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Confidence intervals for quantiles using generalized lambda distributions

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ABSTRACT

Generalized lambda distributions (GLD) can be used to fit a wide range of continuous data. As such, they can be very useful in estimating confidence intervals for quantiles of continuous data. This article proposes two simple methods (Normal–GLD approximation and the analytical-maximum likelihood GLD approach) to find confidence intervals for quantiles. These methods are used on a range of unimodal and bimodal data and on simulated data from ten well-known statistical distributions (Normal, Student's T, Exponential, Gamma, Log Normal, Weibull, Uniform, Beta, F and Chi-square) with sample sizes $n = 10, 25, 50, 100$ for five different quantiles $q = 5\%, 25\%, 50\%, 75\%, 95\%$. In general, the analytical-maximum likelihood GLD approach works better with shorter confidence intervals and has closer coverage probability to the nominal level as long as the GLD models the data with sufficient accuracy. This technique can also be used to find confidence interval for the mode of a continuous data as well as comparing two data sets in terms of quantiles.

1. Introduction

The derivation of confidence interval for quantiles has wide applications especially in finance and hydrology. Most of the recent, general methods for computing confidence intervals for quantiles use nonparametric techniques such as: bootstrapping (Ho and Lee, 2005), fractional statistics (Hutson, 1999) and linear programming to find the shortest "exact" confidence interval based on data (Zieliñski and Zieliñski, 2005). In contrast, little has been done using parametric techniques since it is necessary to estimate the empirical density of the data sufficiently well before these techniques can be applied and most statistical distributions have limited range of shapes which can limit their use in practice. However, some distributions such as the Weibull distribution have a range of different shapes and confidence intervals for quantiles for this distribution have been derived (Heo et al., 2001). Perhaps an even more flexible distribution is the GLD (Generalised Lambda Distribution) and it is possible to fit a comprehensive range of empirical data using this distribution. Recent development in statistics has enabled fitting GLD to data using a range of methods such as maximum likelihood estimation (Su, 2007a,b) and starship method (King and MacGillivray, 1999) or L moments (Asquith, 2007; Karvanen and Nuutinen, 2008). This opens the prospect of a generic, parametric method of constructing confidence interval for quantiles.

2. Method

This section gives a brief overview of the GLD and details the key theory results used in deriving the confidence interval for quantiles.

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2.1. GLD

The Ramberg and Schmeiser (1974) (RS) GLD is an extension of Tukey's lambda distribution (Hastings et al., 1947). It is defined by its inverse distribution function $F^{-1}(u) = \lambda_1 + \frac{u^{\lambda_3} - (1-u)^{\lambda_4}}{\lambda_2}$, $0 \le u \le 1$, $\lambda_2 \neq 0$ and $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are respectively the location, inverse scale and shape parameters of generalized lambda distribution GLD($\lambda_1, \lambda_2, \lambda_3, \lambda_4$). In particular, Karian et al. (1996) noted that GLD is defined if and only if $\frac{\lambda_2}{\lambda_3 u^{\lambda_3 - 1} + \lambda_4 (1-u)^{\lambda_4 - 1}} \ge 0$ for $u \in [0, 1]$.

Another distribution known as FMKL GLD also exists, due to the work of Freimer et al. (1988). The FMKL GLD can be written as $F^{-1}(u) = \lambda_1 + \frac{\frac{u^{\lambda_3}-1}{\lambda_3} - \frac{(1-u)^{\lambda_4}-1}{\lambda_4}}{\lambda_2}$, $0 \le u \le 1$ and $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are consistent with the interpretations in RS GLD, namely λ_1, λ_2 are the location and inverse scale parameters and λ_3, λ_4 are the shape parameters.

The fundamental motivation for the development of FMKL GLD is that the distribution is proper over all λ_3 and λ_4 (Freimer et al., 1988). This adds convenience to users who wish to program this function as there are fewer restrictions on the values of λ_3 and λ_4. The only restriction on FMKL GLD is $\lambda_2 > 0$.

While it is possible to fit both types of GLD using a variety of methods (Asquith, 2007; Karian and Dudewicz, 2000; Karvanen and Nuutinen, 2008; King and MacGillivray, 1999; Lakhany and Massuer, 2000; Ozturk and Dale, 1985; Su, 2005, 2007b), the maximum likelihood method (Su, 2007b) will be used in this article.

2.2. Theory—confidence interval for quantiles

Let X be a continuous random variable with pth quantile $q(p)$ defined by (1) with F_X being the density function of X.

$$q(p) = F_X^{-1}(p). \tag{1}$$

If $X_1, X_2, \ldots, X_n$ are the ordered statistics, then an estimate for $q_n(p)$ is $X_{\lceil np \rceil}$ where $\lceil x \rceil$ is the smallest integer greater than or equal to x. As long as $F_X^{-1}(p)$ is differentiable at $q(p)$ with $F_X'(q(p)) = f_X(q(p)) > 0$, then the central limit theorem specifies that, for large n: (Bahadur, 1966)

$$\hat{q}_n(p) \approx N\left(q(p), \frac{\frac{p(1-p)}{f_X^2(q(p))}}{n}\right). \tag{2}$$

From (2), it is possible to construct a $100(1-\alpha)\%$ confidence interval by estimating the sample standard deviation s to be $\frac{\sqrt{p(1-p)}}{\hat{f}_X(\hat{q}_n(p))}$ with $\hat{q}_n(p)$ being the point estimate derived from fitted distribution $\hat{F}_X^{-1}(p)$. Using (2), an approximate $100(1-\alpha)\%$ confidence interval is constructed in (3), with z being the usual statistics such that $P(-z \le N(0, 1) \le z) = \alpha$.

$$\hat{q}_n(p) \pm \frac{zs}{\sqrt{n}}. \tag{3}$$

This procedure is the Normal–GLD approximation method. Using this method, $\hat{q}_n(p)$ is obtained by fitted GLD and substituted in (3). It is well known that the normal approximation method, in general, only works well for large n due to the reliance on central limit theorem and it has poor coverage probability when estimating extreme quantile such as at 0.01 or 0.95.

If it is possible to find or estimate the "true" density of the empirical data, then it is not necessary to use normal approximation. Cramer (1963) showed that generically, $P\left(X \le X_{\lceil np \rceil}\right)$ or $g(x)$ is:

$$\begin{aligned} g(x) &= \frac{\Gamma(n+1)}{\Gamma(m+1)\Gamma(m+1)\Gamma(n-m)} \left(F_X(x)\right)^m \left(1 - F_X(x)\right)^{n-m-1} f_X(x) \\ \Gamma(y) &= \int_0^\infty u^{y-1} \mathrm{e}^{-u} \mathrm{d}u \\ m &= \lceil np \rceil . \end{aligned} \tag{4}$$

Consequently, to find the confidence interval analytically, all that required is to solve the following equations:

$$\int_0^{\text{UpperLimit}} g(x)\,\mathrm{d}x = 1 - \frac{\alpha}{2} \tag{5}$$

$$\int_0^{\text{LowerLimit}} g(x)\,\mathrm{d}x = \frac{\alpha}{2}. \tag{6}$$

In (5) and (6), $\int_0^{x_0} g(x)\,\mathrm{d}x = B(m+1, n-m)\,|_0^{F_X(x_0)}$ where B is Euler's incomplete beta function normalized by the complete Beta function which is readily available in statistical or mathematical software such as R or Matlab. This procedure is known as the analytical-maximum likelihood GLD approach where GLD is fitted to the empirical data using the maximum likelihood estimation and then substituted to (5) and (6) to find the appropriate upper and lower confidence limits for the quantile estimate.

2.3. Assumptions and limitations

To ensure an accurate estimation of the quantile estimates and its corresponding confidence intervals, the GLD has to approximate the data with sufficient accuracy. The accuracy of the distribution fit can often be gauged by using QQ-plots and using the resample goodness of fit tests described in Su (2007b). In general, the GLDs fit the empirical data quite well but there could be situations where the extreme tails (e.g. quantile at 0.9999 or 0.0001) of the empirical data are not sufficiently well approximated by the GLD. In those cases, the methods described in this article may not provide confidence interval with a coverage probability that is close to the nominal level. The analysis of these extreme tails is a specialized topic and falls outside the scope of this article.

3. Results

This section demonstrates the application of the Normal–GLD approximation and the analytical-maximum likelihood GLD methods on unimodal and bimodal empirical data as well as for ten well-known statistical distributions (Normal, Student's T, Exponential, Gamma, Log Normal, Weibull, Uniform, Beta, F and Chi-square) with sample sizes $n = 10, 25, 50, 100$ for five different quantiles $q = 5\%, 25\%, 50\%, 75\%, 95\%$. Comparison of these two techniques to classical bootstrapping methods is also given using various well-known statistical distributions.

The empirical data demonstration is designed to apply these methods where the true probability density function of the empirical data is unknown. The demonstration on well-known statistical distributions is to show the versatility of these methods in providing a suitable confidence interval even though the GLD may only approximate the true probability density function of the underlying theoretical distribution. The sample quantile used to calculate the coverage probability is the one recommended by Hyndman and Fan (1996), where resulting quantile estimates are approximately median-unbiased for any distribution. This sample quantile computation is available in R.

3.1. Empirical data

3.1.1. Single distribution fit

The data used in this example consists of 100 measurements of the speed of light in air. This classic experiment was carried out by Michelson in 1879 (Dorsey, 1944). Either RS or FMKL GLD can be used since both methods give very similar fits to the data set as can be seen in Fig. 1.

To calculate the 95% confidence interval with $n = 100$, for the 99th quantile, using FMKL GLD under the analytical method gives [299.9936, 300.1412] compared to [299.975, 300.1014] under the Normal–GLD approximation method. To examine the coverage probability, a repeated sample of 100 random data from the fitted distribution[1] is used to calculate how often the simulated 99th quantile falls within the calculated confidence interval over 10,000 times. This happens approximately 95.86% and 95.17% of the time for analytical and approximation method respectively, which is quite close to the nominal level of 95%.

3.1.2. Mixture distribution fit

The data used in this example is the 272 waiting times between eruptions for the Old Faithful geyser in Yellowstone National Park, available in R. Mixtures of GLD can be fitted using either the EM algorithm or the partition maximum likelihood procedure described in Su (2007a). This example fitted RS and FMKL mixture GLDs using the EM algorithm and the results are shown in Fig. 2.

As was done in the unimodal data, to calculate the 95% confidence interval with $n = 100$, for the 80th quantile, the analytical method gives [4.3965, 4.6677] and the Normal–GLD approximation gives [4.3974, 4.6695]. The coverage probabilities over 1000 repeated sampling are 0.95 and 0.954 for analytical and approximation method respectively, which is again quite close to the nominal level.

3.2. GLD estimation and bootstrapping technique

The success of using GLD to find confidence interval depends on the accuracy of GLD estimation to the true distribution. As long as the approximation is sufficiently accurate, the single evaluated confidence interval should give coverage probabilities (based on simulations from true underlying distribution) that are close to the specified nominal levels. This is different to bootstrap, where it is expected, for example, at 95% confidence, 95 out of 100 evaluated confidence intervals would capture the true underlying quantile. For this reason, comparison between GLD estimation and bootstrap method, while of interest, needs to be interpreted under different light. The following examples in Section 3.3 and Section 3.4 evaluate numerous confidence intervals under both methods and report the probability that the true underlying quantile falls inside these confidence intervals. In Section 3.3, an example of estimating quantiles for a heavy tailed distribution G-H (0.2,0.2) is given.

[1] Or the empirical data could be used by sampling with replacement, this gives coverage probability of 0.939 and 0.998 for the analytical and approximation method respectively. This needs to be interpreted with caution unless the empirical data is very large and the fitted GLD approximates the tail sufficiently well. There are only 100 observations in this data set.

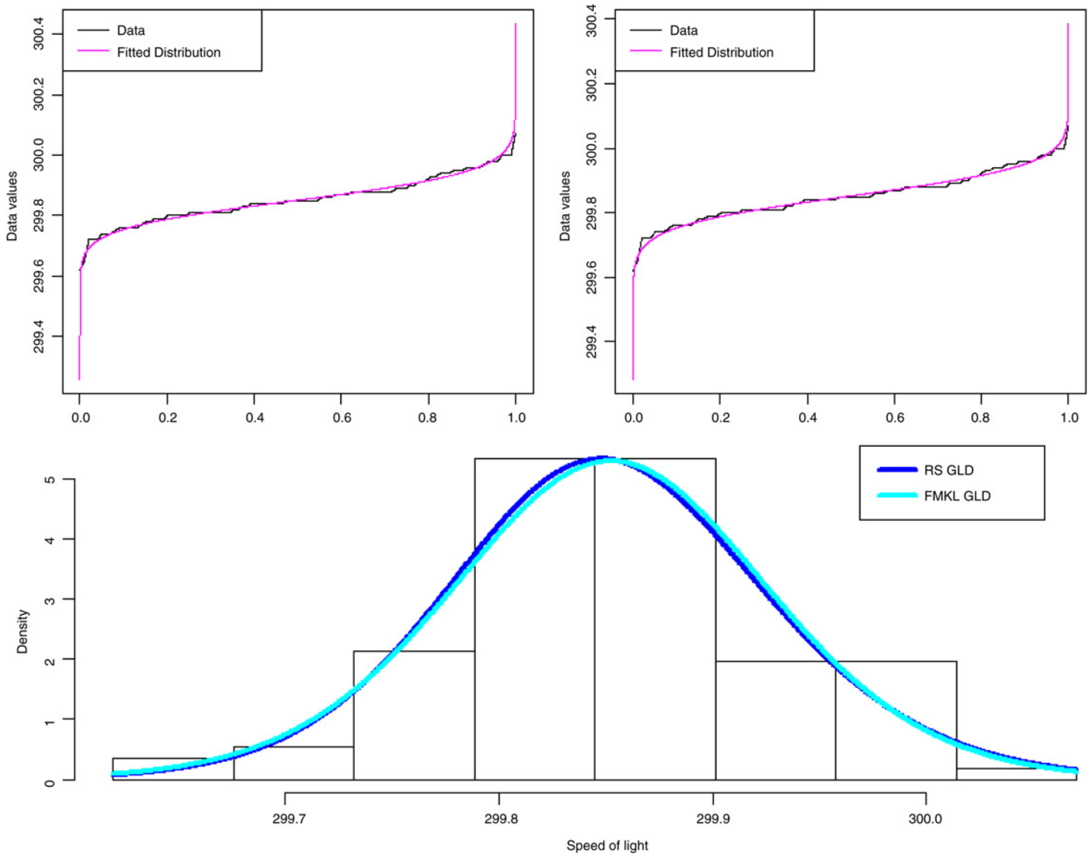

Fig. 1. GLD fits on speed of light data using maximum likelihood estimation.

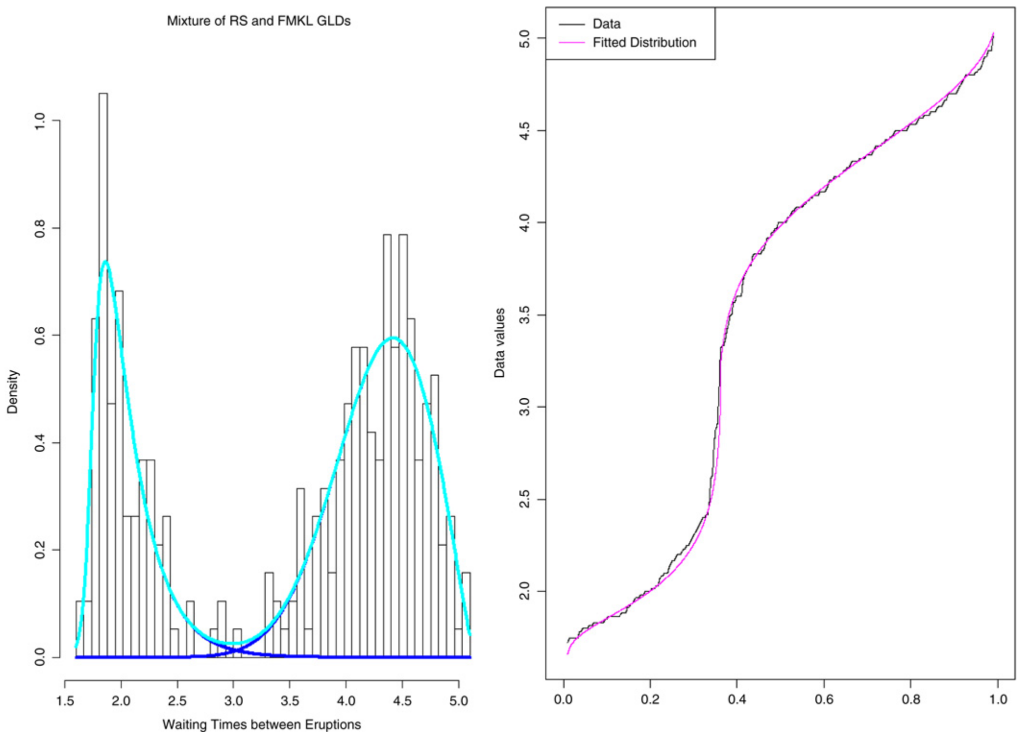

Fig. 2. GLD fits on waiting time between eruptions for Yellowstone Park data using EM algorithm.

Table 1
Coverage out of 100 times for various quantiles under G-H (0.2,0.2) distribution.

Quantile	GLD-Analytical	Normal–GLD approx	Bootstrap	Bootstrap-BCA
0.95	97	95	89	93
0.75	99	99	96	95
0.5	99	99	97	94
0.25	99	99	90	97
0.05	97	98	92	93

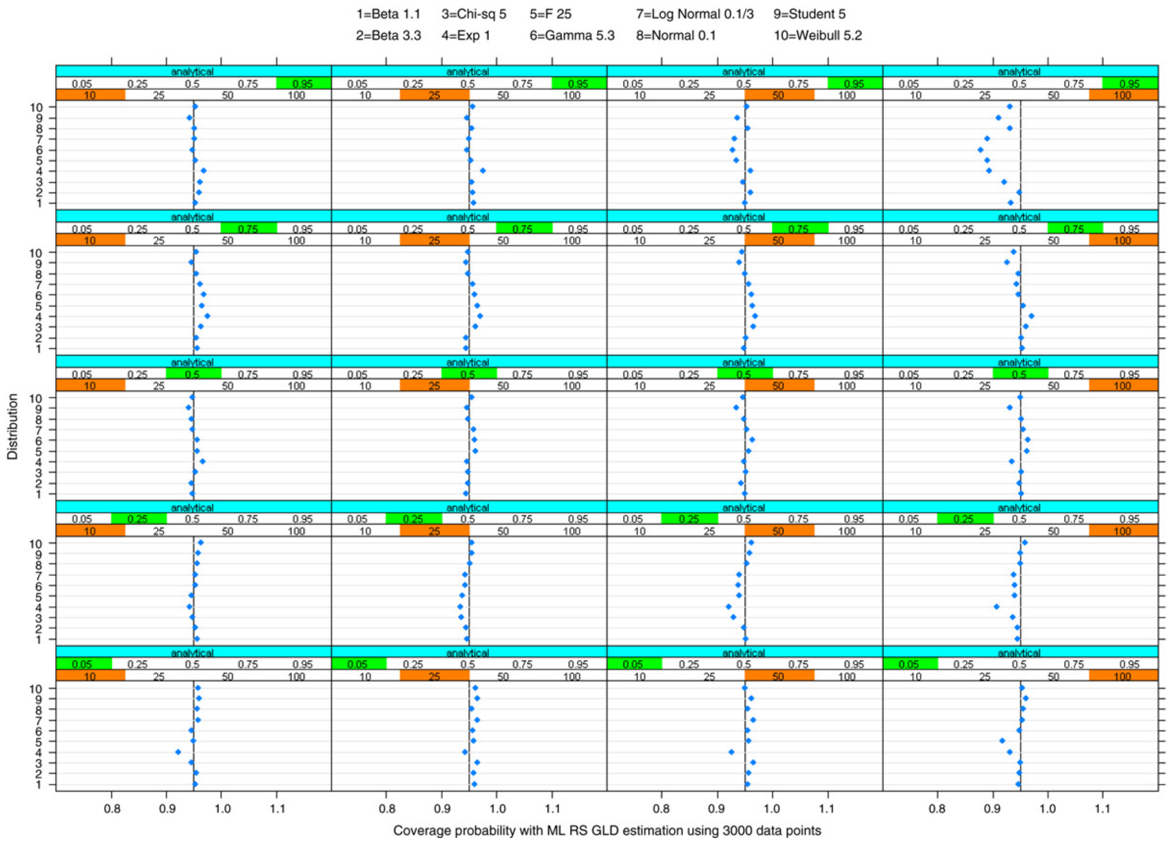

Fig. 3. The coverage probabilities of RS GLD estimation for ten well-known statistical distributions using the analytical method.

In Section 3.4, a case study of using only 100 random observations from various statistical distributions was used to examine the performance of these methods under less than ideal conditions.

3.3. Modeling a heavy tail distribution—G-H distribution (0.2,0.2)

In this example, 1000 random samples were taken from G-H(0.2,0.2) distribution (Hoaglin, 1985), 95% confidence intervals for quantiles at 0.05, 0.25, 0.5, 0.75 and 0.95 were then evaluated using FMKL GLD and the bootstrap (basic and adjusted bootstrap percentile (BCA)) method (Davison and Hinkley, 1997). The entire process was repeated 100 times and the number of times the true underlying quantile falls within the evaluated confidence intervals is reported in Table 1. The coverage statistics for GLD estimation illustrates the degree in which GLD approximates the data, since if GLD provides a reasonable fit to the G-H distribution, the estimated quantiles under GLD would be very close to the true underlying quantile of the G-H distribution. This means the confidence interval under GLD estimation methods would tend to cover the true underling quantile of the G-H distribution as indicated in Table 1. More than 95 times out of 100, the parametric methods cover the true underlying quantile. Nonparametric methods are reasonable, giving coverage ranging from 89% to 97% but tend to give lower coverage compared to parametric methods. For heavy tailed distributions such as G-H, it is important that the number of observations is sufficient for the GLD to approximate the true underlying distribution. For example, if only 100 observations were generated from G-H, there would be too little data at the tails to ensure an accurate confidence interval. It is however, of interest to examine the use of GLD estimation in less than ideal conditions, where the data generated should give a fairly good picture of the true underlying data, but larger variability and subsequently larger

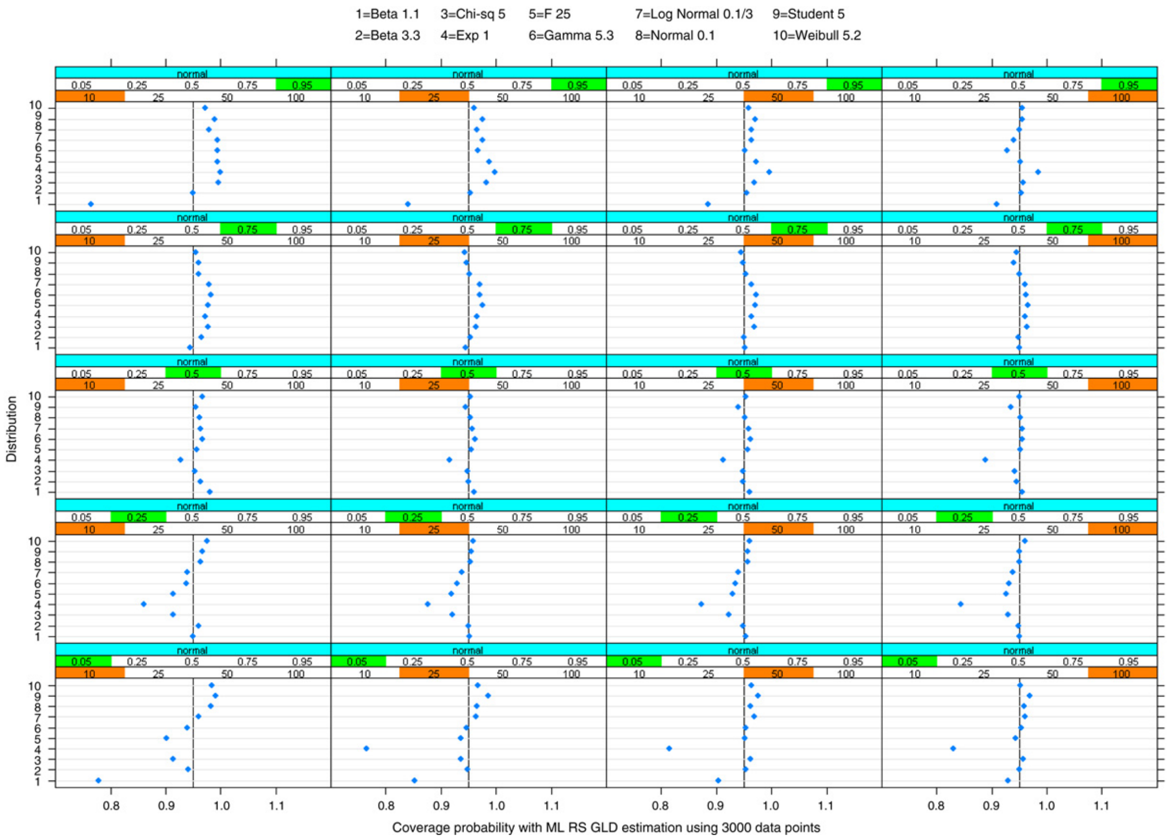

Fig. 4. The coverage probabilities of RS GLD estimation for ten well-known statistical distributions using the normal approximation method.

Table 2
Coverage probabilities out of 5000 simulations for various quantiles using 100 samples from various distributions.

Distribution-quantile	GLD-Analytical	Normal–GLD	Bootstrap	Bootstrap-BCA
Normal(0,1)-0.05	0.92	0.9482	0.9012	0.92
Normal(0,1)-0.5	0.973	0.974	0.921	0.9474
Normal(0,1)-0.95	0.9602	0.9548	0.911	0.9402
Gamma(5,3)-0.05	0.9382	0.9574	0.9082	0.9278
Gamma(5,3)-0.5	0.9754	0.9744	0.9226	0.9496
Gamma(5,3)-0.95	0.9588	0.9452	0.9032	0.9364
Student(5)-0.05	0.9276	0.9458	0.9152	0.9176
Student(5)-0.5	0.968	0.9684	0.9264	0.9418
Student(5)-0.95	0.9548	0.94	0.9066	0.9326
Weibull(5,2)-0.05	0.9308	0.9562	0.9098	0.9246
Weibull(5,2)-0.5	0.9774	0.975	0.9292	0.9528
Weibull(5,2)-0.95	0.952	0.9546	0.9006	0.9328

errors in GLD approximation to the true distribution could lead to less accurate confidence intervals. These examples are indicated in Section 3.4.

3.4. Comparing GLD estimation and bootstrap methods for various statistical distributions

In this example, 100 random samples were taken from Normal(0,1), T(5), Gamma(5,3), Weibull(5,2) distributions, 95% confidence intervals for quantiles at 0.05, 0.5, and 0.95 were then evaluated using FMKL GLD and the bootstrap (basic and adjusted bootstrap percentile (BCA)) method. The entire process was repeated 5000 times and the proportion for which the true underlying quantile falls within the evaluated confidence intervals is reported. This example has the primary focus of examining behavior of both parametric and nonparametric methods where the number of observations sampled is relatively low, hence the coverage probabilities of confidence interval could be less than the nominal level.

Table 2 shows that the both nonparametric methods have less coverage probabilities and are consistently below the nominal level except in Weibull(5,2) at quantile = 0.5. The basic bootstrap is not recommended while the use of bootstrap-BCA yields better results. Parametric method using Normal–GLD approximation method is surprisingly good in this case

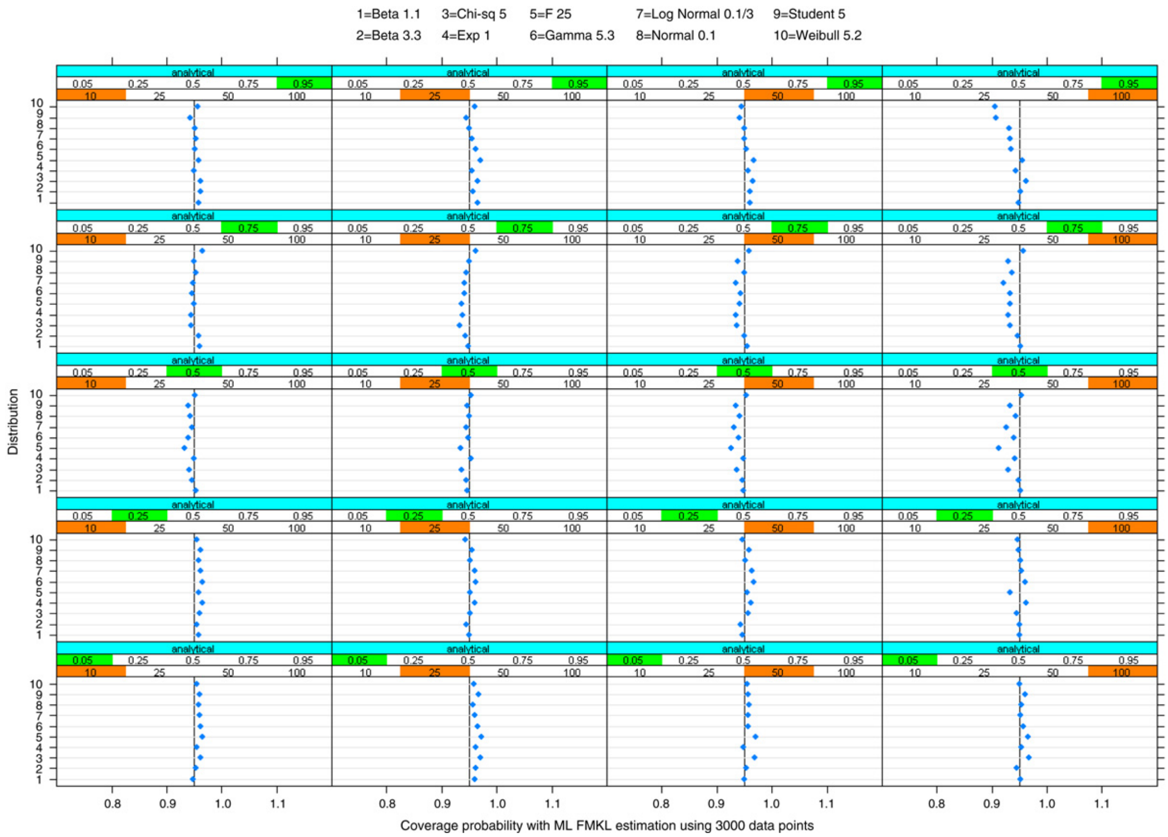

Fig. 5. The coverage probabilities of FMKL GLD estimation for ten well-known statistical distributions using the analytical method.

(with coverage probabilities close to exceed the nominal level of 95%), where as the GLD-Analytical method appears to be slightly weaker. One explanation for the poorer performance of the GLD-Analytical method is that at extreme tails, it is sometimes possible to reach the minimum and maximum support of fitted GLD distribution so the confidence interval may be constrained. While the fitted GLDs do cover the minimum and maximum observations of the sampled data (as they should all do following the method of Su (2007a,b)), a more narrow range of support than that of the true distribution could lead to a more narrow and less accurate confidence intervals. This problem does not exist in Normal–GLD method because the confidence interval evaluation is based on normal approximation, so the boundaries of the confidence interval could exceed the theoretical fitted GLD in these cases.

The comparison of the analytical method over Normal–GLD-approximation method cannot be gauged by a few examples but needs to be more rigorously examined across a range of different distributions for different n and over a number of different quantiles; this is discussed in Section 3.5.

3.5. Approximating well-known statistical distributions

In this example, 3000 data points were simulated from the following distributions: Beta(1,1), Beta (3,3), Chi-squared(5), Exponential(1), F(25), Gamma(5,3), Log Normal(0,1/3), Normal(0,1), Student(5), Weibull(5,2). These data points were then modeled by RS and FMKL GLD using the maximum likelihood estimation and 95% confidence intervals for quantiles ($q = 5\%, 25\%, 50\%, 75\%, 95\%$) were computed for $n = 10, 25, 50, 100$ using both the normal–GLD approximation and the analytical method. The coverage probability is assessed by simulating n samples from a given statistical distribution such as Normal(0,1), then calculate a given quantile at say, 5% and repeat this process 10,000 times and calculate the probability these sample quantiles fall within the calculated 95% confidence interval. These results are shown in Figs. 3–6 and it is clear that the coverage probabilities under the analytical method is consistent and quite close to the nominal level of 95% where as the Normal–GLD approximation method, while convincing in many cases, can perform badly for some distributions at extreme quantiles such as Exponential(1) for $n = 10$ and $q = 0.05$. Additionally, approximately 63% of the time, the length of confidence interval using analytical solution is shorter than the length of confidence interval under Normal–GLD approximation method. These results suggest that the analytical method is preferred over Normal–GLD approximation method in general. However it is not the case that the analytical method is always better than the normal approximation method, in the case of Student(5) distribution with $n = 100$ and $q = 0.95$, the Normal–GLD approximation

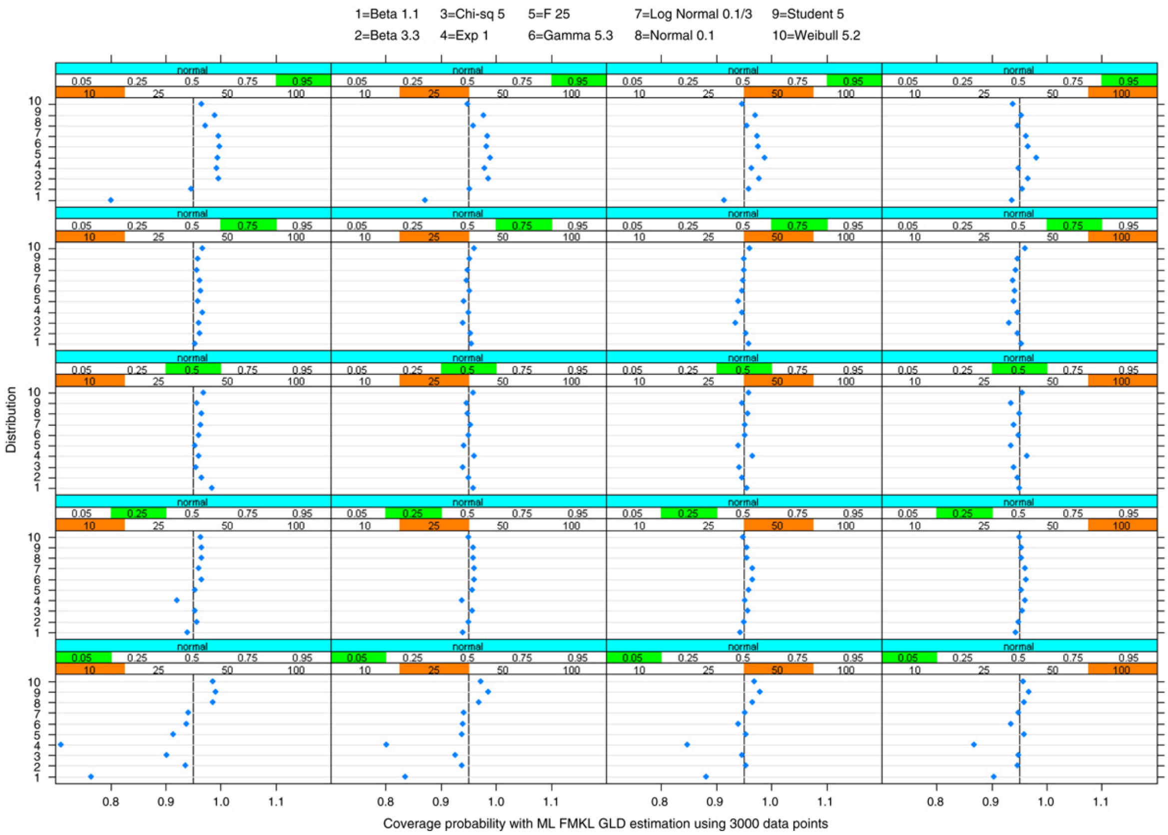

Fig. 6. The coverage probabilities of FMKL GLD estimation for ten well-known statistical distributions using the normal approximation method.

method has better coverage probability. Additionally, in the situations examined in this paper, the FMKL GLD appears to fit these statistical distributions better (contrast Fig. 3 with Fig. 5), as evident by its better coverage of the nominal probabilities at 95% level.

4. Application in data analysis

This example comes from the Body Temperature and Heart Rate for Males and Female Data set (http://www.stat.ucla.edu/datasets/view_data.php?data=20) from UCLA website. The goal is to examine whether heart rate for gender 1 is similar to gender 2 (gender classification not disclosed on web page) and there are 65 heart rate observations in each group. The RS GLD maximum likelihood fits on these data sets are reasonably effective as indicated in QQ plots and comparison of first four moments, so the fitted RS GLDs for gender 1 and 2 are used for illustration. The fitted RS GLD parameters for gender 1 is 72.27328669, 0.05272113, 0.20769612, 0.28625281 and the fitted RS GLD parameters for gender 2 is 82.32088803, 0.03750585, 0.59625679, 0.06432708. The theoretical mean of gender 1 is 73.23 (empirical result 73.37) and the theoretical mean of gender 2 is 73.97 (empirical result 74.15). The GLD fits on gender 1 and 2 are indicated in Fig. 7.

The conventional technique to compare these two variables is to use the Welch test (p-value 0.53). However, this is not without problems. The Welch test comparison is based on the population mean. The population means indicated by GLD are 73.23 and 73.97 for gender 1 and 2 respectively, which clearly missed the modes of the distributions in Fig. 7. The usual philosophy of using the mean is to regard the mean as an overall summary, however, it can be difficult to justify when the mean represents a value that does not appear to have a useful interpretation in the context of the data.

To pursue a more meaningful comparison between genders, two solutions that can be offered under the framework of this article are:

1. Compare the confidence intervals of the mode of the distribution between two groups or
2. Compare the confidence intervals for a range of same quantiles between two groups.

The modes for distributions for heart rate between gender 1 and 2 (72.78 and 77.90) can be found using numerical technique and is a trivial exercise. The quantiles (0.48 for gender 1 and 0.64 for gender 2) associated with these modes can then be found by solving for u from $F^{-1}(u) = \lambda_1 + \frac{u^{\lambda_3}-(1-u)^{\lambda_4}}{\lambda_2}$, (for RS GLD) using Newton–Raphson numerical method. Once found, the confidence interval for the mode can then be found using the technique described in Section 2.2. For 65

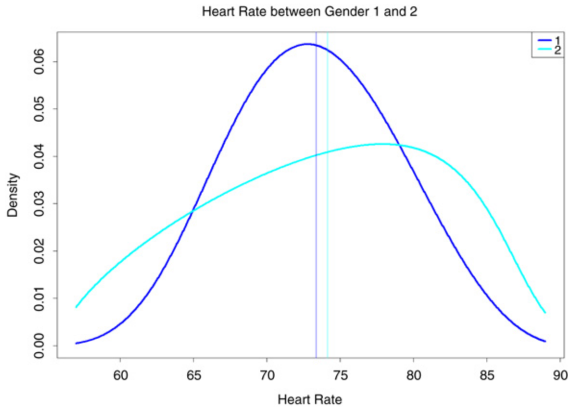

Fig. 7. Comparing GLD fits for gender 1 and 2. The theoretical average values for gender 1 and 2 are indicated by vertical lines.

observations, the 95% confidence interval (using analytical-GLD method) of the mode for gender 1 is [71.02,74.81] and [74.89,80.33] for gender 2. This result indicates that in terms of the mode, the heart rate between genders is not the same at 5% significance level, as indicated by the nonoverlapping confidence intervals.

Instead of choosing a particular point of interest such as the mode of the data, an alternative is to examine whether for the same quantiles, the heart rate between male and female has overlapping confidence intervals. The decision as to how many different quantiles to compare is up to the user, depending on how close the user wants to examine the differences. In the above example, 95% confidence intervals for 1000 equally spaced quantiles ranging from 0.01 to 0.99 were calculated for both genders. Using a sample size of 65, all the confidence intervals overlap, suggesting that overall, the heart rate between male and female are similar. The similarity suggested by the overlapping confidence interval in this example is due to small sample size, if a sample size of 200 is chosen, then only 64% of the quantiles between two groups would be similar. An advantage of using this approach is that when differences are flagged by nonoverlapping confidence intervals, it is easy to examine which quantiles differ between these two variables.

5. Conclusion

This article presents two generic parametric techniques in computing confidence interval for quantiles and in general, the analytical method is preferred over the Normal–GLD approximation method. While the methods described in this appear to be effective in general settings, these methods are subject to the condition that the GLD approximates the data sufficiently well. Once confidence intervals can be reliably estimated, they have useful applications in data analysis. These data analysis methods can be much more informative to decision makers than the traditional and 'standard' use of averages and this article demonstrates the potential usefulness of possessing a distribution that can successfully fit a wide range of empirical data.

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