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1. Question 1:

Solution:
a) $R_2 = \{(a,b)|a = b+1\}$
We take a=1 for example, then b=a+1=1+1=2, so there will be only one non zero entry in the first row where a=1 and that will be in the position(1,2). The same thing happens in all the other rows except for row 1000.because if a=1000 then b must be 1001 and since $A = \{1,...,1000\}$ that cannot happen. So since there is one non zero entry for every row expect row 1000 there are 999 non zero entries in the matrix M_{R2}
b)$R_3 = \{(a,b)|a+b = 1000\}$
In the first row where a=1 b=1000-a=1000-1=999 and that means that there is only one non zero entry in the first row and it is in the location (1,999). That is the case of all the other rows except for the row 1000 where a=1000 and b=0 but since 0 is not in set A there will be no non zero entry in that row. So since there is only one non zero entry for each row except row 1000, it means that there are 999 non zero entries in total in matrix M_{R3}.
c)$R_4 = \{(a,b)|a+b \leq 1001\}$
In the first row where a=1 $\rightarrow b \leq 1001-1 \rightarrow b \leq 1000$ and since $A = \{1,...,1000\}$ that means that the first row has only non zero entries which means 1000 entries. In the second row a=2 and $b \leq 1001-2 \rightarrow b \leq 999$ so there is one less non zero entry in row 2 and this keeps going until row 1000 where there is only one non zero entry because a=1000 and $b \leq 1001-1000 \rightarrow b \leq 1 \rightarrow b = 1$.So the total amount of non zero entries is $\sum_{x=1}^{1000} x$
d)$R_5 = \{(a,b)|a \neq 0\}$
Since$A = \{1,...,1000\}$ in the first row a=1 and in the last row a=1000, a never gets the value 0 so every entry in matrixM_{R5} is non zero. So there are $1000^2 = 1000000$ non zero entries in matrix M_{R5}.

2. Question 2:

Solution:
a)$R_1 \cap R_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$
b)$R_1 \circ R_2 = M_{R_2} \odot M_{R_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \odot \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$
c)$R_2 \circ R_1 = M_{R_1} \odot M_{R_2} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \odot \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

3. Question 3:

Solution: Because it is given that P1 , P2 are partitions of A , B respectively we have the following facts.

1)

$$\widetilde{\bigcup_{A_i \varepsilon P-1}} A_i = A$$

$$\widetilde{\bigcup_{B_i \varepsilon P-2}} B_i = B$$

which means that the partition covers A and B respectively.

In our case we need to prove that :

$$\widetilde{\bigcup_{P_i \varepsilon P}} P_i = (A_1 x B_1)....(A_n x B_n) = AxB$$

and covers the set S = A x B , which is true if we consider the two facts that are separately true we know that every $(A_i x B_i)$ is true . So the first assumption is true.

2)

Also we have as a fact that every subset of P1 , P2 has this capacity : $A_i \bigcap A_j = \varnothing$
and that A is been covert from its partition , the same as B : $B_i \bigcap B_j = \varnothing$

From this we can come to the conclusion that $A_i x B_i \bigcap A_j x B_j = \varnothing$

4. Question 4:

Solution: If R is anti-symmetric and $(x, y) \varepsilon R \bigcap R^{-}1$ then $(x, y) \varepsilon R^{-}1$ implies that $(x, y) \varepsilon R$. Therefore , $(x, y) \varepsilon R$ and $(y, x) \varepsilon R$, and since R is antisymmetric , then x = y and $(x, y) \varepsilon \Delta$. So we have shown that $R \bigcap R^{-}1 \subseteq \Delta$

Conversely , suppose that $R \bigcap R^{-}1 \subseteq \Delta$. If $(x, y) \varepsilon R and (y, x) \varepsilon R, then (x, y) \varepsilon R \bigcap R^{-}1 \subseteq \Delta$, so that x=y , and therefore R is antisymmetric .

5. Question 5:

Solution: The out-degree of a vertex is the number of edges directed out of this vertex in a directed graph.

If there are no cycles in the graph that means that you can not start from a vertex and follow a path which end in this vertex.So at least one vertex has no edges directed out of it.

Let's say that we have to vertexes a,b,c and the relations : (a,b),(b,c) , we do not have an edge directed out of c , out-degree(c) = 0 . If we direct an edge out of c we can do it with three ways.
1) (c,a) , this means that we are making a cycle 2) (c,b) , this means that we are making a cycle as well 3) (c,c) , this is a cycle

We can say that if every vertex has outcome edges this means that for every vertex the out-degree is non-zero , so there must be a cycle in the graph . From this we can say the opposite that if there are no cycles in the graph then at least one vertex has out-degree equals to zero.

6. Question 6:

Solution: First we show the if part:

Suppose for all positive integer $n, Rn \subseteq R$. In particular, $R2 \subseteq R$. For all $a, b, c\varepsilon$ A such that $(a, b), (b, c)\varepsilon R$, we always have $(a, c)\varepsilon R2$. Consequently, $(a, c)\varepsilon R$. This means R is transitive.

Next, we prove the only if part.

Suppose R is transitive. We prove$Rn \subseteq R$ by induction. When $n = 1$,clearly$R1 = R \subseteq R$. For the inductive step, we assume $Rn \subseteq R$ and try to establish that $R_n + 1 \subseteq R$. For any $(a, c)\varepsilon Rn + 1$, there exists b εA such that $(a, b)Rn$ and $(b, c)\varepsilon R$. By the inductive assumption, we get that $(a, b)\varepsilon R$ and $(b, c)\varepsilon R$. Since R is transitive, we have that $(a, c)\varepsilon R$. The above tells us that $R_n + 1\varepsilon R$

7. Question 7:

Solution:

$$W_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Since there are only zeros in row 1 nothing changes from $W_0 \to W_1$.

$W_1 = W_0$

There are 2 non zero entries in row 2 which are 1 and 5 and one non zero entry in column 2 in position 3, so we have to add a one in positions (3,1) and (3,5).

$$W_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

In the third row there are 3 non zero entries in positions 1, 2, 5 and in the third column there is one at position 4 so positions (4,1) (4,2) (4,5) need to be ones.

$$W_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

In row 4 there are non zero entries in positions 1, 2, 3, 5, and in column 4 there is one at position 5, so we need to place ones at positions (5,1) (5,2) (5,3) (5,5).

$$W_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

The fifth row has only non zero entries and the fifth column has ones at positions 2, 3, 4, so we need to add ones at positions (2,2) (2,3) (2,4) (3,3) (3,4) (4,4),

$$W_5 = M_{R^\infty} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

8. Question 8:

Solution: We prove that f is O(g) by finding k and c such that $f(n) \leq c \times g(n)$ for all $n \geq k$
So for c=1 and k=999 we have $n^{100} \leq 2^n$ for all $n \geq 999$
We know that 2^n grows faster than n^{100} so g is not O(f).