Assignment 4 **GIORDANO VITALE ID:14310A**

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QUESTION 1

AR(1) process: $$Y_t = c + a Y_{t-1} + u_t \qquad (1)$$

1) Formula for h-steps-ahead forecast $\hat{y}_{T+h|T}$

For $h=1$, $\hat{Y}_{T+1} = \hat{c} + \hat{a}\, Y_T$

For $h=2$, $\hat{Y}_{T+2} = \hat{c} + \hat{a}\, \hat{Y}_{T+1}$

$$\rightarrow \hat{Y}_{T+2} = \hat{c} + \hat{a}\left(\hat{c} + \hat{a}\, Y_T\right)$$
$$= \hat{c} + \hat{a}\hat{c} + \hat{a}^2 Y_T$$
$$= \hat{c}(1+\hat{a}) + \hat{a}^2 Y_T$$

For $h=3$, $Y_{T+3} = \hat{c} + \hat{a}\, \hat{Y}_{T+2}$

$$= \hat{c} + \hat{a}\left(\hat{c} + \hat{a}\hat{c} + \hat{a}^2 Y_T\right)$$
$$= \hat{c} + \hat{a}\hat{c} + \hat{a}^2\hat{c} + \hat{a}^3 Y_T$$
$$= \hat{c}\left(1 + \hat{a} + \hat{a}^2\right) + \hat{a}^3 Y_T$$

We can spot a pattern and summarize as follows:

$$\boxed{\hat{Y}_{T+h} = c \cdot \sum_{j=0}^{h-1} a^j + a^h Y_T}$$

2) Suppose $c = 0$, $a = 0.7$, $Y_T = 5$

- $h = 1$ $\quad \hat{Y}_{T+1|T} = 0 + 0.7 \cdot 5 = 3.5$
- $h = 2$ $\quad \hat{Y}_{T+2|T} = 0 + (0.7)^2 \cdot 5 = 2.45$
- $h = 30$ $\quad \hat{Y}_{T+30|T} = 0 + (0.7)^{30} \cdot 5 \simeq 0{,}00012$

Predictions are converging to 0, which is the mean.

QUESTION 2

y_t and x_t follow a VAR(1) process with no intercept

$$y_t = a y_{t-1} + b x_{t-1} + u_{1,t}$$

$$x_t = c y_{t-1} + d x_{t-1} + u_{2,t}$$

1) DERIVE THE FORMULA FOR 1-STEP-AHEAD FORECAST FOR y_t AND x_t

Bivariate VAR(1) model in vector notation is:

$$\begin{bmatrix} y_t \\ x_t \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y_{t-1} \\ x_{t-1} \end{bmatrix} + \begin{bmatrix} u_{1,t} \\ u_{2,t} \end{bmatrix} \qquad u \sim WN(0, \Sigma_u)$$

$$\rightarrow Y_t = A_c Y_{t-1} + U_t$$

$$\begin{bmatrix} \hat{y}_{T+1|T} \\ \hat{x}_{T+1|T} \end{bmatrix} = \mathbb{E}_T \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} y_t \\ x_t \end{bmatrix} + \begin{bmatrix} u_{1,T+1} \\ u_{2,T+1} \end{bmatrix} \right)$$

$$= \mathbb{E}_T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} y_t \\ x_t \end{bmatrix}\right) + \underbrace{\mathbb{E}_T\begin{bmatrix} u_{1,T+1} \\ u_{2,T+1} \end{bmatrix}}_{=0}$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \mathbb{E}_T \begin{bmatrix} y_t \\ x_t \end{bmatrix}$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} y_t \\ x_t \end{bmatrix}$$

$$\boxed{\begin{bmatrix} \hat{y}_{T+1|T} \\ \hat{x}_{T+1|T} \end{bmatrix} = \begin{bmatrix} a y_t + b x_t \\ c y_t + d x_t \end{bmatrix}}$$

2) SHOW THAT 2-STEPS-AHEAD FORECAST CAN BE WRITTEN AS

$$\hat{y}_{T+2|T} = \delta_1 y_T + \delta_2 x_T$$

Recall that $y_{T+2} = a y_{T+1} + b x_{T+1} + u_{1,T+1}$

$$\hat{y}_{T+2|T} = \mathbb{E}_T\left(a y_{T+1} + b x_{T+1} + u_{1,T+1}\right)$$

$$= a\,\mathbb{E}_T\left(y_{T+1}\right) + b\,\mathbb{E}_T\left(x_{T+1}\right) + \underbrace{\mathbb{E}_T\left(u_{T,T+1}\right)}_{=0}$$

$$= a\left(a y_T + b x_T\right) + b\left(c y_T + d x_T\right)$$

$$= a^2 y_T + ab x_T + bc y_T + bd x_T$$

$$= y_T\left(a^2 + bc\right) + x_T\left(ab + bd\right)$$

Stating $\boxed{a^2 + bc = \delta_1}$ and $\boxed{ab + bd = \delta_2}$,

$$\boxed{\hat{y}_{T+2|T} = \delta_1 y_T + \delta_2 x_T}$$

3) NOW DERIVE THE EXPRESSION FOR 2-STEP-AHEAD FORECAST FOR THE FOLLOWING MODEL

$$y_t = a y_{t-1} + b x_{t-1} + u_t$$

Recalling that the direct forecast for h is

$$y^{(h)}_{T+h} = a^{(h)} y_T + b_0^{(h)} x_{T+h} + b_1^{(h)} x_T + u^{(h)}_{T+h}$$

We can generate ex-ante forecast only if we drop the term x_{T+h} associated with $b_0^{(h)}$:

$$y^{(h)}_{T+h} = a^{(h)} y_T + b_1^{(h)} x_T + u^{(h)}_{T+h}$$

Thus, for $h=2$

$$\boxed{y^{(2)}_{T+2} = a^{(2)} y_T + b_1^{(2)} x_T}$$

The forecast might differ from the one at point (2) because $a^{(2)}$ can be different from $\delta_1 = a^2 + bc$, as well as $b_1^{(2)}$ can differ from $\delta_2 = ab + bd$.

Moreover, in this model there are not parameters c, d unlike in point (2).