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Impact of family education on online gambling addiction: An age-structured modelling approach

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A B S T R A C T

The rapid growth of the gambling market through online platforms has raised significant concern about its addiction, especially among teenagers. However, our knowledge of how this addiction can affect different age groups and the role an educated family background can play in controlling it is very limited. Therefore, the main objective of this study is to see the impact of family education on individuals of different age groups. We have considered an age-structured online gambling addiction model that incorporates family education and introduces two new classes of individuals called gambling operators and convicted individuals. The semigroup theory approach is used to study the well-posedness of the system. A threshold parameter $\mathcal{R}_0$ is derived to determine the existence and stability of addiction-free and addiction-predominant equilibria. To explore the optimal control strategy, three different control measures, the promotion of family education among uneducated susceptible individuals, the conviction of gambling operators and the rehabilitation of gamblers, are taken. Using sensitivity equations and the adjoint system, the optimality conditions are derived, and the forward-backwards sweep (FBS) method is applied to solve the control problem numerically. Simulations demonstrate that promoting family education is the most effective and superior strategy among the other two measures to control the spread of online gambling addiction.

1. Introduction

Over time, technology has advanced significantly. Although technological advancement has largely benefited mankind, there are some negative aspects as well. The modern tools and devices we use were designed to make our lives more pleasant, yet they are deteriorating humankind's way of life because most of us are now dependent on these gadgets [33]. For instance, as cell phones, gaming computers, and the Internet have advanced, online gambling has emerged as a new kind of entertainment for young people in today's world [9]. The main factors fueling this expansion are the rising Internet adoption rates [1] and the number

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of nations beginning to regulate online gambling marketplaces. Concerns about the consequences of online gambling have grown in comparison to traditional forms of gambling, as it has more engagement, greater concealment, and is more difficult to regulate than traditional gambling [3,17].

In 2003, Griffiths [10] made the first attempt to assess the link between gambling behavior and the Internet. He identified a number of factors that make online gambling potentially addictive, including accessibility, anonymity, cost, ease, interactivity, and disinhibition. Orford [25] argued that online gambling may also be a dangerous medium because of its structural features, which include how winnings are paid out, how quickly games are being played, characteristics that allow players to bet up to a certain amount and the use of stimulating light to create an ambience that may exacerbate addiction. For more literature on the ill effects and ways to prevent gambling, see [29], [16], [7], [15], [23].

Most online games provide in-app purchases with prizes called "loot boxes," which resemble treasure chests. Since players cannot always see what is inside the boxes before clicking to buy, "loot boxes," purchased with real money, are considered gambling because chance plays a major role in gaming. Online casinos and bookmakers, bingo and slot sites and apps, online lottery ticket or scratch card sales, online sports betting, and games using virtual currencies or real-world goods are examples of online games that encourage real-money gambling. Also, there is a particular allure to playing online games, and those with poor willpower tend to get sucked in and play them excessively. In fact, the World Health Organization acknowledged addiction to online gaming as a real disorder, called a gaming disorder, predominantly online, now frequently mentioned as the Internet Gaming Disorder (IGD) [5].

Mathematical models play a very important role in studying the transmission dynamics of diseases. See, for example, [24,32,36,37]. Consequently, involvement in online gaming is also contagious and there is a parallel between the transmission mechanism of infectious diseases and the behavior of indulging in online gaming. Hence, the authors try to study the dynamic behavior of online game addiction using mathematical models. Hiromi Seno [28] considered a mathematical model for online gaming addiction that links the addiction to online gambling with an increase in the consumption of the Internet. Human behaviors are recognized to spread through social interaction these days [22]. Thus, Kong et al. [18] proposed an SHGD (susceptible–hesitator–gambler–disclaimer) gambling model, in which social network topology, anti-gambling laws, and individuals' psychological aspects are considered.

Another major advantage of mathematical models is that they provide a framework for understanding and manipulating systems to achieve desired results effectively; hence, mathematical models are essential for studying optimal control. Very few authors [11], [20], [12] attempt to study optimal control problems in online gaming addiction models under different scenarios. Li and Guo [21] studied the optimal control problem for online gaming addiction under the media effect. They showed that media can play positive as well as negative roles in the spread of online gambling addiction. Another factor that can affect gambling addiction is family education. Viriyapong and Sookpiam [30] consider a mathematical model investigating how education campaigns and family comprehension affect online game addiction.

One of the other major factors that can escalate the problem of online gambling is the involvement of gambling operators in this process. Gambling operators are businesses or groups that offer services or platforms so that people can engage in different kinds of gaming activities. These operators act as intermediaries for players and gambling activities. Salonen et al. [27] studied the impact of gambling operators on gambling. However, these operators must negotiate a complicated legal environment with several regulations designed to protect customers, maintain industry integrity, and minimize the harm from gambling. Operators must abide by these laws in order to conduct business in a legal manner and maintain the confidence of their players and regulators. Not obeying those laws properly has repercussions; they might get sacked for their acts or get convicted.

Age plays a very critical role in the transmission dynamics of some diseases. Therefore, it is evident to consider mathematical models that incorporate the age of an individual as well. For example, Yang et al. [31] considered an age-structured model to study the dynamics of foot and mouth disease. The age of an

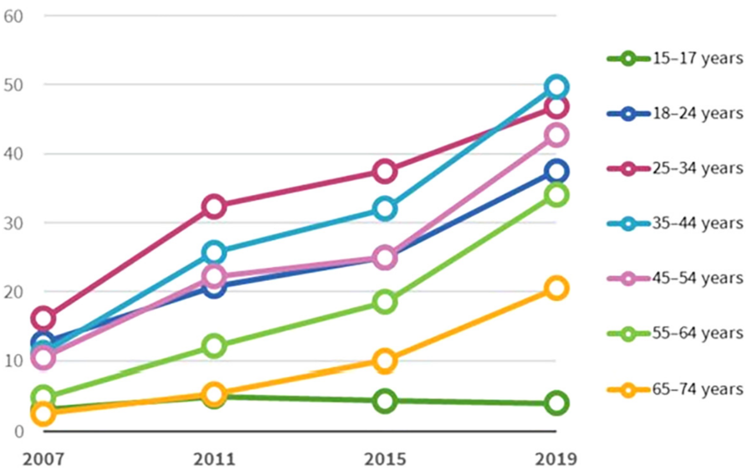


Fig. 1. Prevalence of online gambling, respondents aged between 15 to 74 (by %age)[27].

individual also plays a very important role in online gambling addiction. Many factors can contribute to why adults are more addicted to online gambling. They may be using it to cope with stress or any negative emotions. Another reason may be peer pressure or the acceptance of gambling in their surroundings or social networks. The majority of jurisdictions have strict rules regarding the minimum age to gamble. These rules are designed to protect young people from the possible negative effects of gambling, such as addiction and monetary loss. Therefore, online gambling platforms are legally required to confirm the age of their users to guarantee their adherence to these rules. However, various techniques, such as using a VPN or adult identities, may help a minor tackle this age verification process. Gómez et al. [8] showed that minors not only easily become addicted to online gambling, but also engage in active sexting, cyberbullying, or online communication with strangers. In their scientific report, Salonen et al. [27] studied how participation in online gambling activities varies with the age of an individual (shown in Fig. 1).

Kaihong Zhao [34,35] studied the oscillatory behavior of this addiction by introducing the non-linear diffusion into the gambling addiction model. Guo and Li [14] considered a fraction model of online game addiction. In [13], authors considered an online game addiction model with family education as a measure of control. To the best of our knowledge, no attempt has been made to study the impact of family education on different age groups to control gambling addiction. Further, the previous literature does not discuss the factors that increase this problem among youth. Drawing from the aforementioned literature and investigating real-world issues, we have developed a novel model for online gambling addiction. The main innovation and primary distinctions between this work and earlier work are as follows.

(i) Observing the impacts of age on the gambling process naturally motivates us to consider an online gambling model that also incorporates age structure, making it more realistic.
(ii) We have considered a new class of individuals called the gambling operator class, which escalates this problem, and they are the means of all gambling activities that are being carried out.
(iii) Inclusion of the gambling operators class in our model further inspires us to take the convicted class as well so that all gambling activities are carried out in a peaceful and legal way.
(iv) Apart from the influence of an educated family background on the problem of gambling, we have adopted two more control measures to reduce the problem. One is the conviction of gambling operators, and the other is the rehabilitation of gamblers.

From an analytical point of view, we have shown the existence and uniqueness of the solution of our model using the semigroup theory of linear operators. We show that there exist two equilibrium points. The first is an addiction-free equilibrium, and the other is an addiction-predominant equilibrium. The stability of

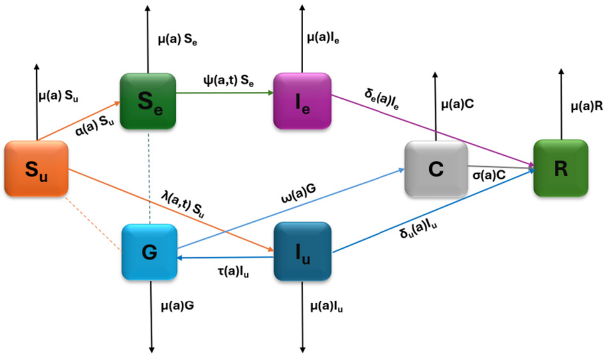


Fig. 2. Flow diagram of age-structured online gambling model.

these equilibrium points is also discussed using the threshold parameter $\mathcal{R}_0$. By including all the control measures defined above, we have studied the optimal control problem. Using the adjoint system and the sensitivity equations, the explicit form of the control variables is also obtained. In addition, we have solved our model numerically to validate our theoretical findings.

The layout of the paper is as follows. In total, there are six main sections in this paper. The formulation of the model and various assumptions to make the model more realistic are given in Section 2. In addition, the existence and uniqueness of the mild solution are also shown. In this section, we have also shown the existence of a global mild solution. Section 3 is devoted to the stability analysis of the two equilibria: addiction-free and addiction-predominant equilibrium. In Section 4, we posed and studied our optimal control problem. Furthermore, numerical simulations are performed in Section 5. Section 6 provides a summary of the findings along with some recommendations.

2. Model formulation

We have classified the total population into seven different classes.

1. **Uneducated susceptible** class S_u, which represents all susceptible individuals who lack family education, can teach them about the demerits of online gambling.
2. **Educated susceptible** class S_e represents all susceptible individuals with an educated family background.
3. **Gambling Operator** class G represents the individuals who affiliate other people to participate in online gambling.
4. **Occasional Gambler** class I_e represents individuals who have family education and, therefore, are less involved in online gambling.
5. **Addicted Gambler** class I_u, which represents all those people who are addicted to online gambling due to the lack of family education.
6. **Convicted** class C represents the individuals who got legal action by breaking some law.
7. **Rehabilitation** class R, the individuals who are no longer addicted to online gambling.

The size of the total population at time t with age a is given by $P(a,t)$ and is the sum of all individuals in the others compartments with age a and at time t, i.e.,

$$P(a,t) = S_u(a,t) + S_e(a,t) + G(a,t) + I_e(a,t) + I_u(a,t) + C(a,t) + R(a,t).$$

Further, the following presumptions are used to represent the dynamics among these classes:

1. Gambling operators are the main and the only reason for introducing gambling into the susceptible environment and their interaction with educated and uneducated susceptible individuals can turn them into occasional and addicted gamblers, respectively.
2. Only addicted gamblers can become gambling operators. This makes sense because they do not get a family education. The other reason is that, because they are addicted to it, they may seek career options in the gambling industry.

These assumptions and flow diagram in Fig. 2 depict the following system of PDEs that describes the spread of gambling addiction

$$\begin{aligned}
\frac{\partial S_u(a,t)}{\partial t} + \frac{\partial S_u(a,t)}{\partial a} &= -\lambda(a,t)S_u(a,t) - \alpha(a)S_u(a,t) - \mu(a)S_u(a,t),\\
\frac{\partial S_e(a,t)}{\partial t} + \frac{\partial S_e(a,t)}{\partial a} &= \alpha(a)S_u(a,t) - \psi(a,t)S_e(a,t) - \mu(a)S_e(a,t),\\
\frac{\partial G(a,t)}{\partial t} + \frac{\partial G(a,t)}{\partial a} &= \tau(a)I_u(a,t) - \omega(a)G(a,t) - \mu(a)G(a,t),\\
\frac{\partial I_e(a,t)}{\partial t} + \frac{\partial I_e(a,t)}{\partial a} &= \psi(a,t)S_e(a,t) - \delta_e(a)I_e(a,t) - \mu(a)I_e(a,t),\\
\frac{\partial I_u(a,t)}{\partial t} + \frac{\partial I_u(a,t)}{\partial a} &= \lambda(a,t)S_u(a,t) - \delta_u(a)I_u(a,t) - \tau(a)I_u(a,t) - \mu(a)I_u(a,t),\\
\frac{\partial C(a,t)}{\partial t} + \frac{\partial C(a,t)}{\partial a} &= \omega(a)G(a,t) - \sigma(a)C(a,t) - \mu(a)C(a,t),\\
\frac{\partial R(a,t)}{\partial t} + \frac{\partial R(a,t)}{\partial a} &= \sigma(a)C(a,t) + \delta_e(a)I_e(a,t) + \delta_u(a)I_u(a,t) - \mu(a)R(a,t),\\
&S_u(0,t) = \Lambda,\ S_e(0,t) = G(0,t) = I_e(0,t) = I_u(0,t) = C(0,t) = R(0,t) = 0,\\
&S_u(a,0) = S_{u_0}(a),\ S_e(a,0) = S_{e_0}(a),\ G(a,0) = G_0(a),\ I_e(a,0) = I_{e_0}(a),\\
&I_u(a,0) = I_{u_0}(a),\ C(a,0) = C_0(a),\ R(a,0) = R_0(a),
\end{aligned} \tag{2.1}$$

where

$$\lambda(a,t) = \int_0^{A_m} r_u(a,b)\frac{G(b,t)}{P(b,t)}db \quad \text{and} \quad \psi(a,t) = \int_0^{A_m} r_e(a,b)\frac{G(b,t)}{P(b,t)}db.$$

Here $\Lambda > 0$ is a positive real number. $S_{u_0}(a)$, $S_{e_0}(a)$, $G_0(a)$, $I_{e_0}(a)$, $I_{u_0}(a)$, $C_0(a)$ and $R_0(a)$ are non negative continuous functions. The age-dependent transmission coefficients $r_u(a,b)$ and $r_e(a,b)$ describe the contact rate between the gambling operators and uneducated and educated susceptible individuals, respectively. The maximum age at which a person can reach is denoted by A_m. $\mu(a)$ denotes the natural mortality rate of age a individuals. $\alpha(a)$ denotes the age-dependent family education rate, i.e. the rate at which uneducated susceptible individuals of age a enter an educated susceptible compartment. $\tau(a)$ is the rate at which addicted gamblers of age a move to the gambling operator class. $\omega(a)$ denotes the age-dependent conviction rate of gambling operators. $\delta_e(a)$, $\delta_u(a)$ and $\sigma(a)$ are the age-dependent rehabilitation rates of occasional gamblers, addicted gamblers and convicted individuals, respectively. Further, to make the model more realistic, we have the following assumptions on the parameters chosen in our model:

Assumption 1. *The functions* $\mu, \alpha, \tau, \omega, \delta_e, \delta_u, \sigma \in L^\infty_+(0, A_m)$.

Assumption 2. *The transmission coefficients* $r_u, r_e \in L^\infty_+((0, A_m) \times (0, A_m))$.

Note that the function $P(a,t)$ satisfies the following equation

$$\frac{\partial P(a,t)}{\partial t} + \frac{\partial P(a,t)}{\partial a} = -\mu(a)P(a,t), \tag{2.2}$$

with age-zero individual inflow

$$P(0,t) = \Lambda,$$

and initial condition

$$P(a,0) = S_{u_0}(a) + S_{e_0}(a) + G_0(a) + I_{e_0}(a) + I_{u_0}(a) + C_0(a) + R_0(a) = P_0(a).$$

Solving the equation (2.2) along the characteristic line $t - a = c$ we get

$$P(a,t) = \begin{cases} \Lambda\pi(a), & a < t, \\ P_0(a-t)\frac{\pi(a)}{\pi(a-t)}, & a \geq t, \end{cases} \tag{2.3}$$

where $\pi(a) = e^{-\int_0^a \mu(\tau)d\tau}$. Therefore, we have the following result:

Theorem 2.1. *The total population $P(a,t)$ is bounded if $P_0(t)$ is bounded.*

We will use the above result while discussing the optimal control problem for our model in section-4. Further, to simplify our model, we perform some transformations, and these are as follows:

$$\begin{aligned} &s_u(a,t) = \frac{S_u(a,t)}{P(a,t)},\ s_e(a,t) = \frac{S_e(a,t)}{P(a,t)},\ g(a,t) = \frac{G(a,t)}{P(a,t)},\ i_e(a,t) = \frac{I_e(a,t)}{P(a,t)}, \\ &i_u(a,t) = \frac{I_u(a,t)}{P(a,t)},\ c(a,t) = \frac{C(a,t)}{P(a,t)},\ r(a,t) = \frac{R(a,t)}{P(a,t)}. \end{aligned}$$

Using these transformations in the system (2.1), we get

$$\begin{aligned} \frac{\partial s_u(a,t)}{\partial t} + \frac{\partial s_u(a,t)}{\partial a} &= -\lambda(a,t)s_u(a,t) - \alpha(a)s_u(a,t), \\ \frac{\partial s_e(a,t)}{\partial t} + \frac{\partial s_e(a,t)}{\partial a} &= \alpha(a)s_u(a,t) - \psi(a,t)s_e(a,t), \\ \frac{\partial g(a,t)}{\partial t} + \frac{\partial g(a,t)}{\partial a} &= \tau(a)i_u(a,t) - \omega(a)g(a,t), \\ \frac{\partial i_e(a,t)}{\partial t} + \frac{\partial i_e(a,t)}{\partial a} &= \psi(a,t)s_e(a,t) - \delta_e(a)i_e(a,t), \\ \frac{\partial i_u(a,t)}{\partial t} + \frac{\partial i_u(a,t)}{\partial a} &= \lambda(a,t)s_u(a,t) - \delta_u(a)i_u(a,t) - \tau(a)i_u(a,t), \\ \frac{\partial c(a,t)}{\partial t} + \frac{\partial c(a,t)}{\partial a} &= \omega(a)g(a,t) - \sigma(a)c(a,t) - \mu(a)c(a,t), \\ \frac{\partial r(a,t)}{\partial t} + \frac{\partial r(a,t)}{\partial a} &= \sigma(a)c(a,t) + \delta_e(a)i_e(a,t) + \delta_u(a)i_u(a,t), \\ & s_u(0,t) = 1,\ s_e(0,t) = g(0,t) = i_e(0,t) = i_u(0,t) = c(0,t) = r(0,t) = 0, \\ & s_u(a,0) = s_{u_0}(a),\ s_e(a,0) = s_{e_0}(a),\ g(a,0) = g_0(a),\ i_e(a,0) = i_{e_0}(a), \\ & i_u(a,0) = i_{u_0}(a),\ c(a,0) = c_0(a),\ r(a,0) = r_0(a), \end{aligned} \tag{2.4}$$

where

$$\lambda(a,t) = \int_0^{A_m} r_u(a,b)g(b,t)db \quad \text{and} \quad \psi(a,t) = \int_0^{A_m} r_e(a,b)g(b,t)db.$$

As the last two equations are decoupled from the above five equations, thus removing these two equations doesn't disrupt the analysis of the model (2.4). Hence, our model becomes

$$\begin{aligned}
&\frac{\partial s_u(a,t)}{\partial t} + \frac{\partial s_u(a,t)}{\partial a} = -\lambda(a,t)s_u(a,t) - \alpha(a)s_u(a,t),\\
&\frac{\partial s_e(a,t)}{\partial t} + \frac{\partial s_e(a,t)}{\partial a} = \alpha(a)s_u(a,t) - \psi(a,t)s_e(a,t),\\
&\frac{\partial g(a,t)}{\partial t} + \frac{\partial g(a,t)}{\partial a} = \tau(a)i_u(a,t) - \omega(a)g(a,t),\\
&\frac{\partial i_e(a,t)}{\partial t} + \frac{\partial i_e(a,t)}{\partial a} = \psi(a,t)s_e(a,t) - \delta_e(a)i_e(a,t),\\
&\frac{\partial i_u(a,t)}{\partial t} + \frac{\partial i_u(a,t)}{\partial a} = \lambda(a,t)s_u(a,t) - \delta_u(a)i_u(a,t) - \tau(a)i_u(a,t),\\
&s_u(0,t) = 1,\ s_e(0,t) = g(0,t) = i_e(0,t) = i_u(0,t) = 0,\\
&s_u(a,0) = s_{u_0}(a),\ s_e(a,0) = s_{e_0}(a),\ g(a,0) = g_0(a),\ i_e(a,0) = i_{e_0}(a), i_u(a,0) = i_{u_0}(a).
\end{aligned} \tag{2.5}$$

To perform the rigorous analysis of the above system, we first define the state space as

$$\mathbb{X} = L^1(0,A_m) \times L^1(0,A_m) \times L^1(0,A_m) \times L^1(0,A_m) \times L^1(0,A_m).$$

It is a normed linear space where norm for any $x = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{X}$ is given by

$$\|x\|_{\mathbb{X}} = \int_0^{A_m} |x_1(a)|da + \int_0^{A_m} |x_2(a)|da + \int_0^{A_m} |x_3(a)|da + \int_0^{A_m} |x_4(a)|da + \int_0^{A_m} |x_5(a)|da.$$

The positive cone of $\mathbb{X}$ denoted by $\mathbb{X}_+$ is given by

$$\mathbb{X}_+ = L^1_+(0,A_m) \times L^1_+(0,A_m) \times L^1_+(0,A_m) \times L^1_+(0,A_m) \times L^1_+(0,A_m),$$

where $L^1_+(0,A_m) = \{f \in L^1(0,A_m) : f(x) \geq 0 \text{ a.e. in } (0,A_m)\}$. Now, we will convert our model into an abstract Cauchy problem. For that define a linear operator $\mathcal{A} : D(\mathcal{A}) \subset \mathbb{X} \to \mathbb{X}$ by

$$\mathcal{A}(\phi) = -\phi' = \Big(-\frac{d}{da}\phi_1(a), -\frac{d}{da}\phi_2(a), -\frac{d}{da}\phi_3(a), -\frac{d}{da}\phi_4(a), -\frac{d}{da}\phi_5(a)\Big)^T, \tag{2.6}$$

where $\phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5) \in \mathbb{X}$ and

$$D(\mathcal{A}) = \{\phi \in \mathbb{X} : \phi_i \in W^{1,1}(0,A_m), \phi(0) = (1,0,0,0,0)^T\}.$$

It is easy to check that $\overline{D(\mathcal{A})} = \mathbb{X}$. We also define a non-linear operator $\mathcal{F} : \mathbb{X} \to \mathbb{X}$ by

$$\mathcal{F}(\phi) = \begin{pmatrix} -(F_1\phi_3)(a)\phi_1(a) - \alpha(a)\phi_1(a) \\ \alpha(a)\phi_1(a) - (F_2\phi_3)(a)\phi_2(a) \\ \tau(a)\phi_5(a) - \omega(a)\phi_3(a) \\ (F_2\phi_3)(a)\phi_2(a) - \delta_e(a)\phi_4(a) \\ (F_1\phi_3)(a)\phi_1(a) - \delta_u(a)\phi_5(a) - \tau(a)\phi_5(a) \end{pmatrix}, \tag{2.7}$$

where

$$(F_1(\phi_3)(\cdot) = \int_0^{A_m} r_u(\cdot, b)\phi_3(b)db \quad \text{and} \quad (F_2(\phi_3)(\cdot) = \int_0^{A_m} r_e(\cdot, b)\phi_3(b)db.$$

So, we can write the system (2.5) in the form of a Cauchy problem defined on an abstract space $\mathbb{X}$ by

$$\begin{cases} x'(t) = \mathcal{A}x(t) + \mathcal{F}x(t), \\ x(0) = x_0 \in \mathbb{X}_+. \end{cases} \tag{2.8}$$

Lemma 2.2. *The linear operator $\mathcal{A}$ defined in (2.6) generates a C_0-semigroup.*

Using the Hille-Yoshida Theorem (see Pazy [26] Example 3.7), it is easy to check that $\mathcal{A}$ generates a C_0-semigroup $\{T(t)\}_{t\in\mathbb{R}_+}$. In fact it is a translational semigroup, i.e. for $x \in \mathbb{X}$,

$$(T(t)x)(s) = x(t+s).$$

Lemma 2.3. *The non-linear operator $\mathcal{F}$ is Lipschitz continuous in $\mathbb{X}$.*

The proof of this lemma is obvious from Assumption 1 and Assumption 2. Combining these two lemmas will give us the local existence of the unique mild solution for the system (2.8), which is stated in the following theorem:

Theorem 2.4. *For the system (2.8) and for each initial data $x_0 \in \mathbb{X}_+$ there exists a maximal interval $[0, t_0)$ and a unique mild solution $t \to x(t, x_0)$ which is continuous from $[0, t_0)$ to $\mathbb{X}$, such that*

$$x(t, x_0) = T(t)x_0 + \int_0^t T(t-\rho)\mathcal{F}(x(\rho, x_0))d\rho \quad \forall\ t \in [0, t_0]. \tag{2.9}$$

Next, we prove the global existence of the solution. For this, define the set Ω that satisfies the following condition:

$$\Omega = \{(s_u, s_e, g, i_e, i_u) \in \mathbb{X} : 0 \le s_u + s_e + g + i_e + i_u \le 1\}.$$

It is easy to see that Ω is a closed convex set and since $\{T(t)\}_{t\in\mathbb{R}_+}$ is a translational semigroup so we have

$$T(t)\Omega \subset \Omega. \tag{2.10}$$

Now, we state the following lemma.

Lemma 2.5. *There exists a constant $\alpha_0 \in (0, 1)$ such that $(I + \alpha_0\mathcal{F})\Omega \subseteq \Omega$.*

Proof. Let the vector $u = (u_1, u_2, u_3, u_4, u_5) \in \Omega$. Defining the vector $v = (v_1, v_2, v_3, v_4, v_5) \in \mathbb{X}$ by

$$(I + \alpha_0\mathcal{F})(u_1, u_2, u_3, u_4, u_5)^T = (v_1, v_2, v_3, v_4, v_5)^T.$$

We need to show that $v \in \Omega$. Since $\sum_{i=1}^5 v_i = \sum_{i=1}^5 (u_i + \alpha_0\mathcal{F}(u_i))$ we get

$$\sum_{i=1}^{5}(u_i + \alpha_0\mathcal{F}(u_i)) = u_1 + u_2 + (1-\alpha_0 w(a))u_3 + (1-\alpha_0\delta_e(a))u_4 + (1-\alpha_0\delta_u(a))u_5.$$

Because of the Assumption 1, we can set $\alpha_+ = \sup\{\alpha(a), a \in (0, A_m)\}$, $\tau_+ = \sup\{\tau(a), a \in (0, A_m)\}$, $\omega_+ = \sup\{\omega(a), a \in (0, A_m)\}$, $\delta_{e_+} = \sup\{\delta_e(a), a \in (0, A_m)\}$, $\delta_{u_+} = \sup\{\delta_u(a), a \in (0, A_m)\}$.

$$0 < \alpha_0 < \min\left\{1, \frac{1}{\omega_+}, \frac{1}{\delta_{e_+}}, \frac{1}{\delta_{u_+}}\right\}. \tag{2.11}$$

With this choice of $\alpha_0 \in (0, 1)$,

$$(I + \alpha_0\mathcal{F})\Omega \subseteq \Omega. \quad \square \tag{2.12}$$

Note that we can rewrite the system (2.8) as

$$\begin{cases} x'(t) = \left(\mathcal{A} - \frac{1}{\alpha_0}I\right)x(t) + \frac{1}{\alpha_0}(I + \alpha_0\mathcal{F})x(t), \\ x(0) = x_0 \in \mathbb{X}_+, \end{cases} \tag{2.13}$$

where $\alpha_0 \in (0, 1)$ is chosen as in (2.11) such that (2.12) holds. Let $\mathcal{A}_1 = \mathcal{A} - \frac{1}{\alpha_0}I$ and $\mathcal{F}_1 = I + \alpha_0\mathcal{F}$. As the operator $\frac{1}{\alpha_0}I$ is a bounded linear operator and $\mathcal{A}$ generates a C_0- semigroup. Thus from [26], $\mathcal{A}_1$ also generates a C_0-semigroup given by $\{e^{-\frac{1}{\alpha_0}t}T(t)\}_{t\geq 0}$, where $\{T(t)\}_{t\geq 0}$ is the semigroup generated by $\mathcal{A}$. Thus, by variation of the constants formula, the solution of the system (2.13) is given by

$$x(t) = e^{-\frac{1}{\alpha_0}t}T(t)x_0 + \frac{1}{\alpha_0}\int_0^t e^{-\frac{1}{\alpha_0}(t-\rho)}T(t-\rho)\mathcal{F}_1(x(\rho, x_0))d\rho.$$

We will start with the standard iterative procedure: Let

$$x^0(t) = x_0 \text{ and } x^{n+1}(t) = e^{-\frac{1}{\alpha_0}t}T(t)x_0 + \frac{1}{\alpha_0}\int_0^t e^{-\frac{1}{\alpha_0}(t-\rho)}T(t-\rho)\mathcal{F}_1(x^n(\rho, x_0))d\rho.$$

Now, because $e^{-\frac{1}{\alpha_0}t} + \frac{1}{\alpha_0}\int_0^t e^{-\frac{1}{\alpha_0}(t-\rho)}d\rho = 1$, whenever $x^n \in \Omega$, from (2.10) and (2.12), $x^{n+1} \in \Omega$. Finally, using the Lemma 2.3, we can say that sequence $x^n(t)$ converges uniformly to $x(t)$. For more details, see Busenberg et al. [4]. Thus, we have the following result.

Theorem 2.6. *For the initial data $x_0 \in \Omega$, the system (2.8) has a unique global mild solution in $\mathbb{X}$.*

3. Stability analysis

3.1. Addiction-free equilibrium

The system (2.5) in a steady-state is given by:

$$
\begin{aligned}
\frac{ds_u}{da} &= -\lambda(a)s_u(a) - \alpha(a)s_u(a),\\
\frac{ds_e}{da} &= \alpha(a)s_u(a) - \psi(a)s_e(a),\\
\frac{dg}{da} &= \tau(a)i_u(a) - \omega(a)g(a),\\
\frac{di_e}{da} &= \psi(a)s_e(a) - \delta_e(a)i_e(a),\\
\frac{di_u}{da} &= \lambda(a)s_u(a) - \delta_u(a)i_u(a) - \tau(a)i_u(a),\\
& s_u(0) = 1,\ \ s_e(0) = g(0) = i_e(0) = i_u(0) = 0,
\end{aligned}
\tag{3.1}
$$

where

$$\lambda(a) = \int_0^{A_m} r_u(a,b)g(b)db \quad \text{and} \quad \psi(a) = \int_0^{A_m} r_e(a,b)g(b)db.$$

Since we are only interested in addiction-free equilibrium $E_0 = (s_u^0(a), s_e^0(a), g^0(a), i_e^0(a), i_u^0(a))$ means that $g^0(a) = i_e^0(a) = i_u^0(a) = 0$. So $s_u^0(a)$ and $s_e^0(a)$ should satisfy

$$
\begin{cases}
\frac{ds_u^0}{da} & = -\alpha(a)s_u^0(a),\\
\frac{ds_e^0}{\partial a} & = \alpha(a)s_u^0(a),\\
s_u^0(0) & = 1, \quad s_e^0(0) = 0.
\end{cases}
\tag{3.2}
$$

Solving (3.2), we get the addiction-free equilibrium $E_0 = (e^{-\int_0^a \alpha(s)ds}, \int_0^a \alpha(\xi)e^{-\int_0^\xi \alpha(s)ds}d\xi, 0, 0, 0)$. To check the local stability of E_0 we will linearize the system (2.5) about E_0. We use the transformations

$$
\begin{aligned}
&\overline{s_u}(a,t) = s_u(a,t) - s_u^0(a),\ \overline{s_e}(a,t) = s_e(a,t) - s_e^0(a),\\
&\overline{g}(a,t) = g(a,t),\ \overline{i_e}(a,t) = i_e(a,t),\ \overline{i_u}(a,t) = i_u(a,t).
\end{aligned}
$$

Then, the linearized system of (2.5) corresponding to these transformations is given by

$$
\begin{aligned}
\frac{\partial\overline{s_u}(a,t)}{\partial t}+\frac{\partial\overline{s_u}(a,t)}{\partial a}&=-\overline{\lambda}(a,t)s_u^0(a)-\alpha(a)\overline{s_u}(a,t),\\
\frac{\partial\overline{s_e}(a,t)}{\partial t}+\frac{\partial\overline{s_e}(a,t)}{\partial a}&=\alpha(a)\overline{s_u}(a,t)-\overline{\psi}(a,t)s_e^0(a),\\
\frac{\partial\overline{g}(a,t)}{\partial t}+\frac{\partial\overline{g}(a,t)}{\partial a}&=\tau(a)\overline{i_u}(a,t)-\omega(a)\overline{g}(a,t),\\
\frac{\partial\overline{i_e}(a,t)}{\partial t}+\frac{\partial\overline{i_e}(a,t)}{\partial a}&=\overline{\psi}(a,t)s_e^0(a)-\delta_e(a)\overline{i_e}(a,t),\\
\frac{\partial\overline{i_u}(a,t)}{\partial t}+\frac{\partial\overline{i_u}(a,t)}{\partial a}&=\overline{\lambda}(a,t)s_u^0(a)-\delta_u(a)\overline{i_u}(a,t)-\tau(a)\overline{i_u}(a,t),\\
\overline{s_u}(0,t)=\overline{s_e}(0,t)=\overline{g}(0,t)&=\overline{i_e}(0,t)=\overline{i_u}(0,t)=0,
\end{aligned}
\tag{3.3}
$$

where

$$\overline{\lambda}(a,t)=\int_0^{A_m} r_u(a,b)\overline{g}(b,t)db \quad \text{and} \quad \overline{\psi}(a,t)=\int_0^{A_m} r_e(a,b)\overline{g}(b,t)db.$$

For some constant $k\in\mathbb{R}$, assume the solution of the above equation has the form

$$\overline{s_u}(a,t)=\overline{s_u}(a)e^{kt},\ \overline{s_e}(a,t)=\overline{s_e}(a)e^{kt},\ \overline{g}(a,t)=\overline{g}(a)e^{kt},\ \overline{i_e}(a,t)=\overline{i_e}(a)e^{kt},\ \overline{i_u}(a,t)=\overline{i_u}(a)e^{kt}.$$

Then $(\overline{s_u}(a),\overline{s_e}(a),\overline{g}(a),\overline{i_e}(a),\overline{i_u}(a))$ satisfies the following system of odes

$$
\begin{aligned}
\frac{d\overline{s_u}(a)}{da}&=-k\overline{s_u}(a)-\overline{\lambda}(a)s_u^0(a)-\alpha(a)\overline{s_u}(a),\\
\frac{d\overline{s_e}(a)}{da}&=-k\overline{s_e}(a)-\overline{\psi}(a)s_e^0(a)+\alpha(a)\overline{s_u}(a),\\
\frac{d\overline{g}(a)}{da}&=-k\overline{g}(a)+\tau(a)\overline{i_u}(a)-\omega(a)\overline{g}(a),\\
\frac{d\overline{i_e}(a)}{da}&=-k\overline{i_e}(a)+\overline{\psi}(a)s_e^0(a)-\delta_e(a)\overline{i_e}(a),\\
\frac{d\overline{i_u}(a)}{da}&=-k\overline{i_u}(a)+\overline{\lambda}(a)s_u^0(a)-\delta_u(a)\overline{i_u}(a)-\tau(a)\overline{i_u}(a),\\
\overline{s_u}(0)=\overline{s_e}(0)&=\overline{g}(0)=\overline{i_e}(0)=\overline{i_u}(0)=0,
\end{aligned}
\tag{3.4}
$$

where

$$\overline{\lambda}(a)=\int_0^{A_m} r_u(a,b)\overline{g}(b)db \quad \text{and} \quad \overline{\psi}(a)=\int_0^{A_m} r_e(a,b)\overline{g}(b)db.$$

Solving the last equation, we obtain

$$\overline{i_u}(a)=\int_0^a e^{-k(a-\eta)}e^{-\int_\eta^a(\delta_u(s)+\tau(s))ds}\overline{\lambda}(\eta)s_u^0(\eta)d\eta.$$

Putting this in the equation concerning $\overline{g}(a)$ we get

$$\overline{g}(a) = \int_0^a \int_0^\xi e^{-k(a-\eta)} e^{-\int_\xi^a \omega(s)ds} e^{-\int_\eta^\xi (\delta_u(s)+\tau(s))ds} \tau(\xi)\overline{\lambda}(\eta) s_u^0(\eta) d\eta d\xi.$$

Using the value of $\overline{\lambda}(a)$, the characteristic equation can be written as

$$G(k) = 1,$$

where

$$G(k) = \int_0^{A_m} \int_0^a \int_0^\xi e^{-k(a-\eta)} e^{-\int_\xi^a \omega(s)ds} e^{-\int_\eta^\xi (\delta_u(s)+\tau(s))ds} \tau(\xi) r_u(\eta, a) s_u^0(\eta) d\eta d\xi da. \tag{3.5}$$

Defining the threshold parameter as $\mathcal{R}_0 = G(0)$. Even though this threshold parameter is not that much biologically relevant, it has a significant role in studying the stability of an addiction-free equilibrium. Before coming to the stability result, we state a lemma regarding the behavior of the function $G(k)$, the proof of which is quite obvious, so we will omit it.

Lemma 3.1. *$G(k)$ is monotonically decreasing function with respect to the parameter k and*

$$lim_{k\to-\infty} G(k) = \infty \quad lim_{k\to\infty} G(k) = 0.$$

Theorem 3.2. *Under the Assumptions 1 and 2, the addiction-free equilibrium of the system (2.5) is locally asymptotically stable if $G(0) = \mathcal{R}_0 < 1$ and unstable if $G(0) = \mathcal{R}_0 > 1$.*

Proof. From the above lemma we can guarantee that there exists k^* such that $G(k^*) = 1$.
Case 1: First suppose $k \in \mathbb{R}$. Now $\mathcal{R}_0 < 1 \iff G(0) < 1$ and from Lemma 3.1 the monotonicity of $G(k)$ implies that $k^* < 0$. Similarly $G(0) > 1 \implies k^* > 0$.
Case 2: If k^* is a complex root. Let us assume $k^* = c_1 + ic_2$ such that $G(c_1 + ic_2) = 1$. Then again because of monotonicity of $G(k)$, for $\mathrm{Re}(k^*) = c_1 \geq 0$, we have $|G(k^*)| \leq |G(0)|$ and hence

$$1 = |G(k^*)| \leq |G(0)| \leq \mathcal{R}_0.$$

Thus if $G(0) = \mathcal{R}_0 < 1$, then $\mathrm{Re}(k^*) < 0$. Therefore, the addiction-free equilibrium is locally asymptotically stable if $G(0) < 1$ and $G(0) > 1$ corresponds to an unstable addiction-free equilibrium. □

3.2. Addiction-predominant equilibrium

For this particular subsection, we take $r_u(a, b) = w_u(a)z(b)$ and $r_e(a, b) = w_e(a)z(b)$. Here $w_u(a)$ and $w_e(a)$ denote the average probability that a susceptible non-gambler of age a from the class $s_u(a, t)$ and $s_e(a, t)$ respectively, involves in gambling after exposure to an individual in the gambling operator class. Also, $z(b)$ represents the probability of exposure per capita for an individual in the gambling operator class of age b. Note that from Assumption 2, $z(b)$ is a bounded function. Giving this particular form to $r_u(a, b)$ and $r_e(a, b)$ not only makes the analysis easier but also makes sense biologically because different age groups of susceptible have different viewpoints regarding the merits and demerits of online gambling due to which they may or may not involve in gambling after contact with gambling operators.

In this subsection, we will try to establish the model's dynamics when gamblers and non-gamblers coexist in a particular population. Let the addiction-predominant equilibrium be represented as $E^* = (s_u^*(a), s_e^*(a), g^*(a), i_e^*(a), i_u^*(a))$. So E^* is the solution of the following equations

$$\begin{aligned}
\frac{ds_u^*}{da} &= -\lambda^*(a)s_u^*(a) - \alpha(a)s_u^*(a),\\
\frac{ds_e^*}{da} &= \alpha(a)s_u^*(a) - \psi^*(a)s_e^*(a),\\
\frac{dg^*}{da} &= \tau(a)i_u^*(a) - \omega(a)g^*(a),\\
\frac{di_e^*}{da} &= \psi^*(a)s_e^*(a) - \delta_e(a)i_e^*(a),\\
\frac{di_u^*}{da} &= \lambda^*(a)s_u^*(a) - \delta_u(a)i_u^*(a) - \tau(a)i_u^*(a),\\
s_u^*(0) = 1,\ & s_e^*(0) = g^*(0) = i_e^*(0) = i_u^*(0) = 0,
\end{aligned} \tag{3.6}$$

where

$$\lambda^*(a) = w_u(a)\int_0^{A_m} z(b)g^*(b)db \quad \text{and} \quad \psi^*(a) = w_e(a)\int_0^{A_m} z(b)g^*(b)db.$$

Letting $\Gamma = \int_0^{A_m} z(a)g^*(a)da$ gives $\lambda^*(a) = w_u(a)\Gamma$ and $\psi^*(a) = w_e(a)\Gamma$. Solving the above system, we get

$$\begin{cases}
s_u^*(a) &= e^{-\int_0^a [\Gamma w_u(s)+\alpha(s)]ds},\\
s_e^*(a) &= \int_0^a e^{-\int_\xi^a \Gamma w_e(s)ds} e^{-\int_0^\xi [\Gamma w_u(s)+\alpha(s)]ds}\alpha(\xi)d\xi,\\
g^*(a) &= \Gamma\int_0^a\int_0^\xi e^{-\int_\xi^a \omega(s)ds} e^{-\int_\eta^\xi [\delta_u(s)+\tau(s)]ds} e^{-\int_0^\eta [\Gamma w_u(s)+\alpha(s)]ds} w_u(\eta)\tau(\xi)d\eta d\xi,\\
i_e^*(a) &= \Gamma\int_0^a\int_0^\eta e^{-\int_\eta^a \delta_e(s)ds} e^{-\int_\xi^\eta \Gamma w_e(s)ds} e^{-\int_0^\xi [\Gamma w_u(s)+\alpha(s)]ds} w_e(\eta)\alpha(\xi)d\xi d\eta,\\
i_u^*(a) &= \Gamma\int_0^a e^{-\int_\eta^a [\delta_u(s)+\tau(s)]ds} e^{-\int_0^\eta [\Gamma w_u(s)+\alpha(s)]ds} w_u(\eta)d\eta.
\end{cases} \tag{3.7}$$

Putting $g^*(a)$ in $\Gamma = \int_0^{A_m} z(a)g^*(a)da$ we get

$$1 = \int_0^{A_m}\int_0^a\int_0^\xi z(a)e^{-\int_\xi^a \omega(s)ds} e^{-\int_\eta^\xi [\delta_u(s)+\tau(s)]ds} e^{-\int_0^\eta [\Gamma w_u(s)+\alpha(s)]ds} w_u(\eta)\tau(\xi)d\eta d\xi da.$$

Define a function

$$F(\Gamma) = \int_0^{A_m}\int_0^a\int_0^\xi z(a)e^{-\int_\xi^a \omega(s)ds} e^{-\int_\eta^\xi [\delta_u(s)+\tau(s)]ds} e^{-\int_0^\eta [\Gamma w_u(s)+\alpha(s)]ds} w_u(\eta)\tau(\xi)d\eta d\xi da. \tag{3.8}$$

Remark 3.3. From equation (3.7), we see that whenever $\Gamma = 0$, the addiction-predominant equilibrium is converted into the addiction-free equilibrium. Also, when $\Gamma = 0$, we get $F(0) = \mathcal{R}_0$.

Note that from the function $F(\Gamma)$, we infer that examining the existence of the solution of the system (3.6) is equivalent to the problem of investigating the roots of equation $F(\Gamma) = 1$. Therefore, we have the following result.

Theorem 3.4. *If $\mathcal{R}_0 > 1$, then under the Assumptions 1 and 2, there exists a unique positive solution to the system (3.6) if and only if Γ that satisfies equation $F(\Gamma) = 1$ is unique.*

Proof. From the definition of $F(\Gamma)$, it is quite evident that $F(\Gamma)$ is strictly monotonically decreasing with respect to Γ and $\lim_{\Gamma\to-\infty} F(\Gamma) = \infty$. Also, since

$$\frac{g^*(a)}{\Gamma} = \int_0^a \int_0^\xi e^{-\int_\xi^a \omega(s)ds} e^{-\int_\eta^\xi [\delta_u(s)+\tau(s)]ds} e^{-\int_0^\eta [\Gamma w_u(s)+\alpha(s)]ds} w_u(\eta)\tau(\xi)d\eta d\xi,$$

we have

$$F(\Gamma) = \frac{1}{\Gamma}\int_0^{A_m} z(a)g^*(a)da.$$

Now boundedness of the functions $z(a)$ and $g^*(a)$ implies that $\lim_{\Gamma\to\infty} F(\Gamma) = 0$. Hence, from these assertions we can say that there exists a unique root Γ^* of the equation $F(\Gamma) = 1$, and if $\mathcal{R}_0 > 1$, then $\Gamma^* > 0$. $\square$

Next, to show the local stability of this addiction-predominant equilibrium, we let

$$\tilde{s_u}(a,t) = s_u(a,t) - s_u^*(a),\ \tilde{s_e}(a,t) = s_e(a,t) - s_e^*(a),\ \tilde{g}(a,t) = g(a,t) - g^*(a),$$
$$\tilde{i_e}(a,t) = i_e(a,t) - i_e^*(a),\ \tilde{i_u}(a,t) = i_u(a,t) - i_u^*(a).$$

Then, the linearized system corresponding to these transformations is

$$\begin{aligned}
\frac{\partial\tilde{s_u}(a,t)}{\partial t} + \frac{\partial\tilde{s_u}(a,t)}{\partial a} &= -\tilde{\lambda}(a,t)s_u^*(a) - \lambda^*(a)\tilde{s_u}(a,t) - \alpha(a)\tilde{s_u}(a,t),\\
\frac{\partial\tilde{s_e}(a,t)}{\partial t} + \frac{\partial\tilde{s_e}(a,t)}{\partial a} &= \alpha(a)\tilde{s_u}(a,t) - \tilde{\psi}(a,t)s_e^*(a) - \psi^*(a)\tilde{s_e}(a,t),\\
\frac{\partial\tilde{g}(a,t)}{\partial t} + \frac{\partial\tilde{g}(a,t)}{\partial a} &= \tau(a)\tilde{i_u}(a,t) - \omega(a)\tilde{g}(a,t),\\
\frac{\partial\tilde{i_e}(a,t)}{\partial t} + \frac{\partial\tilde{i_e}(a,t)}{\partial a} &= \tilde{\psi}(a,t)s_e^*(a) + \psi^*(a)\tilde{s_e}(a,t) - \delta_e(a)\tilde{i_e}(a,t),\\
\frac{\partial\tilde{i_u}(a,t)}{\partial t} + \frac{\partial\tilde{i_u}(a,t)}{\partial a} &= \tilde{\lambda}(a,t)s_u^*(a) + \lambda^*(a)\tilde{s_u}(a,t) - \delta_u(a)\tilde{i_u}(a,t) - \tau(a)\tilde{i_u}(a,t),\\
\tilde{s_u}(0,t) = \tilde{s_e}(0,t) &= \tilde{g}(0,t) = \tilde{i_e}(0,t) = \tilde{i_u}(0,t) = 0,
\end{aligned} \tag{3.9}$$

where

$$\tilde{\lambda}(a,t) = w_u(a)\int_0^{A_m} z(b)\tilde{g}(b,t)db, \quad \lambda^*(a) = w_u(a)\int_0^{A_m} z(b)g^*(b)db,$$

and

$$\tilde{\psi}(a,t) = w_e(a)\int_0^{A_m} z(b)\tilde{g}(b,t)db, \quad \psi^*(a) = w_e(a)\int_0^{A_m} z(b)g^*(b)db.$$

For the above system, we will consider the solution of the nonzero exponential type similar to what we had considered in the case of studying the local stability of addiction-free equilibrium. So let:

$$\tilde{s_u}(a,t) = \tilde{s_u}(a)e^{ht},\ \tilde{s_e}(a,t) = \tilde{s_e}(a)e^{ht},\ \tilde{g}(a,t) = \tilde{g}(a)e^{ht},\ \tilde{i_e}(a,t) = \tilde{i_e}(a)e^{ht},\ \tilde{i_u}(a,t) = \tilde{i_u}(a)e^{ht}.$$

Then $(\tilde{s_u}(a), \tilde{s_e}(a), \tilde{g}(a), \tilde{i_e}(a), \tilde{i_u}(a))$ satisfies the following equations:

$$
\begin{aligned}
\frac{d\tilde{s_u}(a)}{da} &= -h\tilde{s_u}(a) - \widetilde{\Gamma} w_u(a)s_u^*(a) - \Gamma w_u(a)\tilde{s_u}(a) - \alpha(a)\tilde{s_u}(a),\\
\frac{d\tilde{s_e}(a)}{da} &= -h\tilde{s_e}(a) - \widetilde{\Gamma} w_e(a)s_e^*(a) - \Gamma w_e(a)\tilde{s_e}(a) + \alpha(a)\tilde{s_u}(a),\\
\frac{d\tilde{g}(a)}{da} &= -h\tilde{g}(a) + \tau(a)\tilde{i_u}(a) - \omega(a)\tilde{g}(a),\\
\frac{d\tilde{i_e}(a)}{da} &= -h\tilde{i_e}(a) + \widetilde{\Gamma} w_e(a)s_e^*(a) + \Gamma w_e(a)\tilde{s_e}(a) - \delta_e(a)\tilde{i_e}(a),\\
\frac{d\tilde{i_u}(a)}{da} &= -h\tilde{i_u}(a) + \widetilde{\Gamma} w_u(a)s_u^*(a) + \Gamma w_u(a)\tilde{s_u}(a) - \delta_u(a)\tilde{i_u}(a) - \tau(a)\tilde{i_u}(a),\\
\tilde{s_u}(0) &= \tilde{s_e}(0) = \tilde{g}(0) = \tilde{i_e}(0) = \tilde{i_u}(0) = 0,
\end{aligned}
\tag{3.10}
$$

where

$$
\widetilde{\Gamma} = \int_0^{A_m} z(a)\tilde{g}(a)da.
$$

Remark 3.5. From (3.10), we can say that whenever $\widetilde{\Gamma} = 0$, the analysis of the addiction-predominant equilibrium becomes the same as the addiction-free equilibrium. Thus, we take $\widetilde{\Gamma} \neq 0$.

Let $\hat{s_u} = \frac{\tilde{s_u}}{\widetilde{\Gamma}}$, $\hat{s_e} = \frac{\tilde{s_e}}{\widetilde{\Gamma}}$, $\hat{g} = \frac{\tilde{g}}{\widetilde{\Gamma}}$, $\hat{i_e} = \frac{\tilde{i_e}}{\widetilde{\Gamma}}$, $\hat{i_u} = \frac{\tilde{i_u}}{\widetilde{\Gamma}}$, then the system (3.10) becomes

$$
\begin{aligned}
\frac{d\hat{s_u}(a)}{da} &= -h\hat{s_u}(a) - w_u(a)s_u^*(a) - \Gamma w_u(a)\hat{s_u}(a) - \alpha(a)\hat{s_u}(a),\\
\frac{d\hat{s_e}(a)}{da} &= -h\hat{s_e}(a) - w_e(a)s_e^*(a) - \Gamma w_e(a)\hat{s_e}(a) + \alpha(a)\hat{s_u}(a),\\
\frac{d\hat{g}(a)}{da} &= -h\hat{g}(a) + \tau(a)\hat{i_u}(a) - \omega(a)\hat{g}(a),\\
\frac{d\hat{i_e}(a)}{da} &= -h\hat{i_e}(a) + w_e(a)s_e^*(a) + \Gamma w_e(a)\hat{s_e}(a) - \delta_e(a)\hat{i_e}(a),\\
\frac{d\hat{i_u}(a)}{da} &= -h\hat{i_u}(a) + w_u(a)s_u^*(a) + \Gamma w_u(a)\hat{s_u}(a) - \delta_u(a)\hat{i_u}(a) - \tau(a)\hat{i_u}(a),\\
\hat{s_u}(0) &= \hat{s_e}(0) = \hat{g}(0) = \hat{i_e}(0) = \hat{i_u}(0) = 0.
\end{aligned}
\tag{3.11}
$$

Solving system (3.11) we obtain

$$
\begin{aligned}
\hat{s_u}(a) &= -\int_0^a e^{-\int_\xi^a [h+\Gamma w_u(s)+\alpha(s)]ds} w_u(\xi)s_u^*(\xi)d\xi,\\
\hat{s_e}(a) &= -\int_0^a \int_0^\eta e^{-\int_\eta^a [h+\Gamma w_e(s)]ds} e^{-\int_\xi^\eta [h+\Gamma w_u(s)+\alpha(s)]ds} \alpha(\eta)w_u(\xi)s_u^*(\xi)d\xi d\eta\\
&\quad - \int_0^a e^{-\int_\eta^a [h+\Gamma w_e(s)]ds} w_e(\eta)s_e^*(\eta)d\eta,\\
\hat{g}(a) &= \int_0^a \int_0^\rho e^{-\int_\rho^a [h+\omega(s)]ds} e^{-\int_\eta^\rho [h+\delta_u(s)+\tau(s)]ds} \tau(\rho)w_u(\eta)s_u^*(\eta)d\eta d\rho
\end{aligned}
$$

$$-\Gamma\int_0^a\int_0^\rho\int_0^\eta e^{-\int_\rho^a[h+\omega(s)]ds}e^{-\int_\eta^\rho[h+\delta_u(s)+\tau(s)]ds}e^{-\int_\xi^\eta[h+\Gamma w_u(s)+\alpha(s)]ds}\tau(\rho)w_u(\eta)w_u(\xi)s_u^*(\xi)d\xi d\eta d\rho,$$

$$\hat{i_e}(a)=\int_0^a e^{-\int_\rho^a[h+\delta_e(s)]ds}w_e(\rho)s_e^*(\rho)d\rho-\int_0^a\int_0^\rho e^{-\int_\rho^a[h+\delta_e(s)]ds}e^{-\int_\eta^\rho[h+\Gamma w_e(s)]ds}w_e(\eta)s_e^*(\eta)d\eta d\rho$$

$$-\Gamma\int_0^a\int_0^\rho\int_0^\eta e^{-\int_\rho^a[h+\delta_e(s)]ds}e^{-\int_\eta^\rho[h+\Gamma w_e(s)]ds}e^{-\int_\xi^\eta[h+\Gamma w_u(s)+\alpha(s)]ds}w_e(\rho)\alpha(\eta)w_u(\xi)s_u^*(\xi)d\xi d\eta d\rho,$$

$$\hat{i_e}(a)=-\Gamma\int_0^a\int_0^\eta e^{-\int_\eta^a[h+\delta_u(s)+\tau(s)]ds}e^{-\int_\xi^\eta[h+\Gamma w_u(s)+\alpha(s)]ds}w_u(\xi)w_u(\eta)s_u^*(\xi)d\xi d\eta$$

$$+\int_0^a e^{-\int_\eta^a[h+\delta_u(s)+\tau(s)]ds}w_u(\eta)s_e^*(\eta)d\eta.$$

Defining a function

$$K(h)=\int_0^{A_m}z(a)\hat{g}(a)da.$$

Substituting $\hat{g}(a)$ from above we get

$$\begin{aligned}K(h)=&\int_0^{A_m}\int_0^a\int_0^\rho z(a)\Phi_1(\rho,a,h)\Phi_2(\eta,\rho,h)\tau(\rho)w_u(\eta)s_u^*(\eta)d\eta d\rho da\\&-\Gamma\int_0^{A_m}\int_0^a\int_0^\rho\int_0^\eta z(a)\Phi_1(\rho,a,h)\Phi_2(\eta,\rho,h)\Phi_3(\xi,\eta,h)\tau(\rho)w_u(\eta)w_u(\xi)s_u^*(\xi)d\xi d\eta d\rho da,\end{aligned}\tag{3.12}$$

where

$$\begin{aligned}\Phi_1(\rho,a,h)&=e^{-\int_\rho^a[h+\omega(s)]ds},\\\Phi_2(\eta,\rho,h)&=e^{-\int_\eta^\rho[h+\delta_u(s)+\tau(s)]ds},\\\Phi_3(\xi,\eta,h)&=e^{-\int_\xi^\eta[h+\Gamma w_u(s)+\alpha(s)]ds}.\end{aligned}$$

Claim: All the roots of $K(h)=1$ have negative real part if $\mathcal{R}_0>1$.
To show this first note that we evaluate its value at $h=0$,

$$\begin{aligned}K(0)=&\int_0^{A_m}\int_0^a\int_0^\rho z(a)\Phi_1(\rho,a,0)\Phi_2(\eta,\rho,0)\tau(\rho)w_u(\eta)s_u^*(\eta)d\eta d\rho da\\&-\Gamma\int_0^{A_m}\int_0^a\int_0^\rho\int_0^\eta z(a)\Phi_1(\rho,a,0)\Phi_2(\eta,\rho,0)\Phi_3(\xi,\eta,0)\tau(\rho)w_u(\eta)w_u(\xi)s_u^*(\xi)d\xi d\eta d\rho da.\end{aligned}$$

From (3.8) we have

$$\int_0^{A_m}\int_0^{a}\int_0^{\rho} z(a)\Phi_1(\rho,a,0)\Phi_2(\eta,\rho,0)\tau(\rho)w_u(\eta)s_u^*(\eta)d\eta d\rho da = 1,$$

this gives

$$K(0) = 1 - \Gamma\int_0^{A_m}\int_0^{a}\int_0^{\rho}\int_0^{\eta} z(a)\Phi_1(\rho,a,0)\Phi_2(\eta,\rho,0)\Phi_3(\xi,\eta,0)\tau(\rho)w_u(\eta)w_u(\xi)s_u^*(\xi)d\xi d\eta d\rho da.$$

Hence $K(0) < 1$. Let $h = h_1 + ih_2 \in \mathbb{C}$, such that $\mathrm{Re}(h) = h_1 > 0$. Now whenever $\mathcal{R}_0 > 1$, from Theorem 3.4 there exists a unique $\Gamma > 0$ such that $F(\Gamma) = 1$. Now $\Gamma > 0$ and Assumptions (1, 2) imply that

$$K(h) < \mathcal{K}(h),$$

where

$$\mathcal{K}(h) = \int_0^{A_m}\int_0^{a}\int_0^{\rho} z(a)e^{-\int_\rho^a [h+\omega(s)]ds}e^{-\int_\eta^\rho [h+\delta_u(s)+\tau(s)]ds}\tau(\rho)w_u(\eta)s_u^*(\eta)d\eta d\rho da.$$

Note that $\mathcal{K}$ is a monotonically decreasing function of h and since $h_1 > 0$ we have

$$K(h) < \mathcal{K}(h) < \mathcal{K}(0) = F(\Gamma) = 1.$$

Hence $K(h) < 1$ for all $\mathrm{Re}(h) > 0$. Since $\lim_{h\to\infty} K(h) = 0$, it follows that $K(h) = 1$ if and only if $\mathrm{Re}(h) < 1$. Thus, all the roots of $K(h) = 1$ have a negative real part if $\mathcal{R}_0 > 1$. Proving this claim gives us the following theorem, proof of which is similar to Theorem 3.2.

Theorem 3.6. *Under the Assumptions 1 and 2, the addiction-predominant equilibrium is locally asymptotically stable if $\mathcal{R}_0 > 1$.*

Remark 3.7. Since the dynamics of the system are primarily driven by addictive behavior rather than rational decision-making. Better understanding and modelling of such equilibrium in gambling can help policymakers and researchers develop more effective strategies for addressing problem gambling and mitigating its societal impacts.

4. Optimal control

In this section, we discuss the various optimal control strategies to inhibit the problem of online gambling. Basically, we are adopting three measures to control this problem.

- **Promote family education** $\tilde{u_1}(a,t)$- Under this strategy, various ad campaigns are run so that the people who lack family education become aware of the ill effects of online gambling.
- **Rehabilitation of occasional and addicted gamblers** $(\tilde{u_2}(a,t),\ \tilde{u_3}(a,t))$- Under this strategy, the gamblers are motivated to join rehabilitation centres, where they are trained to quit this addiction to online gambling.
- **Conviction of gambling operators** $(\tilde{u_4}(a,t))$- This strategy deals with the conviction of gambling operators, which only applied to them when they breached the laws governing online gambling.

Let $u_1(a,t) = \alpha(a) + \tilde{u_1}(a,t)$, $u_2(a,t) = \delta_e(a) + \tilde{u_2}(a,t)$, $u_3(a,t) = \delta_u(a) + \tilde{u_3}(a,t)$ and $u_1(a,t) = \omega(a) + \tilde{u_4}(a,t)$.

Remark 4.1. Using these new control variables $u_1(a,t), u_2(a,t), u_3(a,t)$ and $u_4(a,t)$ instead of the control variables $\tilde{u_1}(a,t), \tilde{u_2}(a,t), \tilde{u_3}(a,t)$ and $\tilde{u_4}(a,t)$ will not affect the analysis.

Define the age time domain as $D = (0, A_m) \times (0, T)$. Also, for the sake of simplicity, we would add the following assumption to our model.

Assumption 3. *Assume that $\lambda(a,t) = \phi_u(a,t)G(a,t)$ and $\psi(a,t) = \phi_e(a,t)G(a,t)$. Further, we have $\phi_u(a,t), \phi_e(a,t) \in L^\infty(D)$.*

Assumption 4. *Define a set $D_1 = \{h \in L^1_+(0, A_m) : 0 \leq h(a) \leq M\}$. Assume that the initial state of the system (4.1) is bounded, i.e.*

$$S_{u_0}(a), S_{e_0}(a), G_0(a), I_{e_0}(a), I_{u_0}(a), C_0(a), R_0(a) \in D_1$$

We get a new system below by adding these assumptions and the control means to the system (2.1).

$$\begin{aligned}
\frac{\partial S_u(a,t)}{\partial t} + \frac{\partial S_u(a,t)}{\partial a} &= -\phi_u(a,t)G(a,t)S_u(a,t) - \mu(a)S_u(a,t) - u_1(a,t)S_u(a,t),\\
\frac{\partial S_e(a,t)}{\partial t} + \frac{\partial S_e(a,t)}{\partial a} &= u_1(a,t)S_u(a,t) - \phi_e(a,t)G(a,t)S_e(a,t) - \mu(a)S_e(a,t),\\
\frac{\partial G(a,t)}{\partial t} + \frac{\partial G(a,t)}{\partial a} &= \tau(a)I_u(a,t) - \mu(a)G(a,t) - u_4(a,t)G(a,t),\\
\frac{\partial I_e(a,t)}{\partial t} + \frac{\partial I_e(a,t)}{\partial a} &= \phi_e(a,t)G(a,t)S_e(a,t) - \mu(a)I_e(a,t) - u_2(a,t)I_e(a,t),\\
\frac{\partial I_u(a,t)}{\partial t} + \frac{\partial I_u(a,t)}{\partial a} &= \phi_u(a,t)G(a,t)S_u(a,t) - \tau(a)I_u(a,t) - \mu(a)I_u(a,t) - u_3(a,t)I_u(a,t),\\
\frac{\partial C(a,t)}{\partial t} + \frac{\partial C(a,t)}{\partial a} &= u_4(a,t)G(a,t) - \sigma(a)C(a,t) - \mu(a)C(a,t),\\
\frac{\partial R(a,t)}{\partial t} + \frac{\partial R(a,t)}{\partial a} &= \sigma(a)C(a,t) + u_2(a,t)I_e(a,t) + u_3(a,t)I_u(a,t) - \mu(a)R(a,t),\\
&S_u(0,t) = \Lambda,\ S_e(0,t) = G(0,t) = I_e(0,t) = I_u(0,t) = C(0,t) = R(0,t) = 0,\\
&S_u(a,0) = S_{u_0}(a),\ S_e(a,0) = S_{e_0}(a),\ G(a,0) = G_0(a),\ I_e(a,0) = I_{e_0}(a),\\
&I_u(a,0) = I_{u_0}(a),\ C(a,0) = C_0(a),\ R(a,0) = R_0(a).
\end{aligned} \tag{4.1}$$

Our goal is to minimize gambling addiction, keeping in mind to reduce the cost of the various strategies we are implementing. So, we aim to minimize the objective functional:

$$\begin{aligned}
\mathcal{J}(u_1, u_2, u_3, u_4) = \int_0^T \int_0^{A_m} \Big(& c_1 G(a,t) + c_2 I_e(a,t) + c_3 I_u(a,t) + \frac{1}{2} c_4 u_1(a,t)^2 \\
& + \frac{1}{2} c_5 u_2(a,t)^2 + \frac{1}{2} c_6 u_3(a,t)^2 + \frac{1}{2} c_7 u_4(a,t)^2 \Big) da dt,
\end{aligned} \tag{4.2}$$

where $c_i (i = 1, 2, \ldots, 7)$ are the weight factors. Define the control set as

$$\Omega = \{(u_1, u_2, u_3, u_4) : u_i \in L^\infty(D), 0 \leq u_i \leq N_i \text{ for all } i = 1, 2, 3, 4\}. \tag{4.3}$$

4.1. Estimates on the state system

Estimates of the norm of the system's states are needed to support the convergence result and demonstrate the existence of optimal control for our model (see [6] for more details). These types of estimates are given in Theorem 4.2. But before coming to this, we would like to solve the system (4.1) using the method of characteristics; we obtain the solution as

$$S_u(a,t) = \begin{cases} -\int_0^t [\mu(s+a-t) + u_1(s+a-t,s)] S_u(s+a-t,s)ds \\ -\int_0^t \phi_u(s+a-t,s) G(s+a-t,s) S_u(s+a-t,s)ds \\ +S_{u_0}(a-t) \\ \text{if } a > t, \\ -\int_{t-a}^t [\mu(s+a-t) + u_1(s+a-t,s)] S_u(s+a-t,s)ds \\ -\int_{t-a}^t \phi_u(s+a-t,s) G(s+a-t,s) S_u(s+a-t,s)ds + \Lambda \\ \text{if } a < t, \end{cases} \tag{4.4}$$

$$S_e(a,t) = \begin{cases} -\int_0^t \mu(s+a-t) S_e(s+a-t,s)ds \\ -\int_0^t \phi_e(s+a-t,s) G(s+a-t,s) S_e(s+a-t,s)ds \\ +\int_0^t u_1(s+a-t,s) S_u(s+a-t,s)ds + S_{e_0}(a-t) \\ \text{if } a > t, \\ -\int_{t-a}^t \mu(s+a-t) S_e(s+a-t,s)ds \\ -\int_{t-a}^t \phi_e(s+a-t,s) G(s+a-t,s) S_e(s+a-t,s)ds \\ +\int_{t-a}^t u_1(s+a-t,s) S_u(s+a-t,s)ds \\ \text{if } a < t, \end{cases} \tag{4.5}$$

$$G(a,t) = \begin{cases} -\int_0^t [\mu(s+a-t) + u_4(s+a-t,s)] G(s+a-t,s)ds \\ +\int_0^t \tau(s+a-t) I_u(s+a-t,s)ds + G_0(a-t) \\ \text{if } a > t, \\ -\int_{t-a}^t [\mu(s+a-t) + u_4(s+a-t,s)] G(s+a-t,s)ds \\ +\int_{t-a}^t \tau(s+a-t) I_u(s+a-t,s)ds \\ \text{if } a < t, \end{cases} \tag{4.6}$$

$$I_e(a,t) = \begin{cases} -\int_0^t [\mu(s+a-t) + u_2(s+a-t,s)] I_e(s+a-t,s)ds \\ +\int_0^t \phi_e(s+a-t,s) G(s+a-t,s) S_e(s+a-t,s)ds \\ +I_{e_0}(a-t) \\ \text{if } a > t, \\ -\int_{t-a}^t [\mu(s+a-t) + u_2(s+a-t,s)] I_e(s+a-t,s)ds \\ +\int_{t-a}^t \phi_e(s+a-t,s) G(s+a-t,s) S_e(s+a-t,s)ds \\ \text{if } a < t, \end{cases} \tag{4.7}$$

$$I_u(a,t) = \begin{cases} -\int_0^t [\tau(s+a-t)+\mu(s+a-t)] I_u(s+a-t,s)ds \\ -\int_0^t u_3(s+a-t,s) I_u(s+a-t,s)ds \\ +\int_0^t \phi_u(s+a-t,s) G(s+a-t,s) S_u(s+a-t,s)ds \\ +I_{u_0}(a-t) \\ \text{if } a>t, \\ -\int_{t-a}^t [\tau(s+a-t)+\mu(s+a-t)] I_u(s+a-t,s)ds \\ -\int_{t-a}^t u_3(s+a-t,s) I_u(s+a-t,s)ds \\ +\int_{t-a}^t \phi_u(s+a-t,s) G(s+a-t,s) S_u(s+a-t,s)ds \\ \text{if } a<t, \end{cases} \tag{4.8}$$

$$C(a,t) = \begin{cases} -\int_0^t [\mu(s+a-t)+\sigma(s+a-t)] C(s+a-t,s)ds \\ +\int_0^t u_4(s+a-t,s) G(s+a-t,s)ds + C_0(a-t) \\ \text{if } a>t, \\ -\int_{t-a}^t [\mu(s+a-t)+\sigma(s+a-t)] C(s+a-t,s)ds \\ +\int_{t-a}^t u_4(s+a-t,s) G(s+a-t,s)ds \\ \text{if } a<t, \end{cases} \tag{4.9}$$

$$R(a,t) = \begin{cases} -\int_0^t \mu(s+a-t) R(s+a-t,s)ds \\ +\int_0^t u_3(s+a-t,s) I_u(s+a-t,s)ds \\ +\int_0^t u_2(s+a-t,s) I_e(s+a-t,s)ds \\ +\int_0^t \sigma(s+a-t) C(s+a-t,s)ds + R_0(a-t) \\ \text{if } a>t, \\ -\int_{t-a}^t \mu(s+a-t) R(s+a-t,s)ds \\ +\int_{t-a}^t u_3(s+a-t,s) I_u(s+a-t,s)ds \\ +\int_{t-a}^t u_2(s+a-t,s) I_e(s+a-t,s)ds \\ +\int_{t-a}^t \sigma(s+a-t) C(s+a-t,s)ds \\ \text{if } a<t. \end{cases} \tag{4.10}$$

Let $(S_u^U, S_e^U, G^U, I_e^U, I_u^U, C^U, R^U)$ and $(S_u^V, S_e^V, G^V, I_e^V, I_u^V, C^V, R^V)$ be the solutions of the system (4.1) with respect to the control variables $U = (u_1, u_2, u_3, u_4)$ and $V = (v_1, v_2, v_3, v_4)$, respectively. Then, we have the following result.

Theorem 4.2. *If the Assumption 4 holds, then for T sufficiently small, the following estimate holds*

$$\int_D \Big(|S_u^U - S_u^V| + |S_e^U - S_e^V| + |G^U - G^V| + |I_e^U - I_e^V| + |I_u^U - I_u^V| \\ + |C^U - C^V| + |R^U - R^V| \Big) dadt \le L_1 \int_D \Big(|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3| + |u_4 - v_4| \Big) dadt, \tag{4.11}$$

where $L_1 > 0$ is a constant depending on the bounds of initial data, μ, ϕ_u, ϕ_e, and also on the values of τ_+, σ_+. Moreover,

$$\Big(\|S_u^U - S_u^V\|_{\mathbb{Y}} + \|S_e^U - S_e^V\|_{\mathbb{Y}} + \|G^U - G^V\|_{\mathbb{Y}} + \|I_e^U - I_e^V\|_{\mathbb{Y}} + \|I_u^U - I_u^V\|_{\mathbb{Y}} + \|C^U - C^V\|_{\mathbb{Y}} + \|R^U - R^V\|_{\mathbb{Y}}\Big) \leq L_2\Big(\|u_1 - v_1\|_{\mathbb{Y}} + \|u_2 - v_2\|_{\mathbb{Y}} + \|u_3 - v_3\|_{\mathbb{Y}} + \|u_4 - v_4\|_{\mathbb{Y}}\Big), \tag{4.12}$$

where $\mathbb{Y} = L^\infty(D)$ and L_2 is a positive constant.

Proof. First of all Assumption 4 and Theorem 2.1 gives us

$$0 \leq S_u^i, S_e^i, G^i, I_e^i, I_u^i, C^i, R^i \leq M, \tag{4.13}$$

where $i = U, V$. We divide the integral $\int_D \Big|S_u^U - S_u^V|dadt$ into two parts $\int_{D\cap\{a<t\}} \Big|S_u^U - S_u^V|dadt$ and $\int_{D\cap\{a>t\}} \Big|S_u^U - S_u^V|dadt$. From the solution representation of $S_u(a,t)$ we can infer that

$$\int_{D\cap\{a>t\}} |S_u^U - S_u^V|dadt \leq \mu_+ T \int_D |S_u^U - S_u^V|dadt + TM \int_D |u_1 - v_1|dadt + C_1 T \int_D (|S_u^U - S_u^V| + |G^U - G^V|)dadt,$$

where $u_+ = \sup\{\mu(a) : a \in (0, A_m)\}$ and $C_1 = MC_u$, where C_u is the upper bound of ϕ_u. Similarly, for the case of $D \cap \{a < t\}$, the same type of estimate can be obtained. Thus

$$\int_D |S_u^U - S_u^V|dadt \leq \mu_+ T \int_D |S_u^U - S_u^V|dadt + TM \int_D |u_1 - v_1|dadt + C_1 T \int_D (|S_u^U - S_u^V| + |G^U - G^V|)dadt.$$

We can follow the same procedure for the other state variables. After adding all those terms and collecting the like terms, we get

$$\begin{aligned}&\int_D \Big(|S_u^U - S_u^V| + |S_e^U - S_e^V| + |G^U - G^V| + |I_e^U - I_e^V| + |I_u^U - I_u^V| + |C^U - C^V| + |R^U - R^V|\Big)dadt \\ &\leq C_2 \int_D \Big(|u_1 - v_1| + |u_2 - v_2| + |u_3 - v_3| + |u_4 - v_4|\Big)dadt + C_3 T \int_D \Big(|S_u^U - S_u^V| + |S_e^U - S_e^V| \\ &\quad + |G^U - G^V| + |I_e^U - I_e^V| + |I_u^U - I_u^V| + |C^U - C^V| + |R^U - R^V|\Big)dadt.\end{aligned}$$

Here C_2 and C_3 are the constants obtained from the bounds of initial data, μ, ϕ_u, ϕ_e and also from the values of τ_+, σ_+. For sufficiently small T we can obtain (4.11). Now, if we estimate the integration over the age variable only then for every $t \in [0, T]$, we get

$$\int_0^{A_m} |S_u^U - S_u^V|da \leq C_4 T \int_0^{A_m} (|S_u^U - S_u^V| + |G^U - G^V|)da + C_5 T\|u_1 - v_1\|_{L^\infty(D)}.$$

Doing a similar procedure for the other state variables, we get

$$
\begin{aligned}
&\int_0^{A_m} \Big(|S_u^U - S_u^V| + |S_e^U - S_e^V| + |G^U - G^V| + |I_e^U - I_e^V| + |I_u^U - I_u^V| + |C^U - C^V| + |R^U - R^V|\Big) da \\
&\quad \le C_5\Big(\|u_1 - v_1\|_{\mathbb{Y}} + \|u_2 - v_2\|_{\mathbb{Y}} + \|u_3 - v_3\|_{\mathbb{Y}} + \|u_4 - v_4\|_{\mathbb{Y}}\Big) + C_6 T \int_0^{A_m} \Big(|S_u^U - S_u^V| + |S_e^U - S_e^V| \\
&\qquad + |G^U - G^V| + |I_e^U - I_e^V| + |I_u^U - I_u^V| + |C^U - C^V| + |R^U - R^V|\Big) dadt.
\end{aligned}
$$

Since the above inequality holds for every $t \in [0,T]$, for sufficiently small T, we have

$$
\begin{aligned}
&\sup_{t\in[0,T]} \int_0^{A_m} \Big(|S_u^U - S_u^V| + |S_e^U - S_e^V| + |G^U - G^V| + |I_e^U - I_e^V| + |I_u^U - I_u^V| + |C^U - C^V| + |R^U - R^V|\Big) da \\
&\qquad \le C_5\Big(\|u_1 - v_1\|_{\mathbb{Y}} + \|u_2 - v_2\|_{\mathbb{Y}} + \|u_3 - v_3\|_{\mathbb{Y}} + \|u_4 - v_4\|_{\mathbb{Y}}\Big).
\end{aligned} \tag{4.14}
$$

For $a < t$, using the L^∞ estimate we have

$$
\begin{aligned}
|S_u^U - S_u^V| \le & \mu_+ \int_0^{A_m} |S_u^U - S_u^V|(s, s+t-a) ds + C_6 T \|u_1 - v_1\|_{\mathbb{Y}} \\
& + C_7 T (\|S_u^U - S_u^V\|_{\mathbb{Y}} + \|G^U - G^V\|_{\mathbb{Y}}).
\end{aligned}
$$

Similarly we can proceed for $a > t$ case. These equations will estimate $\|S_u^U - S_u^V\|_{\mathbb{Y}}$. Proceeding in the same manner for other state variables will give similar types of inequalities. Combining them all and using (4.14) will give us

$$
\begin{aligned}
&\Big[\|S_u^U - S_u^V\|_{\mathbb{Y}} + \|S_e^U - S_e^V\|_{\mathbb{Y}} + \|G^U - G^V\|_{\mathbb{Y}} + \|I_e^U - I_e^V\|_{\mathbb{Y}} + \|I_u^U - I_u^V\|_{\mathbb{Y}} + \|C^U - C^V\|_{\mathbb{Y}} \\
&\qquad + \|R^U - R^V\|_{\mathbb{Y}}\Big] \ \le \ \mu_+ C_5\Big(\|u_1 - v_1\|_{\mathbb{Y}} + \|u_2 - v_2\|_{\mathbb{Y}} + \|u_3 - v_3\|_{\mathbb{Y}} + \|u_4 - v_4\|_{\mathbb{Y}}\Big) \\
&\qquad + C_8 T\Big(\|S_u^U - S_u^V\|_{\mathbb{Y}} + \|S_e^U - S_e^V\|_{\mathbb{Y}} + \|G^U - G^V\|_{\mathbb{Y}} + \|I_e^U - I_e^V\|_{\mathbb{Y}} + \|I_u^U - I_u^V\|_{\mathbb{Y}} \\
&\qquad + \|C^U - C^V\|_{\mathbb{Y}} + \|R^U - R^V\|_{\mathbb{Y}}\Big).
\end{aligned}
$$

Thus, for sufficiently small T, we obtain the estimate (4.12). $\square$

4.2. Necessary condition for optimal control problem

In this subsection, we will establish the necessary conditions required for optimal control to exist. Let $l_1(\cdot), l_2(\cdot), l_3(\cdot), l_4(\cdot) \in L^\infty(0, A_m)$ be the variation function. For $\epsilon \in (0,1)$, define

$$
\begin{aligned}
& u_1^\epsilon(a,t) = u_1(a,t) + \epsilon l_1(a),\ \ u_2^\epsilon(a,t) = u_2(a,t) + \epsilon l_2(a),\ \ u_3^\epsilon(a,t) = u_3(a,t) + \epsilon l_3(a), \\
& u_4^\epsilon(a,t) = u_4(a,t) + \epsilon l_4(a).
\end{aligned}
$$

Let $U^\epsilon = (u_1^\epsilon, u_2^\epsilon, u_3^\epsilon, u_4^\epsilon)$, $U = (u_1, u_2, u_3, u_4)$ and $L = (l_1, l_2, l_3, l_4)$. To differentiate the objective function $\mathcal{J}$ with respect to the control variables u_1, u_2, u_3 and u_4, we need the differentiability of the map $(u_1, u_2, u_3, u_4) \to (S_u, S_v, G, I_e, I_u, C, R)[u_1, u_2, u_3, u_4]$. From Theorem 4.2 the above map is Lipschitz in

L^∞, hence from Barbu [2] we can conclude that the above map is differentiable and there exist Gateaux derivatives $\overline{S_u}, \overline{S_e}, \overline{G}, \overline{I_e}, \overline{I_u}, \overline{C}$ and $\overline{R}$ such that

$$\lim_{\epsilon\to 0}\left[\frac{(S_u,S_v,G,I_e,I_u,C,R)[U^\epsilon]-(S_u,S_v,G,I_e,I_u,C,R)[U]}{\epsilon}\right]=(\overline{S_u},\overline{S_e},\overline{G},\overline{I_e},\overline{I_u},\overline{C},\overline{R}). \tag{4.15}$$

Then system corresponding to $(\overline{S_u},\overline{S_e},\overline{G},\overline{I_e},\overline{I_u},\overline{C},\overline{R})$ is given by:

$$\begin{aligned}
\frac{\partial\overline{S_u}}{\partial t}+\frac{\partial\overline{S_u}}{\partial a}&=-\phi_u(a,t)(\overline{G}S_u+G\overline{S_u})-\mu(a)\overline{S_u}-u_1(a,t)\overline{S_u}-l_1(a)S_u,\\
\frac{\partial\overline{S_e}}{\partial t}+\frac{\partial\overline{S_e}}{\partial a}&=u_1(a,t)\overline{S_u}+l_1(a)S_u-\phi_e(a,t)(\overline{G}S_e+G\overline{S_e})-\mu(a)\overline{S_e},\\
\frac{\partial\overline{G}}{\partial t}+\frac{\partial\overline{G}}{\partial a}&=\tau(a)\overline{I_u}-\mu(a)\overline{G}-u_4(a,t)\overline{G}-l_4(a)G,\\
\frac{\partial\overline{I_e}}{\partial t}+\frac{\partial\overline{I_e}}{\partial a}&=\phi_e(a,t)(\overline{G}S_e+G\overline{S_e})-\mu(a)\overline{I_e}-u_2(a,t)\overline{I_e}-l_2(a)I_e,\\
\frac{\partial\overline{I_u}}{\partial t}+\frac{\partial\overline{I_u}}{\partial a}&=\phi_u(a,t)(\overline{G}S_u+G\overline{S_u})-\tau(a)\overline{I_u}-\mu(a)\overline{I_u}-u_3(a,t)\overline{I_u}-l_3(a)I_u,\\
\frac{\partial\overline{C}}{\partial t}+\frac{\partial\overline{C}}{\partial a}&=u_4(a,t)\overline{G}+l_4(a)G-\sigma(a)\overline{C}-\mu(a)\overline{C},\\
\frac{\partial\overline{R}}{\partial t}+\frac{\partial\overline{R}}{\partial a}&=\sigma(a)\overline{C}+u_2(a,t)\overline{I_e}+u_3(a,t)\overline{I_u}+l_2(a)I_e+l_3(a)I_u-\mu(a)\overline{R},\\
\overline{S_u}(0,t)&=\overline{S_e}(0,t)=\overline{G}(0,t)=\overline{I_e}(0,t)=\overline{I_u}(0,t)=\overline{C}(0,t)=\overline{R}(0,t)=0,\\
\overline{S_u}(a,0)&=\overline{S_e}(a,0)=\overline{G}(a,0)=\overline{I_e}(a,0)=\overline{I_u}(a,0)=\overline{C}(a,0)=\overline{R}(a,0)=0.
\end{aligned} \tag{4.16}$$

Therefore the directional derivative of $\mathcal{J}$ with respect to U in the direction of L is given by

$$\begin{aligned}
\mathcal{J}'(u_1,u_2,u_3,u_4)=\int_0^T\int_0^{A_m}&(c_1\overline{G}(a,t)+c_2\overline{I_e}(a,t)+c_3\overline{I_u}(a,t)+c_4u_1(a,t)l_1(a)\\
&+c_5u_2(a,t)l_2(a)+c_6u_3(a,t)l_3(a)+c_7u_4(a,t)l_4(a))dadt.
\end{aligned} \tag{4.17}$$

Next, we find the adjoint system corresponding to the system (4.16). Note that

$$\begin{aligned}
&\Big\langle\frac{\partial\overline{S_u}}{\partial t}+\frac{\partial\overline{S_u}}{\partial a}+\phi_u(a,t)(\overline{G}S_u+G\overline{S_u})+\mu(a)\overline{S_u}+u_1(a,t)\overline{S_u}+l_1(a)S_u,S_u^*(a,t)\Big\rangle_{L^1(D)}=0,\\
&\Big\langle\overline{S_u}(a,t),-\Big(\frac{\partial}{\partial t}+\frac{\partial}{\partial a}-\mu(a)-u_1(a,t)-\phi_u(a,t)G\Big)S_u^*\Big\rangle_{L^1(D)}\\
&\quad+\int_0^T\int_0^{A_m}(\phi_u(a,t)\overline{G}(a,t)+l_1(a))S_u(a,t)S_u^*(a,t)dadt=0,
\end{aligned} \tag{4.18}$$

with the conditions

$$S_u^*(A_m,t)=S_u^*(a,T)=0.$$

Now, repeating the same process for the other state variable, we have

$$
\begin{aligned}
&\Big\langle \frac{\partial \overline{S_e}}{\partial t} + \frac{\partial \overline{S_e}}{\partial a} - u_1(a,t)\overline{S_u} - l_1(a)S_u + \phi_e(a,t)(\overline{G}S_e + G\overline{S_e}) + \mu(a)\overline{S_e}, S_e^*(a,t)\Big\rangle_{L^1(D)} = 0,\\
&\Big\langle \overline{S_e}(a,t), -\Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial a} - \mu(a) - \phi_e(a,t)G\Big)S_e^*\Big\rangle_{L^1(D)} - \int_0^T\int_0^{A_m} l_1(a)S_u(a,t)S_e^*(a,t)\,dadt\\
&\quad + \int_0^T\int_0^{A_m} \phi_e(a,t)\overline{G}(a,t)S_e(a,t)S_e^*(a,t)\,dadt - \int_0^T\int_0^{A_m} u_1(a,t)\overline{S_u}(a,t)S_e^*(a,t)\,dadt = 0,
\end{aligned} \tag{4.19}
$$

$$
\begin{aligned}
&\Big\langle \frac{\partial \overline{G}}{\partial t} + \frac{\partial \overline{G}}{\partial a} - \tau(a)\overline{I_u} + \mu(a)\overline{G} + u_4(a,t)\overline{G} + l_4(a)G, G^*(a,t)\Big\rangle_{L^1(D)} = 0,\\
&\Big\langle \overline{G}(a,t), -\Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial a} - \mu(a) - u_4(a,t)\Big)G^*\Big\rangle_{L^1(D)} + \int_0^T\int_0^{A_m} l_4(a)G(a,t)G^*(a,t)\,dadt\\
&\quad - \int_0^T\int_0^{A_m} \tau(a)\overline{I_u}(a,t)G^*(a,t)\,dadt = 0,
\end{aligned} \tag{4.20}
$$

$$
\begin{aligned}
&\Big\langle \frac{\partial \overline{I_e}}{\partial t} + \frac{\partial \overline{I_e}}{\partial a} - \phi_e(a,t)(\overline{G}S_e + G\overline{S_e}) + \mu(a)\overline{I_e} + u_2(a,t)\overline{I_e} + l_2(a)I_e, I_e^*(a,t)\Big\rangle_{L^1(D)} = 0,\\
&\Big\langle \overline{I_e}(a,t), -\Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial a} - \mu(a) - u_2(a,t)\Big)I_e^*\Big\rangle_{L^1(D)} + \int_0^T\int_0^{A_m} l_2(a)I_e(a,t)I_e^*(a,t)\,dadt\\
&\quad - \int_0^T\int_0^{A_m} \phi_e(a,t)\Big(\overline{G}(a,t)S_e(a,t) + G(a,t)\overline{S_e}(a,t)\Big)I_e^*(a,t)\,dadt = 0,
\end{aligned} \tag{4.21}
$$

$$
\begin{aligned}
&\Big\langle \frac{\partial \overline{I_u}}{\partial t} + \frac{\partial \overline{I_u}}{\partial a} - \phi_u(a,t)(\overline{G}S_u + G\overline{S_u}) + \tau(a)\overline{I_u} + \mu(a)\overline{I_u} + u_3(a,t)\overline{I_u} + l_3(a)I_u, I_u^*(a,t)\Big\rangle_{L^1(D)} = 0,\\
&\Big\langle \overline{I_u}(a,t), -\Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial a} - \tau(a) - \mu(a) - u_3(a,t)\Big)I_u^*\Big\rangle_{L^1(D)} + \int_0^T\int_0^{A_m} l_3(a)I_u(a,t)I_u^*(a,t)\,dadt\\
&\quad - \int_0^T\int_0^{A_m} \phi_u(a,t)\Big(\overline{G}(a,t)S_u(a,t) + G(a,t)\overline{S_u}(a,t)\Big)I_u^*(a,t)\,dadt = 0,
\end{aligned} \tag{4.22}
$$

$$
\begin{aligned}
&\Big\langle \frac{\partial \overline{C}}{\partial t} + \frac{\partial \overline{C}}{\partial a} - u_4(a,t)\overline{G} - l_4(a)G + \sigma(a)\overline{C} + \mu(a)\overline{C}, C^*(a,t)\Big\rangle_{L^1(D)} = 0,\\
&\Big\langle \overline{C}(a,t), -\Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial a} - \mu(a) - \sigma(a)\Big)C^*\Big\rangle_{L^1(D)} - \int_0^T\int_0^{A_m} l_4(a)G(a,t)C^*(a,t)\,dadt\\
&\quad - \int_0^T\int_0^{A_m} u_4(a,t)\overline{G}(a,t)C^*(a,t)\,dadt = 0,
\end{aligned} \tag{4.23}
$$

$$\begin{aligned}
&\Big\langle \frac{\partial \overline{R}}{\partial t} + \frac{\partial \overline{R}}{\partial a} - \sigma(a)\overline{C} - u_2(a,t)\overline{I_e} - u_3(a,t)\overline{I_u} - l_2(a)I_e - l_3(a)I_u + \mu(a)\overline{R}, R^*(a,t)\Big\rangle_{L^1(D)} = 0,\\
&\Big\langle \overline{R}(a,t), -\Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial a} - \mu(a)\Big)R^*\Big\rangle_{L^1(D)} - \int_0^T\int_0^{A_m}(l_2(a)I_e(a,t) + l_3(a)I_u(a,t))R^*(a,t)dadt\\
&\quad - \int_0^T\int_0^{A_m}\sigma(a)\overline{C}(a,t)R^*(a,t)dadt - \int_0^T\int_0^{A_m}(u_2(a,t)\overline{I_e}(a,t) + u_3(a,t)\overline{I_u}(a,t))R^*(a,t)dadt = 0.
\end{aligned} \tag{4.24}$$

So, the adjoint system we get is:

$$\begin{aligned}
\frac{\partial S_u^*}{\partial t} + \frac{\partial S_u^*}{\partial a} &= (\mu(a) + u_1(a,t) + \phi_u(a,t)G)S_u^* - u_1(a,t)S_e^* - \phi_u(a,t)GI_u^*,\\
\frac{\partial S_e^*}{\partial t} + \frac{\partial S_e^*}{\partial a} &= (\mu(a) + \phi_e(a,t)G)S_e^* - \phi_e(a,t)GI_e^*,\\
\frac{\partial G^*}{\partial t} + \frac{\partial G^*}{\partial a} &= (\mu(a) + u_4(a,t))G^* + \phi_u(a,t)S_u(S_u^* - I_u^*) + \phi_e(a,t)S_e(S_e^* - I_e^*) - u_4(a,t)C^* + c_1,\\
\frac{\partial I_e^*}{\partial t} + \frac{\partial I_e^*}{\partial a} &= (\mu(a) + u_2(a,t))I_e^* - u_2(a,t)R^* + c_2,\\
\frac{\partial I_u^*}{\partial t} + \frac{\partial I_u^*}{\partial a} &= (\tau(a) + \mu(a) + u_3(a,t))I_u^* - \tau(a)G^* - u_3(a,t)R^* + c_3,\\
\frac{\partial C^*}{\partial t} + \frac{\partial C^*}{\partial a} &= (\mu(a) + \sigma(a))C^* - \sigma(a)R^*,\\
\frac{\partial R^*}{\partial t} + \frac{\partial R^*}{\partial a} &= \mu(a)R^*,
\end{aligned} \tag{4.25}$$

and the transversality conditions are

$$S_u^*(\cdot,T) = S_e^*(\cdot,T) = G^*(\cdot,T) = I_e^*(\cdot,T) = I_u^*(\cdot,T) = C(\cdot,T) = R(\cdot,T) = 0,$$

$$S_u^*(A^m,\cdot) = S_e^*(A^m,\cdot) = G^*(A^m,\cdot) = I_e^*(A^m,\cdot) = I_u^*(A^m,\cdot) = C^*(A^m,\cdot) = R^*(A^m,\cdot) = 0.$$

Now, similar to the Theorem 4.2, we have the following theorem regarding the estimates of the state variables of the adjoint system.

Theorem 4.3. *Let $(S_u^{*,U}, S_u^{*,U}, G^{*,U}, I_u^{*,U}, I_e^{*,U}, C^{*,U}, R^{*,U})$ and $(S_u^{*,V}, S_u^{*,V}, G^{*,V}, I_u^{*,V}, I_e^{*,V}, C^{*,V}, R^{*,V})$ be the solutions of the system (4.25) corresponding to the control variables $U = (u_1,u_2,u_3,u_4)$ and $V = (v_1,v_2,v_3,v_4)$, respectively. Then*

$$\begin{aligned}
&\Big(\|S_u^{*,U} - S_u^{*,V}\|_{\mathbb{Y}} + \|S_e^{*,U} - S_e^{*,V}\|_{\mathbb{Y}} + \|G^{*,U} - G^{*,V}\|_{\mathbb{Y}} + \|I_e^{*,U} - I_e^{*,V}\|_{\mathbb{Y}} + \|I_u^{*,U} - I_u^{*,V}\|_{\mathbb{Y}}\\
&\quad + \|C^{*,U} - C^{*,V}\|_{\mathbb{Y}} + \|R^{*,U} - R^{*,V}\|_{\mathbb{Y}}\Big) \le L_3\Big(\|u_1 - v_1\|_{\mathbb{Y}} + \|u_2 - v_2\|_{\mathbb{Y}} + \|u_3 - v_3\|_{\mathbb{Y}} + \|u_4 - v_4\|_{\mathbb{Y}}\Big),
\end{aligned} \tag{4.26}$$

where L_3 is a positive constant.

The proof of this theorem is similar to Theorem 4.2, so we have omitted the proof here. Now, we are in a position to show the existence of optimal control.

4.3. Existence of optimal control

In this subsection, we have proved the existence of the optimal control. Not only this, but we have also shown the explicit form of the optimal control. We embed our objective functional in the space $L^1(D)$ by defining

$$J(u_1, u_2, u_3, u_4) = \begin{cases} \mathcal{J}(u_1, u_2, u_3, u_4) & \text{if } (u_1, u_2, u_3, u_4) \in \Omega, \\ \infty & \text{otherwise.} \end{cases} \tag{4.27}$$

Therefore, we have the following theorem regarding the existence of the optimal control.

Theorem 4.4. *There exists an optimal control $U^* = (u_1^*, u_2^*, u_3^*, u_4^*) \in \Omega$ for the control problem (4.1) such that*

$$\min_{(u_1,u_2,u_3,u_4)\in\Omega} J(u_1, u_2, u_3, u_4) = J(u_1^*, u_2^*, u_3^*, u_4^*).$$

Proof. From (4.3) the control variables u_1, u_2, u_3 and u_4 are uniformly bounded. Therefore from [19] there exists a minimizing sequence $U_n = (u_{1_n}, u_{2_n}, u_{3_n}, u_{4_n})$ for the objective functional J, i.e.

$$\lim_{n\to\infty} J(u_{1_n}, u_{2_n}, u_{3_n}, u_{4_n}) = \inf_{(u_1,u_2,u_3,u_4)\in\Omega} J(u_1, u_2, u_3, u_4).$$

Next, from Theorem 4.2, we can observe that the bound on S_u, S_e, G, I_e, I_u, C and R depends on the control variable u_1, u_2, u_3 and u_4. Thus, we have $S_{u_n} = S_u(U_n), S_{e_n} = S_e(U_n), G_n = G(U_n), I_{e_n} = I_e(U_n), I_{u_n} = I_u(U_n), C_n = C(U_n), R_n = R(U_n)$. Again from Theorem 4.2, $S_{u_n}, S_{e_n}, G_n, I_{e_n}, I_{u_n}, C_n$ and R_n are uniformly bounded for each n. Therefore, we have $S_u^* = S_u(U^*), S_e^* = S_e(U^*), G^* = G(U^*), I_e^* = I_e(U^*), I_u^* = I_u(U^*), C^* = C(U^*), R^* = R(U^*)$. Also, because of the lower semi-continuity of the control functions $u_{i_n}^2(a,t)$, we have

$$\int_D u_{i_n}^2(a,t)dadt \leq \liminf_{n\to\infty} \int_D u_{i_n}^2(a,t)dadt,$$

for each $i = 1, 2, 3, 4$. Now,

$$\begin{aligned} J(u_1^*, u_2^*, u_3^*, u_4^*) &= \int_D (c_1 G^*(a,t) + c_2 I_e^*(a,t) + c_3 I_u^*(a,t) + c_4 u_1(a,t)^2 \\ &\qquad + c_5 u_2^*(a,t)^2 + c_6 u_3^*(a,t)^2 + c_7 u_4^*(a,t)^2) dadt \\ &\leq \liminf_{n\to\infty} \int_D (c_1 G_n(a,t) + c_2 I_{e_n}(a,t) + c_3 I_{u_n}(a,t) + c_4 u_{1_n}(a,t)^2 \\ &\qquad + c_5 u_{2_n}^*(a,t)^2 + c_6 u_{3_n}^*(a,t)^2 + c_7 u_{4_n}^*(a,t)^2) dadt \\ &= \lim_{n\to\infty} J(u_{1_n}, u_{2_n}, u_{3_n}, u_{4_n}) \\ &= \inf_{(u_1,u_2,u_3,u_4)\in\Omega} J(u_1, u_2, u_3, u_4). \end{aligned} \tag{4.28}$$

Hence $J(U^*) \leq \inf_{U\in\Omega} J(U)$. Therefore, U^* optimizes the objective functional J. $\square$

The next theorem is related to the explicit form of the optimal control.

Theorem 4.5. *Let* $(S_u, S_e, G, I_e, I_u, C, R)$ *and* $(S_u^*, S_e^*, G^*, I_e^*, I_u^*, C^*, R^*)$ *be the solutions of control system (4.1) and its adjoint system (4.25), respectively. If* $U^* = (u_1^*, u_2^*, u_3^*, u_4^*)$ *is an optimal control that minimizes the objective functional* $J(u_1, u_2, u_3, u_4)$*, given in (4.27) then*

$$u_1^*(a,t) = \max\left\{0, \min\left\{N_1, \frac{S_u(S_e^* - S_u^*)}{c_4}\right\}\right\}, \quad u_2^*(a,t) = \max\left\{0, \min\left\{N_2, \frac{I_e(R^* - I_e^*)}{c_5}\right\}\right\},$$
$$u_3^*(a,t) = \max\left\{0, \min\left\{N_3, \frac{I_u(R^* - I_u^*)}{c_6}\right\}\right\}, \quad u_4^*(a,t) = \max\left\{0, \min\left\{N_4, \frac{G(C^* - G^*)}{c_7}\right\}\right\}.$$

Proof. First, note that from the adjoint system (4.25), we can write

$$\begin{aligned}
&\int_0^T\int_0^{A_m}(c_1\overline{G}(a,t) + c_2\overline{I_e}(a,t) + c_3\overline{I_u}(a,t))dadt = \int_0^T\int_0^{A_m}\overline{G}(a,t)\Big[\frac{\partial G^*}{\partial t} + \frac{\partial G^*}{\partial a} - (\mu(a) \\
&\quad + u_4(a,t))G^*(a,t) - \phi_u(a,t)S_u(a,t)(S_u^*(a,t) - I_u^*(a,t)) - \phi_e(a,t)S_e(a,t)(S_e^*(a,t) - I_e^*(a,t)) \\
&\quad + u_4(a,t)C^*(a,t)\Big]dadt + \int_0^T\int_0^{A_m}\overline{I_e}(a,t)\Big[\frac{\partial I_e^*}{\partial t} + \frac{\partial I_e^*}{\partial a} - (\mu(a) + u_2(a,t))I_e^*(a,t) \\
&\quad + u_2(a,t)R^*(a,t)\Big]dadt + \int_0^T\int_0^{A_m}\overline{I_u}(a,t)\Big[\frac{\partial I_u^*}{\partial t} + \frac{\partial I_u^*}{\partial a} - (\tau(a) + \mu(a) + u_3(a,t))I_u^*(a,t) \\
&\quad + \tau(a)G^*(a,t) + u_3(a,t)R^*(a,t)\Big]dadt + \int_0^T\int_0^{A_m}\overline{S_u}(a,t)\Big[\frac{\partial S_u^*}{\partial t} + \frac{\partial S_u^*}{\partial a} \\
&\quad - (\mu(a) + u_1(a,t) + \phi_u(a,t)G(a,t))S_u^*(a,t) + u_1(a,t)S_e^*(a,t) + \phi_u(a,t)G(a,t)I_u^*(a,t)\Big]dadt \\
&\quad + \int_0^T\int_0^{A_m}\overline{S_e}(a,t)\Big[\frac{\partial S_e^*}{\partial t} + \frac{\partial S_e^*}{\partial a} - (\mu(a) + \phi_e(a,t)G(a,t))S_e^*(a,t) + \phi_e(a,t)G(a,t)I_e^*(a,t)\Big]dadt \\
&\quad + \int_0^T\int_0^{A_m}\overline{C}(a,t)\Big[\frac{\partial C^*}{\partial t} + \frac{\partial C^*}{\partial a} - (\mu(a) + \sigma(a))C^*(a,t) + \sigma(a)R^*(a,t)\Big]dadt \\
&\quad + \int_0^T\int_0^{A_m}\overline{R}(a,t)\Big[\frac{\partial R^*}{\partial t} + \frac{\partial R^*}{\partial a} - \mu(a)R^*(a,t)\Big]dadt.
\end{aligned} \tag{4.29}$$

Now using the equations (4.18)-(4.24), we get

$$\begin{aligned}
\int_0^T\int_0^{A_m}(c_1\overline{G}(a,t) + c_2\overline{I_e}(a,t) + c_3\overline{I_u}(a,t))dadt &= \int_0^T\int_0^{A_m} l_1(a)S_u(a,t)(S_u^*(a,t) - S_e^*(a,t))dadt \\
&+ \int_0^{A_m} l_2(a)I_e(a,t)(I_e^*(a,t) - R^*(a,t))dadt + \int_0^{A_m} l_3(a)I_u(a,t)(I_u^*(a,t) - R^*(a,t))dadt \\
&+ \int_0^{A_m} l_4(a)G(a,t)(G^*(a,t) - C^*(a,t))dadt.
\end{aligned} \tag{4.30}$$

From (4.17), we have

$$
\begin{aligned}
J'(u_1,u_2,u_3,u_4) = &\int_0^T \int_0^{A_m} (c_1\overline{G}(a,t) + c_2\overline{I_e}(a,t) + c_3\overline{I_u}(a,t) + c_4u_1(a,t)l_1(a) \\
&+ c_5u_2(a,t)l_2(a) + c_6u_3(a,t)l_3(a) + c_7u_4(a,t)l_4(a))dadt.
\end{aligned}
\tag{4.31}
$$

Using (4.30), we get

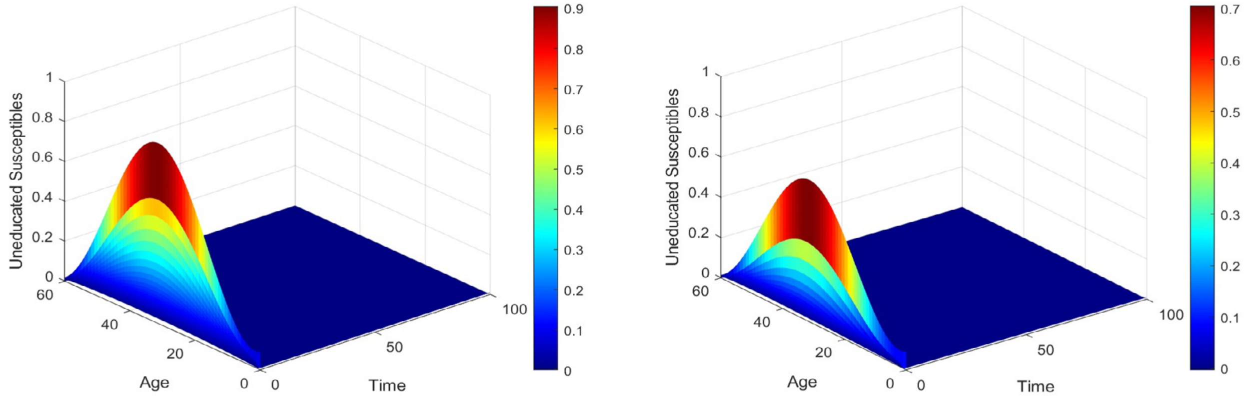
$$
\begin{aligned}
J'(u_1,u_2,u_3,u_4) = &\int_0^T \int_0^{A_m} l_1(a)\Big[S_u(a,t)(S_u^*(a,t) - S_e^*(a,t)) + c_4u_1(a,t)\Big]dadt \\
&+ \int_0^T \int_0^{A_m} l_2(a)\Big[I_e(a,t)(I_e^*(a,t) - R^*(a,t)) + c_5u_2(a,t)\Big]dadt \\
&+ \int_0^T \int_0^{A_m} l_3(a)\Big[I_u(a,t)(I_u^*(a,t) - R^*(a,t)) + c_6u_3(a,t)\Big]dadt \\
&+ \int_0^T \int_0^{A_m} l_4(a)\Big[G(a,t)(G^*(a,t) - C^*(a,t)) + c_7u_4(a,t)\Big]dadt
\end{aligned}
\tag{4.32}
$$

Since, we seek to minimize the objective functional J, thus if $(u_1^*, u_2^*, u_3^*, u_4^*)$ is an optimal control for J, then $J'(u_1^*, u_2^*, u_3^*, u_4^*) \geq 0$. At last, by using the standard optimality arguments, i.e., using the upper bound and lower bound of the controls defined in (4.3), we get

$$
\begin{aligned}
&u_1^*(a,t) = \max\left\{0, \min\left\{N_1, \frac{S_u(S_e^* - S_u^*)}{c_4}\right\}\right\}, \quad u_2^*(a,t) = \max\left\{0, \min\left\{N_2, \frac{I_e(R^* - I_e^*)}{c_5}\right\}\right\}, \\
&u_3^*(a,t) = \max\left\{0, \min\left\{N_3, \frac{I_u(R^* - I_u^*)}{c_6}\right\}\right\}, \quad u_4^*(a,t) = \max\left\{0, \min\left\{N_4, \frac{G(C^* - G^*)}{c_7}\right\}\right\}. \quad \square
\end{aligned}
\tag{4.33}
$$

5. Numerical simulation

In this section, we present some numerical simulations to justify our theoretical findings. We have used an explicit finite difference scheme to solve the original state system without any control. To solve the optimal control problem, we have used a forward-backwards sweep method. In order to use the forward-backwards sweep approach, we must first make a reasonable guess about the initial value of the control variable. The state system must subsequently be solved using forward integration, and the adjoint system must then be solved using backward integration utilizing the solutions that were previously acquired. Control variables must be updated in the last step until the two neighboring optimal solutions are sufficiently close enough. Please refer to [19] for more details regarding this method. The initial age distributions chosen are:



(a) (b)

Fig. 3. Population dynamics of uneducated susceptible individuals ($S_u(a,t)$). Fig. 3(a) without family education ($u_1(a,t)=0$) and Fig. 3(b) with family education ($u_1(a,t)=0.5e^{-0.02a}$).

$$
\begin{aligned}
S_{u_0}(a) &= 0.9\sin(0.05a^2)+0.09e^{-0.09a},\\
S_{e_0}(a) &= 0.7\sin(0.05a^2)+0.01e^{-0.9a},\\
G_0(a) &= 0.9\sin(0.061a^2)+0.1e^{-0.09a},\\
I_{e_0}(a) &= 0.005\sin(0.05a^2)+0.1e^{-0.7a},\\
I_{u_0}(a) &= 0.009\sin(0.05a^2)+0.09e^{-0.01a},\\
C_0(a) &= 0.9\sin(0.05a^2)+0.03e^{-0.1a},\\
R_0(a) &= 0.9\sin(0.05a^2)+0.04e^{-0.1a}.
\end{aligned}
$$

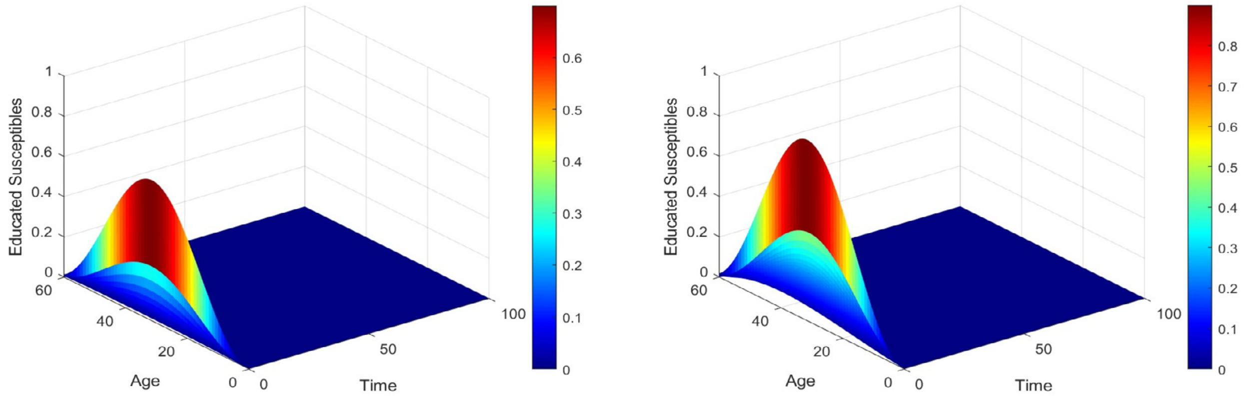
Keeping in mind all assumptions regarding the parameters used in the model, we choose $\mu(a)=\frac{10a}{a+0.01}$, $\phi_u(a,t)=1.98e^{-0.3a}$, $\phi_e(a,t)=1.99e^{-0.4a}$, $\tau(a)=0.5e^{-0.9a}$, $\sigma(a)=\frac{1}{a+0.03}$ and $\Lambda=0.9$.

From Fig. 3 - Fig. 6, we discuss the dynamical behavior of susceptible individuals and gamblers, with and without control measures. For the time being, we restrict ourselves to choosing time-independent control variables. The dynamics of each control variable are shown in Fig. 7.

Fig. 3a, 3b and Fig. 4a, 4b clearly show the decrease in the population density of uneducated susceptible individuals, which further corresponds to an increase in the population density of educated susceptible individuals in the presence of family education control.

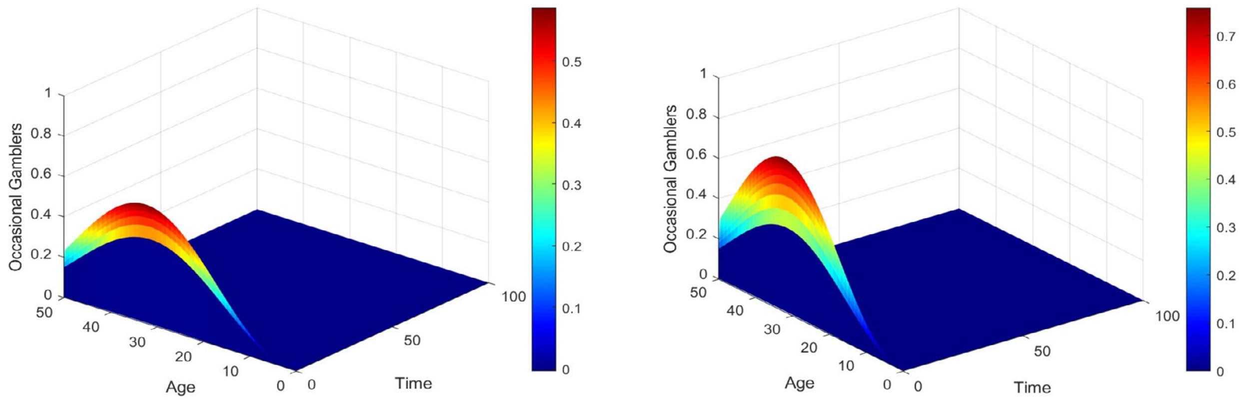
Since introducing family education control has decreased the uneducated susceptible individuals. This means that now the population density of addicted gamblers must also be reduced. This was well shown in Fig. 5 and Fig. 6, where rehabilitation control is also considered in addition to family education control. From Figure- 5b, note that including both control measures increases the density of occasional gamblers, but this should not lead to any problems because the occasional gamblers are well aware of the disadvantages of online gambling and they know up to how much extent online they should involve in online gambling.

Fig. 7 shows the dynamics of the control variables. Observing these figures, we can observe that the control variables u_1, u_2, u_3 and u_4 return zero values for the age between 0 and 17, which makes sense because individuals in this category are less likely to be involved in online gambling. Hence, there is no need to apply control for these individuals, as we also have to minimize the control cost. Furthermore, from Figs. 7b, 7c and 7d, we can observe that the controls u_2, u_3 and u_4 acquire a stable behavior as time passes. But this does not happen in the case of control u_1, as can be seen from Fig. 7a. Initially, as time progresses, the control u_1 first increases, then acquires a stable state between the time interval $[10, 20]$, and from there on, it goes on decreasing. This makes the control variables u_1 superior to the other control variables because the cost of implementing this control will be lower than that of other control measures.



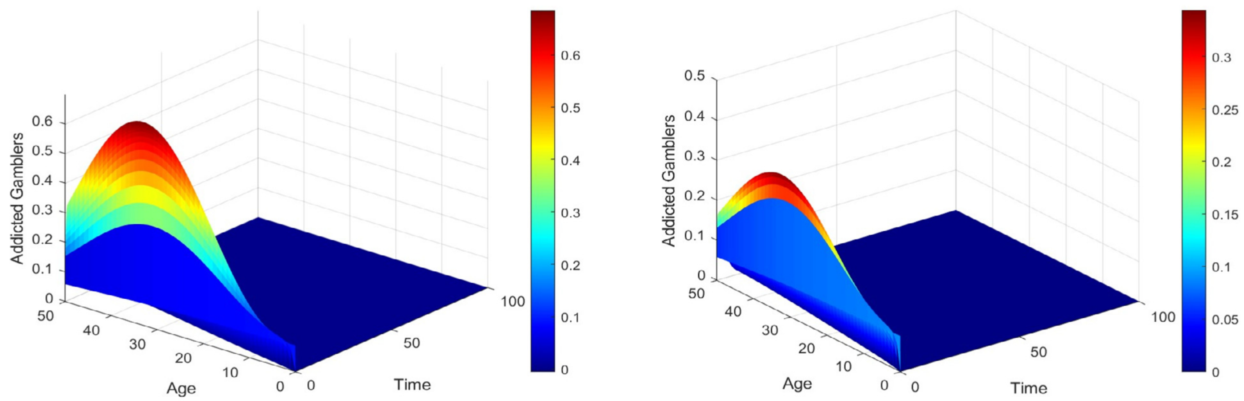
(a) (b)

Fig. 4. Population dynamics of educated susceptible individuals ($S_e(a,t)$). Fig. 4(a) without family education ($u_1(a,t) = 0$) and Fig. 4(b) with family education ($u_1(a,t) = 0.5e^{-0.02a}$).



(a) (b)

Fig. 5. Population dynamics of occasional gamblers. Fig. 5(a) without family education and rehabilitation ($u_1(a,t) = 0, u_2(a,t) = 0$) and Fig. 5(b) with family education and rehabilitation ($u_1(a,t) = 0.5e^{-0.02a}$, $u_2(a,t) = \frac{6}{a+0.06}$).



(a) (b)

Fig. 6. Population dynamics of addicted gamblers. Fig. 6(a) without family education and rehabilitation ($u_1(a,t) = 0, u_3(a,t) = 0$) and Fig. 6(b) with family education and rehabilitation ($u_1(a,t) = 0.5e^{-0.02a}$, $u_3(a,t) = \frac{7}{a+0.0007}$).

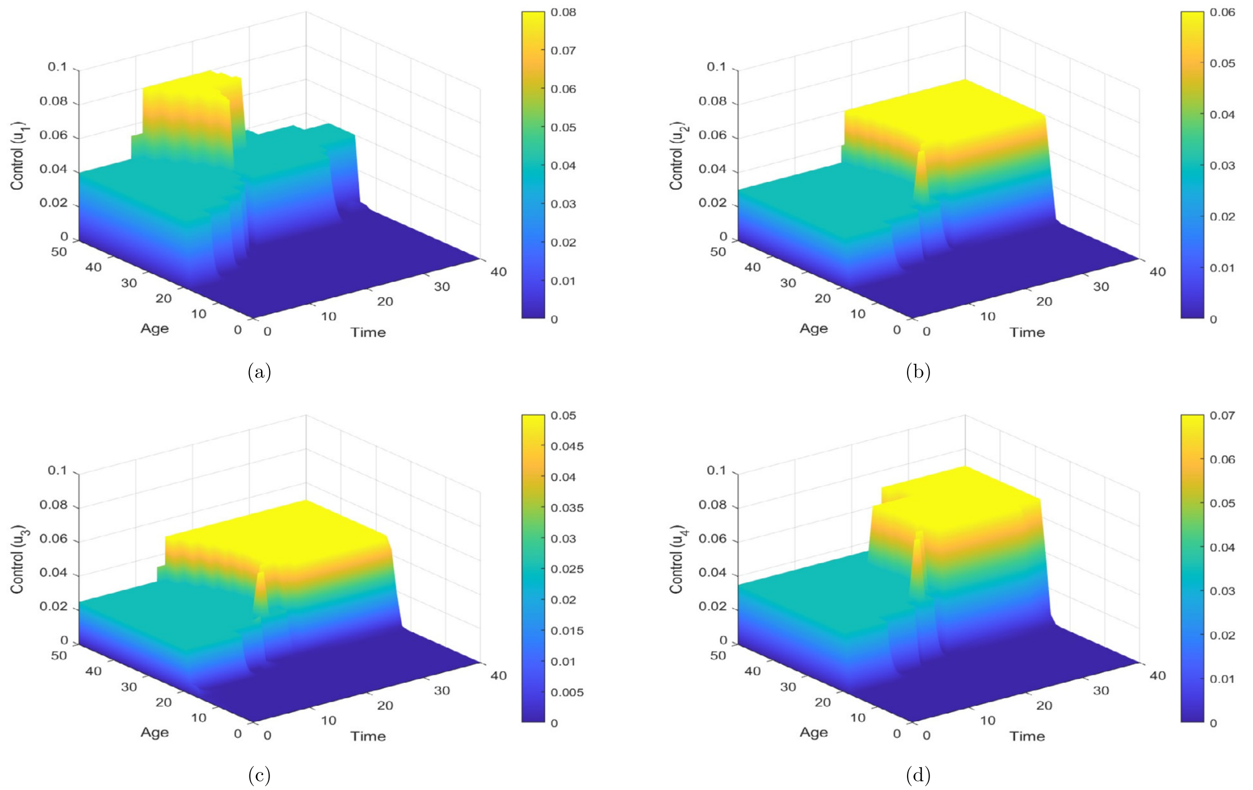


Fig. 7. Dynamics of control variables.

Also, we know that preventing something bad from happening is much better than controlling it later when it has already happened. Thus, by implementing family education control, we are making sure that we are restricting susceptible individuals from becoming addicted gamblers.

Remark 5.1. In numerous previous studies (see, for example, [11–13]), control parameters are typically considered constants or functions of time alone. In contrast, the control parameters used in this work are modelled as functions of both time and age. This approach enhances their applicability to real-life scenarios, as individuals of different ages exhibit varying tendencies to engage in gambling activities, necessitating age-specific control measures. Additionally, we conducted a comparison of the population density of occasional gamblers and addicted gamblers under two scenarios: without any control measures and with two control measures applied simultaneously—family education and rehabilitation. The results demonstrated superior performance compared to other referenced studies, as the latter did not incorporate rehabilitation as a control measure.

Remark 5.2. Unlike other studies that do not explore the dynamics of the control variables, our work not only derives the optimal control forms for all four control parameters but also provides a numerical justification demonstrating the superiority of family education control over other measures.

6. Conclusions

As is well known, there are distinct patterns of gambling behavior and addiction risk across various age groups. Younger people may be more susceptible because of things like less impulse control and more risk-taking behavior, whereas elderly people may be more susceptible because of other things like boredom or financial strain. Further, it is evident that education plays a crucial role in a gambling-prone environment.

Education can help individuals understand the risks involved in gambling activities. People are able to control their involvement in gambling more effectively and make better decisions because of this understanding.

In this work, we proposed a novel age-structured model for online gambling addiction, incorporating family education as well. The purpose of this study was to see the impact of family education on gambling addiction and to investigate the optimal strategy to control this addiction. From the form of threshold parameter $\mathcal{R}_0$, it can be seen that the addictive behavior towards online gambling is mostly driven by addicted gamblers. In addition, it was shown that whenever $\mathcal{R}_0 < 1$, the addiction-free equilibrium is stable and is unstable when $\mathcal{R}_0 > 1$. Moreover, we also showed that whenever $\mathcal{R}_0 > 1$, there exists a unique and positive addiction-predominant equilibrium that is stable.

In the numerical simulations part, it was established that whenever we include family education and rehabilitation control in the model, there is a decrease in the number of addicted gamblers. Finally, by plotting the control variables with respect to age and time, it has been shown that family education is the most efficient way to control the spread of online gambling addiction.

Incorporating family education into an age-structured online gambling addiction model provides a comprehensive approach to preventing and addressing gambling addiction. It emphasizes the critical role of families in early intervention, the effectiveness of tailored strategies, and the broader social and economic benefits of informed and supportive family environments. Because the model's assumptions and outcomes are fairly realistic, policymakers will be better able to comprehend the issue and develop programs that teach parents and guardians how to set limits, keep an eye on their children's online activity, and encourage honest conversations about gambling.

It is open for researchers to incorporate other factors and develop a more advanced model. Factors like the isolation of addicted gamblers (i.e. keeping them away from any source through which they can be involved in gambling activities) and the media effect can have a huge impact on gambling addiction. Also, including delay in the model which represents a cooling-off period (a certain time period where the user cannot gamble after significant wins or losses) is worth considering. We have excluded certain variables such as psychological factors and social influences, which can also be considered. In the simulation part, we have discussed the impact of time-independent control measures on dynamics of state variables. So, the impact of time-dependent controls can also be studied using numerical simulations.

Also, getting real data for the verification and estimation of parameters is difficult, so we have performed a rigorous analysis on the given model. To the best of our knowledge, it is the first attempt to study the impact of family education and some other interventions on online gambling addiction using the age-structure mathematical modelling approach. We encourage other researchers to look into this field.

Contribution

All authors contributed equally.

Declaration of competing interest

Authors declare none.

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Data availability

Data sharing is not applicable as no dataset is used.

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