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Restricted coloring models for timetabling

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This note is dedicated to Claude Berge as a sign of gratitude for having taught to many of us the alphabet of graph theory

> Always **B**erge **C**onjectures **D**id **E**mulate a **F**low of **G**raphs. **H**e **I**nspired the **J**oyful **K**lique of **L**earned **M**athematicians. **N**ew **O**pen **P**roblems and **Q**uestions were **R**aised in **S**timulating **T**alks of **U**nexpected and **V**igorous **W**orth.
>
> **XYZ**

Abstract

We consider some node coloring problems with additional requirements which occur in timetabling and in chromatic scheduling; in these colorings, each node v must get one color chosen in a set $\varphi(v)$ of feasible colors and the cardinalities of the color classes must not exceed some given bounds. We characterize the cases where the constraint matrix is perfect, balanced or totally unimodular and we review some results in the area as well as extensions and variations.

Keywords: Chromatic scheduling; Perfect graphs; Balanced matrices; Totally unimodular matrices; Restricted coloring

1. Introduction

In this note we shall examine some coloring models and we will introduce some types of requirements which are present in several types of timetabling or of chromatic scheduling.

The first extension is the idea of restricted coloring (or list coloring) of a graph: it has allowed the introduction of unavailabilities or of preassignment in graph coloring models dedicated to the solution of timetabling or course scheduling problems.

We shall first consider this type of requirement and we will review mathematical programming formulations of these problems; integrality properties of the solution will then be derived from the structure of the constraint matrix.

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The reader is referred to [1, 3] for all graph-theoretical terms not defined here; some proofs which are not given here can be found in [6, 4].

2. Some preliminaries

Given a graph $G = (V, E)$ and a set C of $k = |C|$ colors, a *node k-coloring* of G is an assignment of one color of C to each node in such a way that adjacent nodes have distinct colors.

For solving course scheduling problems, we associate colors with the periods of the week and courses are represented by the nodes of a graph; conflicting courses (i.e. courses which cannot be scheduled at the same period because they involve the same teacher, or the same students for instance) are linked by edges. A feasible schedule in k periods is then associated with a node k-coloring of the graph.

Some courses may only be scheduled at one of a prespecified set of periods; this additional requirement may be handled via *restricted node colorings*: for each node $v \in V$, let $\varphi(v) \subseteq C$ be a given set of feasible colors for v. The restricted node coloring problem (G, φ) is to find a node coloring such that each node v gets a color $f(v) \in \varphi(v)$.

A $(0, 1)$ matrix is *balanced* if it does not contain a square submatrix of odd order such that each row and each column contains exactly two 1's.

We refer the reader to [2, 8] for the properties of balanced matrices.

Besides balanced matrices, we shall also refer to *totally unimodular* (t.u.) matrices: $a(0, +1, -1)$ matrix A is *t.u.* if the determinant of every square submatrix of A has value 0, $+1$ or -1.

We refer the reader to [12] for properties of polyhedra with t.u. constraint matrices.

Let us now formulate the restricted coloring problem in terms of the clique-node incidence matrix A of $G = (V, E)$ with $|V| = n$ and $C = \{1, 2, \ldots, k\}$. Here the cliques are supposed to be (inclusionwise) maximal. Following the developments in [9, 6], we introduce for each color c a vector $\boldsymbol{x}^c = (x(1, c), \ldots, x(n, c))$ such that $x(v, c) = 1$ if node v gets color c and $x(v, c) = 0$ else.

We may now write

$$\begin{aligned} \max z = {} & \mathbf{1} \cdot \boldsymbol{x}^1 + \cdots + \mathbf{1} \cdot \boldsymbol{x}^k \\ \text{s.t.} \quad & A\boldsymbol{x}^c \leqslant \mathbf{1} \quad (c \in C), \\ & \sum_{c=1}^{k} x(v, c) \leqslant 1 \quad (v \in V), \\ & x(v, c) \in \{0, 1\} \quad (c \in C,\ v \in V). \end{aligned} \tag{2.1}$$

Clearly, (2.1) may be viewed as a problem of finding a maximum stable set of nodes in a graph $\mathscr{G} = G + K_k$ obtained from $G = (V, E)$ by taking k copies $G^1, \ldots, G^k$ of G (G^i has a node set $V_i = \{(v, i) | v \in V\}$ and an edge set $E_i = \{[(v, i), (w, i)]: [v, w] \in E]\}$) and constructing a clique on nodes $(v, 1), (v, 2), \ldots, (v, k)$ for each $v \in V$.

We shall denote by $\mathscr{A}(G,k)$ the constraint matrix of (2.1). Now after removing all variables $x(v,c)$ corresponding to pairs (v,c) for which $c \notin \varphi(v)$, we may call $B(G,\varphi)$ the remaining constraint matrix.

The coloring problem ~~now~~ becomes

$$\max z = \mathbf{1} \cdot \boldsymbol{x}$$

$$\text{s.t.} \quad B(G,\varphi)\boldsymbol{x} \leqslant \mathbf{1}, \quad x \in \{0,1\}^p, \tag{2.2}$$

where

$$p = \sum_{v \in V} |\varphi(v)|.$$

There is a restricted node k-coloring if and only if (2.2) has a feasible $(0,1)$ solution $\boldsymbol{x}$ with $\mathbf{1} \cdot \boldsymbol{x} = |V|$.

By replacing the integrality conditions on $\boldsymbol{x}$ by $\boldsymbol{x} \geqslant \mathbf{0}$ we obtain an LP relaxation of (2.2). Its dual is

$$\min w = \sum_{K \in \mathscr{K}} \sum_{c \in C} y(K,c) + \sum_{v \in V} y(v)$$

$$\text{s.t.} \quad \sum_{K : K \ni v} y(K,c) + y(v) \geqslant 1 \quad \begin{pmatrix} c \in \varphi(v) \\ v \in V \end{pmatrix},$$

$$y(K,c) \geqslant 0 \quad \begin{pmatrix} c \in C \\ K \in \mathscr{K} \end{pmatrix}, \tag{2.3}$$

$$y(v) \geqslant 0 \quad (v \in V).$$

Here, $\mathscr{K}$ is the family of all (inclusionwise maximal) cliques K of G. Let Γ be a subset of pairs (K,c); we shall say that a node v of G is *covered* by Γ if for each color $c \in \varphi(v)$, there is at least one clique K containing node v for which $(K,c) \in \Gamma$. Let $V(\Gamma)$ be the set of nodes of G which are covered by Γ.

A *Hall certificate* of non-colorability for (G,φ) is a set Γ of pairs (K,c) such that $|\Gamma| < |V(\Gamma)|$. In [9,6], characterizations have been given of graphs G with the property that for every possible φ, (G,φ) has either a restricted coloring or a Hall certificate. Notice that there are graphs and feasible sets φ such that (G,φ) has neither a coloring nor a certificate.

Clearly, the restricted coloring problem is generally NP-complete [13] since it contains as a special case the node k-coloring problem (when for each node v we have $\varphi(v) = \{1,\ldots,k\} = C$).

3. Some solvable cases of restricted coloring

Perfect matrices are defined as the clique-node matrices of perfect graphs [3]. It is known that A is perfect if and only if $\{\boldsymbol{x} \mid A\boldsymbol{x} \leqslant \mathbf{1},\ \boldsymbol{x} \geqslant \mathbf{0}\}$ has only integer extreme points. If G is a perfect graph, we may ask when is the restricted k-coloring problem on (G,φ) solvable in polynomial time?

Before recalling results in [9, 14], we define *block graphs* as the graphs where each block (maximal two-connected component) is a clique.

Proposition 3.1. *Let G be a simple graph and $C = \{1, 2, \ldots, k\}$ a set of $k \geqslant 3$ colors. Then the following statements are equivalent:*
(1) *G is a block graph;*
(2) *$\mathscr{A}(G, k)$ (and hence $B(G, \varphi)$) is perfect,*
(3) *for every φ, (G, φ) has either a solution or a Hall certificate.*

The equivalence of (1) and (2) was shown in [11]. With other techniques, it was rediscovered in [5]. In [9], equivalence with (3) was shown. As a consequence of this result, the restricted coloring problem can be solved in polynomial time by linear programming: the LP relaxation of (2.1) has a (0, 1) solution $\boldsymbol{x}$ which is optimal; it can be obtained in polynomial time (see [10]). In [9], Gröflin gives an algorithm in $O(|V|^4|C|)$ which is combinatorial.

We may now give a refinement of a result in [6] by restricting the graph G to a line graph (of a connected graph).

Proposition 3.2. *Let G be a connected line graph and $C = \{1, 2, \ldots, k\}$ a set of $k \geqslant 3$ colors. Then the following statments are equivalent:*
(4) *G is the line graph of a tree;*
(5) *$\mathscr{A}(G, k)$ (and hence $B(G, \varphi)$) is t.u.;*
(6) *for every $\varphi(G, \varphi)$ has either a solution or a Hall certificate.*

The proof can be found in [4].

It is appropriate to recall another statement which is stated in [14].

Proposition 3.3. *For a connected graph G, the following statements are equivalent:*
(7) *G is the line graph of a tree;*
(8) *$\mathscr{A}(G, k)$ is t.u. for any $k \geqslant 3$*
(9) *$\mathscr{A}(G, k)$ is balanced for any $k \geqslant 3$*

It is worth mentioning at this stage that for solving efficiently the restricted coloring problem when G is the line graph of a tree, we have a simple algorithm based on dynamic programming [6] with complexity $O(n'k(\Delta + k)^{2.5})$. Here $n' = 1 +$ number of nodes in the tree with degree > 1, k is the number of colors and Δ is the maximum degree.

Notice that the problem amounts to finding an edge k-coloring of a tree where each edge e may receive one color in a set $\varphi(e)$ of feasible colors (restricted edge k-coloring problem).

It is also important to observe that we may as well consider multitrees (i.e. trees with multiple edges); if in each family of parallel edges any two edges e, f satisfy $\varphi(e) = \varphi(f)$, then the formulation (2.1) may be adapted as follows: A is the node-edge incidence matrix of the tree $T = (V, E)$; each $e \in E$ has 2 multiplicity $m_e \geqslant 1$ (e corresponds to a family of m_e parallel edges).

We introduce vectors $\boldsymbol{x}^c = (x(e_1, c), \ldots, x(e_m, c))$ where

$$x(e,c) = \begin{cases} 1 & \text{if one edge in the family associated to } e \text{ gets color } c, \\ 0 & \text{otherwise.} \end{cases}$$

We have the following:

$$\begin{aligned} &\max z = \mathbf{1} \cdot \boldsymbol{x}^1 + \cdots + \mathbf{1} \cdot \boldsymbol{x}^k \\ &\text{s.t.} \quad A\boldsymbol{x}^c \leqslant \mathbf{1} \quad (c \in C), \\ &\qquad \sum_{c=1}^{k} x(e,c) \leqslant m_e \quad (e \in E), \\ &\qquad x(e,c) \in \{0,1\} \quad \begin{pmatrix} c \in \varphi(e) \\ e \in E \end{pmatrix}. \end{aligned} \tag{3.1}$$

Now $\mathscr{A}(G,k)$ is *t.u.* according to Proposition 3.2; so is $B(G,\varphi)$ and the LP relaxation corresponding to (2.2) with the additional requirement $x(e,c) \leqslant 1$ and some changes on the right-hand side (due to the values m_e) has an optimal solution which is integral.

It corresponds to a restricted edge k-coloring of the multitree if $z = \sum(m_e : e \in E)$; otherwise, there is no such coloring. Observe that the certificate does not have the same simple form as before. The complexity of the algorithm of [6] is now $O(n'k^M(\Delta + k)^{2.5})$ where $M = \max\{m_e : e \in E\}$.

Remark 3.1. If parallel edges are allowed to have different sets of feasible colors, then the above formulation is no longer valid; we must have variables $x(e, c)$ for each edge e and for each color c. It is easy to see (by doubling the middle edge of a chain of three edges) that the resulting matrix $\mathscr{A}(G,k)$ may no longer be perfect.

4. Some remarks on restricted edge colorings

The simplest versions of timetabling problems are often formulated as edge k-coloring problems in bipartite multigraphs $G = (\mathscr{C}, \mathscr{T}, E)$ where $\mathscr{C}$ is a set of classes c_i (a class is a group of students taking the same curriculum), $\mathscr{T}$ a set of teachers t_j and E a collection of meetings $[c_i, t_j]$ involving a class c_i and a teacher t_j.

A family of m_e parallel edges between c_i and t_j represents m_e one-hour meetings of c_i and t_j. In many situations we have a set $C = \{1, \ldots, k\}$ of periods (h) and for each collection e of meetings $[c_i, t_j]$ some set $\varphi(e)$ of periods where they could be scheduled. Notice that these requirements are associated to the edges: they are called *edge-constrained* (EC). So (G, φ) is an EC restricted edge k-coloring problem.

A more special case is when the only restrictions in the colors (i.e. in the possible periods for meetings) are due to the unavailabilities of classes and teachers: each node

v in G has a set $\varphi(v)$ of periods (colors) where its meetings can be scheduled. So for each meeting $[c_i, t_j]$ we have $\varphi([c_i, t_j]) = \varphi(c_i) \cap \varphi(t_j)$; these requirements are called *node-constrained* (NC). So this problem is a NC-restricted edge k-coloring problem. It is a special case of the previous EC problem. For instance, if a graph G has edges ab, bc, cd with $\varphi(ab) = \varphi(cd) = 1$, $\varphi(bc) = 2$, it is EC but not NC. If it were NC, we should have $\varphi(bc) = \{1, 2\}$ because both nodes b and c have 1 as a feasible color ($\varphi(ab) \ni 1$, $\varphi(cd) \ni 1$).

Notice that when all feasible sets $\varphi([c_i, t_j])$ have been derived by taking the intersection $\varphi(c_i) \cap \varphi(t_j)$ in an NC problem, we may replace $\varphi(c_i)$ and $\varphi(t_j)$ by the entire color set $C = \{1, \ldots, k\}$, so we have no more constraints on the nodes.

It follows from results in [7] that the EC-restricted edge k-coloring problem is NP-complete for bipartite multigraphs. Furthermore, using a formulation of timetabling without reference to graphs, it was proved in [7] that the NC restricted edge k-coloring problem is NP-complete even if the constraints bear on the teachers only (i.e. $\varphi(c_i) = C = \{1, \ldots, k\}$ for each class c_i).

We shall strengthen this result. Let us also recall that in [7], the NC-restricted edge k-coloring problem was shown to be solvable in linear time if $\varphi(c_i) = \{1, \ldots, k\}$ for each class c_i and $|\varphi(t_j)| \leqslant 2$ for each teacher t_j.

Let now $G = (\mathscr{C}, \mathscr{T}, E)$ be a bipartite multigraph, let $k \geqslant \Delta(G)$ and let $\varphi(c_i)$, $\varphi(t_j)$ be given sets of feasible colors for each $c_i \in \mathscr{C}$ and each $t_j \in \mathscr{T}$. Is there a NC-restricted edge k-coloring of G?

This problem is called NCEC (k, G, φ).

Proposition 4.1. NCEC (k, G, φ) *is* NP-*complete even if the restrictions bear only on nodes t_j having at most* 2 *neighbors in G.*

Proof. We shall transform the EC-restricted edge k-coloring problem defined on a bipartite multigraph G to a NCEC(k, G', φ) where $\varphi(t_j) \neq C = \{1, \ldots, k\}$ only for some teacher nodes with at most two neighbors in G'.

Consider an edge $e = [c_i, t_j]$ in G with a given set $\varphi(e)$ of feasible colors. Introduce new nodes $t_j^e, c_i^e, \bar{t}_i^e$ and replace edge e by an edge $e' = [c_i, t_j^e]$ and an edge $e'' = [c_i^e, t_j]$. Link t_j^e and c_i^e by $|\varphi(e)| - 1$ parallel edges $[t_j^e, c_i^e]_p$ $(p = 1, \ldots, |\varphi(e)| - 1)$. Introduce furthermore $k - |\varphi(e)|$ parallel edges $[\bar{t}_i^e, c_j^e]_r$ $(r = 1, \ldots, k - |\varphi(e)|)$. The construction is illustrated in Fig. 1.

Define $\varphi'(t_j^e) = \varphi(e)$, $\varphi'(\bar{t}_i^e) = C - \varphi(e)$.

Repeat this for every edge e; the resulting multigraph $G' = (\mathscr{C}', \mathscr{T}', E')$ is bipartite and satisfies $\Delta(G') = k$. We may set $\varphi'(v) = C$ for all nodes v different from the nodes $t_j^e, \bar{t}_i^e$ $(\forall i, j)$. So we have an NCEC (k, G', φ') where the restricted nodes have at most two neighbors.

G' can be constructed in polynomial time: we introduce $3|E|$ new nodes and $4|E|$ families of parallel edges (each one has at most k edges). One may check that in any feasible k-edge coloring of G', the edges $[c_i, t_j^e]$ and $[c_i^e, t_j]$ have the same color. Furthermore, there is a feasible edge k-coloring in G' if and only if there exists one in

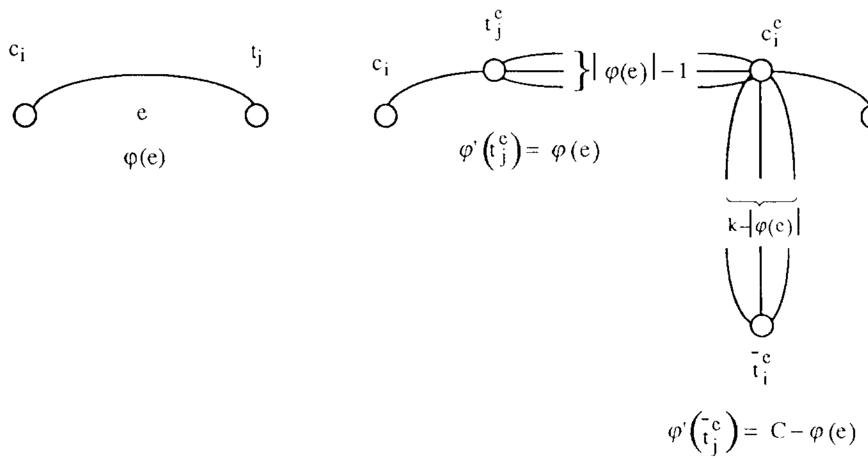

Fig. 1. Construction of G' in the proof of Proposition 4.1.

G. This proves that NCEC(k, G, φ) is NP-complete even with the hypothesis that only teacher nodes with at most two neighbors are restricted. □

Notice that teacher nodes with at most two neighbors may have more than two lectures to give. Compare this with the solvable case in [7] described above.

5. Cardinality constrained color classes

Let us now examine an additional requirement which is often present in chromatic scheduling problems. Given a graph $G = (V, E)$, a set $C = \{1, \ldots, k\}$ of colors and integers $h_1, \ldots, h_k$, we may ask whether there exists a node k-coloring with at most h_c nodes of color c for $c = 1, \ldots, k$. Related results are given in [11].

The problem can now be written as follows:

$$\max z = \mathbf{1}\cdot\boldsymbol{x}^1 + \cdots + \mathbf{1}\cdot\boldsymbol{x}^k$$

$$\text{s.t.} \quad \left.\begin{array}{l} A\boldsymbol{x}^c \leqslant \mathbf{1} \\ \mathbf{1}\cdot\boldsymbol{x}^c \leqslant h_c \end{array}\right\} \quad (c \in C), \tag{5.1}$$

$$\sum_{c=1}^{k} x(v, c) \leqslant 1 \quad (v \in V),$$

$$x(v, c) \in \{0, 1\}^{|V|\cdot|C|} \quad (v \in V,\ c \in C).$$

It is well known that this problem is NP-complete (even for line graphs of bipartite graphs). Let $\mathscr{D}(G, k)$ be the constraint matrix of (5.1). We may consider the restricted version of this problem by giving to each node v a set $\varphi(v)$ of feasible colors. As before let $D(G, \varphi)$ be the constraint matrix of the resulting problem. We may again ask for which graphs is the matrix $\mathscr{D}(G, k)$ balanced. The following result is derived in [4].

Proposition 5.1. *Let G be a simple graph and $k \geqslant 2$ an integer. The following statements are equivalent:*
(1) *G is a collection of node disjoint cliques;*
(2) *$\mathscr{D}(G, k)$ (and hence $D(G, \varphi)$) is t.u.*

In [4] it is shown that $\mathscr{D}(G, k)$ is balanced, total unimodularity follows from the fact that each column has at most three ones (see [4]).

As a consequence, the restricted k-coloring problem with cardinality constraints can be solved in polynomial time when G is a union of node-disjoint cliques. Again, combinatorial algorithms can be devised due to the special structure of the graph. Let us show that it is a flow problem.

Assume $G = (X, E)$ consists of cliques $K_1, K_2, \ldots, K_p$. We have $C = \{1, \ldots, k\}$ and we are given integers $h_1, h_2, \ldots, h_k$. We construct a network N as follows.

Introduce a source s, a sink t, a node x for each node x of G, nodes $\langle K_j, c\rangle$ for each clique K_i and each color c and a node c for each color c. The arcs are constructed as indicated in Table 1. The construction is illustrated in Fig. 2. We notice that there exists a node k-coloring satisfying all requirements if and only if there exists an integral compatible flow from s to t in N. Such a flow can be constructed with one of the classical algorithms. In case no such flow (with value $|X|$) can be found, the algorithm will provide a certificate of non-colorability.

From the max-flow–min-cut theorem (see [1]), we obtain a solution to the coloring problem if and only if every cut separating s and t in N has capacity at least $|X|$ where X is the node set of G. This translates as follows in N.

For any subsets A_i of nodes in K_i and for any subset C^* of colors in C, we have

$$\sum_{i=1}^{p} \left|\bar{A}_i\right| + \sum_{i=1}^{p} \left|\left(\bigcup_{x \in A_i} \varphi(x)\right) \cap (C - C^*)\right| + \sum_{c \in C^*} h_c \geqslant |X|.$$

Here $\bar{A}_i$ is the complement of A_i in K_i. The left-hand side counts the cardinality of subsets $\bar{A}_i$ in each K_i, the maximum number of nodes in all subsets A_i which could be

Table 1

Arcs (u, v)	Capacities $c(u, v)$	Lower bounds $l(u, v)$
(s, v) if $v \in V$	1	0
$(v, \langle K_i, c\rangle)$ if $v \in K_i$ and $c \in \varphi(v)$	∞	0
$(\langle K_i, c\rangle, c)$, $i = 1, \ldots, p$	1	0
(c, t), $c = 1, \ldots, k$	h_c	0

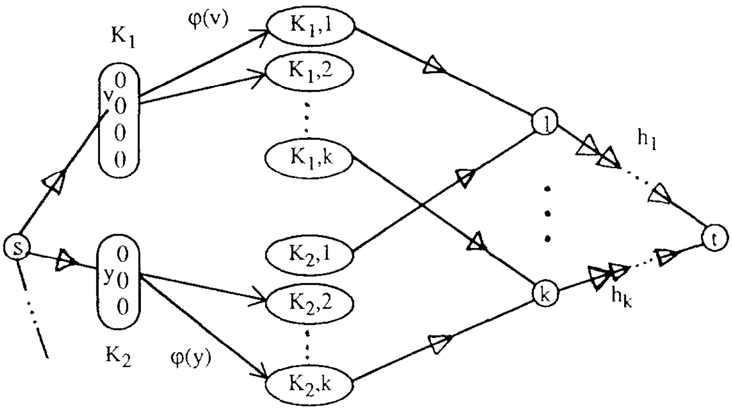

Fig. 2. The network N corresponding to the restricted coloring problem with constraints on cardinalities of color sets.

colored with feasible colors in $C - C^*$ and the maximum number of nodes which could be colored with colors in C^* due to cardinality constraints.

This sum must clearly be at least as large as $|X|$. The max-flow algorithm will give a minimum cut which will provide the certificate in case no solution of the coloring problem exists.

By setting $A = \bigcup_{i=1}^{p} A_i$ we can now state:

Proposition 5.2. *Given a graph $G = (X, E)$ consisting of node-disjoint cliques, feasible color sets $\varphi(x) \subseteq C = \{1, \dots, k\}$ for all nodes x and positive integers $h_1, h_2, \dots, h_k$, there exists a feasible node k-coloring satisfying the cardinality requirements (at most h_i nodes of color i) if and only if for any subset A of nodes and any subset C^* of colors:*

$$\sum_{i=1}^{p} \left| \left(\bigcup_{x \in A_i} \varphi(x) \right) \cap (C - C^*) \right| + \sum (h_c \colon c \in C^*) \geqslant |A|.$$

6. Concluding remarks

Our purpose here was to give a review of some of the mathematical programming formulations which may be useful in characterizing solvable cases of generally NP-complete extensions of coloring problems. It is clear that more direct approaches (based on combinatorial properties) are more powerful. The cases which can be solved in polynomial time are still far from the real situations encountered in timetabling; more solvable cases should be progressively identified. The case of trees or multitrees could give hints in this direction.

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