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L-moments and TL-moments of the generalized lambda distribution

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Abstract

The 4-parameter generalized lambda distribution (GLD) is a flexible distribution capable of mimicking the shapes of many distributions and data samples including those with heavy tails. The method of L-moments and the recently developed method of trimmed L-moments (TL-moments) are attractive techniques for parameter estimation for heavy-tailed distributions for which the L- and TL-moments have been defined. Analytical solutions for the first five L- and TL-moments in terms of GLD parameters are derived. Unfortunately, numerical methods are needed to compute the parameters from the L- or TL-moments. Algorithms are suggested for parameter estimation. Application of the GLD using both L- and TL-moment parameter estimates from example data is demonstrated, and comparison of the L-moment fit of the 4-parameter kappa distribution is made. A small simulation study of the 98th percentile (far-right tail) is conducted for a heavy-tail GLD with high-outlier contamination. The simulations show, with respect to estimation of the 98th-percent quantile, that TL-moments are less biased (more robost) in the presence of high-outlier contamination. However, the robustness comes at the expense of considerably more sampling variability.

Keywords: L-moments; TL-moments; Order statistics; Generalized lambda distribution; Robust estimation

1. Introduction

Probability distributions are used to analyze data in many disciplines, and their use is often complicated by certain intrinsic characteristics of the data such as large range or variation and large skewness. As an example, the analysis of annual peak streamflow, the largest instantaneous streamflow for a given location for a given year, is of major interest to hydrologists (Stedinger et al., 1992). These data can show considerable heavy-tailed characteristics—specifically large positive skewness—and generally have comparatively small sample sizes. Outliers or highly influential values are common.

The data analysis often requires estimation of parameters for a few candidate probability distributions. A classical technique of parameter estimation is the method of moments. This technique generally works well for light-tailed distributions. For heavy-tailed distributions, however, use of the method of moments can be questioned. The method of L-moments is an alternative technique, which is suitable and popular for heavy-tailed distributions. The method of L-moments is particularly useful for distributions, such as the generalized lambda distribution (GLD), that are only expressible in inverse or quantile function form.

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L-moments (Hosking, 1990) are derived from the expectations of order statistics (David, 2003). L-moments are direct analogs to conventional (product) moments; however, L-moments have several advantages including unbiasedness, robustness, and consistency with respect to conventional moments (mean, variance, skew, kurtosis, and so forth). In particular, L-moments might be preferable to conventional moments for characterization of distribution shape as advocated by Hosking (1992). Trimmed L-moments (TL-moments) introduced by Elamir and Seheult (2003) are derived from L-moments, and TL-moments might have additional robust properties compared to the L-moments.

For a dozen or more distributions, including the cauchy, normal, lognormal, generalized extreme value, and pearson type III, L-moments and, to a much lesser degree, TL-moments have been derived and are used in research and nonresearch applications. A particularly useful distribution for hydrologic research applications is the 4-parameter kappa distribution (Hosking, 1994; Hosking and Wallis, 1993, 1997). Because the kappa distribution has four parameters, it can acquire a wider range of shapes than 2- or 3-parameter distributions such as the normal (2 parameter) or generalized extreme value (3 parameter) distributions.

Like the kappa distribution, the GLD, considered by Gilchrist (2000, pp. 160–163), Karian and Dudewicz (2000), and references therein, has four parameters and is a flexible distribution that can be used to approximate a wide range of symmetrical as well as asymmetrical distributions and is useful in modeling heavy-tailed distributions. The quantile function of the GLD is

$$X(F) = \xi + \alpha\left(F^{\kappa} - (1-F)^{h}\right), \tag{1}$$

where $X(F)$ is the quantile for random variable X and nonexceedance probability F, ξ is a location parameter, α is a scale parameter, and κ and h are shape parameters.

The GLD for a given set of parameters is $\mathrm{GLD}_m(\xi, \alpha, \kappa, h)$ in which m will represent a method of parameter estimation. The scale parameter used here is the reciprocal of the scale parameter used by Karian and Dudewicz (2000) ($\alpha = 1/\lambda_2$) so that the scale parameter and location parameter (ξ) will be in the same measurement units. Furthermore, λ is commonly used to designate L-moments, and use of α to designate the scale parameter rather than λ is intended to avoid confusion.

The parameter space of the GLD is defined by Karian and Dudewicz (2000, Theorem 1.3.3); the GLD is valid if and only if

$$\alpha\left(\kappa F^{\kappa-1} + h(1-F)^{h-1}\right) \geqslant 0 \quad \text{for all } F \text{ in } [0, 1]. \tag{2}$$

A comprehensive discussion of the history, properties, uses, and other features of the GLD is provided by Karian and Dudewicz (2000), and the authors also thoroughly discuss parameter estimation of the GLD using the method of moments and the method of percentiles. An alternative least-squares scheme for GLD parameter estimation using order statistics is suggested and demonstrated by Öztürk and Dale (1985). In general, parameter estimation for the GLD is problematic because of complex relations between the two shape parameters; often more than one numerically acceptable solution to κ and h is available. The ξ and α parameters are quickly computed once estimates for κ and h are available.

The 5-parameter GLD briefly is considered by Hosking (1986) in the context of L-moments. The first four L-moments of the 4-parameter GLD were recently considered by Karvanen et al. (2002)—the fifth L-moment is defined here. In Karvanen et al. (2002), the L-moments specifically are used; whereas use of the L-moment ratios is emphasized here. For symmetrical GLD($\kappa = h$), Karvanen et al. (2002, p. 86) show that the parameters are expressible in terms of the 4th L-moment ratio, known as L-kurtosis (τ_4, Eq. (12) of this paper)

$$\kappa = h = \frac{3 + 7\tau_4 + \sqrt{1 + 98\tau_4 + \tau_4^2}}{2\,(1-\tau_4)}. \tag{3}$$

However, for the general case of the GLD, equations to explicitly solve for each parameter from the L-moments cannot be derived. Karvanen et al. (2002) report that the relation between L-moments and parameters is complicated, but numerical solutions using the Newton–Raphson method are possible.

This paper extends the application of the GLD by deriving in parallel the first five L- and TL-moments of the distribution. Algorithms are suggested for the method of L- or TL-moments. The GLD fit by the methods of L- and TL-moments is demonstrated by example, and comparison to the kappa distribution is made. A small simulation study is conducted to compare the properties of L- and TL-moments in the context of far-right-tail quantile estimation for a heavy-tailed GLD with high-outlier contamination.

1.1. L-moments

The theoretical L-moments for a real-valued random variable X with a quantile function $X(F)$ are defined from the expectations of order statistics. The order statistics of X for a sample of size n are formed by the ascending order $X_{1:n} \leqslant X_{2:n} \leqslant \cdots \leqslant X_{n:n}$.The theoretical L-moments are

$$\lambda_r = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k \binom{r-1}{k} E\left[X_{r-k:r}\right], \tag{4}$$

where r is the order of the L-moment, and $E\left[X_{r-k:r}\right]$ is the expectation of the $r-k$ order statistic of a sample of size r. The expectation of an order statistic is

$$E\left[X_{j:r}\right] = \frac{r!}{(j-1)!(r-j)!}\int_0^1 X(F) \times F^{j-1}(1-F)^{r-j}\,\mathrm{d}F. \tag{5}$$

The first four theoretical L-moments are

$$\lambda_1 = \int_0^1 X(F)\,\mathrm{d}F, \tag{6}$$

$$\lambda_2 = \int_0^1 X(F) \times (2F-1)\,\mathrm{d}F, \tag{7}$$

$$\lambda_3 = \int_0^1 X(F) \times \left(6F^2 - 6F + 1\right)\mathrm{d}F, \tag{8}$$

and

$$\lambda_4 = \int_0^1 X(F) \times \left(20F^3 - 30F^2 + 12F - 1\right)\mathrm{d}F. \tag{9}$$

The L-moment ratios are the dimensionless quantities

$$\tau = \lambda_2/\lambda_1 = \text{coefficient of L-variation}, \tag{10}$$

$$\tau_3 = \lambda_3/\lambda_2 = \text{L-skew}, \tag{11}$$

$$\tau_4 = \lambda_4/\lambda_2 = \text{L-kurtosis}, \tag{12}$$

and for $r \geqslant 5$, which are unnamed,

$$\tau_r = \lambda_r/\lambda_2. \tag{13}$$

The sample L-moments are computed from the sample order statistics $x_{1:n} \leqslant x_{2:n} \leqslant \cdots \leqslant x_{n:n}$. The sample L-moments are

$$\hat{\lambda}_r = \frac{1}{r}\sum_{i=1}^{n}\left[\frac{\sum_{j=0}^{r-1}(-1)^j \binom{r-1}{j}\binom{i-1}{r-1-j}\binom{n-i}{j}}{\binom{n}{r}}\right] x_{i:n}. \tag{14}$$

Several L-moments, unlike conventional moments, are bounded (Hosking, 1990, Theorem 2). Two useful examples for L-moment ratios are

$$-1 < \tau_r < 1 \quad \text{for } r \geqslant 3 \tag{15}$$

and

$$\tfrac{1}{4}\left(5\tau_3^2 - 1\right) \leqslant \tau_4 < 1. \tag{16}$$

1.2. TL-moments

TL-moments are computed from conceptual samples larger than the actual sample. The sample size difference is represented by a "trim" level. Symmetrical trimming as considered by Elamir and Seheult (2003) is used here. The $r \geqslant 1$ TL-moment is defined for trimming level $t \geqslant 0$ as follows:

$$\lambda_r^{(t)} = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k \binom{r-1}{k} \frac{(r+2t)!\, I_{r,k}^{(t)}}{(r+t-k-1)!(t+k)!}, \tag{17}$$

where

$$I_{r,k}^{(t)} = \int_0^1 X(F) \times F^{r+t-k-1}(1-F)^{t+k}\,\mathrm{d}F. \tag{18}$$

The ordinary rth L-moment is formed when $t=0$. The first four theoretical TL-moments (Elamir and Seheult, 2003, p. 303) for $t=1$ are

$$\lambda_1^{(1)} = 6\int_0^1 X(F) \times F(1-F)\,\mathrm{d}F, \tag{19}$$

$$\lambda_2^{(1)} = 6\int_0^1 X(F) \times F(1-F)(2F-1)\,\mathrm{d}F, \tag{20}$$

$$\lambda_3^{(1)} = \frac{20}{3}\int_0^1 X(F) \times F(1-F)\left(5F^2 - 5F + 1\right)\mathrm{d}F, \tag{21}$$

and

$$\lambda_4^{(1)} = \frac{15}{2}\int_0^1 X(F) \times F(1-F)\left(14F^3 - 21F^2 + 9F - 1\right)\mathrm{d}F. \tag{22}$$

The sample TL-moments (Elamir and Seheult, 2003, p. 303) are computed by

$$\hat{\lambda}_r^{(t)} = \frac{1}{r}\sum_{i=t+1}^{n-t}\left[\frac{\sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j}\binom{i-1}{r+t-1-j}\binom{n-i}{t+j}}{\binom{n}{r+2t}}\right] x_{i:n}, \tag{23}$$

where t represents the trimming level of the t-smallest and t-largest values, r represents the order of the L-moment, n represents the sample size, and $x_{i:n}$ is the ith sample order statistic ($x_{1:n} \leqslant x_{2:n} \leqslant \cdots \leqslant x_{n:n}$). For the TL-moments considered here, $t=1$. TL-moment ratios $\left(\tau^{(1)} \text{ and } \tau_r^{(1)}\right)$ have the same definition and interpretations as the L-moment ratios (Eqs. (10)–(13)).

2. L-moments and TL-moments of the GLD

2.1. L-moments of the GLD

The L-moments of the GLD are obtained by substitution of Eq. (1) into Eq. (5) using Eq. (4); however, considerable manipulation is required. The first three L-moments (λ_1, λ_2, and λ_3) of the GLD are

$$\lambda_1 = \xi + \alpha\left(\frac{1}{\kappa+1} - \frac{1}{h+1}\right), \tag{24}$$

$$\lambda_2 = \alpha\left(\frac{\kappa}{(\kappa+2)(\kappa+1)} + \frac{h}{(h+2)(h+1)}\right), \tag{25}$$

and

$$\lambda_3 = \alpha\left(\frac{\kappa(\kappa-1)}{(\kappa+3)(\kappa+2)(\kappa+1)} - \frac{h(h-1)}{(h+3)(h+2)(h+1)}\right). \tag{26}$$

The fourth L-moment (λ_4) of the GLD is

$$K_{\lambda_4} = \frac{\kappa(\kappa-2)(\kappa-1)}{(\kappa+4)(\kappa+3)(\kappa+2)(\kappa+1)},$$

$$H_{\lambda_4} = \frac{h(h-2)(h-1)}{(h+4)(h+3)(h+2)(h+1)},$$

and

$$\lambda_4 = \alpha\left(K_{\lambda_4} + H_{\lambda_4}\right). \tag{27}$$

Let L_λ be defined as follows:

$$L_\lambda = \kappa(h+2)(h+1) + h(\kappa+2)(\kappa+1). \tag{28}$$

The L-moment ratio τ_3 of the GLD is

$$K_{\tau_3} = \kappa(\kappa-1)(h+3)(h+2)(h+1),$$

$$H_{\tau_3} = h(h-1)(\kappa+3)(\kappa+2)(\kappa+1),$$

and

$$\tau_3 = \frac{K_{\tau_3} - H_{\tau_3}}{(\kappa+3)(h+3)L_\lambda}. \tag{29}$$

The L-moment ratio τ_4 of the GLD is

$$K_{\tau_4} = \kappa(\kappa-2)(\kappa-1)(h+4)(h+3)(h+2)(h+1),$$

$$H_{\tau_4} = h(h-2)(h-1)(\kappa+4)(\kappa+3)(\kappa+2)(\kappa+1),$$

and

$$\tau_4 = \frac{K_{\tau_4} + H_{\tau_4}}{(\kappa+4)(h+4)(\kappa+3)(h+3)L_\lambda}. \tag{30}$$

Higher L-moment ratios are readily obtained because of the symmetrical pattern of increase in K and H terms and the alternating pattern of addition and subtraction of the K and H terms. Thus, for example, the fifth L-moment ratio τ_5 is

$$K_{\tau_5} = \kappa(\kappa-3)(\kappa-2)(\kappa-1)(h+5)(h+4)(h+3)(h+2)(h+1),$$

$$H_{\tau_5} = h(h-3)(h-2)(h-1)(\kappa+5)(\kappa+4)(\kappa+3)(\kappa+2)(\kappa+1),$$

and

$$\tau_5 = \frac{K_{\tau_5} - H_{\tau_5}}{(\kappa+5)(h+5)(\kappa+4)(h+4)(\kappa+3)(h+3)L_\lambda}. \tag{31}$$

Finally, the L-moments are potentially defined for

$$\kappa > -1 \quad \text{and} \quad h > -1. \tag{32}$$

2.2. TL-moments of the GLD

As for the L-moments, the TL-moments with $t=1$ of the GLD are obtained by substitution of Eq. (1) using Eq. (17) through Eq. (18). The first two TL-moments $\left(\lambda_1^{(1)} \text{ and } \lambda_2^{(1)}\right)$ of the GLD are

$$\lambda_1^{(1)} = \xi + 6\alpha\left(\frac{1}{(\kappa+3)(\kappa+2)} - \frac{1}{(h+3)(h+2)}\right) \tag{33}$$

and

$$\lambda_2^{(1)} = 6\alpha\left(\frac{\kappa}{(\kappa+4)(\kappa+3)(\kappa+2)} + \frac{h}{(h+4)(h+3)(h+2)}\right). \tag{34}$$

The third TL-moment ($\lambda_3^{(1)}$) of the GLD is

$$K_{\lambda_3^{(1)}} = \frac{\kappa(\kappa-1)}{(\kappa+5)(\kappa+4)(\kappa+3)(\kappa+2)},$$

$$H_{\lambda_3^{(1)}} = \frac{h(h-1)}{(h+5)(h+4)(h+3)(h+2)},$$

and

$$\lambda_3^{(1)} = \tfrac{20}{3}\alpha\left(K_{\lambda_3^{(1)}} - H_{\lambda_3^{(1)}}\right). \tag{35}$$

The fourth TL-moment $\left(\lambda_4^{(1)}\right)$ of the GLD is

$$K_{\lambda_4^{(1)}} = \frac{\kappa(\kappa-2)(\kappa-1)}{(\kappa+6)(\kappa+5)(\kappa+4)(\kappa+3)(\kappa+2)},$$

$$H_{\lambda_4^{(1)}} = \frac{h(h-2)(h-1)}{(h+6)(h+5)(h+4)(h+3)(h+2)},$$

and

$$\lambda_4^{(1)} = \tfrac{15}{2}\alpha\left(K_{\lambda_4^{(1)}} + H_{\lambda_4^{(1)}}\right). \tag{36}$$

The fifth TL-moment $\left(\lambda_5^{(1)}\right)$ of the GLD is

$$K_{\lambda_5^{(1)}} = \frac{\kappa(\kappa-3)(\kappa-2)(\kappa-1)}{(\kappa+7)(\kappa+6)(\kappa+5)(\kappa+4)(\kappa+3)(\kappa+2)},$$

$$H_{\lambda_5^{(1)}} = \frac{h(h-3)(h-2)(h-1)}{(h+7)(h+6)(h+5)(h+4)(h+3)(h+2)},$$

and

$$\lambda_5^{(1)} = \tfrac{42}{5}\alpha\left(K_{\lambda_5^{(1)}} - H_{\lambda_5^{(1)}}\right). \tag{37}$$

Let $L_{\lambda^{(1)}}$ be defined as follows:

$$L_{\lambda^{(1)}} = \kappa(h+4)(h+3)(h+2) + h(\kappa+4)(\kappa+3)(\kappa+2). \tag{38}$$

The TL-moment ratio $\tau_3^{(1)}$ of the GLD is

$$K_{\tau_3^{(1)}} = \kappa(\kappa-1)(h+5)(h+4)(h+3)(h+2),$$

$$H_{\tau_3^{(1)}} = h(h-1)(\kappa+5)(\kappa+4)(\kappa+3)(\kappa+2),$$

and

$$\tau_3^{(1)} = \frac{10}{9}\left(\frac{K_{\tau_3^{(1)}} - H_{\tau_3^{(1)}}}{(\kappa+5)(h+5)L_{\lambda^{(1)}}}\right). \tag{39}$$

The TL-moment ratio $\tau_4^{(1)}$ of the GLD is

$$K_{\tau_4^{(1)}} = \kappa(\kappa-2)(\kappa-1)(h+6)(h+5)(h+4)(h+3)(h+2),$$

$$H_{\tau_4^{(1)}} = h(h-2)(h-1)(\kappa+6)(\kappa+5)(\kappa+4)(\kappa+3)(\kappa+2),$$

and

$$\tau_4^{(1)} = \frac{5}{4}\left(\frac{K_{\tau_4^{(1)}} + H_{\tau_4^{(1)}}}{(\kappa+6)(h+6)(\kappa+5)(h+5)L_{\lambda^{(1)}}}\right). \tag{40}$$

Finally, the TL-moment ratio $\tau_5^{(1)}$ of the GLD is

$$K^1_{\tau_5^{(1)}} = \kappa(\kappa-3)(\kappa-2)(\kappa-1),$$

$$K^2_{\tau_5^{(1)}} = (h+7)(h+6)(h+5)(h+4)(h+3)(h+2),$$

$$H^1_{\tau_5^{(1)}} = h(h-3)(h-2)(h-1),$$

$$H^2_{\tau_5^{(1)}} = (\kappa+7)(\kappa+6)(\kappa+5)(\kappa+4)(\kappa+3)(\kappa+2),$$

and

$$\tau_5^{(1)} = \frac{7}{5}\left(\frac{K^1_{\tau_5^{(1)}}K^2_{\tau_5^{(1)}} - H^1_{\tau_5^{(1)}}H^2_{\tau_5^{(1)}}}{(\kappa+7)(h+7)(\kappa+6)(h+6)(\kappa+5)(h+5)L_{\lambda^{(1)}}}\right). \tag{41}$$

Finally, the TL-moments are potentially defined for

$$\kappa > -2 \quad \text{and} \quad h > -2. \tag{42}$$

2.3. GLD parameter estimation

The estimation of κ and h is difficult for both L- and TL-moment characterizations of the GLD. Optimization methods are appealing, but solutions depend on starting values of κ and h. The following objective function, which is analogous to square error, could be minimized to provide estimates of κ and h:

$$\varepsilon = \left(\hat{\tau}_3 - \tilde{\tau}_{3(\kappa,h)}\right)^2 + \left(\hat{\tau}_4 - \tilde{\tau}_{4(\kappa,h)}\right)^2, \tag{43}$$

where ε is the error to be minimized, $\tilde{\tau}_{r(\kappa,h)}$ is the estimated value of the L-moment ratio for a candidate pairing of κ and h of the GLD, and $\hat{\tau}_r$ is the value of the L-moment ratio computed from the sample. The objective function by analogy for the TL-moments is

$$\varepsilon^{(1)} = \left(\hat{\tau}_3^{(1)} - \tilde{\tau}^{(1)}_{3(\kappa,h)}\right)^2 + \left(\hat{\tau}_4^{(1)} - \tilde{\tau}^{(1)}_{4(\kappa,h)}\right)^2. \tag{44}$$

The two objective functions are used for parameter estimation for the example application shown in this paper. The search algorithm must also accommodate the parameter (Eqs. (2), (32), and (42)) and L-moment (Eqs. (15) and (16)) restrictions. Selection of a preferable solution from two or more solutions that are acceptable by the objective function could be based on having $\tilde{\tau}_{5(\kappa,h)}$ of the candidate GLD closest to $\hat{\tau}_5$ of the sample. A similar idea is suggested by Hosking (1986), for a 5-parameter GLD and using the 6th-order L-moment (λ_6) for comparison.

For given data, the suggested L-moment-based GLD-estimation algorithm is

(1) Compute L-moments $\left(\hat{\lambda}_1,\ \hat{\lambda}_2,\ \hat{\tau}_3,\ \hat{\tau}_4, \text{ and } \hat{\tau}_5\right)$ of the data.
(2) Select initial values for κ and h, use numerical methods to minimize Eq. (43), and retain the values $\tilde{\kappa}$ and $\tilde{h}$ and corresponding $\tilde{\tau}_3$ and $\tilde{\tau}_4$ that minimized ε.
(3) Test that $\tilde{\tau}_3$ and $\tilde{\tau}_4$ are valid by Eqs. (15) and (16); if not, repeat step 2 with different initial values for κ and h.
(4) Substitute $\hat{\lambda}_2$, $\tilde{\kappa}$, and $\tilde{h}$ in Eq. (25) and solve for $\tilde{\alpha}$.
(5) Test that $\tilde{\alpha}$, $\tilde{\kappa}$, and $\tilde{h}$ are valid by Eq. (2); if not, repeat step 2 with different initial values of κ and h.
(6) Substitute $\tilde{\kappa}$ and $\tilde{h}$ in Eq. (31) and compute $\tilde{\tau}_5$.
(7) Let $\Delta\tau_5 = |\tilde{\tau}_5 - \hat{\tau}_5|$.
(8) Repeat step 2 with different initial values for κ and h until some minimum $\Delta\tau_5$ is found.
(9) Finally, substitute $\tilde{\alpha}$, $\tilde{\kappa}$, and $\tilde{h}$ in Eq. (24) and solve for $\tilde{\xi}$.

For given data, the TL-moment-based algorithm varies slightly. The suggested TL-algorithm is

(1) Compute TL-moments $\left(\hat{\lambda}_1^{(1)},\ \hat{\lambda}_2^{(1)},\ \hat{\tau}_3^{(1)},\ \hat{\tau}_4^{(1)},\ \text{and } \hat{\tau}_5^{(1)}\right)$ of the data.
(2) Select initial values for κ and h, use numerical methods to minimize Eq. (44), and retain the values $\tilde{\kappa}$ and $\tilde{h}$ that minimized $\varepsilon^{(1)}$.
(3) Using $\tilde{\kappa}$ and $\tilde{h}$ compute $\tilde{\tau}_3$ and $\tilde{\tau}_4$ (not $\tilde{\tau}_3^{(1)}$and $\tilde{\tau}_4^{(1)}$) by Eqs. (29) and (30). (This step is specific to the method of TL-moments.)
(4) Test that $\tilde{\tau}_3$ and $\tilde{\tau}_4$ are valid by Eqs. (15) and (16); if not, repeat step 2 with different initial values for κ and h.
(5) Substitute $\hat{\lambda}_2^{(1)}$, $\tilde{\kappa}$, and $\tilde{h}$ in Eq. (34) and solve for $\tilde{\alpha}$.
(6) Test that $\tilde{\alpha}$, $\tilde{\kappa}$, and $\tilde{h}$ are valid by Eq. (2); if not, repeat step 2 with different initial values of κ and h.
(7) Substitute $\tilde{\kappa}$ and $\tilde{h}$ in Eq. (41) and compute $\tilde{\tau}_5^{(1)}$.
(8) Let $\Delta\tau_5^{(1)} = \left|\tilde{\tau}_5^{(1)} - \hat{\tau}_5^{(1)}\right|$.
(9) Repeat step 2 with different initial values for κ and h until some minimum $\Delta\tau_5^{(1)}$ is found.
(10) Finally, substitute $\tilde{\alpha}$, $\tilde{\kappa}$, and $\tilde{h}$ in Eq. (33) and solve for $\tilde{\xi}$.

For both algorithms, steps 7 and 8 (L-moments) or steps 8 and 9 (TL-moments) constitute the component that accommodates multiple acceptable GLD solutions. The most favorable solution, in the event of two or more acceptable solutions, should have the smallest $\Delta\tau_5$ or $\Delta\tau_5^{(1)}$. Finally, choices of initial κ and h values for execution of each step 2 is not straightforward. Potentially useful starting values for κ and h are provided by Eq. (3).

3. Example application and simulation of the GLD

3.1. Example application

An example application of the GLD using the method of L- and TL-moments is demonstrated using annual peak streamflow data. Such data often are used for flood-plain-management purposes and emphasis by many hydrologists is in the far-right tail ($F \gtrsim 0.9$) of fitted probability distributions. This emphasis is made because streamflow values in the far-right tail represent floods, which can cause loss of life and property damage. Thus, distribution fit in the far-right tail is paramount for reliable risk representation.

The annual peak streamflows for U.S. Geological Survey (USGS) streamflow-gaging station 08150000 Llano River at Junction, Texas, are listed in Table 1. The sample comprises 88 years of record for water years 1916–2005.

Table 1
Annual peak streamflows in cubic meters per second for U.S. Geological Survey streamflow-gaging station 08150000 Llano River at Junction, Texas, water years 1916–2005

Date	Value	Date	Value	Date	Value
1916-05-22	314.3	1944-10-04	92.60	1973-10-13	2769
1917-05-11	5.44	1946-09-26	36.53	1974-10-31	118.4
1918-04-14	421.9	1947-05-18	128.6	1976-09-01	727.7
1919-09-24	1011	1948-06-24	3455	1977-05-11	1178
1920-05-14	387.9	1949-02-23	342.6	1978-08-02	2172
1921-03-19	24.92	1949-10-24	16.08	1979-08-11	106.8
1922-04-03	455.9	1951-06-10	150.6	1980-09-08	3936
1923-04-25	1710	1952-05-18	71.92	1981-06-17	563.5
1923-10-29	2421	1953-05-11	142.4	1981-10-13	3653
1925-05-29	2178	1954-06-29	87.50	1982-11-26	4.62
1925-10-16	158.6	1955-09-24	1042	1983-11-06	6.03
1927-07-23	17.19	1955-10-03	28.88	1984-12-31	3002
1927-10-01	906.1	1957-05-27	1147	1985-10-19	1036
1929-05-28	59.47	1957-10-14	1801	1987-06-03	360.0
1930-04-24	78.44	1959-06-26	495.5	1988-07-11	1538
1930-10-06	2540	1960-08-15	1005	1989-05-17	9.17
1932-09-01	3002	1961-06-18	1577	1990-09-17	1424
1933-05-25	447.4	1961-10-11	10.90	1991-09-16	680.6
1934-04-04	133.7	1963-05-06	46.16	1991-12-20	152.1
1935-06-14	9033	1964-09-24	747.6	1992-12-13	8.18
1936-09-16	4474	1965-05-18	231.6	1994-05-13	162.0
1936-10-29	56.63	1966-09-09	390.8	1995-09-22	62.30
1938-07-22	3879	1967-07-20	18.63	1998-08-23	1504
1939-07-13	2107	1968-01-18	470.1	1999-06-22	11.58
1939-10-10	169.9	1969-08-27	45.59	2000-06-10	17.41
1941-04-27	205.3	1970-05-15	611.6	2000-11-03	4475
1942-08-22	1218	1971-08-11	560.7	2001-11-15	1407
1943-06-05	283.2	1971-10-20	247.8	2002-10-23	122.1
1944-05-01	244.6	1973-07-31	186.0	2003-10-12	317.1
				2004-11-17	3398

Table 2
Sample L-moments of example data

L-moment	Value
$\hat{\lambda}_1$	1001
$\hat{\lambda}_2$	663.1
$\hat{\tau}_3$	0.4863
$\hat{\tau}_4$	0.2152
$\hat{\tau}_5$	0.1100

(Data for the 1996 and 1997 water years are not available.) The USGS defines a water year as the period October 1 through September 30. The sample L-moments of the data are listed in Table 2.

The relation between the annual peak streamflows and their respective nonexceedance probabilities is shown in Fig. 1. The nonexceedance probabilities are estimated by the plotting-position formula $F_i = i/(n+1)$ for the ith largest value of sample of size n. General convention is to plot the logarithms of streamflow for visualization of peak-streamflow quantile functions; as a result, differences in the left tail are exaggerated relative to those in the right tail. To clarify, the logarithms of streamflow for sample L- or TL-moment computation are not used in this paper.

The data, in general, have considerable heavy-tailed characteristics and contain an exceptionally large peak (9033 cubic meters per second for water year 1935). The 1935 peak streamflow is believed to be the largest for at least the

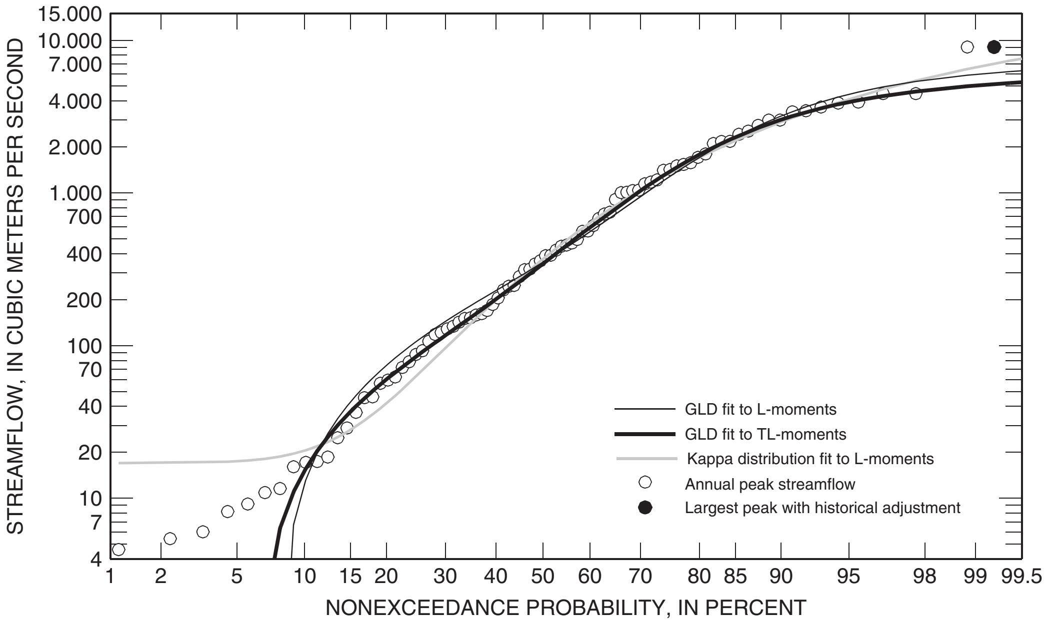

Fig. 1. Sample distribution of annual peak streamflows and three fitted distributions.

131 years (1875–2005)—so an alternative plotting position for this peak ($F = 131/132 = 0.9924$) is shown in Fig. 1 as a filled circle.

The R-function (R Development Core Team, 2005, function "optim") was used to estimate the parameters of the GLD using the L-moments (GLD$_\lambda$) by minimizing Eq. (43) to $\varepsilon < 4 \times 10^{-6}$. The GLD$_\lambda$(4379.1, 4422.3, 6.7809, 0.12045) is shown in Fig. 1. It is evident that the GLD$_\lambda$ generally fits the data with some room for interpretation far into the left and right tails. Distribution fit in the left tail is not considered further in this paper.

Because the GLD has four parameters, comparison of the fit of the 4-parameter kappa distribution using the method of L-moments is informative. The quantile function of the kappa distribution [kappa$_m(\xi, \alpha, \kappa, h)$] is

$$X(F) = \xi + \frac{\alpha}{\kappa}\left(1 - \left(\frac{1 - F^h}{h}\right)^{\kappa}\right), \tag{45}$$

where all the parameters have similar interpretations as those of the GLD and m represents the method of parameter estimation. A restriction on τ_4 for the kappa distribution (Hosking and Wallis, 1997, p. 197 and 203) is that

$$\tau_4 \leqslant \frac{1 + 5\tau_3^2}{6}. \tag{46}$$

Analysis of both distributions shows that the GLD can have larger τ_4 values for a given τ_3 than the kappa. For example, the L-moment ratio pair $(\tau_3, \tau_4) = (0.45, 0.60)$ produces at least one valid GLD solution $[(\kappa, h) = (-0.510, -0.743)]$. The kappa distribution does not exist for these τ_3 and τ_4 values. On this basis, the GLD might have a larger range of τ_3 and τ_4 than the kappa distribution.

Returning to the example data, kappa$_\lambda(-2600.4, 2702.5, 0.13675, 2.8262)$ was estimated by the method of L-moments (Hosking, 1996), and the distribution is shown in Fig. 1. In addition to graphical comparison, the fit of GLD$_\lambda$ and kappa$_\lambda$ can be assessed using τ_5. The data have $\hat{\tau}_5 = 0.110$, the GLD has GLD$_\lambda$-$\tilde{\tau}_5 = 0.080$, and the kappa distribution has a kappa$_\lambda$-$\tilde{\tau}_5 = 0.116$. Because kappa$_\lambda$-$\tilde{\tau}_5$ is closer to $\hat{\tau}_5$ than GLD$_\lambda$-$\tilde{\tau}_5$, a tentative conclusion is that the kappa$_\lambda$ generally provides a more favorable fit. However, for purposes of illustration, additional discourse is useful.

In general, large peak streamflows often have much greater uncertainty than small peaks; the reasons are related to discipline-specific technical details of high-magnitude streamflow measurement and estimation. Compared to the other data values, the 1935 peak appears to be a high outlier. It is important to stress that the 1935 peak was not

Table 3
Sample TL-moments of example data

TL-moment	Value
$\hat{\lambda}_1^{(1)}$	678.1
$\hat{\lambda}_2^{(1)}$	312.2
$\hat{\tau}_3^{(1)}$	0.3805
$\hat{\tau}_4^{(1)}$	0.1210
$\hat{\tau}_5^{(1)}$	0.03085

directly measured (Aragon Long et al., 2005, pp. 172–173). However, the 1935 peak is believed to be the largest in the sample and has a larger nonexceedance probability than indicated by the sample size alone—an exceptionally large flood did occur in 1935. Because of its magnitude, the 1935 peak is an influential value. To clarify, the 1935 peak is believed to have a larger nonexceedance probability than indicated by the sample size. It is documented in anecdotal historical (nonstreamflow) records that there is a period preceding the systematic record (1875–1915) in which no higher (measured by elevation of water surface not actual volumetric streamflow rate) event occurred.

Although the L-moments are robust and have considerable resistance to the presence of exceptional values, the fit of a distribution using the entire sample remains affected. Specifically, the GLD_λ and kappa_λ exceed many of the larger data values. If the 1935 peak is erroneous (perhaps considerably overestimated), application of the TL-moments might be preferable.

The sample TL-moments of the data are listed in Table 3. The GLD was fit to the TL-moments by minimizing Eq. (44) to $\varepsilon^{(1)} < 10^{-8}$. The fitted distribution is $\mathrm{GLD}_{\lambda^{(1)}}(3842.2, 3886.7, 5.0782, 0.097883)$ and also is shown in Fig. 1. By inspection of the figure, it is evident that the $\mathrm{GLD}_{\lambda^{(1)}}$ shows further flattening far into the right tail of the data. The author judges that the $\mathrm{GLD}_{\lambda^{(1)}}$ has a more favorable fit to the seventh through second-largest peaks than either GLD_λ or kappa_λ.

The TL-moments shown here provide an objective framework that de-emphasizes the smallest and largest data values by theoretically adjusting weights on the sample data rather than simply throwing out the smallest and largest values. The example demonstrates the utility using both L- and TL-moments of the GLD.

3.2. Simulation

A small simulation study of L- and TL-moment properties for quantile estimation for a "flood-like" GLD parent with high-outlier contamination is made. The GLD parent was derived from the L-moments of the data listed in Table 1 with the exclusion of the 1935 peak and is $\mathrm{GLD}_\lambda(3692.8, 3703.1, 4.7172, 0.081076)$, which is abbreviated as GLD^P. The study concerns the $F = 0.98$ quantile of the distribution: $\mathrm{GLD}^P_{(F=0.98)} = 4663$ cubic meters per second. The error $\left(\varepsilon^i_m\right)$ for the ith simulation is computed as

$$\varepsilon^i_m = \mathrm{GLD}^{S_i}_{m(F=0.98)} - 4663, \tag{47}$$

where S reflects that the GLD was fit to a simulated sample from GLD^P and m continues to represent parameter estimation by L- or TL-moments.

The metrics of focus are the $\lambda_1\left(\varepsilon^i_m\right)$ (mean) and $\lambda_2\left(\varepsilon^i_m\right)$ values of 500 simulations $(1 \leqslant i \leqslant 500)$ for each 98th-percent quantile estimate: $\mathrm{GLD}^{S_i}_{\lambda(F=0.98)}$ and $\mathrm{GLD}^{S_i}_{\lambda^{(1)}(F=0.98)}$. The $\lambda_1\left(\varepsilon^i_m\right)$ represents bias and $\lambda_2\left(\varepsilon^i_m\right)$ represents sampling variability. For each simulated sample of size $n-1$, a data point of 9033 cubic meters per second was added to each sample to represent high-outlier contamination. The results of the simulation are listed in Table 4.

The $\lambda_1\left(\varepsilon^i_m\right)$ and $\lambda_2\left(\varepsilon^i_m\right)$ statistics in Table 4 (columns 2 and 3) show that considerable positive bias (overestimation) of $\mathrm{GLD}^P_{(F=0.98)}$ occurs for both $\mathrm{GLD}^S_{\lambda(F=0.98)}$ and $\mathrm{GLD}^S_{\lambda^{(1)}(F=0.98)}$. For example, the bias for a sample of size 20 for the 98th-percent quantile using L-moments is about 3325 cubic meters per second and the bias using TL-moments is about 2310 cubic meters per second. In general, inspection of the table shows that the TL-moments produce less biased estimates in the presence of contamination than the L-moments. The $\lambda_2\left(\varepsilon^i_m\right)$ values (columns 4 and 5) show that the

Table 4
Simulation study of 98th-percent quantile of $\mathrm{GLD}_\lambda(3692.8,\ 3703.1,\ 4.7172,\ 0.081076)$ using 500 simulations for each sample size with contamination by a high-outlier of 9033

Sample size	$\lambda_1\left(\varepsilon^i_\lambda\right)$	$\lambda_1\left(\varepsilon^i_{\lambda^{(1)}}\right)$	$\lambda_2\left(\varepsilon^i_\lambda\right)$	$\lambda_2\left(\varepsilon^i_{\lambda^{(1)}}\right)$
10	5991	6861	1224	5391
20	3325	2310	601.6	1440
30	2415	1121	472.8	812.6
40	1984	814.5	383.3	623.1
60	1490	553.0	286.2	432.6
80	1138	314.8	229.7	345.7
100	957.1	275.0	203.0	312.2
150	626.0	152.4	156.7	255.7
200	484.1	131.1	134.3	201.3

L-moments consistently have smaller sampling variability—the $\lambda_2\left(\varepsilon^i_\lambda\right)$ are considerably less than the $\lambda_2\left(\varepsilon^i_{\lambda^{(1)}}\right)$ values. The simulations show, with respect to estimation of the 98th-percent quantile, that TL-moments are less biased (more robost) in the presence of high-outlier contamination. However, the robustness comes at the expense of considerably more sampling variability.

4. Conclusions

The first five L- and TL-moments of the GLD are derived. Therefore, distributional analysis using the theory of L-moments and the GLD is extended. Although analytical solutions exist for the L- and TL-moments in terms of the parameters, there are no simple expressions for the GLD parameters in terms of the $r \geqslant 3$ order L-moments. Algorithms for GLD parameter estimation are suggested; however, evaluation of formal algorithmic procedures for parameter estimation is left for further study. The GLD and the kappa distributions each have four parameters; however, the GLD has a larger τ_3 and τ_4 space than the kappa distribution. Precise specification of the τ_3 and τ_4 space of the GLD also is left for further study. By brief example, the GLD is shown to complement the kappa distribution and is useful for analysis of heavy-tailed data. A small simulation study of the 98th percentile (far-right tail) is conducted for a heavy-tail GLD with high-outlier contamination. The simulations show, with respect to estimation of the 98th-percent quantile, that TL-moments are less biased (more robost) in the presence of high-outlier contamination. However, the robustness comes at the expense of considerably more sampling variability.

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