An Introduction to List Colorings of Graphs

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Thesis submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

Master of Science
in
Mathematics

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April 30, 2009
Blacksburg, Virginia

Keywords: Graph, List Coloring, Channel Assignment Problem

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(ABSTRACT)

One of the most popular and useful areas of graph theory is graph colorings. A graph coloring is an assignment of integers to the vertices of a graph so that no two adjacent vertices are assigned the same integer. This problem frequently arises in scheduling and channel assignment applications. A list coloring of a graph is an assignment of integers to the vertices of a graph as before with the restriction that the integers must come from specific lists of available colors at each vertex. For a physical application of this problem, consider a wireless network. Due to hardware restrictions, each radio has a limited set of frequencies through which it can communicate, and radios within a certain distance of each other cannot operate on the same frequency without interfering. We model this problem as a graph by representing the wireless radios by vertices and assigning a list to each vertex according to its available frequencies. We then seek a coloring of the graph from these lists.

In this thesis, we give an overview of the last thirty years of research in list colorings. We begin with an introduction of the list coloring problem, as defined by Erdős, Rubin, and Taylor in [6]. We continue with a study of variations of the problem, including cases when all the lists have the same length and cases when we allow different lengths. We will briefly mention edge colorings and overview some restricted list colors such as game colorings and $L(p, q)$-labelings before concluding with a list of open questions.

Acknowledgments

I would like to thank my parents for their unwavering support. I would also like to thank Dr. Rossi, Dr. Shimozono, and Dr. Brown for agreeing to be on my committee, but especially Dr. Brown for all of his help as an advisor. Finally, I would like to thank Dr. Matthews for introducing me to graph colorings as an undergraduate.

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Chapter 1

Introduction

Let $G = (V, E)$ be a graph. We denote the vertex set of G by $V(G)$ and the edge set of G by $E(G)$. The number of vertices is denoted $|V(G)|$ and similarly for the number of edges, $|E(G)|$.

A popular area of graph theory is the study of graph colorings. A *proper coloring* of a graph is a function $f : V(G) \to \mathbb{Z}^+$ such that for all $u, v \in V(G)$, $f(u) \neq f(v)$ if $uv \in E(G)$. We define the *chromatic number* of G, $\chi(G)$, to be the least positive integer k such that G has a proper coloring assigning the integers $\{1, 2, \ldots, k\}$ to $V(G)$. Furthermore, if k is any integer such that G has a proper coloring from the colors $\{1, 2, ..., k\}$, we say that G is *k-colorable*.

Let C be a set of colors, and for each $v \in V(G)$, let $L : V(G) \to 2^C$ be a function assigning to each vertex $v \in V(G)$ a list of colors $L(v) \subseteq C$. If there is a function $f : V(G) \to C$ such that $f(v) \in L(v)$ for all $v \in V(G)$ and $f(u) \neq f(v)$ for $uv \in E(G)$, then G is said to be *L-colorable* [4]. This defines the notion of a *list coloring*. Note that a graph coloring as previously defined is a special case of list colorings, namely the case when all the lists are the sets $\{1, 2, \ldots, \chi(G)\}$.

If k is a positive integer, the function L is such that $|L(v)| = k$ for all v in $V(G)$, and the graph G has a proper list coloring, then we say G is *k-choosable*, and we define the *choice number*, $\chi_L(G)$, to be the minimum such k so that G has a proper list coloring no matter what lists are assigned to the vertices of G. Note that k-choosable implies k-colorable, but not conversely as we will soon see. Much of the research in list colorings involves finding $\chi_L(G)$ for particular types of graphs or given various restrictions on the coloring rules. However, we can use any function $L : V(G) \to 2^C$ to assign lists to the vertices of a graph G, as we will see in Chapter 3.

List colorings were introduced in [6] by Erdős, Rubin, and Taylor, who were inspired by the following problem presented by Jeffrey Dinitz at the 10^{th} Southeastern Conference on Combinatorics, Graph Theory and Computing in 1979:

> "Given an $m \times m$ array of m-sets, is it always possible to choose one from each set, keeping the chosen elements distinct in every row, and distinct in every column?" [6]

We can model this problem as a list coloring problem by considering the graph G with $|V(G)| = m^2$, where each vertex corresponds to an entry in the array described above. Then the m-sets become the lists on each vertex, and for vertices u and v in G, uv is an edge in G if either the entries corresponding to u and v are in the same row or the same column. The goal is to find a proper coloring of G such that the color assigned to v comes from the m-set assigned to v. If a proper coloring can be chosen from the lists no matter what value m takes or what the m-sets are, then the answer to Dinitz's question is, "Yes." As it stands, however, this question is still unanswered.

The goal of this thesis is to introduce the reader to a variety of research done on the subject of list colorings. We begin with some fundamental results in the subject and some applications of list colorings. We then discuss graphs with known values of k for which they are k-

choosable. We continue with a consideration of non-constant L-functions and choosability and give a list coloring version of Brooks's Theorem. Finally, we close with an overview of restricted list coloring problems such as $L(p, q)$-labelings in the list coloring setting and a list of open problems.

1.1 Basic Results in List Colorings

We define a *bipartite graph*, $G[X, Y]$, to be a graph whose vertices are partitioned into two sets, X and Y, such that no two vertices of X share an edge, nor do any two vertices of Y; i.e. all edges of $G[X, Y]$ are of the form xy for some x in X and y in Y. See Figure 1.1 for an example. A *complete graph*, denoted K_n, is a simple graph with n vertices such that any two distinct vertices are adjacent. A *complete bipartite graph* is a bipartite graph, $G[X, Y]$, in which each x in X is adjacent to every y in Y.

As noted in [6], since the usual graph colorings are special cases of list colorings, we have $\chi(G) \leq \chi_L(G)$ for all graphs G. Erdős et al. proved with Figure 1.1 that strict inequalities exist. For all bipartite graphs, $\chi(G) = 2$, but in Figure 1.1, we see that each vertex has a list of length two as shown on the vertex, but there is no proper coloring possible from the assigned lists. Thus, for this graph, $\chi_L(G) > \chi(G)$. In fact, the authors also show that there is no general bound for how much $\chi_L(G)$ can exceed $\chi(G)$.

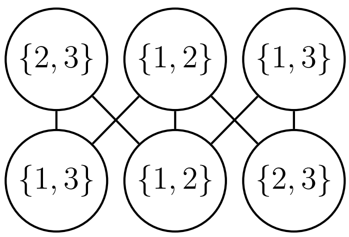


Figure 1.1: A bipartite graph that is not 2-choosable

Theorem 1 (Erdős, Rubin and Taylor, [6]). *There is no bound on how much* $\chi_L(G)$ *can* *exceed $\chi(G)$ as $|V(G)|$ increases.*

Proof. Let k be a positive integer. Let $m = \binom{2k-1}{k}$, and let $K_{m,m}$ be the complete bipartite graph with m vertices in each part. We will show that $K_{m,m}$ is not k-choosable.

$m = \binom{2k-1}{k}$ is the number of k-subsets of a set of size $2k-1$. We construct the complete bipartite graph $G = [X, Y]$ with m vertices in X and m vertices in Y. Assign each of the k-subsets to one vertex in X and to one vertex in Y.

We begin choosing colors for the vertices in X. We must use at least k colors for these vertices. For, if j is a nonnegative integer with $j < k$, and if we only use j colors on the vertices in X, we have $2k - 1 - j > 2k - 1 - k = k - 1$ colors left. Thus, we have at least k colors that we have not used, and these k colors are the list on some vertex in X, meaning that vertex remains uncolored.

Now the k colors we have used in X are precisely the k colors in the list of some vertex $y \in Y$. So we cannot complete the coloring. Thus $K_{m,m}$ is not k-choosable, as claimed.

Since $\chi(K_{m,m}) = 2$, and we have shown a construction that gives $\chi_L(K_{m,m}) > k$ for any positive integer k (and thus m) we choose, there is no bound on how much $\chi_L(G)$ can exceed $\chi(G)$ as $|V(G)|$ increases. □

A natural question, and perhaps the most common question, to ask is, "Given a graph G, for what values of k is G k-choosable?" This question is quite difficult to answer, as Rubin proves in [6] that finding a list coloring for a graph is NP-hard.

1.2 Applications of List Colorings

Graph colorings are useful to solve various problems ranging from scheduling [3] to the channel assignment problem. List colorings in particular arise naturally from applications with restrictions on which values can be assigned to certain objects. In this section, we offer a few examples of these problems and how list colorings are used to solve them.

In [12], Zeitlhofer and Wess describe using list colorings to determine register assignments for computing processes. They discuss a situation with multiple functional units–some capable of addition and multiplication, and some capable of only addition. List coloring is then used for instruction scheduling. For example, if an operation requires a multiplication, it cannot be assigned to an addition-only unit. Thus, each operation is assigned a list of units that can process it, and a list coloring problem emerges. A second list coloring problem arises when assigning registers to store intermediate values. The operations are assigned lists of registers depending on which registers the necessary computation units access, i.e. if an operation requires an addition followed by a multiplication, the intermediate value of the addition should not be stored in a register that is not accessed by a multiplication-capable unit. Solving these problems then determines an appropriate register assignment and instruction schedule for the desired computation. Because the list coloring problem is NP-hard, Zeitlhofer and Wess take advantage of certain properties of the graphs emerging from their registry assignment problem in order to find a coloring.

Another application of list colorings is the channel assignment problem. As Ramachandran, Belding, Almeroth, and Buddhikot discuss in [9], wireless networks near each other often interfere. Thus, to limit the interference and to satisfy hardware requirements, one must limit the frequencies available to a router. This situation is modeled as a list coloring problem. Ramachandran et al. describe assigning frequencies to a wireless mesh network built on a mixture of multi-radio and single-radio routers. Multi-radio routers have multiple wireless radios which can be tuned to nonoverlapping channels and communicate with several other radios. Single-radio routers, on the other hand, can only be tuned to one channel. In representing this problem as a graph theory problem, it is assumed that the hardware for the network is already in place, so that a network topology is given. Each radio then corresponds to a vertex. Thus, if a router has three radios, that router corresponds to three vertices. Each edge represents a wireless link between radios in the given network topology. Call this graph G. The authors then construct the Multi-radio Conflict Graph (MCG) where each edge in G becomes a vertex. Then if two wireless links in G interfere with each other, an edge is drawn between the vertices in MCG that represent those links. Lists of available frequencies are assigned to each vertex of MCG, and a proper coloring is sought. Ramachandran et al. cope with the difficulty of the list coloring problem by using a breadth-first search algorithm to assign channels to the vertices of MCG. Each network has a gateway router where the network connects to an external network. This router is chosen as the starting point of the breadth-first search since its connections are assumed to host the most traffic. One of the vertices of MCG corresponding to a wireless link with the gateway router will be the first colored, and the coloring will be extended to the rest of MCG in a way that minimizes interference [9]. Note that the list coloring problem presented here is actually a list coloring of the edges of G. Edge colorings will be discussed in more detail in Chapter 4.

Chapter 2

k-Choosability of Graphs

The simplest functions to understand are constant functions. For this reason, we first analyze k-choosability, for lists of constant length k, before moving on to list functions of nonconstant length in the next chapter. In this chapter, we completely characterize 2-choosable graphs and show that planar graphs are 5-choosable. As is often the case in graph theory, many of the proofs to follow are lengthy constructive arguments. We ask for the reader's patience, for the results are quite beautiful, and there is much to learn from the techniques of these proofs.

2.1 2-Choosable Graphs

In this section, we provide a characterization of all 2-choosable graphs which can be found in [6]. First, though, we must provide a few definitions.

Recall that *k-choosable* means that the list on each vertex has length k, and that from any such set of lists, the graph G may be properly colored.

If G has adjacent vertices u and v such that u and v have more than one edge between them, we call these edges *parallel edges*. Note that some texts use the term *multiple edges* and call G a *multigraph*. A *loop* is an edge that is self-incident. See Figure 2.1 for examples of these concepts. We say that G is *simple* if G has no parallel edges or loops.

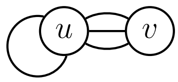


Figure 2.1: Parallel edges and a loop

We define the *core* of a graph G to be G with all vertices of degree one recursively removed. Note that these vertices may always be colored from lists of length two since they only have one neighbor to conflict with a coloring, leaving at least one color available for each of these vertices. Thus, if we can show the core of G is 2-choosable, we are done.

We define a *θ graph* to be a graph with two distinct vertices, u and v, with three vertex-disjoint paths between them. We identify a θ graph by the lengths of these paths. See Figure 2.2 for an example.

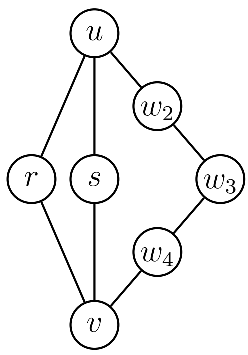


Figure 2.2: $\theta_{2,2,4}$

We are now ready to give a complete characterization of 2-choosable graphs. The following theorem and proof are due to Rubin and may be found in [6]. Note that G is only 2-choosable if every component of G is 2-choosable. Thus, we may assume G is a connected graph.

Theorem 2 (Rubin, [6]). *A graph G is 2-choosable if and only if the core of G is K_1, an even cycle, or of the form $\theta_{2,2,2k}$, where k is a positive integer.*

Proof. Sufficiency

We begin by showing K_1, even cycles, and $\theta_{2,2,2k}$ graphs are 2-choosable. First, since K_1 has only one vertex, it is certainly 2-choosable. Secondly, K_2 with a parallel edge forms a cycle of length two and is clearly 2-choosable. Thus, we assume we only have even cycles of length four or greater. We will next show that $\theta_{2,2,2k}$ is 2-choosable. Since even cycles are subgraphs of these θ graphs, the result for even cycles will follow.

We label the vertices of $\theta_{2,2,2k}$ as in Figure 2.2. Let r and s be the vertices on the two paths of length two between u and v. We will label the vertices on the path of length $2k$ with w_i, where i ranges from 1 to $2k+1$. Note that the vertices u and v correspond with w_1 and w_{2k+1}, respectively. We have two cases: one when all the lists are the same, and one when the lists are different.

Case 1 The lists on $\{w_i : 1 \leq i \leq 2k+1\}$ are all the same. In this case, let $L(w_i) = \{a, b\}$. Then choose a for w_i when i is odd, and choose b for even i. Note that both u and v will be colored with a, and both $L(r)$ and $L(s)$ must contain a color other than a (since $|L(v)| = 2$ for all v in $V(G)$) so that we may color r and s as well.

Case 2 If the lists on $\{w_i : 1 \leq i \leq 2k+1\}$ are not all the same, then there are values j and $j+1$ for which $w_j w_{j+1}$ is an edge in G, but $L(w_j) \neq L(w_{j+1})$. We choose a value a_j from $L(w_j) \setminus L(w_{j+1})$. We then choose a value a_{j-1} from $L(w_{j-1}) \setminus \{a_j\}$ and continue in this manner until we have colored $u = w_1$ with a_1. Now, suppose $L(w_{2k+1}) = L(v) = \{r_1, s_1\}$. If $L(r) \neq \{a_1, r_1\}$ or $L(s) \neq \{a_1, s_1\}$, then we can choose colors for r, s, and w_{2k+1} and continue the coloring from w_{2k+1} back to w_{j+1}.

On the other hand, if $L(r) = \{a_1, r_1\}$ and $L(s) = \{a_1, s_1\}$, then we cannot color w_{2k+1} because we are forced to choose r_1 for r and s_1 for s, leaving no choice for w_{2k+1}, as in Figure 2.3. In this case, we return to w_{j+1} and begin again. We choose a_{j+1} from $L(w_{j+1}) \setminus L(w_{j+2})$ and continue by choosing a_{j+2} from $L(w_{j+2}) \setminus \{a_{j+1}\}$, etc. Since $L(w_{2k+1}) = \{r_1, s_1\}$ and $L(r) = \{a_1, r_1\}$, we know $r_1 \neq a_1$. Thus, we choose one of r_1

or s_1 for w_{2k+1}, and a_1 for both r and s. Then there is a color left in $L(w_1) \setminus \{a_1\}$ and we can continue the coloring.

Thus, $\theta_{2,2,2k}$ graphs are 2-choosable, and so are even cycles of at least four vertices.

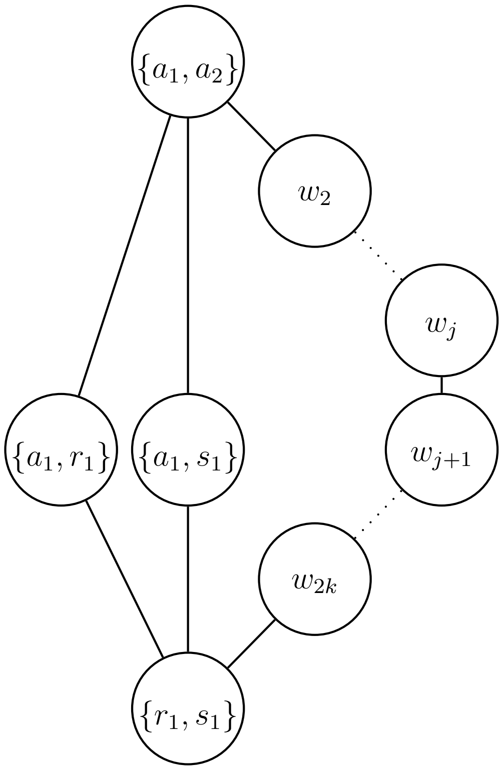


Figure 2.3: Not all w_i have the same lists

Necessity

We must now show that the only graphs which are 2-choosable have cores that are K_1, even cycles, or $\theta_{2,2,2k}$. We proceed with a proof by contradiction. Suppose G is a 2-choosable graph whose core is not K_1, an even cycle, or $\theta_{2,2,2k}$. Since vertices of degree 1 are always 2-choosable, we will assume G has been trimmed down to its core.

G must contain a cycle, else G's core is K_1. If H is an odd cycle, then $\chi(H) = 3$. Thus, G cannot contain an odd cycle since $\chi_L(G) \geq \chi(G)$ and we have assumed $\chi_L(G) = 2$.

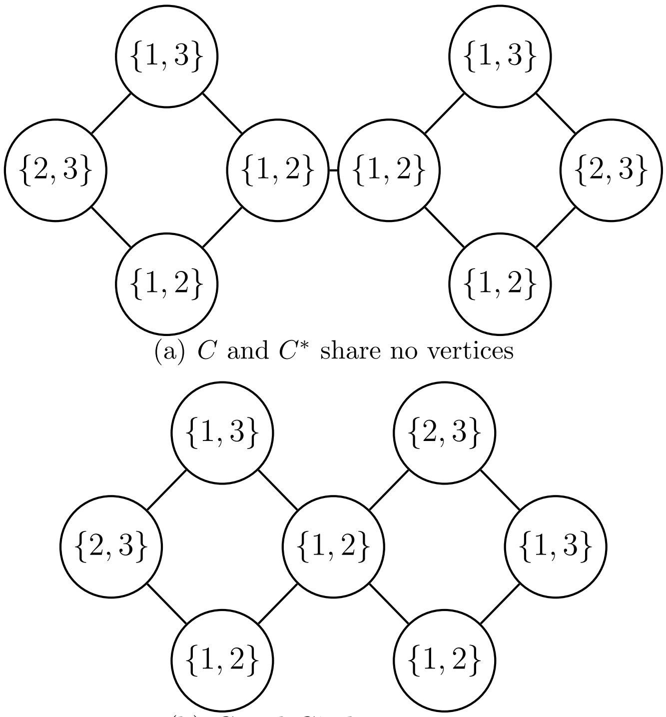
By assumption, G is not an even cycle. So, if C is a shortest cycle in G, then there is an edge of G that is not in C. This edge must lie on another cycle or on a path connecting cycles to satisfy degree requirements. If C^* is another cycle in G, then since G is connected, either there is a path between C and C^*, or C and C^* share vertices. Suppose C and C^* share at most one vertex. We may apply the following reduction technique. We delete a vertex x from G, and merge the vertices that were adjacent to x, removing any parallel edges this creates since parallel edges do not affect vertex colorings. We repeat this procedure until we are left with either Figure 2.4(a) or 2.4(b). The reader may verify that Figures 2.4(a) and 2.4(b) are not 2-choosable with the given lists.

Now we may assume that C and C^* share at least two vertices. But this means that there is a path which is edge-disjoint from C and connects two distinct vertices of C. Let P be a shortest such path. If $C \cup P$ is not of the form $\theta_{2,2,2k}$, then it must be a $\theta_{a,b,c}$ graph where $a \neq 2$ and $b \neq 2$. Using the same reduction technique as above, we can reduce $C \cup P$ to Figure 2.5, which is not 2-choosable.

So now suppose $C \cup P$ is a $\theta_{2,2,2k}$ graph. Note that this means C is a 4-cycle since C was a shortest cycle in G. We label the vertices of C as in Figure 2.6 for clarity. We have assumed G is not a $\theta_{2,2,2k}$ graph. Thus, by a similar argument as before, we must have another shortest path P^*, which is edge-disjoint from $C \cup P$ and connects two distinct vertices of $C \cup P$. We have six possible cases:

Case 1 The endpoints of P^* are interior vertices of P. If this case, then we have two edge-disjoint cycles connected by a path, which we can reduce with our method to Figure 2.4(a).

Case 2 One endpoint of P^* is c_1, and the other is an interior vertex of P. We then have two edge-disjoint cycles sharing a vertex, which we can reduce to Figure 2.4(b).



(a) C and C^* share no vertices

(b) C and C^* share 1 vertex

Figure 2.4: Cycles joined by at most 1 vertex

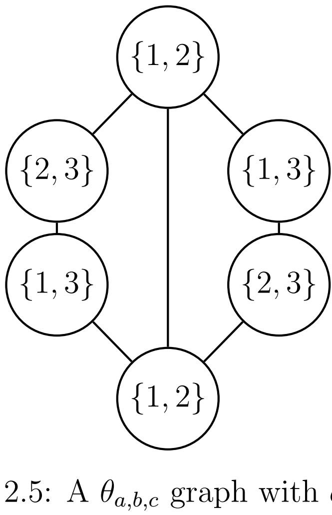


Figure 2.5: A $\theta_{a,b,c}$ graph with $a, b \neq 2$

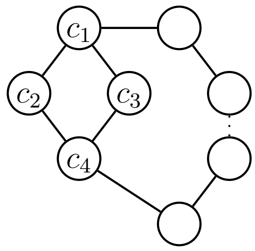


Figure 2.6: $C \cup P$

Case 3 One endpoint of P^* is c_3, and the other is an interior vertex u of P. Then the edge c_1c_3, the path $c_1c_2c_4c_3$, and the path from c_1 to u through P joined with the path from u to c_3 through P^* forms a $\theta_{a,b,c}$ graph with $a \neq 2$ and $b \neq 2$. We can reduce this graph to Figure 2.5.

Case 4 One endpoint of P^* is c_1, and the other is c_3. This is the same as Case (3) since now P^* and the paths $c_1c_2c_4c_3$ and c_1c_3 give us a $\theta_{a,b,c}$ graph with $a \neq 2$ and $b \neq 2$.

Case 5 The endpoints of P^* are c_1 and c_4. If P has length two, then we have a graph of the form shown in Figure 2.7. Note that this graph can be reduced to Figure 2.8, which is not 2-choosable. If the length of P is greater than two, then the length of P^* is also greater than two. Then the edge c_1c_3 and the paths P^* and $c_1c_2c_4c_3$ form a $\theta_{a,b,c}$ graph with $a \neq 2$ and $b \neq 2$ as in Cases (3) and(4).

Case 6 The endpoints of P^* are c_2 and c_3. In this case we remove edge c_1c_3. Then the edge c_2c_4, P^*, and the path from c_2 to c_4 through the edge c_2c_1 and P form a $\theta_{a,b,c}$ graph where $a \neq 2$ and $b \neq 2$.

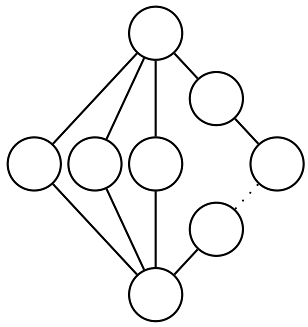


Figure 2.7: A θ graph with an extra 2-path

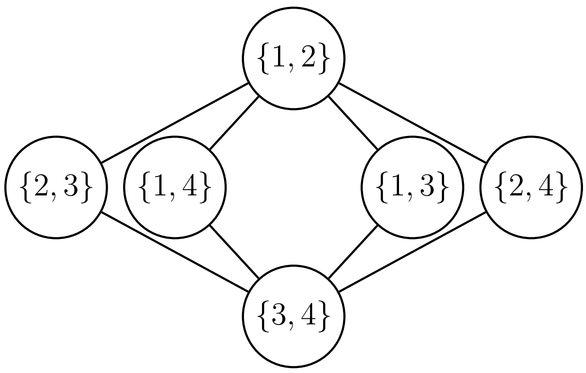


Figure 2.8: Reduction of a θ graph with an extra 2-path

One important detail we have omitted proving until this point is that if G' is a reduction of G obtained by our method and G' is not 2-choosable, then neither is G. We prove this now. Suppose G' is a reduction of G and G' is not 2-choosable. Let x be the vertex we deleted from G to obtain G'. Undo the reduction by unmerging the vertices of G that were adjacent to x and adding x back into the graph. Assign to all of these vertices the same list, $\{a, b\}$, that was on the merged vertex in G'. If we choose a for x, then we must choose b for every vertex adjacent to x. But then if this choice creates a proper coloring for G, it would have created a proper coloring for G' as well, which is a contradiction. Thus, G is not 2-choosable, and our reductions in Figures 2.4(a), 2.4(b), 2.5, and 2.8 show that the only graphs which are 2-choosable have cores that are K_1, even cycles, or $\theta_{2,2,2k}$ graphs. $\square$

2.2 Planar Graphs

In [6], Erdős et al. conjectured that all planar graphs were 5-choosable. Fourteen years later, this conjecture was proven by Carsten Thomassen in [10]. We present the proof here, but first we must present a definition and a lemma.

A *near-triangulation* is a simple planar graph consisting of an outer cycle whose interior has been divided up into triangles through the addition of vertices and edges [10].

Lemma 1. *For every simple planar graph G, there is a near-triangulation that contains an isomorphic copy of G as a subgraph.*

Proof. The proof proceeds by induction on $n = |V(G)|$.

Let G be a simple planar graph on n vertices. If $n = 1$ or $n = 2$, then K_1 and K_2 are subgraphs of K_3, which is a near-triangulation.

Now assume the statement is true for $1 \leq |V(G)| \leq n-1$. We will show it is true for $|V(G)| = n$.

Let x be in $V(G)$. Consider $G \backslash \{x\}$. $|V(G \setminus \{x\})| = n-1$. Thus, by the induction hypothesis, there is a near-triangulation T such that $G \setminus \{x\}$ is isomorphic to a subgraph of T. Let G' be a planar embedding of this subgraph of T and $\phi : G \to G'$ be an isomorphism.

Now consider the subgraph H of G induced by edges incident with x. H looks like the spokes of a wheel, as shown in Figure 2.9.

Since G is planar, H must fit inside some face F' of a planar embedding of $G \setminus \{x\}$, i.e. H fits inside a face F' of G'. Let v_1, v_2, ..., v_k be the k neighbors of x in G, as shown in Figure 2.9. Since G is simple, G' is simple, and every face is bounded by a cycle of length at least three. Let u_1, u_2,..., u_p be the vertices of the cycle C bounding F'.

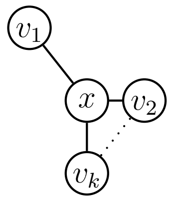


Figure 2.9: Subgraph induced by edges incident with x

In T, erase all edges and vertices inside $F^{'}$ that are not part of $G^{'}$. Call this new graph $T^{'}$. We will make $T^{'}$ a near-triangulation including x as follows:

Case 1 Suppose for all $1 \leq i \leq k$, $\phi(v_i) = u_j$ for some $1 \leq j \leq p$. Since H must fit inside $F^{'}$, we must have $k \leq p$. If $k < p$, add edges so that xu_i is an edge for all $1 \leq i \leq p$. We have now created a wheel and spokes, which is clearly triangulated, as seen in Figure 2.10. Since we have not altered any other faces of $T^{'}$, our new graph is a near-triangulation that contains a graph isomorphic to G as a subgraph.

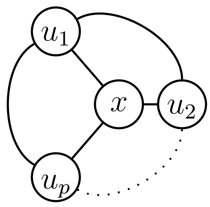


Figure 2.10: Wheel and spokes

Case 2 Now assume not all of v_1, v_2,..., v_k correspond under ϕ to vertices on C. Let $\{v_1^{'}, v_2^{'}, ..., v_r^{'}\}$ be the subset of $\{v_1, v_2, ...v_k\}$ that do not correspond to vertices on C. No $v_i^{'}$ can be in or on a face of $G^{'}$ other than $F^{'}$ because H must fit inside $F^{'}$. Also, each $v_i^{'}$ has at most one neighbor on C, or we have two smaller faces formed from $F^{'}$ separated by $u_1v_1^{'}$ and $v_1^{'}u_2$, without loss of generality. Then we must fit H into (ambiguously) one of these faces instead of into $F^{'}$. So each $v_i^{'}$ has at most one neighbor on C, as claimed.

a) If for all $\{v_i^{'} : 1 \leq i \leq r\}$, $v_i^{'}$ has only x as a neighbor, then each $v_i^{'}$ is an isolated vertex inside $F^{'}$. Draw edges connecting x to each u_j on C, triangulating $F^{'}$ as in Case 1. Place $v_1^{'}$ inside the triangle formed by u_1u_2, u_2x, and xu_1. Add the edges $xv_1^{'}$, $v_1^{'}u_1$,

$v_1^{'}u_2$ to triangulate the face bounded by u_1u_2, u_2x, and xu_1, as in Figure 2.11. Repeat this construction with $v_i^{'}$ for each $i = 2, ..., r$.

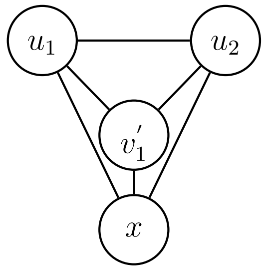


Figure 2.11: Triangulation of $F^{'}$, case (a)

b) Suppose some $v_i^{'}$ has x and a u_j as neighbors. Then connect x to each u_j on C as before. Each $v_i^{'}$ will lie inside exactly one triangle created by the edges u_jx, $1 \leq j \leq p$.

Consider the triangle formed by the edges u_1u_2, u_2x, and xu_1. Without loss of generality, let $v_1^*, v_2^*, ..., v_s^*$ be the $\{v_i^{'} : v_i^{'}$ is adjacent to $u_1\}$ and $v_{s+1}^*, v_{s+2}^*, ..., v_t^*$ be the $\{v_i^{'} : v_i^{'}$ is adjacent to $u_2\}$. Add the edges v_i^*x for all $1 \leq i \leq t$. Then add the edges $v_i^*v_{i+1}^*$ for $1 \leq i \leq t-1$. Finally, add the edge $v_s^*u_2$ to triangulate the face formed by u_1u_2, u_2x, and xu_1, as in Figure 2.12.

Repeat this construction for each triangle u_ju_{j+1}, u_jx, xu_j for $1 \leq j \leq p-1$ with a $v_i^{'}$ inside it and adjacent to u_j or u_{j+1}. Then treat the $v_i^{'}$ that are not adjacent to any u_j on C as in case (a). The resulting graph will be a triangulation of $F^{'}$.

Again, since we have not altered any face in $T^{'}$ other than $F^{'}$, our new graph is a near triangulation that contains a graph isomorphic to G.

Thus, the lemma holds for $|V(G)| = n$, and by induction, every simple planar graph is isomorphic to a subgraph of a near-triangulation. □

We may now state and prove Thomassen's theorem that planar graphs are 5-choosable. Note that a *chord* is an edge joining two nonadjacent vertices on a cycle.

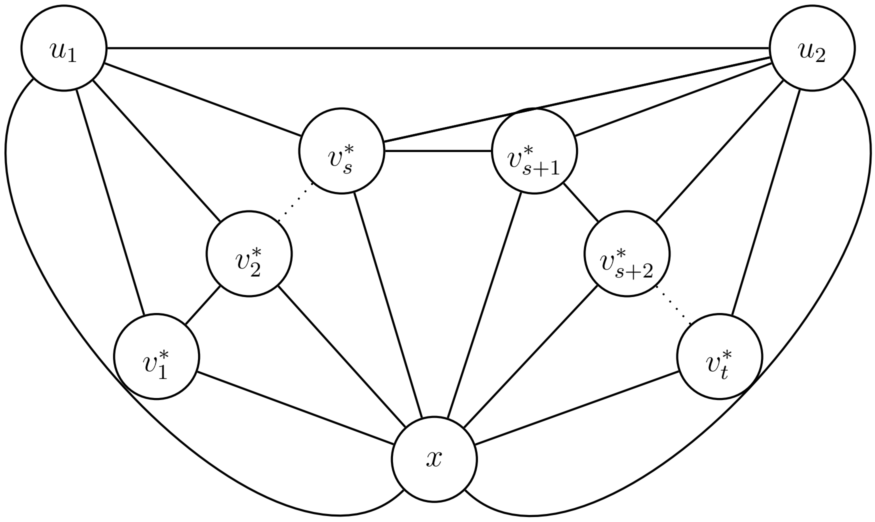


Figure 2.12: Triangulation of $F^{'}$, case (b)

Theorem 3 (Thomassen, [10]). *Let G be a near-triangulation with outer cycle $C = v_1v_2...v_kv_1$. Assume that $L(v)$ is a list of at least three colors if v is in C and at least five colors if v is in $G \setminus C$. Without loss of generality, choose colors 1 and 2 for v_1 and v_2, respectively. Then the coloring of v_1 and v_2 can be extended to a list coloring of G.*

Proof. The proof is by induction on $n = |V(G)|$. If $n = 3$, then $G = C$, and since $|L(v_3)| = 3$, there is a color other than 1 and 2 to use on v_3.

So assume that the theorem holds for all planar graphs G with $1 \leq |V(G)| \leq n-1$. We will show the theorem holds for $|V(G)| = n$.

Suppose there are vertices v_i and v_j on C, with $2 \leq i \leq j-2 \leq k-1$ such that v_iv_j is a chord of C. Then the subgraphs formed by the cycles $v_1v_2...v_iv_jv_{j+1}...v_kv_1$, $v_iv_{i+1}...v_jv_i$, and their interiors have fewer than n vertices, and we apply the induction hypothesis to show that G is 5-choosable. So we assume C does not have a chord.

Consider the neighbors of v_k, $\{v_1, u_1, u_2, ..., u_r, v_{k-1}\}$. Since G is a near-triangulation, G must contain the path $P = v_1u_1u_2...u_rv_{k-1}$. Consider $P \cup (C \setminus v_k)$. Since C has no chord, the edge $v_{k-1}v_1$ is not in G, and hence, $P \cup (C \setminus v_k)$ is a cycle, $C^{'} = v_1v_2...v_{k-1}u_ru_{r-1}...u_1v_1$. Recall that $|L(v_k)| = 3$. Thus, $L(v_k) \setminus \{1\}$ has at least two colors, say a and b. Since $|L(u_j)| = 5$ for $1 \leq j \leq r$, we can assign new lists $L^{'}(u_j) = L(u_j) \setminus \{a, b\}$ to the u_j. If v is a vertex of G such that v is not in $\{u_1, ..., u_r\}$, let $L^{'}(v) = L(v)$. Now apply the induction hypothesis to $C^{'}$ and its interior with the new list assignment $L^{'}$. All that remains to be colored is v_k, but v_1 is colored, and the colors $\{a, b\}$ are left at v_k. So we may choose one of these so that v_k and v_{k-1} are assigned distinct colors, completing the coloring.

Now, since we have shown that every near-triangulation is 5-choosable, by the lemma we have every simple planar graph is 5-choosable. Then since adding parallel edges and loops will not affect a vertex-coloring, every planar graph is 5-choosable. $\square$

Erdős et al. also conjectured in [6] that there is a planar graph which is not 4-choosable, meaning that if this graph can be shown to exist, then Thomassen's bound of 5 for $\chi_L(G)$ for planar graphs G is sharp. Margit Voigt constructed such a graph in [11]. The construction involves an elaborate nesting of triangulated graphs such that the final graph contains 238 vertices.

Chapter 3

Non-constant List Functions

Lest the reader begin to believe that all research in list colorings involves lists of constant length, we now examine non-constant list functions. We first build to a proof of a list-coloring version of Brooks's Theorem, and then outline an algebraic approach to obtain conditions for choosability.

3.1 D-Choosable Graphs

We define the *degree* of a vertex v to be the number of edges incident with v in a graph G, and denote this quantity by $deg_G(v)$, or by $deg(v)$ if the context is clear. Note that loops are usually counted twice. Most of the graphs we consider will be simple, however, so this will not be an issue. We define the *maximum degree* of G to be $\Delta(G) = max(deg_G(v))$, and write Δ if the context is clear.

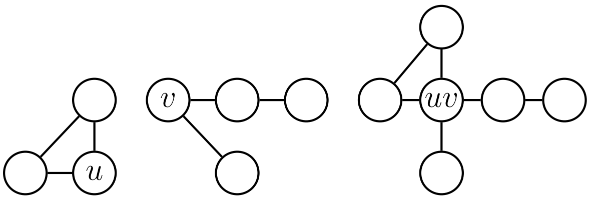
We will consider the list function L such that $|L(v)| = deg(v)$. Keeping with the notation of Erdős et al., we will let $D(v) = L(v)$, and describe conditions under which a graph is

D-choosable. Note that D is a non-constant list function if the vertices of G have varying degrees.

Recall that in a complete graph K_n, each vertex is adjacent to every other vertex. Hence, each vertex has $deg(v) = n - 1$. Clearly each vertex must have a unique color, requiring n colors. Thus, K_n is not D-choosable since if we assign the list $\{1, 2, ..., n-1\}$ to each vertex, we will not be able to complete the coloring.

Now consider odd cycles. Each vertex of a cycle has degree 2, but as noted in Chapter 2 in the proof of Theorem 2, odd cycles are not 2-choosable, and hence, are not D-choosable.

We show that all graphs which are not D-choosable can be constructed from odd cycles and complete graphs. First, though, we define a way to merge graphs by "gluing" vertices. Let G and H be graphs containing vertices u and v, respectively. Denote by $G \odot H$ the graph formed by merging the vertices u and v into one vertex, as in Figure 3.1. We now show that if G and H are not D-choosable, then $G \odot H$ is not D-choosable, either.



(a) G (b) H (c) $G \odot H$

Figure 3.1: Gluing together graphs

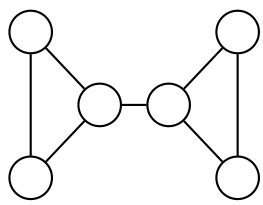
Lemma 2 (Erdős et al., [6]). *Suppose G and H are not D-choosable. Then neither is $G \odot H$.*

Proof. Let G and H be graphs with list assignments such that G and H are not D-choosable. Let u be a vertex in G and v be a vertex in H. Create $G \odot H$ by merging u and v into one vertex, uv.

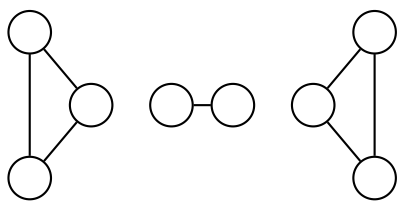
Let $C_u = L(u)$ be the colors used in the list assignment of u in G, and similarly, $C_v = L(v)$ be the colors used in the list assignment of H. We may assume that $C_u \cap C_v$ is empty, because otherwise we have fewer colors from which to choose, and we will show there are not enough colors available to color uv in $G \odot H$.

Note that $deg_{G \odot H}(uv) = deg_G(u) + deg_H(v)$. Assume $C_u \cup C_v$ is the list assigned to uv, and that the other lists are the same as their original assignments in G and H. If we choose a color from C_u for uv, then since G is not D-choosable, and u is the only vertex of G that has been altered, we cannot complete the coloring in the G subgraph of $G \odot H$. A similar result follows if we try to choose a color from C_v for uv. Thus, if G and H are not D-choosable, neither is $G \odot H$. $\square$

We make a few more definitions and state a theorem due to Rubin before we completely characterize graphs that are not D-choosable. If a graph has two subgraphs which are connected and share only a single vertex, this vertex is called a *separating vertex.* A graph is *separable* if it contains separating vertices or is disconnected; otherwise, it is *nonseparable.* The *blocks* of G are the maximal nonseparable subgraphs of G. For an example of these concepts, see Figures 3.2(a) and 3.2(b). A *cut vertex* is a vertex which if deleted will disconnect G. Note that a cut vertex is also a separating vertex. We call the maximal connected subgraphs of G *components.* Hence, a cut vertex increases the number of components of G when deleted [3].



(a) A Separable Graph



(b) The Blocks of G

Figure 3.2: A separable graph G and its blocks

The following theorem may be found in [6] and is due to Rubin. Its four-page proof is by exhaustion, and we omit it here, but the proof may be found in [6] as well.

Theorem 4 (Rubin, [6])**.** *Suppose G has no cut vertex. Then G is a complete graph, G is an odd cycle, or G must contain as a vertex-induced subgraph an even cycle with at most one chord.*

Let $\mathcal{F}$ be the family of graphs consisting of odd cycles, complete graphs, and any combinations of these formed by merging vertices as above. By Lemma 2, graphs in $\mathcal{F}$ are not D-choosable. Erdős et al. proved that the converse is also true:

Theorem 5 (Erdős et al., [6])**.** *If G is a connected graph, then G is not D-choosable if and only if G is in $\mathcal{F}$.*

Proof. We assume G is simple since loops and parallel edges do not affect vertex colorings.

Necessity

Let G be a graph and consider the blocks of G. If each block of G is a complete graph or an odd cycle, then G is in $\mathcal{F}$, and by Lemma 2, since the blocks of G are not D-choosable, G is not D-choosable.

Sufficiency

Suppose G has a block that is neither a complete graph nor an odd cycle, so that G is not in $\mathcal{F}$. Then by Theorem 4, this block contains an even cycle C with at most one chord as a vertex-induced subgraph (since a block contains no separating vertex, and hence, no cut vertex). We show that this subgraph is D-choosable, and then show that G is also D-choosable. Let L be a list assignment on G such that $|L(v)| = deg(v)$.

Suppose C has no chord. Then by Chapter 2, we know C is 2-choosable, and since the degree of every vertex in a cycle is 2, C is D-choosable.

Now assume C has a chord. Then C with this chord forms a θ graph. We prove a more general result, showing that *every* θ graph is D-choosable. Label the vertices of a θ graph as in Figure 3.3. We begin by coloring v_1. Since $|L(v_t)| = 2$ and $|L(v_1)| = 3$, there is a color in $L(v_1) \setminus L(v_t)$. Choose this color for v_1. Continue the coloring around the cycle in the order shown in Figure 3.3. When we reach each vertex $v_1, v_2, ..., v_{t-1}$, that vertex has in its list at least one more color than it has colored neighbors, and thus may be colored as well. The vertex v_t is the vertex of interest. $L(v_t)$ has two colors and v_t has two neighbors who were previously colored. We chose the color on v_1, though, so that it was not a color in $L(v_t)$. Thus, v_t has at least one color available, and we may complete the coloring. So θ graphs are D-choosable, and as a result, C with a chord is D-choosable.

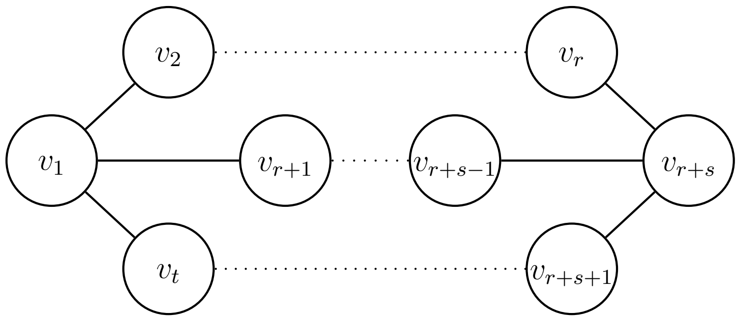


Figure 3.3: General θ graph

We must now show that if G has a vertex-induced D-choosable subgraph, then G is D-choosable. We use induction on $|V(G)| = n$.

If $n = 1$, there is nothing to show. So let $n > 1$; we assume the result is true for $|V(G)| = 1, ..., n-1$ and show it is true for $|V(G)| = n$.

Let H be a vertex-induced D-choosable subgraph of G. If $G = H$, we are done. So we assume $G \setminus H$ is not empty, and let v be a vertex of $G \setminus H$ at maximal distance from H. Note that if $G \setminus \{v\}$ is disconnected, then there must be some vertex w in a component of $G \setminus \{v\}$ other than the component containing H. Then this w is at a greater distance from H than v is, which is a contradiction. Thus $G \setminus \{v\}$ is connected.

Choose a color c_v for v from its list. For every neighbor r of v, let $L^{'}(r) = L(r) \setminus \{c_v\}$. If s is not a neighbor of v, let $L^{'}(s) = L(s)$. Since $L^{'}$ is a list assignment of $G \setminus \{v\}$ such that $\left|L^{'}(u)\right| = deg(u)$ for all u in $V(G \setminus \{v\})$ and since $G \setminus \{v\}$ has $n-1$ vertices, by the induction hypothesis, $G \setminus \{v\}$ is D-choosable. Then when we add back v, G is also D-choosable.

Therefore by induction, if G has a vertex-induced D-choosable subgraph, G is also D-choosable. Thus, if G is not in $\mathcal{F}$, G is D-choosable. $\square$

We have now completely characterized graphs that are not D-choosable. For those who prefer to think in terms of when a graph *is* D-choosable, we combine Theorems 4 and 5 and offer the following restatement of Theorem 5.

Theorem 6 (Erdős et al., [6]). *A connected graph G is D-choosable if and only if G contains an even cycle or a θ graph as a vertex-induced subgraph.*

We now further discuss D-choosability of infinite graphs, which will aid us in proving Brooks's Theorem. Note that Figure 3.4 is not D-choosable under the given list assignment.

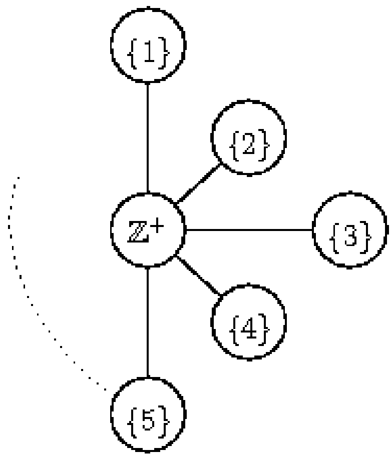


Figure 3.4: Infinite asterisk

We prove that if no vertex of G has infinitely many neighbors, then G is D-choosable.

Theorem 7 (Erdős et al., [6]). *If G is a countably infinite connected graph with $\Delta < \infty$, then G is D-choosable.*

Proof. Let G be a countably infinite connected graph, and suppose L is a list assignment such that $|L(v)| = deg(v)$ for all vertices v in G. For convenience, index the vertices of G with the positive integers; this is possible because G is countable. Then $V(G) = \{v_1, v_2, ..., v_n, ...\}$. We will choose colors for the vertices by the following procedure, letting i be the smallest index of an uncolored vertex.

Case 1 Suppose deleting vertex v_i does not disconnect a finite component of G from the rest of the graph. Then we choose any color from $L(v_i)$ and remove this color from the lists on v_i's neighbors. We will no longer consider v_i.

Case 2 Suppose deleting v_i disconnects a finite component of G from the rest of the graph. In this case, we will color each finite component before coloring v_i. Find an uncolored vertex v at maximal distance from v_i. As in the proof of Theorem 5, v will not disconnect its component. Then we apply Case 1 to v, and repeat. Note that since G is connected, there is a path from v_i to v for all vertices v in this finite component. Hence, we can uncolor any vertices on this path as necessary and re-color the path from v to v_i, because for each vertex u in this path, there will always be some neighbor of u that remains uncolored (since v_i is uncolored), and thus, some color available at u. Eventually, we will color every vertex in each finite component except for v_i.

Since G is infinite and connected, v_i has a neighbor v_j that will lie in an infinite component of G when v_i is deleted. If $j > i$, then v_j has not been colored yet, and v_i has one more color left in its list. Choose this color for v_i. Assume $j < i$. If v_j is uncolored, then as before, v_i has a color available. So we assume all the neighbors of v_i have been colored. If v_i still has a color left in its list, we can choose this color. So we assume $L(v_i)$ is precisely the list of colors used on the neighbors of v_i. Since G is infinite, there is some vertex k in G with $k > i$ and k the least such integer. Then since G is connected, there is some shortest path P from v_i to v_k. Uncolor the vertices on this path. Now every vertex in P has an uncolored neighbor, meaning every vertex in P has a color available since $|L(u)| = deg(u)$. Color these vertices in order from v_i to v_k along P until every vertex that was previously colored has been colored again. Thus, v_i can be colored.

Since every vertex in G must fall into one of these cases, we can find a proper coloring of G from the list assignment L, and G is D-choosable. □

3.2 Brooks's Theorem

Brooks's Theorem is a fundamental result in graph colorings. It states that $\chi \leq \Delta$ if G is a connected graph and neither an odd cycle nor a complete graph; in these cases, $\chi = \Delta + 1$.

Theorem 8 (Brooks's Theorem, [3]). *Let G be a connected graph. If G is neither an odd cycle nor a complete graph, then $\chi \leq \Delta$; otherwise, $\chi = \Delta + 1$.*

Erdős et al. proved a version of this for list colorings. We now state this result.

Theorem 9 (Erdős et al., [6]). *Let G be a connected graph. If G is neither an odd cycle nor a complete graph, then $\chi_L \leq \Delta$; otherwise, $\chi_L = \Delta + 1$.*

Proof. Recall that $\mathcal{F}$ is the family of graphs which are not D-choosable and that $\mathcal{F}$ contains all complete graphs, all odd cycles, and any combination of them formed by gluing together vertices. We have seen that complete graphs and odd cycles have $\chi_L = \Delta + 1$ (see p. 21).

Assume G is neither an odd cycle nor a complete graph. If G is not in $\mathcal{F}$, then we are done since G is D-choosable implies that G is Δ-choosable. Thus we assume G is in $\mathcal{F}$ and let

L be a list assignment with $|L(v)| = deg_G(v)$ for all vertices v in $V(G)$. Since G is neither an odd cycle nor a complete graph, G cannot be $\Delta(G)$-regular. Thus, there is some vertex u with $deg_G(u) < \Delta(G)$. Create a new graph G' by attaching an infinite tail to u as shown in Figure 3.5. Let $L'(v) = L(v)$ for v in $V(G \setminus \{u\})$ and add one color to $L(u)$ so that $|L'(u)| = deg_G(u) + 1 = deg_{G'}(u)$. Let $L'(x)$ be lists of length 2 for each vertex x on the infinite tail. By Theorem 7, G' is D-choosable and this list assignment L' yields a proper coloring, $c : V(G') \to \mathbb{Z}$. The only vertex of G that we have altered, though, is u. So we will use our coloring c to color G as well. Add colors to the lists $L(v)$ as necessary so that $|L(v)| = \Delta(G)$ for every vertex v in G. For v in $V(G \setminus \{u\})$, color v with $c(v)$. Then u has $\Delta(G)$ colors in its list, but by the way we chose u, it has fewer than $\Delta(G)$ neighbors. Thus, there is a color available at u, and we can complete the coloring of G. Hence, G is Δ-choosable, and the theorem holds. $\square$

Since $\chi < \chi_L$, the original version of Brooks's Theorem follows from Theorem 9.

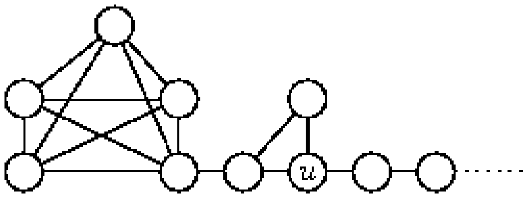


Figure 3.5: G with an infinite tail

3.3 An Algebraic Approach

In [1] Alon and Tarsi prove results similar to what we have seen in Sections 1 and 2, but use a delightful algebraic approach. We detail their methods here.

A *directed graph* D is a graph whose edges $\vec{uv}$ are understood to be oriented from u to v. The *tail* of a directed edge is the vertex at which it starts, and the *head* is the vertex at which it terminates. The *outdegree* of u, $d_D^+(u)$, is the number of directed edges of D whose tail is u. Likewise, the *indegree* $d_D^-(u)$ is the number of directed edges of D whose head is u [3].

We say that a subgraph H of a directed graph is *Eulerian* if $d_H^+(v) = d_H^-(v)$ for every v in $V(H)$. We also call H an *odd* Eulerian subgraph if it has an odd number of edges; otherwise H is an *even* Eulerian subgraph. In keeping with Alon and Tarsi's notation, we denote the number of even Eulerian subgraphs of D by $EE(D)$. Similarly, we denote by $EO(D)$ the number of odd Eulerian subgraphs of D.

Define a list assignment function such that $|L(v)| = d_D^+(v)$ for v in $V(G)$. Alon and Tarsi prove in [1] that if $EE(D) \neq EO(D)$, then D is L-choosable. We will first prove a few lemmas and then state and prove this theorem.

Lemma 3 (Alon and Tarsi, [1])**.** *Let $P = P(x_1, x_2, ..., x_n)$ be a polynomial in n variables over $\mathbb{Z}$. Let $\{d_i\}$ be integers such that the degree of P as a polynomial in x_i is at most d_i. Let $S_i \subset \mathbb{Z}$ be sets of $d_i + 1$ distinct integers. If $P(s_1, s_2, ..., s_n) = 0$ for all n-tuples $(s_1, s_2, ..., s_n)$ in $S_1 \times S_2 \times ... \times S_n$, then P is the zero polynomial.*

Proof. The proof is by induction on n. If $n = 1$, then $P = P(x)$ with degree at most d. Let S be a set of $d + 1$ distinct integers such that $P(s) = 0$ for all s in S. Then since the degree of P is at most d, P has at most d roots in $\mathbb{Z}$, but we have found $d + 1$ roots. Thus, P must be the zero polynomial. So assume $n > 1$.

Let $P = P(x_1, x_2, ..., x_n)$ be a polynomial in n variables over $\mathbb{Z}$, $\{d_i\}$ be integers such that P as a polynomial in x_i has degree at most d_i, and S_i be sets of $d_i + 1$ distinct integers such that $P(s_1, s_2, ..., s_n) = 0$ for all n-tuples $(s_1, s_2, ..., s_n)$ in $S_1 \times S_2 \times ... \times S_n$.

We can write $P = \sum_{i=0}^{d_n} P_i(x_1, ..., x_{n-1})x_n^i$, where each polynomial P_i has degree in x_j at most d_j. Now we have P as a polynomial in x_n.

Fix an $(n-1)$-tuple $(s_1, s_2, ..., s_{n-1})$ in $S_1 \times S_2 \times ... \times S_{n-1}$. By assumption, $P(s_1, s_2, ..., s_{n-1}, t_n) = 0$ for every t_n in S_n. Now P is a polynomial in the variable x_n with degree at most d_n, and we have found $d_n + 1$ roots. Thus, we must have that $P \equiv 0$ for this choice of $(n-1)$-tuple, which means $P_i(s_1, s_2, ..., s_{n-1}) = 0$ for $i = 1, 2, ..., d_n$. But $(s_1, s_2, ..., s_{n-1})$ was arbitrary; so $P_i(s_1, s_2, ..., s_{n-1}) = 0$ for every $(n-1)$-tuple in $S_1 \times S_2 \times ... \times S_{n-1}$ and for all $i = 1, ..., d_n$. Then by the induction hypothesis, $P_i \equiv 0$ for all $i = 1, ..., d_n$, which means $P \equiv 0$. $\square$

We define the *graph polynomial* f_G of an undirected graph G by

$$f_G(x_1, x_2, ..., x_n) = \prod\{(x_i - x_j) : i < j, v_iv_j \in E(G)\},$$

where $n = |V(G)|$. We call D an *orientation* of G if D is a directed graph whose underlying graph is G. We also define a *decreasing edge* to be a directed edge $v_i\vec{v}_j$ where $i > j$. Let $v_i\vec{v}_j$ be a directed edge of D and define the *weight* of $v_i\vec{v}_j$ by

$$w(v_iv_j) = \begin{cases} x_i, & \text{if } v_i\vec{v}_j \text{ is increasing} \\ -x_i, & \text{if } v_i\vec{v}_j \text{ is decreasing} \end{cases}. \tag{3.1}$$

We define the weight of D by $w(D) = \prod_{v_iv_j \in E(G)} w(v_iv_j)$.

Let G be an undirected graph and D an orientation of G. Consider $f_G = \prod\{(x_i - x_j) : i < j, v_iv_j \in E(G)\}$. Since G is undirected, each edge appears once in the product as a binomial $(x_i - x_j)$, with $i < j$. Then for $k = 1, ..., n$, x_k appears $deg_G(x_k)$ times. Now let $v_i\vec{v}_j$ be the directed edge in D corresponding to the edge in G with endpoints v_i and v_j. If $i < j$, then $(x_i - x_j)$ is a factor in f_G and x_i has the sign of $w(v_iv_j)$; otherwise, the factor is $(x_j - x_i)$ and x_i still has the same sign as $w(v_iv_j)$.

Expanding $\prod\{(x_i - x_j) : i < j, v_iv_j \in E(G)\}$, we see that each monomial has total degree equal to the number of edges of G. Thus, f_G is homogeneous. Note that each monomial corresponds to an orientation D of G. If x_k^m appears in the monomial, then we can consider it as m edges oriented out from x_k. Recall that $w(v_i\vec{v}_j) = \pm x_i$ for the directed edge $v_i\vec{v}_j$, with the sign given as previously explained. Thus, we see that since $w(D) = \prod_{v_iv_j \in E(G)} w(v_iv_j)$, we have $f_G = \sum_{\text{D an orientation of G}} w(D)$.

Recall that an Eulerian subgraph H of a directed graph D is even if it has an even number of edges and odd otherwise. We abuse the terminology calling an orientation D of a graph G *even* if D has an even number of decreasing edges and *odd* otherwise. For clarity we will always state whether we mean even as an orientation or as an Eulerian subgraph.

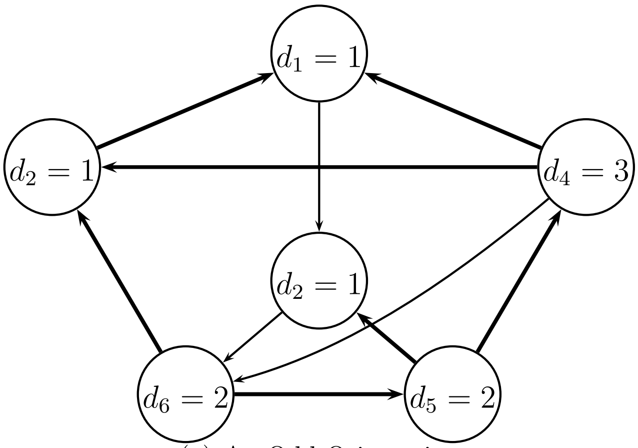
Let G be a graph with n vertices, $\{v_1, v_2, ..., v_n\}$, and $\{d_1, d_2, ..., d_n\}$ a set of nonnegative integers. Denote by $DE(d_1, d_2, ..., d_n)$ the set of all even orientations of G with $d^+(v_i) = d_i$. Similarly, denote by $DO(d_1, d_2, ..., d_n)$ the set of all odd orientations of G with $d^+(v_i) = d_i$. See Figure 3.6 for an example of these concepts. Note that decreasing edges are bold and that Figure 3.6 demonstrates a case where the same underlying graph has even and odd orientations with fixed $d^+(v_i)$ for all vertices v_i. We are now ready to prove our next lemma.

Lemma 4 (Alon and Tarsi, [1]).
$f_G(x_1, x_2, ..., x_n) = \sum_{d_1,...,d_n \geq 0} (|DE(d_1, ..., d_n)| - |DO(d_1, ..., d_n)|) \prod_{i=1}^{n} x_i^{d_i}$.

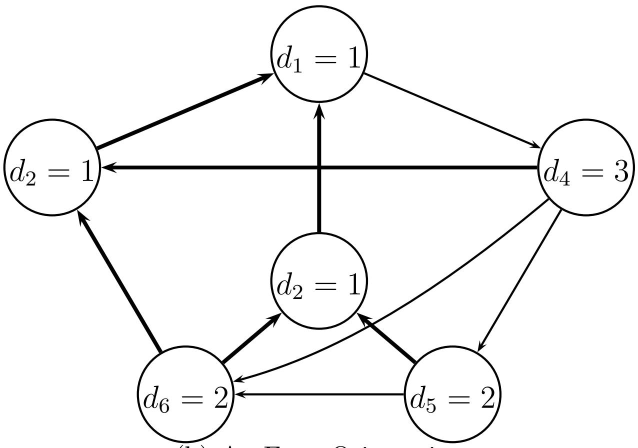
Proof. Recall that $f_G = \sum w(D)$. We have shown that each orientation D corresponds to a monomial of the form $\prod_{i=1}^{n} x_i^{d_i}$. This monomial will appear with a positive sign $|DE(d_1, ..., d_n)|$ times and with a negative sign $|DO(d_1, ..., d_n)|$ times by the definitions of $w(D)$ and $w(v_iv_j)$. Thus, the lemma follows. □

Now let G be a graph on n vertices and fix nonnegative integers $d_1, d_2, ..., d_n$. Choose two orientations D_1 and D_2 in $DO(d_1, d_2, ..., d_n) \cup DE(d_1, d_2, ..., d_n)$. We define $D_1 \oplus D_2$ to be the subgraph of D_1 induced by edges in D_1 who are directed the opposite way in D_2. For example, if D_1 is Figure 3.6(a) and D_2 is Figure 3.6(b), then $D_1 \oplus D_2$ is given by Figure 3.7.

Since $d_1, d_2, ..., d_n$ are fixed, for each $i = 1, 2, ..., n$ we have $d^+_{D_1}(v_i) = d^+_{D_2}(v_i)$. Thus, we know that for every edge e in $E(D_1 \oplus D_2)$ with its tail at v_i, there is an edge e' in $E(D_1 \oplus D_2)$



(a) An Odd Orientation



(b) An Even Orientation

Figure 3.6: Odd and even orientations of G

with its head at v_i in order to maintain the outdegree of v_i from D_1 to D_2. It follows that $d^+_{D_1 \oplus D_2}(v_i) = d^-_{D_1 \oplus D_2}(v_i)$ and $D_1 \oplus D_2$ is an Eulerian subgraph. Once again, recall that an Eulerian subgraph is even if it has an even number of edges and odd otherwise. We now give conditions under which $D_1 \oplus D_2$ is an even Eulerian subgraph.

Lemma 5 (Alon and Tarsi, [1])**.** *$D_1 \oplus D_2$ is even as an Eulerian subgraph if and only if D_1 and D_2 are both even or are both odd orientations (have an even or odd number of decreasing edges).*

Proof. Sufficiency

Suppose $D_1 \oplus D_2$ is an even Eulerian subgraph. Then $|E(D_1 \oplus D_2)| = 2k$ for some nonneg-

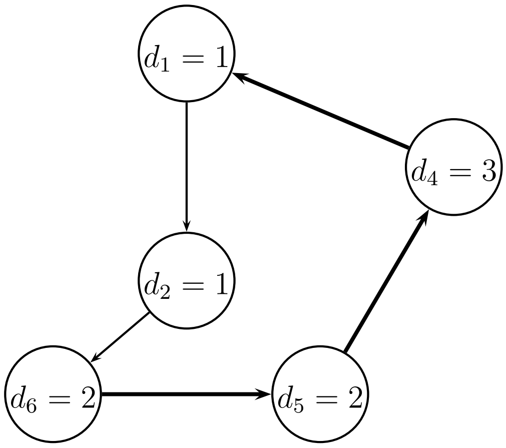


Figure 3.7: $D_1 \oplus D_2$

ative integer k. This means that there is an even number of edges in D_2 that are oriented in the opposite direction from their orientation in D_1.

Suppose there are k_1 increasing edges in $D_1 \oplus D_2$. Then these are k_1 decreasing edges in D_2. Also assume there are k_2 decreasing edges in $D_1 \oplus D_2$. Then these are k_2 increasing edges in D_2. Note that $k_1 + k_2 = 2k$.

If D_1 is an even orientation, then the number of decreasing edges in D_1 is $2n$ for some nonnegative integer n. So we must have $2n - k_2 = p$ edges that are decreasing in both D_1 and D_2. We have two cases.

a) If p is even, then k_2 is also even. Since $k_1 + k_2 = 2k$, k_1 is even. So $k_1 + p$ is even, but $k_1 + p$ is the number of decreasing edges in D_2. Hence, D_2 is an even orientation.

b) If p is odd, then k_2 is also odd. Since $k_1 + k_2 = 2k$, k_1 must be odd as well. Thus, $k_1 + p$ is an even number of decreasing edges in D_2. Thus, D_2 is an even orientation.

If D_1 is an odd orientation, a similar argument holds, and we see that D_1 and D_2 are either both even or are both odd orientations.

Necessity

Now suppose D_1 and D_2 are both even orientations or are both odd orientations. Let us assume D_1 and D_2 are both even orientations. Then D_1 has $2k_1$ decreasing edges and D_2 has $2k_2$ decreasing edges for some integers k_1 and k_2.

Consider $D_1 \oplus D_2$ with n edges. If e is one of these edges, then e is either increasing or decreasing in D_1, and is oriented in the opposite direction in D_2. Let n_i (respectively n_d) be the number of edges in $D_1 \oplus D_2$ that are increasing (respectively decreasing) in D_1. Then $n_i + n_d = n$.

Let $c_1 = 2k_1 - n_d$ be the number of decreasing edges in D_1 that are not edges in $D_1 \oplus D_2$. Then there are c_1 edges that are decreasing in both D_1 and D_2. Similarly, let $c_2 = 2k_2 - n_i$ be the number of decreasing edges in D_2 that are not edges in $D_1 \oplus D_2$. Note that this means $c_1 = c_2$. So we have $2k_1 - c_1 + 2k_2 - c_2 = n_d + n_i = n$, which means $2(k_1 + k_2) - (c_1 + c_2) = n$. Then if n is odd, $c_1 + c_2$ is odd, which forces only one of c_1 or c_2 to be odd, but this is a contradiction since $c_1 = c_2$. Thus, n is even, and $D_1 \oplus D_2$ is even as an Eulerian subgraph. A similar proof works if D_1 and D_2 are both odd orientations. □

It is easily seen that the map $\phi : D_2 \to D_1 \oplus D_2$ is a bijection from $DE(d_1, ..., d_n) \cup DO(d_1, ..., d_n)$ to the set of all Eulerian subgraphs of D_1. Furthermore, by the previous lemma, if D_1 is even, then ϕ maps even orientations to even Eulerian subgraphs and odd orientations to odd Eulerian subgraphs. Thus, $||DE(d_1, ..., d_n)| - |DO(d_1, ..., d_n)|| = |EE(D_1) - EO(D_1)|$.

Lemma 6 (Alon and Tarsi, [1]). *Let D be an orientation of an undirected graph G on n vertices, $\{v_1, ..., v_n\}$. Let $d_i = d_D^+(v_i)$ for $i = 1, ..., n$. Then the absolute value of the coefficient of the monomial $\prod_{j=1}^{n} x_j^{d_j}$ in the standard representation of $f_G = f_G(x_1, ..., x_n)$ as a linear combination of monomials is $|EE(D) - EO(D)|$. In particular, if $EE(D) \neq EO(D)$, then this coefficient is nonzero.*

Proof. This lemma is a direct result of Lemma 4 and the fact that
$|\,|DE(d_1,...,d_n)| - |DO(d_1,...,d_n)|\,| = |EE(D_1) - EO(D_1)|$. □

We are now ready to prove our main theorem for this section.

Theorem 10 (Alon and Tarsi, [1])**.** *Let D be a directed graph. For each vertex v, let $S(v)$ be a set of $d_D^+(v)+1$ distinct integers. If $EE(D) \neq EO(D)$, then there is a proper coloring $c : V(D) \to \mathbb{Z}$ such that $c(v)$ is in $S(v)$ for every vertex v. That is, if L is a list assignment such that $L(v) = d_D^+(v)+1$ for all vertices v in D, then D is L-choosable.*

Proof. Suppose D is a directed graph on n integers, $\{v_1,...,v_n\}$, $d_i = d_D^+(v_i)$ for $i = 1,...,n$, and $S_i = S(v_i)$ is a set of $d_i + 1$ distinct integers for each $i = 1,...,n$. Assume $EE(D) \neq EO(D)$ and there is no proper coloring $c : V(D) \to \mathbb{Z}$ such that $c(v_i)$ is in S_i for all i. We will find a contradiction.

Let G be the underlying graph of D and $f_G = f_G(x_1,...,x_n)$ its graph polynomial. Then by assumption there is no proper coloring from the lists S_i; so for each n-tuple $(c_1,c_2,...,c_n)$ in $S_1 \times S_2 \times ... \times S_n$, there must be adjacent vertices v_i and v_j with $i < j$ such that $c_i = c_j$. This means

$$f_G(c_1,c_2,...,c_n) = f_G(c_1,c_2,...,c_n) = \prod\{(c_i - c_j) : i < j, v_iv_j \in E(G)\} = 0. \tag{3.2}$$

For each $i = 1,...,n$, define polynomials $Q_i(x_i) = \prod\limits_{s\in S_i}(x_i - s) = x_i^{d_i+1} - \sum\limits_{j=0}^{d_i} q_{ij}x_i^j$. Note that if c_i is in S_i, $Q_i(c_i) = 0$, i.e.,

$$c_i^{d_i+1} = \sum_{j=0}^{d_i} q_{ij}c_i^j \tag{3.3}$$

.

Now let $\bar{f_G}$ be a polynomial obtained from $f_G(c_1,c_2,...,c_n)$ by writing f_G as a linear combination of monomials and replacing $c_i^{e_i}$, where $e_i > d_i$, using equation (3.3). Then $\bar{f_G}$ is a polynomial of degree at most d_i in c_i for each $i = 1,...,n$ and $\bar{f_G}(c_1,c_2,...,c_n) = f_G(c_1,c_2,...,c_n)$

for all n-tuples $(c_1, c_2, ..., c_n)$ in $S_1 \times S_2 \times ... \times S_n$. Thus, by equation (3.2), $\bar{f_G} = 0$ for all n-tuples $(c_1, c_2, ..., c_n)$ in $S_1 \times S_2 \times ... \times S_n$. We now apply Lemma 3 to see that $\bar{f_G} \equiv 0$.

But since $EE(D) \neq EO(D)$ by assumption, Lemma 6 gives us that the coefficent of $\prod_{i=1}^{n} c_i^{d_i}$ in f_G is nonzero. Let a be this coefficient. Since in $\prod_{i=1}^{n} c_i^{d_i}$, the exponent of each c_i is at most d_i, our substitution did not affect this term, and $a\prod_{i=1}^{n} c_i^{d_i}$ is a term in $\bar{f_G}$. So in order for $\bar{f_G} \equiv 0$ to hold, there must be monomials in $\bar{f_G}$ summing to $-a\prod_{i=1}^{n} c_i^{d_i}$. We show this cannot happen.

Recall that f_G is homogeneous with total degree on each term equal to the number of edges in G. We reduce the total degree on each monomial to which we apply the substitution in 3.3. Thus, our new monomial will not have degree equal to the number of edges in G, and the substitution cannot result in $-a\prod_{i=1}^{n} c_i^{d_i}$. Thus, the coefficient of $\prod_{i=1}^{n} c_i^{d_i}$ is $a \neq 0$, which is a contradiction to the fact that $\bar{f_G} \equiv 0$. Hence, there must be a proper coloring from the lists S_i. $\square$

Chapter 4

Restricted List Colorings

4.1 List Edge Coloring

Thus far we have examined only vertex colorings. A natural question to ask is, "What happens when we color edges instead of vertices?" We define an *edge coloring* of a graph similarly to a vertex coloring; that is, an edge coloring is a function $c : E(G) \to \mathbb{Z}^+$ such that if any two edges e_1, e_2 share a vertex between them, then $c(e_1) \neq c(e_2)$. We note that only loopless graphs are edge colorable since a loop is self-incident, but can only be assigned one color [3]. We define the *chromatic index* of G, $\chi^{'}(G)$, to be the least integer k such that G has an edge coloring using k colors; if the context is clear, we will write $\chi^{'}$. If G has a proper edge coloring from the set $\{1, 2, ..., k\}$, we say that G has a *k-edge-coloring*. Note $k \geq \chi^{'}$.

If G is a graph, the *line graph* of G is constructed by treating the edges of G as vertices with e_1 and e_2 being adjacent if they are incident with a common vertex in G [3]. One approach to the edge coloring problem is to perform a vertex coloring on the line graph of G. We saw an example of this technique in Chapter 1 (see p. 6).

In this section, we will study list edge colorings. To define these formally, let G be a graph, $C \subset \mathbb{Z}$ be a set of colors, and $L : E(G) \to 2^C$ be a function assigning a list of colors to each edge of G. If there is a function $c : E(G) \to \mathbb{Z}$ such that $c(e)$ is in $L(e)$ for each edge e in $E(G)$, we say G has a proper *list edge coloring*. Let L be a function and k be a constant such that $|L(e)| = k$ for every edge e. If G has an edge coloring from any such list assignment, we say that G is *k-edge-choosable*. We define the *list chromatic index*, $\chi'_L(G)$, to be the smallest such k that guarantees G is k-edge-choosable; if the context is clear, we will write χ'_L. Similarly to vertex colorings, we have $\chi'_L(G) \geq \chi'(G)$ for all graphs G. The difference with vertex colorings, however, is that equality may hold, which is the statement of the famous List Edge Coloring Conjecture:

Conjecture 1 (List Edge Coloring Conjecture)**.** *Let G be a loopless graph. Then $\chi'_L(G) = \chi'(G)$.*

Much work has gone into proving the List Edge Coloring Conjecture (which remains open). We refer the reader to [7], where Häggkvist and Chetwynd mention that it has been proven for trees, snarks (graphs which are 3-regular, but not 3-edge-colorable), 2-regular graphs, and the complete bipartite graphs $K_{3,3}$, $K_{4,4}$, $K_{6,6}$ and $K_{r,n}$ with $7r \leq 20n$ using some of the techniques we saw in the last two chapters to color the line graph of G. Since any edge coloring problem can be translated to a vertex coloring problem via the line graph, and since we have already extensively discussed vertex colorings, we opt now to consider restrictions on the edge coloring problem.

Let G be a graph and q and k be nonnegative integers. Let $C = \{1, 2, ..., k + q\}$ and $L : E(G) \to 2^C$ be a list assignment function such that $|L(e)| = n$ for all edges e in $E(G)$. Denote by $\chi'_{L,q}(G)$ the least value k can take and still assure G will be L-choosable. We call $\chi'_{L,q}(G)$ the *q-restricted list chromatic index* of G [7]. The name reflects the idea that we are restricting each edge from taking q values by assigning to its list only k colors from a set of $k + q$ colors. If we know that the q forbidden colors come from the set of colors

$\{1, 2, ..., r\}$ ($r \geq q$, of course), then we write $\chi'_{L,q,r}(G)$. Häggkvist and Chetwynd prove some upper bounds for these values in [7]. In order to give the reader a taste of list edge colorings, we will present some of their bounds. First, however, we must define a few more terms.

Let G be a graph and $k(e)$ be a color we wish to prohibit the edge e from taking. Let c be a proper edge coloring of G. If $c(e) = k(e)$, we say that the edge e is a *violation*, and define the *graph of violations* to be the subgraph induced by such edges.

Proposition 1 (Häggkvist and Chetwynd, [7])**.** *Let G be a loopless graph (possibly with parallel edges) on n vertices. Let r be a nonnegative integer and suppose G has an r-edge-coloring, c. Assume as well that one color from the set $\{1, 2, ..., r\}$ is prohibited at each edge. Let t be the least integer such that $n < t!$. Then there is a proper edge-coloring c' of G obtained from c by a permutation of the set $\{1, 2, ..., r\}$ and by switching colors on parallel edges so that the graph of violations is simple and has maximum degree at most $t - 1$.*

Proof. Let G be a loopless graph and $n = |V(G)|$. For each edge e in $E(G)$, let $k(e)$ be a color we wish to prohibit at e. Assume c is a proper r-edge-coloring of G. Further, let t be an integer and T be a set of t edges incident with some vertex v.

Since c is a proper r-edge-coloring, each edge in T is assigned a different color. Thus, if we permute the colors used on G and every edge e in T is then colored with $k(e)$, the prohibited colors in T must be distinct. We call such a permutation a *bad permutation for T*. It follows that the number of bad permutations for T is $(r - t)!$ because in order to have a bad permutation, there are t colors we must fix. The vertex v has at most $\binom{\Delta}{t}$ t-sets. Since $n = |(V(G)|$, the total number of bad permutations in G is at most $n\binom{\Delta}{t}(r - t)!$.

Now if $n\binom{\Delta}{t}(r - t)! < r!$, there is some good permutation where all the vertices of G have at most $t - 1$ violations. Note that in order for a graph to have a proper r-edge-coloring, we must have $r \geq \Delta$. Thus, if equality holds, we can be assured of a good permutation if $n\binom{\Delta}{t}(\Delta - t)! < \Delta!$. Since $n\binom{\Delta}{t}(\Delta - t)! = n\frac{\Delta!}{t!(\Delta - t)!}(\Delta - t)!$, this simplifies to $n < t!$. If $r > \Delta$, we use the same proof with the fact $n\binom{\Delta}{t}(r-t)! < n\binom{r}{t}(r-t)!$. So, if t is the smallest integer such that $n < t!$, we have a coloring $c^{'}$ given by applying a good permutation to c and the graph of violations of G under $c^{'}$ has maximum degree $t-1$.

Now consider vertices v and w with parallel edges $e_1, e_2, ..., e_m$ between them. Since c is a proper edge coloring, the colors used on these edges are distinct. If, say, e_1 and e_2 are both colored with their prohibited colors, then $k(e_1)$ and $k(e_2)$ are distinct. Thus, we can switch the colors $c(e_1)$ and $c(e_2)$ and reduce the number of violations. The result is a graph of violations which is simple. The proposition follows. □

One of the most famous results in edge colorings is Vizing's Theorem. We state it as given in [3], but omit the proof; instead we refer the reader to [3] for a proof of the original and generalized versions of Vizing's Theorem.

Theorem 11 (Vizing's Theorem, [3])**.** *Let G be a simple graph. Then $\chi^{'}(G) \leq \Delta + 1$.*

In 1964, Vizing extended his theorem to graphs with parallel edges. We define the *multiplicity* of G, $\mu(G)$, to be the maximum number of parallel edges between any two vertices of G; if the context is clear, we write μ [3]. Then the generalization of Vizing's Theorem is as follows:

Theorem 12 (Vizing, [3])**.** $\chi^{'} \leq \Delta + \mu$.

We can apply this version of Vizing's Theorem to Proposition 1 to obtain an upper bound on $\chi^{'}_{L,1,r}(G)$.

Theorem 13 (Häggkvist and Chetwynd, [7])**.** *Let G be a loopless graph on n vertices and let t be the least integer such that $n < t!$. Then taking $r = \chi^{'}$, we have $\chi^{'}_{L,1,r} \leq r + t$.*

Proof. We proceed as in the proof of Proposition 1. In that proof, our graph of violations had $\mu = 1$ and $\Delta \leq t-1$. So the graph of violations can be colored with t colors by

Vizing's Theorem. Take these colors to be $\{r+1, r+2, ..., r+t\}$. Thus, since the prohibited colors come from the set $\{1, ..., r\}$, we can find a proper $(r+t)$-edge-coloring of G without violations. $\square$

We can generalize this theorem to prohibiting a set of q colors at each edge as follows:

Theorem 14 (Häggkvist and Chetwynd, [7]). *Let G be a loopless graph on n vertices and let q and t be positive integers such that $nq^t < t!$. Set $r = \chi'$. Then $\chi'_{L,q,r} \leq r + t + \mu - 1$.*

Proof. We follow similar reasoning as in the proof of Proposition 1. This time, assume we have a set of q prohibited colors assigned to each edge. G has a proper r-coloring since $r = \chi'$. We let T be a set of t edges incident with a vertex v as before. There are q^t possible assignments of colors to T resulting in each of the t edges having a prohibited color. Since v has at most $\binom{\Delta}{t}$ t-sets, v has at most $q^t\binom{\Delta}{t}$ possible sets having bad color assignments. Thus, there are at most $nq^t\binom{\Delta}{t}(r-t)!$ possible bad permutations of the colors $\{1, ..., r\}$. So if $nq^t\binom{\Delta}{t}(r-t)! < r!$, we are assured a good permutation. As before, this simplifies to $nq^t < t!$. So if $nq^t < t!$, we have a good permutation in which the maximum degree of the graph of violations is $t-1$. Note that the multiplicity of the graph of violations is at most μ, the multiplicity of G. By Vizing's Theorem, the graph of violations has a proper edge coloring from the set $\{r+1, r+2, ..., r+t+\mu-1\}$. Thus, G can be edge colored with $r+t+\mu-1$ colors and $\chi'_{L,q,r} \leq r+t+\mu-1$. $\square$

If C is a set of colors and G is a graph, we can imagine forming a list assignment on $E(G)$ by specifying a set of colors $S(e)$ we do *not* want to allow on an edge e, and then placing $C \setminus S(e)$ in the list on e. Thus, the above results reflect list edge coloring problems with the added restriction that we know a little information about the list assignment; specifically, we know a value r for which every color greater than r is guaranteed to be in $L(e)$. We would like to, however, find an upper bound for $\chi'_{L,q}$. That is, we want to find an upper bound for the minimum k necessary to ensure a list coloring from lists of size k taken from the set $\{1, 2, ..., k+q\}$. To do this, we continue our line of reasoning from the previous theorems.

Theorem 15 (Häggkvist and Chetwynd, [7]). *Let G be a loopless graph on n vertices and let q and t be positive integers with $nq^t < t!$. Then $\chi'_{L,q} \leq \chi' + 2t + q - 4$.*

Proof. Let G be a loopless graph on n vertices, and let q and t be positive integers such that $nq^t < t!$. Set $r = \chi'$. As in the preceding proofs, since $nq^t < t!$, there is a good permutation of the colors $\{1, 2, ..., r\}$ such that the graph of violations G_v has $\Delta(G_v) \leq t-1$ and is simple. Let e be an edge in G_v. Then e is assigned one of its prohibited colors, and since the color assignment is from the set $\{1, 2, ..., r\}$, we know one of e's q prohibited colors is at most r. So e has at most $q-1$ prohibited colors in the set $\{r+1, r+2, ...\}$. We also know that e is incident with exactly two vertices in G_v, and each of these vertices is of degree at most $t-1$. Thus, e shares a vertex with at most $2((t-1)-1) = 2t-4$ other edges. Then let p be an integer such that $p \geq 2t-3$ and assign a list of p allowed colors to each edge e in G_v. There is a proper edge coloring of G_v such that $c(e)$ is in the set of p colors assigned to e since every edge has more colors than it has incident edges. Thus, if we have lists of size $q-1+2t-3 = q+2t-4$, we are guaranteed to have a proper coloring of G_v with no violations. Hence, we need at most $r+2t+q-4 = \chi' + 2t + q - 4$ colors in the list on each edge of G to have a proper edge coloring, which implies $\chi'_{L,q} \leq \chi' + 2t + q - 4$. $\square$

4.2 Game List Coloring

For those more whimsical readers, we now introduce game list colorings. Borowiecki, Sidorowicz, and Tuza give a brief history of game colorings in [4] before detailing the game we now outline. We define a two-player game with players Alice and Bob. Because Bob is a gentleman, Alice will have the first move. The playing surface will be a simple graph G with a list assignment $L : V(G) \rightarrow 2^{\mathbb{Z}^+}$. Alice and Bob take turns choosing an uncolored vertex and assigning it a color from its list, with Alice having the ultimate goal of producing a proper coloring of G, and Bob attempting to block her. We say that G is *game list colorable* if G and L are such that Alice can always produce a proper coloring, and we define the *game choice number* of G, $ch_g(G)$, to be the least integer k such that G is game list choosable with lists of length k; if the context is clear, we will write ch_g. From our previous results and the definition of $ch_g(G)$, we have the following inequalities: $\chi \leq \chi_L \leq ch_g$. Note that if every vertex is assigned a list of length $deg(v) + 1$, then there is always one more color available at v than v has colored neighbors. Thus, Alice will always find a proper coloring with such a list assignment, and we can add an upper bound for ch_g to our list of inequalities: $\chi \leq \chi_L \leq ch_g \leq \Delta + 1$.

Based on the results on D-choosability found in [6] (see Chapter 3), we may prove the following theorem from [4]. It states that if G is a graph with a list assignment L such that $|L(v)| \geq deg(v)$ for all vertices v and at least one vertex u has a list of length strictly greater than $deg(u)$, then G is L-choosable. We will show that this theorem does not extend to game choosability.

Theorem 16. *Let G be a connected graph and L a list assignment to the vertices of G such that:*

1) $|L(v)| \geq deg(v)$ for all vertices v in $V(G)$.

2) There is a vertex u in G with $|L(u)| \geq deg(u) + 1$.

Then G is L-choosable.

Proof. If G is D-choosable, then we are done. Thus, we assume G is not D-choosable. Then G consists of odd cycles and complete graphs glued together as in Chapter 3.

First we assume that G is an odd cycle. Then every vertex of G has at least two colors in its list, with some vertex u having at least three colors. Let x and y be the neighbors of u.

Begin coloring G with x and continue around the cycle toward y, leaving u to be the last vertex colored. Since each vertex in the path from x to y will have one more color than it has colored neighbors, this is possible. Finally, since u has at least three colors, the coloring can be completed.

Now assume G is a complete graph on n vertices. As before, let u be a vertex with at least $n = deg(u) + 1$ colors in its list. Begin coloring G with any vertex in $V(G) \setminus \{u\}$, leaving u to be the last vertex colored. Then every vertex in $V(G) \setminus \{u\}$ will have u left as an uncolored neighbor and hence, will have at least one color available in its list. Choose this color for v. We note that u has a remaining color, and we can finish the coloring.

Now assume that G is some combination of odd cycles and complete graphs, and that u is a vertex with $|L(u)| \geq deg(u) + 1$. We begin coloring by finding a vertex v at maximal distance from u, and coloring its block as above, with the exception that we leave for last the separating vertex y joining this block with the next block in G. Since each vertex in v's block that is not y will then have at least one uncolored neighbor, we can color each vertex in the block as above. Note that since y joins the block with an uncolored block, y can be colored as well. We repeat this procedure until every block except the one containing u is colored. The block containing u is either an odd cycle or a complete graph and may be colored as above. Thus, G is L-choosable. $\square$

Before we show that this theorem does not extend to game choosability, we must make a definition and introduce some notation. We define an *ear* of G to be a path of length at least one with its endpoints in G, but its internal vertices not in G. See Figure 4.1 for an example. In Figure 4.1, the vertices on the path $h_1p_1p_2p_3p_4h_3$ form an ear.

Let G be a graph and L be a list assignment such that $|L(v)| \geq deg(v)$ for all vertices v in $V(G)$. Let X be the set of vertices with $|L(v)| = deg(v)$ and Y be the set of vertices with $|L(v)| \geq deg(v) + 1$.

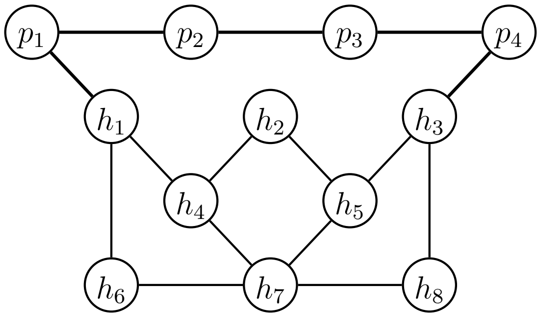


Figure 4.1: P is an ear of H

Proposition 2 (Borowiecki et al., [4])**.** *If n is a positive integer, then there is a connected graph G and a list assignment L with $|L(v)| \geq deg(v)$ for all vertices v in $V(G)$ such that $|Y| - |X| > n$ and G is not game list choosable.*

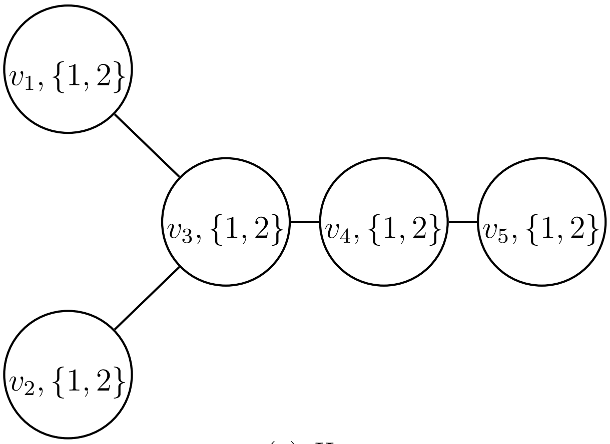
Proof. Let n be a positive integer, and let H be a connected graph on an even number m of vertices, such that $m > n+8$. Let L_H be a list assignment on H such that $|L_H(v)| = deg(v)+1$ for every vertex v in $V(H)$. Let P_4 be a path on four vertices and L_P be a list assignment on P_4 such that $L(u) = \{1, 2\}$ for all u in $V(P_4)$. Create the graph $G = H \cup P_4$ by attaching P_4 as an ear to H, as in Figure 4.1. We will show that G is the graph we want.

First note that $|Y| \geq |V(H)| - 2 > n + 6$ since we are adding edges to two vertices in H, which may increase the degrees on these vertices to equality with their list lengths. Similarly, $|X| \leq |V(P_4)| + 2 = 6$ since every vertex v of P_4 has $|L(v)| = deg(v)$ after the addition of edges to connect the endpoints with H, and as mentioned, we may be adding two vertices of H to the set X. Then $|Y| - |X| > n + 6 - 6 = n$. It remains to show that G is not game list choosable.

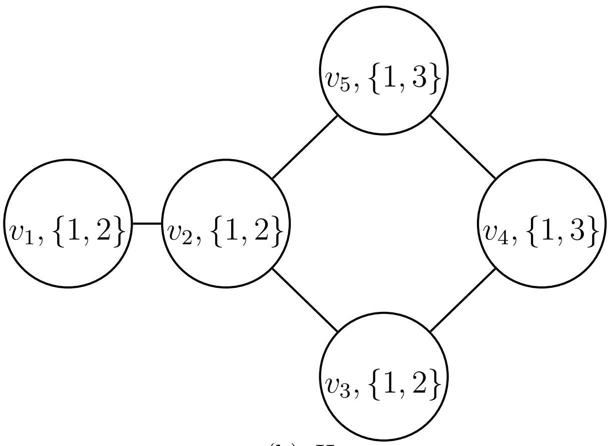
Since H has an even number of vertices, and Bob moves on even turns, Bob can force Alice to be the first to color a vertex on the P_4 subgraph of G. Without loss of generality, suppose Alice chooses the color 1 for a vertex u on the P_4 subgraph. Bob's next move is to color the vertex on the P_4 subgraph at distance two from u with the color 2, thus preventing Alice from finding a proper coloring of G and ending the game. $\square$

Borowiecki et al. also completely characterize graphs with $ch_g = 2$. We proceed similarly to our characterization of D-choosable graphs by first specifying which graphs do not have $ch_g = 2$ and then stating which graphs do.

Lemma 7 (Borowiecki et al., [4])**.** *Let H_1 and H_2 be as in Figure 4.2. Define the set $\mathcal{S} = \{C_{2k+1}, H_1, H_2, P_5, K_{2,3}\}$. If $ch_g(G) = 2$, then G does not contain any member of $\mathcal{S}$ as a vertex-induced subgraph.*



(a) H_1



(b) H_2

Figure 4.2: H_1 and H_2

Proof. Let G be a graph with $ch_g(G) = 2$. We show that G cannot have a vertex-induced subgraph in the set $\mathcal{S}$.

We know $\chi \leq \chi_L \leq ch_g$ for all graphs G and $\chi(C_{2k+1}) = 3$. Thus, if $ch_g(G) = 2$, G contains no odd cycles as vertex-induced subgraphs.

Suppose now that G contains H_1 as a vertex-induced subgraph. Label the vertices of H_1 with v_i as in Figure 4.2(a). Let L be the list assignment on G such that $L(v_i) = \{1, 2\}$ for vertices v_i in H_1 as in Figure 4.2(a) and $L(u) = \{3, 4\}$ for u in $V(G \setminus H_1)$. We have two cases:

Case 1 Suppose Alice's first move is to color a v_i in the H_1 subgraph of G. Without loss of generality, Alice colors this vertex with 1. Then Bob chooses a vertex v_j in the H_1 subgraph of G such that v_i and v_j share a neighbor, v_k. Bob colors v_j with 2, which prevents Alice from achieving a proper coloring at v_k.

Case 2 Suppose in her first move, Alice does not color a vertex in the H_1 subgraph of G. Then Bob colors v_1 with 1. We have the following cases:

a) Alice colors a v_j, for $j = 2, 3, 4$ on H_1. Then Bob chooses v_k so that v_k and v_j share a neighbor. See Figure 4.2(a) to see that this is possible. He colors v_k with the opposite color from that used on v_j. The neighbor of v_j and v_k cannot be colored.

b) Alice colors v_5. Bob chooses v_2 and colors it 2. Now v_3 cannot be colored.

c) Alice does not color a v_j on H_1. Then Bob colors v_2 with 2 and v_3 cannot be colored as in case (b).

Thus, G cannot contain H_1 as a vertex-induced subgraph.

Suppose G contains H_2 as a vertex-induced subgraph. Let L be a list assignment on G with $L(v_i)$ as given in Figure 4.2(b) for all v_i in $V(H_2)$ and $L(u) = \{3, 4\}$ for u in $V(G \setminus H_2)$. Again, we have two cases:

Case 1 Alice's first move is to color a v_i in H_2. We have the following cases:

a) If Alice colors v_1 or v_3, then Bob colors the other one with the opposite color, making v_2 uncolorable.

b) Alice colors v_2 with 1. Then Bob colors v_4 with 1, making v_5 uncolorable. The case where Alice colors v_2 with 2 is symmetric.

c) Alice colors v_i for $i = 4, 5$. If she colors v_i with 1, then Bob chooses a vertex v_j at distance two from v_i and colors it with $L(v_j) \setminus \{1\}$. Then a common neighbor of v_i and v_j cannot be colored. The case where Alice colors v_i with 3 is symmetric.

Case 2 In her first move, Alice does not color a v_i in H_2. Bob colors v_1 with 1. We have two cases:

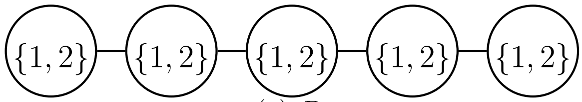
a) Alice colors v_j from $\{v_2, v_3\}$. Bob chooses a vertex v_k at distance two from v_j. If Alice colored v_j with 1, then Bob colors v_k with $L(v_k) \setminus \{1\}$, and a common neighbor of v_j and v_k cannot be colored. If Alice colors v_j with 2, then Bob colors v_k with 1, and again, a common neighbor of v_j and v_k cannot be colored.

b) Alice does not color v_2 or v_3. Then Bob colors v_3 with 2 and v_2 cannot be colored.

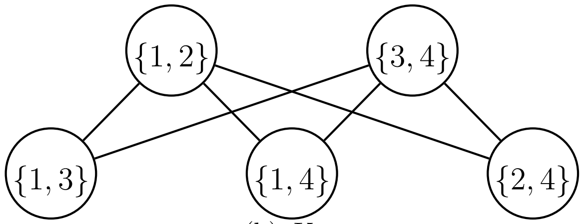
Thus G cannot contain H_2 as a vertex-induced subgraph.

The proofs that P_5 and $K_{2,3}$ cannot be vertex-induced subgraphs of G are similar. We provide the reader with appropriate list assignments for these subgraphs in Figure 4.3 and leave the details of the proof to the reader. □

A *star*, S_n, is a complete bipartite graph of the form $K_{1,n}$ [5]. See Figure 4.4 for an example. Noting that P_2 is the same as S_1 and P_3 is the same as S_2, it is clear that the following lemma is a direct consequence of Lemma 7.



(a) P_5



(b) $K_{2,3}$

Figure 4.3: P_5 and $K_{2,3}$

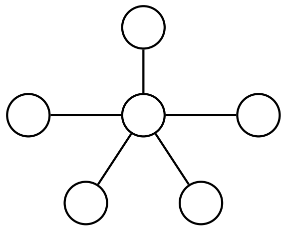


Figure 4.4: S_5

Lemma 8 (Borowiecki et al., [4])**.** *Let G be a simple graph with $ch_g(G) = 2$. Then the components of G are all isomorphic to P_4, C_4, or S_n for some $n \geq 1$.*

We now introduce some terminology and provide conditions under which Alice can produce proper colorings of the components of G. Suppose a graph G is partially colored. Let v be an uncolored vertex with k uncolored neighbors. Remove from $L(v)$ all colors used on the colored neighbors of v. Call this new list $L^{'}(v)$. If $\left|L^{'}(v)\right| \geq k$, then we call v *safe*. It is clear that if at any point in the game all remaining vertices are safe, then Alice will win the game.

Lemma 9 (Borowiecki et al., [4])**.** *If Bob is the first to color a P_4 component of G, then Alice can achieve a proper coloring of this component.*

Proof. Let $v_1v_2v_3v_4$ be a path on 4 vertices. By symmetry, we can reduce the proof to two simple cases:

Case 1 Suppose Bob colors v_1 with c.

a) Assume $L(v_2) = L(v_3)$. If c is in $L(v_3)$, then Alice colors v_3 with c, and v_2 and v_4 are safe. If c is not in $L(v_3)$, then c is not in $L(v_2)$. Alice chooses any color she likes for v_3, and v_2 and v_4 are safe.

b) Assume $L(v_2) \neq L(v_3)$. Then there is a color d in $L(v_3) \setminus L(v_2)$, and Alice colors v_3 with d. Then v_2 and v_4 are safe.

Case 2 Suppose Bob colors v_2 with c. $L(v_3) \setminus \{c\}$ has at least one color, d. Alice colors v_3 with d, and v_1 and v_4 are safe.

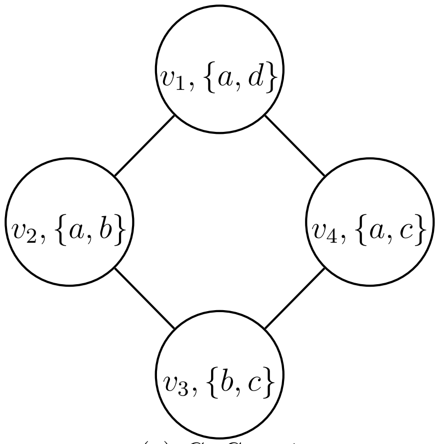
Since Alice can achieve a situation where all the remaining vertices of P_4 are safe, Alice can properly color the P_4 component once Bob begins the coloring. $\square$

Consider P_4 with the list $\{1, 2\}$ on each vertex. If Alice is the first to color a vertex, then Bob can easily block her by choosing a vertex at distance two and coloring it the opposite color as in the proof of Lemma 7. Thus, if $ch_g(G) = 2$ and G has a P_4 component, $|V(G) \setminus V(P_4)|$ must be odd so that Alice can force Bob to be the first to color a vertex of the P_4 component.

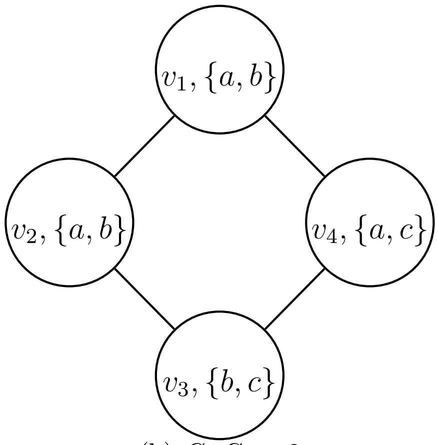
We now study what happens when G contains a C_4 component, showing situations in which Alice can win the game and situations in which Bob can win the game.

Lemma 10 (Borowiecki et al., [4])**.** *Let G be a graph with a C_4 component and L be a list assignment on this component given by Figure 4.5. If Alice begins coloring this component, then she can complete the coloring. If, however, Bob begins coloring this component, then he can block Alice.*

Proof. Let C be a 4-cycle with one of the list assignments given by Figure 4.5. Then we have one of two cases:



(a) C_4 Case 1



(b) C_4 Case 2

Figure 4.5: C_4 with two list assignments

Case 1 Alice is the first to color a vertex on C. Alice chooses v_2 and colors it with a. If Bob chooses v_3, then Alice next colors v_1 with $L(v_1) \setminus \{a\}$, and v_4 can be colored with a. If Bob chooses v_1 or v_4, then v_1 and v_3 are both safe and Alice can complete the coloring.

Case 2 Bob is the first to color a vertex on C. Bob chooses v_1 and colors it with a. Then v_3 cannot be colored because v_2 must be colored b and v_4 must be colored c.

Thus, if Alice is the first to color C, she may complete the coloring, but if Bob is the first to color C, he can block Alice, as claimed. $\square$

We now show that if G has a C_4 component with a list assignment L such that $|L(v)| = 2$ for every v in $V(C_4)$ and L is different from the list assignments in Figure 4.5, then Alice can complete the coloring as long as Bob is the first to color this component.

Lemma 11 (Borowiecki et al., [4])**.** *Let C be a 4-cycle and L be a list assignment on C such that $|L(v)| = 2$ for all v in $V(C)$, but L is not one of the assignments given by Figure 4.5. If Bob is the first to color a vertex of C, then Alice can achieve a proper coloring of C.*

Proof. Let $C = v_1v_2v_3v_4v_1$ be a 4-cycle and L be a list assignment satisfying the criteria of the lemma. Bob chooses a vertex, say v_1, and colors it with a color in its list, c_1. We have two cases:

Case 1 The color c_1 is in $L(v_3)$. Then Alice colors v_3 with c_1 and v_2 and v_4 are safe.

Case 2 The color c_1 is not in $L(v_3)$. Let $L(v_3) = \{b, c\}$. We will show that either Alice wins or we have a contradiction. If Alice does not win, then no matter which color she chooses on v_3, one of v_2 or v_4 is uncolorable. That is, without loss of generality, $L(v_2) = \{a, b\}$ and $L(v_4) = \{a, c\}$. But this means that L is one of the assignments given in Figure 4.5, and we have a contradiction.

Thus if Bob colors C first, Alice can complete the coloring. □

Now let C be the graph given in Figure 4.6. If Alice is the first to color a vertex v on C, then Bob chooses the vertex at distance two on the cycle from v and colors it the opposite color, blocking Alice from achieving a proper coloring. Therefore, we can say that if $ch_g(G) = 2$, then G does not contain two vertex-disjoint 4-cycles. As a result of Lemmas 10 and 11, we also know that if $ch_g(G) = 2$ and G contains a 4-cycle, C, then $|V(G) \setminus V(C)|$ is odd.

We define a *star forest* to be a graph G whose components are all stars. We show that star

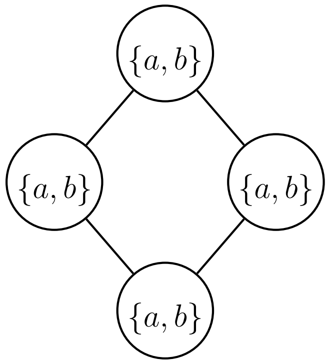


Figure 4.6: Two 4-cycles with $ch_g = 2$

forests are always 2-game-choosable and conclude this section with a complete characterization of graphs with $ch_g = 2$.

Lemma 12. *Any star forest, G, has $ch_g(G) = 2$.*

Proof. Suppose G is a star forest and S is an n-star component of G. Then S is a tree with root v and n leaves attached to v. Assume G has a list assignment such that $|L(u)| = 2$ for every vertex u in $V(G)$. If Alice is the first to color a vertex of S, then Alice chooses v and colors it with any color in its list. Note that the leaves of S are safe. If Bob is the first to color a vertex of S and colors v, then as before, the leaves are safe. So Bob colors a leaf of S. Alice colors v since $|L(v)| = 2$, and the remaining leaves are safe. Since S was an arbitrary star in G, every star in G is 2-game-choosable and $ch_g(G) = 2$ as claimed. □

We may now completely characterize graphs with $ch_g = 2$, noting that the theorem follows immediately from our discussion of the components of such a graph.

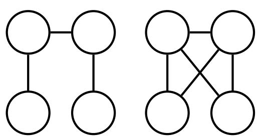
Theorem 17 (Borowiecki et al., [4]). *If G is a nontrivial graph with $ch_g(G) = 2$, then $G = k_1P_4 \cup k_2C_4 \cup H$, where k_1 is a nonnegative integer, k_2 is either 0 or 1, and H is a star forest (of odd order if k_1 or k_2 is nonzero).*

4.3 The Balanced List Channel Assignment Problem

In a basic graph coloring, the only rule is that colors assigned to adjacent vertices must be different. In practical applications, however, the colors assigned to vertices must frequently satisfy additional restrictions. A common restriction is that colors assigned to adjacent vertices must differ by some constant p, and colors assigned to vertices at distance two must differ by another constant q. That is, if uv is an edge in G and $c : V(G) \to \mathbb{Z}^+$ is a coloring of G, then $|c(u) - c(v)| \geq p$, and in addition, if x and y are vertices at distance two in G, then $|c(x) - c(y)| \geq q$. This is called an $L(p, q)$*-labeling* and arises in the channel assignment problem.

We define a *weight function* on a graph G to be a function $w : E(G) \to \mathbb{Z}^+$ assigning a positive value to each edge in G and the *weighted degree* of v, $d_w(v)$, to be the sum of the weights of all edges incident with v. The *channel assignment problem* is the problem of finding a coloring of G, $c : V(G) \to \mathbb{Z}^+$ such that $|c(u) - c(v)| \geq w(uv)$ for all edges uv in G. We will keep with our previous terminology and call such a coloring of G *proper*.

The *square* of G, G^2, is G with edges added between vertices at distance two from one another. See Figure 4.7 for an example. We can set up the $L(p, q)$-labeling problem as follows:
Let G be a graph and G^2 be its square. In G^2, let $w(uv) = p$ for all uv in $E(G)$ and $w(uv) = q$ for all uv in $E(G^2) \setminus E(G)$. We seek a coloring $c : V(G) \to \mathbb{Z}^+$ so that $|c(u) - c(v)| \geq w(uv)$ for every edge uv in $E(G^2)$.



(a) G (b) G^2

Figure 4.7: A graph and its square

We are interested in the situation in which $c(v)$ must come from a list $L(v)$. This is called the *list channel assignment problem*, and denoted by the triple (G, L, w). If G is a graph, w is a weight function on G, and L is a list assignment function on G such that $|L(v)| = d_w(v)$, then we call the triple (G, L, w) a *balanced list channel assignment problem* [8].

We call two paths P_1 and P_2 *disjoint* if no vertex on P_1 is a vertex on P_2. We say P_1 and P_2 are *internally disjoint* if they share endpoints, but do not share internal vertices. If G is a graph so that any two vertices u and v have at least k pairwise internally disjoint paths, then we say G is *k-connected* [3]. We will be interested in 2-connected graphs.

We introduce a greedy algorithm for list coloring a graph. Given a list channel assignment problem, (G, L, w), and an ordering of the vertices of G, Algorithm 1 returns a (partial) coloring $c : V(G) \to \mathbb{Z}^+$ such that $c(v)$ is in $L(v)$ for all vertices v that receive colors.

Algorithm 1 Greedy Algorithm [8]

Input: an ordering of the vertices of G, $v_1, v_2, ..., v_n$, weight function w, lists of colors $L(v_i)$
Output: a (partial) coloring c of G

```
color ← minimum color in the lists L(v_i)
max ← maximum color in the lists L(v_i)
while color < max do
  for j = 1 to n do
    if v_j is uncolored and color is in L(v_j) then
      if for every colored neighbor v_k of v_j, |c(v_k) − color| ≥ w(v_k v_j) then
        c(v_j) = color
      end if
    end if
  end for
  color ← color + 1
end while
```

We let the reader verify that in the case Algorithm 1 returns a full coloring of G, it is a proper coloring. It is also straightforward to see that if Algorithm 1 does not assign the color k to a vertex v_j, then one of the following must hold:

(i) The vertex v_j has been assigned a color k' with $k' < k$.

(ii) The color k has been assigned to a neighbor v_i with $i < j$ in the given ordering.

(iii) Some neighbor v_i of v_j has been assigned a color k' such that $\left|k' - k\right| < w(v_i v_j)$.

We now state a few lemmas building to a theorem that allows us to characterize when (G, L, w) has a proper coloring for 2-connected graphs G. In the following lemmas, we slightly abuse interval notation. By the interval $[c(v_i), c(v_i) + w(v_i v_n) - 1]$, we mean the integers $\{c(v_i), c(v_i) + 1, ..., c(v_i) + w(v_i v_n) - 1\}$.

Lemma 13 (Král' and Škrekovski, [8])**.** *Let G be a connected graph, (G, L, w) be a balanced list channel assignment problem, and $v_1, v_2, ..., v_n$ be an ordering of the vertices of G such that for every v_i with $1 \leq i \leq n-1$, there is a $j > i$ in the ordering with v_j a neighbor of v_i. Apply Algorithm 1 to this ordering. If v_n is not colored, then*

$$L(v_n) = \bigcup_{v_i v_n \in E(G)} [c(v_i), c(v_i) + w(v_i v_n) - 1]$$

and the intervals $[c(v_i), c(v_i) + w(v_i v_n) - 1]$ are disjoint.

Proof. Let G be a connected graph, (G, L, w) be a balanced list channel assignment problem, and $v_1, v_2, ..., v_n$ be an ordering of the vertices of G such that for $i = 1, ..., n-1$, v_i has a neighbor v_j with $j > i$. Suppose that v_n is not colored after applying Algorithm 1.

Then by the previous statement, since v_n is uncolored, v_n must fall into case (ii) or case (iii) above. So for every k in $L(v_n)$, k is either assigned to some v_i, $i < n$, and v_i is a neighbor of v_n or k is such that $|c(v_i) - k| < w(v_i v_k)$ for some neighbor v_i of v_k. Thus, we have

$$L(v_n) \subseteq \bigcup_{v_i v_n \in E(G)} [c(v_i), c(v_i) + w(v_i v_n) - 1].$$

Now, (G, L, w) is balanced, so $|L(v_n)| = d_w(v_n)$. Note that the number of colors in the interval $[c(v_i), c(v_i)+w(v_iv_n)-1]$ is $c(v_i)+w(v_iv_n)-1-c(v_i)+1 = w(v_iv_n)$. If any intervals overlap, we will have fewer than $d_w(v_n)$ colors in the union, meaning that some color is still available at v_n. Hence, the intervals are disjoint. Let $|[c(v_i), c(v_i)+w(v_iv_n)-1]|$ denote the number of colors in the interval $[c(v_i), c(v_i)+w(v_iv_n)-1]$. Since $|L(v_n)| = d_w(v_n) = \sum_{v_iv_n \in E(G)} |[c(v_i), c(v_i)+w(v_iv_n)-1]|$, we must have equality and

$$L(v_n) = \bigcup_{v_iv_n \in E(G)} [c(v_i), c(v_i)+w(v_iv_n)-1].$$

□

We now prove that the minimum and maximum colors in the lists must be in every list for 2-connected graphs, or (G, L, w) admits a proper coloring.

Lemma 14 (Král' and Škrekovski, [8]). *Let G be a 2-connected graph and (G, L, w) be a balanced list channel assignment problem. Let m be the minimum color and M be the maximum color in the lists on $V(G)$. If there is some vertex v in $V(G)$ such that either m or M is not in $L(v)$, then there is a proper list coloring of (G, L, w).*

Proof. Let G be a 2-connected graph and let (G, L, w) be a balanced list channel assignment problem. Assume that m is not in every list $L(v)$. Since G is connected, we must have neighbors v_1 and v_n such that m is in $L(v_1)$, but m is not in $L(v_n)$. Order the vertices of G beginning with v_1 and ending with v_n so that for every $i = 1, ..., n-1$, v_i has a neighbor v_j with $i < j$ (This can be achieved by post-ordering a depth-first search of $G \setminus \{v_1\}$ beginning at v_n). Apply Algorithm 1 and Lemma 13 to see that all vertices $v_1, ..., v_{n-1}$ are colored. Note that v_1 will be colored with m. If v_n is not colored, then Lemma 13 says that m must be in $L(v_n)$, a contradiction to our assumption. Thus, G has a proper coloring.

The case when M is not in $L(v)$ for some vertex v is symmetric. Let N be a sufficiently large integer, and assign to each vertex a new list $L^{'}(v) = \{N - k : k \in L(v)\}$. Then the minimum of these lists is $N - M$, and the proof of the previous case gives us a proper coloring $c^{'}$. The coloring we want is then given by $c(v) = N - c^{'}(v)$ for all v in $V(G)$. □

We define a *k-fan* to be a set of internally disjoint paths from a vertex x to a set of vertices Y whose endpoints in Y are distinct [3]. A famous and quite useful result in graph theory is the Fan Lemma (Proposition 3). We state it here as it is found in [3] without proof. The proof is relatively simple, and the reader is encouraged to try it as an exercise. We will use the Fan Lemma to prove our next proposition about the structure of graphs which are neither complete graphs nor cycles.

Proposition 3 (Fan Lemma, [3])**.** *Let G be a k-connected graph, v be a vertex in G, and Y be a subset of $V(G) \setminus \{v\}$ such that $|Y| \geq k$. Then there is a k-fan from v to Y in G.*

Proposition 4 ([8])**.** *Let G be a 2-connected graph that is neither a complete graph nor a cycle. Then G contains vertices u, v, and w such that uw and vw are edges in G, uv is not an edge in G, and $G \setminus \{u, v\}$ is connected.*

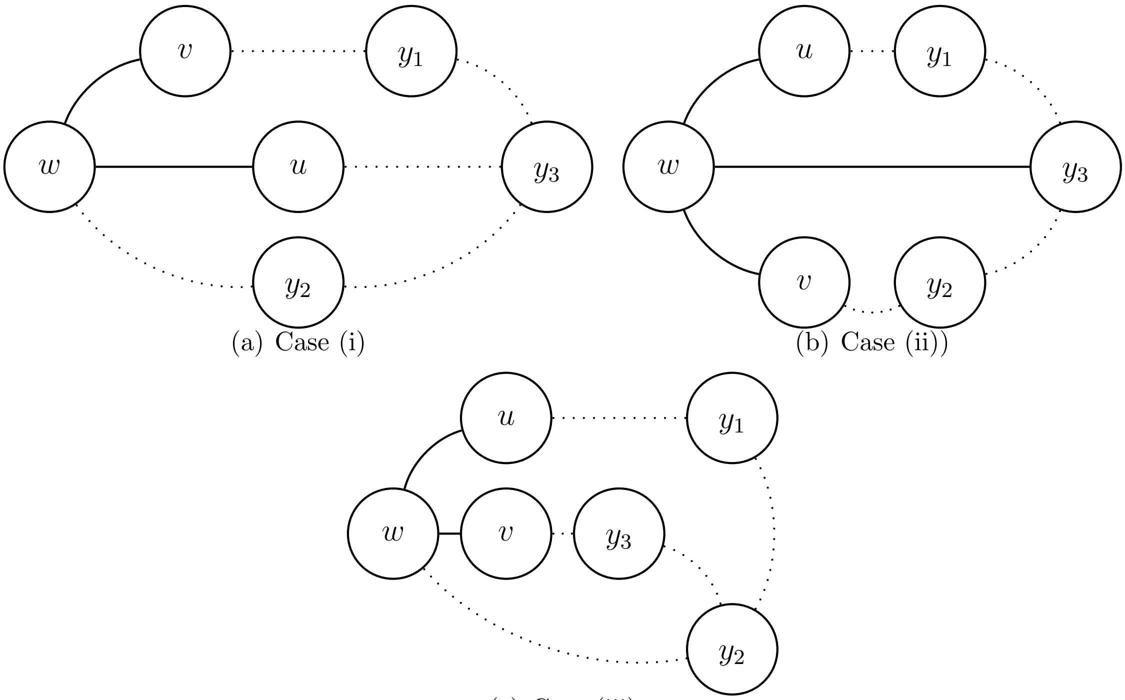
Proof. Let G be a 2-connected graph that is neither a complete graph nor a cycle and let w be a vertex in G. By the Fan Lemma, G contains a 2-fan from w to a set Y such that the endpoints in Y are distinct. Let y_1 and y_2 be these endpoints. Since G is 2-connected, there must be two vertex disjoint paths from y_1 to y_2. Thus, there must be a path from y_1 to y_2 not through w and the paths given by the 2-fan from w to y_1, y_2. Then we have a cycle C from w to y_1 to y_2 and back to w. But G is not a cycle, so there must be more to G. We have the following cases shown in Figure 4.8:

(i) There is a vertex y_3 on C and a path of length at least 2 from w to y_3. See Figure 4.8(a). Let v be a vertex adjacent to w on the path from w to y_1 (we could even have $v = y_1$) and u be a vertex adjacent to w on the path from w to y_3.

(ii) There is a vertex y_3 on C and wy_3 is a chord. See Figure 4.8(b). Let u be a vertex adjacent to w on the path from w to y_1 and let v be a vertex adjacent to w on the path from w to y_2.

(iii) There is a vertex y_3 not on C and a path from w to y_3 (since G is connected). Since G is 2-connected, there is a path from y_3 to y_2 not through w. See Figure 4.8(c). Let u be a vertex adjacent to w on the path from w to y_1 and v be a vertex adjacent to w on the path from w to y_3.

In each of the above cases, we have chosen u, v, and w so that uw and vw are edges in G, but uv is not, and so that $G \setminus \{u, v\}$ is connected. □



(a) Case (i)

(b) Case (ii))

(c) Case (iii)

Figure 4.8: The possibilities of Proposition 4

We can now begin to characterize 2-connected graphs which admit a proper coloring under a balanced list channel assignment.

Theorem 18 (Král' and Škrekovski, [8]). *Suppose G is 2-connected and neither a complete graph nor an odd cycle. Also assume (G, L, w) is a balanced list channel assignment problem. Then (G, L, w) has a proper coloring.*

Proof. Let m be the minimum color and M be the maximum color in the lists $L(v)$. By Lemma 14, we may assume that m and M are in $L(v)$ for every vertex v in G. First we assume that G is an even cycle with $2k$ vertices. Order these vertices around the cycle, $v_1 v_2 ... v_{2k} v_1$. Color the cycle by alternating m and M. Using the facts that G is a cycle, (G, L, w) is balanced and $w(v_{i-1}v_i)$ is positive, we have

$$\begin{aligned} |c(v_i) - c(v_{i+1})| &= M - m \\ &\geq |L(v_i)| - 1 \\ &= w(v_{i-1}v_i) + w(v_i v_{i+1}) - 1 \\ &\geq w(v_i v_{i+1}). \end{aligned}$$

Thus, the coloring c is proper.

So we assume that G is not an even cycle. Then by assumption, G is not a cycle; nor is it a complete graph. We apply Lemma 4 to obtain vertices x, y, and z so that x and y are each adjacent to z, but x is not adjacent to y and $G \setminus \{x, y\}$ is connected. We obtain an ordering of the vertices of G: x, y, $v_3, v_4, ..., v_n = z$ so that for $i = 3, ..., n-1$, v_i has a neighbor v_j with $j > i$ by postordering a depth-first search of $G \setminus \{x, y\}$ beginning with z. We apply Algorithm 1 to this ordering and obtain a partial coloring c of G. Since x and y are not adjacent and m is in $L(x)$ and $L(y)$, Algorithm 1 will return $c(x) = c(y) = m$. If z is uncolored, then by Lemma 13, we know

$$L(z) = \bigcup_{v_i z \in E(G)} [c(v_i), c(v_i) + w(v_i z) - 1]$$

and the intervals are disjoint. This is a contradiction, though, because $c(x) = c(y)$ and the fact that xz and yz are edges implies the intervals $[c(x), c(x) + w(xz) - 1]$ and $[c(y), c(y) + w(yz) - 1]$ are not disjoint. Thus, z must be colored and G has a proper coloring. □

Král' and Škrekovski continue [8] by characterizing list assignments on odd cycles and complete graphs that result in a balanced list channel assignment problem (G, L, w) not having a proper coloring. We include them here for completeness, but refer the reader to [8] for the proofs which require multiple lemmas, have several cases, and run to four pages in length.

Theorem 19 (Král' and Škrekovski, [8])**.** *Assume G is an odd cycle, and let (G, L, w) be a balanced list channel assignment problem. Let m be the minimum color in any of the lists and M be the maximum color. Then (G, L, w) does not have a proper coloring if and only if there are integers $a, b,$ and k such that $1 \leq a \leq b$, $1 \leq k$, and one of the following is true for every vertex v and its incident edges, e_1 and e_2:*

$$L(v) = \begin{cases} [k, k+a-1] \cup [k+b, k+a+b-1] & \text{if } w(e_1) = w(e_2) = a \\ [m, M] & \text{if } w(e_1) = a < b = w(e_2). \end{cases}$$

Theorem 20 (Král' and Škrekovski, [8])**.** *Assume G is a complete graph on the n vertices $v_1, ..., v_n$. Let (G, L, w) be a balanced list channel assignment problem. Then (G, L, w) does not have a proper coloring if and only if one of the following is true:*

(i) There are positive integers a, k_i, for $i = 1, ..., n-1$, such that $a \geq 1$, $1 \leq k_1 < k_2 < ... < k_{n-1}$, and for $i = 1, ..., n-2$, $k_i + a \leq k_{i+1}$. We must also have $w(e) = a$ for every edge e and $L(v_i) = \bigcup_{1 \leq j \leq n-1} [k_j, k_j + a - 1]$.

(ii) There are positive integers $a, b,$ and k such that $1 \leq a < b$, $k > 1$, and after reordering vertices, $w(v_i v_j) = b$ for $i, j = 1, ..., n-1$ and $w(v_i v_n) = a$ for $i = 1, ..., n-1$. We must also have $L(v_i) = [k, k + b(n-2) + a - 1]$ for $i = 1, ..., n-1$ and $L(v_n) = \bigcup_{0 \leq j \leq n-2} [k + bj, k + bj + a - 1]$.

The reader may find Theorem 18 useful for determining upper bounds on the list sizes necessary to obtain $L(p, q)$-labelings of 2-connected graphs. We consider an example using generalized Petersen graphs. The Petersen graph is given by Figure 4.9. A *generalized*

Petersen graph (GPG) consists of two cycles of length $n \geq 3$, an inner cycle, and an outer cycle with each vertex of the outer cycle being adjacent to precisely one vertex of the inner cycle. The Petersen graph is such a graph with $n = 5$. We define a particular type of GPG, an *n-star* by taking n to be odd and ordering the vertices of the inner cycle clockwise by $v_0, v_2, ..., v_{n-1}$, and requiring for $i = 0, ..., n-1$, $v_{\frac{(n-1)i}{2}}$ is adjacent to $v_{\frac{(n-1)(i+1)}{2}}$, where subscripts are taken modulo n. These definitions may be found in the author's undergraduate thesis, [2]. See Figure 4.10 for an example of a 7-star.

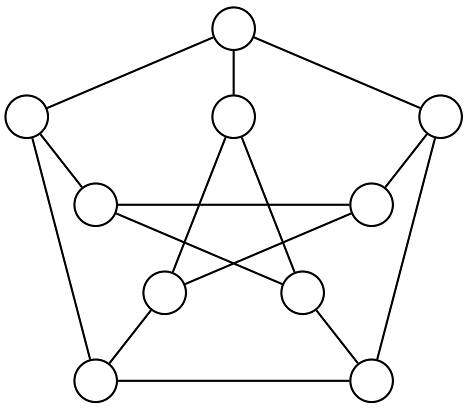


Figure 4.9: The Petersen Graph

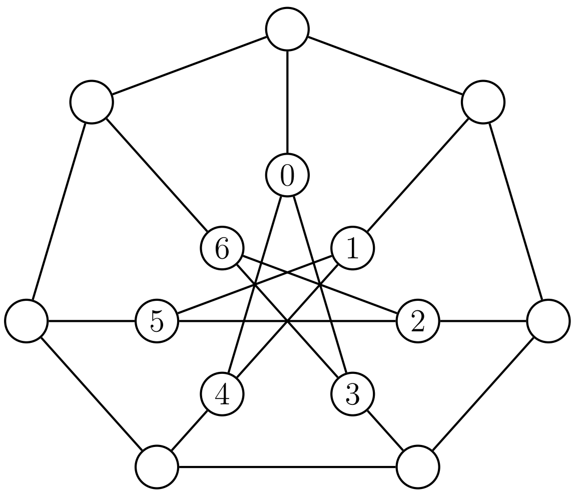


Figure 4.10: 7-star

Now note that for an n-star S with $n \geq 7$, each vertex in S has three neighbors and is at distance two from six other vertices. So, in S^2, S has nine neighbors. Then since S^2 has

$2n \geq 14$ vertices, S^2 is not a complete graph. Thus, with the weight function on S^2 given by

$$w(e) = \begin{cases} 2, & \text{if } e \text{ is an edge in } S \\ 1, & \text{if } e \text{ is an edge in } S^2 \setminus E(S), \end{cases}$$

we see by Theorem 18 that at most $2 * 3 + 6 = 12$ colors are necessary in each list on S^2 to find a list $L(2,1)$-labeling of S. The author shows in [2] that any n-star on more than five vertices has an $L(2,1)$-labeling with six colors. Let $\chi_{L,2}(G)$ denote the minimum k such that a list assignment satisfying $|L(v)| = k$ will guarantee a list $L(2,1)$-labeling of G. Then the next theorem follows:

Theorem 21. *If S is an n-star with $n \geq 7$, then $6 \leq \chi_{L,2}(G) \leq 12$.*

Since the square of the Petersen graph is a complete graph, we cannot apply Theorem 18. We leave bounds on this value as an open question, though we do note that ten colors are required to have a proper $L(2,1)$-labeling, as shown in [2].

Chapter 5

Open Questions

We have given an overview of the past thirty years of research in list colorings, but there is still much research to be done. As promised, we now present some open questions.

1) Can we characterize graphs which are 3-choosable? Alon and Tarsi show in [1] that all planar bipartite graphs are 3-choosable.

2) Does $\chi_L^{'}(G) = \chi^{'}(G)$ for all loopless graphs G? This is the List Edge Coloring Conjecture.

3) What is an upper bound for $\chi_{L,2}$ for the Petersen graph? Can we determine $\chi_{L,2}$ for all n-stars?

The question posed by Dinitz which started all of this still remains open as well:

> "Given an $m \times m$ array of m-sets, is it always possible to choose one from each set, keeping the chosen elements distinct in every row, and distinct in every column?" [6]

Bibliography

[1] N. Alon and M. Tarsi. Colorings and orientations of graphs. *Combinatorica*, 12(2):125–134, 1992.

[2] C. Baber. Distance-2 colorings of certain Generalized Petersen Graphs. Undergraduate Thesis, Clemson University, 2007 (unpublished).

[3] J.A. Bondy and U.S.R. Murty. *Graph Theory*, volume 244 of *Graduate Texts in Mathematics.* Springer, New York, 2008.

[4] M. Borowiecki, E. Sidorowicz, and Zs. Tuza. Game list colouring of graphs. *Electron. J. Combin.*, 14(1):1–11, 2007.

[5] R. Diestel. *Graph Theory*, volume 173 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1997.

[6] P. Erdős, A.L. Rubin, and H. Taylor. Choosability in graphs. In *Proceedings of the West Coast Conference on Combinatorics, Graph Theory and Computing, Congres. Num. 26*, pages 125–157, 1979.

[7] R. Häggkvist and A. Chetwynd. Some upper bounds on the total and list chromatic numbers of multigraphs. *J. Graph Theory*, 16(5):503–516, 1992.

[8] D. Král' and R. Škrekovski. A theorem about the channel assignment problem. *SIAM J. Discrete Math.*, 16(3):426–437, 2003.

[9] K. Ramachandran, E. Belding, K. Almeroth, and M. Buddhikot. Interference-aware channel assignment in multi-radio wireless mesh networks. In *INFOCOM 2006. 25th IEEE International Conference on Computer Communications. Proceedings*, pages 1–12, 2006.

[10] C. Thomassen. Every planar graph is 5-choosable. *J. of Combin. Theory Ser. B*, 62(1):180–181, 1994.

[11] M. Voigt. List colourings of planar graphs. *Discrete Math.*, 120:215–219, 1993.

[12] T. Zeitlhofer and B. Wess. List-coloring of interval graphs with application to register assignment for heterogenous register-set architectures. *Signal Process.*, 83:1411–1425, 2003.