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Characterizing the generalized lambda distribution by L-moments

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Abstract

The generalized lambda distribution (GLD) is a flexible four parameter distribution with many practical applications. L-moments of the GLD can be expressed in closed form and are good alternatives for the central moments. The L-moments of the GLD up to an arbitrary order are presented, and a study of L-skewness and L-kurtosis that can be achieved by the GLD is provided. The boundaries of L-skewness and L-kurtosis are derived analytically for the symmetric GLD and calculated numerically for the GLD in general. Additionally, the contours of L-skewness and L-kurtosis are presented as functions of the GLD parameters. It is found that with an exception of the smallest values of L-kurtosis, the GLD covers all possible pairs of L-skewness and L-kurtosis and often there are two or more distributions that share the same L-skewness and the same L-kurtosis. Examples that demonstrate situations where there are four GLD members with the same L-skewness and the same L-kurtosis are presented. The estimation of the GLD parameters is studied in a simulation example where method of L-moments compares favorably to more complicated estimation methods. The results increase the knowledge on the distributions that belong to the GLD family and can be utilized in model selection and estimation.

Keywords: Skewness; Kurtosis; L-moment ratio diagram; Method of moments; Method of L-moments

1. Introduction

The generalized lambda distribution (GLD) is a four parameter distribution that has been applied to various problems where a flexible parametric model for univariate data is needed. The GLD provides fit for a large range of skewness and kurtosis values and can approximate many commonly used distributions such as normal, exponential and uniform. The applications of the GLD include, e.g. option pricing (Corrado, 2001), independent component analysis (Karvanen et al., 2002), statistical process control (Pal, 2005), analysis of fatigue of materials (Bigerelle et al., 2005), measurement technology (Lampasi et al., 2005) and generation of random variables (Ramberg and Schmeiser, 1974; Karvanen, 2003; Headrick and Mugdadib, 2006).

The price for the high flexibility is high complexity. Probability density function (pdf) or cumulative distribution function (cdf) of the GLD do not exist in closed form but the distribution is defined by the inverse distribution function

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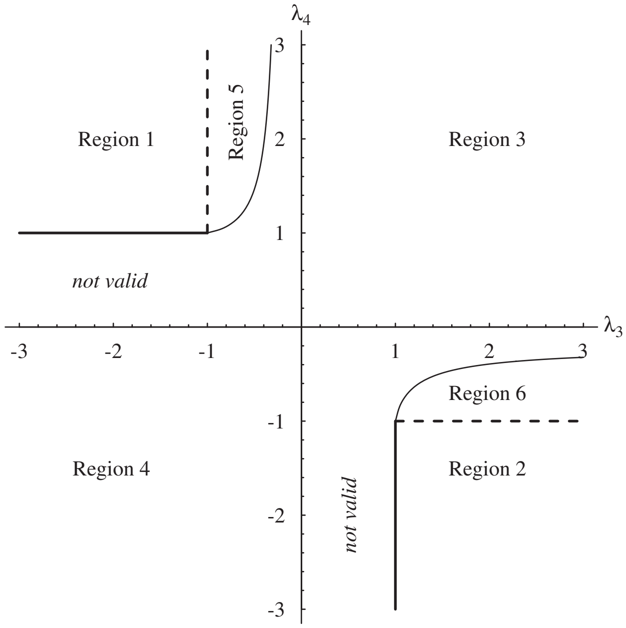

Fig. 1. GLD regions as defined in (Karian et al., 1996). The distributions are bounded on the right in regions 1 and 5, bounded on the left in regions 2 and 6, bounded in region 3 and unbounded in region 4.

(Ramberg and Schmeiser, 1974):

$$F^{-1}(u) = \lambda_1 + \frac{u^{\lambda_3} - (1-u)^{\lambda_4}}{\lambda_2}, \tag{1}$$

where $0 \leqslant u \leqslant 1$ and λ_1, λ_2, λ_3 and λ_4 are the parameters of the GLD. Eq. (1) defines a distribution if and only if (Karian et al., 1996)

$$\frac{\lambda_2}{\lambda_3 u^{\lambda_3 - 1} + \lambda_4 (1-u)^{\lambda_4 - 1}} \geqslant 0 \quad \text{for all } u \in [0, 1]. \tag{2}$$

Karian et al. (1996) divide the GLD into six regions on the basis of the parameters λ_3 and λ_4 that control skewness and kurtosis of the distribution. These regions are presented in Fig. 1. The regions have different characteristics: the distributions in region 3 are bounded whereas the distributions in region 4 are unbounded; the distributions in regions 1 and 5 are bounded on the right and the distributions in regions 2 and 6 are bounded on the left. The boundaries of the domain in each region are described, e.g. by Karian and Dudewicz (2000) and Fournier et al. (2007).

The members of the GLD family have been traditionally characterized by the central moments, especially by the central moment skewness and kurtosis (Ramberg and Schmeiser, 1974; Karian et al., 1996). L-moments (Hosking, 1990), defined as linear combinations of order statistics, are attractive alternatives for the central moments. Differently from the central moments, the L-moments of the GLD can be expressed in closed form, which allows us to derive some analytical results and makes it straightforward to perform numerical analysis. The parameters of the GLD can be estimated by method of L-moments (Karvanen et al., 2002; Asquith, 2007). Method of L-moments can be used independently or together with other estimation methods, analogously to what was done in Su (2007), Fournier et al. (2007) and Lakhany and Mausser (2000) with numerical maximum likelihood, method of moments, method of percentiles (Karian and Dudewicz, 1999), the least square method (Öztürk and Dale, 1985) and the starship method (King and MacGillivray, 1999).

Karian and Dudewicz (2003) report that method of percentiles gives superior fits compared to method of moments. They also study method of L-moments in some of their examples where the overall performance looks comparable to the overall performance of the percentile method. Method of percentiles and method of L-moments are related in sense that they both are based on order statistics. There are, however, some differences: L-moments are linear combinations of all order statistics whereas method of percentiles uses only a limited number of order statistics that need to be explicitly chosen. Fournier et al. (2007) study the choice of the order statistics for the percentile method and conclude that there is no trivial rule for choosing them. Such problems are avoided in method of L-moments.

In this paper we analyze the GLD using L-moments and consider the estimation by method of L-moments. In Section 2 we present the L-moments of the GLD up to an arbitrary order. In Section 3 we analyze the symmetric case ($\lambda_3 = \lambda_4$) and derive the boundaries of L-kurtosis analytically and in Section 4 we consider the general case ($\lambda_3 \neq \lambda_4$) and calculate the boundaries of the GLD in the terms of L-skewness and L-kurtosis using numerical methods. The boundaries are calculated separately for each GLD region. We also calculate the contours of L-skewness and L-kurtosis as functions of λ_3 and λ_4. The examples presented illustrate the multiplicity of the distributional forms of the GLD. In Section 5 we present an example on the estimation by method of L-moments and compare the results to the results by Fournier et al. (2007). Section 6 concludes the paper.

2. L-moments and the GLD

The L-moment of order r can be expressed as (Hosking, 1990)

$$L_r = \int_0^1 F^{-1}(u) P^*_{r-1}(u)\,\mathrm{d}u, \tag{3}$$

where

$$P^*_{r-1}(u) = \sum_{k=0}^{r-1} (-1)^{r-k-1} \binom{r-1}{k} \binom{r+k-1}{k} u^k \tag{4}$$

is the shifted Legendre polynomial of order $r-1$. All L-moments of a real-valued random variable exists if and only if the random variable has a finite mean and furthermore, a distribution whose mean exists, is uniquely determined by its L-moments (Hosking, 1990, 2006).

Similarly to the central moments, L_1 measures location and L_2 measures scale. The higher order L-moments are usually transformed to L-moment ratios

$$\tau_r = \frac{L_r}{L_2}, \quad r = 3, 4, \ldots . \tag{5}$$

L-skewness τ_3 is related to the asymmetry of the distribution and L-kurtosis τ_4 is related to the peakedness of the distribution. Differently from the central moment skewness and kurtosis, τ_3 and τ_4 are constrained by the conditions (Hosking, 1990; Jones, 2004)

$$-1 < \tau_3 < 1 \tag{6}$$

and

$$(5\tau_3^2 - 1)/4 \leqslant \tau_4 < 1. \tag{7}$$

Distributions are commonly characterized using an L-moment ratio diagram in which L-skewness is on the horizontal axis and L-kurtosis is on the vertical axis.

The L-moments of the GLD have been presented in the literature up to order 5 (Bergevin, 1993; Karvanen et al., 2002; Asquith, 2007). We generalize the results for an arbitrary order r. The derivation is straightforward if we first note that

$$\int_0^1 u^{\lambda_3} P^*_{r-1}(u)\,\mathrm{d}u = \sum_{k=0}^{r-1} \frac{(-1)^{r-k-1}\binom{r-1}{k}\binom{r+k-1}{k}}{k+1+\lambda_3} \tag{8}$$

and

$$\int_0^1 (1-u)^{\lambda_4} P^*_{r-1}(u)\,\mathrm{d}u = -\int_0^1 (u)^{\lambda_4} P^*_{r-1}(1-u)\,\mathrm{d}u$$
$$= (-1)^r \sum_{k=0}^{r-1} \frac{(-1)^{r-k-1}\binom{r-1}{k}\binom{r+k-1}{k}}{k+1+\lambda_4}, \tag{9}$$

where the last equality follows from the property $P^*_{r-1}(1-u) = (-1)^{r-1}P^*_{r-1}(u)$. We obtain for $r \geqslant 2$

$$\lambda_2 L_r = \sum_{k=0}^{r-1}(-1)^{r-k-1}\binom{r-1}{k}\binom{r+k-1}{k}\left(\frac{1}{k+1+\lambda_3} + \frac{(-1)^r}{k+1+\lambda_4}\right). \tag{10}$$

The explicit formulas for the first six L-moments of the GLD are

$$L_1 = \lambda_1 - \frac{1}{\lambda_2}\left(\frac{1}{1+\lambda_4} - \frac{1}{1+\lambda_3}\right), \tag{11}$$

$$L_2\lambda_2 = -\frac{1}{1+\lambda_3} + \frac{2}{2+\lambda_3} - \frac{1}{1+\lambda_4} + \frac{2}{2+\lambda_4}, \tag{12}$$

$$L_3\lambda_2 = \frac{1}{1+\lambda_3} - \frac{6}{2+\lambda_3} + \frac{6}{3+\lambda_3} - \frac{1}{1+\lambda_4} + \frac{6}{2+\lambda_4} - \frac{6}{3+\lambda_4}, \tag{13}$$

$$\begin{aligned} L_4\lambda_2 = & -\frac{1}{1+\lambda_3} + \frac{12}{2+\lambda_3} - \frac{30}{3+\lambda_3} + \frac{20}{4+\lambda_3} \\ & -\frac{1}{1+\lambda_4} + \frac{12}{2+\lambda_4} - \frac{30}{3+\lambda_4} + \frac{20}{4+\lambda_4}, \end{aligned} \tag{14}$$

$$\begin{aligned} L_5\lambda_2 = & \frac{1}{1+\lambda_3} - \frac{20}{2+\lambda_3} + \frac{90}{3+\lambda_3} - \frac{140}{4+\lambda_3} + \frac{70}{5+\lambda_3} \\ & -\frac{1}{1+\lambda_4} + \frac{20}{2+\lambda_4} - \frac{90}{3+\lambda_4} + \frac{140}{4+\lambda_4} - \frac{70}{5+\lambda_4}, \end{aligned} \tag{15}$$

and

$$\begin{aligned} L_6\lambda_2 = & -\frac{1}{1+\lambda_3} + \frac{30}{2+\lambda_3} - \frac{210}{3+\lambda_3} + \frac{560}{4+\lambda_3} - \frac{630}{5+\lambda_3} + \frac{252}{6+\lambda_3} \\ & -\frac{1}{1+\lambda_4} + \frac{30}{2+\lambda_4} - \frac{210}{3+\lambda_4} + \frac{560}{4+\lambda_4} - \frac{630}{5+\lambda_4} + \frac{252}{6+\lambda_4}. \end{aligned} \tag{16}$$

Note that the results in (Asquith, 2007) and (Bergevin, 1993) are the same but expressed in a different form.

The mean of GLD and therefore also all L-moments exist if $\lambda_3, \lambda_4 > -1$. This implies that the characterization by L-moments covers regions 3, 5 and 6 and the subset of region 4 where $\lambda_3, \lambda_4 > -1$.

3. The L-moments of the GLD in the symmetric case

In this section we consider the special case $\lambda_3 = \lambda_4$, which defines a symmetric distribution. The symmetric distributions can be bounded (in region 3) or unbounded (in region 4). Applying the condition $\lambda_3 = \lambda_4$ to Eqs. (13) and (14) we find that $\tau_3 = 0$ and

$$\tau_4 = \frac{\lambda_3^2 - 3\lambda_3 + 2}{\lambda_3^2 + 7\lambda_3 + 12}, \quad \lambda_3 = \lambda_4 > -1. \tag{17}$$

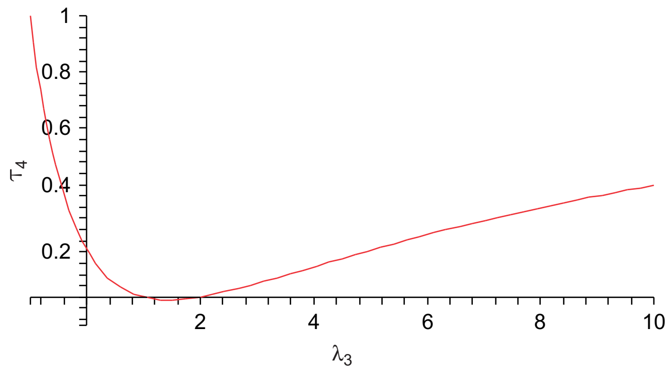

Fig. 2. L-kurtosis $\tau_4(\lambda_3)$ for symmetric distributions $\lambda_3 = \lambda_4$.

In Fig. 2 τ_4 is plotted as a function of λ_3. The value of τ_4 approaches to 1 as λ_3 approaches to infinity. The minimum of τ_4 is found to be

$$\frac{12-5\sqrt{6}}{12+5\sqrt{6}} \approx -0.0102, \tag{18}$$

and is obtained when $\lambda_3 = -1+\sqrt{6}$.

In the symmetric case it is also possible to solve λ_3 and λ_4 as functions of τ_4 (Karvanen et al., 2002)

$$\lambda_4 = \lambda_3 = \frac{3+7\tau_4 \pm \sqrt{1+98\tau_4+\tau_4^2}}{2(1-\tau_4)}. \tag{19}$$

It follows that if $\tau_4 > \frac{1}{6}$ there are symmetric GLD members available from both regions 3 and 4. If

$$\frac{12-5\sqrt{6}}{12+5\sqrt{6}} < \tau_4 \leqslant \frac{1}{6}, \tag{20}$$

there are two GLD members available from region 3, and if

$$\tau_4 < \frac{12-5\sqrt{6}}{12+5\sqrt{6}}, \tag{21}$$

there are no symmetric GLD members available.

Examples of symmetric GLDs sharing the same τ_4 are presented in Fig. 3. In Fig. 3(a) the value of τ_4 is set to be equal to the τ_4 of the normal distribution. Both GLDs are from region 3 and are bounded. The GLD for the more peaked distribution in Fig. 3(a) has $\tau_6 \approx 0.0004$ whereas the other GLD has $\tau_6 \approx 0.043$, which is very near to the τ_6 of the normal distribution. For comparison the pdf of the normal distribution is plotted (dotted line). The differences with the normal distribution are appreciable only in the far tails. In Fig. 3(b) $\tau_4 = 0.25$, which means that Eq. (17) has one positive and one negative root. The GLD with $\tau_6 \approx 0.016$ and the higher peak is from region 3 and has a bounded domain. The other GLD with $\tau_6 \approx 0.121$ is from region 4 and has unbounded domain.

In addition to the symmetric case we may consider some other special cases. If $\lambda_3 = 0$ we obtain

$$\tau_3(\lambda_4) = \frac{1-\lambda_4}{\lambda_4+3} \tag{22}$$

and

$$\tau_4(\lambda_4) = \frac{\lambda_4^2-3\lambda_4+2}{\lambda_4^2+7\lambda_4+12}. \tag{23}$$

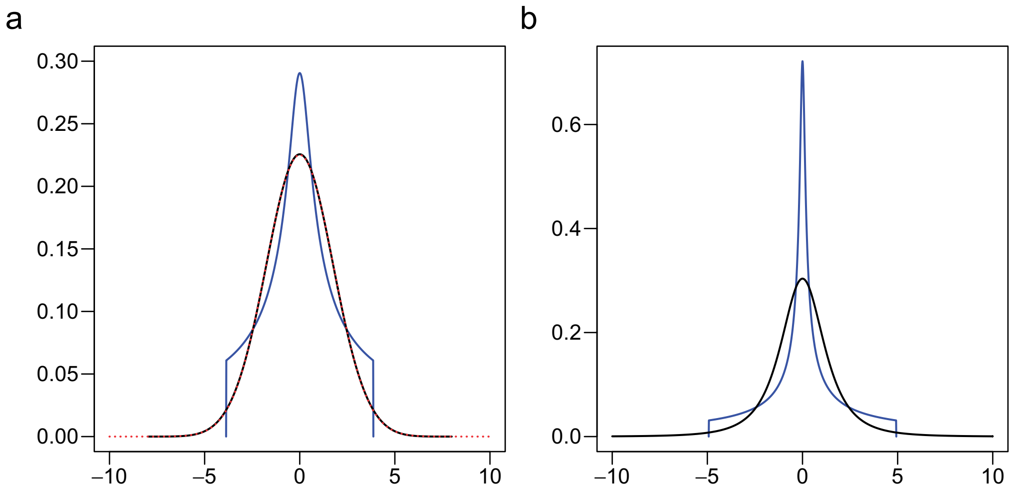

Fig. 3. Examples of symmetric GLDs sharing the same τ_4: (a) $\tau_4 \approx 0.12260$ (τ_4 of normal distribution) and (b) $\tau_4 = 0.25$. The pdfs of the GLDs are plotted with solid line and the pdf of normal distribution is plotted with dotted line. All distributions have $\lambda_1 = 0$, $\lambda_2 = 1$ and $\tau_3 = 0$.

If $\lambda_4 = 0$ we obtain

$$\tau_3(\lambda_3) = \frac{\lambda_3 - 1}{\lambda_3 + 3} \tag{24}$$

and

$$\tau_4(\lambda_3) = \frac{\lambda_3^2 - 3\lambda_3 + 2}{\lambda_3^2 + 7\lambda_3 + 12}. \tag{25}$$

Interestingly, Eqs. (23) and (25) are identical to Eq. (17), which indicates that the (λ_3, λ_4) pairs (λ, λ), $(\lambda, 0)$ and $(0, \lambda)$ lead to the same value of τ_4.

4. Boundaries of the L-moment ratios of the GLD

In the general case it is difficult to derive analytical results and therefore we resort to numerical methods. Because the L-moments of the GLD are available in closed form, we may choose a straightforward approach and calculate the values of τ_3 and τ_4 for a large number of (λ_3, λ_4) pairs. Naturally, the values of (λ_3, λ_4) need to be chosen such a way that the essential properties of the GLD are revealed. In region 4, we use a grid of one million points where both λ_3 and λ_4 are equally spaced in the interval $[-1 + 10^{-10}, 10^{-10}]$. In region 3 we use an unequally spaced grid of 4 104 676 points. The grid is dense near the origin (λ_3 and λ_4 have minimum 10^{-10}) and sparse for large values (λ_3 and λ_4 have maximum 10^{12}). In region 5, λ_3 is equally spaced in the interval $[-1 + 10^{-10}, 0]$ and λ_4 is unequally spaced starting from -1. The grid contains 1 362 429 valid points. The calculations thus result in a large number of (τ_3, τ_4) pairs. The results for region 6 can be obtained from the results for region 5 by swapping λ_3 and λ_4. The problem is to find the boundaries of the GLD in the (τ_3, τ_4) space on the basis of these data. Let $(\tau_3[i], \tau_4[j])$ be the image of the grid point $(\lambda_3[i], \lambda_4[j])$, where i and j are the grid indexes. If the point $(\tau_3[i], \tau_4[j])$ is not located inside the polygon defined by points $(\tau_3[i-1], \tau_4[j])$, $(\tau_3[i], \tau_4[j+1])$, $(\tau_3[i+1], \tau_4[j])$ and $(\tau_3[i], \tau_4[j-1])$ it is considered to be a potential boundary point.The images of the points that are located on the boundary of the grid in the (λ_3, λ_4) space are also potential boundary points in the (τ_3, τ_4) space. The actual boundaries are found combining the potential boundaries and finally it is checked that all (τ_3, τ_4) pairs are located inside the boundaries. The calculations are made using R (R Development Core Team, 2006) and the contributed R packages `PBSmapping` (Schnute et al., 2006) and `gld` (King, 2006) are utilized.

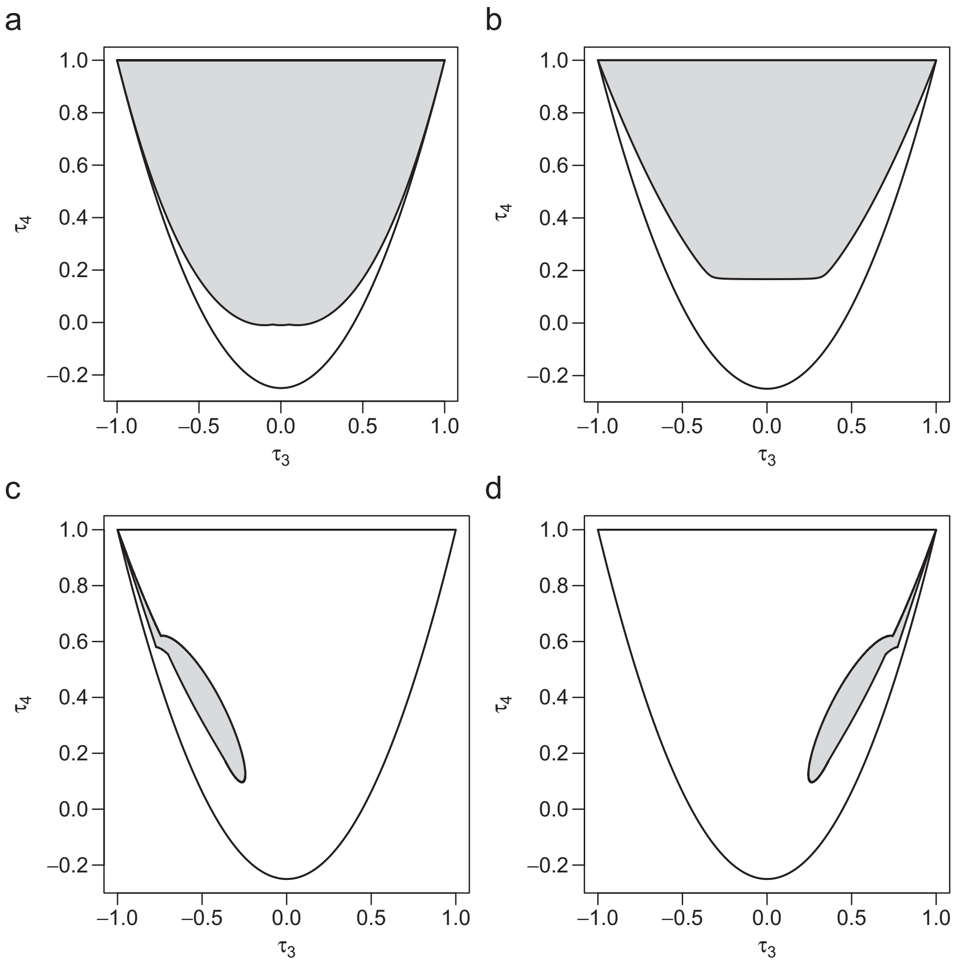

Fig. 4. Boundaries of the GLD in the (τ_3, τ_4) space: (a) region 3; (b) region 4; (c) region 5; (d) region 6. The shading shows the values of L-skewness and L-kurtosis that are achievable by the GLD. The outer limits present the boundaries for all distributions.

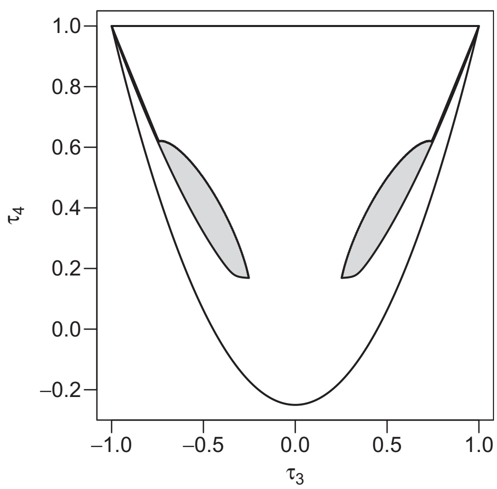

Fig. 5. The (τ_3, τ_4) area where there exist GLD members from regions 3, 4 and 5 (negative L-skewness), or from regions 3, 4 and 6 (positive L-skewness).

Table 1
Four different GLD members with $L_1 = 0$, $L_2 = 1$, $\tau_3 = 0.4$ and $\tau_4 = 0.25$

Region	λ_1	λ_2	λ_3	λ_4	τ_5	τ_6
3(a)	5.322	0.138	21.526	0.286	0.163	0.103
3(b)	−1.168	0.124	5.417	92.608	−0.029	0.067
4	−1.62	−0.157	−0.014	−0.212	0.158	0.121
6	−7.04	−0.194	11.905	−0.306	0.204	0.180

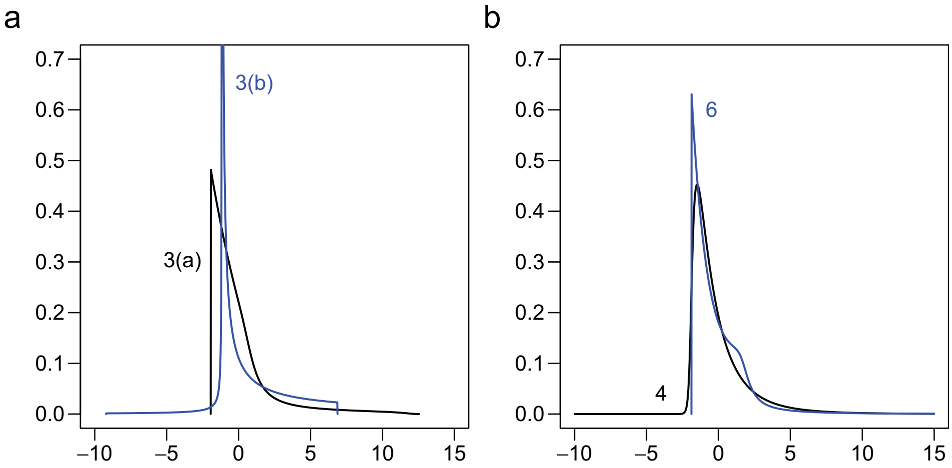

Fig. 6. Pdfs of four GLDs with $\tau_3 = 0.4$ and $\tau_4 = 0.25$: (a) from regions 3 and (b) from regions 4 and 6. The parameters of the distributions are presented in Table 1.

The boundaries are shown in Fig. 4. It can be seen that region 3 covers most of the (τ_3, τ_4) space; only the smallest values of τ_4 are unattainable. Region 4 largely overlaps with region 3 but cannot achieve τ_4 smaller than $\frac{1}{6}$. Region 5 and its counterpart, region 6, cover a rather small area of the (τ_3, τ_4) space.

Fig. 5 shows the (τ_3, τ_4) area where there exist GLD members from regions 3, 4 and 5, or from regions 3, 4 and 6. To illustrate these distributions we fix $\lambda_1 = 0$, $\lambda_2 = 1$, $\tau_3 = 0.4$ and $\tau_4 = 0.25$ and seek for the solutions in regions 3, 4 and 6. Table 1and Fig. 6 present the four distributions that are found. Although the first four L-moments of the distributions are the same, the differences of the pdfs are clearly visible. Distribution 3(a) has bounded domain with sharp left limit. Distribution 3(b) is also bounded but characterized by the high peak. Distribution 4 has unbounded domain but is otherwise almost similar to distribution 3(a). Distribution 6 is bounded from the left and characterized by a minor 'peak' around $x = 2$.

The contours of τ_3 and τ_4 in different GLD regions are shown in Figs. 7–9. In region 3 the contours are rather complicated. From Fig. 7(a) and 7(c) it can be seen that there are three separate areas where τ_3 has negative values. It is also seen that besides the line $\lambda_3 = \lambda_4$, there are two curves where $\tau_3 = 0$. Naturally, the distributions defined by the curves are not symmetric; they just have zero L-skewness.

5. Estimation by method of L-moments

In method of L-moments, we first calculate L-moments $\hat{L}_1$, $\hat{L}_2$, $\hat{\tau}_3$ and $\hat{\tau}_4$ from the data. Then we numerically find parameters $\hat{\lambda}_3$ and $\hat{\lambda}_4$ that minimize some objective function that measures the distance between the L-skewness and L-kurtosis of the data $(\hat{\tau}_3, \hat{\tau}_4)$ and the L-skewness and L-kurtosis of the estimated model $(\tau_3(\hat{\lambda}_3, \hat{\lambda}_4), \tau_4(\hat{\lambda}_3, \hat{\lambda}_4))$. After that estimates $\hat{\lambda}_1$ and $\hat{\lambda}_2$ can be solved from Eqs. (11) and (12). A natural objective function is the sum of squared

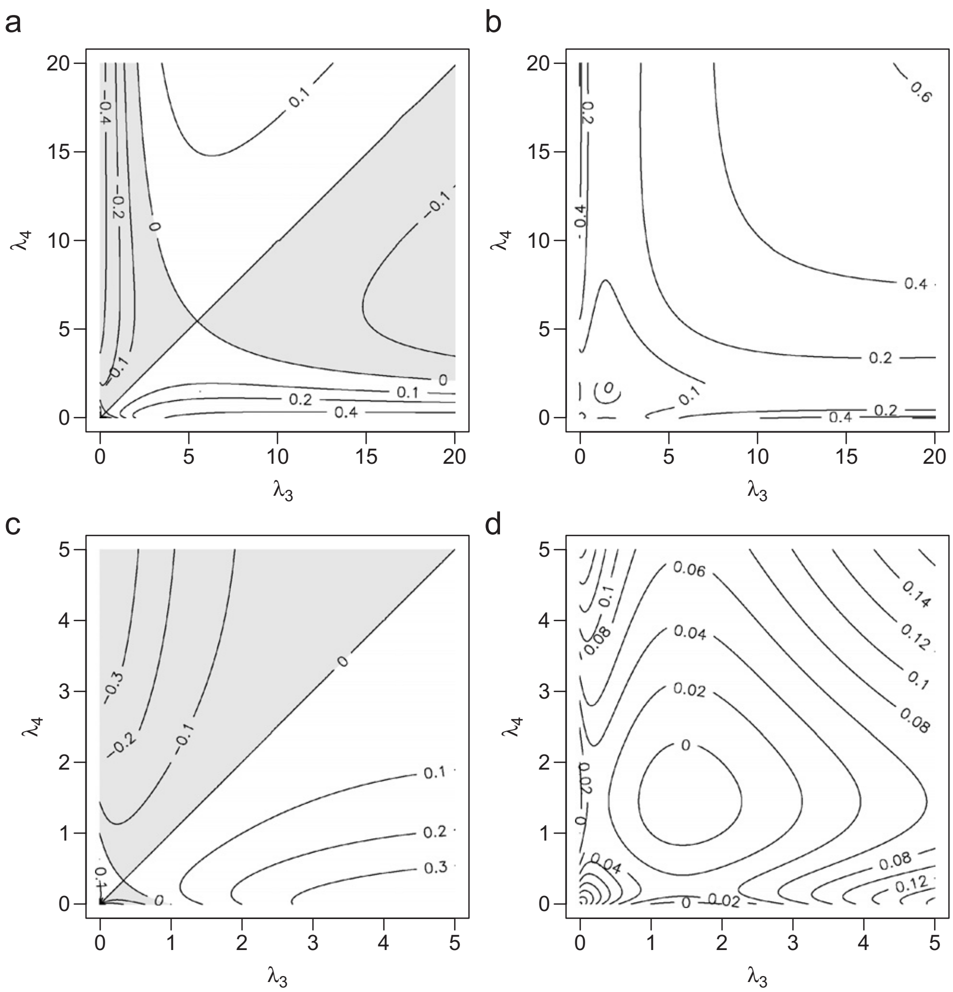

Fig. 7. Contours of τ_3 and τ_4 in region 3: (a) contours of L-skewness τ_3; (b) contours of L-kurtosis τ_4; (c) contours of L-skewness τ_3 (zoomed); and (d) contours of L-kurtosis τ_4 (zoomed). The shading indicates negative values of τ_3.

distances

$$(\hat{\tau}_3 - \tau_3(\hat{\lambda}_3, \hat{\lambda}_4))^2 + (\hat{\tau}_4 - \tau_4(\hat{\lambda}_3, \hat{\lambda}_4))^2, \tag{26}$$

which was used also by Asquith (2007).

We study the same simulation example that was studied by Fournier et al. (2007). They compare five estimators in situation where a sample of 1000 observations is generated from $GLD(0, 0.19, 0.14, 0.14)$, which is a symmetric distribution close to the standard normal distribution. We apply method of L-moments to the same problem and present the results in a comparable form. Before carrying out the simulations, we analyze the example using the results given in the preceding sections. Because $\lambda_3 = \lambda_4 = 0.14$, $GLD(0, 0.19, 0.14, 0.14)$ is a symmetric distribution from region 3. The theoretical L-moments $L_1 = 0$, $L_2 \approx 0.60407$, $\tau_3 = 0$ and $\tau_4 \approx 0.12305$ of $GLD(0, 0.19, 0.14, 0.14)$ are obtained from Eqs. (11)–(14). Because $\tau_3 = 0$ we may apply Eq. (19) that gives the 'correct' solution $\lambda_3 = \lambda_4 = 0.14$ and another solution $\lambda_3 = \lambda_4 \approx 4.26316$ also from region 3. By inspecting the contours in Fig. 7 we find that there exist also two asymmetric GLDs from region 3 with $\tau_3 = 0$ and $\tau_4 \approx 0.12305$. The numerical minimization of objective function (26) with suitable initial values gives solutions ($\lambda_3 \approx 1.98$, $\lambda_4 \approx 22.59$) and ($\lambda_3 \approx 22.59$, $\lambda_4 \approx 1.98$). Fig. 10 shows the contours of objective function (26) when $\hat{\tau}_3 = 0$ and $\hat{\tau}_4 \approx 0.12305$. It can be clearly seen that the achieved numerical solution depends on the initial values of λ_3 and λ_4.

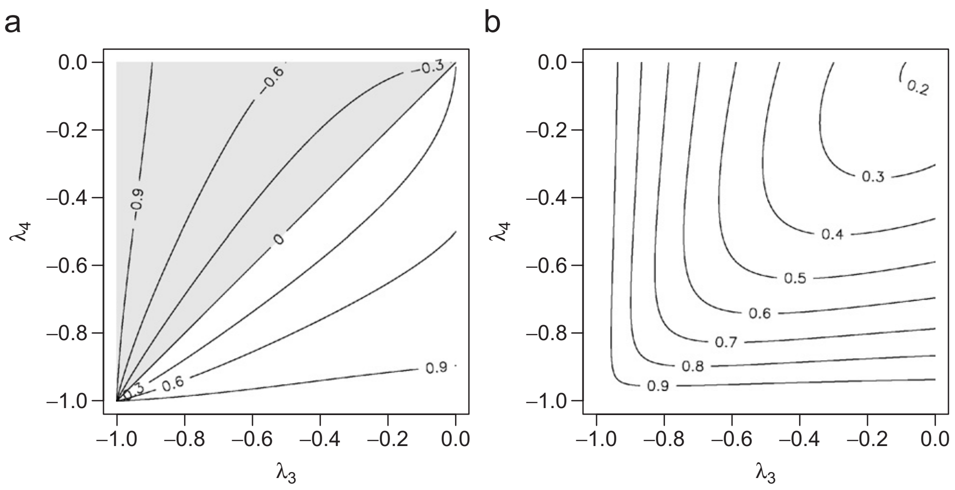

Fig. 8. Contours of τ_3 and τ_4 in region 4: (a) contours of L-skewness τ_3 and (b) contours of L-kurtosis τ_4. The shading indicates negative values of τ_3.

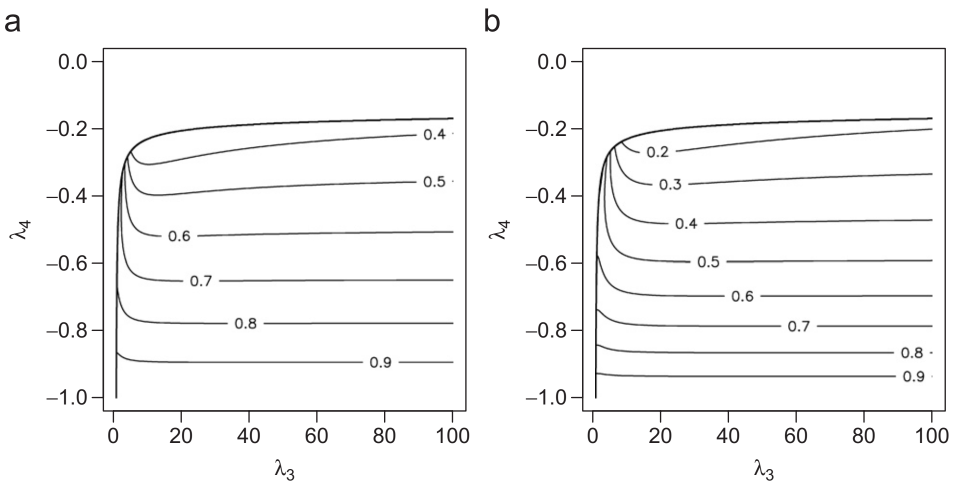

Fig. 9. Contours of τ_3 and τ_4 in region 6: (a) contours of L-skewness τ_3 and (b) contours of L-kurtosis τ_4.

In the simulation we choose a strategy where $\hat{\tau}_4$ computed from the data and the two solutions of Eq. (19) are tried as initial values of λ_3 and λ_4. The numerical optimization of the objective function (26) is carried out by the Nelder–Mead method (Nelder and Mead, 1965) in R (R Development Core Team, 2006) and the computer used for the simulation is comparable to the computer used by Fournier et al. (2007). The results from 10 000 data sets of 1000 observations generated from $GLD(0, 0.19, 0.14, 0.14)$ are presented in Table 2. The first column of the table reports the L-moment estimates corresponding to the smaller initial solution which leads to the estimates corresponding to the generating GLD. The last three columns are copied from (Fournier et al., 2007) and are based on 100 data sets of 1000 observations generated from the same GLD. The L-moment estimates of λ_1, λ_3 and λ_4 are unbiased and λ_2 is slightly underestimated. Method of L-moments is about 100 times faster than the Fournier et al. method and over 7000 times faster than the starship methods. In terms of the Kolmogorov–Smirnov statistics E_{KS} between the observed data and the fitted model, method of L-moments gave slightly better fit than the starship methods with the used settings.

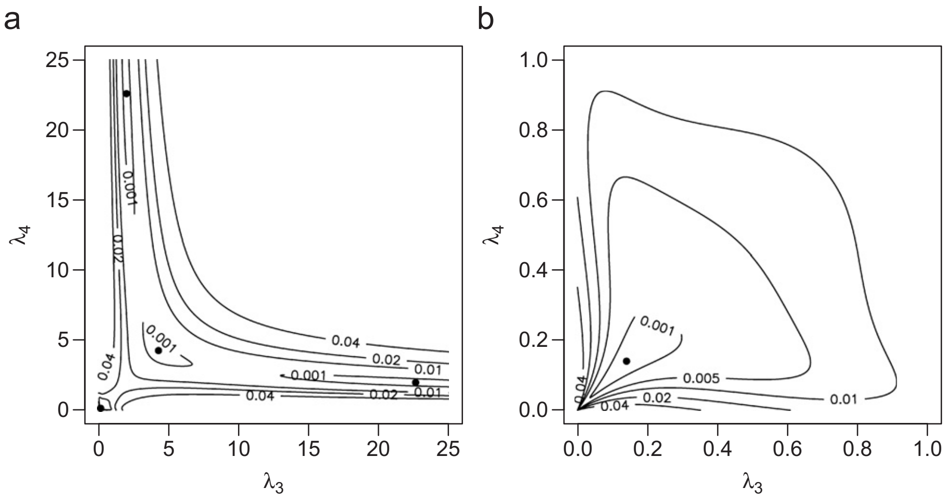

Fig. 10. Contours of objective function (26) when $\hat{\tau}_3 = 0$ and $\hat{\tau}_4 \approx 0.12305$: (a) Contours and the four solutions and (b) contours in the neighborhood of $GLD(0, 0.19, 0.14, 0.14)$. The dots show the two symmetric and the two asymmetric solutions.

Table 2
Comparison of estimation methods

Quantity		L-moments	Fournier et al.	Starship KS	Starship AD
λ_1	Mean	0.00153	−0.00030	0.25010	0.00720
	Std error	0.10287	0.05010	0.10000	0.07300
λ_2	Mean	0.18795	0.19960	0.20020	0.19400
	Std error	0.03625	0.05650	0.04760	0.03020
λ_3	Mean	0.14012	0.15110	0.15530	0.14600
	Std error	0.03602	0.05150	0.04630	0.03170
λ_4	Mean	0.13981	0.15060	0.14750	0.14350
	Std error	0.03573	0.05030	0.04510	0.02860
Time (s)	Mean	0.02767	3.19010	206.99000	213.47000
	Std error	0.00575	0.15020	40.42010	42.15210
E_{KS}	Mean	0.01597	0.01850	0.01680	0.01856
	Std error	0.00319	0.00430	0.00270	0.00360

The mean and the estimated standard error of the L-moment estimators are computed from 1000 data sets generated from $GLD(0, 0.19, 0.14, 0.14)$. The results for the other methods are based 100 generated data sets from the same GLD and are copied from Fournier et al. (2007). Fournier et al. method is based on the percentile method and the minimization of the Kolmogorov–Smirnov distance in the (λ_3, λ_4) space. Starship KS and starship AD methods correspond to the starship method (King and MacGillivray, 1999) using the Kolmogorov–Smirnov or the Anderson–Darling distance, respectively.

6. Conclusion

We have presented the L-moments of the GLD up to an arbitrary order and studied which values of L-skewness τ_3 and L-kurtosis τ_4 can be achieved by the GLD. For the symmetric case the boundaries were derived analytically and in the general case numerical methods were used. It was found that with an exception of the smallest values of τ_4, the GLD covers all possible (τ_3, τ_4) pairs and often there are two or more distributions sharing the same τ_3 and τ_4. The example in Section 4 demonstrates a situation where there are four GLD members sharing the same τ_3 and τ_4.

We argue that L-moments are natural descriptive statistics for the GLD because they, differently from the central moments, can be expressed in closed form. The existence of the closed form presentation of the GLD L-moments follows from the fact that the inverse distribution function of the GLD is available in closed form and can be integrated. The relation between L-moments and distributions defined by the inverse distribution function is not restricted only to the GLD (Karvanen, 2006). The existence of the first four central moments of the GLD requires that $\lambda_3, \lambda_4 > -\frac{1}{4}$ whereas the existence of L-moments of any order requires only that $\lambda_3, \lambda_4 > -1$. Thus, using the L-moments we can characterize wider subset of the GLD than using the central moments. Asquith (2007) concluded that even wider subset of the GLD could be characterized using the trimmed L-moments (Elamir and Seheult, 2003).

The results presented in this paper can be utilized in model selection and estimation. The characterization by L-moments gives an insight into which τ_3 and τ_4 are available in each GLD region. For instance, there are symmetric GLD members available from both region 3 and 4, if $\tau_4 > \frac{1}{6}$. This kind of information is useful when making the decision whether the GLD is a potential model for certain data. The results are also useful in the estimation of the GLD parameters. The parameters can be estimated directly by method of L-methods or the L-moment estimates can be used as starting values for other estimation methods. In both cases, we can use the L-moment ratio boundaries to specify the potential GLD regions where we should search for the parameter estimates. The choice between alternative solutions can be based on the type of the domain (bounded/unbounded) or some other additional criterion such as the Kolmogorov–Smirnov statistic or L-moment ratios τ_5 and τ_6.

In the simulation example, method of L-moments compared favorably to more complicated estimation methods. Estimation by method of L-moments was the fastest in the comparison and the estimates were unbiased or nearly unbiased. The goodness of fit measured by the Kolmogorov–Smirnov statistic was the same or better than with the alternative estimation methods. More simulations are needed to find out how general these results are.

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