MASTER'S DEGREE IN PHYSICS
Academic Year 2019-2020
Introduction to Many Body Theory

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HOMEWORK 2

First Exercise

To compute the Green function for a non interacting system we must start from the general expression for the green function of an homogeneous system:

$$G_{\alpha\beta}(\mathbf{k},\omega) = \hbar V \left[\frac{\langle\phi_0|\hat{\psi}_\alpha(0)|n,\mathbf{k}\rangle\ \langle n,\mathbf{k}|\hat{\psi}^\dagger_\beta(0)|\phi_0\rangle}{\hbar\omega - \mu - \mathcal{E}^{(N+1)}_{\mathbf{k}} + i\eta} + \frac{\langle\phi_0|\hat{\psi}^\dagger_\beta(0)|n,-\mathbf{k}\rangle\ \langle n,-\mathbf{k}|\hat{\psi}_\alpha(0)|\phi_0\rangle}{\hbar\omega - \mu - \mathcal{E}^{(N-1)}_{\mathbf{k}} - i\eta} \right] \tag{1}$$

where the fields are defined as follows:

$$\hat{\psi}_\alpha(0) = \sum_{\mathbf{k}'} \varphi_{\mathbf{k}'}(0)\hat{c}_{\mathbf{k}',\alpha} \qquad\qquad \hat{\psi}^\dagger_\beta(0) = \sum_{\mathbf{k}''} \varphi_{\mathbf{k}''}(0)\hat{c}^\dagger_{\mathbf{k}',\alpha}. \tag{2}$$

We notice that for a non interacting system a complete set of eigenstates for the momentum operator $\hat{\vec{P}}$ is also a complete set of eigenstates for the Hamiltonian $\hat{H}$; the excited states of the system are identified just by the $\mathbf{k}$ index and thus the sum over n in (1) is unnecessary and can be omitted.
Let's proceed computing the matrix elements in (1). The first semplification of the expression is given by the dyad $|\mathbf{k}\rangle\langle\mathbf{k}|$, which "selects" in the summations over $\mathbf{k}'$ and $\mathbf{k}''$ present in the definitions of the fields only the terms with momentum equal to $\mathbf{k}$:

$$\begin{aligned}
\langle\phi_0|\hat{\psi}_\alpha(0)|\mathbf{k}\rangle\ \langle\mathbf{k}|\hat{\psi}^\dagger_\beta(0)|\phi_0\rangle &= \sum_{\mathbf{k}'\mathbf{k}''} \langle\phi_0|\varphi_{\mathbf{k}'}(0)\hat{c}_{\mathbf{k}',\alpha}|\mathbf{k}\rangle\ \langle\mathbf{k}|\varphi^\dagger_{\mathbf{k}''}(0)\hat{c}^\dagger_{\mathbf{k}'',\beta}|\phi_0\rangle\,\Theta(k-k_F) && (3)\\
&= \sum_{\mathbf{k}',\mathbf{k}''} \varphi_{\mathbf{k}'}(0)\varphi^\dagger_{\mathbf{k}''}(0)\,\langle\mathbf{k}',\alpha|\mathbf{k}\rangle\,\langle\mathbf{k}|\mathbf{k}'',\beta\rangle\,\Theta(k-k_F) && (4)\\
&= \sum_{\mathbf{k}',\mathbf{k}''} \varphi_{\mathbf{k}'}(0)\varphi^\dagger_{\mathbf{k}''}(0)\delta_{\alpha,\beta}\delta_{\mathbf{k}',\mathbf{k}}\delta_{\mathbf{k}'',\mathbf{k}}\Theta(k-k_F) && (5)\\
&= |\varphi_{\mathbf{k}}(0)|^2\delta_{\alpha,\beta}\Theta(k-k_F). && (6)
\end{aligned}$$

In (3) we introduced the $\Theta(k-k_F)$: this is due to the fact that the field operators are acting in both the matrix elements as creation operators over a filled Fermi sphere: in this situation all contributions to the sum like $\hat{c}^\dagger_{\mathbf{k}}\,|\mathbf{k}\rangle$ for $k < k_F$ are annihilated due to the exclusion principle. The second couple of matrix elements in (1) is computed similarly:

$$\langle\phi_0|\hat{\psi}^\dagger_\beta(0)|n,-\mathbf{k}\rangle\ \langle n,-\mathbf{k}|\hat{\psi}_\alpha(0)|\phi_0\rangle = |\varphi_{-\mathbf{k}}(0)|^2\delta_{\alpha,\beta}\Theta(k_F-k); \tag{7}$$

in this case the limitation to the sum is applied to all the values of k above the Fermi level, since the ladder operators act in both the matrix element as destruction operators on the ground state $|\phi_0\rangle$.

The denominators both contain an addend which is the excitation energy of the system with $N+1$ particles ($N-1$ respectively). This term can be rewtitten as a function of the momentum k, since in a non interacting system it indicizes the excited states:

$$\begin{aligned}
\mathcal{E}^{(N+1)}_{\mathbf{k}} &= E^{(N+1)}_{\mathbf{k}} - E^{(N+1)} && (8)\\
&= E^{(N+1)}_{\mathbf{k}} - E^{(N)} - \left(E^{(N+1)} - E^{(N)}\right) && (9)\\
&= \mathcal{E}^0_{\mathbf{k}} - \mathcal{E}^0_F && (10)\\
&= \frac{\hbar^2}{2m}\left(k^2 - k_F^2\right) && (11)
\end{aligned}$$

and similarly for the $\mathcal{E}_{\mathbf{k}}^{(N-1)}$ term:

$$\mathcal{E}_{-\mathbf{k}}^{(N-1)} = \frac{\hbar^2}{2m}\left(k_F^2 - k^2\right). \tag{12}$$

The chemical potential μ is assumed equal for the $N \to N+1$ and $N \to N-1$ excitations:

$$\mu = E^{(N+1)} - E^{(N)} = E^{(N-1)} - E^{(N)} = \frac{\hbar^2 k_F^2}{2m} \tag{13}$$

1 Second Exercise