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Solutions Manual

Digital Signal Processing

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Arnhem, 17 juli 1998

Contents

Introductory Notes

The book contains problems at the end of each chapter, with an emphasis on practising and revisiting basic concepts, analysis and design methods. This Solutions Manual is fairly self-contained, although occasional reference is made to figures, equations, Worked Examples and discussion in the main text — which should therefore be used as a companion to it.

Some problems, or parts of problems, involve using the book's computer programs. This is particularly the case with later chapters. Although we have decided not to offer detailed solutions in such cases (every instructor's approach to programming being different!), we have indicated the level of computing/programming skill required by placing one of the following symbols against the problem number:

(p) indicates the need for elementary computing/programming skills — typically running the program and entering appropriate data (in response to screen prompts), or making a very minor modification to the program.

(P) indicates the need for intermediate skills — typically modifying a program to cope with a slightly different application, or a design change.

We believe that skills at either level will be well within the competence of any student who has understood the text and has taken an introductory course in computer programming.

P. A. L.
W. F.
1994

Chapter 1

Q1.1

Maximum signal frequency = 6 MHz.
Hence minimum sampling rate = 12 MHz.

For a quantization error < 1%, the ADC must have at least 50 discrete amplitude levels or "slots" (we assume that coding produces a maximum error equal to plus or minus half a slot width).

Now $2^5 = 32$ and $2^6 = 64$. Hence we need at least a 6-bit code to represent each sample value, giving a binary pulse rate of:

$$12 \times 6 = 72 \text{ Mbit/second.}$$

Q1.2

The solution follows from that for Q1.1. If only 20% of the ADC range is filled, we need 5 times as many amplitude slots as before (that is, 250 slots), to preserve the signal-to-noise ratio.

Now $2^8 = 256$, hence we need an 8-bit code. The minimum transmission rate is now:

$$12 \times 8 = 96 \text{ Mbit/second.}$$

(Note: In earlier editions, the answer given at the back of the book was incorrect)

Q 1.3

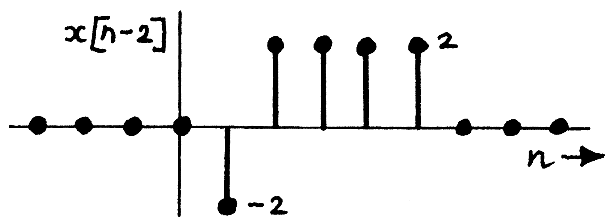
The maximum frequency in the EKG is 200 Hz, so we require a sampling rate of at least 400 samples/second. The minimum rate of information transmission is therefore:

$$400 \times 6 = 2400 \text{ bits/second.}$$

Q 1.4

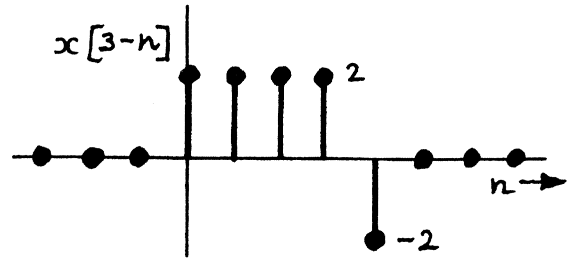
(a) The signal is the same as shown in the book, but shifted in time by two sampling intervals. (the value $x[0]$ now occurs when $[n-2]=0$ or $n=2$).

Sketch:



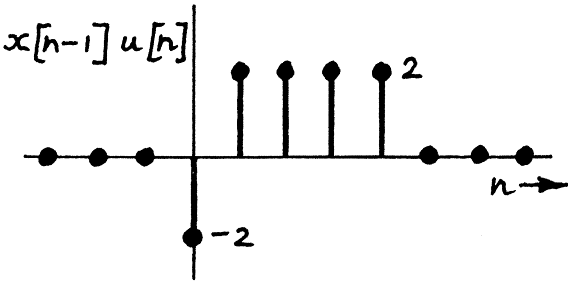
(b) This version of the signal is both shifted and time-reversed (the value $x[0]$ now occurring when $[3-n]=0$ or $n=3$).

Sketch:



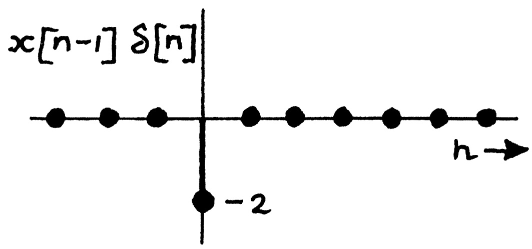
(c) This version of the signal is shifted by one sampling interval, and modulated (multiplied) by the unit step function. Actually, in this case the step function has no effect since the signal $x[n-1]$ is already zero for all $n<0$.

Sketch:



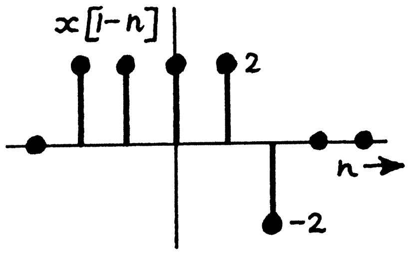
(d) Here, $x[n-1]$ is multiplied by the unit impulse function $\delta[n]$. The effect is to preserve the value where the impulse occurs (at $n=0$), reducing all other values to zero.

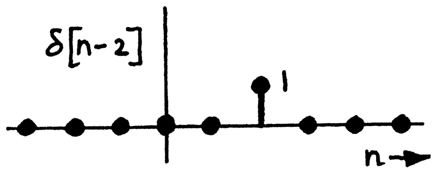
Sketch:



(e) $x[1-n]$ is a shifted, time-reversed, version of $x[n]$. $\delta[n-2]$ is a unit impulse occurring at $n=2$.

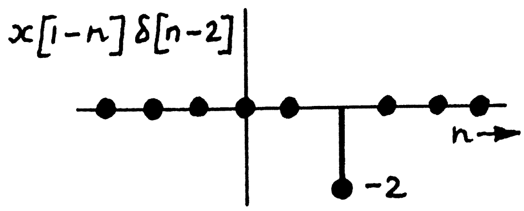
Sketches:





Their product is a weighted impulse, of value -2, occurring at $n=2$.

Sketch:



Q 1.5

There are various ways of showing the required results (which are intuitively fairly obvious). For example we may denote the successive values of an even signal and an odd signal by:

even: $\cdots \; c \quad b \quad a \quad b \quad c \; \cdots$

odd: $\cdots \; -C \quad -B \quad 0 \quad B \quad C \; \cdots$

(a) Their product is clearly:

$$-cC \quad -bB \quad 0 \quad bB \quad cC$$

which is odd. Similarly, (b) the square of the odd signal is clearly even, and (c) the sum of its values over all n is zero.

Q 1.6

(a) The signal is a weighted (multiplied by 3) unit step function, shifted by 1 sampling interval. Hence:

$$x[n] = 3\,u[n-1]$$

(b) The signal is a weighted (multiplied by -2) unit impulse function, shifted by two sampling intervals. Hence:

$$x[n] = -2\,\delta[n+2]$$

(c) There are several alternative ways of describing this signal as a set of superposed impulses, steps, or ramps. For example we may consider it as built up from:

(i) a unit ramp starting at $n=-4$ ($r[n+4]$), plus:
(ii) a ramp starting at $n=0$ with a downward slope ($-r[n]$), plus:
(iii) a ramp starting at $n=7$ with a double downward slope ($-2r[n-7]$), plus:
(iv) a final, "cancelling" ramp starting at $n=9$, with a double downward slope ($2r[n-9]$).

Hence:

$$x[n] = r[n+4] - r[n] - 2r[n-7] + 2r[n-9]$$

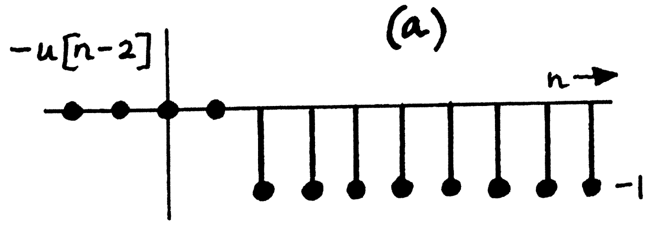
(Note: In earlier editions, the answer given at the back of the book was incorrect).

Q 1.7

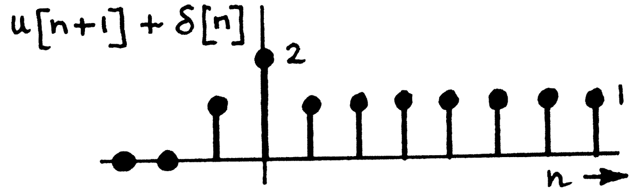
By this stage, students should be finding the description and manipulation of impulse, step, and ramp functions quite straightforward.

Sketches:

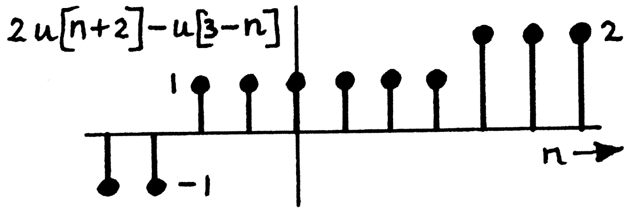
(a)



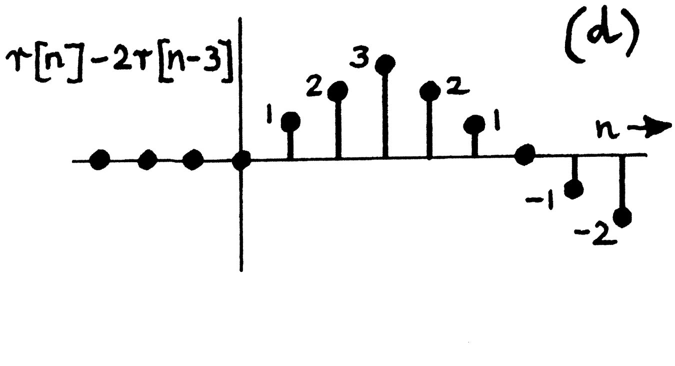
(b)



(c)



(d)



Q 1.8

A digital sinusoid $\sin(n\Omega)$ is only strictly periodic if there is an integer m such that:

$$\Omega/2\pi = m/N, \quad \text{or} \quad N = 2\pi m/\Omega$$

(a) In this case $\Omega = \pi/9$ giving $N = 18\pi m/\pi = 18m$. Putting $m=1$, we see that the signal is truly periodic with a period equal to 18 sampling intervals.

(b) In this case $N = 2m/\pi$ giving $m = N\pi/2$, which cannot be an integer. Hence the signal is not periodic.

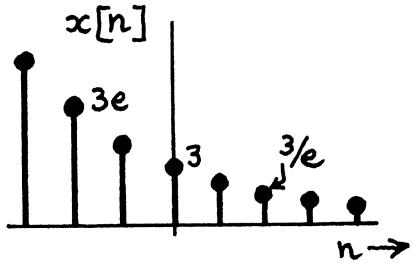
(c) Here $\Omega = n\pi/15$, showing that the frequency varies with time. Hence the signal is not periodic.

(d) The signal contains sine and cosine components. Both are strictly periodic, with periods 10 and 20 sampling intervals respectively. Hence the composite signal is also periodic with period 20 sampling intervals.

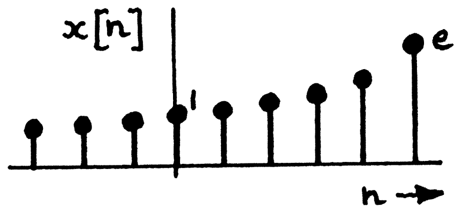
Q1.9

Typical signal portions are sketched as follows:

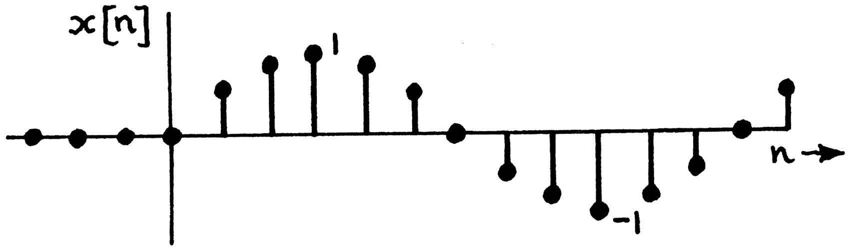
(a)



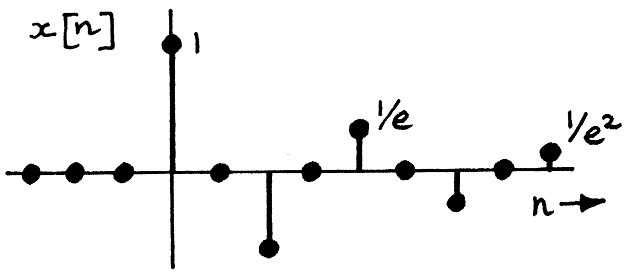
(b)



(c)



(d)



Q 1.10

This problem requires careful intuitive reasoning as much as formal proof. Some of the answers are far from obvious and may lead to differences of opinion even among experts!

(a) This is a strange system, in which forward time at the output corresponds to reverse time at the input. Thus whereas output value $y[5]$ depends on input value $x[0]$, the "later" output $y[100]$ depends on the "earlier" value $x[-95]$, and so on.

The system is not time-invariant because it exhibits increasing memory as time progresses. But it is linear, causal, and stable.

(b) Each output value is the product of two inputs. This makes the system nonlinear. However it is time-invariant (properties do not vary as time progresses); causal (output depends only on present and previous inputs); stable (output is finite and bounded for a finite, bounded, input); and it possesses memory (output depends on previous input(s)).

(c) Each output value is a simple weighted sum of the present and previous inputs. Hence the system exhibits all the properties mentioned.

(d) Each output value is a simple weighted sum of a previous input and output. Hence the system exhibits all the properties mentioned.

(e) The coefficients by which inputs are multiplied is the time variable n. Hence the system is not time-invariant. Each output depends only on the present input. Hence the system possesses no memory. But it is linear and causal. An interesting discussion can be had about whether or not it is stable!

Q 1.11

(a) Invertible, since each output value is uniquely related to a single input value.

(b) Not invertible, since many combinations of values $x[n]$ and $x[n-1]$ would produce the same $y[n]$. For example, the input sample sequence could be inverted (multiplied by -1) without altering the output sequence.

(c) Not invertible, since many combinations of n and $x[n]$ would produce the same $y[n]$. For example, all products $n\,x[n]$ equal to $0, 2\pi, 4\pi \ldots$ would give zero output.

Q 1.12

The difference equation of the system is:

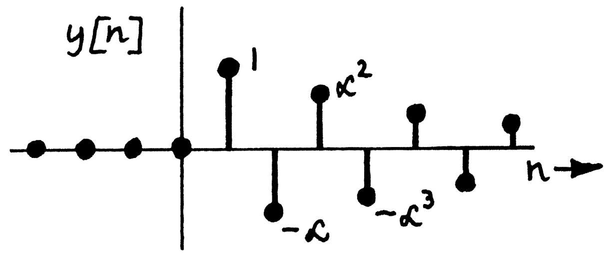
$$y[n] = x[n-1] - \alpha\, y[n-1]$$

(a) When the input is the unit impulse function, we may evaluate the output sample sequence term-by-term:

$$\begin{aligned}
y[0] &= x[-1] - \alpha\, y[-1] = 0 \\
y[1] &= x[0] - \alpha\, y[0] = 1 - 0 = 1 \\
y[2] &= x[1] - \alpha\, y[1] = 0 - \alpha = -\alpha \\
y[3] &= x[2] - \alpha\, y[2] = 0 - \alpha(-\alpha) = \alpha^2
\end{aligned}$$

etc.

Sketch:



(b)

We could similarly evaluate the output sequence for a unit step function input. Alternatively, we note that just as a unit step is the running sum of a unit impulse, so the output must be the running sum of the signal sketched above in part (a). Hence successive outputs, starting at $n=0$, must be: $0, 1, (1-\alpha), (1-\alpha+\alpha^2), (1-\alpha+\alpha^2-\alpha^3)$ etc.

Clearly, if $\alpha > 1$ or $\alpha < -1$, the magnitudes of successive output values would grow without limit. This means that the system would be unstable.

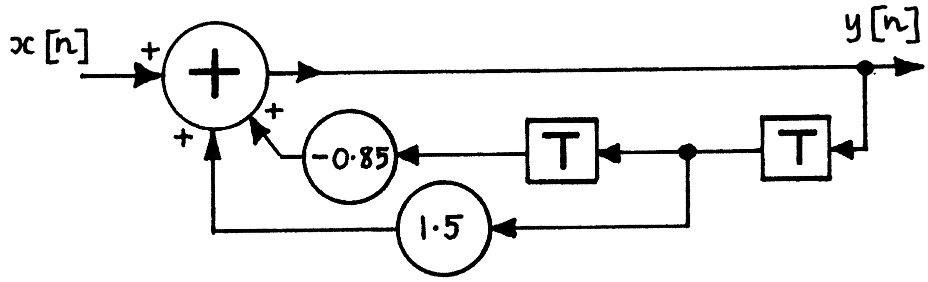
Q 1·13

Equation 1.4 in the main text is:

$$y[n] = 1{\cdot}5\, y[n-1] - 0{\cdot}85\, y[n-2] + x[n]$$

The block diagram therefore requires two multiplier units (1.5 and −0.85), two delay units, and one adder.

Sketch:



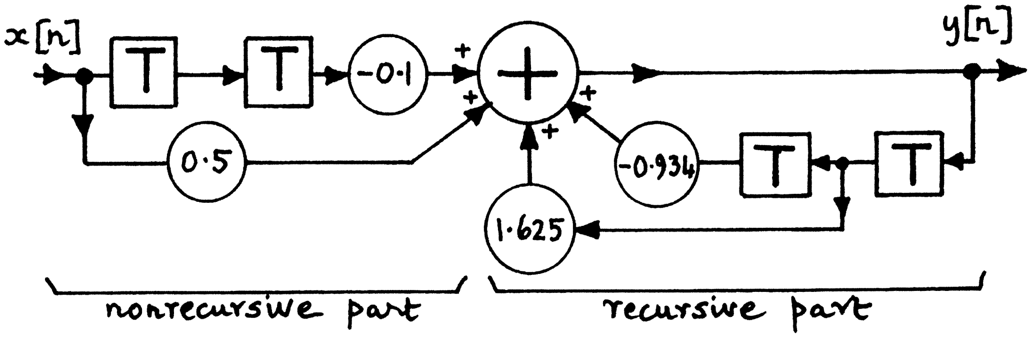
Q 1·14

The recurrence formula (difference equation) is:

$$y[n] = 1{\cdot}625\, y[n-1] - 0{\cdot}934\, y[n-2] + 0{\cdot}5\, x[n] - 0{\cdot}1\, x[n-2]$$

This requires 4 multipliers, 4 delay units and 1 adder:

Sketch:



Chapter 2

Q 2.1

The answers to this question may be written down by inspection, remembering that a unit impulse occurring at $n = m$ is denoted by $\delta[n-m]$. Hence:

(a) $x[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$

(b) $x[n] = 2\delta[n+2] + 2\delta[n] - 0.5\delta[n-1] + \delta[n-4]$

Q 2.2

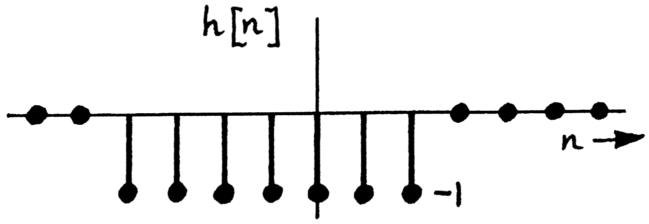
(a)

The impulse response consists of two sample values (impulses) at $n=2$ and $n=4$. The system is therefore causal and stable. It is also LTI, because the output is a simple weighted sum of inputs.

(b)

The impulse response equals the difference between two step functions, one starting at $n=-5$, the other at $n=3$.

Sketch:



Since it begins before $n=0$, the system is not causal. However $h[n]$ is finite and bounded, so the system is stable. It is also LTI because the output is a simple weighted sum of inputs.

(c) The impulse response consists of an oscillatory cosine, starting at $n=0$ and ending at $n=19$. The system is therefore causal, stable, and LTI.

(d) The impulse response consists of an exponentially-decaying step function and is zero before $n=0$. As in (c), the system is causal, stable, and LTI.

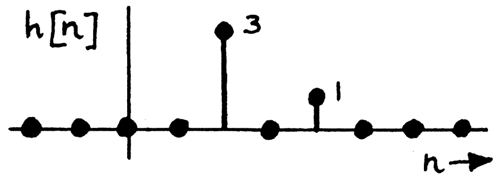
(e) The impulse response consists of an exponentially-increasing sine function, multiplied by a unit step function (which makes it zero before $n=0$). The system is therefore unstable, but causal and LTI.

To summarize: (a), (c), and (d) are causal, stable, and LTI.

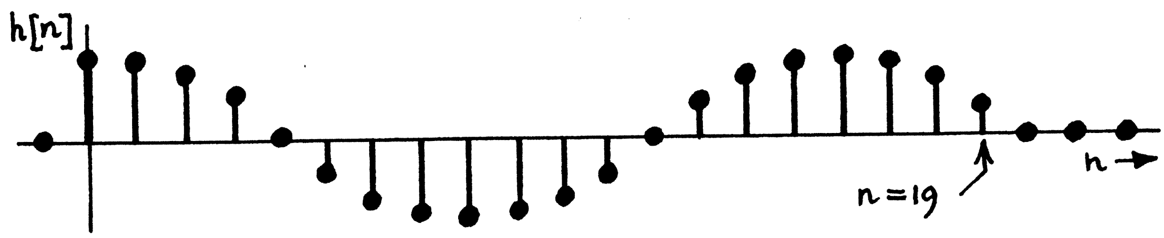
Q 2·3

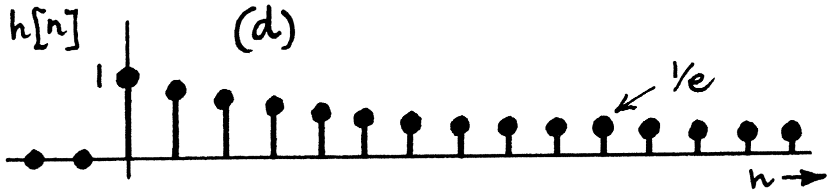
The sketches below should be self-explanatory. (Note: the answer to part (b) has already been sketched in Q 2·2 above).

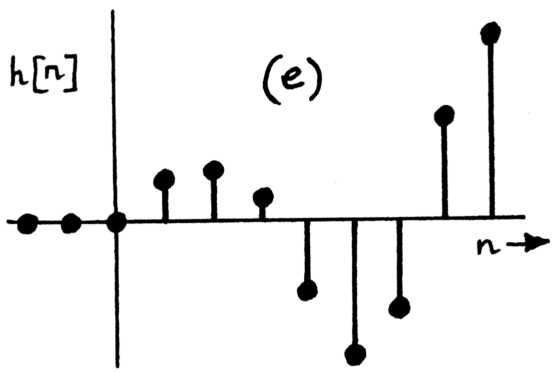
(a)



(c)







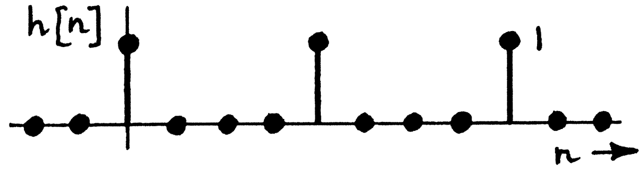
(Note: it is hard to sketch this rapidly-expanding function accurately!)

<u>Q 2·4</u>

(a) Since the filter or system is nonrecursive, its impulse response may be simply found by substituting $\delta[n]$ for $x[n]$ in its recurrence formula, and so on. $y[n]$ then becomes $h[n]$. In this case:

$$h[n] = \delta[n] + \delta[n-4] + \delta[n-8]$$

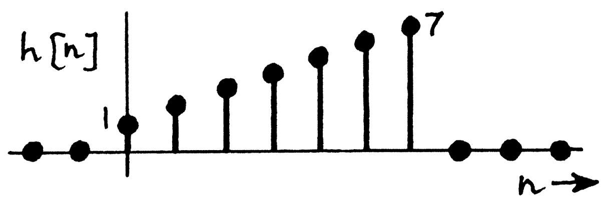
Sketch:



(b) Using the same approach as in (a) above:

$$h[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 4\delta[n-3] + 5\delta[n-4] + 6\delta[n-5] + 7\delta[n-6]$$

Sketch:



(c) Using the same approach as in (a) and (b) above:

$$h[n] = h[n-1] + \delta[n] - \delta[n-8]$$

The essential difference in this case is that $h[n]$ is in part generated recursively. The input impulse terms only contribute at $n=0$ and $n=8$. Successive values of $h[n]$ are found as follows:

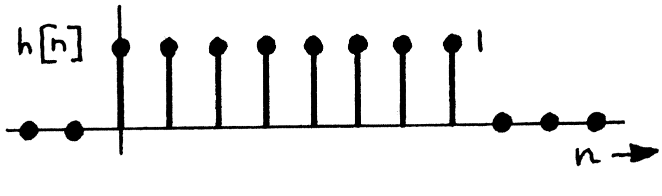
$$h[0] = h[-1] + \delta[0] - \delta[-8] = 0 + 1 - 0 = 1$$
$$h[1] = h[0] + \delta[1] - \delta[-7] = 1 + 0 - 0 = 1$$

The next values are similarly unity, until:

$$h[8] = h[7] + \delta[8] - \delta[0] = 1 + 0 - 1 = 0$$

All subsequent values of $h[n]$ are also zero.

Sketch:



(d) Using similar substitutions to those in part (c):

$$h[n] = h[n-1] - 0.5\,h[n-2] + \delta[n]$$

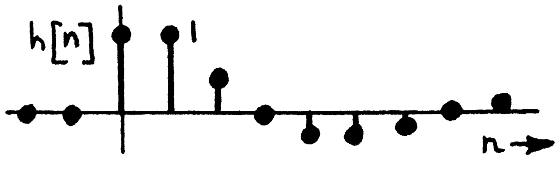
In this case the input impulse only contributes at $n=0$. All subsequent values of $h[n]$ are generated recursively. We may tabulate as follows:

$$h[0] = h[-1] - 0.5\,h[-2] + \delta[0] = 0 - 0 + 1 = 1$$
$$h[1] = h[0] - 0.5\,h[-1] + \delta[1] = 1 - 0 + 0 = 1$$

Similarly:

$$h[2] = 1 - 0.5 = 0.5$$
$$h[3] = 0.5 - 0.5 = 0$$
$$h[4] = 0 - 0.25 = -0.25$$
$$h[5] = -0.25 - 0 = -0.25$$
$$h[6] = -0.25 + 0.125 = -0.125$$
$$h[7] = -0.125 + 0.125 = 0$$
$$h[8] = 0 + 0.0625 = 0.0625 \quad \text{etc.}$$

Sketch:



Q 2.5 (p)

Q 2.6

Substituting $\delta[n]$ for $x[n]$ and so on, and $h[n]$ for $y[n]$, we obtain the following equation for the impulse response:

$$h[n] = 1.8523\, h[n-1] - 0.94833\, h[n-2] + \delta[n] - 1.9021\, \delta[n-1] + \delta[n-2]$$

Care is needed in calculating the first ten values of $h[n]$. It is probably best to generate a table as follows:

$1.8523\, h[n-1]$	$-0.94833\, h[n-2]$	$h[n]$	n
		1	0
1.8523	0	−0.0498	1
−0.0922	−0.9483	−0.0406	2
−0.0752	0.0472	−0.0279	3
−0.0517	0.0385	−0.0133	4
−0.0246	0.0265	0.0019	5
0.0036	0.0126	0.0162	6
0.0299	−0.0018	0.0281	7
0.0520	−0.0153	0.0367	8
0.0680	−0.0266	0.0414	9

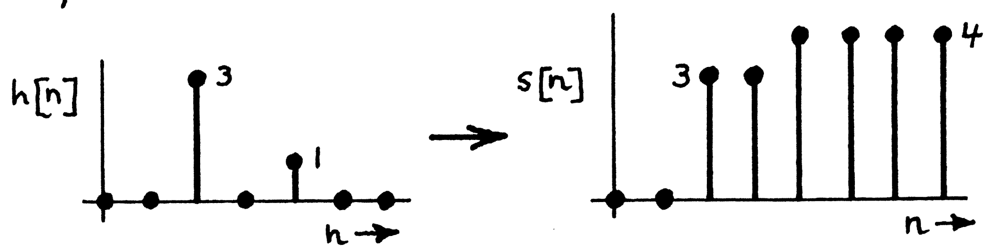
Although students cannot be expected to understand the form of $h[n]$, it actually consists of a unit impulse at $n=0$ representing an "all-pass" characteristic; plus the start of an inverted, gently decaying, oscillation with 20 samples per period representing the filter's notch frequency.

(Note: in early editions the answer given at the back of the book was inaccurate).

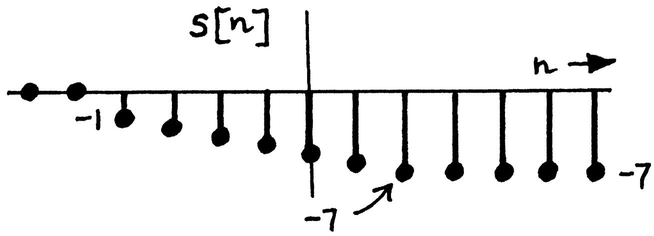
Q 2.7

The step response of an LTI system is the running sum of its impulse response. Hence:

(a)



(b) The impulse response is shown in the solution to Q 2.2(b) above. The running sum is:



Q 2.8

The relevant impulse responses are shown in the solution to Q 2.4 above. In each case the step

Q 2.10

(a) The graphical interpretation of convolution involves laying a reversed version of $h[n]$ below $x[n]$, followed by cross-multiplication and summation. Rather than sketching the functions, it is convenient to work with the equivalent number sequences.

In this case $h[n]$ is causal and $x[n]$ begins at $n=2$. The first nonzero output value therefore occurs at $n=2$. The relevant number sequences are:

$x[n]$:	0 0 0 1 1 1 1 1 1 1 0 0 0
reversed $h[n]$:	1 1 1

Cross-multiplying and summing, we get $y[2]=1$. Moving the reversed $h[n]$ step-by-step to the right and repeating the process, we obtain:

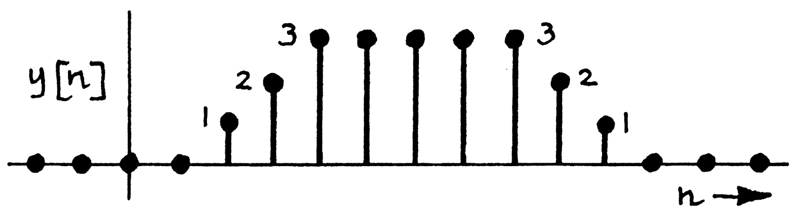
$$y[3]=2, \quad y[4]=y[5]=y[6]=y[7]=y[8]=3$$

When we reach $n=9$ the reversed impulse response starts to "move out of" the input sequence, giving:

$$y[9]=2, \quad y[10]=1, \quad y[11]=0$$

All subsequent output values are also zero.

Sketch:



(b) The approach is similar to part (a) above. The first nonzero output value occurs at $n=0$, for which the number sequences are:

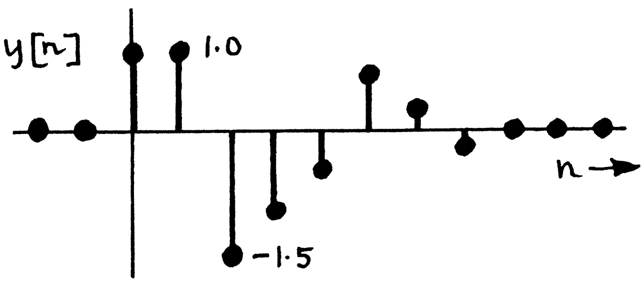
$x[n]$:	0	1	2	0·5	−0·5	−1	−0·2	0·1	0	0
reversed $h[n]$:	−1	1								

Hence:

$$y[0] = 1.0$$
$$y[1] = 1(2) - 1(1) = 1.0$$
$$y[2] = 1(0.5) - 1(2) = -1.5$$
$$y[3] = 1(-0.5) - 1(0.5) = -1.0$$
$$y[4] = 1(-1) - 1(-0.5) = -0.5$$
$$y[5] = 1(-0.2) - 1(-1) = 0.8$$
$$y[6] = 1(0.1) - 1(-0.2) = 0.3$$
$$y[7] = 1(0) - 1(0.1) = -0.1$$
$$y[8] = 1(0) - 1(0) = 0$$

All subsequent output values are zero.

Sketch:



Q 2.11

The impulse response $h[n]$ is only nonzero for $n = 0, 1, 2$. Hence the function $h[k]$ is only nonzero for $k = 0, 1, 2$. The summation may therefore be performed over this reduced range of k. (Note also that $x[n]$ is only nonzero for $n = 2$ to $n = 8$. Hence $x[n-k]$ is only nonzero, given $k = 0, 1, 2$, for $n = 2$ to 10. All nonzero values of $y[n]$ therefore fall in the range $n = 2$ to 10).

Q 2.12 (P)

Whatever the details of the computer program, it must calculate a 7-term moving-average. Thus each smoothed output value must equal the average of the coincident input and the 3 inputs to either side of it.

In the case of $y[0]$, 3 raw data (input) values are missing. If we take them as zero, the program will calculate:

$$y[0] = \frac{1}{7}(9 + 10 + 8 + 10) = 5.29$$

The other requested values are:

$$y[6] = \frac{1}{7}(10 + 12 + 11 + 9 + 11 + 10 + 16) = 11.29$$
$$y[20] = \frac{1}{7}(10 + 9 + 10 + 16 + 17 + 18 + 14) = 13.43$$

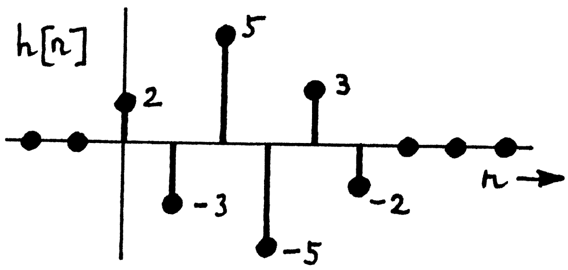
There are transients at the beginning and end of the filtered data, equal in duration to the impulse response of the filter (7 sampling intervals). The low value calculated for $y[0]$ is one consequence of this. There is no satisfactory way round the difficulty, which arises whenever time-limited data is processed with a linear filter (we could "guess" a few raw data values prior to $n=0$ and after $n=29$ but this would not add to the true information content of the signal).

Q 2·13

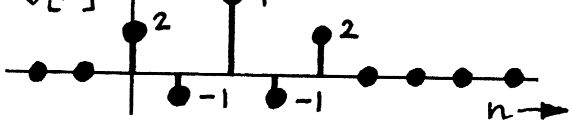
The impulse response may be found by delivering a unit impulse to the input. The resulting difference equation is:

$$h[n] = 2\delta[n] - 3\delta[n-1] + 5\delta[n-2] - 5\delta[n-3] + 3\delta[n-4] - 2\delta[n-5]$$

Sketch:



The step response is the running sum of the impulse response:



The step response settles to zero because the sum of all impulse response terms is zero. This implies zero output in response to a continuing, sampled, DC-level. In other words the system has zero response to DC, and must therefore be some form of high-pass or bandpass filter.

Q 2·14

(a) We may derive the equivalent nonrecursive formula by first finding the system's impulse response. Successive impulse response terms will then give the required multipliers. Replacing $x[n]$ by $\delta[n]$, $y[n]$ by $h[n]$ and so on, we obtain:

$$h[n] = h[n-1] + \delta[n] - \delta[n-7]$$

Evaluating term-by-term:

$$h[0] = 0+1-0 = 1$$
$$h[1] = 1+0-0 = 1$$

Further terms are also unity until:

$$h[7] = h[6] + \delta[7] - \delta[0] = 1+0-1 = 0$$

Subsequent terms are also zero.
Therefore the impulse response is given by:

$$h[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \cdots \delta[n-6]$$

giving the equivalent nonrecursive formula:

$$y[n] = x[n] + x[n-1] + x[n-2] + \cdots x[n-6]$$

The recursive version requires 8 delays and 2 additions/subtractions; the nonrecursive version, 6 delays and 6 additions.

(b) Using a similar approach to part (a), we have:

$$h[n] = 0.9\,h[n-1] + \delta[n]$$

Evaluating term-by-term:

$$h[0] = 0.9(0) + 1 = 1$$
$$h[1] = 0.9(1) + 0 = 0.9$$
$$h[2] = 0.9(0.9) + 0 = 0.81 \quad \text{etc.}$$

We see that $h[n]$ takes the form of a decaying real exponential. In theory it goes on for ever. The equivalent nonrecursive formula is:

$$y[n] = x[n] + 0.9\,x[n-1] + 0.81\,x[n-2] + \cdots$$

and involves (theoretically) an infinite number of terms. In practice this could not be implemented exactly, but we could obtain a good approximation by ignoring very small terms corresponding to the "tail" of $h[n]$.

<u>Q 2.15</u>

The first processor's impulse response begins at $n=1$ and has the nonzero values:

1, 0.5, 0.3333

The second processor's impulse response also begins at $n=1$ and has the nonzero values:

1, 2, 3

We convolve these two number sequences to find the impulse response of the overall system. This is done by laying a reversed version of one sequence below the other sequence, then cross-multiplying and summing terms. Thus for $n=1$ we have:

$$\begin{array}{cccccc} & & 1 & 0.5 & 0.3333 \\ 3 & 2 & 1 & & \end{array}$$

Hence $h[1] = 1(1) = 1$. Moving the lower sequence step-by-step to the right, we obtain:

$$h[2] = 0.5(1) + 1(2) = 2.5$$
$$h[3] = 0.3333(1) + 0.5(2) + 1(3) = 4.3333$$
$$h[4] = 0.3333(2) + 0.5(3) = 2.167$$
$$h[5] = 0.3333(3) = 1$$

The overall impulse response, starting at $n=1$, is therefore:

$$1,\ 2.5,\ 4.3333,\ 2.167,\ 1,\ 0,\ 0\ \ldots\ldots$$

Q 2.16

The first processor's impulse response is given by:

$$h[n] = \delta[n] + 2\,\delta[n-1] + 3\,\delta[n-2]$$

and therefore has sample values (starting at $n=0$):

$$1,\ 2,\ 3$$

The second processor's impulse response is given by:

$$h[n] = 0.8\,h[n-1] + \delta[n]$$

Evaluating term-by-term:

$$h[0] = 0 + 1 = 1$$
$$h[1] = 0.8(1) + 0 = 0.8$$
$$h[2] = 0.8(0.8) + 0 = 0.64$$

Similarly:

$$h[3] = 0.512 \qquad h[4] = 0.4096$$
$$h[5] = 0.3277 \qquad h[6] = 0.2621$$
$$h[7] = 0.2097 \qquad h[8] = 0.1678 \quad \text{etc.}$$

With parallel systems, the overall impulse response equals the sum of the individual responses. Hence successive values (starting at $n = 0$):

$$2,\ 2.8,\ 3.64,\ 0.512,\ 0.4096,\ 0.3277,\ 0.2621 \quad \text{etc.}$$

Q 2.17 (P)

Q 2.18

The processor's step response is given by:

$$s[n] = 0.8\, s[n-1] + 0.2\, u[n]$$

Evaluating term-by-term:

$$s[0] = 0.8(0) + 0.2(1) = 0.2$$
$$s[1] = 0.8(0.2) + 0.2(1) = 0.36$$
$$s[2] = 0.8(0.36) + 0.2(1) = 0.488$$

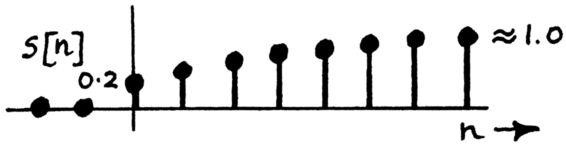
Suppose the final value reached is m. Then:

$$m = 0.8(m) + 0.2(1)$$

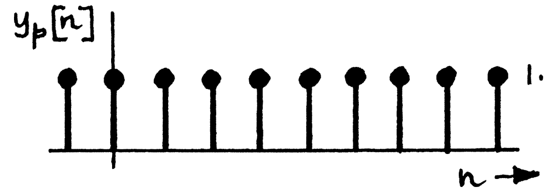
$$\therefore \; 0.2m = 0.2 \quad \text{or} \quad m = 1$$

The step response therefore has the form:

Sketch:



The particular component of the response is the steady-state response to the ongoing unit step, and is clearly:



Extended back to $n=-1$, it gives the value 1.0, which must be counteracted by a homogeneous component equal to -1.0. The homogeneous solution obeys the equation:

$$y_h[n] = 0.8\, y_h[n-1]$$

Hence with $y_h[-1] = -1.0$ we have:

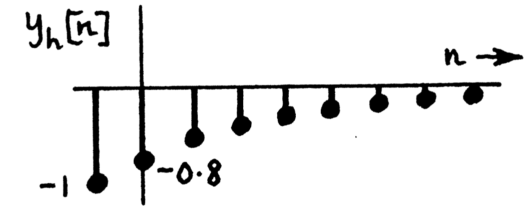
$$y_h[0] = 0.8(-1.0) = -0.8$$

$$y_h[1] = -0.64$$

$$y_h[2] = -0.512 \quad \text{etc.}$$

The homogeneous part of the response is therefore:

Sketch:

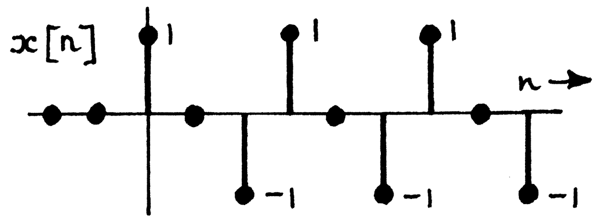


It is clear that adding together the particular and homogeneous components produces the complete step response $s[n]$.

We are told that the processor has low-pass filtering properties. This ties in with the form of the particular component, which shows a continuing unit-height output in response to an ongoing (DC) unit-height input. The homogeneous component, on the other hand, takes the same general form as the processor's impulse response (a decaying real exponential).

Q 2·19

The input signal has the form:



etc.

The impulse response is given by:

$$h[n] = -h[n-1] - 0.5\, h[n-2] + \delta[n]$$

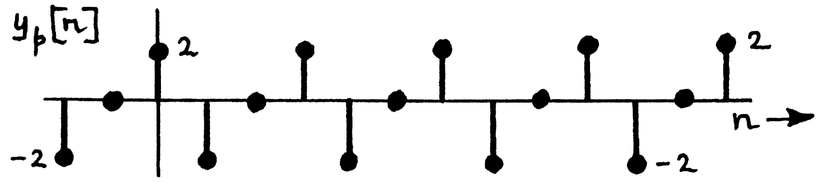
Evaluating term-by-term we obtain:

$h[0]=1 \qquad h[1]=-1 \qquad h[2]=0.5 \quad$ etc.

Laying a reversed version of $h[n]$ below the input signal, cross-multiplying and summing terms, we obtain the following values for $y[n]$, $n=0$ to 12:

$y[0]=1$	$y[1]=-1$	$y[2]=-0.5$
$y[3]=2$	$y[4]=-1.75$	$y[5]=-0.25$
$y[6]=2.125$	$y[7]=-2$	$y[8]=-0.0625$
$y[9]=2.0625$	$y[10]=-2.03125$	$y[11]=0$
$y[12]=2.0156$		

As n becomes large, we see the particular solution emerging, with the repeating sequence $2, -2, 0$. We infer that the particular solution is:



The homogeneous solution must therefore contribute:

$$y_h[-2]=2 \quad \text{and} \quad y_h[-1]=0$$

It obeys the equation:

$$y_h[n]=-y_h[n-1]-0.5\,y_h[n-2]$$

Hence: $\quad y_h[0]=-(0)-0.5(2)=-1$

$\qquad\quad y_h[1]=1-0.5(0)=1$

$$y_h[2] = -1 - 0.5(-1) = -0.5$$
$$y_h[3] = 0.5 - 0.5(1) = 0 \qquad \text{etc.}$$

We may tabulate as follows:

n	$y[n]$	$y_p[n]$	$y_h[n]$
0	1	2	−1
1	−1	−2	1
2	−0.5	0	−0.5
3	2	2	0
4	−1.75	−2	0.25
5	−0.25	0	−0.25
6	2.125	2	0.125
7	−2	−2	0
8	−0.0625	0	−0.0625
9	2.0625	2	0.0625
10	−2.03125	−2	−0.03125
11	0	0	0
12	2.0156	2	0.0156

It is clear that the particular and homogeneous components superimpose to give the total output $y[n]$.

Chapter 3

Q 3.1

(a) $$x[n] = 5 + \sin(n\pi/2) + \cos(n\pi/4)$$

$$= 5 + \frac{1}{2j}\left\{\exp(jn\pi/2) - \exp(-jn\pi/2)\right\}$$

$$+ \frac{1}{2}\left\{\exp(jn\pi/4) + \exp(-jn\pi/4)\right\}$$

$$= -\frac{1}{2j}\exp(-2jn\pi/4) + \frac{1}{2}\exp(-jn\pi/4) + 5$$

$$+ \frac{1}{2}\exp(jn\pi/4) + \frac{1}{2j}\exp(2jn\pi/4)$$

The Fourier Series coefficients are therefore:

$a_{-2} = -\frac{1}{2j} = 0.5j$; $a_{-1} = 0.5$; $a_0 = 5$; $a_1 = 0.5$;
and $a_2 = \frac{1}{2j} = -0.5j$

Note that $x[n]$ has 8 samples per period, so its Fourier Series must repeat every 8 harmonics. Our analysis gives only 5 finite terms, so the other 3 must be zero. That is:

$$a_3 = -a_3 = 0 \quad ; \quad a_4 = 0$$

(Note that $a_{-1} = a_7$, $a_{-2} = a_6$, and $a_{-1} = a_5$)

(b) $$x[n] = \cos\left(\frac{n\pi}{2} - \frac{\pi}{4}\right)$$

$$= \frac{1}{2}\left\{\exp\left(\frac{jn\pi}{2} - \frac{j\pi}{4}\right) + \exp\left(-\frac{jn\pi}{2} + \frac{j\pi}{4}\right)\right\}$$

Now: $\exp\left(\frac{j\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) + j\sin\left(\frac{\pi}{4}\right) = (1+j)/\sqrt{2}$

and: $\exp\left(-\frac{j\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) - j\sin\left(\frac{\pi}{4}\right) = (1-j)/\sqrt{2}$

Hence: $a_{-1} = (1+j)/2\sqrt{2} = a_3$

$a_0 = 0$

$a_1 = (1-j)/2\sqrt{2}$

<u>Q 3.2</u> (p)

Program no. 7 yields:

$a_0 = 0$ $\quad a_1 = 0.0107 + j\,0.5622$

$a_2 = 0.5671 + j\,0.4632$ $\quad a_3 = 0.4223 + j\,1.4033$

$a_4 = 0.4223 - j\,1.4033$ $\quad a_5 = 0.5671 - j\,0.4632$

$a_6 = 0.0107 - j\,0.5622$

a_0 is zero because the sum of all signal sample values over one period is zero, in other words there is no DC component.

The signal has 7 samples per period. Sampled at 1 MHz, each period occupies 7 μs. The coefficient a_2 denotes the second harmonic of the series, which therefore has a period of 3.5 μs. Its frequency is therefore $10^6/3.5$ Hz = 286 kHz.

Q 3.3 (p)

Q 3.4 (p)

Q 3.5

There is a DC component of value 1.0; a cosine of amplitude 1.0 at the fundamental frequency; and a sine of amplitude 1.0 at 8th-harmonic frequency. Recalling that a unit-amplitude sine (or cosine) gives two exponential coefficients of amplitude 0.5, we have:

$$a_0 = 1$$

$$|a_1| = |a_{-1}| = |a_{63}| = 0.5 \quad (\text{phase} = 0)$$

$$|a_8| = |a_{-8}| = |a_{56}| = 0.5 \quad (\text{phase} = \pm \pi/2)$$

Q 3.6 (p)

Referring to figures 3.5 and 3.6 in the main text, we see that whereas the phase spectrum of a unit impulse train having an impulse at $n=0$ is zero, when the impulse occurs at $n=1$ there is a phase lag proportional to frequency. The phase changes through 2π as index k changes from 0 to 32.

In this problem we also have a signal period equal to 64 sampling intervals. An impulse occurs at $n=5$. We therefore expect a rate of change of phase 5 times as great, passing through 10π (seen as 5 cycles of 2π) as k changes from 0 to 32.

<u>Q 3.7</u>

The periodic signal defined in Q 3.2 has values:

$$2, -4, 1, -2, 3, -2, 2$$

Their sum of squares, corresponding to the l.h.s. of equation 3.9, is:

$$\frac{1}{7}(4+16+1+4+9+4+4) = 6$$

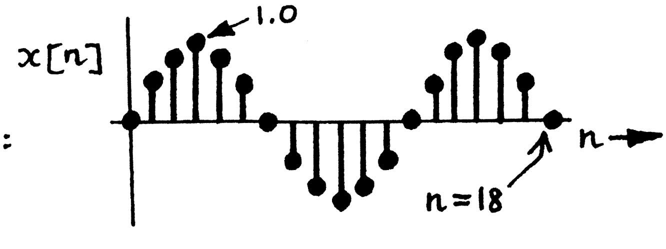
The sum of squares of the spectral coefficient magnitudes, corresponding to the r.h.s. of equation 3.9, is:

$$
\begin{aligned}
& 2(0.0107^2 + 0.5622^2) \\
+\ & 2(0.5671^2 + 0.4632^2) \\
+\ & 2(0.4223^2 + 1.4033^2) \\
=\ & 0.6324 + 1.072 + 4.297 = 6.000
\end{aligned}
$$

Hence Parseval's theorem is obeyed.

<u>Q 3.8</u>

Sketch:



The differentiation property states:

If $x[n] \leftrightarrow a_k$

Then $x[n]-x[n-1] \leftrightarrow a_k\{1-\exp(-j2\pi k/N)\}$

The signal in this case is a sinusoid with 12 samples per period. This is the fundamental, and there are no harmonics, nor a DC term. Therefore the only nonzero spectral coefficient is a_1 (and a_{11}). With $N=12$ and $k=1$ we have:

$$1-\exp(-j2\pi k/N) \rightarrow 1-\exp(-j\pi/6) = 1-\exp(-j\,30^\circ)$$

$$= 1-(\cos\pi/6 - j\sin\pi/6) = (1-0.8661)+j0.5$$

This has the magnitude:

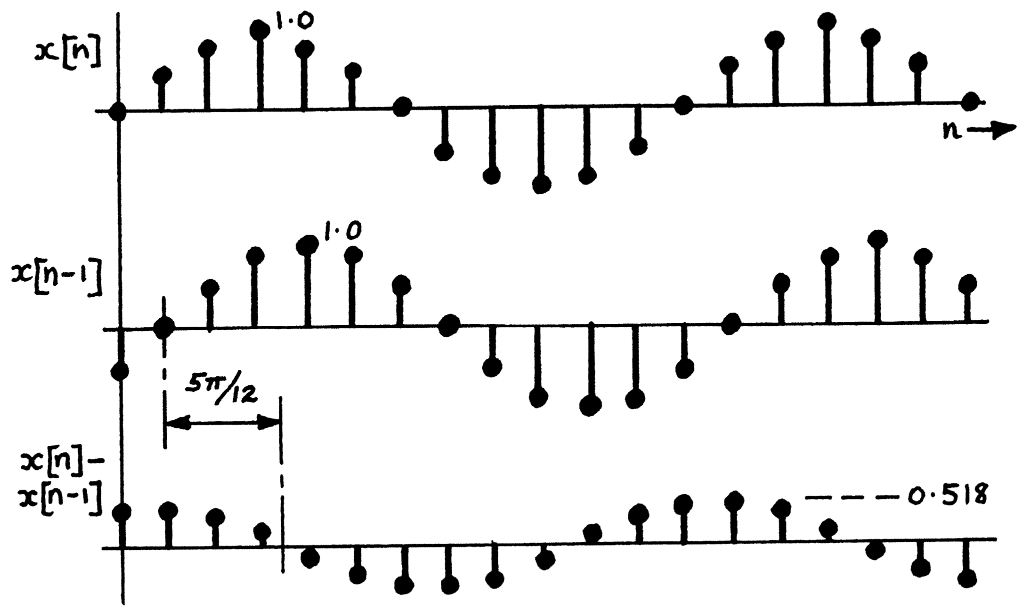
$$(0.1339^2+0.5^2)^{1/2} = 0.518$$

and a phase:

$$\tan^{-1} 0.5/0.1339 = 75^\circ$$

Hence we expect the magnitude of the first-order difference to be 0.518, with a phase shift of 75°, or $5\pi/12$, compared with $x[n]$.

We now sketch $x[n]$, $x[n-1]$, and their difference:



<u>Q3·9</u> (p)

A reversed version of one period of the second signal is laid below the first signal, followed by cross-multiplication and summation. Thus for $n=0$ the sequences are:

1 0 0·5 0 0 0·5 0 1 0 0·5 0 0 0·5 0

1 0 0 0 0 1 0

Denoting the periodic convolution by $y[n]$ we obtain:

$y[0]=0$ $y[1]=1{\cdot}5$ $y[2]=0$ $y[3]=0{\cdot}5$

$y[4]=0{\cdot}5$ $y[5]=0$ $y[6]=1{\cdot}5$

The 3 signals may now be used as data for Program no. 7.

Q 3.11

When a signal is expressed as a set of weighted impulses, we may use the sifting property of the unit impulse function (see Worked Example 3.2 in the main text) to write down its spectrum directly. Thus:

(a) $X(\Omega) = 1 + 2\exp(-j\Omega) + \exp(-j2\Omega)$

(b) $X(\Omega) = \exp(j\Omega) - \exp(-j\Omega) = 2j\sin\Omega$

(c) We first note that:

$$u[n+3] = \delta[n+3] + \delta[n+2] + \delta[n+1] + \cdots$$
$$u[n-4] = \delta[n-4] + \delta[n-5] + \delta[n-6] + \cdots$$

Hence:

$$u[n+3] - u[n-4] = \delta[n+3] + \delta[n+2] + \cdots \delta[n-3]$$

giving:

$$X(\Omega) = \exp(j3\Omega) + \cdots 1 + \cdots \exp(-j3\Omega)$$
$$= 1 + 2\cos\Omega + 2\cos 2\Omega + 2\cos 3\Omega$$

Q 3.12

If we express an impulse response as a set of weighted impulses, we may use the sifting property of the unit impulse function to write down its frequency response directly.

(a) $H(\Omega) = 4 + 2\exp(-j\Omega) + \exp(-j2\Omega)$

(b) $h[n] = h[n-1] + \delta[n] - \delta[n-7]$

Evaluating $h[n]$ term-by-term we obtain:

$$h[0] = 0 + 1 - 0 = 1$$
$$h[1] = 1 + 0 - 0 = 1$$

further terms are also unity until we reach:

$$h[7] = 1 + 0 - 1 = 0$$

All subsequent values are zero. Hence the frequency response is given by:

$$H(\Omega) = 1 + \exp(-j\Omega) + \exp(-j2\Omega) + \cdots \exp(-j6\Omega)$$

(c) $h[n] = 0.8\, h[n-1] + \delta[n]$

Evaluating $h[n]$ term-by-term we obtain:

$h[0] = 1 \quad h[1] = 0.8 \quad h[2] = 0.64 \quad h[3] = 0.512$

and so on (infinite series). Hence:

$$H(\Omega) = 1 + 0.8\exp(-j\Omega) + 0.64\exp(-j2\Omega) + \cdots\cdots$$
$$= \frac{1}{1 - 0.8\exp(-j\Omega)}$$

Q 3·13

The difference equation is:

$$y[n] = -0.9\,y[n-1] + 0.1\,x[n]$$

The impulse response is therefore given by:

$$h[n] = -0.9\,h[n-1] + 0.1\,\delta[n]$$

Evaluating term-by-term:

$$h[0] = 0.1 \qquad h[1] = -0.9(0.1) = -0.09$$

$$h[2] = 0.081 \quad \text{etc. (infinite series)}$$

Hence:

$$H(\Omega) = 0.1 - 0.09\exp(-j\Omega) + 0.081\exp(-j2\Omega) - \cdots$$

$$= \frac{0.1}{1 + 0.9\exp(-j\Omega)}$$

(a) When $\Omega = 0$, $H(\Omega) = \dfrac{0.1}{1+0.9} = 0.0526$

(b) When $\Omega = \pi$, $H(\Omega) = \dfrac{0.1}{1-0.9} = 1.0$

Q 3·14

By inspection we see that the recurrence formula (difference equation) of the band~~stop~~pass filter is:

$$y[n] = 0.9\,y[n-1] - 0.8\,y[n-2] + x[n]$$

$$\therefore \quad y[n] - 0.9\,y[n-1] + 0.8\,y[n-2] = x[n]$$

Hence $$H(\Omega) = \frac{1}{1 - 0.9\exp(-j\Omega) + 0.8\exp(-j2\Omega)}$$

The magnitude is given by:

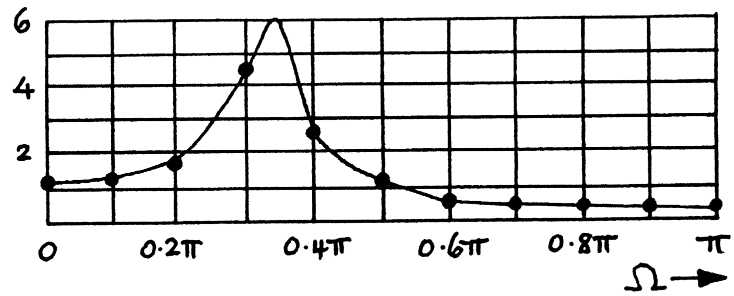
$$|H(\Omega)| = \left\{(1 - 0.9\cos\Omega + 0.8\cos 2\Omega)^2 + (0.9\sin\Omega - 0.8\sin 2\Omega)^2\right\}^{-1/2}$$

Express this in the form $(A^2 + B^2)^{-1/2}$. Then we may tabulate:

Ω	A^2	B^2	$\lvert H(\Omega)\rvert = (A^2+B^2)^{-1/2}$
0	0.81	0	1.111
0.1π	0.6262	0.0369	1.228
0.2π	0.2695	0.0537	1.759
0.3π	0.0500	0.0011	4.424
0.4π	0.0056	0.1488	2.545
0.5π	0.0400	0.8100	1.085
0.6π	0.3980	1.758	0.681
0.7π	1.643	2.217	0.509
0.8π	3.902	1.664	0.424
0.9π	6.266	0.5600	0.383
π	7.29	0	0.370

Sketch:

$|H(\Omega)|$



Q 3·15

The frequency response of the filter is given by equation 3·50 in the main text:

$$H(\Omega) = \left\{ \frac{1 - 1\cdot9021 \exp(-j\Omega) + \exp(-j2\Omega)}{1 - 1\cdot8523 \exp(-j\Omega) + 0\cdot9483 \exp(-j2\Omega)} \right\}$$

(a) Putting $\Omega = 0$, we obtain:

$$H(\Omega) = \frac{1 - 1\cdot9021 + 1}{1 - 1\cdot8523 + 0\cdot9483} = \frac{0\cdot0979}{0\cdot09603} = 1\cdot02$$

(b) $$H(\Omega) = \frac{(1 - 1\cdot9021 \cos\Omega + \cos 2\Omega) + j(1\cdot9021 \sin\Omega - \sin 2\Omega)}{(1 - 1\cdot8523 \cos\Omega + 0\cdot9483 \cos 2\Omega) + j(1\cdot8523 \sin\Omega - 0\cdot9483 \sin 2\Omega)}$$

Putting $\Omega = 0\cdot1\pi$, this becomes:

$$H(\Omega) = \frac{(1 - 1\cdot8091 + 0\cdot8090) + j(0\cdot5877 - 0\cdot5878)}{(1 - 1\cdot7617 + 0\cdot7672) + j(0\cdot5724 - 0\cdot5574)}$$

$$= \frac{-0\cdot0001 - j(0\cdot0001)}{0\cdot0055 + j(0\cdot015)}$$

Hence $$|H(\Omega)| = \frac{0\cdot00014}{0\cdot01598} = 0\cdot009$$

(Note: we have worked to 4 decimal places. This is hardly sufficient since we are dealing with differences between relatively large numbers which are nearly equal. Computing to 6 decimal figures gives the answer 0·0104 as quoted at the end of the book. Clearly, the numerator more or less vanishes, giving the required notch effect at $\Omega = 0\cdot1\pi$).

Q 3·16 (P)

Chapter 4

Q 4.1

(a) By inspection, we have:

$$X(z) = 3z^{-3} + 3\alpha z^{-4} + 3\alpha^2 z^{-5} + \cdots$$
$$= \frac{3}{z^2}\left(z^{-1} + \alpha z^{-2} + \alpha^2 z^{-3} + \cdots\right)$$
$$= \frac{3}{z^2}(z-\alpha)^{-1} = \frac{3}{z^2(z-\alpha)}$$

(b)
$$X(z) = 2 + 2z^{-1} + 2z^{-2} + \cdots 2z^{-7}$$
$$= 2(1 + z^{-1} + z^{-2} + \cdots) - 2(z^{-8} + z^{-9} + z^{-10} + \cdots)$$
$$= (2 - 2z^{-8})(1 + z^{-1} + z^{-2} + \cdots)$$
$$= 2z^{-8}(z^8 - 1)\,z(z-1)^{-1}$$
$$= \frac{2(z^8-1)}{z^7(z-1)}$$

Q 4.2

(a)
$$X(z) = \frac{1}{(z-0.5)} = (z-0.5)^{-1}$$
$$= z^{-1} + 0.5z^{-2} + 0.25z^{-3} + 0.125z^{-4} + \cdots$$

Hence the first 5 sample values (starting at $n=0$) are: 0, 1, 0.5, 0.25, 0.125

(b) $$X(z) = \frac{z}{(z+1.1)} = z(z+1.1)^{-1}$$
$$= z(z^{-1} - 1.1z^{-2} + 1.21z^{-3} - 1.331z^{-4} + 1.4641z^{-5} - \ldots)$$
$$= 1 - 1.1z^{-1} + 1.21z^{-2} - 1.331z^{-3} + 1.4641z^{-4} \ldots$$

Hence the first 5 sample values (starting at $n=0$) are: 1, −1.1, 1.21, −1.331, 1.4641

(c) $$X(z) = \frac{(z+1)}{(z-1)} = (z+1)(z-1)^{-1}$$
$$= (z+1)(z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5} + \cdots)$$
$$= (1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + \cdots) + (z^{-1} + z^{-2} + z^{-3} + z^{-4} + \cdots)$$
$$= 1 + 2z^{-1} + 2z^{-2} + 2z^{-3} + 2z^{-4} + \cdots$$

Hence the first 5 sample values (starting at $n=0$) are: 1, 2, 2, 2, 2

Q 4.3

The signal may be made up by superposing a delayed version of the signal shown in Figure Q 4.1(b), and an inverted, delayed, and scaled version of the signal shown in Figure Q 4.1(a).

As shown above, the answer to Q 4.1(b) is:

$$X(z) = \frac{2(z^8-1)}{z^7(z-1)}$$

Delaying the signal by 2 sampling intervals is equivalent to multiplying its z-transform by z^{-2}. Hence the required function is:

$$X(z) = \frac{2(z^8-1)}{z^9(z-1)}$$

The answer given to Q 4·1 (a) is:

$$X(z) = \frac{3}{z^2(z-\alpha)}$$

We need to scale this signal by 2/3, invert it, delay it by 5 sampling intervals (so that it begins at $n=8$), and put $\alpha = 1/2$. Its z-transform now becomes:

$$X(z) = \frac{-2}{z^7(z-0{\cdot}5)}$$

Hence the z-transform of the signal shown in Figure Q 4·3 must be:

$$X(z) = \frac{2(z^8-1)}{z^9(z-1)} - \frac{2}{z^7(z-0{\cdot}5)}$$

Q 4·4

By inspection we may write:

$$X_1(z) = 1 - z^{-2} + z^{-3}$$
$$X_2(z) = 2z^{-1} + z^{-2} - z^{-3}$$

The signals have sample values (starting at $n=0$):

$$1, 0, -1, 1 \qquad \text{and} \qquad 0, 2, 1, -1$$

We convolve the signals by laying a reversed version of the second signal below the first signal, cross-multiplying and summing terms. Thus for $n=0$ the sequences are:

$$\begin{array}{ccccccc} & & & 1 & 0 & -1 & 1 \\ -1 & 1 & 2 & 0 & & & \end{array}$$

giving:

$$x_3[0] = 1(0) = 0$$
$$x_3[1] = 0(0) + 1(2) = 2$$
$$x_3[2] = -1(0) + 0(2) + 1(1) = 1$$

Similarly: $x_3[3] = -3$, $x_3[4] = 1$, $x_3[5] = 2$, $x_3[6] = -1$

Therefore the sample values of $x_3[n]$, starting at $n=0$, are:

$$0, 2, 1, -3, 1, 2, -1$$

Its z-transform is:

$$X_3(z) = 2z^{-1} + z^{-2} - 3z^{-3} + z^{-4} + 2z^{-5} - z^{-6}$$

If we multiply $X_1(z)$ by $X_2(z)$ we obtain:

$$\begin{array}{lllllll}
 & 1 & & -z^{-2} & +z^{-3} & & \\
 & 2z^{-1} & & +z^{-2} & -z^{-3} & & \\
\hline
2z^{-1} & & -2z^{-3} & +2z^{-4} & & & \\
 & z^{-2} & & -z^{-4} & +z^{-5} & & \\
 & & -z^{-3} & & +z^{-5} & -z^{-6} & \\
\hline
2z^{-1} & +z^{-2} & -3z^{-3} & +z^{-4} & +2z^{-5} & -z^{-6} & \\
\hline
\end{array}$$

which checks with the above result.

<u>Q 4.5</u>

Laying a reversed version of $h[n]$ below $x[n]$ we obtain, for $n=0$:

$$\begin{array}{ccccccccc}
 & & 1 & 2 & 3 & 1 & -1 & 1 \\
1 & 1 & 1 & & & & &
\end{array}$$

Giving:

$$y[0] = 1(1) = 1$$
$$y[1] = 2(1) + 1(1) = 3$$
$$y[2] = 3(1) + 2(1) + 1(1) = 6$$

Similarly $y[3]=6$, $y[4]=3$, $y[5]=1$, $y[6]=0$, $y[7]=1$

Therefore the z-transform of $y[n]$ is:

$$Y(z) = 1 + 3z^{-1} + 6z^{-2} + 6z^{-3} + 3z^{-4} + z^{-5} + z^{-7}$$

It is easy to show that this equals the product of the transforms of $x[n]$ and $h[n]$.

Q 4.6

(a) $$X(z) = \frac{0.5z}{z^2 - z + 0.5}$$

The last transform pair in the Table is relevant. If we put:

$a = \frac{1}{\sqrt{2}}$ and $\cos\Omega_0 = \frac{1}{\sqrt{2}}$, giving $\Omega_0 = \pi/4$ and

$\sin\Omega_0 = \frac{1}{\sqrt{2}}$, then the spectrum is as required.

The corresponding signal, found from the Table, is:

$$x[n] = \left(\frac{1}{\sqrt{2}}\right)^n \sin\frac{n\pi}{4}\; u[n]$$

(b) $$X(z) = \frac{z - 0.5}{z(z-0.8)(z-1)}$$

The Table does not include this transform, but it does contain (pairs 2 and 4) the transforms:

$$\frac{z}{(z-1)} \quad \text{and} \quad \frac{z}{(z-\alpha)}$$

Let us therefore write:

$$X(z) = \frac{1}{z}\left\{\frac{B}{(z-0.8)} + \frac{D}{(z-1)}\right\} = \frac{1}{z}\left\{\frac{B(z-1)+D(z-0.8)}{(z-0.8)(z-1)}\right\}$$

$$\therefore\ X(z) = \frac{1}{z}\left\{\frac{z(B+D)-(B+0.8D)}{(z-0.8)(z-1)}\right\}$$

Comparing with the original expression for $X(z)$ we see that:

$$B+D=1 \quad \text{and} \quad B+0.8D=0.5$$

giving: $\quad B=-1.5 \quad \text{and} \quad D=2.5$

Hence we may write:

$$X(z) = \frac{1}{z}\left\{\frac{-1.5}{(z-0.8)} + \frac{2.5}{(z-1)}\right\}$$
$$= z^{-2}\left\{\frac{-1.5z}{(z-0.8)} + \frac{2.5z}{(z-1)}\right\}$$

Referring to the Table we note that the bracketed expression is equivalent to the time function:

$$-1.5(0.8)^n u[n] + 2.5\,u[n]$$
$$= \{2.5 - 1.5(0.8)^n\}\,u[n]$$

The z^{-2} multiplier has the effect of delaying this function by 2 sampling intervals. The required signal is therefore:

$$x[n] = \{2.5 - 1.5(0.8)^{n-2}\}\,u[n-2]$$

Q4.7 (P)

Q4.8

(a) $$H(z) = \frac{z^2 - z - 2}{z^2 - 1.3z + 0.4} = \frac{(z-2)(z+1)}{(z-0.5)(z-0.8)}$$

Therefore we have zeros at $z=2$ and $z=-1$, and poles at $z=0.5$ and $z=0.8$.

The poles are inside the unit circle, so the system is stable. Since the order of the denominator polynomial (at least) equals that of the numerator, the impulse response must begin at (or after) $n=0$, so the system is causal.

(b) $$H(z) = \frac{(z^2 - z + 1)}{(z^2+1)} = \frac{(z-0.5+j0.866)(z-0.5-j0.866)}{(z+j)(z-j)}$$

Hence there are zeros at $z = 0.5 \pm j0.866$, and poles at $z = \pm j$. Since the poles are on the unit circle, the system is (marginally) unstable.

(c) $$H(z) = \frac{(z^3 - z^2 + z - 1)}{(z^2 - 0.25)} = \frac{(z-1)(z^2+1)}{(z+0.5)(z-0.5)}$$

Hence there are zeros at $z=1$ and $z=\pm j$, and poles at $z=\pm 0.5$. Since the denominator polynomial is of lower order than the numerator, the system is noncausal.

(d) $$H(z) = \frac{(z^9 - 1)}{(z-1)\, z^8}$$

The roots of $z^9 = 1$ are 9 points equally spaced around the unit circle, one of them being at $z=1$. These are the zero locations. There is also a pole at $z=1$ and an eighth-order pole at the origin.

The pole at $z=1$ is on the unit circle and may therefore appear to make the system (marginally) unstable. However its effect is exactly cancelled by a coincident zero. The system is therefore stable. It is also causal, since both numerator and denominator contain terms in z^9.

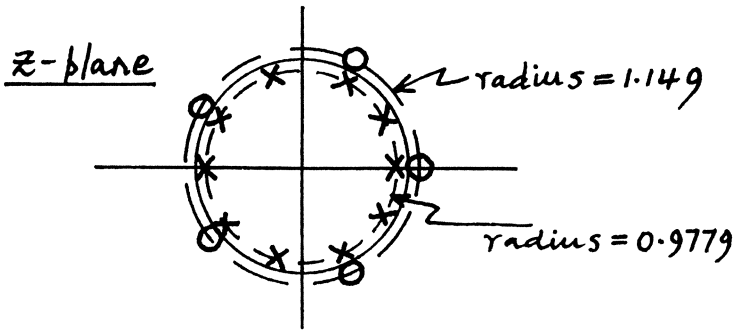
Q 4.9

(a) $$X(z) = \frac{z^5 - 2}{z^{10} - 0.8}$$

The roots of the numerator are 5 equally-spaced points on a circle of radius $2^{0.2} = 1.149$. The roots of the denominator are 10 equally-spaced points on a circle of radius $0.8^{0.1} = 0.9779$. The poles are all inside the unit circle, so the signal does not grow without limit as $n \to \infty$.

The pole-zero configuration of the signal is:

Sketch:



(b) $$X(z) = \frac{(z^2 + 1.5z + 0.9)}{(z^2 - 1.5z + 1.1)}$$

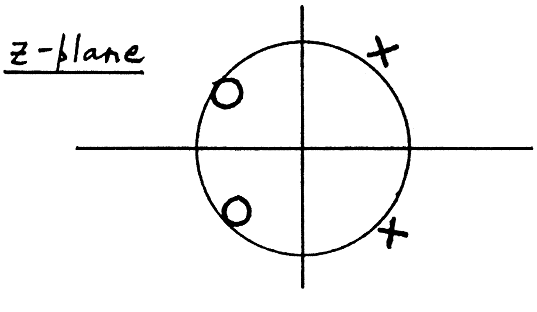
The roots of the numerator polynomial give the zero locations:

$$z = \frac{-1.5 \pm \{2.25 - 4(0.9)\}^{0.5}}{2} = -0.75 \pm j\,0.581$$

The roots of the denominator polynomial give the pole locations:

$$z = \frac{1.5 \pm \{2.25 - 4(1.1)\}^{0.5}}{2} = 0.75 \pm j\,0.733$$

Sketch:



Note that the pole radius is $(0.75^2 + 0.733^2)^{0.5} = 1.049$. The poles are therefore just outside the unit circle, and the signal will grow without limit as $n \to \infty$.

Q 4.10

(a) From the figure we see that (apart from a possible scale factor) the z-transform is given by:

$$X(z) = \frac{z}{(z+0.8)(z-1)}$$

This corresponds to the 5^{th} pair in the Table, with $\alpha = -0.8$. Hence, apart from a possible scale factor) the signal is:

$$x[n] = \left(1-(-0.8)^n\right)\ u[n]$$

(b) The pole-zero configuration corresponds to the 6^{th} entry in the Table, with $\Omega_0 = 2\pi/3$. Therefore apart from a possible scale factor the signal is:

$$x[n] = \left(\cos 2\pi n/3\right)\ u[n]$$

(c) The pole-zero configuration corresponds to the last entry in the Table with $\Omega_0 = \pi/2$, and $\alpha = 0.8$. Therefore apart from a possible scale factor the signal is:

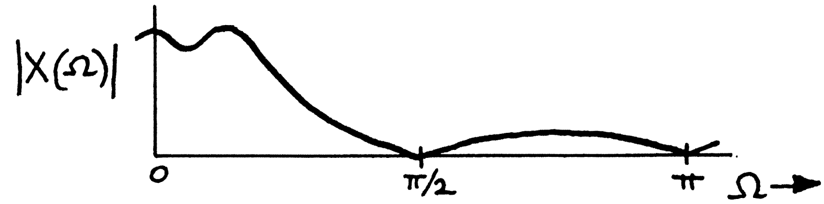
$$x[n] = \left(0.8^n \sin n\pi/2\right)\ u[n]$$

Q 4·11

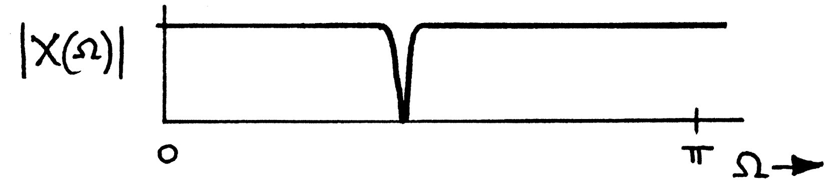
Sketches:

(a)

(b)



(c)



If these were filters rather than signals, their types would be (a) high-pass, (b) lowpass, and (c) bandstop, or notch.

Q 4·12

$$H(z) = \frac{z^3 - z^2 + 0\cdot 8z - 0\cdot 8}{z^3 + 0\cdot 8 z^2}$$

In general, we would resort to standard formulae to factorize the numerator cubic. However in this case, since the coefficients of z^3 and z^2 are ± 1, and of z^1 and z^0 are $\pm 0\cdot 8$, it is clear that $(z-1)$ is a likely factor. Long-division confirms this:

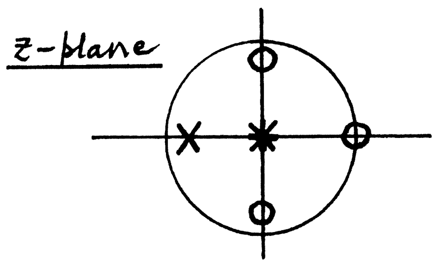
$$
\begin{array}{r|l}
 & z^2 + 0.8 \\
\hline
z - 1 & z^3 - z^2 + 0.8z - 0.8 \\
 & \underline{z^3 - z^2} \\
 & \qquad\quad 0.8z - 0.8 \\
 & \qquad\quad \underline{\underline{0.8z - 0.8}}
\end{array}
$$

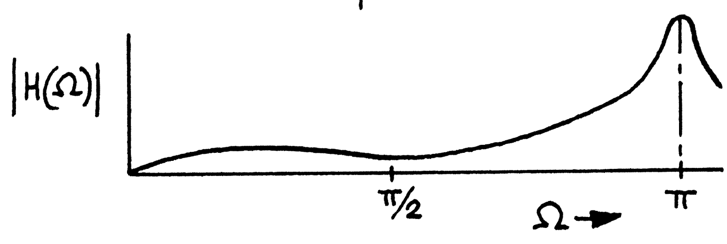
Hence: $H(z) = \dfrac{(z-1)(z^2+0.8)}{z^2(z+0.8)}$

$$= \frac{(z-1)(z+j\sqrt{0.8})(z-j\sqrt{0.8})}{z^2(z+0.8)}$$

We therefore have zeros at $z=1$ and $\pm j\,0.894$, a pole at $z=-0.8$ and a second-order pole at the origin.

Sketch:





Q 4.13

The stated performance can be met by the following:

(a) a zero at $z=1$

(b) a complex conjugate zero pair at $z = \exp(\pm j\pi/3)$

(c) a complex conjugate pole pair at $z = 0.9 \exp(\pm j\frac{2\pi}{3})$

(d) extra poles at the origin (if necessary) to equalize the total numbers of poles and zeros.

Thus the filter's transfer function is:

$$H(z) = \frac{(z-1)(z - \exp(j\pi/3))(z - \exp(-j\pi/3))}{(z - 0.9\exp(j2\pi/3))(z - 0.9\exp(-j2\pi/3))z}$$

$$= \frac{(z-1)(z^2 - (2\cos\pi/3)z + 1)}{(z^2 - (1.8\cos 2\pi/3)z + 0.81)z}$$

$$= \frac{(z^3 - 2z^2 + 2z - 1)}{(z^2 + 0.9z + 0.81)z} = \frac{Y(z)}{X(z)}$$

Hence:

$$X(z)(z^3 - 2z^2 + 2z - 1) = Y(z)(z^3 + 0.9z^2 + 0.81z)$$

giving the difference equation:

$$x[n+3] - 2x[n+2] + 2x[n+1] - x[n] = y[n+3] + 0.9y[n+2] + 0.81y[n+1]$$

Subtracting 3 from all terms in brackets and rearranging:

$$y[n] = -0.9y[n-1] - 0.81y[n-2] + x[n] - 2x[n-1] + 2x[n-2] - x[n-3]$$

The impulse response is given by:

$$h[n] = -0.9\,h[n-1] - 0.81\,h[n-2] + \delta[n] - 2\delta[n-1] + 2\delta[n-2] - \delta[n-3]$$

Evaluating term-by-term, we obtain the first 6 values:

$$h[0] = 1$$
$$h[1] = -0.9(1) - 2 = -2.9$$
$$h[2] = -0.9(-2.9) - 0.81(1) + 2 = 3.8$$
$$h[3] = -0.9(3.8) - 0.81(-2.9) - 1 = -2.071$$
$$h[4] = -0.9(-2.071) - 0.81(3.8) = -1.2141$$
$$h[5] = -0.9(-1.2141) - 0.81(-2.071) = 2.7702$$

Q 4.14

The transfer function, after modification as specified, becomes:

$$H(z) = \frac{(z+1)}{(z-0.8)} = \frac{Y(z)}{X(z)}$$

(a) We may write:

$$(z+1)\,X(z) = (z-0.8)\,Y(z)$$

Therefore the difference equation is:

$$x[n+1] + x[n] = y[n+1] - 0.8\,y[n]$$

Subtracting 1 from all terms in brackets and rearranging:

$$y[n] = 0.8\,y[n-1] + x[n] + x[n-1]$$

(b) The impulse response is given by:

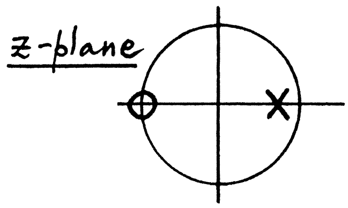
$$h[n] = 0.8\, h[n-1] + \delta[n] + \delta[n-1]$$

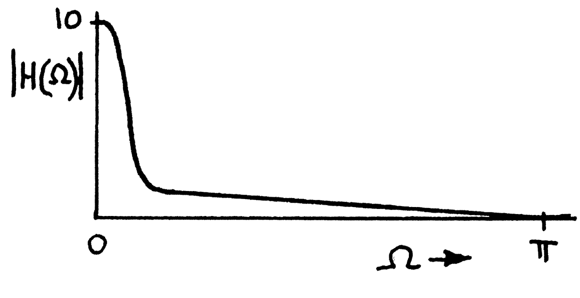
Evaluating the first 5 values term-by-term we obtain:

$$h[0] = 1$$
$$h[1] = 0.8(1) + 1 = 1.8$$
$$h[2] = 0.8(1.8) = 1.44$$
$$h[3] = 0.8(1.44) = 1.152$$
$$h[4] = 0.8(1.152) = 0.9216$$

(c) The pole-zero configuration and inferred frequency response are as follows:

Sketches:





Q 4.15

$$y[n] = \alpha\, y[n-1] - \beta\, y[n-2] + x[n]$$

Adding 2 to all terms in brackets and rearranging:

$$y[n+2] - \alpha\, y[n+1] + \beta\, y[n] = x[n+2]$$

Taking the z-transform of both sides:

$$Y(z)(z^2 - \alpha z + \beta) = X(z)\, z^2$$

This gives the transfer function:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^2}{(z^2 - \alpha z + \beta)}$$

Stability requires poles to be inside the unit circle. It is helpful to use polar coordinates. Thus a pair of poles at radius r and angles $\pm\theta$ gives a denominator equal to:

$$\{z - r\exp(j\theta)\}\{z - r\exp(-j\theta)\} = z^2 - (2r\cos\theta)z + r^2$$

For stability r must be less than unity. Hence we require that $0 \leq \beta < 1$. Note also that:

$$\alpha = 2r\cos\theta$$

Hence if $0 \leq r < 1$, and given that $\cos\theta$ is always in the range ± 1, it follows that $-2 < \alpha < 2$.

If β is close to unity, we have poles close to the unit circle. These will give a sharply-defined passband at the frequency corresponding to their angular position.

(a) For a passband at $\Omega = \pi/3$ we have:

$$\alpha = 2r\cos\theta \approx 2\cos\pi/3 = 1$$

(b) For a passband at $\Omega = 2\pi/3$ we have:
$$\alpha = 2r\cos\theta \approx 2\cos 2\pi/3 = -1$$

<u>Q 4·16</u> (P)

<u>Q 4·17</u>

$$y[n] = y[n-1] - 0{\cdot}8\, y[n-2] + x[n]$$

If $y[-1]=1$ and $y[-2]=1$, the following output contribution will be caused by the initial (auxiliary) conditions:

$$\begin{aligned}
y[0] &= 1 - 0{\cdot}8(1) = 0{\cdot}2 \\
y[1] &= 0{\cdot}2 - 0{\cdot}8(1) = -0{\cdot}6 \\
y[2] &= -0{\cdot}6 - 0{\cdot}8(0{\cdot}2) = -0{\cdot}76 \\
y[3] &= -0{\cdot}76 - 0{\cdot}8(-0{\cdot}6) = -0{\cdot}28 \\
y[4] &= -0{\cdot}28 - 0{\cdot}8(-0{\cdot}76) = 0{\cdot}328 \\
y[5] &= 0{\cdot}328 - 0{\cdot}8(-0{\cdot}28) = 0{\cdot}552 \\
y[6] &= 0{\cdot}552 - 0{\cdot}8(0{\cdot}328) = 0{\cdot}2896 \\
y[7] &= 0{\cdot}2896 - 0{\cdot}8(0{\cdot}552) = -0{\cdot}152 \quad \text{etc.}
\end{aligned}$$

The contribution takes the same general form as the filter's impulse response (in fact, analysis of the filter's pole locations shows that it has a passband at about $\Omega = \pi/3$. The above values represent the first cycle or so of an oscillation at this frequency). It is often desirable to ensure zero initial conditions, so that the response to a "new" input can be distinguished from that due to a "previous" one.

Q 4·18

By inspection, the processor's difference equation is:

$$y[n] + 0{\cdot}5\, y[n-1] + 0{\cdot}5\, y[n-2] = x[n] - x[n-1]$$

Using equations 4·50 and 4·51 in the main text, we take z-transforms of both sides, allowing for nonzero initial conditions:

$$Y(z) + 0{\cdot}5\{y[-1] + z^{-1}Y(z)\} + 0{\cdot}5\{y[-2] + y[-1]z^{-1} + z^{-2}Y(z)\}$$
$$= X(z) - \{x[-1] + z^{-1}X(z)\}$$

This gives:

$$Y(z) = \frac{X(z)\{1 - z^{-1}\} - x[-1] - 0{\cdot}5\, y[-1] - 0{\cdot}5\{y[-2] + y[-1]z^{-1}\}}{(1 + 0{\cdot}5 z^{-1} + 0{\cdot}5 z^{-2})}$$

(a) $y[-1] = y[-2] = 0$, so the initial conditions are zero. If we apply a unit impulse $\delta[n]$, the term $x[-1]$ is also zero, and $X(z) = 1$. Hence:

$$Y(z) = \frac{1 - z^{-1}}{(1 + 0{\cdot}5 z^{-1} + 0{\cdot}5 z^{-2})} = \frac{z^2 - z}{(z^2 + 0{\cdot}5 z + 0{\cdot}5)}$$

This also represents the true transfer function of the processor.

(b) $y[-1] = -2$, $y[-2] = 2$. In this case the transform of the output, for an impulse input, is:

$$Y(z) = \frac{\{1 - z^{-1}\} - 0.5(-2) - 0.5(2 - 2z^{-1})}{1 + 0.5z^{-1} + 0.5z^{-2}}$$

$$= \frac{1}{(1 + 0.5z^{-1} + 0.5z^{-2})} = \frac{z^2}{(z^2 + 0.5z + 0.5)}$$

The initial conditions which will ensure zero output, when the input is a unit impulse, must cause the denominator of $Y(z)$ to vanish. Hence:

$$1 - z^{-1} - 0.5\,y[-1] - 0.5\,y[-2] - 0.5\,y[-1]\,z^{-1} = 0$$

So that: $\quad 1 - 0.5\,y[-1] - 0.5\,y[-2] = 0$

and: $\quad -1 - 0.5\,y[-1] = 0$, giving $y[-1] = -2$

Substituting: $\quad 1 - 0.5(-2) - 0.5\,y[-2] = 0$, giving $y[-2] = 4$

Thus the required initial conditions are:

$$y[-1] = -2 \quad \text{and} \quad y[-2] = 4$$

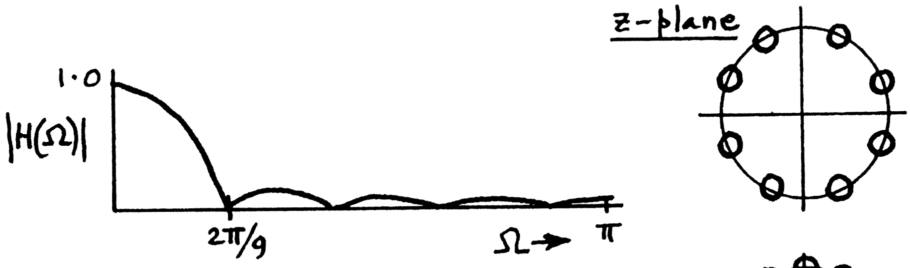
These conditions ensure zero output for all $n \geq 0$ since they give rise to a transient which is equal and opposite to the response caused by the input impulse.

Chapter 5

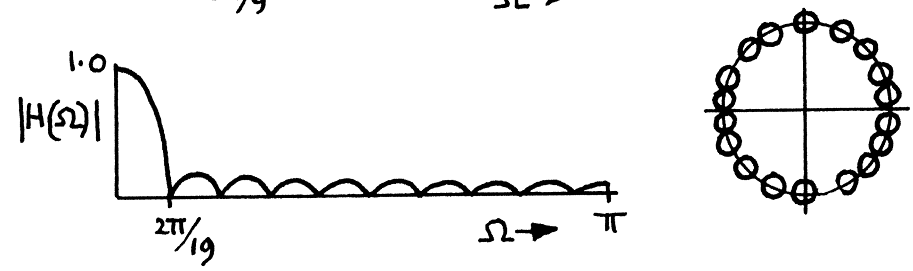
Q 5.1 (p)

These filters are of the type described in Section 5.2 of the main text, and illustrated in Figure 5.5. An m-term filter has a frequency response with $m/2$ lobes (including the main lobe) in the range $0 < \Omega < \pi$. Hence we expect the responses to be as follows. The zero configurations are also shown.

(a)



(b)



Q 5·2 (P)

Q 5·3

Equation 5·11 in the main text gives the inverse transform relationship for an ideal low-pass filter:

$$h[n] = \frac{1}{n\pi} \sin(n\Omega_1)$$

(a) With $\Omega_1 = 0.4\pi$ we have:

$$h[n] = \frac{1}{n\pi} \sin(0.4\pi n)$$

Using l'Hospital's rule to find $h[0]$:

$$h[0] = \left.\frac{\frac{d}{dn}(\sin 0.4\pi n)}{\frac{d}{dn}(n\pi)}\right|_{n=0} = \frac{0.4\pi \cos 0.4\pi n}{\pi} = 0.4$$

The next few values are readily calculated thus:

$$h[1] = \frac{1}{\pi} \sin 0.4\pi = 0.30273$$

Similarly:

$h[2] = 0.09355$ $\quad h[3] = -0.06237$ $\quad h[4] = -0.07568$

$h[5] = 0$ $\quad h[6] = 0.05046$ $\quad h[7] = 0.02673$

(b) For a high-pass filter, we may use equation 5.12 in the main text with $\Omega_0 = \pi$. Note that the modulating cosine $\cos(n\Omega_0)$ now has successive values $1, -1, 1, -1 \cdots$ Thus it simply inverts alternate impulse response values of the low-pass prototype.

In this case $\Omega_1 = 0.8\pi$, which gives the same bandwidth as a low-pass filter with $\Omega_1 = 0.2\pi$. This has already been explored in part (a) of Worked Example 5.1 in the main text, so the impulse response must be the same — but with alternate values inverted. We conclude (see p.140 in the book) that:

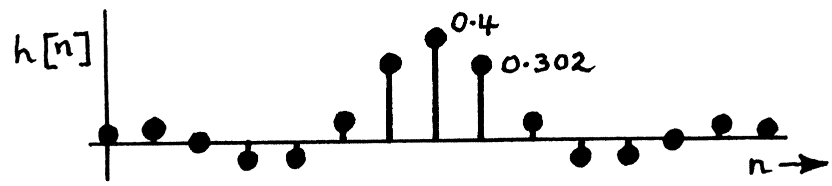
$h[0] = 0.2$ $h[1] = -0.187098$ $h[2] = 0.151365$
$h[3] = -0.10091$ $h[4] = 0.046774$ $h[5] = 0$
$h[6] = -0.031183$ $h[7] = 0.043247$

(Of course, these values may also be found by calculation using equation 5.12).

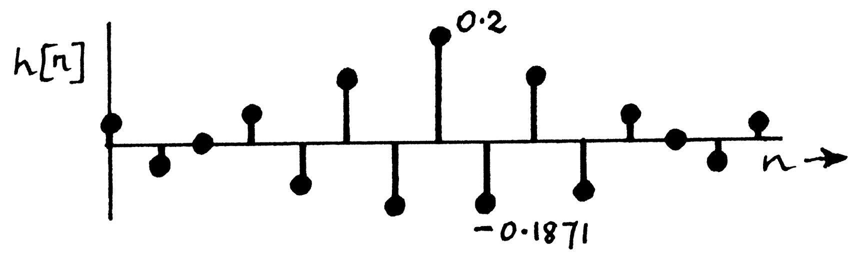
Causal, linear-phase, versions of the two filters have the following impulse responses (truncated to 15 terms):

Sketches:

(a)



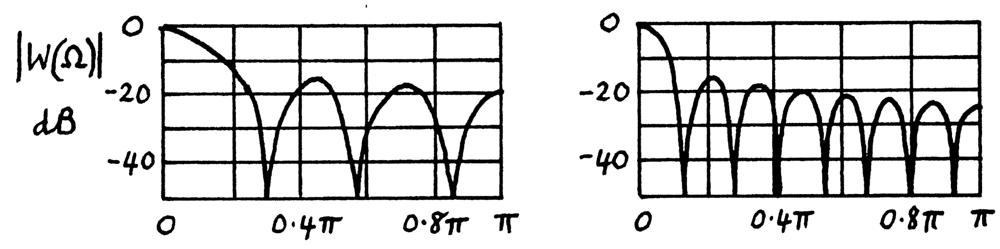
(b)



Q 5.4 (p)

Q 5·5 (p)

Sketches:



(a) 7 terms   (b) 15 terms

Q 5·6 (P)

Q 5·7

(a) A von Hann window with 11 terms ($M=5$) is defined by (see equation 5.21 in the main text):

$$w[n] = 0{\cdot}5 + 0{\cdot}5\cos\left(\frac{n\pi}{6}\right), \quad -5 \leq n \leq 5$$

Estimating "one half" of the window, $w[0]$ to $w[5]$ inclusive:

$$w[0] = 0{\cdot}5 + 0{\cdot}5\cos(0) = 1{\cdot}0$$
$$w[1] = 0{\cdot}5 + 0{\cdot}5(0{\cdot}8660) = 0{\cdot}93301$$
$$w[2] = 0{\cdot}5 + 0{\cdot}5(0{\cdot}5) = 0{\cdot}75$$
$$w[3] = 0{\cdot}5 + 0 = 0{\cdot}5$$
$$w[4] = 0{\cdot}5 + 0{\cdot}5(-0{\cdot}5) = 0{\cdot}25$$
$$w[5] = 0{\cdot}5 + 0{\cdot}5(-0{\cdot}8660) = 0{\cdot}06699$$

(b) A Hamming window with 13 terms ($M=6$) is defined by (see equation 5.22 in the main text):

$$w[n] = 0.54 + 0.46 \cos\left(\frac{n\pi}{6}\right), \quad -6 \le n \le 6$$

Estimating "one half" of the window, $w[0]$ to $w[6]$ inclusive, we obtain:

$$w[0] = 0.54 + 0.46(1) = 1.0$$
$$w[1] = 0.54 + 0.46(0.8660) = 0.93837$$
$$w[2] = 0.54 + 0.46(0.5) = 0.77$$
$$w[3] = 0.54 + 0 = 0.54$$
$$w[4] = 0.54 + 0.46(-0.5) = 0.31$$
$$w[5] = 0.54 + 0.46(-0.8660) = 0.14163$$
$$w[6] = 0.54 + 0.46(-1) = 0.08$$

Q 5.8 (p)

Q 5.9 (P)

Q 5.10 (p)

Q 5.11

The ripple level in decibels is:

$$A = -20 \log_{10} \delta = 53.98$$

Since $A > 50$, α is given by:

$$\alpha = 0.1102(A - 8.7) = 4.9899$$

The transition width (expressed as a fraction of 2π) is:

$$\Delta = 0.05$$

Hence:

$$M \geq \frac{53.98 - 7.95}{28.72\,(0.05)} = 32.05$$

Rounding M up to the next integer, we have $M = 33$, giving a window length of $(2M+1) = 67$.

<u>Q 5.12</u>

The ripple level in decibels is:

$$A = -20 \log_{10} \delta = 46.02$$

Since $21 < A < 50$, α is given by:

$$\begin{aligned}\alpha &= 0.5842(A-21)^{0.4} + 0.07886(A-21) \\ &= 0.5842(3.6251) + 1.9731 \\ &= 4.0910\end{aligned}$$

The transition width (expressed as a fraction of 2π) is:

$$\Delta = 0.075$$

Hence:

$$M \geq \frac{46.02 - 7.95}{28.72\,(0.075)} = 17.67$$

Rounding M up to the nearest integer gives $M=18$ and a window length of $(2M+1) = 37$.

<u>Q 5.13</u> (b)

Q 5.14

The first-order difference (FOD) differentiator has a frequency response:

$$H(\Omega) = 1 - \exp(-j\Omega) = 1 - \cos\Omega + j\sin\Omega$$

giving the magnitude response (see equation 5.36 in the main text):

$$|H(\Omega)| = 2\sin\left(\frac{\Omega}{2}\right)$$

At $\Omega = 0.2\pi$ we have:

$$|H(\Omega)| = 2\sin\left(\frac{0.2\pi}{2}\right) = 0.61803$$

The alternative central-difference (CD) differentiator has a frequency response:

$$\begin{aligned} H(\Omega) &= 0.5(1 - \exp(-j2\Omega)) \\ &= 0.5\{(1 - \cos 2\Omega) + j\sin 2\Omega\} \end{aligned}$$

giving the magnitude response:

$$\begin{aligned} |H(\Omega)| &= 0.5\{(1-\cos 2\Omega)^2 + \sin^2 2\Omega\}^{1/2} \\ &= 0.5\{1 - 2\cos 2\Omega + \cos^2 2\Omega + \sin^2 2\Omega\}^{1/2} \\ &= 0.5(2 - 2\cos 2\Omega)^{1/2} \\ &= 0.7071(1 - \cos 2\Omega)^{1/2} \end{aligned}$$

At $\Omega = 0.2\pi$ we have:

$$|H(\Omega)| = 0.7071(1-\cos 0.4\pi)^{1/2} = 0.5878$$

The ideal differentiator would give:

$$|H(\Omega)| = 0.2\pi = 0.62832$$

The relative response at $\Omega = 0.2\pi$ is therefore:

(a) For the FOD differentiator:

$$\frac{0.6180}{0.6283} = 0.9836 \quad \text{or} \quad -0.143 \text{ dB}$$

(b) For the CD differentiator:

$$\frac{0.5878}{0.6283} = 0.9355 \quad \text{or} \quad -0.579 \text{ dB}$$

Q 5.15

The frequency response is given by:

$$H(\Omega) = j\,B(\Omega) = j\Omega, \quad -\pi/2 < \Omega < \pi/2$$

Note that the inverse Fourier Transform to be carried out is identical to that in Worked Example 5.2 in the main text, except that the limits of integration are $\pm\pi/2$ rather than $\pm\pi$. Hence:

$$h[n] = \frac{1}{2\pi}\left[\exp(j\Omega n)\left\{\frac{\Omega}{n} - \frac{1}{jn^2}\right\}\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{2\pi}\left[\exp\left(jn\frac{\pi}{2}\right)\left\{\frac{\pi}{2n} + \frac{j}{n^2}\right\} - \exp\left(-jn\frac{\pi}{2}\right)\left\{-\frac{\pi}{2n} + \frac{j}{n^2}\right\}\right]$$

Hence:

$$h[1] = \frac{1}{2\pi}\left\{j\left(\frac{\pi}{2}+j\right) - (-j)\left(-\frac{\pi}{2}+j\right)\right\} = -\frac{1}{\pi}$$

$$h[2] = \frac{1}{2\pi}\left\{(-1)\left(\frac{\pi}{4}+\frac{j}{4}\right) - (-1)\left(-\frac{\pi}{4}+\frac{j}{4}\right)\right\} = \frac{1}{2\pi}\left(\frac{-2\pi}{4}\right) = -\frac{1}{4}$$

$$h[3] = \frac{1}{2\pi}\left\{(-j)\left(\frac{\pi}{6}+\frac{j}{9}\right) - (j)\left(-\frac{\pi}{6}+\frac{j}{9}\right)\right\} = \frac{1}{2\pi}\left(\frac{2}{9}\right) = \frac{1}{9\pi}$$

$$h[4] = \frac{1}{2\pi}\left\{(1)\left(\frac{\pi}{8}+\frac{j}{16}\right) - (1)\left(-\frac{\pi}{8}+\frac{j}{16}\right)\right\} = \frac{1}{2\pi}\left(\frac{2\pi}{8}\right) = \frac{1}{8}$$

and so on.

Furthermore, as in Worked Example 5.2, $h[0]=0$.
(Note: the above results tie in with the answers given at the end of the book, where general formulae for $h[n]$ are quoted).

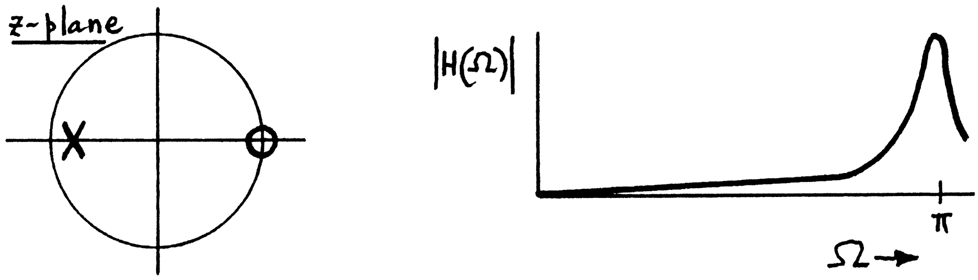
<u>Q5.16</u> **(P)**

Chapter 6

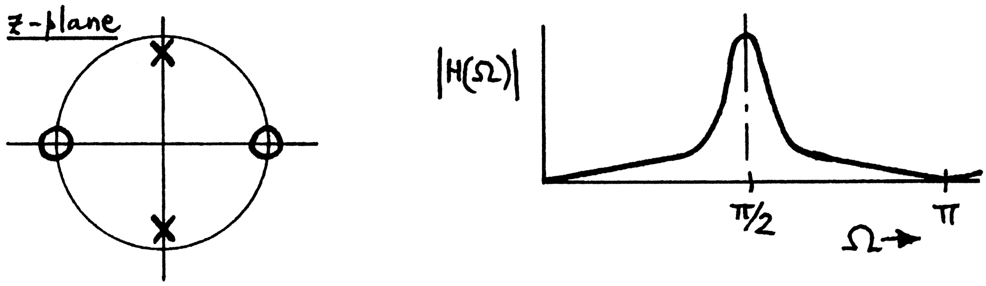
Q 6·1

Sketches :

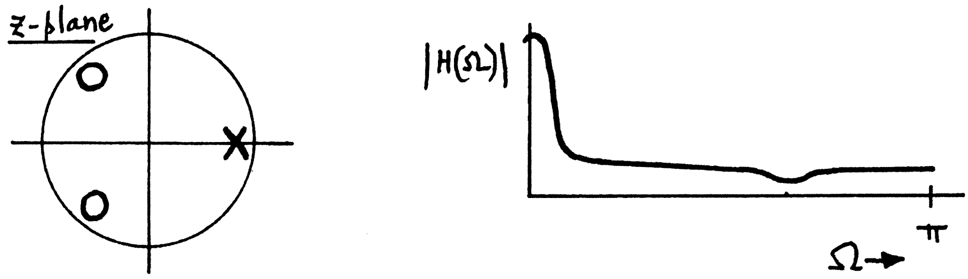
(a)



(b)



(c)



Q 6.2 (p)

Q 6.3

(a) $dB = 20 \log_{10} 10 = 20(1) = 20$

(b) $dB = 20 \log_{10} 1 = 20(0) = 0$

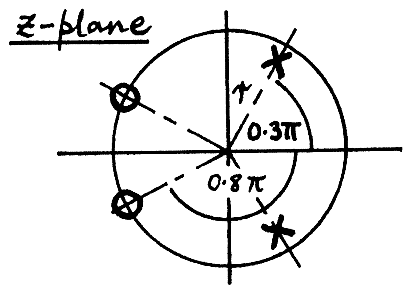
(c) $dB = 20 \log_{10} 0.001 = 20(-3) = -60$

(d) $dB = 20 \log_{10} 0.0004 = 20(-3.398) = -67.96$

(e) $dB = 20 \log_{10} 0 = 20(-\infty) = -\infty$

Q 6.4 (p)

A suitable pole-zero configuration is:



Note: $0.3\pi^c = 54^0$
$0.8\pi^c = 144^0$

The bandwidth requirement is similar to that explained in Worked Example 6.1 in the main text. Thus:

$$2(1-r) = 0.03\pi, \text{ giving } r = 0.95288$$
(r = pole radius)

Assume that the peak response occurs at the frequency $\Omega = \theta$. Providing r is close to unity, at this frequency the pole and zero vector lengths drawn from the relevant point on the unit circle are as follows:

(a) vector to nearby pole: $(1-r)$
(b) vector to distant pole: $\approx 2r\sin\theta$
(c) vector to zero: $\approx r\sin\theta$

Hence to a good approximation the peak response is given by:

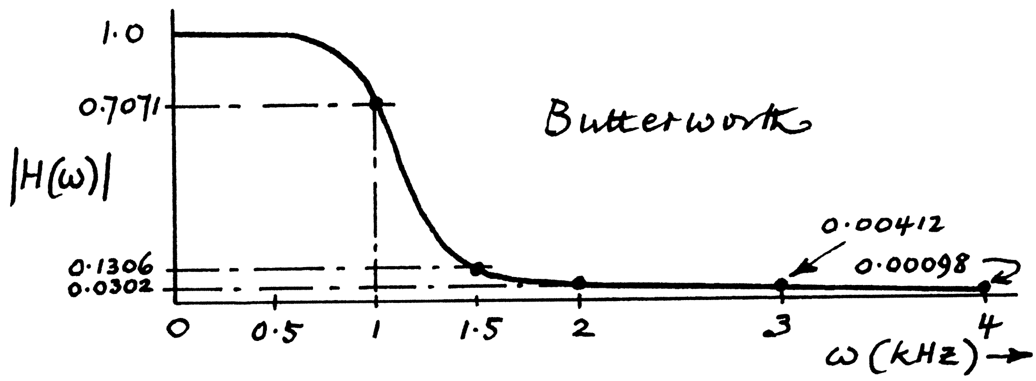
$$|H(\Omega)|_{max} = \frac{r\sin\theta}{(1-r)\,2r\sin\theta} = \frac{1}{2(1-r)}$$

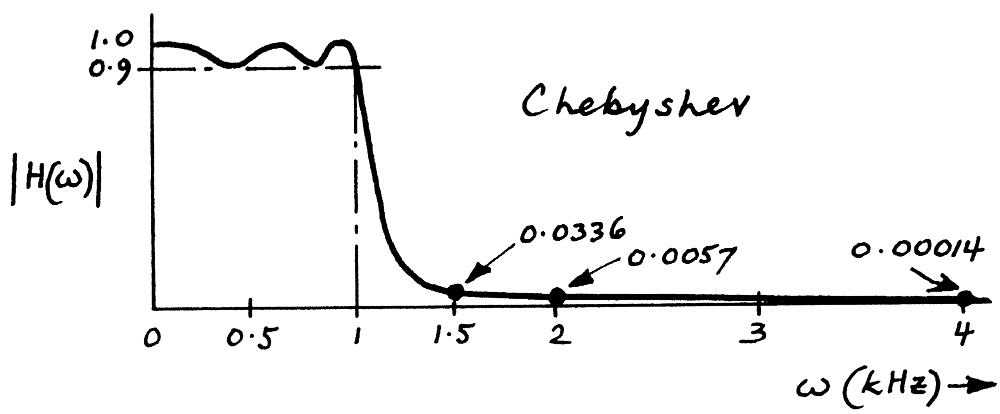
which is independent of the center-frequency θ.

Q 6.7

Using equations 6.10 to 6.14 in the main text we obtain the following characteristics:

Sketches:

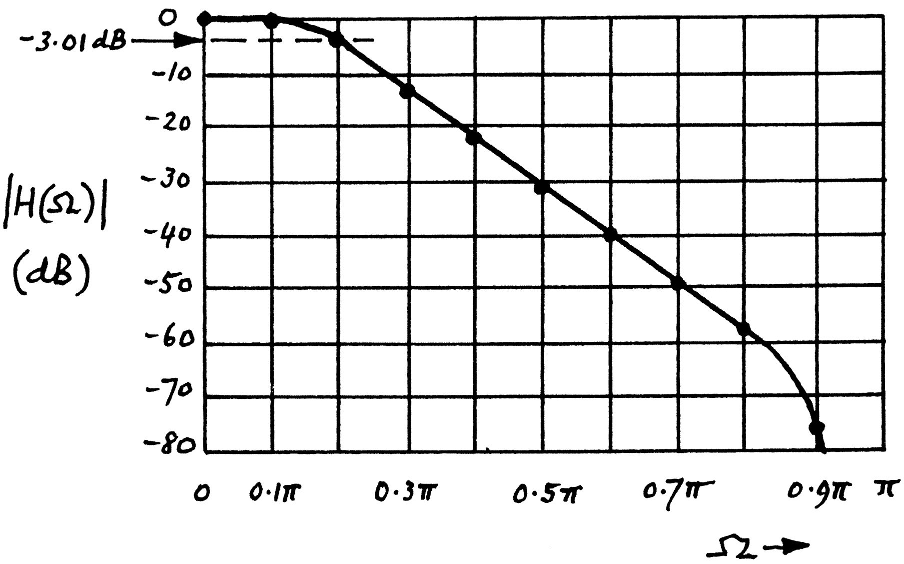




Q 6.8

Using equation 6.18 in the main text we may derive:

Sketch:



Note: the response at $\Omega=\pi$ is zero ($-\infty$ dB).

Q 6.9 (p)

Q 6.10

We assume a design based on the bilinear transformation. Also, a high-pass filter has a pole-zero configuration (and a frequency response) which is the "mirror-image" of the equivalent low-pass design. It is therefore simplest to

Consider a low-pass filter using equation 6.18 in the main text, with the following design parameters:

cut-off frequency $= (\pi - 0.7\pi) = 0.3\pi$
response > 30 dB down at $(\pi - 0.5\pi) = 0.5\pi$
response > 50 dB down at $(\pi - 0.3\pi) = 0.7\pi$

Writing $\left\{\frac{\tan \Omega/2}{\tan \Omega_1/2}\right\}$ as α, equation 6.18 gives:

$$|H(\Omega)| = \frac{1}{(1+\alpha^{2n})^{1/2}}$$

Now -30dB is equivalent to 0.03162, and -50 dB to 0.003162. Hence at $\Omega = 0.5\pi$:

$$\frac{1}{(1+\alpha^{2n})^{1/2}} \leq 0.03162 \quad \text{and} \quad \alpha = \frac{\tan \pi/4}{\tan 0.15\pi} = 1.9626$$

$$\therefore \frac{1}{(1+1.9626^{2n})^{1/2}} \leq 0.03162, \quad \text{giving} \quad n \geq 5.12$$

Also, at $\Omega = 0.7\pi$:

$$\frac{1}{(1+\alpha^{2n})^{1/2}} \leq 0.003162 \quad \text{and} \quad \alpha = \frac{\tan 0.35\pi}{\tan 0.15\pi} = 3.8518$$

$$\therefore \frac{1}{(1+3.8518^{2n})^{1/2}} \leq 0.003162, \quad \text{giving} \quad n \geq 4.27$$

The cut-off required at $\Omega = 0.5\pi$ is therefore the more stringent. n must be an integer. We therefore choose $n = 6$.

Q 6.11 (b)

The program gives the polar coordinates of poles as:

$$r = 0.90985, \ \theta = 180^\circ \ \text{(first-order real pole)}$$
$$r = 0.95479, \ \theta = \pm 163.71^\circ \ \text{(complex pair)}$$

There is also a 3rd order zero at $z = 1$. The peak gain of the filter is 1126.2 (61.03 dB), and the cut-off at $\Omega = 0.8\pi$ is about -29 dB.

The transfer function is therefore:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{(z-1)^3}{(z+0.90985)(z - r\exp(j\theta))(z - r\exp(-j\theta))}$$

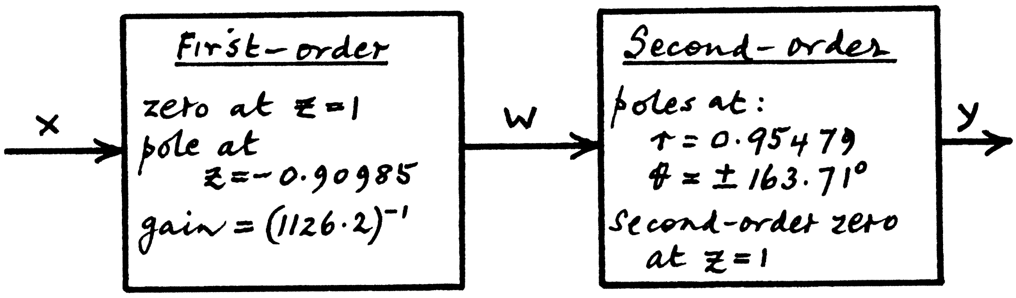
$$= \frac{z^3 - 3z^2 + 3z - 1}{z^3 + 2.7428 z^2 + 2.5793 z + 0.8294}$$

Hence the difference equation is:

$$y[n] = -2.7428\, y[n-1] - 2.5793\, y[n-2] - 0.8294\, y[n-3] + x[n] - 3x[n-1] + 3x[n-2] - x[n-3]$$

Q 6.12

Referring to the solution to Q 6.11 we see that the cascaded first and second-order subfilters will be as follows:



The transfer function of the first-order subfilter is:

$$H_1(z) = \frac{W(z)}{X(z)} = \frac{(z-1)(1126\cdot 2^{-1})}{(z+0.90985)}$$

giving the difference equation:

$$w[n] = -0.90985\, w[n-1] + 0.000888\{x[n] - x[n-1]\}$$

The transfer function of the second-order subfilter is:

$$H_2(z) = \frac{Y(z)}{W(z)} = \frac{(z-1)^2}{z^2 - 2r\cos\theta(z) + r^2}$$

$$= \frac{z^2 - 2z + 1}{z^2 + 1.8329z + 0.9116}$$

giving the difference equation:

$$y[n] = -1.8329\, y[n-1] - 0.9116\, y[n-2] + w[n] \\ - 2w[n-1] + w[n-2]$$

Q 6.13 (p)

The analog transfer function is:

$$H(s) = \frac{s}{(s+1)(s+2)}$$

Performing a partial fraction expansion, we have

$$H(s) = \frac{A}{(s+1)} + \frac{B}{(s+2)} = \frac{s(A+B)+(2A+B)}{(s+1)(s+2)}$$

Hence $(2A+B)=0$ and $(A+B)=1$, giving:

$$A=-1 \text{ and } B=2$$

We now write $H(s)$ as:

$$H(s) = \frac{-1}{(s+1)} + \frac{2}{(s+2)}$$

The impulse-invariant equivalent, in terms of digital subfilters, is:

$$H(z) = \frac{(-1)z}{z-\exp(-T)} + \frac{2z}{z-\exp(-2T)}$$

If $T=0.1$ we obtain:

$$H(z) = \frac{-z}{(z-0.90484)} + \frac{2z}{(z-0.81873)}$$

$$= \frac{-z(z-0.81873)+2z(z-0.90484)}{(z-0.90484)(z-0.81873)}$$

$$= \frac{z^2 - 0.9909z}{z^2 - 1.7236z + 0.74082} = \frac{Y(z)}{X(z)}$$

The difference equation is therefore:

$$y[n] = 1.7236\, y[n-1] - 0.74082\, y[n-2] + x[n] - 0.9909\, x[n-1]$$

The above equations show that there are poles at $z = 0.90484$ and $z = 0.81873$; and zeros at $z = 0$ and $z = 0.9909$. Program no. 20 confirms a simple bandpass characteristic with a response about -19 dB at $\Omega = \pi$.

Q 6.14

Referring to the solution for Q 6.13, and putting $T = 0.3$, we obtain:

$$H(z) = \frac{-z}{z - \exp(-0.3)} + \frac{2z}{z - \exp(-0.6)}$$

$$= \frac{-z}{z - 0.7708} + \frac{2z}{z - 0.5488}$$

Which is rearranged as:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z(z - 0.9928)}{z^2 - 1.3196z + 0.4230}$$

The difference equation is therefore:

$$y[n] = 1.3196\,y[n-1] - 0.4230\,y[n-2] + x[n] - 0.9928\,x[n-1]$$

There are poles at $z = 0.7708$ and $z = 0.5488$, and zeros at $z = 0$ and $z = 0.9928$.

Program no. 20 confirms an increased aliasing effect, with a response about -7 dB at $\Omega = \pi$.

$\left.\begin{array}{l}\underline{Q\ 6.15}\\ \underline{Q\ 6.16}\end{array}\right\}$ These two problems are essentially as worked Example 6.6 in the main text. It is helpful to note that the transfer function given may be recast in terms of the four denominator factors:

$$s + 0.9239 \pm j\,0.3827$$
$$s + 0.3827 \pm j\,0.9239$$

Q 6.17 (P)

The comb filter needs $360/4 = 90$ zeros placed at radius 0.998 in the z-plane. Its transfer function is therefore:

$$H(z) = \frac{W(z)}{X(z)} = \frac{z^{90} - 0.998^{90}}{z^{90}} = 1 - 0.83512\,z^{-90}$$

The resonator equation takes the general form:

$$y[n] = 2r\cos\theta \; y[n-1] - r^2 y[n-2] + x[n]$$

Here, $r = 0.998$ and the various angles θ are $40°, 44°, 48°, 52°, 56°$ and $60°$. The two sample weights of 0.5 at 56° and 60° may be included as gain factors in their respective equations. Thus, for the resonator at $\theta = 40°$:

$$p[n] = 2(0.998)\cos 40° \, p[n-1] - 0.998^2 p[n-2] + w[n]$$

giving:

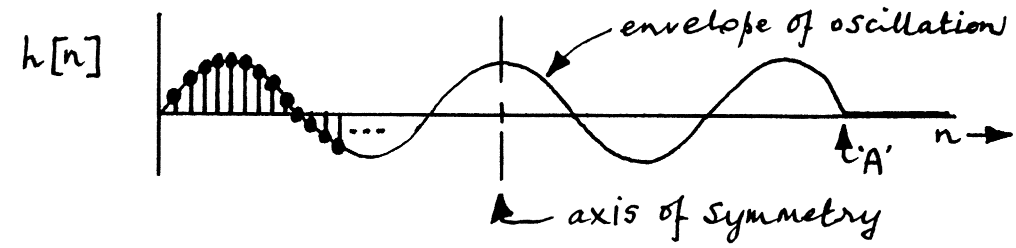
$$p[n] = 1.5290\, p[n-1] - 0.99600\, p[n-2] + w[n]$$

The other resonator equations are similarly determined – remembering that alternate input weights must be inverted. The complete set of comb filter, resonator and adder equations is:

$$w[n] = x[n] - 0.83512\, x[n-90]$$
$$p[n] = 1.5290\, p[n-1] - 0.99600\, p[n-2] + w[n]$$
$$q[n] = 1.43580\, q[n-1] - 0.99600\, q[n-2] - w[n]$$
$$r[n] = 1.33558\, r[n-1] - 0.99600\, r[n-2] + w[n]$$
$$s[n] = 1.22886\, s[n-1] - 0.99600\, s[n-2] - w[n]$$
$$t[n] = 1.11615\, t[n-1] - 0.99600\, t[n-2] + 0.5\, w[n]$$
$$u[n] = 0.99800\, u[n-1] - 0.99600\, u[n-2] - 0.5\, w[n]$$
$$y[n] = p[n] + q[n] + r[n] + s[n] + t[n] + u[n]$$

Q 6.18 (P)

(a) If each resonator has an impulse response consisting of an odd number of half-cycles of oscillation, then typically:



Because h[n] is symmetrical in form, it represents a pure linear-phase characteristic (phase shift proportional to frequency).

(b) To stop the oscillation at point 'A' above, the comb filter must supply a second positive-going impulse. If its impulse response consists of two positive-going impulses, it must have a finite response at zero frequency, and cannot therefore have a zero at $z=1$.

The redesigned filter therefore has a comb filter specified by:

$$w[n] = x[n] + 0.886867\, x[n-120]$$

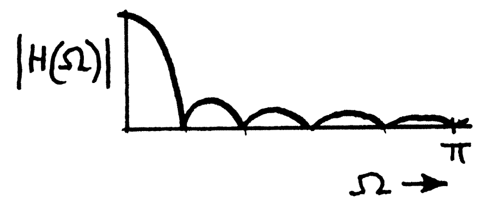
All resonator equations are as before except that the angle θ of each is reduced by 1.5°.

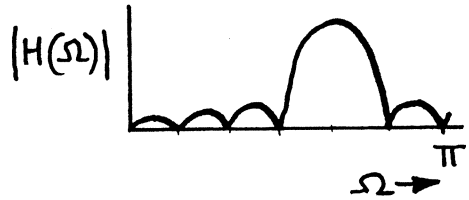
The angles θ are therefore $73.5°$ to $91.5°$ inclusive. The set of difference equations is:

$$
\begin{aligned}
w[n] &= x[n] + 0.886867\, x[n-120] \\
p[n] &= 0.56746\, p[n-1] - 0.9980\, p[n-2] + 0.5\, w[n] \\
q[n] &= 0.46642\, q[n-1] - 0.9980\, q[n-2] - w[n] \\
r[n] &= 0.36411\, r[n-1] - 0.9980\, r[n-2] + w[n] \\
s[n] &= 0.26079\, s[n-1] - 0.9980\, r[n-2] - w[n] \\
t[n] &= 0.15676\, t[n-1] - 0.9980\, t[n-2] + w[n] \\
u[n] &= 0.05230\, u[n-1] - 0.9980\, u[n-2] - 0.6667\, w[n] \\
v[n] &= -0.05230\, v[n-1] - 0.9980\, v[n-2] + 0.3333\, w[n] \\
y[n] &= p[n] + q[n] + r[n] + s[n] + t[n] + u[n] + v[n]
\end{aligned}
$$

<u>Q 6.19</u>

(a)





(b) The low-pass filter's transfer function is:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{(z^{10}-1)}{z^9(z-1)} = \frac{(1-z^{-10})}{(1-z^{-1})}$$

Its difference equation is therefore:

$$y[n] = y[n-1] + x[n] - x[n-10]$$

The bandpass filter's transfer function is:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{12}-1}{\left(z^2 - 2\left(\cos\frac{2\pi}{3}\right)z + 1\right)} = \frac{1-z^{-12}}{1+z^{-1}+z^{-2}}$$

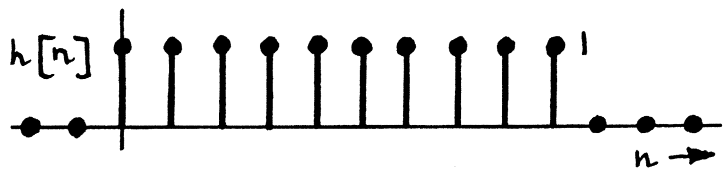
Its difference equation is therefore:

$$y[n] = -y[n-1] - y[n-2] + x[n] - x[n-12]$$

(c) The impulse response of the low-pass filter is given by:

$$h[n] = h[n-1] + \delta[n] - \delta[n-10]$$

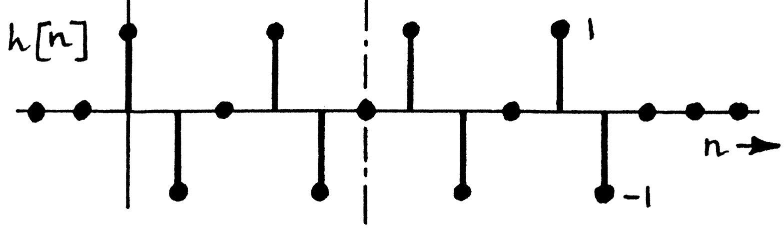
Evaluating term-by-term we easily derive:



The impulse response of the bandpass filter is given by:

$$h[n] = -h[n-1] - h[n-2] + \delta[n] - \delta[n-12]$$

Evaluating term-by-term we derive:



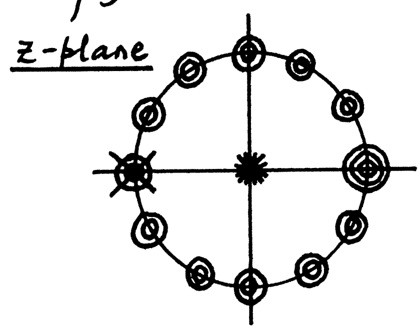
The impulse response of the low-pass filter is symmetric in form, therefore it has a pure linear-phase characteristic.

The impulse response of the bandpass filter is antisymmetric in form (about the dotted line). Therefore it is linear-phase plus a constant $\pi/2$ phase shift at all frequencies.

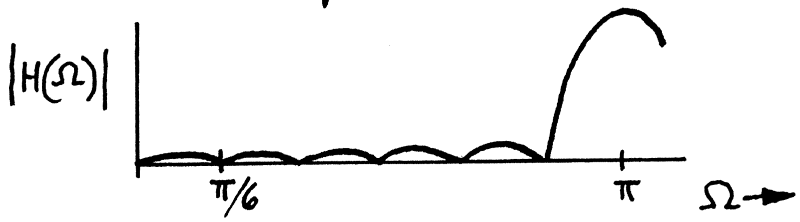
Q 6.20

(a) $$H(z) = \frac{(z^{12}-1)^2 (z-1)}{z^{23}(z+1)^2}$$

The pole-zero configuration is:



The second-order pole at $z=-1$ cancels the coincident second-order zero to give a passband. The filter is therefore high-pass. Its frequency response (magnitude) takes the form:



(b) $$H(z) = \frac{(z^{12}-1)^2(z-1)}{z^{23}(z+1)^2}$$

$$= \frac{(1-z^{-1}-2z^{-12}+2z^{-13}+z^{-24}-z^{-25})}{(1+2z^{-1}+z^{-2})} = \frac{Y(z)}{X(z)}$$

The difference equation is therefore:

$$y[n] = -2y[n-1] - y[n-2] + x[n] - x[n-1] - 2x[n-12] + 2x[n-13] + x[n-24] - x[n-25]$$

Since all multipliers are small integers ($\pm 1, \pm 2$), if input samples are expressed in integer form the whole filtering operation can be carried out using integer arithmetic.

(c) The impulse response may be found by substituting $h[n]$ for $y[n]$, and $\delta[n]$ for $x[n]$ etc. in the above equation, then evaluating the output term-by-term. Thus:

$$h[0] = 1$$
$$h[1] = -2(1) - 1 = -3$$
$$h[2] = -2(-3) - 1 = 5$$
$$h[3] = -2(5) - (-3) = -7 \quad \text{and so on.}$$

The complete impulse response sequence is:

1, −3, 5, −7, 9, −11, 13, −15, 17, −19, 21, −23,
23, −21, 19, −17, 15, −13, 11, −9, 7, −5, 3, −1, 0, 0 ...

The equivalent nonrecursive design (in which the impulse response is convolved directly with the input signal) therefore involves 24 multipliers ranging in value from ± 1 to ± 23. This compares with only 8 multipliers in the recursive version (5 of which are actually just add/subtract). The recursive version is therefore much more computationally efficient.

Q 6.22

For the running-sum integrator, we have:

$$H(\Omega) = \frac{\exp(j\Omega)}{\exp(j\Omega) - 1}$$

Hence:

$$|H(\Omega)| = \frac{1}{\{(\cos\Omega - 1)^2 + \sin^2\Omega\}^{1/2}}$$

For the trapezoid:

$$H(\Omega) = \frac{\exp(j\Omega) + 1}{2\{\exp(j\Omega) - 1\}} = \frac{\cos\Omega + j\sin\Omega + 1}{2(\cos\Omega + j\sin\Omega - 1)}$$

Hence:

$$|H(\Omega)| = \frac{\{(\cos\Omega + 1)^2 + \sin^2\Omega\}^{1/2}}{2\{(\cos\Omega - 1)^2 + \sin^2\Omega\}^{1/2}}$$

For Simpson's rule:

$$H(\Omega) = \frac{\exp(2j\Omega) + 4\exp(j\Omega) + 1}{3\{\exp(2j\Omega) - 1\}}$$

$$= \frac{\cos 2\Omega + j\sin 2\Omega + 4\cos\Omega + 4j\sin\Omega + 1}{3(\cos 2\Omega + j\sin 2\Omega - 1)}$$

Hence:

$$|H(\Omega)| = \frac{\{(\cos 2\Omega + 4\cos\Omega + 1)^2 + (\sin 2\Omega + 4\sin\Omega)^2\}^{1/2}}{3\{(\cos 2\Omega - 1)^2 + \sin^2 2\Omega\}^{1/2}}$$

We need to evaluate these various magnitude functions at $\Omega = 0.2\pi$, 0.5π, and 0.9π. This is probably best done by writing a short computer program. Results are as follows:

Ω	Running sum (dB)	Trapezoid (dB)	Simpson (dB)	Ideal (dB)
0.2π	9.624	8.621	9.312	9.294
0.5π	−6.931	−13.863	−8.109	−9.032
0.9π	−13.615	−50.718	2.471	−20.787

Hence compared with the ideal integrator, responses at the 3 frequencies are:

Running sum : +0.33, +2.10, +7.17 dB
Trapezoid : −0.67, −4.83, −29.9 dB
Simpson : +0.02, +0.92, +23.26 dB

Chapter 7

Q7.1

The DFT and IDFT involve an essentially similar set of computations. The DFT is defined in the problem, and the IDFT (see equation 7.2 in the main text) is given by:

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\,W_N^{-kn} = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\exp(j2\pi kn/N)$$

The only differences between them are the $1/N$ scaling factor and the change of sign of the exponent. Simple modifications to a DFT program are therefore sufficient to convert it to the IDFT. Note that we may write:

$$W_N^{kn} = \exp(-j2\pi kn/N) = \cos(2\pi kn/N) - j\sin(2\pi kn/N)$$

for the DFT, and

$$W_N^{-kn} = \exp(j2\pi kn/N) = \cos(2\pi kn/N) + j\sin(2\pi kn/N)$$

for the IDFT.

Hence, assuming the DFT program evaluates the "exp" function in terms of a cosine and sine, it is only necessary to change the sign of all sine (imaginary) terms, to convert from DFT to IDFT.

Q 7·2 (Note: the points raised by this problem are fully discussed in Section 7.2.1 of the main text).

Q 7·3

If $x[n] \longleftrightarrow X[k]$, then:

(a) $3x[n] \longleftrightarrow 3X[k]$ (linearity property).

(b) $x[n-2] \longleftrightarrow X[k]\,W_N^{2k}$ (time-shifting property).

(c) $2x[n] + x[n+1] \longleftrightarrow 2X[k] + X[k]\,W_N^{-k}$

$$= X[k]\{2 + W_N^{-k}\}$$

(linearity and time-shifting properties).

(d) $x[n]\,x[n-1] \longleftrightarrow \frac{1}{N}\sum_{m=0}^{N-1} X[m]\,X[k-m]\,W_N^{k-m}$

(time-shifting and modulation properties).

Q 7·4

(a) Real part of $X[k]$ is even, imaginary part is odd, and coefficients form a "mirror image" sequence.

(b) $X[k]$ is also real and even.

(c) $X[k]$ is imaginary and odd.

(d) $X[k]$ is complex.

Q 7.5

An N-point signal $x[n]$ gives an N-point DFT. If $x[n]$ is real, half the set of coefficients $X[k]$ is sufficient to define the DFT since the other half forms a "mirror-image". Note however that the periodicity in $N/2$ is not perfect, in the sense that the mirror-image half-set has the same magnitudes but equal and opposite phases (or, its real parts are the same, but its imaginary parts have opposite signs). Providing this point is remembered, we may say that the minimal condition is that $x[n]$ be real.

Q 7.6

(a) $x[n] = 1, -1$

$$X[k] = \sum_{n=0}^{N-1} x[n] \exp(-j2\pi kn/N) = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

In this case:

$$X[k] = \sum_{n=0}^{1} x[n]\, W_N^{kn}$$

Where: $\quad W_N = \exp(-j2\pi/N) = \exp(-j\pi) = -1$

If $k=0$ we have:

$$X[0] = \sum_{n=0}^{1} x[n](-1)^0 = x[0] + x[1] = 0$$

and $$X[1] = \sum_{n=0}^{1} x[n](-1)^n = x[0] + x[1](-1) = 2$$

Hence:

$$X[0] = 0 \quad \text{and} \quad X[1] = 2$$

(b) $$X[k] = \sum_{n=0}^{N-1} x[n]\, W_4^{kn} \quad \text{where } W_4 = \exp(-j\tfrac{\pi}{2}) = -j$$

Hence:

$$X[0] = \sum_{n=0}^{3} x[n] = 1+2+1+3 = 7$$

$$X[1] = \sum_{n=0}^{3} x[n]\, W_4^n$$

$$= x[0]W_4^0 + x[1]W_4^1 + x[2]W_4^2 + x[3]W_4^3$$

$$= 1(1) + 2(-j) + 1(-1) + 3(j) = 0 + j = j$$

$$X[2] = \sum_{n=0}^{3} x[n]\, W_4^{2n}$$

$$= x[0]W_4^0 + x[1]W_4^2 + x[2]W_4^4 + x[3]W_4^6$$

$$= 1(1) + 2(-1) + 1(1) + 3(-1) = -3$$

$$X[3] = \sum_{n=0}^{3} x[n]\, W_4^{3n}$$

$$= x[0]\,W_4^0 + x[1]\,W_4^3 + x[2]\,W_4^6 + x[3]\,W_4^9$$

$$= x[0] + x[1]\,W_4^3 + x[2]\,W_4^2 + x[3]\,W_4$$

$$= 1 + 2(j) + 1(-1) + 3(-j) = -j$$

Thus, to summarise:

$$X[0] = 7, \quad X[1] = j, \quad X[2] = -3, \quad X[3] = -j$$

Q7.7

The spectral coefficients $X[k]$ form a periodic sequence, thus:

$$X[-2] \quad X[-1] \quad X[0] \quad X[1] \; - - - - - - \; X[15] \quad X[16] \quad X[17]$$

etc.

$X[-1] = X[15]$, $X[0] = X[16]$, and so on.

Clearly the results $X[0]$ to $X[7]$ are given as expected. And $X[-8]$ to $X[-1]$, which are the ones quoted, are the same as expected values $X[8]$ to $X[15]$ inclusive.

Note also that since the signal is real, these two "halves" of the coefficient set will be "mirror images", having equal magnitudes but opposite phases.

Q7.8

(a) We first note that a signal $\cos(2\pi n/N)$ or $\sin(2\pi n/N)$ represents the "fundamental" of an N-point DFT, because it completes one cycle as n varies between 0 and N (i.e. one cycle per period of the DFT itself).

Hence with $N=40$, a signal $\cos(18\pi n/40)$ represents the 9^{th} harmonic. The coefficient $X[1]$ corresponds to the fundamental, and $X[9]$ to the 9^{th} harmonic. Furthermore, the signal is both real and even (a cosine), so coefficients $X[0]$ to $X[20]$ will be an exact "mirror-image" of coefficients $X[21]$ to $X[40]$. Thus $X[9]$ will be the same as $X[31]$.

We conclude that $X[9]$ and $X[31]$ have the largest magnitudes, representing the cosine.

(b) The cosine signal is even, and goes through 9 complete cycles as n varies between 0 and 40. There is therefore no "DC component", and $X[0]=0$. Neither is there any other frequency present because there are no truncation or "switch-off" effects. The cosine is an exact harmonic. Hence $X[39]$ is also zero.

Q7.9 (P)

Q7.10 (P)

Q 7.11

$$W_4^{kn} = \exp(-j2\pi kn/4) = \exp(-j\pi kn/2)$$

As the product kn varies from 0 to 3, the values of W_4^{kn} are: $1, -j, -1, j$

For larger values of the product kn, W_4^{kn} may be evaluated modulo-4. Hence:

Value of k \ Value of n	0	1	2	3
0	1	1	1	1
1	1	$-j$	-1	j
2	1	-1	1	-1
3	1	j	-1	$-j$

(Note: this is a matrix with diagonal symmetry).

Q 7.12

$$x_1[n] = x[2n], \quad n = 0, 1, 2 \cdots (N/2 - 1)$$
$$= x[0], x[2], x[4] \cdots x[N-2]$$

$$x_2[n] = x[2n+1], \quad n = 0, 1, 2 \cdots (N/2 - 1)$$
$$= x[1], x[3], x[5] \cdots x[N-1]$$

Now if:

$$x_1[n] \leftrightarrow X_1[k] \quad \text{and} \quad x_2[n] \leftrightarrow X_2[k],$$

we first note that $x[n]$ is formed by superposition of $x_1[n]$ and a time-shifted version of $x_2[n]$ (delayed by one sampling interval). Hence we may use the time-shifting property of the DFT to write:

$$x[n] \leftrightarrow X[k] = X_1[k] + X_2[k]\, W_N^k$$

(Note: this is the same approach and result as developed in more detail in equations 7·20 to 7·22 of the main text).

<u>Q 7.13</u> (Note: this problem is covered by discussion in the main text).

<u>Q 7.14</u>

If multiplications are the main factor limiting speed, we expect the speed advantage of an FFT to be approximately $N/\log_2 N$. Hence:

(a) $N/\log_2 N = 128/7 = 18 \cdot 3$

(b) $N/\log_2 N = 1024/10 = 102 \cdot 4$

(c) $N/\log_2 N = 65536/16 = 4096$

Q 7.15

$$n = 8n_1 + 4n_2 + 2n_3 + n_4$$
$$k = k_1 + 2k_2 + 4k_3 + 8k_4$$

Index maps for n and k may now be constructed as follows (compare with Worked Example 7.1 in the main text):

n_1	n_2	n_3	n_4	n
0	0	0	0	0
1	0	0	0	8
0	1	0	0	4
1	1	0	0	12
0	0	1	0	2
1	0	1	0	10
0	1	1	0	6
1	1	1	0	14
0	0	0	1	1
1	0	0	1	9
0	1	0	1	5
1	1	0	1	13
0	0	1	1	3
1	0	1	1	11
0	1	1	1	7
1	1	1	1	15

k_1	k_2	k_3	k_4	k
0	0	0	0	0
1	0	0	0	1
0	1	0	0	2
1	1	0	0	3
0	0	1	0	4
1	0	1	0	5
0	1	1	0	6
1	1	1	0	7
0	0	0	1	8
1	0	0	1	9
0	1	0	1	10
1	1	0	1	11
0	0	1	1	12
1	0	1	1	13
0	1	1	1	14
1	1	1	1	15

Comparing the two maps, we see that if k is taken through the natural-order sequence 0–15, then the corresponding shuffled input sequence is:

0, 8, 4, 12, 2, 10, 6, 14, 1, 9, 5, 13, 3, 11, 7, 15

(b) Using the bit-reversal technique, we have:

Natural-order data	4-bit code	bit-reversed code	Shuffled data
$x[0]$	0000	0000	$x[0]$
$x[1]$	0001	1000	$x[8]$
$x[2]$	0010	0100	$x[4]$
$x[3]$	0011	1100	$x[12]$
$x[4]$	0100	0010	$x[2]$
$x[5]$	0101	1010	$x[10]$
$x[6]$	0110	0110	$x[6]$
$x[7]$	0111	1110	$x[14]$
$x[8]$	1000	0001	$x[1]$
$x[9]$	1001	1001	$x[9]$
$x[10]$	1010	0101	$x[5]$
$x[11]$	1011	1101	$x[13]$
$x[12]$	1100	0011	$x[3]$
$x[13]$	1101	1011	$x[11]$
$x[14]$	1110	0111	$x[7]$
$x[15]$	1111	1111	$x[15]$

Q7.16 The appropriate table for a 32-point FFT:

Natural order	5-bit code	bit-reversed code	Shuffled order
0	00000	00000	0
1	00001	10000	16
2	00010	01000	8
3	00011	11000	24
4	00100	00100	4
5	00101	10100	20
6	00110	01100	12
7	00111	11100	28
8	01000	00010	2
9	01001	10010	18
10	01010	01010	10
11	01011	11010	26
12	01100	00110	6
13	01101	10110	22
14	01110	01110	14
15	01111	11110	30
16	10000	00001	1
17	10001	10001	17
18	10010	01001	9
19	10011	11001	25
20	10100	00101	5
21	10101	10101	21
22	10110	01101	13
23	10111	11101	29
24	11000	00011	3
25	11001	10011	19
26	11010	01011	11
27	11011	11011	27
28	11100	00111	7
29	11101	10111	23
30	11110	01111	15
31	11111	11111	31

Q7.17 (Note: it is intended that the student checks signal flow paths through the two figures to ensure that the computations are equivalent).

Q7.18

(a) The twiddle factors follow the butterflies, hence this is a decimation-in-frequency algorithm (4-point, radix 2, shuffled input, natural order output).

(b) The twiddle factors precede the butterflies, hence this is a decimation-in-time algorithm (4-point, radix 2, natural order input, shuffled output).

Q7.19

The index map equations in Worked Example 7.2 are:

$$n = n_1 + 2n_2 + 4n_3 \quad \text{and} \quad k = 4k_1 + 2k_2 + k_3$$

Substituting in the 8-point DFT equation:

$$X = \sum_{n=0}^{7} x W_8^{kn} = \sum_{n_3=0}^{1} \sum_{n_2=0}^{1} \sum_{n_1=0}^{1} x W_8^{(4k_1+2k_2+k_3)(n_1+2n_2+4n_3)}$$

The index of W_8 multiplies out to give:

$$\left(16n_3k_1 + 8n_2k_1 + 8n_3k_2 + 4n_1k_1 + 4n_2k_2 + 4n_3k_3 + 2n_1k_2 + 2n_2k_3 + n_1k_3\right)$$

Evaluated modulo-8, the first 3 terms are always unity and do not affect the computation. We therefore have:

$$X = \sum_{n_3=0}^{1}\sum_{n_2=0}^{1}\sum_{n_1=0}^{1} x W_8^{4k_1n_1} W_8^{4k_2n_2} W_8^{4k_3n_3} W_8^{2k_2n_1} W_8^{2k_3n_2} W_8^{k_3n_1}$$

$$= \sum_{n_3=0}^{1} W_8^{4k_3n_3} \sum_{n_2=0}^{1} W_8^{4k_2n_2} W_8^{2k_3n_2} \sum_{n_1=0}^{1} x W_8^{4k_1n_1} W_8^{n_1(2k_2+k_3)}$$

The 3 summations correspond to the 3 stages of processing. In each case there are transmittances equal to a power of W_8^4, which always have values ± 1 and are incorporated as part of the FFT butterflies. The extra terms represent the twiddle factors.

Q7.20

It is helpful to bear in mind the essential "duality" of the decimation-in-time and decimation-in-frequency approaches. The signal flow graphs have flow directions reversed and inputs and outputs interchanged.

A decimation-in-frequency FFT with natural order input and output therefore has a flow graph which is a reversed version of that of its decimation-in-time equivalent. The required flow graph is therefore a reversed version of Figure 7.10(b) in the main text, with inputs and outputs interchanged. Such reversal produces twiddle factors which follow the butterflies.

Q 7.21 (Note: the relevant points are all discussed in the main text).

Q 7.22

The DFT equation is:

$$X[k] = \sum_{n=0}^{N-1} x[n]\, W_N^{kn} = \sum_{n=0}^{63} x[n]\, W_{64}^{kn}$$

Writing $X[k]$ as X, and $x[n]$ as x:

$$X = \sum_{n_3=0}^{3} \sum_{n_2=0}^{3} \sum_{n_1=0}^{3} x\, W_{64}^{(n_1+4n_2+16n_3)(16k_1+4k_2+k_3)}$$

The index of W_{64} multiplies out to give:

$$(16k_1n_1 + 64k_1n_2 + 256k_1n_3 + 4k_2n_1 + 16k_2n_2 + 64k_2n_3 + k_3n_1 + 4k_3n_2 + 16k_3n_3)$$

Terms $64k_1n_2$, $256\,k_1n_3$ and $64\,k_2n_3$, evaluated modulo-64, always equal unity and do not affect the computation. Hence the nested summations may be written as:

$$X = \sum_{n_3=0}^{3} W_{64}^{16k_3n_3} \sum_{n_2=0}^{3} W_{64}^{16k_2n_2} W_{64}^{4k_3n_3} \sum_{n_1=0}^{3} x\, W_{64}^{16k_1n_1+4k_2n_1+k_3n_1}$$

Now $W_{64}^{16} = W_4$, so that:

$$X = \sum_{n_3=0}^{3} W_4^{k_3n_3} \sum_{n_2=0}^{3} W_4^{k_2n_2} W_{64}^{4k_3n_3} \sum_{n_1=0}^{3} x\, W_4^{k_1n_1} W_{64}^{n_1(4k_2+k_3)}$$

The W_4 terms are associated with the basic 4-point DFT's, and the W_{64} terms represent twiddle factors.

Chapter 8

Q 8.1 (P)

Q 8.2 (P)

Q 8.3 (P)

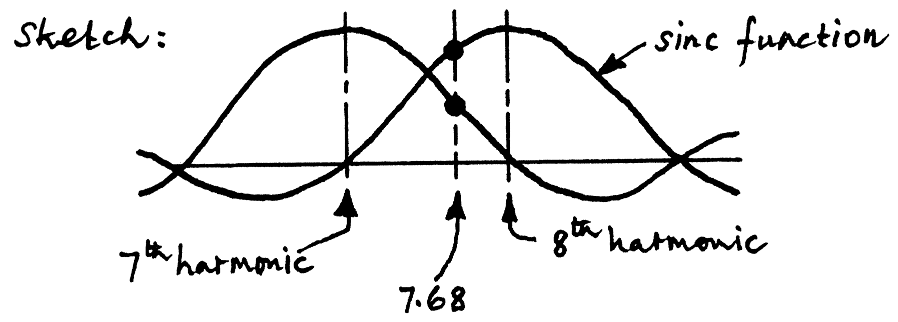
Q 8.4

(a) The transform has 64 spectral coefficients in the range 0 – 500 Hz. Hence the spacing between adjacent FFT harmonics is $500/64 = 7.8125$ Hz.

60 Hz therefore corresponds to harmonic "number" $60/7.8125 = 7.68$ (that is, between the 7th and 8th harmonics, but closer to the latter). The 8th (and also the 56th) spectral coefficient will therefore be the largest.

(b) Following on from part (a), the next largest spectral coefficient will be the 7th (and the 57th). The relative sizes of the 7th and 8th coefficients may be found by constructing sinc, or $\sin x/x$, functions around the 7th and 8th harmonic frequencies as follows:

Sketch:



Note that an interval of one harmonic along the frequency axis corresponds to π radians for the sinc function. Hence an interval of 0.32 harmonic is equivalent to $0.32\pi^c$, and so on.

The magnitude of the 8th coefficient is given by:

$$\frac{\sin 0.32\pi}{0.32\pi} = 0.8399$$

The magnitude of the 7th coefficient is given by:

$$\frac{\sin 0.68\pi}{0.68\pi} = 0.3952$$

The relative size of the 7th coefficient, compared with the 8th coefficient, is therefore:

$$\frac{0.3952}{0.8399} = 0.4705$$

Expessed in decibels, this is: $20\log_{10} 0.4705 = -6.55$ dB

(c) The spacing between harmonics is 7.8125 Hz. The harmonic closest to 60 Hz is the 8th, which has exact frequency $8 \times 7.8125 = 62.5$ Hz.

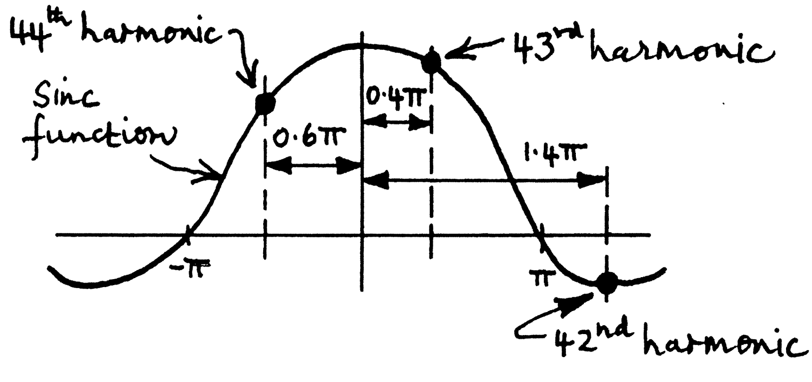
For the 8^{th} harmonic to coincide exactly with 60 Hz, we must therefore reduce the sampling frequency to:

$$500 \times \frac{60}{62.5} = 480 \text{ samples/second.}$$

Q 8.5 (p)

(a) $$x[n] = \sin\left(\frac{2\pi n}{256}\, 43.4\right)$$

The frequency of this signal lies between the 43^{rd} and 44^{th} harmonics of a 256-point FFT — but closer to the 43^{rd}. The 3 largest spectral coefficients will therefore be the 42^{nd}, 43^{rd} and 44^{th}. Their relative magnitudes may be found by considering the relevant 'offset' of each from the peak of a sinc function, as follows:



The magnitudes of the 3 coefficients are given by:

42^{nd}: $\frac{\sin 1.4\pi}{1.4\pi} = (-)0.21624$ $\qquad$ 43^{rd}: $\frac{\sin 0.4\pi}{0.4\pi} = 0.75683$

$$44^{th}: \quad \frac{\sin 0.6\pi}{0.6\pi} = 0.50455$$

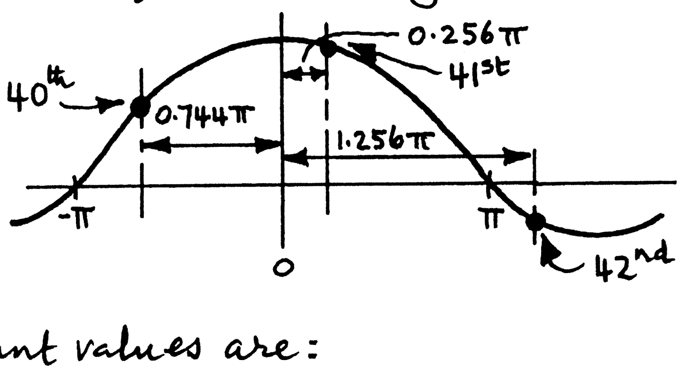
Or, taking the 43^{rd} as unity, their relative magnitudes are:

42^{nd}	43^{rd}	44^{th}
0.286	1	0.667

(b)

$$x[n] = \sin[n] = \sin\left(\frac{2\pi n}{256}\,\frac{256}{2\pi}\right) = \sin\left(\frac{2\pi n}{256}\,40.744\right)$$

Therefore the signal lies between the 40^{th} and 41^{st} harmonics of a 256-point FFT – but closer to the 41^{st}. The 3 largest spectral coefficients will therefore be the 40^{th}, 41^{st}, and 42^{nd}. The relevant sinc function diagram is:



The relevant values are:

$$40^{th}: \quad \frac{\sin 0.744\pi}{0.744\pi} = 0.30817$$

$$41^{st}: \quad \frac{\sin 0.256\pi}{0.256\pi} = 0.89563$$

$$42^{nd}: \quad \frac{\sin 1.256\pi}{1.256\pi} = (-)\,0.18255$$

Taking the largest of these as unity, their relative magnitudes are:

40^{th}	41^{st}	42^{nd}
0.344	1	0.204

Q 8.6

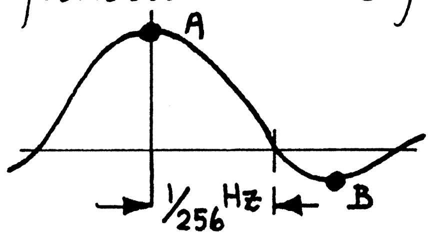
(a) With 44 kHz sampling, a 2048-point transform gives frequency resolution of about:

$$\frac{44\text{ kHz}}{2048} = 21.48\text{ Hz}$$

(b) A 256-point DFT or FFT may be thought of as providing 256 elementary bandpass filters in the frequency range $0 \leq \Omega \leq 2\pi$. Hence the filter separation is $2\pi/256$ radian, and the main lobe width of each sinc function is twice as great, that is:

$$\frac{2\pi}{256}(2) = 0.0491 \text{ radian}$$

(c) Each sinc function has the form:



Hence the separation between points A and B is:

$$\frac{1}{256}(1.5) = 0.00586\text{ Hz}.$$

Q 8.7 (P)

Q 8.8

Equation 8.4 in the main text defines the Hamming window. If $N=16$ we have:

$$w[n] = 0.54 + 0.46 \cos\left\{\frac{(2n-15)\pi}{16}\right\}, \quad 0 \leq n \leq 15$$

This is a symmetric window, with center-line midway between $n=7$ and $n=8$. The values $w[7]$ and $w[8]$ are therefore the largest (and equal to one another), and have the value:

$$0.54 + 0.46 \cos(\pi/16) = 0.99116$$

Q 8.9 (P)

Q 8.10 (P)

Q 8.11 (P)

Q 8.12 (P)

Q 8.13 (P)

Q 8·14

Circular convolution may be visualized as the placing of the N samples of one function around the circumference of a cylinder, and the N samples of the other function in reverse order around a second, concentric, cylinder. One cylinder is rotated, and coincident samples are multiplied together and summed. The result is periodic.

On a 2-D surface such as this sheet of paper, the same result is obtained by moving a single period of one function over a repetitive version of the other function, with one of the functions reversed. Hence in this case:

$x[n]$	1	2	0	-1	1	1	1	2	0	-1	1	1	1	2	···
$h[n]$ (reversed)	1	0	-1	1	-1	1	→ move step-by-step								
summed cross-product						·· 0	4	1	-3	1	1	0	4	1	··
							one period								

Therefore one period of the circular convolution has the values: 4, 1, −3, 1, 1, 0

Linear convolution of $x[n]$ and $h[n]$, assuming them to be aperiodic (with zero values on either end of the given samples) is performed as follows:

$x[n]$		1 2 0 −1 1 1
$h[n]$ (reversed)	1 0 −1 1 −1 1 $\longrightarrow$ move step-by-step	
Summed cross-product		1 1 −1 0 0 0 3 0 −2 1 1

Giving the sequence:

1, 1, −1, 0, 0, 0, 3, 0, −2, 1, 1

<u>Q 8·15</u> (P)

<u>Q 8·16</u>
<u>Q 8·17</u>

Note: Solutions to these problems involve careful examination of Figures 8.14 and 8.15 in the main text.