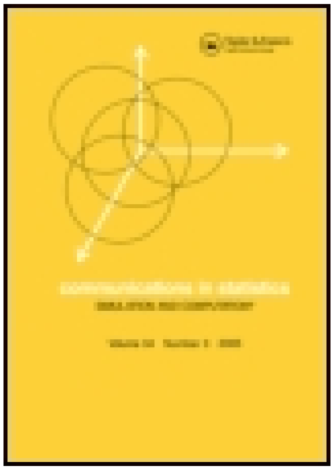

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Estimating the Parameters of the Generalized Lambda Distribution: Which Method Performs Best?

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Estimating the Parameters of the Generalized Lambda Distribution: Which Method Performs Best?

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Generalized Lambda Distribution (GLD) is a flexible distribution that can represent a wide variety of distributional shapes. This property of the GLD has made it very popular in simulation input modeling in recent years, and several fitting methods for estimating the parameters of the GLD have been proposed. Nevertheless, there appears to be a lack of insights about the performances of these fitting methods in estimating the parameters of the GLD for a variety of distributional shapes and input data. Our primary goal in this paper is to compare the goodness-of-fits of the popular fitting methods in estimating the parameters of the GLD introduced in Freimer et al. (1988), i.e., FMKL GLD, and provide guidelines to the simulation practitioner about when to use each method. We further describe the use of the genetic algorithm for the FMKL GLD, and investigate the performances of the suggested methods in modeling the daily exchange rates of eight currencies.

Key words: generalized lambda distribution; parameter estimation; least-squares; method of matching percentiles; genetic algorithm

1 Introduction

An important problem in the design of stochastic simulations is to appropriately choose an input model (i.e., probability distribution) to represent the available historical data. The common approach is to use a standard input model such as normal, exponential, and beta (Law, 2007). However, the shapes that are represented by these standard distributions are limited and therefore they may fail to capture the distributional characteristics of the historical data set.

This limitation of the standard distributions led researchers to build flexible distributions that have the ability to represent a wide variety of distributional shapes. Examples of flexible distributions are the curves proposed by Pearson (1895), the Johnson translation system (Johnson 1949), the generalized lambda distribution (Ramberg and Schmeiser, 1974; Freimer et al., 1988), the four-parameter distribution introduced by Schmeiser and Deutsch (1977), and the generalized beta family of distributions (Kuhl et al., 2009). In this paper, we focus on the generalized lambda distribution (GLD) developed by Freimer et al. (1988). The GLD has attracted particular attention in the simulation community due to its ease of use in generating random variates and flexibility (Johnson and Mollaghasemi, 1994). It is easy to generate random variates from the GLD as it is defined by an inverse cumulative distribution function. It is a quite flexible distribution in the sense that it can represent unimodal, U-shaped, S-shaped, and monotone density shapes that standard distributions may not represent. Additionally, it can approximate several standard distributions including normal, uniform, Student's *t*, exponential, chi-square, Gamma, Weibull, lognormal, and beta distribution. We refer the reader to Karian and Dudewicz (2000) for a complete list of distributions that the GLD can represent and their corresponding parameters.

In addition to the simulation applications (e.g., generation of random number and variance reduction in simulation design; see Ramberg and Schmeiser, 1974; Wilcox, 2002; Karvanen, 2003; Headrick and Mugdadi, 2006; Stengos and Wu, 2006; Willink, 2009 for example studies), GLD has also received attention from other fields including meteorology, statistical process control, supply chain management, economics, and finance. Öztürk and Dale (1982), and also Perera and Jayasekara (2013) use the GLD to fit solar radiation data, which typically exhibit negative skewness. Negiz and Çınar (1997), Pal (2005), and Fournier et al. (2006) use the GLD for statistical process control. Kumaran and Achary (1996) and Poojari et al. (2008) model the demand data with the GLD. Tarsitano (2004) and Beena and Kumaran (2010) model the personal income data of a population with the GLD, while Corrado (2001) uses the GLD to model security price distributions and Chalabi et al. (2010) utilizes GLD to model financial returns. Later in Section 4, we use the

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GLD to model the daily exchange rates of eight major currencies.

Due to its wide use in practice, several fitting methods for estimating the parameters of the GLD have been developed; e.g., the method of matching moments (Ramberg et al., 1979), the least-squares estimation method (Öztürk and Dale, 1985), the method of matching percentiles (Karian and Dudewicz, 1999), the starship method (King and MacGillivray, 1999), the percentile K-S method (Fournier et al., 2007), the maximum likelihood estimation method (Su, 2007a; Su, 2007b), the method of L-moments (Asquith, 2007; Karvanen and Nuutinen, 2008), the genetic algorithm (Poojari et al., 2008), the Bayesian method (Allingham et al., 2009), and the quantile matching method (Su, 2010). A limited number of studies in the literature compares the performance of the parameter estimation methods. For instance, the method of matching percentiles is found to produce better estimates compared to the method of matching moments and the method of L-moments, while there is no difference between the method of matching moments and the least-squares estimation method (Öztürk and Dale, 1985; Karian and Dudewicz, 2003). The maximum likelihood estimation method produces parameter estimates with lower variability than the method of matching moments, the method of L-moments, and the starship method for a variety of GLD shapes as well as sample sizes (Su, 2007a). More recently, Su (2010) compares the performance of the quantile matching method with respect to the performances of the method of matching moments, the method of L-moments, the starship method, and the maximum likelihood estimation method, and finds that no method is superior to others in terms of the mean absolute bias. However, most of these studies are limited in the density shapes considered, and how they evaluate the goodness-of-fit is not consistent with each other. Furthermore, the performance of the genetic algorithm is not clear with respect to other parameter estimation methods. Therefore, it appears that there is a lack of insights about how the goodness-of-fits of different parameter estimation methods of the GLD compare with each other.

Our goal in this paper is to investigate the relative performances of the method of matching percentiles, the method of L-moments, the quantile matching method, the least-squares estimation method, the maximum likelihood estimation method, the starship method, and the genetic algorithm in estimating the parameters of the GLD of Freimer et al. (1988), and provide guidelines to the simulation practitioner on when to use each method. We do this by conducting a comprehensive numerical study. More specifically, we assume the availability of finite, independent and identically distributed data of various lengths generated from the GLD with several distributional characteristics, and estimate the parameters of the GLD using the methods of interest of this paper. We assess the goodness-of-fits via the Kolmogorov-Smirnov (KS) test statistic (Chakravant et al., 1967) and the Anderson-Darling (AD) test statistic (Anderson and Darling, 1954). Furthermore, we investigate the relative performances of the suggested methods in fitting the daily exchange rates of eight currencies, which exhibit skewed and heavy-tailed behavior, using the AD test statistic, visual plots, and Value-at-Risk (VaR).

Our results show that for unimodal densities with continuous tails (Class-I shape presented in Figure 2), the maximum likelihood estimation method can be used in the presence of less than 50 data points or more than 200 data points. For the number of data points between 50 and 200, the quantile matching method performs better. For monotone-shaped densities (Class-II shape presented in Figure 2), the maximum likelihood estimation method can be used for small data sets of length 25 and 50, while the starship method performs better with increasing data length. Similarly, for U-shaped and S-shaped densities (Class-III and Class-IV shapes presented in Figure 2, respectively), the maximum likelihood estimation method and the starship method perform better than all the other methods. In particular, for U-shaped densities the maximum likelihood estimation method can be used when the number of data points is less than 200, and the starship method can be used for larger data sets. For S-shaped densities, the maximum likelihood estimation method outperforms other methods for data sets with less than 400 observations, while the starship method can be used for larger data sets. For unimodal densities with truncated tails (Class-V shape presented in Figure 2), the method of L-moments can be used in the presence of less than 50 data points, while either the quantile matching method or the method of L-moments can be used in the presence of more than 400 observations. For the number of data points between 50 and 200, the quantile matching method performs better. On the other hand, for representing the tail behavior of all classes of the GLD, the starship method outperforms all other methods. This observation is supported with our findings in fitting the major daily exchange-rate data to the GLD using the starship method, the quantile matching method, and the method of L-moments. Furthermore, we find that in terms of the VaR performance, all three methods perform similar to each other.

We organize the remainder of the paper as follows. In Section 2 we introduce the GLD and review the fitting methods proposed for the GLD. Section 3 presents our comprehensive numerical study and discusses our findings. We illustrate the performances of the fitting methods in modeling the daily exchange rate currencies in Section 4, and we conclude in Section 5 with future research directions.

2 Generalized Lambda Distribution

We introduce the GLD and discuss its application areas in Section 2.1. In Section 2.2 we review the fitting methods in the literature to estimate the parameters of the GLD, while we present the implementation of the method of matching percentiles, the least-squares estimation method, and the genetic algorithm for estimating the parameters of the FMKL GLD in Section 2.3.

2.1 Description

The generalized lambda distribution (Filliben, 1975; Joiner and Rosenblatt, 1971; Ramberg and Schmeiser, 1974), which is an extension of Tukey's lambda distribution (Hastings et al., 1947) is defined by the following inverse cumulative distribution function (cdf):

$$F^{-1}(u;\lambda_1,\lambda_2,\lambda_3,\lambda_4) = \lambda_1 + \frac{u^{\lambda_3} - (1-u)^{\lambda_4}}{\lambda_2}, \tag{1}$$

where $0 \leq u \leq 1$; λ_1 is the location parameter, λ_2 is the scale parameter, λ_3 and λ_4 are related to skewness and kurtosis, respectively. This representation is denoted Ramberg-Schmeiser Generalized Lambda Distribution (RS GLD) referring to the parameterization of Ramberg and Schmeiser (1974). The probability density function (pdf) related to (1) is given by

$$f(u; \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \frac{\lambda_2}{\lambda_3 u^{\lambda_3 - 1} + \lambda_4 (1-u)^{\lambda_4 - 1}}. \tag{2}$$

The representation in (2) does not specify a proper pdf for all combinations of the shape parameters λ_3 and λ_4 (see Fournier et al. (2007) for the six-regions in which a proper pdf is well defined). This limitation of the RS GLD becomes problematic especially when estimating the parameters of the GLD. More specifically, any (λ_3, λ_4) pair estimate that is not part of the specified six-regions would not produce a valid pdf. In order to avoid this problem, Freimer et al. (1988) propose a different parameterization for the GLD denoted Freimer-Mudholkar-Kollia-Lin Generalized Lambda Distribution (FMKL GLD), which is given by

$$F^{-1}(u; \lambda_1, \lambda_2, \lambda_3, \lambda_4) = \lambda_1 + \frac{1}{\lambda_2}\left(\frac{u^{\lambda_3} - 1}{\lambda_3} - \frac{(1-u)^{\lambda_4} - 1}{\lambda_4}\right). \tag{3}$$

This parameterization is well defined for all parameter values; the only restriction is that $\lambda_2 > 0$. Also, in order to have a finite k^{th} moment, the additional requirement is $min(\lambda_3, \lambda_4) > -1/k$. The definitions of the FMKL GLD parameters are similar to those of the RS GLD. Both of these representations can present a wide variety of shapes and therefore are utilized in practice; however, it is generally the FMKL GLD preferred due to the ease in its use. In this paper, we also use the FMKL GLD representation.

2.2 Review of Fitting Methods

The method of matching moments, which is based on the matching of the sample moments of the data with the theoretical moments of the GLD, has been a widely used method for estimating the parameters of the GLD due to the existence of tables that tabulate the corresponding estimates from the given sample moments (Ramberg et al., 1979). When the sample moments are not identified in the tables, numerical methods are used to obtain the parameter estimates. Despite its popularity, the method of matching moments may not work well for heavy tailed distributions (Asquith, 2007). Another shortcoming of this method is that different values of the shape parameters give rise to the same values of the third and fourth moments. There are also cases where the third and the fourth sample moments (i.e., skewness and kurtosis) yield shape parameters that do not exist. Due to these drawbacks, we do not consider the method of matching moments in this paper.

The method of L-moments proposed by Petersen (2001) and studied in Karian and Dudewicz (2003), Asquith (2007), and Karvanen and Nuutinen (2008) is an alternative method that is based on order statistics, and it has several advantages with respect to the conventional method of matching moments including unbiasedness, robustness, and consistency.

An alternative method that matches the sample percentiles to the theoretical percentiles of the GLD is the method of matching percentiles. It is related to the method of L-moments; one key difference is that the L-moments are linear combinations of all order statistics, while the method of matching percentiles uses only a limited number of order statistics. Karian and Dudewicz (2003) perform a comparison of the method of matching moments, the method of matching percentiles, and the method of L-moments in fitting the GLD, and conclude the superiority of the method of matching percentiles when the quality of the fit is assessed by the L^2-norm. We compare the performances of the method of L-moments and the method of matching percentiles using the KS and AD test statistics.

As another alternative to the method of matching moments, Öztürk and Dale (1985) describe the least-squares estimation method, whose origin dates back to Ramberg et al. (1979). The least-squares estimation method minimizes the total squared differences between the data and the expected values of the order statistics, and solves the corresponding optimization problem using the Nelder-Mead downhill simplex algorithm (Nelder and Mead, 1965). This method does not suffer from the particular limitations of the method of matching moments.

A closely related method to the least-squares estimation method is the quantile matching method introduced in Su (2010). The quantile matching method minimizes the sum of squared deviations between the fitted quantiles and the corresponding sample quantiles. Su (2010) compares this method with the method of matching moments, the method of L-moments, the starship method, and the maximum likelihood estimation method (Su, 2007a) for a variety of GLD shapes as well as sample sizes, and finds no major difference between these methods in terms of the mean absolute bias. However, the maximum likelihood estimation method produces parameter estimates with lower variability.

The common practice in using the previous fitting methods is to assess the resulting fits via goodness-of-fit tests (e.g. KS goodness-of-fit test), and several trials may be required before finding an acceptable solution. Motivated by that, King and MacGillivray (1999) propose the starship method that minimizes a goodness-of-fit criterion. One shortcoming of this method is that it is extremely slow especially with large sample sizes. In order to reduce the execution time, Lakhany and Mausser (2000) propose a method based on successive simplex from random starting points that runs until the goodness-of-fit tests stop rejecting the model. Despite the computational speed gained, this method does not produce the best fit and the bias and standard errors can be quite high (Fournier et al., 2007). Fournier et al. (2007) devise a method that combines both the method of matching percentiles and the starship method. Specifically, the method first identifies the two shape parameters by minimizing a goodness-of-fit criteria. Then, the closed form formulas obtained under the method of matching percentiles that relate the shape parameters to the location and scale parameters are used to obtain the location and scale parameters. Compared to the starship method and the method in Lakhany and Mausser (2000), this hybrid method is less biased; it is computationally faster than the starship method but slightly slower than the method in Lakhany and Mausser (2000). We use the starship method in King and MacGillivray (1999) due to its popularity and ease in its use compared to the method in Fournier et al. (2007). The starship method minimizes the AD test statistic; thus, it is expected that it will outperform other methods when the comparison is performed using the AD test statistic. However, we still include this method in our analysis as its performance is not clear under the KS test statistic.

Allingham et al. (2009) develop a Bayesian model to estimate the parameters of the GLD but the method is computationally slow and its implementation is generally problem specific. In particular, the expert opinion that is available for each parameter of the GLD; i.e, choice of the prior distributions, which are important components in the implementation of the Bayesian method, may change from one instance to another, and this may affect the performance of the algorithm. Thus, we do not consider this method in this paper.

Poojari et al. (2008) is the first to use the genetic algorithm in estimating the parameters of the GLD; however, the performance of this method has not been compared to the performances of the other methods. In this paper, we perform a comprehensive numerical study to compare the performances of the widely used fitting methods including the least-squares estimation method, the method of matching percentiles, the method of L-moments, the maximum likelihood estimation method, the quantile matching method, the starship method, and the genetic algorithm in estimating the parameters of the FMKL GLD. Our main contribution is to provide guidelines to the simulation practitioner on when to use each method. The algorithms are coded in the R software (R Development Core Team, 2008). Since GLD has been used extensively in practice, many of these fitting methods are available in the GLDEX package of the R software. To the best of our knowledge, however, genetic algorithm for estimating the parameters of the RS GLD and the FMKL GLD is not available in R. Our secondary contribution is to develop the genetic algorithm and the FMKL representations of the method of matching percentiles and the least-squares estimation method in the R software. The related algorithms are described in the following section, and both the codes and the guidelines for implementation are available upon request. Our third contribution is to illustrate the performances of the best method(s) in fitting the daily exchange rate returns, which exhibit skewness and fat-tails.

2.3 Implementation of Fitting Methods

The method of L-moments, the quantile matching method, the starship method, and the maximum likelihood estimation method are available in the GLDEX package of the R software for fitting the FMKL GLD. However, the method of matching percentiles and the least-squares estimation method are available for fitting only RS GLD. This section adapts the method of matching percentiles (Section 2.3.1) and the least-squares estimation method (Section 2.3.2) for the FMKL presentation of the GLD, and presents for the first time the genetic algorithm (Section 2.3.3) for fitting the FMKL GLD.

2.3.1 Method of Matching Percentiles for FMKL GLD

The method of matching percentiles seeks to determine the parameters that equate the first four estimated percentiles of the data to their theoretical counterparts. More specifically, letting ρ_i denote the theoretical quantiles and $\tilde{\rho}_i$ denote the sample quantiles, the method of matching percentiles identifies the parameters λ_i, $i = 1, 2, 3, 4$ by solving the equations $\tilde{\rho}_i = \rho_i$, $i = 1, 2, 3, 4$. Using the FMKL quantile function in (3) we obtain the following theoretical quantiles for the FMKL GLD, where u is a number between 0 and 0.25:

$$
\begin{aligned}
\rho_1 &= \lambda_1 + \frac{1}{\lambda_2}\left(\frac{0.5^{\lambda_3} - 1}{\lambda_3} - \frac{0.5^{\lambda_4} - 1}{\lambda_4}\right) \\
\rho_2 &= \frac{1}{\lambda_2}\left(\frac{(1-u)^{\lambda_3} - u^{\lambda_3}}{\lambda_3} - \frac{u^{\lambda_4} - (1-u)^{\lambda_4}}{\lambda_4}\right) \\
\rho_3 &= \frac{\frac{0.5^{\lambda_3} - u^{\lambda_3}}{\lambda_3} - \frac{0.5^{\lambda_4} - (1-u)^{\lambda_4}}{\lambda_4}}{\frac{(1-u)^{\lambda_3} - 0.5^{\lambda_3}}{\lambda_3} - \frac{u^{\lambda_4} - 0.5^{\lambda_4}}{\lambda_4}} \\
\rho_4 &= \frac{\frac{0.75^{\lambda_3} - 0.25^{\lambda_3}}{\lambda_3} - \frac{0.25^{\lambda_4} - 0.75^{\lambda_4}}{\lambda_4}}{\frac{(1-u)^{\lambda_3} - u^{\lambda_3}}{\lambda_3} - \frac{u^{\lambda_4} - (1-u)^{\lambda_4}}{\lambda_4}}
\end{aligned}
$$

Letting $\tilde{\pi}_p$ denote the $(100p)^{th}$ percentile of the data, Karian and Dudewicz (1999) define $\tilde{\rho}_1 = \tilde{\pi}_{0.5}$, $\tilde{\rho}_2 = \tilde{\pi}_{1-u} - \tilde{\pi}_u$, $\tilde{\rho}_3 = (\tilde{\pi}_{0.5} - \tilde{\pi}_u)/(\tilde{\pi}_{1-u} - \tilde{\pi}_{0.5})$, and $\tilde{\rho}_4 = (\tilde{\pi}_{0.75} - \tilde{\pi}_{0.25})/\tilde{\rho}_2$.

These statistics can be interpreted as the sample median, the range between the u^{th} percentile and the $(1-u)^{\text{th}}$ percentile, a measure of relative tail weights, and the tail weight factor, respectively, which individuate the relevant characteristics of the distributions (Lampasi et al., 2006). A natural question to ask is how we choose the value of u, which is set to 0.1 in Karian and Dudewicz (1999). We experimented with several values of u and found that $u = 0.1$ gives smaller KS and AD test statistics; thus, we set $u = 0.1$ in our numerical study (Section 3).

What is important to recognize here is that the theoretical quantiles ρ_3 and ρ_4 are only functions of parameters λ_3 and λ_4. Therefore, we first solve for λ_3 and λ_4 from the equations $\rho_3 = \tilde{\rho}_3$ and $\rho_4 = \tilde{\rho}_4$ using the Nelder-Mead method. Once we obtain λ_3 and λ_4, we use these parameters to estimate λ_2 and λ_1 using the following characterizations:

$$\begin{aligned}
\lambda_2 &= \frac{\frac{(1-u)^{\lambda_3}-u^{\lambda_3}}{\lambda_3} - \frac{u^{\lambda_4}-(1-u)^{\lambda_4}}{\lambda_4}}{\tilde{\rho}_2} \\
\lambda_1 &= \tilde{\rho}_1 - \frac{1}{\lambda_2}\left(\frac{0.5^{\lambda_3}-1}{\lambda_3} - \frac{0.5^{\lambda_4}-1}{\lambda_4}\right)
\end{aligned}$$

2.3.2 Least-Squares Estimation Method for FMKL GLD

The FMKL presentation of the least-squares estimation method was first introduced in Lakhany and Mausser (2000). Assuming the availability of independent and identically distributed sample x_i, $i = 1, 2, \ldots, n$ of size n from the FMKL GLD, we let $x_{(i)}$ denote the i^{th} order statistic and $U_{(i)}$ denote the corresponding uniformly distributed random variable. The i^{th} order statistic in the sample is given by $X_{(i)} = \mathrm{E}(\mathrm{X}_{(\mathrm{i})}) + \epsilon_{\mathrm{i}}$, where E denotes the expectation operator and ϵ represents the deviation of the i^{th} order statistic from its expected value; i.e., the error around the i^{th} order statistic. We assume that $\mathrm{E}(\epsilon_{\mathrm{i}}) = 0$ for $i = 1, 2, \ldots, n$ and the errors have a finite variance. Using the relation

$$x_u = \lambda_1 + \frac{1}{\lambda_2}\left(\frac{u^{\lambda_3}-1}{\lambda_3} - \frac{(1-u)^{\lambda_4}-1}{\lambda_4}\right),$$

we can represent the i^{th} order statistic $X_{(i)}$ as

$$X_{(i)} = \lambda_1 + \frac{1}{\lambda_2}\left(\frac{\mathrm{E}(\mathrm{U}_{(\mathrm{i})})^{\lambda_3} - 1}{\lambda_3} - \frac{\mathrm{E}(1 - \mathrm{U}_{(\mathrm{i})})^{\lambda_4} - 1}{\lambda_4}\right) + \epsilon_i,$$

where $\mathrm{E}(\mathrm{U}_{(\mathrm{i})})^{\lambda_3} = \Gamma(\mathrm{n}+1)\Gamma(\mathrm{i}+\lambda_3)/\Gamma(\mathrm{i})\Gamma(\mathrm{n}+\lambda_3+1)$ and $\mathrm{E}(1-\mathrm{U}_{(\mathrm{i})})^{\lambda_4} = \Gamma(\mathrm{n}+1)\Gamma(\mathrm{n}-\mathrm{i}+\lambda_4+1)/\Gamma(\mathrm{n}-\mathrm{i}+1)\Gamma(\mathrm{n}+\lambda_4+1)$ (Mykytka and Ramberg, 1979). In here, Γ is the Gamma function.

The least-squares method minimizes the difference between the observed and the predicted order statistics; i.e.,

$$\begin{aligned} S(\lambda_1,\lambda_2,\lambda_3,\lambda_4) &= \sum_{i=1}^{n}(x_{(i)} - \mathrm{E}(\mathrm{X}_{(\mathrm{i})}))^2 \\ &= \sum_{i=1}^{n}(x_{(i)} - \lambda_1 - Q_i/\lambda_2)^2 \end{aligned} \tag{4}$$

where

$$Q_i = \frac{1}{\lambda_3}\left(\mathrm{E}(\mathrm{U}_{(\mathrm{i})})^{\lambda_3} - 1\right) - \frac{1}{\lambda_4}\left(\mathrm{E}(1 - \mathrm{U}_{(\mathrm{i})})^{\lambda_4} - 1\right).$$

Notice that $S(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ is a nonlinear function of λ_3 and λ_4. For computational efficiency, the common practice is to first minimize $S(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ with respect to λ_1 and λ_2 assuming λ_3 and λ_4 are given. Then, estimated λ_1 and λ_2 are substituted into the minimization function to compute λ_3 and λ_4. This produces $\hat{\lambda}_2 = \sum_{i=1}^{n}(Q_i - \mu_Q)^2/\sum_{i=1}^{n}(x_{(i)} - \mu_x)(Q_i - \mu_Q)$ and $\hat{\lambda}_1 = \mu_x - (\sum_{i=1}^{n}(x_{(i)} - \mu_x)(Q_i - \mu_Q)/\sum_{i=1}^{n}(Q_i - \mu_Q)^2\mu_Q)$, where $\mu_x = \sum_{i=1}^{n} x_{(i)}/n$ and $\mu_Q = \sum_{i=1}^{n} Q_i/n$. Inserting $\hat{\lambda}_1$ and $\hat{\lambda}_2$ into Equation (4) leads to

$$S(\lambda_3,\lambda_4) = (1 - r_{xQ}^2(\lambda_3,\lambda_4))\sum_{i=1}^{n}(x_{(i)} - \mu_x),$$

where $r_{xQ}(\lambda_3,\lambda_4)$ is the correlation coefficient between x_i and Q_i given by

$$r_{xQ} = \frac{(\sum_{i=1}^{n} x_{(i)}Q_i) - n\mu_x\mu_Q}{(\sum_{i=1}^{n} x_{(i)}^2 - n\mu_x^2)(\sum_{i=1}^{n} Q_i^2 - n\mu_Q^2)}.$$

Since $x_{(i)}$ and μ_x are not the functions of λ_3 and λ_4, it would be sufficient to minimize $-r_{xQ}^2(\lambda_3,\lambda_4)$ in order to minimize $S(\lambda_3,\lambda_4)$. We do this by using the Nelder-Mead algorithm. Then, the estimated λ_3 and λ_4 are inserted into $\hat{\lambda}_1$ and $\hat{\lambda}_2$ to find the estimates of λ_1 and λ_2.

It is important to note that the presence of Gamma functions in Q_i makes it computationally demanding to evaluate Q_i. This is especially true with increasing number of data points. To avoid this computational difficulty, following the approach in Öztürk and Dale (1985) we replace Q_i with

$$R_i = \frac{(i/(n+1))^{\lambda_3} - 1}{\lambda_3} - \frac{((n-i+1)/(n+1))^{\lambda_4} - 1}{\lambda_4},$$

especially when the number of data points is greater than 100.

2.3.3 The Genetic Algorithm for FMKL GLD

Genetic algorithms refer to a family of algorithms that imitate the evolution in searching for optimal solutions. A typical genetic algorithm is composed of initialization, selection, genetic operators such as crossover and mutation, and termination phases. We summarize the steps of a genetic algorithm in Figure 1.

In the initialization phase, we create a population of size 20. This is done by generating pairs of (λ_3, λ_4) values. The individuals are then completed by determining corresponding λ_1 and λ_2 values using the method of matching moments as described in Lakhany and Mausser (2000). In the selection phase, our selection function favors individuals having better fitness values, where fitness of an individual is calculated based on the difference between the fitted distribution function and the empirical distribution function. The selected individual is then subject to mutation with a mutation rate of 0.1. Mutation randomly changes the values of the shape parameters of the individual and recalculates the corresponding location parameters. Then, the individual (generally referred as parent within the context of crossover operation) is subjected to crossover with a crossover rate of 0.7. To accomplish this, a second parent is selected. The shape parameters of the offspring lie between the shape parameters of the two parents. The performance of the new offspring is evaluated based on the discrepancies between the fitted distribution function and the empirical distribution function. The number of offspring created is equal to the half of the initial population. The rest of the new generation is filled by selecting members from the old generation using the selection

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method. After a new population is created, the best solution is compared against the incumbent solution, which is replaced if necessary. We terminate the algorithm after 400 replications.

The crossover probability, the mutation rate, and the termination condition are problem-specific elements of a genetic algorithm. We investigate the sensitivity of each element to our solutions, and we confirm that these set of parameters give a good combination for our algorithm.

3 Numerical Study

In this section our goal is to evaluate the performances of the method of matching percentiles, the method of L-moments, the least-squares estimation method, the quantile matching method, the maximum likelihood estimation method, the starship method, and the genetic algorithm for a variety of FMKL GLD shapes. Section 3.1 presents our design of experiments, and Section 3.2 discusses our findings.

3.1 Design of Experiments

Freimer et al. (1988) classify the FMKL GLD shapes to five categories depending on the variety of distributions that can be represented by the several combinations of the shape parameters λ_3 and λ_4. In particular, Class-I ($\lambda_3 < 1$, $\lambda_4 < 1$) represents unimodal densities with continuous tails, Class-II ($\lambda_3 > 1$, $\lambda_4 < 1$) represents monotone pdfs similar to the exponential distribution, Class-III ($1 < \lambda_3 < 2$, $1 < \lambda_4 < 2$) represents U-shaped densities with truncated tails, Class-IV ($\lambda_3 > 2$, $1 < \lambda_4 < 2$) represents S-shaped densities, and Class-V ($\lambda_3 > 2$, $\lambda_4 > 2$) represents unimodal densities with truncated tails. In this paper, we use a representative density from each class. Table 1 presents the parameters we use from each class, while Figure 2 provides the shapes that are represented by the parameters indicated in Table 1. These are also the parameter values used in Su (2007a).

We let the number of data points n take values 25, 50, 100, 200, 400, 1000, and 5000. To calculate the goodness-of-fit, we use the KS test statistic and the AD test statistic. Both test statistics summarize the difference between the fitted distribution function (i.e., the cdf that is evaluated with the estimated parameters of the FMKL GLD) and the empirical distribution function of the FMKL GLD. However, the KS test statistic focuses on the discrepancies in the middle of the distribution functions, while the AD test statistic emphasizes the discrepancies in the tails of the distribution functions. In each experiment of the following section, we determine the number of replications by ensuring that the absolute error of the KS and AD test statistics is no more than 1%, which corresponds to at least 2500 replications of the simulation study.

3.2 Results

We abbreviate the fitting methods of interest as follows: The quantile matching method (QM), the method of L-moments (LM), the maximum likelihood estimation method (MLE), the starship method (STAR), the method of matching percentiles (MoP), the genetic algorithm (GEN), and the least-squares estimation method (LS). The KS and AD test statistics obtained for Class-I, Class-II, Class-III, Class-IV, and Class-V distributions with parameters listed in Table 1 are provided in Table 2, Table 3, Table 4, Table 5, and Table 6, respectively. The entries with * correspond to the column minimum.

First, we present our findings for the Class-I distributions. The comparison of the KS statistics reveals that when $n < 50$ and $n \geq 200$, the MLE outperforms all other methods, while the QM is superior when the number of data points is between 50 and 200; i.e., $50 \leq n \leq 200$ (Table 2). The AD statistics, as expected, favor the STAR for all data sizes. However, it is important to note that the QM and the LM appear to be alternatives to the STAR for large data sizes (i.e., $n \geq 1000$). Later, we investigate the performances of the STAR, the QM, and the LM in fitting the daily exchange-rate returns.

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For the Class-II distributions, the MLE and the STAR appear to be promising methods. The KS statistics favor the MLE when $n \leq 50$ (for smaller data sizes the QM performs similar to the MLE) and the STAR for larger data sizes. The comparison of the AD test statistics reveals the STAR as the best method for all data sizes (Table 3). The superiority of the MLE and the STAR also holds for the Class-III and the Class-IV distributions. More specifically, for the Class-III distributions, the comparison of the KS statistic shows that the MLE outperforms all other methods when $n \leq 200$, while the STAR is a good choice when $n \geq 200$ (Table 4). Similarly, for the Class-IV distributions, the MLE is identified as the best fitting method when $n \leq 400$, while the STAR can be used when $n > 400$ (Table 5). For both classes the AD test statistics reveal the STAR as the best fitting method regardless of the number of data points.

The superiority of the MLE and the STAR for the Class-II, Class-III, and Class-IV distributions does not carry itself to the Class-V distributions. Comparing the KS test statistics we find that the LM and the QM outperform other methods for the Class-V distributions (Table 6). In particular, the LM appears to be the best method when $n < 50$, and the QM appears to be the best method when $50 \leq n < 200$. Either the LM or the QM can be used when $n \geq 200$. The AD test statistics favor the STAR for all sample sizes.

4 Real Life Data: Application to Modeling Daily Exchange Rates

Our objective is to investigate the performances of the best fitting methods of the previous section in fitting the daily exchange-rate return data. Section 4.1 describes the statistical properties of the daily exchange-rate data, while Section 4.2 compares the performances of the STAR, the LM, and the QM for each exchange-rate currency by using the AD test statistics, frequency plots, and in-sample VaR levels.

4.1 Daily Exchange Rate Data

Our data consists of eight daily exchange-rate currencies in terms of the US dollar. These currencies include the Brazilian real (BRA), the euro (EURO), the Swiss franc (CHF), the Canadian dollar (CAD), the Japanese Yen (YEN), sterling (GBP), the Australian dollar (AUD), and the Turkish lira (TL). All data sets are closing exchange rates from DataStream and cover a six-year period from 2 January 2006 to 30 December 2011 with a total of 1566 observations.

We transform the data set into daily logarithmic returns, $X_t = \log(S_t/S_{t-1})$ where S_t is the level of the daily exchange rate at time t. Table 7 presents preliminary statistics on the data. The second and the third column of Table 7 present the mean and the standard deviation of the log-returns, while the fourth and the fifth columns give the skewness and the excess kurtosis. The skewness and excess kurtosis are, respectively, given by $s = m_3/m_2^{3/2}$ and $k = (m_4/m_2^2) - 3$, where m_i for $i = 2, 3, 4$ is the estimate of the i^{th} moment around the mean. We find evidence of significant skewness for all currencies except the CAD, the EURO, and the GBP. Furthermore, all currencies have kurtosis significantly greater than three, implying that the distributions deviate from the normal distribution. This observation is supported by the Bera-Jarque (BJ) statistic (Bera and Jarque, 1981) tabulated at the sixth column of Table 7, which is a commonly used method in finance to test any departure from normality. The BJ test statistic is given by $n(s^2/6 + k^2/24)$, and under the null hypothesis of normality, it is distributed as a chi-squared distribution with two degrees of freedom. Under 1% level of significance, the BJ test statistic is 9.21, indicating that all of the eight exchange rate currencies are nonnormal. Finally, we investigate the tail behavior of the log-return series at the last column of Table 7 using the measure $[U(0.2) - L(0.2)]/[U(0.5) - L(0.5)]$, where $U(p)$ is the average of the largest $100p$% of the sample and $L(p)$ is the average of the smallest $100p$% of the sample (Hogg, 1974). The population measure associated with the normal distribution is 1.75; thus, we conclude that all currencies have heavier tails than the normal distribution. This is why we use the AD test statistic and the in-sample VaR in the following section to test the performances of the fitting methods in fitting the currencies.

4.2 Fitting GLD to Daily Exchange Rate Data

We first fit the FMKL GLD to each currency with different fitting methods, and find that for each currency $\lambda_3 < 1$ and $\lambda_4 < 1$; thus, each currency falls under the Class-I family of the FMKL GLD. Since each currency exhibits heavy-tailed behavior (as shown in the previous section), we focus on the STAR, the LM, and the QM, which are the methods that are favored under the AD test statistic of the Class-I distributions when the number of data points is larger than 1000. Table 8 presents the parameter estimates under each three fitting method for each currency. Then, we investigate the behavior of each method at the tails using the AD test statistic, frequency plots as well as in-sample VaR.

Table 9 presents the AD test statistics for each currency. We find that the STAR performs better than the QM and the LM in all cases. Nevertheless, focusing on Figure 3, which presents the histogram for each currency versus the FMKL GLD shapes obtained under each fitting method, we observe that the performances of the QM and the LM are quite close to the performance of the STAR in modeling the tail behavior of the currencies. Since the performances of these three fitting methods are very close to each other, the fitting lines are almost indistinguishable from each other in the figure. This observation is consistent with our findings in Section 3.2.

We are now ready to present the in-sample VaR test results. In particular, we use the fitted distributions to determine the risk for long and short positions of each currency at levels $\alpha \in \{0.005, 0.01, 0.05, 0.95, 0.99, 0.995\}$. The first three levels are used to measure the risk of long positions, while the last three levels are used to measure the risk of short positions. We apply Kupiec likelihood ratio test (Kupiec, 1995) to test the hypothesis that the expected proportion of violations is equal to α. The likelihood ratio statistic is given by $2\log((t/n)^t(1-t/n)^{n-t}) - 2\log(\alpha^t(1-\alpha)^{n-t})$, where t is the number of times the observed returns are above (for short positions) or below (for long positions) the theoretical VaR value, and n is the length of the data. Under the null hypothesis, this statistic is distributed as a chi-squared distribution with one degree of freedom.

Table 10 presents the p-values of the likelihood ratio statistic for both short and long positions. Given that we use a 5% level for the test, the null hypothesis is rejected only twice under the STAR, while it is never rejected under the QM and the LM. Therefore, in terms of the VaR performance, all three methods seem to yield good performance.

5 Conclusion

In this paper, we consider the problem of estimating the parameters of the FMKL GLD by using a variety of fitting methods including the method of matching percentiles, the least-squares estimation method, the method of L-moments, the quantile matching method, the maximum likelihood estimation method, the starship method, and the genetic algorithm. Our main contribution to the discrete-event stochastic simulation literature is to compare the performances of these fitting methods in terms of the goodness-of-fit for historical data sets of finite lengths and various GLD shapes. We also develop the FMKL representations of the method of matching percentiles and the least-squares estimation method, and we describe the implementation of the genetic algorithm for fitting the parameters of the FMKL GLD. Furthermore, we evaluate the performances of the favored methods in fitting the daily exchange-rate data.

Our results suggest that the maximum likelihood estimation method, the quantile matching method, the method of L-moments, and the starship method are the favored methods when the comparison is performed using the KS test statistic. The starship method, on the other hand, outperforms other methods when the comparison is performed using the AD test statistic. The superiority of the starship method also holds in fitting the daily exchange-rate data to the FMKL GLD. In terms of the VaR performance, however, the starship method, the quantile matching method, or the method of L-moments can be used to model the daily exchange-rate data.

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Many of the fitting methods considered in this paper (e.g., the method of L-moments, the least-squares estimation method, and the method of matching percentiles) follow a two-step procedure for estimating the four parameters of the FMKL GLD. In the first step, the two parameters λ_1 and λ_2 are represented in terms of the parameters λ_3 and λ_4. In the second step, an optimization problem is solved to estimate the parameters λ_3 and λ_4. Once λ_3 and λ_4 are found, it is trivial to obtain λ_1 and λ_2 using the relationship in the first step. Although the optimization problem solved in the second step changes from one fitting method to another, it is important to note that in each of these methods the objective function of the optimization problem is neither linear nor convex. In the literature, the main methods used to solve such optimization problems are the Nelder-Mead algorithm and the Newton-Raphson method, which produce local optimal solutions. Replacing these techniques with algorithms that provide global optimal solutions such as controlled random search, simulated annealing, and genetic algorithms is the topic of future research.

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Table 1: Properties of five FMKL GLDs used in numerical experiments

	λ_1	λ_2	λ_3	λ_4
Class-I	0	1	0.5	0.6
Class-II	0	1	2	0.5
Class-III	0	1	1.5	1.5
Class-IV	0	1	2.5	1.5
Class-V	0	1	3	3

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Table 2: The KS and AD test statistics for Class-I distributions

The KS test statistics

Method	$n = 25$	$n = 50$	$n = 100$	$n = 200$	$n = 400$	$n = 1000$	$n = 5000$
QM	0.0999	0.0664*	0.0452*	0.0312*	0.0222	0.0140	0.0061
LM	0.0991	0.0677	0.0466	0.0323	0.0231	0.0146	0.0064
MLE	0.0977*	0.0699	0.0481	0.0312*	0.0213*	0.0130*	0.0055*
STAR	0.1005	0.0675	0.0460	0.0317	0.0224	0.0140	0.0061
MoP	0.1170	0.0861	0.0637	0.0465	0.0345	0.0223	0.0097
GEN	0.1046	0.0741	0.0525	0.0367	0.0261	0.0165	0.0084
LS	0.1321	0.0998	0.0752	0.0679	0.0708	0.0724	0.0668

The AD test statistics

Method	$n = 25$	$n = 50$	$n = 100$	$n = 200$	$n = 400$	$n = 1000$	$n = 5000$
QM	0.1449	0.0715	0.0387	0.0231	0.0151	0.0091	0.0040*
LM	0.0729	0.0514	0.0326	0.0213	0.0147	0.0092	0.0042*
MLE	0.2396	0.1012	0.0467	0.0278	0.0194	0.0124	0.0056
STAR	0.0442*	0.0332*	0.0245*	0.0180*	0.0131*	0.0085*	0.0039*
MoP	0.2456	0.2849	0.2927	0.2844	0.2418	0.1588	0.0551
GEN	0.1251	0.1091	0.0916	0.0692	0.0471	0.0259	0.0288
LS	0.3500	0.5313	0.7412	1.4580	2.3656	4.2998	8.4874

Table 3: The KS and AD test statistics for Class-II distributions

The KS test statistics

Method	n = 25	n = 50	n = 100	n = 200	n = 400	n = 1000	n = 5000
QM	0.0974*	0.0662	0.0467	0.0334	0.0245	0.0159	0.0071
LM	0.0982	0.0676	0.0476	0.0337	0.0245	0.0158	0.0073
MLE	0.0972*	0.0647*	0.0454	0.0320	0.0243	0.0177	0.0127
STAR	0.0984	0.0654	0.0448*	0.0310*	0.0222*	0.0141*	0.0066*
MoP	0.1150	0.0838	0.0603	0.0427	0.0309	0.0200	0.0092
GEN	0.1036	0.0724	0.0508	0.0360	0.0257	0.0167	0.0099
LS	0.1120	0.0890	0.0756	0.0693	0.0644	0.0582	0.0531

The AD test statistics

Method	n = 25	n = 50	n = 100	n = 200	n = 400	n = 1000	n = 5000
QM	0.1179	0.0675	0.0420	0.0297	0.0221	0.0153	0.0084
LM	0.0757	0.0555	0.0385	0.0281	0.0210	0.0143	0.0081
MLE	0.2858	0.1381	0.0768	0.0492	0.0351	0.0260	0.0267
STAR	0.0499*	0.0381*	0.0278*	0.0207*	0.0155*	0.0105*	0.0065*
MoP	0.1951	0.1872	0.1557	0.1205	0.0906	0.0634	0.0379
GEN	0.1492	0.1195	0.0902	0.0691	0.0524	0.0394	0.0322
LS	0.3107	0.3532	0.4738	0.91 63	1.0203	1.2968	0.5586

Table 4: The KS and AD test statistics for Class-III distributions

The KS test statistics

Method	$n = 25$	$n = 50$	$n = 100$	$n = 200$	$n = 400$	$n = 1000$	$n = 5000$
QM	0.0976	0.0660	0.0461	0.0328	0.0240	0.0155	0.0070
LM	0.0962	0.0671	0.0475	0.0339	0.0248	0.0160	0.0072
MLE	0.0896*	0.0593*	0.0416*	0.0299*	0.0225	0.0147	0.0067
STAR	0.0973	0.0642	0.0435	0.0300*	0.0212*	0.0132*	0.0058*
MoP	0.1156	0.0842	0.0613	0.0446	0.0334	0.0224	0.0104
GEN	0.1042	0.0730	0.0516	0.0363	0.0259	0.0164	0.0084
LS	0.1424	0.1238	0.1136	0.1062	0.0983	0.0925	0.0796

The AD test statistics

Method	$n = 25$	$n = 50$	$n = 100$	$n = 200$	$n = 400$	$n = 1000$	$n = 5000$
QM	0.1170	0.0654	0.0401	0.0274	0.0202	0.0141	0.0077
LM	0.0759	0.0557	0.0397	0.0289	0.0220	0.0153	0.0082
MLE	0.3382	0.1630	0.0866	0.0512	0.0331	0.0195	0.0078
STAR	0.0529*	0.0405*	0.0298*	0.0220*	0.0164*	0.0107*	0.0051*
MoP	0.2041	0.2022	0.1742	0.1468	0.1191	0.0878	0.0506
GEN	0.1136	0.0936	0.0763	0.0630	0.0495	0.0371	0.0265
LS	0.3260	0.4644	0.6756	1.1044	1.3388	2.1733	1.6082

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Table 5: The KS and AD test statistics for Class-IV distributions

The KS test statistics

Method	$n = 25$	$n = 50$	$n = 100$	$n = 200$	$n = 400$	$n = 1000$	$n = 5000$
QM	0.0972	0.0654	0.0458	0.0325	0.0241	0.0164	0.0096
LM	0.0959	0.0663	0.0469	0.0334	0.0248	0.0170	0.0104
MLE	0.0893*	0.0585*	0.0408*	0.0295*	0.0227*	0.0164	0.0108
STAR	0.0971	0.0646	0.0445	0.0315	0.0232	0.0158*	0.0089*
MoP	0.1150	0.0840	0.0617	0.0456	0.0352	0.0255	0.0170
GEN	0.1033	0.0726	0.0515	0.0364	0.0264	0.0177	0.0099
LS	0.1313	0.1076	0.0973	0.0903	0.0921	0.0829	0.0616

The AD test statistics

Method	$n = 25$	$n = 50$	$n = 100$	$n = 200$	$n = 400$	$n = 1000$	$n = 5000$
QM	0.1237	0.0671	0.0399	0.0257	0.0180	0.0129	0.0120
LM	0.0759	0.0542	0.0375	0.0261	0.0191	0.0139	0.0131
MLE	0.3150	0.1457	0.0747	0.0444	0.0301	0.0207	0.0198
STAR	0.0501*	0.0376*	0.0273*	0.0202*	0.0151*	0.0112*	0.0096*
MoP	0.2042	0.2125	0.1992	0.1841	0.1677	0.1578	0.1849
GEN	0.1231	0.1045	0.0876	0.0760	0.0670	0.0616	0.0757
LS	0.2843	0.4610	0.6011	1.1881	2.0087	3.5215	6.1574

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Table 6: The KS and AD test statistics for Class-V distributions

The KS test statistics							
Method	n = 25	n = 50	n = 100	n = 200	n = 400	n = 1000	n = 5000
QM	0.1015	0.0690*	0.0489*	0.0359*	0.0276*	0.0201*	0.0132*
LM	0.1004*	0.0695	0.0496	0.0362*	0.0276*	0.0199*	0.0133*
MLE	0.1059	0.0784	0.0601	0.0467	0.0380	0.0311	0.0257
STAR	0.1027	0.0706	0.0500	0.0366	0.0281	0.0204	0.0135
MoP	0.1173	0.0870	0.0649	0.0487	0.0390	0.0318	0.0284
GEN	0.1054	0.0752	0.0539	0.0387	0.0289	0.0206	0.0133
LS	0.1278	0.0936	0.0683	0.0516	0.0417	0.0280	0.0169
The AD test statistics							
Method	n = 25	n = 50	n = 100	n = 200	n = 400	n = 1000	n = 5000
QM	0.1386	0.0669	0.0373	0.0246	0.0195	0.0180	0.0260
LM	0.0740	0.0476	0.0307	0.0224	0.0189	0.0186	0.0285
MLE	0.2573	0.1223	0.0707	0.0525	0.0459	0.0503	0.0868
STAR	0.0444*	0.0338*	0.0257*	0.0205*	0.0177*	0.0172*	0.0255*
MoP	0.2279	0.2393	0.2047	0.1503	0.0873	0.0568	0.0959
GEN	0.1235	0.0997	0.0744	0.0469	0.0289	0.0214	0.0276
LS	0.3220	0.4879	0.6304	0.6560	0.6301	0.2186	0.0620

Table 7: Statistical properties of the log-return series

Currency	Mean	Standard Deviation	Skewness	Excess Kurtosis	Bera-Jarque	Heavy-Tailness
AUD	0.0002	0.0105	-1.0058	11.4628	8832.0008	2.02
BRA	0.0001	0.0102	-0.5848	13.6094	12167.0046	2.05
CAD	0.0001	0.0070	-0.1445	3.1836	666.3677	1.94
CHF	0.0002	0.0075	-0.7379	13.7407	12453.8787	2.17
EURO	0.0001	0.0068	0.1516	3.6169	859.0920	1.99
GBP	-0.0001	0.0069	-0.0575	4.3617	1241.4591	2.02
YEN	0.0002	0.0071	0.3545	4.3946	1292.1399	2.28
TL	-0.0002	0.0095	-0.5571	7.4040	3655.6662	1.87

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Table 8: Parameter estimates of the log-returns

Currency/ Method	λ_1 ($\times 10^{-4}$)	λ_2 ($\times 10^{2}$)	λ_3 ($\times 10^{-1}$)	λ_4 ($\times 10^{-1}$)
AUD				
STAR	7.146422	2.653951	-2.415666	-1.444612
QM	6.841588	2.943572	-2.910505	-2.123447
LM	6.545432	2.716194	-2.499134	-1.759301
BRA				
STAR	5.911200	3.389061	-3.489961	-2.737936
QM	5.391568	3.115934	-2.927649	-2.203763
LM	5.959094	3.283675	-3.218486	-2.455041
CAD				
STAR	3.275358	3.537780	-2.220500	-1.554907
QM	2.877329	3.079846	-1.172795	-0.621767
LM	3.034814	3.340958	-1.661063	-1.122475
CHF				
STAR	0.605826	2.862875	-0.424353	-0.996887
QM	2.564520	2.966276	-0.906730	-0.868708
LM	1.343861	2.933420	-0.830646	-1.035374
EURO				
STAR	1.652361	3.509614	-1.883594	-1.479870
QM	-0.261110	3.111438	-0.740838	-0.816887
LM	1.147255	3.323975	-1.279259	-1.141854
GBP				
STAR	1.021768	3.064824	-1.001933	-0.519422
QM	0.467986	3.275705	-1.319020	-1.017092
LM	0.810841	3.127049	-1.123513	-0.752114
YEN				
STAR	1.685737	3.126828	-0.923469	-1.198517
QM	1.470302	3.257400	-1.115776	-1.407822
LM	1.752188	3.126392	-0.934164	-1.179684
TL				
STAR	3.717446	2.743220	-2.441447	-1.233794
QM	2.442628	2.729027	-2.187030	-1.308514
LM	3.165309	2.716730	-2.243250	-1.267283

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Table 9: The AD test statistics of the currencies

Method	AUD	BRA	CAD	CHF	EURO	GBP	YEN	TL
STAR	0.318	0.870	0.435	0.249	0.478	0.472	0.351	0.451
QM	0.807	1.102	0.963	0.676	1.022	0.683	0.462	0.569
LM	0.366	0.895	0.536	0.344	0.593	0.491	0.352	0.482

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Table 10: p-values from the Kupiec test

	Significance Levels					
Currency/ Method	0.005	0.01	0.05	0.95	0.99	0.995
AUD						
STAR	0.950	0.670	0.619	0.510	0.128	0.166
QM	0.763	0.488	0.977	0.666	0.735	0.763
LM	0.950	0.670	0.704	0.977	0.560	0.950
BRA						
STAR	0.763	0.670	0.666	0.538	0.212	0.681
QM	0.681	0.560	0.585	0.619	0.670	0.681
LM	0.681	0.929	0.585	0.704	0.488	0.681
CAD						
STAR	0.130	0.032	0.666	0.751	0.212	0.130
QM	0.495	0.560	0.510	0.510	0.929	0.950
LM	0.130	0.670	0.440	0.510	0.488	0.130
CHF						
STAR	0.455	0.670	0.884	0.440	0.066	0.130
QM	0.763	0.670	0.274	0.096	0.212	0.763
LM	0.763	0.670	0.274	0.376	0.066	0.130
EURO						
STAR	0.013	0.124	0.704	0.884	0.334	0.278
QM	0.455	0.560	0.267	0.884	0.488	0.763
LM	0.130	0.868	0.376	0.884	0.488	0.763
GBP						
STAR	0.455	0.560	0.088	0.224	0.560	0.681
QM	0.763	0.735	0.145	0.145	0.929	0.495
LM	0.455	0.560	0.088	0.088	0.735	0.495
YEN						
STAR	0.284	0.735	0.931	0.619	0.334	0.681
QM	0.284	0.735	0.931	0.619	0.124	0.950
LM	0.284	0.735	0.977	0.619	0.334	0.681
TL						
STAR	0.495	0.334	0.977	0.463	0.560	0.950
QM	0.950	0.670	0.440	0.393	0.929	0.495
LM	0.763	0.670	0.510	0.393	0.929	0.495

Initialization: Choose an initial population

while termination condition not satisfied

repeat

Selection: Select individuals for reproduction

Genetic operators: Apply genetic operators to generate second generation population:

Mutation: Mutate the individuals

Crossover: Create offsprings by crossing individuals

Evaluation: Evaluate the fitness of the offspring

until sufficient offspring is produced

Select new population

endwhile

Figure 1: Steps of a genetic algorithm

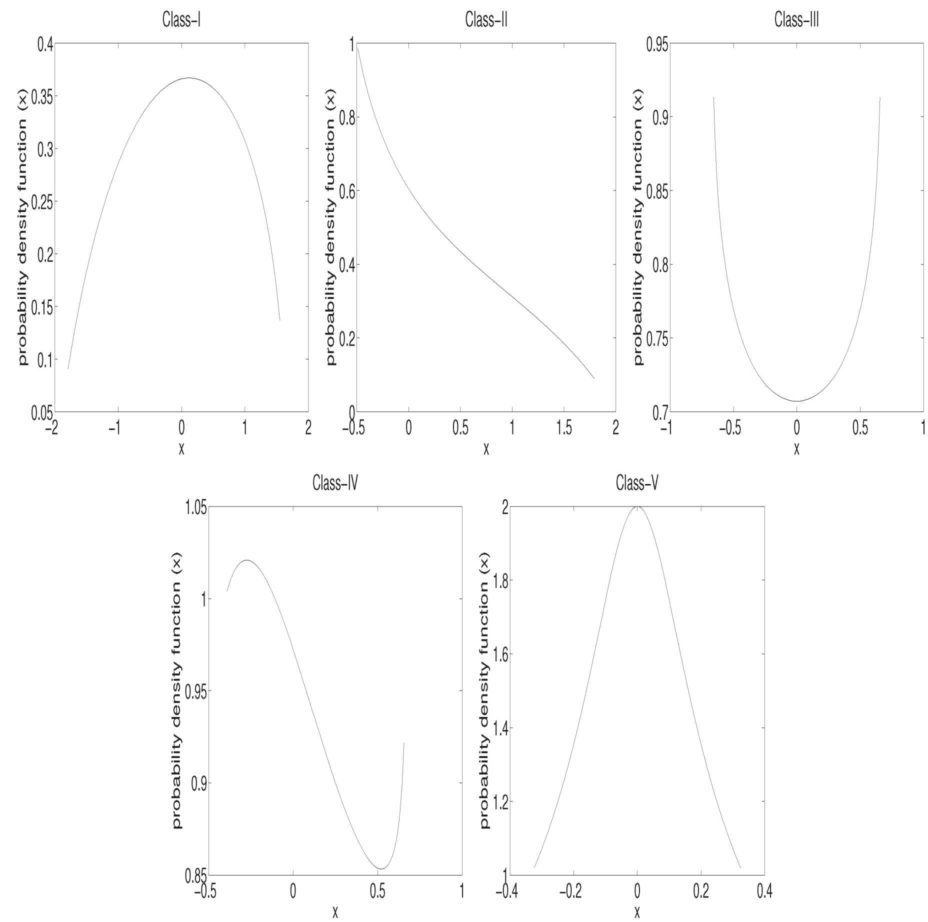

Figure 2: Pdfs of FMKL GLD distributions

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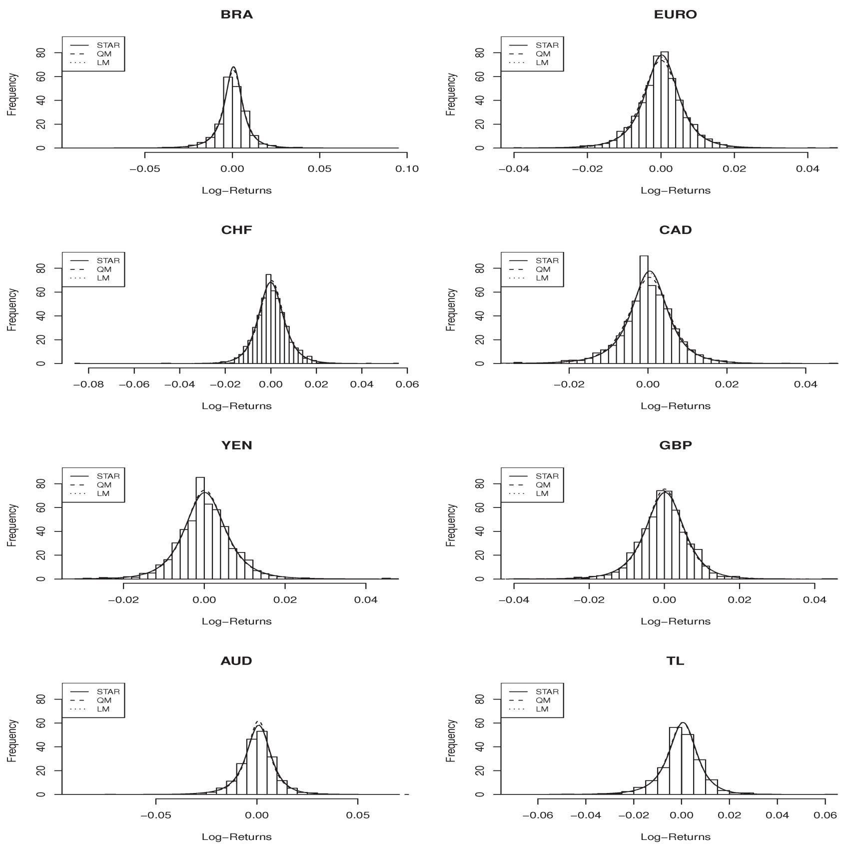

Figure 3: Histogram of log-returns versus fitted distributions with STAR, QM, and LM.