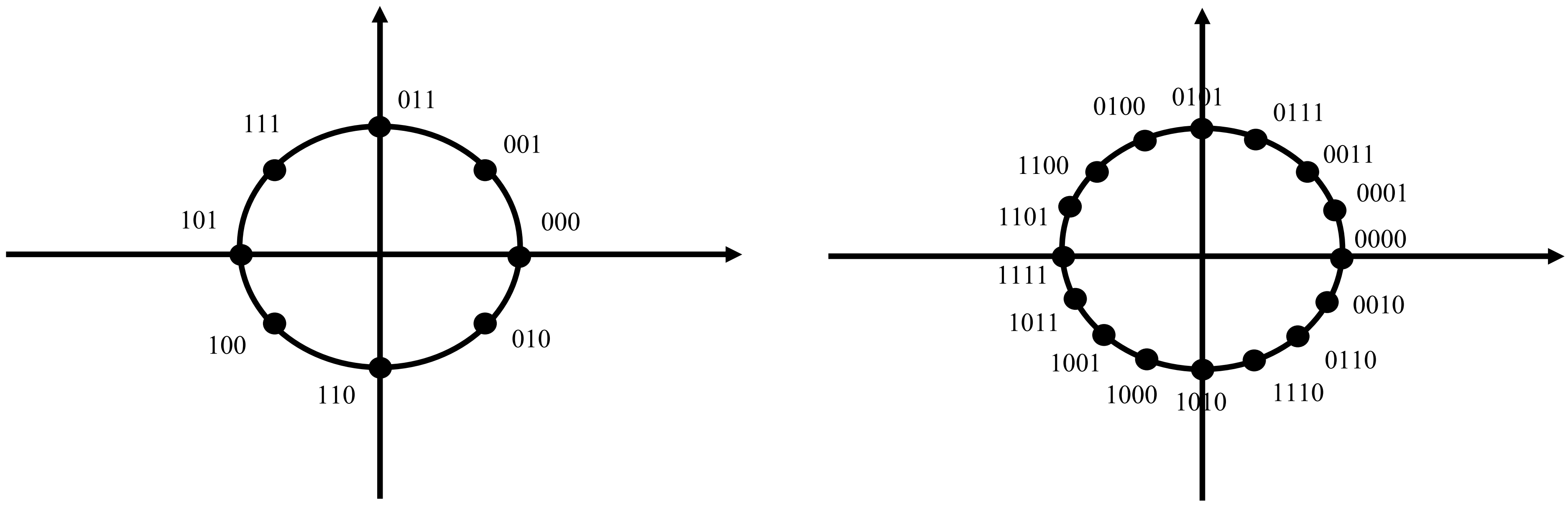
$$M=\{\underline{s}_1=(A,0)\,,\underline{s}_2=(0.924A,0.383A)\,,\underline{s}_3=(A/\sqrt{2},A/\sqrt{2})\,,\underline{s}_4=(0.383A,0.924A),$$
$$\underline{s}_5=(0,A)\,,\underline{s}_6=(-0.383A,0.924A,)\,,\underline{s}_7=(-A/\sqrt{2},A/\sqrt{2})\,,\underline{s}_8=(-0.924A,0.383A),$$
$$\underline{s}_9=(-A,0)\,,\underline{s}_{10}=(-0.924A,-0.383A)\,,\underline{s}_{11}=(-A/\sqrt{2},-A/\sqrt{2})\,,\underline{s}_{12}=(-0.383A,-0.924A),$$
$$\underline{s}_{13}=(0,-A)\,,\underline{s}_{14}=(0.383A,-0.924A,)\,,\underline{s}_{15}=(A/\sqrt{2},-A/\sqrt{2})\,,\underline{s}_{16}=(0.924A,-0.383A)\}\subseteq R^2$$



***m-PSK**: binary labeling*

$$e : H_k \leftrightarrow M$$

It is always possible to build Gray labelings



m-PSK: *transmitted waveform*

$k = \log_2 m$

$\boxed{T = kT_b}$

$\boxed{R = \frac{R_b}{k}}$

Each symbol has duration T
Each symbol component (α and β) lasts for T second

Transmitted waveform

$$s(t) = \underbrace{\left[\sum_n \alpha[n]p(t-nT)\right]}_{i(t)}\cos(2\pi f_0 t) + \underbrace{\left[\sum_n \beta[n]p(t-nT)\right]}_{q(t)}\sin(2\pi f_0 t)$$

I component (in phase)

Q component (in quadrature)

m-PSK: *analytic signal*

$$s(t) = \underbrace{\left[\sum_n \alpha[n]p(t-nT)\right]}_{i(t)}\cos(2\pi f_0 t) + \underbrace{\left[\sum_n \beta[n]p(t-nT)\right]}_{q(t)}\sin(2\pi f_0 t)$$

$$s(t) = \mathrm{Re}\left[\dot{s}(t)\right] = \mathrm{Re}\left[\tilde{s}(t)e^{j2\pi f_0 t}\right]$$

$$\tilde{s}(t) = i(t) - jq(t) = \sum_n \gamma[n]p(t-nT) \qquad \gamma[n] = \alpha[n] - j\beta[n]$$

m-PSK: *bandwidth and spectral efficiency*

Transmitted waveform

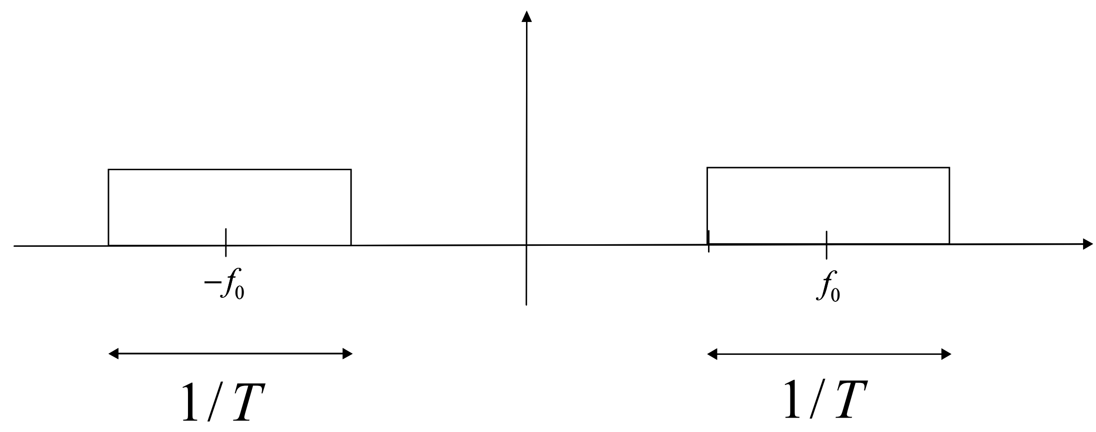
$$s(t) = \left[\sum_n \alpha[n] p(t - nT)\right] \cos(2\pi f_0 t) + \left[\sum_n \beta[n] p(t - nT)\right] \sin(2\pi f_0 t)$$

$$G_s(f) = z\left[\left|P(f - f_0)\right|^2 + \left|P(f + f_0)\right|^2\right] \qquad z \in R$$

Each symbol $\alpha[n]$ and $\beta[n]$ has time duration $T = kT_b$

m-PSK: *bandwidth and spectral efficiency*

Case 1: $p(t)$ = ideal low pass filter



Total bandwidth
(ideal case)

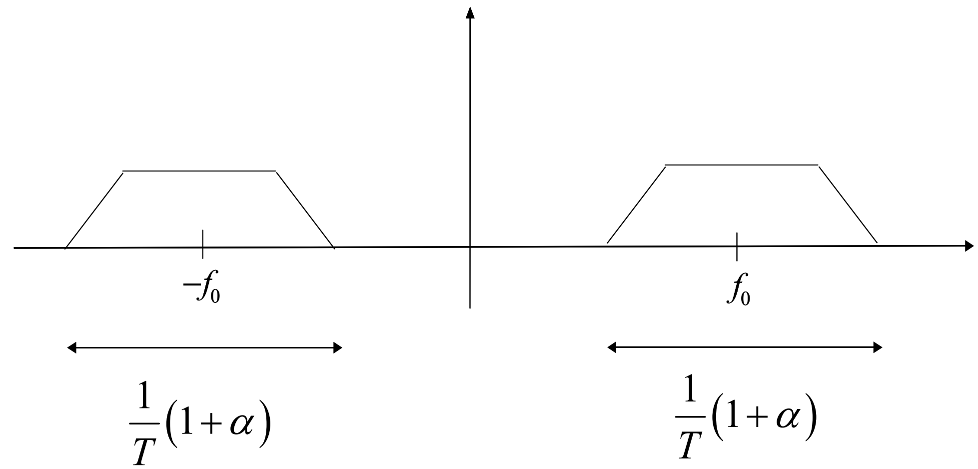
$$\boxed{B_{id} = R = \frac{R_b}{k}}$$

Spectral efficiency
(ideal case)

$$\boxed{\eta_{id} = \frac{R_b}{B_{id}} = k \ bps / Hz}$$

m-PSK: *bandwidth and spectral efficiency*

Case 2: $p(t)$ = RRC filter with roll off α



Total bandwidth $$B = R(1+\alpha) = \frac{R_b}{k}(1+\alpha)$$

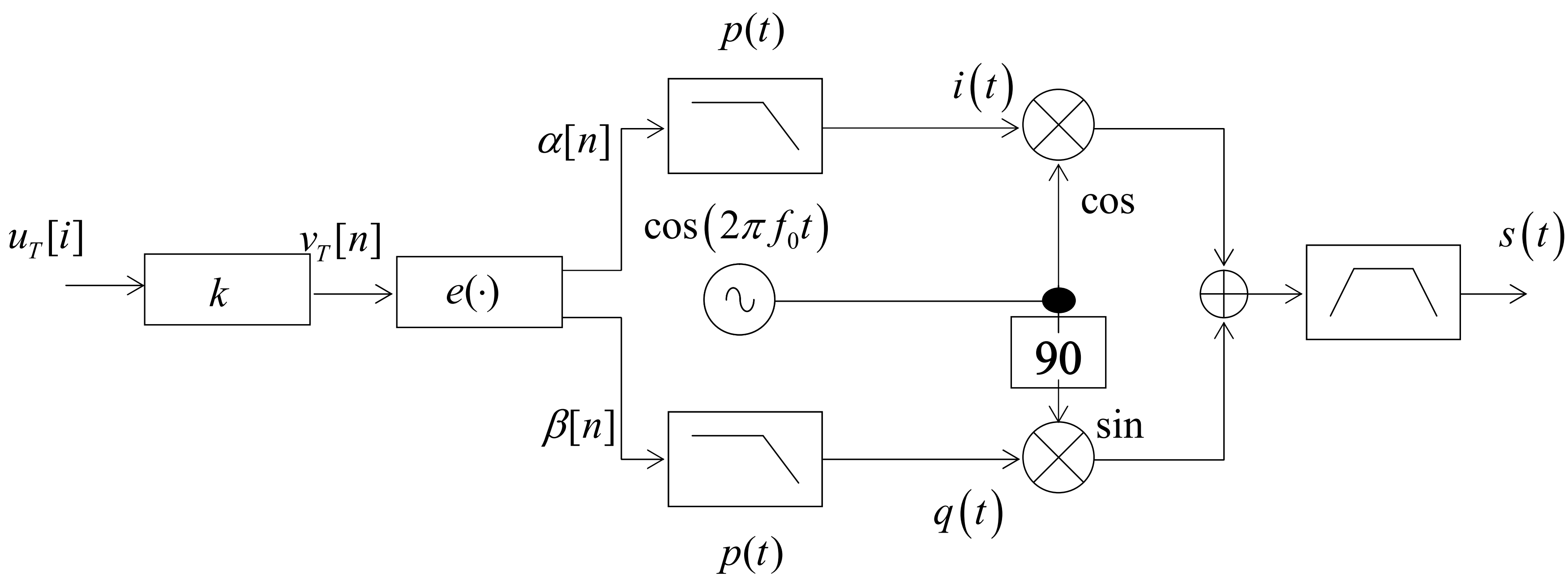
Spectral efficiency $$\boxed{\eta = \frac{R_b}{B} = \frac{k}{(1+\alpha)}\, bps / Hz}$$

Exercize

Given a bandpass channel with bandwidth B = 4000 Hz, centred around f_0=2 GHz, compute the maximum bit rate R_b we can transmit over it with an 8-PSK constellation or a 16-PSK constellation in the two cases:

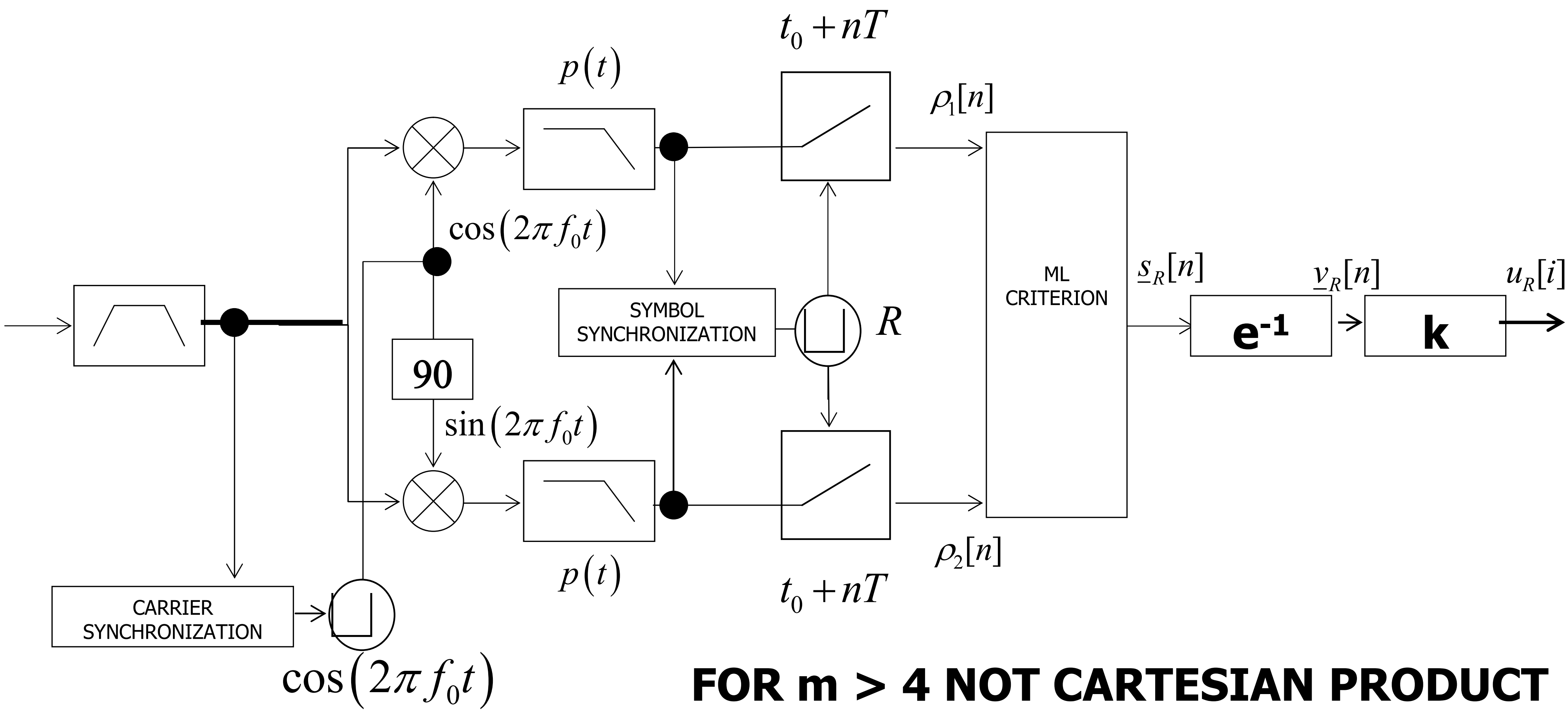
- Ideal low pass filter
- RRC filter with α=0.25

m-PSK: modulator



FOR m > 4 NOT CARTESIAN PRODUCT

m-PSK: demodulator



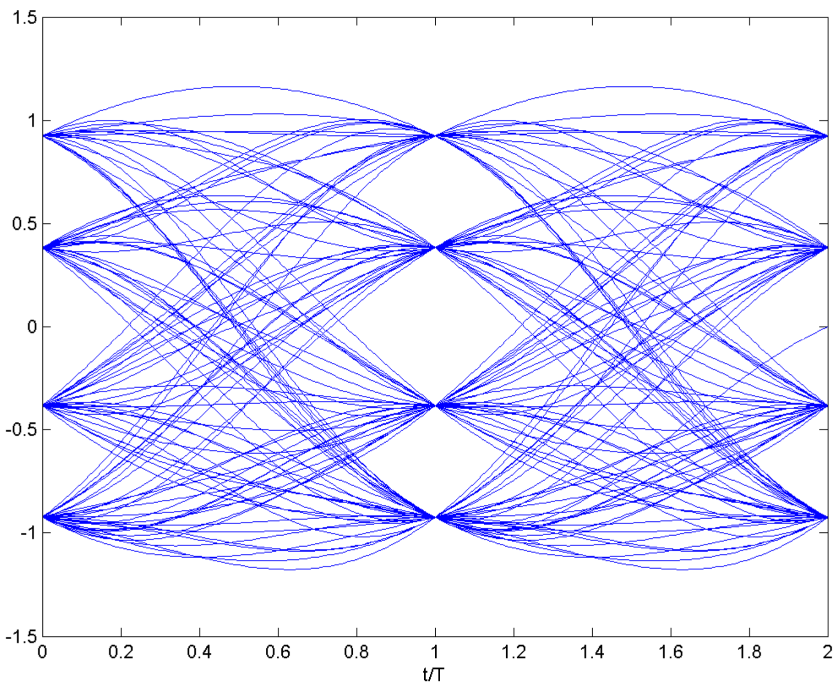
FOR m > 4 NOT CARTESIAN PRODUCT

Voronoi regions = plane sectors

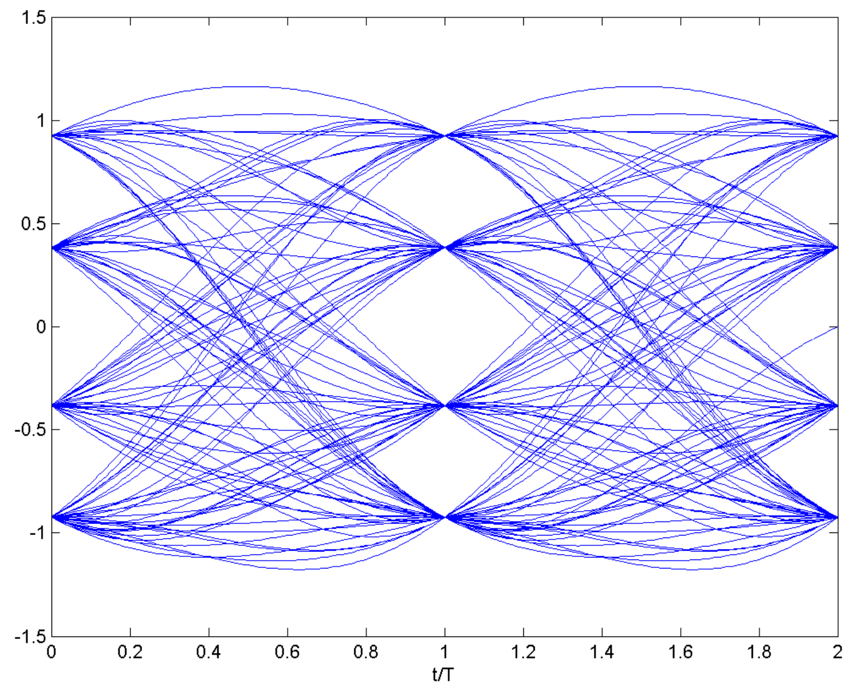
m-PSK: *eye diagram*

8-PSK constellation with RRC filter (α=0.5)

[α and β components = 0.924,0.383,-0.383,-0.924]



Channel I

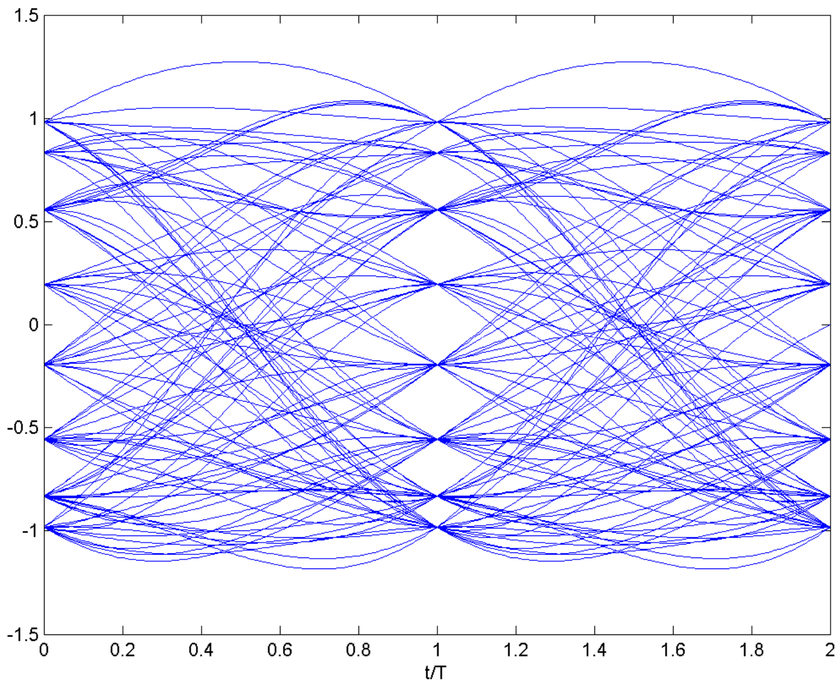


Channel Q

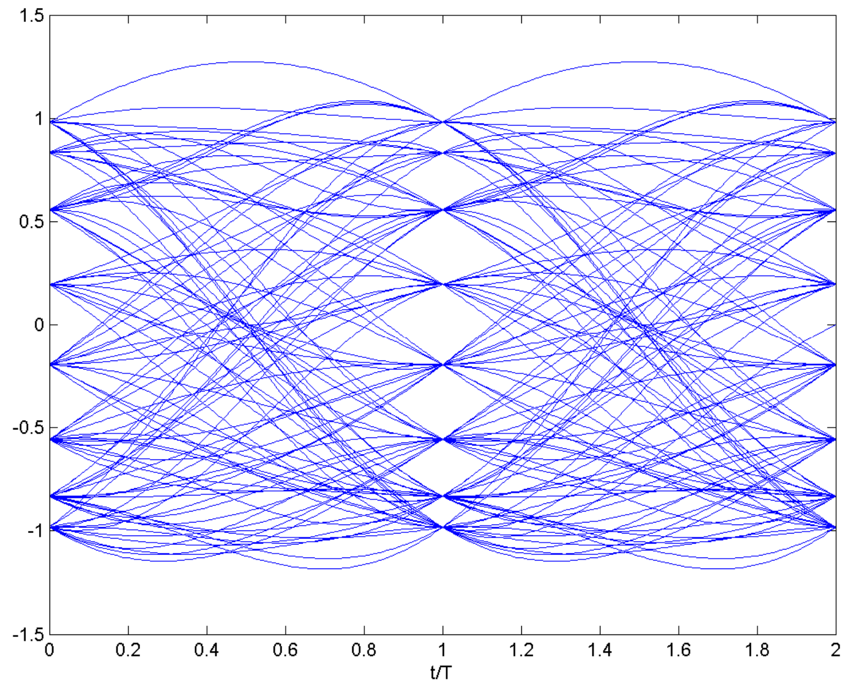
***m-PSK**: eye diagram*

16-PSK constellation with RRC filter (α=0.5)

[α and β components = 0.981,0.832,0.556,0.195,-0.195,-0.556,-0.832,-0.981]



Channel I



Channel Q

m-PSK constellation: *error probability*

By applying the asymptotic approximation we can obtain

$$\boxed{P_b(e) \approx \frac{1}{k} erfc\left(\sqrt{k \frac{E_b}{N_0} \sin^2\left(\frac{\pi}{m}\right)}\right)}$$

The performance decrease for increasing *m*

(minimum distance decreases)

***m*-PSK constellation**: *error probability*

4-PSK: $P_b(e) \approx \frac{1}{2} erfc\left(\sqrt{\frac{E_b}{N_0}}\right)$

8-PSK: $P_b(e) \approx \frac{1}{3} erfc\left(\sqrt{0.439\frac{E_b}{N_0}}\right)$ - 3.6 dB with respect to 4-PSK

16-PSK: $P_b(e) \approx \frac{1}{4} erfc\left(\sqrt{0.152\frac{E_b}{N_0}}\right)$ - 4.6 dB with respect to 8-PSK

No one uses m-PSK for m > 16: very poor BER performance

m-PSK constellation: *error probability*