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An Advanced Signature Scheme:

Schnorr Algorithm and its Benefits to the Bitcoin Ecosystem

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Introduction

The Elliptic Curve Digital Signature Algorithm (ECDSA) is used in the Bitcoin protocol as signature scheme, but it has some problems:

1. Limited efficiency (DER encoding, no batch validation);
2. Poor implementation of higher level constructions (low privacy and fungibility[1], limited scalability);
3. Not provably secure (malleable).

[1]Fungibility is the property of a good or a commodity whose individual units are indistinguishable, thus interchangeable.

Outline

Hash functions ($\simeq$ Random oracles)

Input		Output
"Politecnico di Milano"	$\xrightarrow{\text{SHA-256}}$ $\not\leftarrow$	52c61456bd3d60c08599a8c61e113e7b c39118b75d06b2643c55957ece91269b
	$\xrightarrow{\text{SHA-256}}$ $\not\leftarrow$	1f12f8384d5941e8e273926635be646a f786405a8662e0ba6dd1fd0526ec0528
	$\xrightarrow{\text{SHA-256}}$ $\not\leftarrow$	721a14d89cb463b51cf20ecc707aa290 5b7c8f32d9114dc19fe353e611494bf1

Elliptic curve cryptography

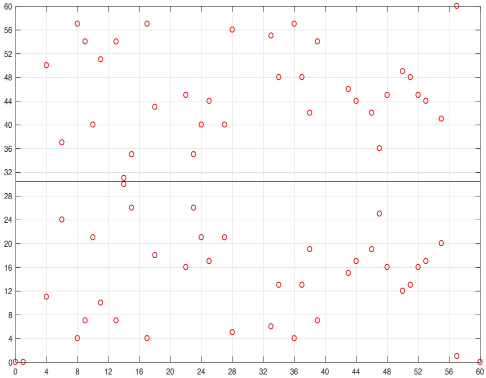
An elliptic curve over a finite field is defined by the equation:

$$E: \; y^2 = x^3 + ax + b \pmod{p}.$$

Bitcoin: $a = 0, b = 7, p \simeq 2^{256}$.

It is possible to define:

- Addition:
  $Q_3 := Q_1 + Q_2$,
  $\forall Q_1, Q_2 \in E$;
- Scalar multiplication:
  $qG := G + \ldots + G, \forall G \in E$,
  $\forall q \in \mathbb{N}$.



The curve $y^2 = x^3 - x$ over $\mathbb{F}_{61}$.

Discrete logarithm problem

Fixed $G \in E$, we can define $Q = qG \ \ \forall q \in \{1, ..., n-1\}$, where n is the smallest integer such that $nG = \infty$, the identity element of the sum:

- The direct operation $q \mapsto Q$ is efficient (double and add algorithm);
- The inverse operation $Q \mapsto q$ is computationally infeasible for certain groups.

Asymmetric cryptography: $\{q, Q\}$ is a key pair whose elements have complementary roles.

- q: private key (signature);
- Q: public key (verification).

Outline

Digital signature

Alice Bob

Digital signature

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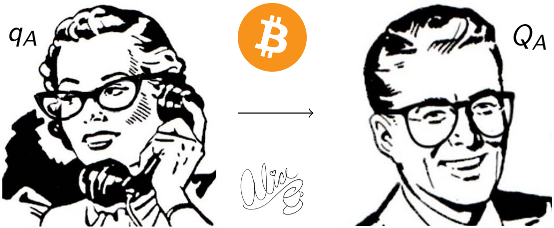

Alice Bob

- Authentication: the recipient is confident that the funds come from the owner;

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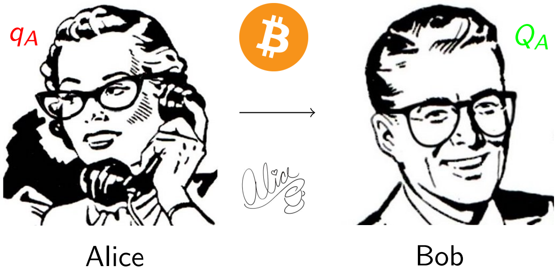

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- Non repudiation: the sender cannot deny having sent the funds;

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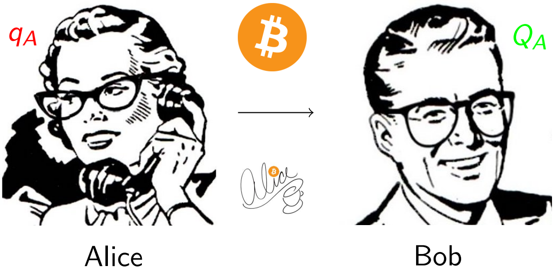
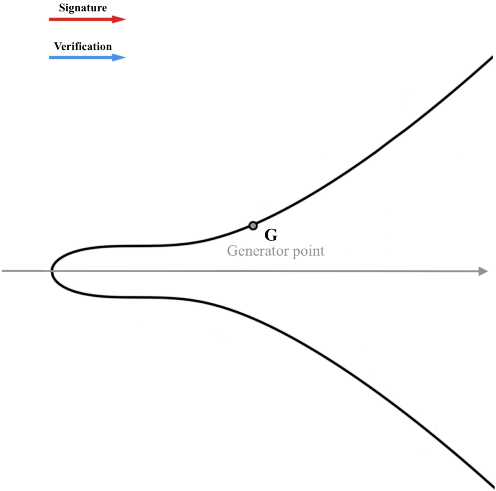


- Authentication: the recipient is confident that the funds come from the owner;
- Non repudiation: the sender cannot deny having sent the funds;
- Integrity: ensures that the transaction has not been altered during transmission.

Elliptic curve digital signature algorithm

ECDSA_SIG(m, q):



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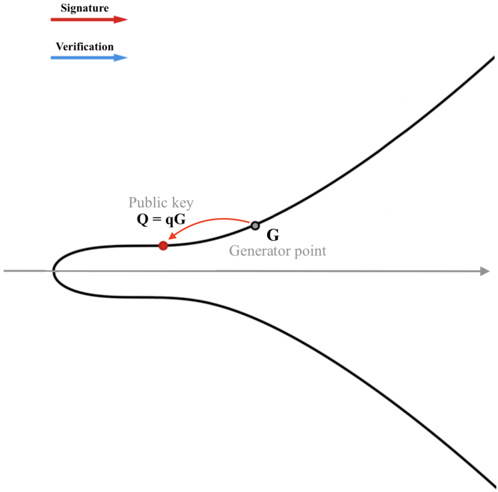

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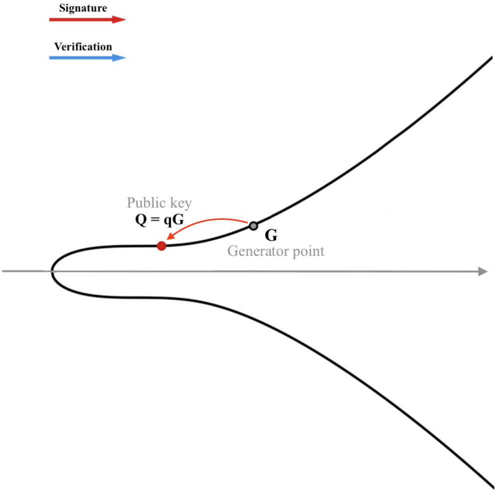

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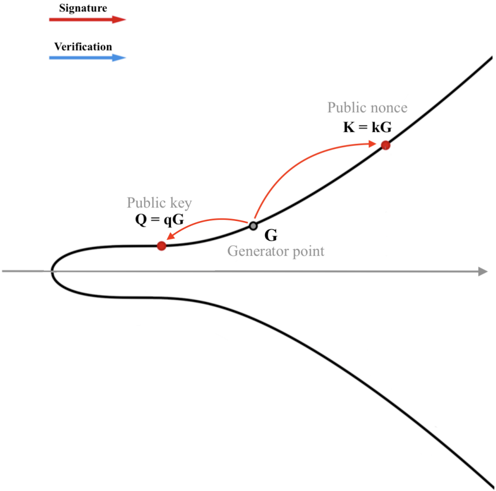

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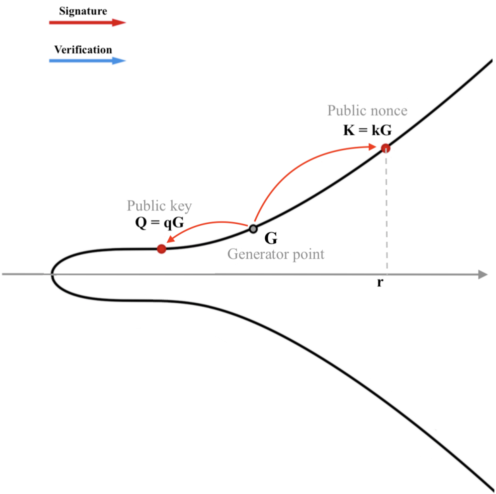

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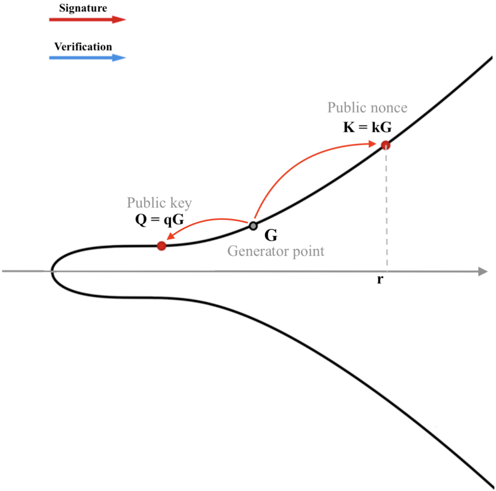

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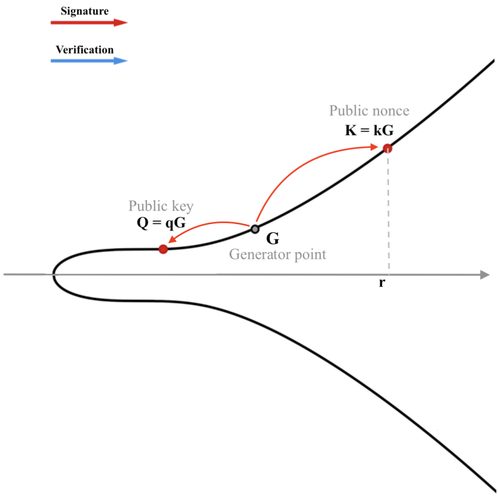

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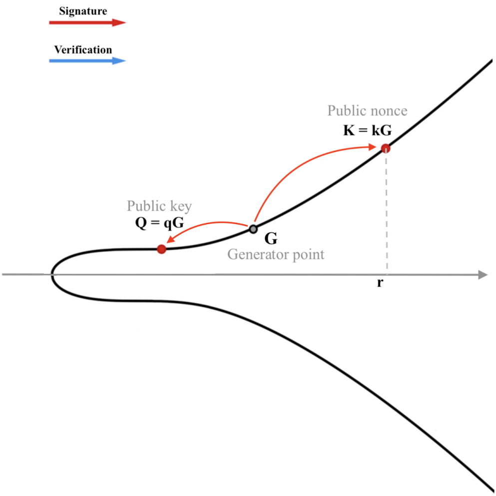

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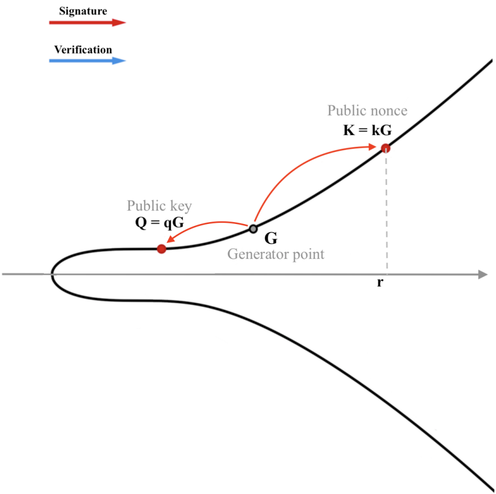

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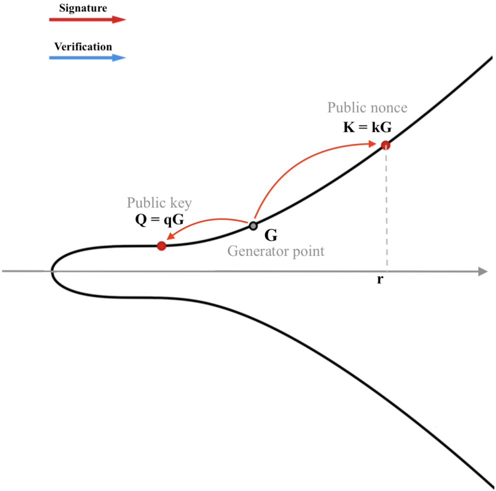

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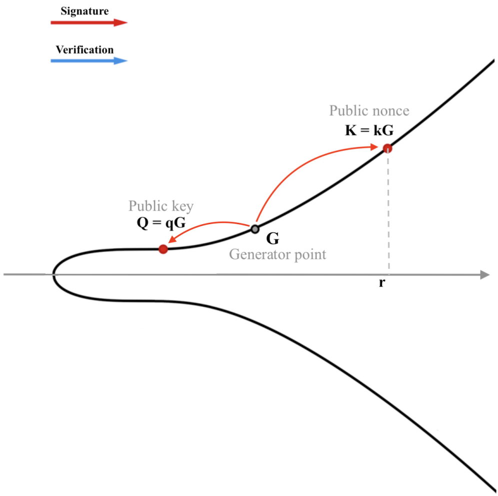

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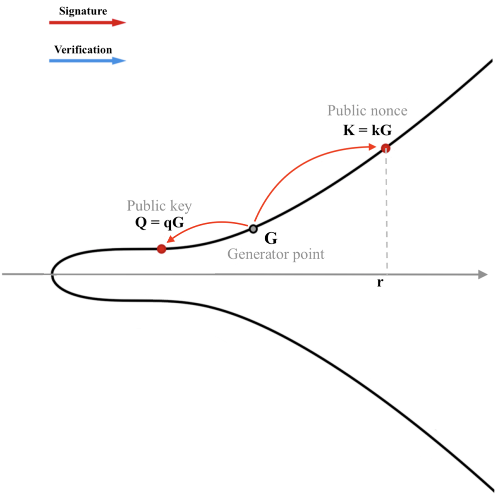

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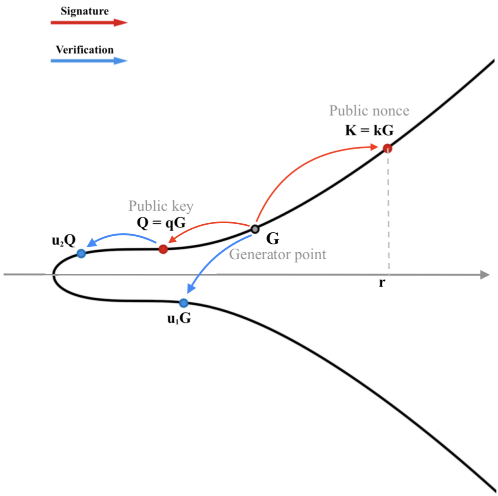

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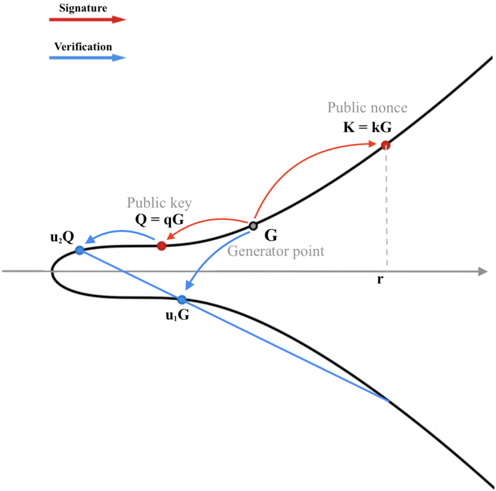

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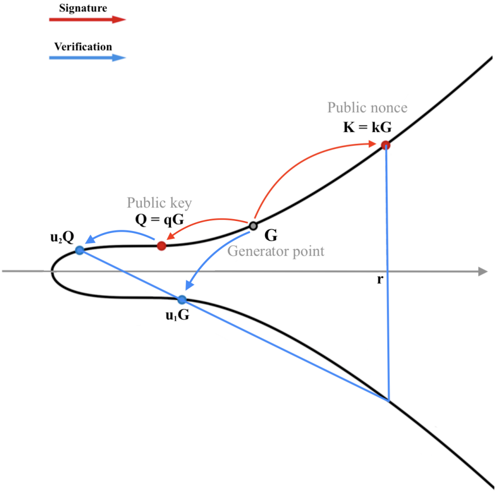

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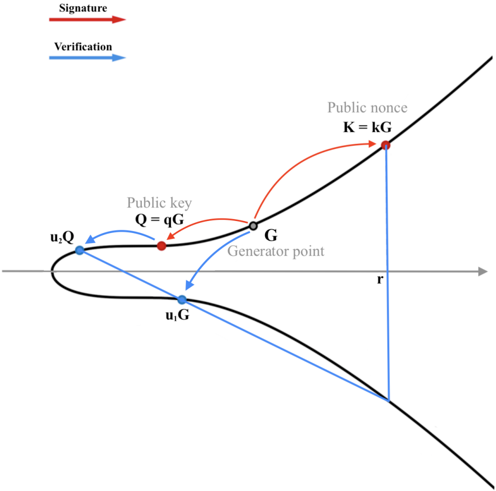

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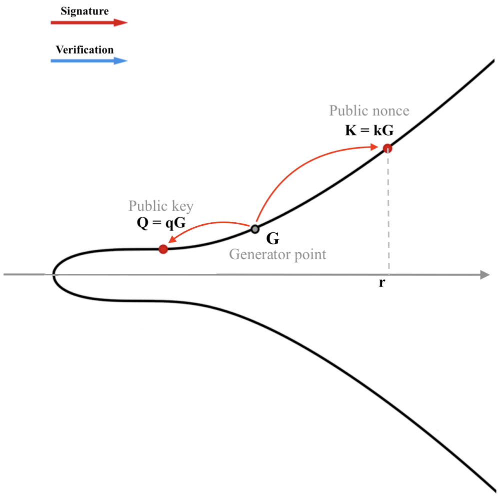

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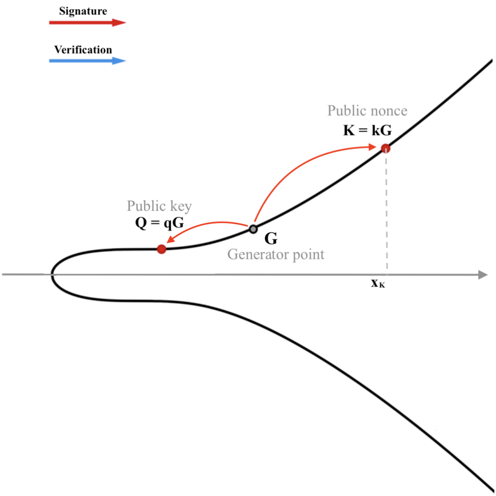

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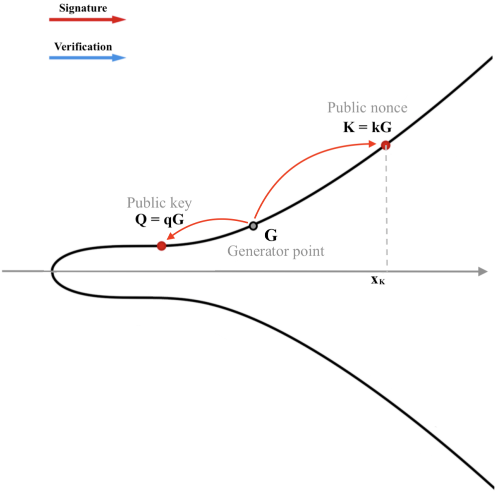

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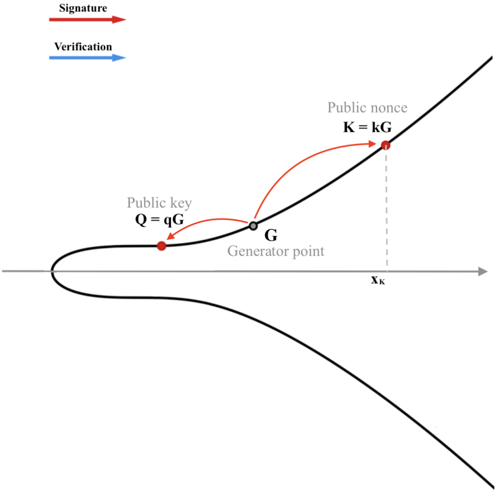

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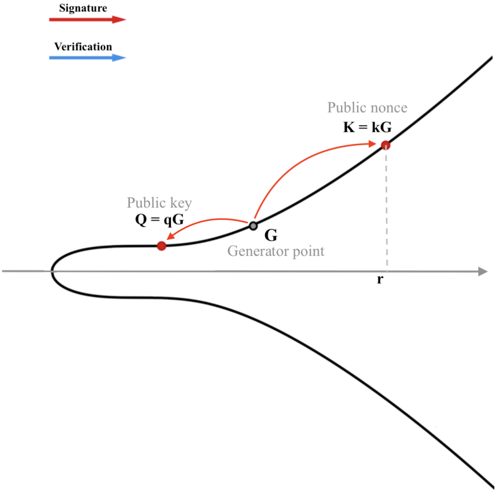

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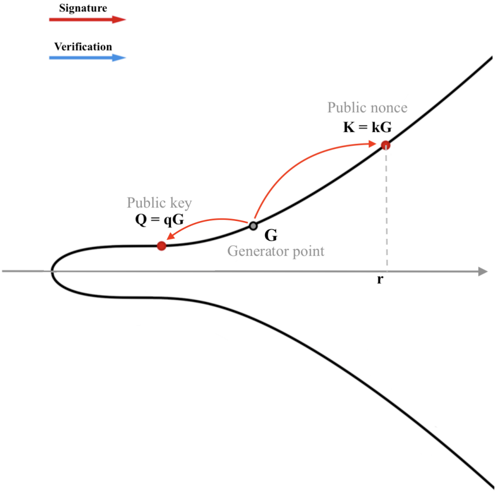

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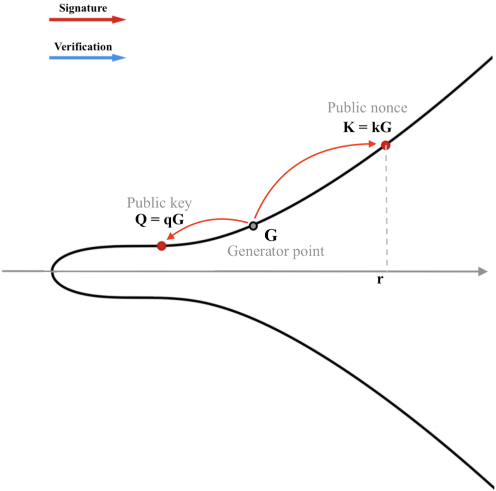

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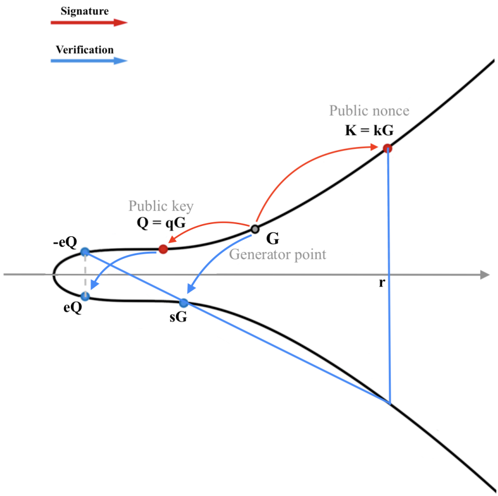

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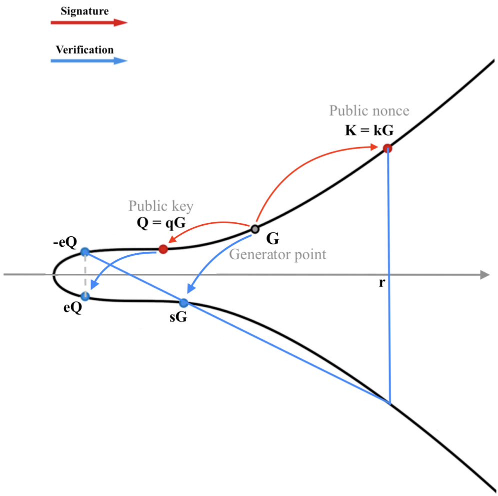

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- Batch validation scales logarithmically;
- Linear: easier higher level constructions.

Outline

Bitcoin's smart contracts

Bitcoin has been conceived as programmable money: the funds are locked by a smart contract that embeds the spending conditions.

In conjunction with signatures, they enforce the property right in the digital realm: this is why signatures are typically necessary to spend bitcoins and are required by the unlocking sripts.

An easy example: Pay-to-Public-Key (P2PK)

- Locking script: <pubKey> OP_CHECKSIG
- Unlocking script: <sig>

Multi-signature schemes

Multi-signature schemes allow a group of users to cooperate to sign a single message: they are fundamental in real life applications.

ECDSA multi-signature (*t*-of-*m*)

- Locking script: t <pubKey1> <pubKey2> ... <pubKeym> m OP_CHECKMULTISIG
- Unlocking script: 0 <sig1> <sig2> ... <sigt>

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Schnorr multi-signature (2-of-2) implemented naively:

- Alice ($\{q_A, Q_A\}$) and Bob ($\{q_B, Q_B\}$) generate K_A and K_B;
- They exchange them and set the public nonce at $K = K_A + K_B$. The joint public key is set at $Q = Q_A + Q_B$;
- Their partial signatures are: $s_i = k_i + \text{hash}(x_K || Q || msg) q_i \pmod{n}, \ i \in \{A, B\}$;
- The signature $(x_K, s_A + s_B \pmod{n})$ is valid for msg and Q.

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INSECURE: rogue key attack!

MuSig: compact m-of-m signature scheme

To solve the problem without resorting to the Knowledge Of Secret Key (KOSK) assumption, the idea is to introduce a “random factor” (ROM) $a_i = \text{hash}(\{Q_1, ..., Q_m\}||Q_i)$ per public key Q_i:

$$Q = \quad Q_1 + \quad Q_2 + ... + \quad Q_m.$$

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$$Q = a_1 Q_1 + a_2 Q_2 + ... + a_m Q_m.$$

$$s_i = k_i + \text{hash}(x_K||Q||msg) a_i q_i \pmod{n}, \;\; i = 1, ..., m.$$

The signature $(x_K, s = \sum_{i=1}^{m} s_i \pmod{n})$ can be verified as a simple Schnorr signature against Q:

$$sG = \left(\sum_{i=1}^{m} k_i + \text{hash}(x_K||Q||msg) \sum_{i=1}^{m} a_i q_i \right) G =$$

$$= \sum_{i=1}^{m} K_i + \text{hash}(x_K||Q||msg) \sum_{i=1}^{m} a_i Q_i =$$

$$= K + \text{hash}(x_K||Q||msg) Q.$$

MuSig: compact m-of-m signature scheme

- Compact: same size as the single user case;
- Secure in the plain public key model: allows signature aggregation at transaction level;
- Interactive: affects usability, prevents signature aggregation at block level;
- Key aggregation: signature indistinguishable from the single user case.

The multi-signature policy is completely hidden: this is a huge improvement for both privacy and efficiency.

Threshold signature scheme (t-of-m)

- Pedersen verifiable secret sharing scheme:
  - A dealer chooses secret $r \in \{1, ..., n-1\}$;
  - He embeds it in a random polynomial of degree $t-1$: $f(u) = r + f_1 u + ... + f_{t-1} u^{t-1} \pmod{n}$, $f_1, ..., f_{t-1} \xleftarrow{\$} \{0, ..., n-1\}$;
  - Each participant $i = 1, ..., m$ of the scheme receives the share $s_i = f(i)$;
  - The secret is reconstructed by any coalition $\mathcal{P}$ with $|\mathcal{P}| = t$ via Lagrange's interpolation formula: $r = f(0) = \sum_{i \in \mathcal{P}} s_i \prod_{h \in \mathcal{P}, h \neq i} \frac{h}{h-i} \pmod{n}$.

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    $r = f(0) = \sum_{i \in \mathcal{P}} s_i \prod_{h \in \mathcal{P}, h \neq i} \frac{h}{h-i} \pmod{n}$.
- Protocol for the generation of a shared random secret: every participant acts as the dealer in the previous protocol with secret r_i and sharing polynomial $f_i(u)$:
  - Shared secret: $r = \sum_{i=1}^{m} r_i \pmod{n} \implies \{r, R\}$;
  - Global sharing polynomial: $F(u) = \sum_{i=1}^{m} f_i(u) \pmod{n}$;
  - Share of the secret belonging to i:
    $s_i = \sum_{j=1}^{m} f_j(i) \pmod{n} = F(i)$.

Threshold signature scheme (t-of-m)

- The protocol for the generation of a shared random secret is run twice to establish a key pair $\{q, Q\}$ (shares $\alpha_i = F_1(i)$, $i = 1, ..., m$) and a nonce pair $\{k, K\}$ (shares $\beta_i = F_2(i)$, $i = 1, ..., t$);
- Each signer $i = 1, ..., t$ computes his partial signature as: $\gamma_i = \beta_i + e\alpha_i \pmod{n}$, $e = \text{hash}(x_K||Q||msg) \pmod{n}$;
- The signature is (x_K, s), with $s = \sum_{i=1}^{t} \gamma_i \prod_{h \neq i} \frac{h}{h-i} \pmod{n}$.

s computed in this way satisfies $s = k + eq \pmod{n}$:

$$F_3(u) := F_2(u) + eF_1(u) \implies$$

$$\implies s := F_3(0) = F_2(0) + eF_1(0) = k + eq \pmod{n}.$$

$$F_3(i) = F_2(i) + eF_1(i) = \beta_i + e\alpha_i \pmod{n} = \gamma_i.$$

ECDSA vs. ECSSA (multi-signature)

ECDSA

- Locking script: t <pubKey1> <pubKey2> ... <pubKeym> m OP_CHECKMULTISIG
- Unlocking script: 0 <sig1> <sig2> ... <sigt>

$\Longrightarrow$ 33 bytes * m + 70 bytes * t.

ECSSA

- Locking script: <jointPubKey> OP_SCHNORR
- Unlocking script: <jointSig>

$\Longrightarrow$ 33 bytes + 64 bytes.

Outline

Conclusion

We have seen that Schnorr would result in huge privacy (fungibility) and efficiency (scalability) improvements for Bitcoin:

- Batch validation: a group of signatures could be validated much faster;
- Smaller key size, aggregation at transaction level, and hidden policies;
- Improved security, being Schnorr provably secure.

Conclusion

We have seen that Schnorr would result in huge privacy (fungibility) and efficiency (scalability) improvements for Bitcoin:

- Batch validation: a group of signatures could be validated much faster;
- Smaller key size, aggregation at transaction level, and hidden policies;
- Improved security, being Schnorr provably secure.

But there is even more:

- Adaptor signatures: they aim at reducing the verbose scripting semantics to a fixed size signature with huge benefits for privacy and efficiency in protocols like atomic swaps and the Lightning Network;
- Taproot: built on top of the concepts of Merkelized Abstract Syntax Tree (MAST) and Pay-To-Contract, its aim is to make any arbitrary script to look the same as a single signer transaction in the cooperative case.

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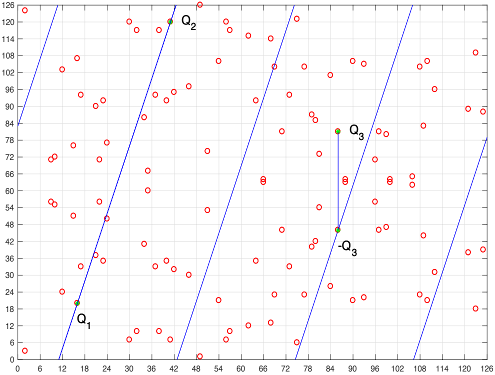
Point addition - Geometric interpretation

The intuition to add points belonging to an EC defined over a finite field is the same presented for the curve defined over the real numbers: draw the line passing through the points (that repeats along the plane in this case) until it intersects a third point: then reflect this point with respect to the line $y = \frac{p}{2}$, i.e. apply the transformation $(x_3, y_3) \rightarrow (x_3, p - y_3)$.



The curve $y^2 = x^3 - x + 3$ over $\mathbb{F}_{127}$.

Point addition - Algebraic formulas

There are some cases to be considered:

- $Q_2 = -Q_1$: by definition we have $Q_1 + Q_2 = \infty$, where ∞ is the identity element of the addition operation;
- $Q_2 = \infty$: $Q_1 + Q_2 = Q_1$;
- $Q_2 = Q_1$: $x_3 = m^2 - 2x_1$ and $y_3 = m(x_1 - x_3) - y_1$, where $m = \frac{3x_1^2 + a}{2y_1}$;
- $Q_2 \neq \pm Q_1$: $x_3 = m^2 - x_1 - x_2$ and $y_3 = m(x_1 - x_3) - y_1$, where $m = \frac{y_2 - y_1}{x_2 - x_1}$.

Double and add algorithm

Scalar multiplication is the core of ECC due to its computational asymmetry:

- The direct operation $q \mapsto Q$ can be made efficiently (i.e. there exist some polynomial time algorithms);
- The inverse operation $Q \mapsto q$ in general cannot be made efficiently (i.e. do not exist sub-exponential algorithms).

Double and add algorithm: $q = 41$

We decompose q according to its binary representation:

$$41 = 1 + 8 + 32 \implies 41G = G + 8G + 32G.$$

5 point doubling and 2 additions vs. 40 additions.

Batch validation

A signature (K, s) is valid if $K = sG - \text{hash}(x_K \mid\mid Q \mid\mid m)Q$.
Thus, two valid signatures (K_0, s_0) and (K_1, s_1) satisfies:

$K_0 + K_1 = (s_0 + s_1)G - \text{hash}(x_{K_0} \mid\mid Q_0 \mid\mid m_0)Q_0 - \text{hash}(x_{K_1} \mid\mid Q_1 \mid\mid m_1)Q_1.$

Insecure (cancelation attack): introduction of random factors.

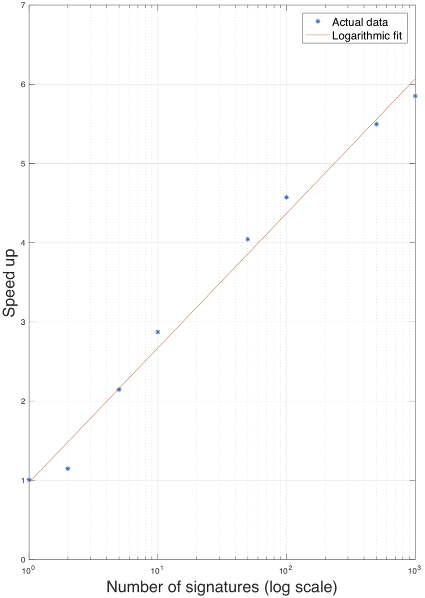
$$a_0 K_0 + a_1 K_1 =$$

$= (a_0 s_0 + a_1 s_1)G - a_0\text{hash}(x_{K_0} \mid\mid Q_0 \mid\mid m_0)Q_0 - a_1\text{hash}(x_{K_1} \mid\mid Q_1 \mid\mid m_1)Q_1.$

Batch validation - Bos-Coster's algorithm

$$a_0 K_0 + a_1 K_1 =$$
$$= (a_0 - a_1)K_0 + a_1(K_0 + K_1).$$

- Sort the tuples (a_i, K_i) according to a_i in descending order;
- While the list has length larger than one:
  - Substitute (a_0, K_0) and (a_1, K_1) with $(a_0 - a_1, K_0)$ and $(a_1, K_0 + K_1)$;
  - Sort the list again;
- When only one element remains, with very large probability it will be of the form $(1, K)$, otherwise it will be of the form (a, K).



Adaptor signatures

Building block for *scriptless script*: aim at encapsulating the flexibility of script semantics in fixed size signatures.

The idea is to add to the public nonce K a random $T = tG$ (public adaptor) but still consider k as private nonce: this results in an invalid signature, however learning t (private adaptor) is equivalent to learn a valid signature:

$$(x_T, x_{K+T}, \tilde{s}),\ \tilde{s} = k + \mathrm{hash}(x_{K+T}||Q||m)q \pmod{n}.$$

$$\text{Consistency equation: } \tilde{s}G = K + \mathrm{hash}(x_{K+T}||Q||m)Q.$$

Adaptor signatures can be used in conjunction with the MuSig protocol: a signer generates $s_i' = k_i + \mathrm{hash}(x_K||Q||msg)a_i q_i \pmod{n}$ after having used $K_i + T$ as public nonce.

Cross-chain atomic swaps

Exchange of different crypto-currencies among two distrustful users in an atomic and decentralized way.

Nowadays, this is achieved via Hashed TimeLock Contract:

HTLC

- Locking script:

  OP_IF

      OP_HASH256 <digest> OP_EQUALVERIFY OP_DUP

      OP_HASH160 <Bob address>

  OP_ELSE

      <num> OP_CHECKSEQUENCEVERIFY OP_DROP

      OP_DUP OP_HASH160 <Alice address>

  OP_ENDIF

  OP_EQUALVERIFY OP_CHECKSIG
- Unlocking script:
  - <Bob sig> <Bob pubkey> <preimage> 1
  - <Alice sig> <Alice pubkey> 0

Cross-chain atomic swaps via adaptor signature

- Alice and Bob agree on a pair of transactions which are secured by the pairs of keys (Q_1^A, Q_1^B) and (Q_2^A, Q_2^B), aggregated through the MuSig protocol;
- They engage the MuSig protocol to spend from the two transactions, but Bob generates in parallel adaptor signatures for both transactions: Alice has to verify them, in particular she needs to ensure that he used the same t value;
- This results in two invalid signatures: but Bob, knowing t, can build the corresponding valid signature;
- When he finally takes Alice's coins, he publish this valid signature on chain: Alice, that has to monitor the blockchain, learns it. But since she has the corresponding adaptor signature she can extract t and take Bob's coins.

Cross-chain atomic swaps via adaptor signature

Efficiency and privacy gains:

- We were able to condensate the verbose script semantics of the HTLC in a signature;
- The policy is indistinguishable from the single user setting (adaptor signatures are deniable: for every signature on the blockchain one can come up with some t and construct an adaptor signature);
- It is impossible to link the two transactions.