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DS Group No. 29

Answers

1. 12 points Let $\Sigma = \{a, b\}$ be an alphabet.

   (a) List the elements of L_1, the language consisting of all strings in Σ^* that are palindromes and have length less than or equal to four.

   (b) List the elements of L_2, the language consisting of all strings in Σ^* of length less than or equal to three in which all a's appear to the left of all b's.

Solution:
$L_1 = \{a, b, aa, bb, aba, bab, aaa, bbb, abba, baab, aaaa, bbbb\}$
$L_2 = \{ab, abb, aab\}$

2. 12 points Use mathematical induction to prove the following statement:

Suppose real numbers a and b, such that

$$A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

Then, for every positive integer n, it holds:

$$A^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$$

Solution:
Base case: for $n = 0$ $\quad A^0 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} a^0 & 0 \\ 0 & b^0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$Inductioncase: fork \geq 0 \quad A^k = \begin{pmatrix} a^k & 0 \\ 0 & b^k \end{pmatrix}$
Need to prove that $A^{k+1} = \begin{pmatrix} a^{k+1} & 0 \\ 0 & b^{k+1} \end{pmatrix}$
$A^{k+1} = A^k \times A = \begin{pmatrix} a^k & 0 \\ 0 & b^k \end{pmatrix} \times \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a^k \times a + 0 \times 0 & a^k \times 0 + 0 \times b \\ 0 \times a + b^k \times 0 & 0 \times 0 + b^k \times b \end{pmatrix} = \begin{pmatrix} a^{k+1} & 0 \\ 0 & b^{k+1} \end{pmatrix}$
Because $A^n = \begin{pmatrix} a^n & 0 \\ 0 & b^n \end{pmatrix}$ is true and $A^1 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ is also true , so the outcome is true.

3. 12 points Use mathematical induction to prove the following statement:

Let $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ be sets such that $A_i \subseteq B_i$ for every $i \in \{1, \ldots, n\}$. Then

$$\bigcup_{i=1}^{n} A_i \subseteq \bigcup_{i=1}^{n} B_i$$

Solution:
Base case: $i = 1 \quad A_1 \subseteq B_1$
Induction case: for $k \geq 0 \quad \bigcup_{i=1}^{k} A_i \subseteq \bigcup_{i=1}^{k} B_i$
We need to prove that $\bigcup_{i=1}^{k+1} A_i \subseteq \bigcup_{i=1}^{k+1} B_i$
$\bigcup_{i=1}^{k+1} A_i = \bigcup_{i=1}^{k} A_i \cup A_{k+1} \subseteq \bigcup_{i=1}^{k+1} B_i = \bigcup_{i=1}^{k} B_i \cup B_{k+1}$
We know that $\bigcup_{i=1}^{k} A_i \subseteq \bigcup_{i=1}^{k} B_i$
also $A_{k+1} \subseteq B_{k+1}$
so $\bigcup_{i=1}^{k+1} A_i \subseteq \bigcup_{i=1}^{k+1} B_i$ is true.

4. $\boxed{\text{12 points}}$ Use mathematical induction to prove the following statement:

The product of any 3 consecutive integers is divisible by 6.

Solution: Base case: $1 \times 2 \times 3 = 6$ and 6 is divisible by 6.
Induction case: $k \times (k+1) \times (k+2) = 6a$
We need to prove for k+1 $\quad (k+1) \times (k+2) \times (k+3) = 6b$
$k \times (k+1) \times (k+2) + 3 \times (k+1) \times (k+2)$
$6a + 3 \times (k+1) \times (k+2)$
The product of $(k+1) \times (k+2)$ is even beacuse one of them is always odd and the other is even and the product of an odd and even number is even.
Let c be a number, then since we know that $(k+1) \times (k+2)$ is even we write 2c in its place.
$6a + 3 \times 2c = 6a + 6c = 6(a+c)$
so $(k+1) \times (k+2) \times (k+3)$ is divisible by 6

5. $\boxed{\text{12 points}}$ Obtain an explicit formula for the following recurrence relations:

(a) $c_n = 6c_{n-1} - 9c_{n-2}$, with $c_1 = 2.5$, $c_2 = 4.7$.
(b) $b_n = -3b_{n-1} - 2b_{n-2}$, with $b_1 = -2$, $b_2 = 4$.

Solution: a) $a = 6 \quad b = -9$
$r^2 = ar + b \rightarrow r^2 - 6r + 9 = 0$
$r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{6 \pm \sqrt{36 - 4 \times 9}}{2} = \dfrac{6 \pm 0}{2} = 3$
case b $r_1 = r_2 = 3$
$C_n = c_1 \times r^n + c_2 \times n \times r^n = r^n(c_1 + n \times c_2)$
$C_1 = 2,5 = c_1 \times 3 + c_2 \times 1 \times 3 \rightarrow 2,5 = 3(c_1 + c_2) \rightarrow \dfrac{2,5}{3} = c_1 + c_2 \rightarrow c_1 = \dfrac{2,5 \times c_2}{3}$
$C_2 = 4,7 = c_1 \times 3^2 + c_2 \times 2 \times 3^2 \rightarrow 4,7 = 9 \times c_1 + 18 \times c_2 = 9()c_1 + 2 \times c_2 \rightarrow \dfrac{4,7}{9} = c_1 + 2 \times c_2 =$
$\dfrac{2,5}{3} - c_2 + 2 \times c_2 \rightarrow c_2 = \dfrac{4,7 - 7,5}{9} = \dfrac{2,8}{9}$
$c_1 = \dfrac{2,5}{3} - \dfrac{2,8}{9} = \dfrac{7,5 - 2,8}{9} = \dfrac{4,7}{9}$
$C_n = \dfrac{4,7}{9} \times 3^n + \dfrac{2,8}{9} \times n \times 3^n$

b)$a = -3 \quad b = -2 \quad r^2 + 3r + 2 = 0 \quad r = \dfrac{-3 \pm \sqrt{9-8}}{2} = \dfrac{-3 \pm 1}{2} \rightarrow r_1 \dfrac{-4}{2} = -2 \quad r_2 = \dfrac{-2}{2} = -1$
case a $r_1 = -2 \quad r_2 = -1 \quad r_1 \neq r_2$

$b_n = c_1 \times r_1 + c_2 \times r_2^n$
$b_1 = -2 = -2 \times c_1 + -c_2 \rightarrow c_2 = 2 - 2 \times r_1$
$b_2 = 4 = 4 \times c_1 + c_2 = 4 \times c_1 + 2 - 2 \times c_1 \rightarrow 2 = 2 times c_1 \rightarrow c_1 = 1$
$c_2 = 2 - 2 \times 1 = 0$
$b_n = (-2)^n + 0 \times (-1)^n = (-2)^n$

6. $\boxed{\text{12 points}}$ Ten people volunteer for a three-person committee:

(a) How many three-person committees can be formed from the 10 volunteers?
(b) Assume that every possible committee of three that can be formed from these ten names is written on a slip of paper, one slip for each possible committee, and the slips are put in ten hats. Show that at least one hat contains 12 or more slips of paper.

Solution: a) $\dfrac{10!}{3!(10-3)!} = \dfrac{10!}{3! \times 7!} = \dfrac{10 \times 9 \times 8}{6} = 120$
b) 120 slips in 10 hats
one hat must contain at least $\dfrac{120-1}{10} + 1$ slips$\rightarrow 11 + 1 = 12$
at least 12 slips per hat.

7. $\boxed{\text{12 points}}$ Consider the sequences of integers whose first six elements are as follows:

(i) $3, 6, 12, 24, 48, 96, \ldots$
(ii) $2, 16, 54, 128, 250, 432, \ldots$

In each case:

(a) Find an explicit formula that generates the elements of each sequence.
(b) Check that your formula is correct by finding the next three elements of the sequence.

Solution: i) a)
$a_n = a_{n-1} \times 2$
$a_n = a_{n-2} \times 4$
$a_n = 8 \times a_{n-3}$
$a_n = 3 \times 2^n$
b)
$a_0 = 1 \times 3 = 3$
$a_1 = 2 \times 3 = 6$
$a_2 = 2^2 \times 3 = 4 \times 3 = 12$
ii) a)
$a_n = 2 \times n^3 \quad \text{for} \quad n > 0$
b)
$a_1 = 2 \times 1^3 = 2$
$a_2 = 2 \times 2^3 = 16$
$a_3 = 2 \times 3^3 = 54$

8. $\boxed{\text{12 points}}$ Consider the recurrence relation given by

$$a_n = -8a_{n-1} - 16a_{n-2}$$

(a) Find an explicit formula for the sequence defined by a_n, with $a_1 = \alpha$ and $a_2 = \beta$, where α and β are two given constants.

(b) Use your solution to part (a) to obtain an explicit formula for the sequence defined by a_n, with initial conditions $a_1 = -5$ and $a_2 = 9$.

Solution: $r^2 + 8r + 16 = 0$

$r = \frac{-8 \pm \sqrt{64-60}}{2} = \frac{-8}{2} = -6$

case b $r_1 = r_2 = -4$

$a_n = c_1(-4)^n + c_2 \times n(-4)^n = (-4)^n(c_1 + c_2 \times n)$

$a_1 = \alpha = -4 \times c_1 - 4 \times c_2$

$a_2 = \beta = (-4)^2(c_1 + c_2 \times 2) = 14(c_1 + 2 \times c_2) = 16 \times c_1 + 31 \times c_2$

$\alpha = -4 \times c_1 - 4 \times c_2$

$c_1 = -\frac{\alpha}{4} - c_2$

$\beta = 16(-\frac{\alpha}{4} - c_2) + 32 \times c_2 = -4\alpha - 16c_2 + 32c_2 = -4\alpha + 16c_2$

$16c_2 = \beta + 4\alpha \to c_2 = \frac{4\alpha + \beta}{16}$

$c_1 = -\frac{\alpha}{4} - \frac{\beta + 4\alpha}{16} = \frac{-4\alpha - \beta - 4\alpha}{16} = -\frac{8\alpha + \beta}{16}$

$a_n = 9 - 40^n(-\frac{8\alpha + \beta}{16} + \frac{\beta + 4\alpha}{16} \times n)$

$\alpha = -5 \quad \beta = 9$

$a_n = (-4)^n(-\frac{8(-5) + 9}{16} + \frac{9 + 4() - 5}{16} \times n) = (-4)^n(\frac{40 - 9}{16} + \frac{9 - 20}{16} \times n) = (-4)^n(\frac{31}{16} - \frac{11}{16} \times n)$