The N-Gamma-Ti-Te model

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1 Introduction

The equations considered are:

- Continuity equation
$$\partial_t n + \nabla\cdot(nu\boldsymbol{b}) - \nabla\cdot(D\nabla_\perp n) = \hat{S}_n. \tag{1}$$

- Momentum equation
$$\partial_t(m_i nu) + \nabla\cdot(m_i nu^2\boldsymbol{b}) + \nabla_\parallel(k_b n(T_e + T_i)) - \nabla\cdot(\mu\nabla_\perp(m_i nu)) = \hat{S}_\Gamma. \tag{2}$$

- Total ions energy equation
$$\begin{aligned}&\partial_t(\frac{3}{2}k_b nT_i + \frac{1}{2}m_i nu^2) + \nabla\cdot((\frac{5}{2}k_b nT_i + \frac{1}{2}m_i nu^2)u\boldsymbol{b}) - nueE_\parallel + \\ &- \nabla\cdot\left(\frac{3}{2}k_b(TiD\nabla_\perp n + n\chi_i\nabla_\perp T_i)\right) - \nabla\cdot(-\frac{1}{2}m_i u^2 D\nabla_\perp n + \frac{1}{2}m_i\mu n\nabla_\perp u^2) + \\ &- \nabla\cdot(k_{\parallel i}T_i^{5/2}\nabla_\parallel T_i\boldsymbol{b}) + \frac{3}{2}\frac{k_b n}{\hat{\tau_{ie}}}(T_e - T_i) = \hat{S}_{E_i}.\end{aligned} \tag{3}$$

- Total electrons energy equation
$$\begin{aligned}&\partial_t(\frac{3}{2}k_b nT_e) + \nabla\cdot(\frac{5}{2}k_b nT_e u\boldsymbol{b}) + nueE_\parallel - \nabla\cdot\left(\frac{3}{2}k_b(T_e D\nabla_\perp n + n\chi_e\nabla_\perp T_e)\right) + \\ &- \nabla\cdot(k_{\parallel e}T_e^{5/2}\nabla_\parallel T_e\boldsymbol{b}) - \frac{3}{2}\frac{k_b n}{\hat{\tau_{ie}}}(T_e - T_i) = \hat{S}_{E_e}.\end{aligned} \tag{4}$$

The ion and electron pressures are defined as
$$\hat{p}_i = nT_i k_b \quad \hat{p}_e = nT_e k_b, \tag{5}$$
while the parallel electric field is given by
$$neE_\parallel = -\nabla_\parallel(k_b nT_e) = -\nabla_\parallel\hat{p}_e. \tag{6}$$
Finally the temperature exchange coefficient $\hat{\tau_{ie}} \sim [s]$ is written as
$$\hat{\tau_{ie}} = \tau_{ie}\frac{T_e^{3/2}}{n}, \tag{7}$$
where τ_{ie} can be computed as
$$\tau_{ie} = \frac{3\sqrt{2}}{e^4}\frac{{\varepsilon_0}^2}{\Lambda}\pi^{\frac{3}{2}}\frac{m_i}{m_e}\sqrt{m_e}\;e^{\frac{3}{2}}, \tag{8}$$
and $\Lambda \approx 12$.

1.1 Some assumptions and arrangements

The equations presented are re-elaborated:

- Continuity equation: this equation is left unchanged.
- Momentum equation: this equation is divided by m_i and the specific pressures $p_i, p_e \sim [m^{-1}s^{-2}]$ are introduced

  $$p_i = \frac{\hat{p}_i}{m_i}, \quad p_e = \frac{\hat{p}_e}{m_i}, \tag{9}$$

  hence it becomes

  $$\partial_t(nu) + \boldsymbol{\nabla}\cdot(nu^2\boldsymbol{b} + nu\boldsymbol{u}_\perp) + \boldsymbol{\nabla}_\parallel(p_i + p_e) - \boldsymbol{\nabla}\cdot(\mu\boldsymbol{\nabla}_\perp nu) = S_\Gamma, \tag{10}$$

  where $S_\Gamma = \hat{S}_\Gamma / m_i$.
- Total ions energy equation: this equation is divided by m_i and the specific total energy for ions $E_i \sim [m^2s^{-2}]$ is introduced

  $$E_i = \frac{3}{2}\frac{k_b}{m_i}T_i + \frac{1}{2}u^2 = \frac{3}{2}\frac{p_i}{n} + \frac{1}{2}u^2. \tag{11}$$

  The assumption $D = \mu = \chi_i$ is made, hence the equation becomes

  $$\begin{aligned}\partial_t(nE_i) + \boldsymbol{\nabla}\cdot\big((nE_i + p_i)u\boldsymbol{b} + nE_i\boldsymbol{u}_\perp\big) - \boldsymbol{\nabla}\cdot(\chi_i\boldsymbol{\nabla}_\perp nE_i) - \boldsymbol{\nabla}\cdot(\frac{k_{\parallel i}}{m_i}T_i^{5/2}\boldsymbol{\nabla}_\parallel T_i\boldsymbol{b}) + u\boldsymbol{\nabla}_\parallel p_e \\ + \frac{3}{2}\frac{n^2k_b}{\tau_{ie}m_iT_e^{3/2}}(T_e - T_i) = S_{E_i},\end{aligned}$$

  (12)
  where $S_{E_i} = \hat{S}_{E_i}/m_i$.
- Total electrons energy equation: this equation is divided by m_i and the specific total energy for electrons $E_e \sim [m^2s^{-2}]$ is introduced

  $$E_e = \frac{3}{2}\frac{k_b}{m_i}T_e = \frac{3}{2}\frac{p_i}{n}. \tag{13}$$

  The assumption $D = \mu = \chi_e$ is made, hence the equation becomes

$$\begin{aligned}\partial_t(nE_e) + \boldsymbol{\nabla}\cdot\big((nE_e + p_e)u\boldsymbol{b} + nE_e\boldsymbol{u}_\perp\big) - \boldsymbol{\nabla}\cdot(\chi_e\boldsymbol{\nabla}_\perp nE_e) - \boldsymbol{\nabla}\cdot(\frac{k_{\parallel e}}{m_i}T_e^{5/2}\boldsymbol{\nabla}_\parallel T_e\boldsymbol{b}) - u\boldsymbol{\nabla}_\parallel p_e \\ - \frac{3}{2}\frac{n^2k_b}{\tau_{ie}m_iT_e^{3/2}}(T_e - T_i) = S_{E_e},\end{aligned} \tag{14}$$

where $S_{E_e} = \hat{S}_{E_e}/m_i$.

2 System of dimensional equations

The system of equations is

$$
\begin{cases}
\partial_t n + \boldsymbol{\nabla}\cdot(nu\boldsymbol{b} + n\boldsymbol{u}_\perp) - \boldsymbol{\nabla}\cdot(D\boldsymbol{\nabla}_\perp n) = S_n, \\
\partial_t(nu) + \boldsymbol{\nabla}\cdot(nu^2\boldsymbol{b} + nu\boldsymbol{u}_\perp) + \boldsymbol{\nabla}_\parallel(p_i + p_e) - \boldsymbol{\nabla}\cdot(\mu\boldsymbol{\nabla}_\perp nu) = S_\Gamma, \\
\partial_t(nE_i) + \boldsymbol{\nabla}\cdot\big((nE_i + p_i)u\boldsymbol{b} + nE_i\boldsymbol{u}_\perp\big) - \boldsymbol{\nabla}\cdot(\chi_i\boldsymbol{\nabla}_\perp nE_i) - \boldsymbol{\nabla}\cdot(\frac{k_{\parallel i}}{m_i}T_i^{5/2}\boldsymbol{\nabla}_\parallel T_i\boldsymbol{b}) + u\boldsymbol{\nabla}_\parallel p_e \\
\qquad + \frac{3}{2}\frac{n^2 k_b}{\tau_{ie} m_i T_e^{3/2}}(T_e - T_i) = S_{E_i}, \\
\partial_t(nE_e) + \boldsymbol{\nabla}\cdot\big((nE_e + p_e)u\boldsymbol{b} + nE_e\boldsymbol{u}_\perp\big) - \boldsymbol{\nabla}\cdot(\chi_e\boldsymbol{\nabla}_\perp nE_e) - \boldsymbol{\nabla}\cdot(\frac{k_{\parallel e}}{m_i}T_e^{5/2}\boldsymbol{\nabla}_\parallel T_e\boldsymbol{b}) - u\boldsymbol{\nabla}_\parallel p_e \\
\qquad - \frac{3}{2}\frac{n^2 k_b}{\tau_{ie} m_i T_e^{3/2}}(T_e - T_i) = S_{E_e},
\end{cases}
\tag{15}
$$

where the ionic and electronic specific pressures are

$$
p_i = \frac{nT_i k_b}{m_i}, \quad p_e = \frac{nT_e k_b}{m_i}, \tag{16}
$$

and the ionic and electronic energies are

$$
E_i = \frac{1}{2}u^2 + \frac{3}{2}\frac{p_i}{n}, \quad E_e = \frac{3}{2}\frac{p_e}{n}. \tag{17}
$$

The Bohm boundary condition is expressed in terms of parallel flux of energies for ions and electrons, that is

$$
\begin{aligned}
q_\parallel^i &= \gamma_i u p_i + \frac{1}{2}nu^3, \\
q_\parallel^e &= \gamma_e u p_e,
\end{aligned}
$$

which provides, after replacing the expression of the parallel energy fluxes,

$$
\begin{aligned}
(nE_i + p_i)u - \frac{k_{\parallel i}}{m_i}T_i^{5/2}\boldsymbol{\nabla}_\parallel T_i &= \gamma_i u p_i + \frac{1}{2}nu^3, \\
(nE_e + p_e)u - \frac{k_{\parallel e}}{m_i}T_e^{5/2}\boldsymbol{\nabla}_\parallel T_e &= \gamma_e u p_e.
\end{aligned}
\tag{18}
$$

3 Non-dimensionalization

A set of reference values is defined

Density:	n_0,
Length:	L_0,
Time:	t_0,
Velocity:	$u_0 = \dfrac{L_0}{t_0}$,
Temperature:	T_0,

and the non-dimensional quantities and operators are

$$\text{Density:} \qquad n^* = \frac{n}{n_0},$$
$$\text{Velocity:} \qquad u^* = \frac{u}{u_0},$$
$$\text{Energy:} \qquad E^* = \frac{E}{{u_0}^2},$$
$$\text{Time derivative:} \qquad \partial_t^* = \frac{\partial_t}{t_0},$$
$$\text{Nabla:} \qquad \boldsymbol{\nabla}^* = \frac{\boldsymbol{\nabla}}{L_0}.$$

3.1 The continuity equation

The continuity equation

$$\partial_t n + \boldsymbol{\nabla}\cdot(nu\boldsymbol{b} + n\boldsymbol{u}_\perp) - \boldsymbol{\nabla}\cdot(D\boldsymbol{\nabla}_\perp n) = S_n,$$

is rewritten using the reference values

$$\frac{n_0}{t_0}\partial_t^* n^* + \frac{n_0 u_0}{L_0}\boldsymbol{\nabla}\cdot^*(n^* u^* \boldsymbol{b} + n^* \boldsymbol{u}_\perp^*) - \frac{n_0}{{L_0}^2}\boldsymbol{\nabla}\cdot^*(D\boldsymbol{\nabla}_\perp^* n^*) = S_n,$$

and rearranging and dropping the stars gives

$$\partial_t n + \boldsymbol{\nabla}\cdot(nu\boldsymbol{b} + n\boldsymbol{u}_\perp) - \frac{t_0}{{L_0}^2}\ \boldsymbol{\nabla}\cdot(D\boldsymbol{\nabla}_\perp n) = \frac{t_0}{n_0}S_n, \tag{19}$$

which allows to define the non-dimensional perpendicular coefficient for the density

$$\boxed{D = D\frac{t_0}{{L_0}^2}}, \tag{20}$$

and the non-dimensional source of density

$$S_n = \frac{t_0}{n_0}S_n.$$

3.2 The momentum equation

From definition of the specific pressures

$$p_i = nT_i k_b/m_i, \quad p_e = nT_e k_b/m_i,$$

the non-dimensional specific pressures is obtained

$$p_i^* = n^* T_i^*, \quad p_e^* = n^* T_e^*,$$

hence

$$p_i = \frac{n_0 T_0 k_b}{m_i}p_i^*, \quad p_e = \frac{n_0 T_0 k_b}{m_i}p_e^*, \tag{21}$$

The momentum equation

$$\partial_t(nu) + \boldsymbol{\nabla}\cdot(nu^2\boldsymbol{b} + nu\boldsymbol{u}_\perp) + \boldsymbol{\nabla}_\parallel(p_i + p_e) - \boldsymbol{\nabla}\cdot(\mu\boldsymbol{\nabla}_\perp nu) = S_\Gamma$$

is rewritten using the reference values

$$\frac{n_0 u_0}{t_0}\partial_t^*(n^* u^*)+\frac{n_0 {u_0}^2}{t_0}\boldsymbol{\nabla}\cdot^*(n^* {u^*}^2\boldsymbol{b}+n^* u^*\boldsymbol{u}_\perp)+\frac{n_0 T_0 k_b}{L_0 m_i}\boldsymbol{\nabla}_\parallel^*(p_i^*+p_e^*)-\frac{n_0 u_0}{{L_0}^2}\boldsymbol{\nabla}\cdot^*(\mu\boldsymbol{\nabla}_\perp n^* u^*)=S_\Gamma.$$

Defining the reference Mach (squared)

$$\boxed{M_{ref}=\frac{T_0 k_b}{m_i {u_0}^2}}$$

rearranging the terms and dropping the stars, we obtain

$$\partial_t(nu)+\boldsymbol{\nabla}\cdot(nu^2\boldsymbol{b}+nu\boldsymbol{u}_\perp)+M_{ref}\boldsymbol{\nabla}_\parallel(p_i+p_e)-\frac{t_0}{{L_0}^2}\boldsymbol{\nabla}\cdot(\mu\boldsymbol{\nabla}_\perp nu)=\frac{t_0}{n_0 u_0}S_\Gamma, \tag{22}$$

which allows to define the non-dimensional perpendicular diffusion coefficient for the momentum,

$$\boxed{\mu=\mu\frac{t_0}{{L_0}^2}}, \tag{23}$$

and the non-dimensional source of momentum

$$S_\Gamma=\frac{t_0}{n_0 u_0}S_\Gamma.$$

3.3 The ions energy equation

From the ion energy definition, using (21), we obtain

$$E_i=\frac{1}{2}u^2+\frac{3}{2}\frac{p_i}{n}={u_0}^2\frac{1}{2}{u^*}^2+\frac{\cancel{n_0}T_0 k_b}{m_i\cancel{n_0}}\frac{3}{2}\frac{p_i^*}{n^*},$$

and therefore the non-dimensional energy for the ions is

$$E_i^*=\frac{E_i}{{u_0}^2}=\frac{1}{2}{u^*}^2+\frac{3}{2}\frac{p_i^*}{n^*}M_{ref}.$$

The ions energy equation

$$\partial_t(nE_i)+\boldsymbol{\nabla}\cdot\big((nE_i+p_i)u\boldsymbol{b}+nE_i\boldsymbol{u}_\perp\big)-\boldsymbol{\nabla}\cdot(\chi_i\boldsymbol{\nabla}_\perp nE_i)-\boldsymbol{\nabla}\cdot(k_{\parallel i}T_i^{5/2}\boldsymbol{\nabla}_\parallel T_i\boldsymbol{b})+u\boldsymbol{\nabla}_\parallel p_e+\frac{3}{2}\frac{n^2}{\tau_{ie}T_e^{3/2}}(T_e-T_i)=S_{E_i}$$

becomes then

$$\begin{aligned}\frac{n_0 {u_0}^2}{t_0}\partial_t^*(n^* E_i^*)+\frac{1}{L_0}\boldsymbol{\nabla}\cdot^*\Big((n_0 {u_0}^2 n^* E_i^*+\frac{n_0 T_0 k_b}{m_i}p_i^*)u_0 u\boldsymbol{b}\Big)-\frac{n_0 {u_0}^2}{{L_0}^2}\boldsymbol{\nabla}\cdot^*(\chi_i\boldsymbol{\nabla}_\perp^*(n^* E_i^*))+\\ -\frac{{T_0}^{7/2}}{{L_0}^2}\boldsymbol{\nabla}\cdot^*(\frac{k_{\parallel i}}{m_i}{T_i^*}^{5/2}\boldsymbol{\nabla}_\parallel^* T_i^*\boldsymbol{b})+\frac{u_0 n_0 T_0 k_b}{L_0 m_i}u^*\boldsymbol{\nabla}_\parallel(p_e^*)+\frac{3}{2}\frac{k_b {n_0}^2 T_0}{m_i\tau_{ie}{T_0}^{3/2}}\frac{{n^*}^2}{{T_e^*}^{3/2}}(T_e^*-T_i^*)=S_{E_i}.\end{aligned}$$

Rearranging the terms and dropping the stars we obtain

$$\begin{aligned}\partial_t(nE_i)+\boldsymbol{\nabla}\cdot\big((nE_i+M_{ref}p_i)u\boldsymbol{b}+nE_i\boldsymbol{u}_\perp\big)-\frac{t_0}{{L_0}^2}\boldsymbol{\nabla}\cdot(\chi_i\boldsymbol{\nabla}_\perp nE_i)-\frac{{t_0}^3{T_0}^{7/2}}{{L_0}^4 n_0}\boldsymbol{\nabla}\cdot(\frac{k_{\parallel i}}{m_i}T_i^{5/2}\boldsymbol{\nabla}_\parallel T_i\boldsymbol{b})\\ +M_{ref}u\boldsymbol{\nabla}_\parallel p_e+\frac{3}{2}\frac{t_0 n_0 k_b}{\tau_{ie}m_i{T_0}^{1/2}{u_0}^2}\frac{n^2}{T_e^{3/2}}(T_e-T_i)=\frac{t_0}{n_0{u_0}^2}S_{E_i},\end{aligned} \tag{24}$$

which allows to define the non-dimensional perpendicular diffusion coefficient for the ion energy

$$\boxed{\chi_i = \chi_i \frac{t_0}{{L_0}^2}}, \tag{25}$$

the non-dimensional parallel diffusion coefficient for the temperature

$$\boxed{k_{\parallel i} = k_{\parallel i} \frac{{t_0}^3 {T_0}^{7/2}}{{L_0}^4 n_0 m_i}}, \tag{26}$$

the non-dimensional relaxation time for ion-energy temperatures

$$\boxed{\tau_{ie} = \frac{2}{3} \frac{\tau_{ie} {T_0}^{1/2} m_i {u_0}^2}{n_0 t_0 k_b}}, \tag{27}$$

and the non-dimensional source of energy

$$S_E = \frac{t_0}{n_0 {u_0}^2} S_E.$$

3.4 The electron energy equation

From the electron energy definition, using (21), we obtain

$$E_e = \frac{3}{2}\frac{p_e}{n} = \frac{\cancel{n_0} T_0 k_b}{m_i \cancel{n_0}} \frac{3}{2} \frac{p_e^*}{n^*},$$

and therefore the non-dimensional energy for the electrons is

$$E_e^* = \frac{E_e}{{u_0}^2} = \frac{3}{2}\frac{p_e^*}{n^*} M_{ref},$$

The electron energy equation

$$\partial_t (nE_e) + \boldsymbol{\nabla}\cdot\big((nE_e + p_e)u\boldsymbol{b} + nE_e \boldsymbol{u}_\perp\big) - \boldsymbol{\nabla}\cdot(\chi_e \boldsymbol{\nabla}_\perp nE_e) - \boldsymbol{\nabla}\cdot(\frac{k_{\parallel e}}{m_i} T_e^{5/2} \boldsymbol{\nabla}_\parallel T_e \boldsymbol{b}) - u\boldsymbol{\nabla}_\parallel p_e - \frac{3}{2}\frac{n^2 k_b}{\tau_{ie} m_i T_e^{3/2}}(T_e - T_i) = S_{E_e}$$

becomes then

$$\frac{n_0 {u_0}^2}{t_0} \partial_t^* (n^* E_e^*) + \frac{1}{L_0} \boldsymbol{\nabla}\cdot^* \Big((n_0 {u_0}^2 n^* E_e^* + \frac{n_0 T_0 k_b}{m_i} p_e^*) u_0 u \boldsymbol{b} \Big) - \frac{n_0 {u_0}^2}{{L_0}^2} \boldsymbol{\nabla}\cdot^* (\chi_e \boldsymbol{\nabla}_\perp^* (n^* E_e^*)) +$$
$$- \frac{{T_0}^{7/2}}{{L_0}^2} \boldsymbol{\nabla}\cdot^* (\frac{k_{\parallel e}}{m_i} {T_e^*}^{5/2} \boldsymbol{\nabla}_\parallel^* T_e^* \boldsymbol{b}) - \frac{u_0 n_0 T_0 k_b}{L_0 m_i} u^* \boldsymbol{\nabla}_\parallel (p_e^*) - \frac{3}{2} \frac{{n_0}^2 k_b T_0}{\tau_{ie} m_i {T_0}^{3/2}} \frac{{n^*}^2}{{T_e^*}^{3/2}} (T_e^* - T_i^*) = S_{E_e}.$$

Rearranging the terms and dropping the stars we obtain

$$\partial_t (nE_e) + \boldsymbol{\nabla}\cdot\big((nE_e + M_{ref} p_e)u\boldsymbol{b} + nE_e \boldsymbol{u}_\perp\big) - \frac{t_0}{{L_0}^2} \boldsymbol{\nabla}\cdot(\chi_e \boldsymbol{\nabla}_\perp nE_e) - \frac{{t_0}^3 {T_0}^{7/2}}{{L_0}^4 n_0 m_i} \boldsymbol{\nabla}\cdot(k_{\parallel e} T_e^{5/2} \boldsymbol{\nabla}_\parallel T_e \boldsymbol{b})$$
$$- M_{ref} u \boldsymbol{\nabla}_\parallel p_e - \frac{3}{2} \frac{t_0 n_0 k_b}{\tau_{ie} m_i {T_0}^{1/2} {u_0}^2} \frac{n^2}{T_e^{3/2}} (T_e - T_i) = \frac{t_0}{n_0 {u_0}^2} S_{E_e}, \tag{28}$$

which allows to define the non-dimensional perpendicular diffusion coefficient for the electron energy

$$\boxed{\chi_e = \chi_e \frac{t_0}{{L_0}^2}}, \tag{29}$$

and the non-dimensional parallel diffusion coefficient for the electron temperature

$$\boxed{k_{\parallel e} = k_{\parallel e} \frac{{t_0}^3 {T_0}^{7/2}}{{L_0}^4 n_0 m_i}}, \tag{30}$$

and the non-dimensional source of energy

$$S_{E_e} = \frac{t_0}{n_0 {u_0}^2} S_{E_e}.$$

3.5 The Bohm boundary condition

Using (21), the Bohm boundary condition (18)

$$\begin{aligned}
(nE_i + p_i)u - \frac{k_{\parallel i}}{m_i} T_i^{5/2} \nabla_\parallel T_i &= \gamma_i u p_i + \frac{1}{2} n u^3, \\
(nE_e + p_e)u - \frac{k_{\parallel e}}{m_i} T_e^{5/2} \nabla_\parallel T_e &= \gamma_e u p_e.
\end{aligned}$$

is written as

$$\begin{aligned}
(n_0 {u_0}^2 n^* E_i^* + \frac{n_0 T_0 k_b}{m_i} p_i^*) u_0 u^* - \frac{{T_0}^{7/2}}{L_0 m_i} k_{\parallel i} {T_i^*}^{5/2} \nabla_\parallel T_i^* &= \gamma_i \frac{n_0 T_0 k_b}{m_i} u_0 u^* p_i^* + n_0 {u_0}^3 \frac{1}{2} n^* {u^*}^3, \\
(n_0 {u_0}^2 n^* E_e^* + \frac{n_0 T_0 k_b}{m_i} p_e^*) u_0 u^* - \frac{{T_0}^{7/2}}{L_0 m_i} k_{\parallel e} {T_e^*}^{5/2} \nabla_\parallel T_e^* &= \gamma_e \frac{n_0 T_0 k_b}{m_i} u_0 u^* p_e^*.
\end{aligned}$$

Rearranging the terms and dropping the stars we obtain

$$\begin{aligned}
(nE_i + \frac{T_0 k_b}{m_i {u_0}^2} p_i) u - \frac{{t_0}^3 {T_0}^{7/2}}{{L_0}^4 n_0 m_i} k_{\parallel i} {T_i}^{5/2} \nabla_\parallel T_i &= \gamma_i \frac{T_0 k_b}{m_i {u_0}^2} u p_i + \frac{1}{2} n u^3, \\
(nE_e + \frac{T_0 k_b}{m_i {u_0}^2} p_e) u - \frac{{t_0}^3 {T_0}^{7/2}}{{L_0}^4 n_0 m_i} k_{\parallel e} {T_e}^{5/2} \nabla_\parallel T_e &= \gamma_e \frac{T_0 k_b}{m_i {u_0}^2} u p_e.
\end{aligned}$$

which gives

$$\begin{aligned}
\Big(nE_i + M_{ref}(1-\gamma_i) p_i\Big) u - k_{\parallel i} {T_i}^{5/2} \nabla_\parallel T_i - \frac{1}{2} n u^3 &= 0, \\
\Big(nE_e + M_{ref}(1-\gamma_e) p_e\Big) u - k_{\parallel e} {T_e}^{5/2} \nabla_\parallel T_e &= 0.
\end{aligned} \tag{31}$$

3.6 Reference values and physical parameters

The choice of the reference values is

L_0	$1.901\ 10^{-3}$	$[m]$
t_0	$1.3736\ 10^{-7}$	$[s]$
n_0	10^{19}	$[m^{-3}]$
u_0	$1.3839\ 10^4$	$[ms^{-1}]$
T_0	50	$[eV]$

Other useful physical parameters are

Boltzmann constant	k_b : $1.38\ 10^{-23}$	$[kg\ m^2 s^{-2} K^{-1}]$
Ionic mass	m_i : $3.35\ 10^{-27}$	$[kg]$
Electronic mass	m_e : $9.11\ 10^{-31}$	$[kg]$
Vacuum permeability	ε_0 : $8.85\ 10^{-12}$	$[\mathrm{C\ N}^{-1} m^{-1}]$
Electron charge	e : $1.60\ 10^{-19}$	$[\mathrm{C}\ ]$

Considering that the conversion between Kelvin K and electronvolt eV is

$$T_K = T_{eV} \frac{e}{k_b},$$

the non-dimensional values are computed next.

The reference Mach is

$$M_{ref} = \frac{T_0 k_b}{m_i u_0{}^2} = \frac{T_0[eV] e \cancel{k_b}}{m_i \cancel{k_b} u_0{}^2} \approx 12.5.$$

The perpendicular diffusion coefficients are chosen as

$$D = \mu = \chi_i = \chi_e = 1\ [m^2 s^{-1}],$$

which gives

$$D = \mu = \chi_i = \chi_e = 1 * \frac{t_0}{L_0{}^2} = 0.038.$$

The parallel diffusion coefficients are taken from [Ref: Stangeby]. For the ions is

$$k_{\parallel i} = 33,$$

which gives

$$k_{\parallel i} = k_{\parallel i} \frac{t_0{}^3 T_0{}^{7/2}}{L_0{}^4 n_0 m_i} = 1.74\ 10^5$$

while for the electron it is

$$k_{\parallel e} = 2000,$$

which gives

$$k_{\parallel e} = k_{\parallel e} \frac{t_0{}^3 T_0{}^{7/2}}{L_0{}^4 n_0 m_i} \approx 10^7.$$

The exchange temperature term is

$$\tau_{ie} = \frac{3\sqrt{2}}{e^4}\frac{{\varepsilon_0}^2}{\Lambda}\pi^{\frac{3}{2}}\frac{m_i}{m_e}\sqrt{m_e}\ e^{\frac{3}{2}} \approx 5.27\ 10^{13},^{1}$$

which gives

$$\tau_{ie} = \frac{2}{3}\frac{\tau_{ie}{T_0}^{1/2}m_i{u_0}^2}{n_0 t_0 k_b} \approx 8.4\ 10^6.$$

4 Conservative form of the non-dimensional system

The non-dimensional equations (19),(22) and (28) are rewritten as

$$\begin{aligned}
&\partial_t n + \boldsymbol{\nabla}\cdot(nu\boldsymbol{b} + n\boldsymbol{u}_\perp) - \boldsymbol{\nabla}\cdot(D\boldsymbol{\nabla}_\perp n) = S_n,\\
&\partial_t(nu) + \boldsymbol{\nabla}\cdot(nu^2\boldsymbol{b} + nu\boldsymbol{u}_\perp) + M_{ref}\boldsymbol{\nabla}_\parallel(p_i+p_e) - \boldsymbol{\nabla}\cdot(\mu\boldsymbol{\nabla}_\perp nu) = S_\Gamma,\\
&\partial_t(nE_i) + \boldsymbol{\nabla}\cdot\big((nE_i + M_{ref}p_i)u\boldsymbol{b} + nE_i\boldsymbol{u}_\perp\big) - \boldsymbol{\nabla}\cdot(\chi_i\boldsymbol{\nabla}_\perp nE_i) - \boldsymbol{\nabla}\cdot(k_{\parallel i}T_i^{5/2}\boldsymbol{\nabla}_\parallel T_i\boldsymbol{b})\\
&\qquad + M_{ref}u\boldsymbol{\nabla}_\parallel p_e + \frac{n^2}{\tau_{ie}}\frac{(T_e-T_i)}{T_e^{3/2}} = S_{E_i},\\
&\partial_t(nE_e) + \boldsymbol{\nabla}\cdot\big((nE_e + M_{ref}p_e)u\boldsymbol{b} + nE_e\boldsymbol{u}_\perp\big) - \boldsymbol{\nabla}\cdot(\chi_e\boldsymbol{\nabla}_\perp nE_e) - \boldsymbol{\nabla}\cdot(k_{\parallel e}T_e^{5/2}\boldsymbol{\nabla}_\parallel T_e\boldsymbol{b})\\
&\qquad - M_{ref}u\boldsymbol{\nabla}_\parallel p_e - \frac{n^2}{\tau_{ie}}\frac{(T_e-T_i)}{T_e^{3/2}} = S_{E_e}
\end{aligned}$$

The momentum equation is rewritten in the following form

$$\partial_t(nu) + \boldsymbol{\nabla}\cdot(nu^2\boldsymbol{b} + nu\boldsymbol{u}_\perp + M_{ref}(p_i+p_e)\boldsymbol{b}) - M_{ref}(p_i+p_e)\boldsymbol{\nabla}\cdot\boldsymbol{b} - \boldsymbol{\nabla}\cdot(\mu\boldsymbol{\nabla}_\perp nu) = S_\Gamma,$$

using the relation

$$\boldsymbol{\nabla}_\parallel(p_i+p_e) = \boldsymbol{\nabla}(p_i+p_e)\cdot\boldsymbol{b} = \boldsymbol{\nabla}\cdot\Big((p_i+p_e)\boldsymbol{b}\Big) - (p_i+p_e)\boldsymbol{\nabla}\cdot\boldsymbol{b},$$

while the drift term is simplified considering a divergence-free drift velocity, $\boldsymbol{\nabla}\cdot\boldsymbol{u}_\perp = 0$. This leads to a new form of the system

$$\boxed{\begin{aligned}
&\partial_t n + \boldsymbol{\nabla}\cdot(nu\boldsymbol{b}) + \boldsymbol{u}_\perp\cdot\boldsymbol{\nabla}n - \boldsymbol{\nabla}\cdot(D\boldsymbol{\nabla}_\perp n) = S_n,\\
&\partial_t(nu) + \boldsymbol{\nabla}\cdot(nu^2\boldsymbol{b} + M_{ref}(p_i+p_e)\boldsymbol{b}) + \boldsymbol{u}_\perp\cdot\boldsymbol{\nabla}(nu) - M_{ref}(p_i+p_e)\boldsymbol{\nabla}\cdot\boldsymbol{b} - \boldsymbol{\nabla}\cdot(\mu\boldsymbol{\nabla}_\perp nu) = S_\Gamma,\\
&\partial_t(nE_i) + \boldsymbol{\nabla}\cdot\big((nE_i + M_{ref}p_i)u\boldsymbol{b}\big) + \boldsymbol{u}_\perp\cdot\boldsymbol{\nabla}(nE_i) - \boldsymbol{\nabla}\cdot(\chi_i\boldsymbol{\nabla}_\perp nE_i) - \boldsymbol{\nabla}\cdot(k_{\parallel i}T_i^{5/2}\boldsymbol{\nabla}_\parallel T_i\boldsymbol{b})\\
&\qquad + M_{ref}u\boldsymbol{\nabla}_\parallel p_e + \frac{n^2}{\tau_{ie}}\frac{(T_e-T_i)}{T_e^{3/2}} = S_{E_i},\\
&\partial_t(nE_e) + \boldsymbol{\nabla}\cdot\big((nE_e + M_{ref}p_e)u\boldsymbol{b}\big) + \boldsymbol{u}_\perp\cdot\boldsymbol{\nabla}(nE_e) - \boldsymbol{\nabla}\cdot(\chi_e\boldsymbol{\nabla}_\perp nE_e) - \boldsymbol{\nabla}\cdot(k_{\parallel e}T_e^{5/2}\boldsymbol{\nabla}_\parallel T_e\boldsymbol{b})\\
&\qquad - M_{ref}u\boldsymbol{\nabla}_\parallel p_e - \frac{n^2}{\tau_{ie}}\frac{(T_e-T_i)}{T_e^{3/2}} = S_{E_e},
\end{aligned}}$$

(32)

[1]Note that using the reference values n_0 and T_0, we obtain $\hat{\tau_{ie}} = \tau_{ie}\frac{{T_0}^{3/2}}{n_0} \approx 1.9\ 10^{-3}s$

with the following added relations

$$\boxed{\begin{aligned} E_i = \frac{1}{2}u^2 + \frac{3}{2}M_{ref}\frac{p_i}{n} \rightarrow p_i = \frac{2}{3}\frac{n}{M_{ref}}(E_i - \frac{1}{2}u^2), \\ E_e = \frac{3}{2}M_{ref}\frac{p_e}{n} \rightarrow p_e = \frac{2}{3}\frac{n}{M_{ref}}E_e, \\ p_i = nT_i \rightarrow T_i = \frac{2}{3M_{ref}}(E_i - \frac{1}{2}u^2), \\ p_e = nT_e \rightarrow T_e = \frac{2}{3M_{ref}}E_e, \end{aligned}}$$

and the Bohm boundary conditions

$$\boxed{\begin{aligned} \Big(nE_i + M_{ref}(1-\gamma_i)p_i\Big)u - k_{\parallel i}T_i^{5/2}\nabla_\parallel T_i - \frac{1}{2}nu^3 = 0, \\ \Big(nE_e + M_{ref}(1-\gamma_e)p_e\Big)u - k_{\parallel e}T_e^{5/2}\nabla_\parallel T_e = 0. \end{aligned}}$$

System (32) is recast in first order conservative form

$$\boxed{\partial_t \boldsymbol{U} + (\boldsymbol{u}_\perp \cdot \boldsymbol{\nabla})\boldsymbol{U} + \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}} - \boldsymbol{\nabla}\cdot(D_f\boldsymbol{\mathcal{Q}}) + \boldsymbol{\nabla}\cdot(D_f\boldsymbol{\mathcal{Q}}\boldsymbol{b}\otimes\boldsymbol{b}) - \boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{F}}_t + \boldsymbol{f}_{E_\parallel} + \boldsymbol{f}_{EX} - \boldsymbol{g} = \boldsymbol{s}} \tag{33}$$

using the following definition of the conservative variables,

$$\boldsymbol{U} = \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix}, = \begin{Bmatrix} n \\ nu \\ nE_i \\ nE_e \end{Bmatrix}$$

and the tensor of the conservative variable derivatives,

$$\boldsymbol{\mathcal{Q}} = \boldsymbol{\nabla}\boldsymbol{U} = \begin{bmatrix} U_{1,x} & U_{1,y} \\ U_{2,x} & U_{2,y} \\ U_{3,x} & U_{3,y} \\ U_{4,x} & U_{4,y} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\nabla}U_1^T \\ \boldsymbol{\nabla}U_2^T \\ \boldsymbol{\nabla}U_3^T \\ \boldsymbol{\nabla}U_4^T \end{bmatrix} = \begin{bmatrix} \mathcal{Q}_{11} & \mathcal{Q}_{12} \\ \mathcal{Q}_{21} & \mathcal{Q}_{22} \\ \mathcal{Q}_{31} & \mathcal{Q}_{32} \\ \mathcal{Q}_{41} & \mathcal{Q}_{42} \end{bmatrix}.$$

The pressure and temperature are written as

$$\begin{aligned} p_i &= \frac{2}{3M_{ref}}\Big(U_3 - \frac{1}{2}\frac{U_2^2}{U_1}\Big), \\ p_e &= \frac{2}{3M_{ref}}U_4, \\ T_i &= \frac{2}{3M_{ref}}\Big(\frac{U_3}{U_1} - \frac{1}{2}\frac{U_2^2}{U_1^2}\Big), \\ T_e &= \frac{2}{3M_{ref}}\frac{U_4}{U_1}. \end{aligned}$$

The convective flux is written as

$$\boldsymbol{\mathcal{F}} = \begin{Bmatrix} nu \\ (nu^2 + M_{ref}(p_i + p_e)) \\ (nE_i + M_{ref}p_i)u \\ (nE_e + M_{ref}p_e)u \end{Bmatrix} \otimes \boldsymbol{b}^T = \begin{Bmatrix} U_2 \\ \frac{U_2^2}{U_1} + \frac{2}{3}\Big(U_3 + U_4 - \frac{1}{2}\frac{U_2^2}{U_1}\Big) \\ \Big(U_3 + \frac{2}{3}(U_3 - \frac{1}{2}\frac{U_2^2}{U_1})\Big)\frac{U_2}{U_1} \\ \Big(U_4 + \frac{2}{3}U_4\Big)\frac{U_2}{U_1} \end{Bmatrix} \otimes \boldsymbol{b}^T.$$

The ions temperature gradient is written as

$$\boldsymbol{\nabla} T_i = \frac{2}{3M_{ref}} \boldsymbol{\nabla}\Big(\frac{U_3}{U_1} - \frac{1}{2}\frac{U_2^2}{U_1^2}\Big) = \frac{2}{3M_{ref}}\Big(\boldsymbol{\nabla} U_1 (\frac{U_2^2}{U_1^3} - \frac{U_3}{U_1^2}) + \boldsymbol{\nabla} U_2(-\frac{U_2}{U_1^2}) + \boldsymbol{\nabla} U_3(\frac{1}{U_1})\Big),$$

and can be simplified using the following definition

$$\boldsymbol{V}_i(\boldsymbol{U}) = \begin{Bmatrix} \frac{U_2^2}{U_1^3} - \frac{U_3}{U_1^2} \\ -\frac{U_2}{U_1^2} \\ \frac{1}{U_1} \\ 0 \end{Bmatrix} \tag{34}$$

as

$$\boldsymbol{\nabla} T_i = \frac{2}{3M_{ref}} \boldsymbol{\mathcal{Q}}_t \boldsymbol{V}_i(\boldsymbol{U}),$$

where the transpose of the variable gradient has been introduced, $\boldsymbol{\mathcal{Q}}_t = \boldsymbol{\mathcal{Q}}^T$.

The electrons temperature gradient is written as

$$\boldsymbol{\nabla} T_e = \frac{2}{3M_{ref}} \boldsymbol{\nabla}\Big(\frac{U_4}{U_1}\Big) = \frac{2}{3M_{ref}}\Big(\boldsymbol{\nabla} U_1(-\frac{U_4}{U_1^2}) + \boldsymbol{\nabla} U_4(\frac{1}{U_1})\Big),$$

and can be simplified using the following definition

$$\boldsymbol{V}_e(\boldsymbol{U}) = \begin{Bmatrix} -\frac{U_4}{U_1^2} \\ 0 \\ 0 \\ \frac{1}{U_1} \end{Bmatrix} \tag{35}$$

as

$$\boldsymbol{\nabla} T_e = \frac{2}{3M_{ref}} \boldsymbol{\mathcal{Q}}_t \boldsymbol{V}_e(\boldsymbol{U}).$$

Hence, using the definition of the parallel gradient, we have

$$\boldsymbol{\nabla}_\parallel T_i = \boldsymbol{\nabla} T_i \cdot \boldsymbol{b} = \frac{2}{3M_{ref}} \boldsymbol{\mathcal{Q}}_t \boldsymbol{V}_i(\boldsymbol{U}) \cdot \boldsymbol{b},$$
$$\boldsymbol{\nabla}_\parallel T_e = \boldsymbol{\nabla} T_e \cdot \boldsymbol{b} = \frac{2}{3M_{ref}} \boldsymbol{\mathcal{Q}}_t \boldsymbol{V}_e(\boldsymbol{U}) \cdot \boldsymbol{b},$$

and the energy flux related to the parallel diffusion of the temperature is written as

$$\boldsymbol{\mathcal{F}}_t = \begin{Bmatrix} 0 \\ 0 \\ k_{\parallel i} T_i^{5/2} \boldsymbol{\nabla}_\parallel T_i \\ k_{\parallel e} T_e^{5/2} \boldsymbol{\nabla}_\parallel T_e \end{Bmatrix} \otimes \boldsymbol{b}^T = \begin{Bmatrix} 0 \\ 0 \\ k_{\parallel i}\Big(\frac{2}{3M_{ref}}\Big)^{7/2}\Big(\frac{U_3}{U_1} - \frac{1}{2}\frac{U_2^2}{U_1^2}\Big)^{5/2} \boldsymbol{\mathcal{Q}}_t \boldsymbol{V}_i(\boldsymbol{U}) \cdot \boldsymbol{b} \\ k_{\parallel e}\Big(\frac{2}{3M_{ref}}\Big)^{7/2}\Big(\frac{U_4}{U_1}\Big)^{5/2} \boldsymbol{\mathcal{Q}}_t \boldsymbol{V}_e(\boldsymbol{U}) \cdot \boldsymbol{b} \end{Bmatrix} \otimes \boldsymbol{b}^T.$$

The vector related to the parallel electric field $\boldsymbol{f}_{E_\parallel}$ is

$$\boldsymbol{f}_{E_\parallel} = M_{ref} u \boldsymbol{\nabla}_\parallel p_e \begin{Bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{Bmatrix} = \frac{2}{3}\frac{U_2}{U_1} \boldsymbol{\nabla} U_4 \cdot \boldsymbol{b} \begin{Bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{Bmatrix}$$

and can be rewritten as

$$\boldsymbol{f}_{E_\parallel} = \frac{2}{3}\mathcal{Q}_t \boldsymbol{W}(\boldsymbol{U}) \cdot \boldsymbol{b} \begin{Bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{Bmatrix}$$

having defined the vector

$$\boldsymbol{W}(\boldsymbol{U}) = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \frac{U_2}{U_1} \end{Bmatrix}.$$

The vector of temperature exchange between ions and electrons is $\boldsymbol{f}_{EX}$ is

$$\boldsymbol{f}_{EX} = \frac{n^2}{\tau_{ie}} \frac{(T_e - T_i)}{T_e^{3/2}} \begin{Bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{Bmatrix} = \frac{1}{\tau_{ie}} \left(\frac{2}{3M_{ref}}\right)^{-1/2} \frac{{U_1}^{5/2}}{{U_4}^{3/2}} \left(U_3 - U_4 + \frac{1}{2}\frac{U_2^2}{U_1}\right) \begin{Bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{Bmatrix}.$$

Finally, the curvature term $\boldsymbol{g}$ is

$$\boldsymbol{g} = \begin{Bmatrix} 0 \\ (p_i + p_e)\boldsymbol{\nabla}\cdot\boldsymbol{b} \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \frac{2}{3}\left(U_3 + U_4 - \frac{1}{2}\frac{U_2^2}{U_1}\right)\boldsymbol{\nabla}\cdot\boldsymbol{b} \\ 0 \\ 0 \end{Bmatrix}.$$

The Bohm boundary conditions are re-written using the conservative variables and (35),(34) as

$$\begin{aligned} &\frac{5-2\gamma_i}{3}\frac{U_2}{U_1}\left(U_3 - \frac{1}{2}\frac{U_2^2}{U_1}\right) - k_{\parallel i}\left(\frac{2}{3M_{ref}}\right)^{7/2}\left(\frac{U_3}{U_1} - \frac{1}{2}\frac{U_2^2}{U_1^2}\right)^{5/2} \mathcal{Q}_t \boldsymbol{V}_i(\boldsymbol{U}) \cdot \boldsymbol{b} = 0, \\ &\frac{5-2\gamma_e}{3}\frac{U_2 U_4}{U_1} - k_{\parallel e}\left(\frac{2}{3M_{ref}}\right)^{7/2}\left(\frac{U_4}{U_1}\right)^{5/2} \mathcal{Q}_t \boldsymbol{V}_e(\boldsymbol{U}) \cdot \boldsymbol{b} = 0, \end{aligned}$$

and are re-cast in a vector form using the boundary vector

$$\boldsymbol{B} = \begin{Bmatrix} 0 \\ 0 \\ \frac{5-2\gamma_i}{3}\frac{U_2}{U_1}\left(U_3 - \frac{1}{2}\frac{U_2^2}{U_1}\right) - k_{\parallel i}\left(\frac{2}{3M_{ref}}\right)^{7/2}\left(\frac{U_3}{U_1} - \frac{1}{2}\frac{U_2^2}{U_1^2}\right)^{5/2} \mathcal{Q}_t \boldsymbol{V}_i(\boldsymbol{U}) \cdot \boldsymbol{b} \\ \frac{5-2\gamma_e}{3}\frac{U_2 U_4}{U_1} - k_{\parallel e}\left(\frac{2}{3M_{ref}}\right)^{7/2}\left(\frac{U_4}{U_1}\right)^{5/2} \mathcal{Q}_t \boldsymbol{V}_e(\boldsymbol{U}) \cdot \boldsymbol{b} \end{Bmatrix}$$

as

$$\boldsymbol{B} = 0, \text{ on } \partial\Omega_{\text{Bohm}}.$$

5 Treatment of the non-linear terms

The convective flux $\boldsymbol{\mathcal{F}}$, the parallel diffusion flux $\boldsymbol{\mathcal{F}}_t$ and the vectors $\boldsymbol{f}_{E_\parallel}$, $\boldsymbol{f}_{EX}$ and $\boldsymbol{g}$ are non-linear terms. In a Newton-Raphson (NR) framework, the bilinear forms related to these terms are linearized using a second-order approximation. The linearization used for a generic term $\boldsymbol{f}$ is the following

$$\boldsymbol{f}(\boldsymbol{w}_1^k, \boldsymbol{w}_2^k, ...) = \boldsymbol{f}(\boldsymbol{w}_1^{k-1}, \boldsymbol{w}_2^{k-1}, ...) + \frac{d}{d\varepsilon} f(\boldsymbol{w}_1^{k-1} + \varepsilon \boldsymbol{dw}_1, \boldsymbol{w}_2^{k-1} + \varepsilon \boldsymbol{dw}_2, ...)\big|_{\varepsilon=0} + \mathcal{O}(\boldsymbol{dw}_1^2, \boldsymbol{dw}_2^2, ...), \tag{36}$$

where k is the NR iteration and $\boldsymbol{dw}_i = \boldsymbol{w}_i^k - \boldsymbol{w}_i^{k-1}$.

5.1 Linearization of the convective term

The convective term

$$\boldsymbol{\mathcal{F}} = \begin{Bmatrix} U_2 \\ \frac{2}{3}\left(U_3 + U_4 + \frac{U_2^2}{U_1}\right) \\ \frac{5}{3}\frac{U_3 U_2}{U_1} - \frac{1}{3}\frac{U_2^3}{U_1^2} \\ \frac{5}{3}\frac{U_4 U_2}{U_1} \end{Bmatrix} \otimes \boldsymbol{b}^T$$

can be written as

$$\boldsymbol{\mathcal{F}} = \frac{d\boldsymbol{\mathcal{F}}}{d\boldsymbol{U}}\boldsymbol{U} = \mathbb{A}(\boldsymbol{U})\boldsymbol{U}, \tag{37}$$

where the Jacobian third order tensor has been introduced

$$\mathbb{A} = \frac{d\boldsymbol{\mathcal{F}}}{d\boldsymbol{U}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{2}{3}\frac{U_2^2}{U_1^2} & \frac{4}{3}\frac{U_2}{U_1} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3}\frac{U_2^3}{U_1^3} - \frac{5}{3}\frac{U_3 U_2}{U_1^2} & \frac{5}{3}\frac{U_3}{U_1} - \frac{U_2^2}{U_1^2} & \frac{5}{3}\frac{U_2}{U_1} & 0 \\ -\frac{5}{3}\frac{U_4 U_2}{U_1^2} & \frac{5}{3}\frac{U_4}{U_1} & 0 & \frac{5}{3}\frac{U_2}{U_1} \end{bmatrix} \otimes \boldsymbol{b}^T = \boldsymbol{\mathcal{A}} \otimes \boldsymbol{b}^T,$$

where $\boldsymbol{\mathcal{A}}$ is a second order tensor. Deriving (37) with respect to $\boldsymbol{U}$ we obtain

$$\frac{d\boldsymbol{\mathcal{F}}}{d\boldsymbol{U}} = \mathbb{A}(\boldsymbol{U}) + \frac{d\mathbb{A}(\boldsymbol{U})}{d\boldsymbol{U}}\boldsymbol{U} \rightarrow \frac{d\mathbb{A}(\boldsymbol{U})}{d\boldsymbol{U}}\boldsymbol{U} = \boldsymbol{0}. \tag{38}$$

Applying now (36) to the convective flux, we obtain

$$\begin{aligned} \boldsymbol{\mathcal{F}}(\boldsymbol{U}^k) &= \mathbb{A}(\boldsymbol{U}^k)\boldsymbol{U}^k = \mathbb{A}(\boldsymbol{U}^{k-1})\boldsymbol{U}^{k-1} + \frac{d}{d\varepsilon}\Big(\mathbb{A}(\boldsymbol{U}^{k-1} + \varepsilon \boldsymbol{dU})(\boldsymbol{U}^{k-1} + \varepsilon \boldsymbol{dU})\Big)\Big|_{\varepsilon=0} + \mathcal{O}(\boldsymbol{dU}^2) \\ &= \mathbb{A}(\boldsymbol{U}^{k-1})\boldsymbol{U}^{k-1} + \mathbb{A}(\boldsymbol{U}^{k-1} + \varepsilon \boldsymbol{dU})\Big|_{\varepsilon=0}\boldsymbol{dU} + \frac{d\mathbb{A}}{d\boldsymbol{U}}\Big|_{k-1}\boldsymbol{dU}(\boldsymbol{U}^{k-1} + \varepsilon \boldsymbol{dU})\Big|_{\varepsilon=0} + \mathcal{O}(\boldsymbol{dU}^2). \end{aligned}$$

Substituting $\varepsilon = 0$ and $\boldsymbol{dU} = \boldsymbol{U}^k - \boldsymbol{U}^{k-1}$ we obtain

$$\begin{aligned} \boldsymbol{\mathcal{F}}(\boldsymbol{U}^k) &= \mathbb{A}(\boldsymbol{U}^{k-1})\boldsymbol{U}^{k-1} + \mathbb{A}(\boldsymbol{U}^{k-1})(\boldsymbol{U}^k - \boldsymbol{U}^{k-1}) + \frac{d\mathbb{A}}{d\boldsymbol{U}}\Big|_{k-1}(\boldsymbol{U}^k - \boldsymbol{U}^{k-1})\boldsymbol{U}^{k-1} + \mathcal{O}(\boldsymbol{dU}^2) \\ &= \cancel{\mathbb{A}(\boldsymbol{U}^{k-1})\boldsymbol{U}^{k-1}} + \mathbb{A}(\boldsymbol{U}^{k-1})\boldsymbol{U}^k - \cancel{\mathbb{A}(\boldsymbol{U}^{k-1})\boldsymbol{U}^{k-1}} + \left(\frac{d\mathbb{A}}{d\boldsymbol{U}}\Big|_{k-1}\boldsymbol{U}^k\right)\boldsymbol{U}^{k-1} - \left(\cancelto{0}{\frac{d\mathbb{A}}{d\boldsymbol{U}}\Big|_{k-1}\boldsymbol{U}^{k-1}}\right)\boldsymbol{U}^{k-1} \\ &\quad + \mathcal{O}(\boldsymbol{dU}^2). \end{aligned}$$

where eq.(38) has been used in the last expression. Now, it results that, [see appendix]

$$\left(\frac{d\mathbb{A}}{d\boldsymbol{U}}\Big|_{k-1}\boldsymbol{U}^k\right)\boldsymbol{U}^{k-1} = \boldsymbol{0}$$

hence

$$\boxed{\boldsymbol{\mathcal{F}}(\boldsymbol{U}^k) = \mathbb{A}(\boldsymbol{U}^{k-1})\boldsymbol{U}^k + \mathcal{O}(\boldsymbol{dU}^2)}. \tag{39}$$

5.2 Linearization of the curvature term $\boldsymbol{g}$

The curvature term is

$$\boldsymbol{g} = \left\{ \begin{matrix} 0 \\ \frac{2}{3}\left(U_3 + U_4 - \frac{1}{2}\frac{U_2^2}{U_1}\right)\boldsymbol{\nabla}\cdot\boldsymbol{b} \\ 0 \\ 0 \end{matrix} \right\}.$$

Applying (36) we get

$$\boldsymbol{g}(\boldsymbol{U}^k) = \boldsymbol{g}(\boldsymbol{U}^{k-1}) + \frac{d}{d\varepsilon}\left(\boldsymbol{g}(\boldsymbol{U}^{k-1} + \varepsilon d\boldsymbol{U})\right)\Big|_{\varepsilon=0} + \mathcal{O}(d\boldsymbol{U}^2) = \boldsymbol{g}(\boldsymbol{U}^{k-1}) + \frac{d\boldsymbol{g}}{d\boldsymbol{U}}\Big|_{k-1}(\boldsymbol{U}^k - \boldsymbol{U}^{k-1}) + \mathcal{O}(d\boldsymbol{U}^2).$$

We also have

$$\frac{d\boldsymbol{g}}{d\boldsymbol{U}} = \frac{2}{3}\boldsymbol{\nabla}\cdot\boldsymbol{b}\begin{bmatrix} 0 & 0 & 0 & 0 \\ \frac{1}{2}\frac{U_2^2}{U_1^2} & -\frac{U_2}{U_1} & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

which verifies

$$\boldsymbol{g} = \frac{d\boldsymbol{g}}{d\boldsymbol{U}}\boldsymbol{U}.$$

Therefore, the linearization of the $\boldsymbol{g}$ term results

$$\boxed{\boldsymbol{g}(\boldsymbol{U}^k) = \frac{d\boldsymbol{g}}{d\boldsymbol{U}}\Big|_{k-1}\boldsymbol{U}^k + \mathcal{O}(d\boldsymbol{U}^2)}.$$

5.3 Linearization of the parallel diffusion flux

The parallel diffusion flux is rewritten as

$$\boldsymbol{\mathcal{F}}_t = \left\{ \begin{matrix} 0 \\ 0 \\ k_{\parallel i}\left(\frac{2}{3M_{ref}}\right)^{7/2}\left(\frac{U_3}{U_1} - \frac{1}{2}\frac{U_2^2}{U_1^2}\right)^{5/2}\boldsymbol{\mathcal{Q}}_t\boldsymbol{V}_i(\boldsymbol{U})\cdot\boldsymbol{b} \\ k_{\parallel e}\left(\frac{2}{3M_{ref}}\right)^{7/2}\left(\frac{U_4}{U_1}\right)^{5/2}\boldsymbol{\mathcal{Q}}_t\boldsymbol{V}_e(\boldsymbol{U})\cdot\boldsymbol{b} \end{matrix} \right\}\otimes\boldsymbol{b}^T = \left\{ \begin{matrix} 0 \\ 0 \\ k_{\parallel i}\left(\frac{2}{3M_{ref}}\right)^{7/2}\boldsymbol{f}_i(\boldsymbol{U},\boldsymbol{\mathcal{Q}}_t)\cdot\boldsymbol{b} \\ k_{\parallel e}\left(\frac{2}{3M_{ref}}\right)^{7/2}\boldsymbol{f}_e(\boldsymbol{U},\boldsymbol{\mathcal{Q}}_t)\cdot\boldsymbol{b} \end{matrix} \right\}\otimes\boldsymbol{b}^T,$$

where the vector function $\boldsymbol{f}_i(\boldsymbol{U},\boldsymbol{\mathcal{Q}}_t)$ and $\boldsymbol{f}_e(\boldsymbol{U},\boldsymbol{\mathcal{Q}}_t)$ are

$$\boldsymbol{f}_i(\boldsymbol{U},\boldsymbol{\mathcal{Q}}_t) = r_i(\boldsymbol{U})\boldsymbol{\mathcal{Q}}_t\boldsymbol{V}_i(\boldsymbol{U}), \tag{40}$$

$$\boldsymbol{f}_e(\boldsymbol{U},\boldsymbol{\mathcal{Q}}_t) = r_e(\boldsymbol{U})\boldsymbol{\mathcal{Q}}_t\boldsymbol{V}_e(\boldsymbol{U}), \tag{41}$$

and the scalar functions $r_i(\boldsymbol{U})$ and $r_e(\boldsymbol{U})$ are

$$r_i(\boldsymbol{U}) = \left(\frac{U_3}{U_1} - \frac{1}{2}\frac{U_2^2}{U_1^2}\right)^{5/2}, \tag{42}$$

$$r_e(\boldsymbol{U}) = \left(\frac{U_4}{U_1}\right)^{5/2}. \tag{43}$$

The linearization of $\boldsymbol{\mathcal{F}}_t$ reduces to the linearization of $\boldsymbol{f}(\boldsymbol{U}, \boldsymbol{\mathcal{Q}}_t)$ $(= \boldsymbol{f}_{i,e}(\boldsymbol{U}, \boldsymbol{\mathcal{Q}}_t))$. Applying (36) to $\boldsymbol{f}(\boldsymbol{U}, \boldsymbol{\mathcal{Q}}_t)$ we have

$$\begin{aligned}\boldsymbol{f}(\boldsymbol{U}^k, \boldsymbol{\mathcal{Q}}_t^k) &= \boldsymbol{f}(\boldsymbol{U}^{k-1}, \boldsymbol{\mathcal{Q}}_t^{k-1}) + \frac{d}{d\varepsilon}\boldsymbol{f}(\boldsymbol{U}^{k-1} + \varepsilon \boldsymbol{dU}, \boldsymbol{\mathcal{Q}}_t^{k-1} + \varepsilon \boldsymbol{d\mathcal{Q}}_t)\Big|_{\varepsilon=0} + \mathcal{O}(\boldsymbol{dU}^2, \boldsymbol{d\mathcal{Q}}_t^2) \\ &= r(\boldsymbol{U}^{k-1})\boldsymbol{\mathcal{Q}}_t^{k-1}\boldsymbol{V}(\boldsymbol{U}^{k-1}) + \frac{d}{d\varepsilon}\Big(r(\boldsymbol{U}^{k-1} + \varepsilon \boldsymbol{dU})(\boldsymbol{\mathcal{Q}}_t^{k-1} + \varepsilon \boldsymbol{d\mathcal{Q}}_t)\boldsymbol{V}(\boldsymbol{U}^{k-1} + \varepsilon \boldsymbol{dU})\Big)\Big|_{\varepsilon=0} + \mathcal{O}(\boldsymbol{dU}^2, \boldsymbol{d\mathcal{Q}}_t^2).\end{aligned} \tag{44}$$

Developing the derivative term $\dfrac{d}{d\varepsilon}(\ldots)$ we obtain

$$\begin{aligned}\frac{d}{d\varepsilon}(\ldots)_{\varepsilon=0} = &\frac{dr}{d\boldsymbol{U}}\Big|_{k-1} \cdot \boldsymbol{dU}(\boldsymbol{\mathcal{Q}}_t^{k-1} + \varepsilon \boldsymbol{d\mathcal{Q}}_t)\Big|_{\varepsilon=0} \boldsymbol{V}(\boldsymbol{U}^{k-1} + \varepsilon \boldsymbol{dU})\Big|_{\varepsilon=0} \\ &+ r(\boldsymbol{U}^{k-1} + \varepsilon \boldsymbol{dU})\Big|_{\varepsilon=0} \boldsymbol{d\mathcal{Q}}_t \boldsymbol{V}(\boldsymbol{U}^{k-1} + \varepsilon \boldsymbol{dU})\Big|_{\varepsilon=0} \\ &+ r(\boldsymbol{U}^{k-1} + \varepsilon \boldsymbol{dU})\Big|_{\varepsilon=0} (\boldsymbol{\mathcal{Q}}_t^{k-1} + \varepsilon \boldsymbol{d\mathcal{Q}}_t)\Big|_{\varepsilon=0} \frac{d\boldsymbol{V}}{d\boldsymbol{U}}\Big|_{k-1} d\boldsymbol{U} \\ &= \frac{dr}{d\boldsymbol{U}}\Big|_{k-1} \boldsymbol{dU}\,\boldsymbol{\mathcal{Q}}_t^{k-1}\boldsymbol{V}(\boldsymbol{U}^{k-1}) + r(\boldsymbol{U}^{k-1})\boldsymbol{d\mathcal{Q}}_t\boldsymbol{V}(\boldsymbol{U}^{k-1}) + r(\boldsymbol{U}^{k-1})\boldsymbol{\mathcal{Q}}_t^{k-1}\frac{d\boldsymbol{V}}{d\boldsymbol{U}}\Big|_{k-1} d\boldsymbol{U},\end{aligned} \tag{45}$$

where the derivatives are

$$\frac{dr_i}{d\boldsymbol{U}} = \frac{5}{2}\Big(\frac{U_3}{U_1} - \frac{1}{2}\frac{U_2^2}{U_1^2}\Big)^{3/2} \begin{Bmatrix} -\frac{U_3}{U_1^2} + \frac{U_2^2}{U_1^3} \\ -\frac{U_2}{U_1^2} \\ \frac{1}{U_1} \\ 0 \end{Bmatrix},$$

$$\frac{dr_e}{d\boldsymbol{U}} = \frac{5}{2}\Big(\frac{U_4}{U_1}\Big)^{3/2} \begin{Bmatrix} -\frac{U_4}{U_1^2} \\ 0 \\ 0 \\ \frac{1}{U_1} \end{Bmatrix},$$

and

$$\frac{d\boldsymbol{V}_i}{d\boldsymbol{U}} = \begin{bmatrix} -3\frac{U_2^2}{U_1^4} + 2\frac{U_3}{U_1^3} & 2\frac{U_2}{U_1^3} & -\frac{1}{U_1^2} & 0 \\ 2\frac{U_2}{U_1^3} & -\frac{1}{U_1^2} & 0 & 0 \\ -\frac{1}{U_1^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\frac{d\boldsymbol{V}_e}{d\boldsymbol{U}} = \begin{bmatrix} 2\frac{U_4}{U_1^3} & 0 & 0 & -\frac{1}{U_1^2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{U_1^2} & 0 & 0 & 0 \end{bmatrix},$$

Replacing $d\boldsymbol{U} = \boldsymbol{U}^k - \boldsymbol{U}^{k-1}$ and $d\boldsymbol{\mathcal{Q}}_t = \boldsymbol{\mathcal{Q}}_t^k - \boldsymbol{\mathcal{Q}}_t^{k-1}$ and plugging (45) into (44) we obtain

$$
\begin{aligned}
\boldsymbol{f}(\boldsymbol{U}^k, \boldsymbol{\mathcal{Q}}_t^k) = & \cancel{r(\boldsymbol{U}^{k-1})\boldsymbol{\mathcal{Q}}_t^{k-1}\boldsymbol{V}(\boldsymbol{U}^{k-1})} + \frac{dr}{d\boldsymbol{U}}\Big|_{k-1} \boldsymbol{U}^k \boldsymbol{\mathcal{Q}}_t^{k-1}\boldsymbol{V}(\boldsymbol{U}^{k-1}) - \xcancel{\frac{dr}{d\boldsymbol{U}}\Big|_{k-1} \cdot \boldsymbol{U}^{k-1}\boldsymbol{\mathcal{Q}}_t^{k-1}\boldsymbol{V}(\boldsymbol{U}^{k-1})}^{\;0} \\
& + r(\boldsymbol{U}^{k-1})\boldsymbol{\mathcal{Q}}_t^{k}\boldsymbol{V}(\boldsymbol{U}^{k-1}) - \cancel{r(\boldsymbol{U}^{k-1})\boldsymbol{\mathcal{Q}}_t^{k-1}\boldsymbol{V}(\boldsymbol{U}^{k-1})} \\
& + r(\boldsymbol{U}^{k-1})\boldsymbol{\mathcal{Q}}_t^{k-1}\frac{d\boldsymbol{V}}{d\boldsymbol{U}}\Big|_{k-1}\boldsymbol{U}^k - r(\boldsymbol{U}^{k-1})\boldsymbol{\mathcal{Q}}_t^{k-1}\frac{d\boldsymbol{V}}{d\boldsymbol{U}}\Big|_{k-1}\boldsymbol{U}^{k-1} \\
& + \mathcal{O}(d\boldsymbol{U}^2, d\boldsymbol{\mathcal{Q}}_t^2)
\end{aligned}
$$

where we have used the fact that

$$
\frac{dr}{d\boldsymbol{U}}\Big|_{k-1} \cdot \boldsymbol{U}^{k-1} = 0.
$$

Finally, the linearization of $\boldsymbol{f}(\boldsymbol{U}, \boldsymbol{\mathcal{Q}}_t)$ results

$$
\boxed{
\begin{aligned}
\boldsymbol{f}(\boldsymbol{U}^k, \boldsymbol{\mathcal{Q}}_t^k) = & \Big(\frac{dr}{d\boldsymbol{U}}\Big|_{k-1} \cdot \boldsymbol{U}^k\Big)\boldsymbol{\mathcal{Q}}_t^{k-1}\boldsymbol{V}(\boldsymbol{U}^{k-1}) + r(\boldsymbol{U}^{k-1})\boldsymbol{\mathcal{Q}}_t^{k}\boldsymbol{V}(\boldsymbol{U}^{k-1}) \\
& + r(\boldsymbol{U}^{k-1})\boldsymbol{\mathcal{Q}}_t^{k-1}\frac{d\boldsymbol{V}}{d\boldsymbol{U}}\Big|_{k-1}\boldsymbol{U}^k - r(\boldsymbol{U}^{k-1})\boldsymbol{\mathcal{Q}}_t^{k-1}\frac{d\boldsymbol{V}}{d\boldsymbol{U}}\Big|_{k-1}\boldsymbol{U}^{k-1} \\
& + \mathcal{O}(d\boldsymbol{U}^2, d\boldsymbol{\mathcal{Q}}_t^2).
\end{aligned}
}
$$

The parallel diffusion flux can be written as

$$
\boldsymbol{\mathcal{F}}_t(\boldsymbol{U}^k, \boldsymbol{\mathcal{Q}}_t^k) = \boldsymbol{\mathcal{F}}_t^{\boldsymbol{\mathcal{Q}}}(\boldsymbol{U}^{k-1}, \boldsymbol{\mathcal{Q}}_t^k) + \boldsymbol{\mathcal{F}}_t^{\boldsymbol{U}}(\boldsymbol{U}^k, \boldsymbol{\mathcal{Q}}_t^{k-1}) + \boldsymbol{\mathcal{F}}_t^{0}(\boldsymbol{U}^{k-1}, \boldsymbol{\mathcal{Q}}_t^{k-1}) + \mathcal{O}(d\boldsymbol{U}^2, d\boldsymbol{\mathcal{Q}}_t^2)
$$

where the terms are defined as

$$
\boldsymbol{\mathcal{F}}_t^{\boldsymbol{\mathcal{Q}}} = \Big(\frac{2}{3M_{ref}}\Big)^{7/2}
\begin{Bmatrix}
0 \\
0 \\
k_{\parallel i} r_i(\boldsymbol{U}^{k-1})\boldsymbol{\mathcal{Q}}_t^{k}\boldsymbol{V}_i(\boldsymbol{U}^{k-1}) \cdot \boldsymbol{b} \\
k_{\parallel e} r_e(\boldsymbol{U}^{k-1})\boldsymbol{\mathcal{Q}}_t^{k}\boldsymbol{V}_e(\boldsymbol{U}^{k-1}) \cdot \boldsymbol{b}
\end{Bmatrix} \otimes \boldsymbol{b}^T,
$$

$$
\boldsymbol{\mathcal{F}}_t^{\boldsymbol{U}} = \Big(\frac{2}{3M_{ref}}\Big)^{7/2}
\begin{Bmatrix}
0 \\
0 \\
k_{\parallel i}\Big(\Big(\frac{dr_i}{d\boldsymbol{U}}\Big|_{k-1} \cdot \boldsymbol{U}^k\Big)\boldsymbol{\mathcal{Q}}_t^{k-1}\boldsymbol{V}_i(\boldsymbol{U}^{k-1}) + r_i(\boldsymbol{U}^{k-1})\boldsymbol{\mathcal{Q}}_t^{k-1}\frac{d\boldsymbol{V}_i}{d\boldsymbol{U}}\Big|_{k-1}\boldsymbol{U}^k\Big) \cdot \boldsymbol{b} \\
k_{\parallel e}\Big(\Big(\frac{dr_e}{d\boldsymbol{U}}\Big|_{k-1} \cdot \boldsymbol{U}^k\Big)\boldsymbol{\mathcal{Q}}_t^{k-1}\boldsymbol{V}_e(\boldsymbol{U}^{k-1}) + r_e(\boldsymbol{U}^{k-1})\boldsymbol{\mathcal{Q}}_t^{k-1}\frac{d\boldsymbol{V}_e}{d\boldsymbol{U}}\Big|_{k-1}\boldsymbol{U}^k\Big) \cdot \boldsymbol{b}
\end{Bmatrix} \otimes \boldsymbol{b}^T,
$$

$$
\boldsymbol{\mathcal{F}}_t^{0} = -\Big(\frac{2}{3M_{ref}}\Big)^{7/2}
\begin{Bmatrix}
0 \\
0 \\
k_{\parallel i} r_i(\boldsymbol{U}^{k-1})\boldsymbol{\mathcal{Q}}_t^{k-1}\frac{d\boldsymbol{V}_i}{d\boldsymbol{U}}\Big|_{k-1}\boldsymbol{U}^{k-1} \cdot \boldsymbol{b} \\
k_{\parallel e} r_e(\boldsymbol{U}^{k-1})\boldsymbol{\mathcal{Q}}_t^{k-1}\frac{d\boldsymbol{V}_e}{d\boldsymbol{U}}\Big|_{k-1}\boldsymbol{U}^{k-1} \cdot \boldsymbol{b}
\end{Bmatrix} \otimes \boldsymbol{b}^T.
$$

5.4 Linearization of the parallel current vector

The parallel current vector is re-written as

$$\boldsymbol{f}_{E_\parallel} = \frac{2}{3}\boldsymbol{\mathcal{Q}}_t \boldsymbol{W}(\boldsymbol{U}) \cdot \boldsymbol{b} \begin{Bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{Bmatrix}, = \frac{2}{3}\boldsymbol{h}(\boldsymbol{U}) \cdot \boldsymbol{b} \begin{Bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{Bmatrix},$$

having defined the vector function $\boldsymbol{h}(\boldsymbol{U}, \boldsymbol{\mathcal{Q}}_t) = \boldsymbol{\mathcal{Q}}_t \boldsymbol{W}(\boldsymbol{U})$. The linearization of $\boldsymbol{f}_{E_\parallel}$ reduces to the linearization of $\boldsymbol{h}(\boldsymbol{\mathcal{Q}}_t, \boldsymbol{U})$ which is

$$\begin{aligned}
\boldsymbol{h}(\boldsymbol{U}^k, \boldsymbol{\mathcal{Q}}_t^k) = \boldsymbol{h}(\boldsymbol{U}^{k-1}, \boldsymbol{\mathcal{Q}}_t^{k-1}) + \frac{d}{d\varepsilon}\boldsymbol{h}(\boldsymbol{U}^{k-1} + \varepsilon \boldsymbol{dU}, \boldsymbol{\mathcal{Q}}_t^{k-1} + \varepsilon \boldsymbol{d\mathcal{Q}}_t)\Big|_{\varepsilon=0} + \mathcal{O}(\boldsymbol{dU}^2, \boldsymbol{d\mathcal{Q}}_t^2) = \\
\boldsymbol{\mathcal{Q}}_t^{k-1}\boldsymbol{W}(\boldsymbol{U}^{k-1}) + \frac{d}{d\varepsilon}\Big((\boldsymbol{\mathcal{Q}}_t^{k-1} + \varepsilon \boldsymbol{d\mathcal{Q}}_t)\boldsymbol{W}(\boldsymbol{U}^{k-1} + \varepsilon \boldsymbol{dU})\Big)\Big|_{\varepsilon=0} + \mathcal{O}(\boldsymbol{dU}^2, \boldsymbol{d\mathcal{Q}}_t^2) = \\
\boldsymbol{\mathcal{Q}}_t^{k-1}\boldsymbol{W}(\boldsymbol{U}^{k-1}) + \boldsymbol{d\mathcal{Q}}_t \boldsymbol{W}(\boldsymbol{U}^{k-1}) + \boldsymbol{\mathcal{Q}}_t^{k-1}\frac{d\boldsymbol{W}}{d\boldsymbol{U}}\Big|_{k-1} \boldsymbol{dU} + \mathcal{O}(\boldsymbol{dU}^2, \boldsymbol{d\mathcal{Q}}_t^2) = \\
\cancel{\boldsymbol{\mathcal{Q}}_t^{k-1}\boldsymbol{W}(\boldsymbol{U}^{k-1})} + \boldsymbol{\mathcal{Q}}_t^{k}\boldsymbol{W}(\boldsymbol{U}^{k-1}) - \cancel{\boldsymbol{\mathcal{Q}}_t^{k-1}\boldsymbol{W}(\boldsymbol{U}^{k-1})} + \boldsymbol{\mathcal{Q}}_t^{k-1}\frac{d\boldsymbol{W}}{d\boldsymbol{U}}\Big|_{k-1} \boldsymbol{U}^k - \cancelto{0}{\boldsymbol{\mathcal{Q}}_t^{k-1}\frac{d\boldsymbol{W}}{d\boldsymbol{U}}\Big|_{k-1} \boldsymbol{U}^{k-1}} \\
+ \mathcal{O}(\boldsymbol{dU}^2, \boldsymbol{d\mathcal{Q}}_t^2).
\end{aligned}$$

Hence, the linearization of $\boldsymbol{h}(\boldsymbol{U}^k, \boldsymbol{\mathcal{Q}}_t^k)$ results

$$\boxed{\boldsymbol{h}(\boldsymbol{U}^k, \boldsymbol{\mathcal{Q}}_t^k) = \boldsymbol{\mathcal{Q}}_t^{k}\boldsymbol{W}(\boldsymbol{U}^{k-1}) + \boldsymbol{\mathcal{Q}}_t^{k-1}\frac{d\boldsymbol{W}}{d\boldsymbol{U}}\Big|_{k-1} \boldsymbol{U}^k + \mathcal{O}(\boldsymbol{dU}^2, \boldsymbol{d\mathcal{Q}}_t^2)},$$

where the derivative $\dfrac{d\boldsymbol{W}}{d\boldsymbol{U}}$ is

$$\frac{d\boldsymbol{W}}{d\boldsymbol{U}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{U_2}{U_1^2} & \frac{1}{U_1} & 0 & 0 \end{bmatrix}.$$

The parallel current vector can be written as

$$\boldsymbol{f}_{E_\parallel} = \boldsymbol{f}_{E_\parallel}^{\boldsymbol{\mathcal{Q}}} + \boldsymbol{f}_{E_\parallel}^{\boldsymbol{U}},$$

where the two terms are defined as

$$\boldsymbol{f}_{E_\parallel}^{\boldsymbol{\mathcal{Q}}} = \frac{2}{3}\boldsymbol{\mathcal{Q}}_t^k \boldsymbol{W}(\boldsymbol{U}^{k-1}) \cdot \boldsymbol{b} \begin{Bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{Bmatrix},$$

$$\boldsymbol{f}_{E_\parallel}^{\boldsymbol{U}} = \frac{2}{3}\boldsymbol{\mathcal{Q}}_t^{k-1}\frac{d\boldsymbol{W}}{d\boldsymbol{U}}\Big|_{k-1} \boldsymbol{U}^k \cdot \boldsymbol{b} \begin{Bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{Bmatrix},$$

5.5 Linearization of the temperature exchange vector

The vector of temperature exchange between ions and electrons is $\boldsymbol{f}_{EX}$ is

$$\boldsymbol{f}_{EX} = \frac{1}{\tau_{ie}}\Big(\frac{2}{3M_{ref}}\Big)^{-1/2}\frac{{U_1}^{5/2}}{{U_4}^{3/2}}\Big(U_4 - U_3 + \frac{1}{2}\frac{U_2^2}{U_1}\Big)\begin{Bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{Bmatrix} = \frac{1}{\tau_{ie}}\Big(\frac{2}{3M_{ref}}\Big)^{-1/2} s(\boldsymbol{U})\begin{Bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{Bmatrix},$$

having defined the scalar function

$$s(\boldsymbol{U}) = \frac{{U_1}^{5/2}}{{U_4}^{3/2}}\Big(U_4 - U_3 + \frac{1}{2}\frac{U_2^2}{U_1}\Big).$$

Hence, the linearization of the term $\boldsymbol{f}_{EX}$ reduces to the linearization of the function $s(\boldsymbol{U})$, that is

$$\begin{aligned} s(\boldsymbol{U}^k) =& s(\boldsymbol{U}^{k-1}) + \frac{d}{d\varepsilon}\Big(s(\boldsymbol{U}^{k-1} + \varepsilon \boldsymbol{dU})\Big)\Big|_{\varepsilon=0} + \mathcal{O}(\boldsymbol{dU}^2) = \\ & s(\boldsymbol{U}^{k-1}) + \frac{ds}{d\boldsymbol{U}}\Big|_{k-1} d\boldsymbol{U} + \mathcal{O}(\boldsymbol{dU}^2) = \\ & s(\boldsymbol{U}^{k-1}) + \frac{ds}{d\boldsymbol{U}}\Big|_{k-1} \boldsymbol{U}^k - \frac{ds}{d\boldsymbol{U}}\Big|_{k-1} \boldsymbol{U}^{k-1} + \mathcal{O}(\boldsymbol{dU}^2). \end{aligned}$$

It can be shown that

$$\frac{ds}{d\boldsymbol{U}}\Big|_{k-1} \boldsymbol{U}^{k-1} = 2s(\boldsymbol{U}^{k-1}),$$

therefore the linearization of $s(\boldsymbol{U})$ finally gives

$$\boxed{s(\boldsymbol{U}^k) = \frac{ds}{d\boldsymbol{U}}\Big|_{k-1} \cdot \boldsymbol{U}^k - s(\boldsymbol{U}^{k-1}) + \mathcal{O}(\boldsymbol{dU}^2)},$$

where the derivative $\frac{ds}{d\boldsymbol{U}}$ is

$$\frac{ds}{d\boldsymbol{U}} = \begin{Bmatrix} \frac{5}{2}\Big(\frac{U_1}{U_4}\Big)^{3/2}(U_4 - U_3 + \frac{1}{2}\frac{U_2^2}{U_1}) - \frac{1}{2}\frac{U_1^{1/2}U_2^2}{U_4^{3/2}} \\ U_2(\frac{U_1}{U_4})^{3/2} \\ -\frac{{U_1}^{5/2}}{{U_4}^{3/2}} \\ -\frac{3}{2}(\frac{U_1}{U_4})^{5/2}(U_4 - U_3 + \frac{1}{2}\frac{U_2^2}{U_1}) + \frac{{U_1}^{5/2}}{U_4^{3/2}} \end{Bmatrix}.$$

The temperature exchange vector can be written as

$$\boldsymbol{f}_{EX} = \boldsymbol{f}_{EX}^{\boldsymbol{U}} + \boldsymbol{f}_{EX}^{0},$$

where the two terms are

$$\boldsymbol{f}_{EX}^{\boldsymbol{U}} = \frac{1}{\tau_{ie}}\Big(\frac{2}{3M_{ref}}\Big)^{-1/2}\frac{ds}{d\boldsymbol{U}}\Big|_{k-1} \cdot \boldsymbol{U}^k \begin{Bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{Bmatrix},$$

$$\boldsymbol{f}_{EX}^{0} = -\frac{1}{\tau_{ie}}\Big(\frac{2}{3M_{ref}}\Big)^{-1/2} s(\boldsymbol{U}^{k-1})\begin{Bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{Bmatrix},$$

5.6 Linearization of the Bohm boundary conditions

The Bohm boundary conditions for ions and electrons are

$$\frac{5-2\gamma_i}{3}\frac{U_2}{U_1}\Big(U_3-\frac{1}{2}\frac{U_2^2}{U_1}\Big)-k_{\parallel i}\Big(\frac{2}{3M_{ref}}\Big)^{7/2}\Big(\frac{U_3}{U_1}-\frac{1}{2}\frac{U_2^2}{U_1^2}\Big)^{5/2}\boldsymbol{\mathcal{Q}}_t\boldsymbol{V}_i(\boldsymbol{U})\cdot\boldsymbol{b}=0,$$
$$\frac{5-2\gamma_e}{3}\frac{U_2U_4}{U_1}-k_{\parallel e}\Big(\frac{2}{3M_{ref}}\Big)^{7/2}\Big(\frac{U_4}{U_1}\Big)^{5/2}\boldsymbol{\mathcal{Q}}_t\boldsymbol{V}_e(\boldsymbol{U})\cdot\boldsymbol{b}=0.$$

and can be re-written as, using (40), as

$$\boldsymbol{B}=0,\ \text{on } \partial\Omega_{\text{BOHM}}.$$

where

$$\boldsymbol{B}=\boldsymbol{B}_c-\boldsymbol{B}_t$$

and the terms $\boldsymbol{B}_c$ and $\boldsymbol{B}_t$ are

$$\boldsymbol{B}_c=\frac{1}{3}\left\{\begin{matrix}0\\0\\(5-2\gamma_i)\frac{U_2}{U_1}\Big(U_3-\frac{1}{2}\frac{U_2^2}{U_1}\Big)\\(5-2\gamma_e)\frac{U_2U_4}{U_1}\end{matrix}\right\}$$

and

$$\boldsymbol{B}_t=\left\{\begin{matrix}0\\0\\k_{\parallel i}\Big(\frac{2}{3M_{ref}}\Big)^{7/2}\boldsymbol{f}_i(\boldsymbol{U},\boldsymbol{\mathcal{Q}}_t)\cdot\boldsymbol{b}\\k_{\parallel e}\Big(\frac{2}{3M_{ref}}\Big)^{7/2}\boldsymbol{f}_e(\boldsymbol{U},\boldsymbol{\mathcal{Q}}_t)\cdot\boldsymbol{b}\end{matrix}\right\}.$$

The linearization of the Bohm boundary conditions reduces to the linearization of $\boldsymbol{f}_i(\boldsymbol{U},\boldsymbol{\mathcal{Q}}_t)$ and $\boldsymbol{f}_i(\boldsymbol{U},\boldsymbol{\mathcal{Q}}_t)$, already threated in 5.3, and to the linearization of $\boldsymbol{B}_c$. It results

$$\boldsymbol{B}_c(\boldsymbol{U})=\frac{d\boldsymbol{B}_c(\boldsymbol{U})}{d\boldsymbol{U}}\boldsymbol{U},$$

and therefore, similarly to 5.1, it gives

$$\boldsymbol{B}_c(\boldsymbol{U}^k)=\frac{d\boldsymbol{B}_c(\boldsymbol{U})}{d\boldsymbol{U}}\Big|_{k-1}\boldsymbol{U}^k+\mathcal{O}(d\boldsymbol{U}^2),$$

where the Jacobian is

$$\frac{d\boldsymbol{B}_c(\boldsymbol{U})}{d\boldsymbol{U}}=\frac{1}{3}\begin{bmatrix}0&0&0&0\\0&0&0&0\\-(5-2\gamma_i)\Big(\frac{U_2U_3}{U_1^2}-\frac{U_2^3}{U_1^3}\Big)&(5-2\gamma_i)\Big(\frac{U_3}{U_1}-\frac{3}{2}\frac{U_2^2}{U_1^2}\Big)&(5-2\gamma_i)\frac{U_2}{U_1}&0\\-(5-2\gamma_e)\frac{U_2U_4}{U_1^2}&(5-2\gamma_e)\frac{U_4}{U_1}&0&(5-2\gamma_e)\frac{U_2}{U_1}\end{bmatrix}.$$

From the linearization of the parallel diffusion term, we have

$$\boldsymbol{B}_t(\boldsymbol{U}^k,\boldsymbol{\mathcal{Q}}_t^k)=\boldsymbol{B}_t^{\boldsymbol{\mathcal{Q}}}(\boldsymbol{U}^{k-1},\boldsymbol{\mathcal{Q}}_t^k)+\boldsymbol{B}_t^{\boldsymbol{U}}(\boldsymbol{U}^k,\boldsymbol{\mathcal{Q}}_t^{k-1})+\boldsymbol{B}_t^0(\boldsymbol{U}^{k-1},\boldsymbol{\mathcal{Q}}_t^{k-1})+\mathcal{O}(d\boldsymbol{U}^2,d\boldsymbol{\mathcal{Q}}_t^2)$$

where

$$\boldsymbol{B}_t^{\boldsymbol{\mathcal{Q}}} = \Big(\frac{2}{3M_{ref}}\Big)^{7/2}\begin{Bmatrix} 0 \\ 0 \\ k_{\parallel i} r_i(\boldsymbol{U}^{k-1})\boldsymbol{\mathcal{Q}}_t^k \boldsymbol{V}_i(\boldsymbol{U}^{k-1})\cdot\boldsymbol{b} \\ k_{\parallel e} r_e(\boldsymbol{U}^{k-1})\boldsymbol{\mathcal{Q}}_t^k \boldsymbol{V}_e(\boldsymbol{U}^{k-1})\cdot\boldsymbol{b} \end{Bmatrix},$$

$$\boldsymbol{B}_t^{\boldsymbol{U}} = \Big(\frac{2}{3M_{ref}}\Big)^{7/2}\begin{Bmatrix} 0 \\ 0 \\ k_{\parallel i}\Big(\big(\frac{dr_i}{d\boldsymbol{U}}\Big|_{k-1}\cdot\boldsymbol{U}^k\big)\boldsymbol{\mathcal{Q}}_t^{k-1}\boldsymbol{V}_i(\boldsymbol{U}^{k-1}) + r_i(\boldsymbol{U}^{k-1})\boldsymbol{\mathcal{Q}}_t^{k-1}\frac{d\boldsymbol{V}_i}{d\boldsymbol{U}}\Big|_{k-1}\boldsymbol{U}^k\Big)\cdot\boldsymbol{b} \\ k_{\parallel e}\Big(\big(\frac{dr_e}{d\boldsymbol{U}}\Big|_{k-1}\cdot\boldsymbol{U}^k\big)\boldsymbol{\mathcal{Q}}_t^{k-1}\boldsymbol{V}_e(\boldsymbol{U}^{k-1}) + r_e(\boldsymbol{U}^{k-1})\boldsymbol{\mathcal{Q}}_t^{k-1}\frac{d\boldsymbol{V}_e}{d\boldsymbol{U}}\Big|_{k-1}\boldsymbol{U}^k\Big)\cdot\boldsymbol{b} \end{Bmatrix},$$

$$\boldsymbol{B}_t^{0} = -\Big(\frac{2}{3M_{ref}}\Big)^{7/2}\begin{Bmatrix} 0 \\ 0 \\ k_{\parallel i} r_i(\boldsymbol{U}^{k-1})\boldsymbol{\mathcal{Q}}_t^{k-1}\frac{d\boldsymbol{V}_i}{d\boldsymbol{U}}\Big|_{k-1}\boldsymbol{U}^{k-1}\cdot\boldsymbol{b} \\ k_{\parallel e} r_e(\boldsymbol{U}^{k-1})\boldsymbol{\mathcal{Q}}_t^{k-1}\frac{d\boldsymbol{V}_e}{d\boldsymbol{U}}\Big|_{k-1}\boldsymbol{U}^{k-1}\cdot\boldsymbol{b} \end{Bmatrix},$$

Therefore, it results

$$\boldsymbol{B}(\boldsymbol{U}^k,\boldsymbol{\mathcal{Q}}_t^k) = \frac{d\boldsymbol{B}_c(\boldsymbol{U})}{d\boldsymbol{U}}\Big|_{k-1}\boldsymbol{U}^k - \boldsymbol{B}_t^{\boldsymbol{\mathcal{Q}}}(\boldsymbol{U}^{k-1},\boldsymbol{\mathcal{Q}}_t^k) - \boldsymbol{B}_t^{\boldsymbol{U}}(\boldsymbol{U}^k,\boldsymbol{\mathcal{Q}}_t^{k-1}) - \boldsymbol{B}_t^0(\boldsymbol{U}^{k-1},\boldsymbol{\mathcal{Q}}_t^{k-1}) + \mathcal{O}(\boldsymbol{dU}^2,\boldsymbol{d\mathcal{Q}}_t^2)$$

6 Weak form of the system

6.1 The local problems

Using (33) and the definition of the variable gradient, the system to solve is

$$\begin{aligned} &\boldsymbol{\mathcal{Q}} - \boldsymbol{\nabla}\boldsymbol{U} = 0 \\ &\partial_t\boldsymbol{U} + \boldsymbol{\nabla}\cdot(\boldsymbol{\mathcal{F}} - D_f\boldsymbol{\mathcal{Q}} + D_f\boldsymbol{\mathcal{Q}}\boldsymbol{b}\otimes\boldsymbol{b} - \boldsymbol{\mathcal{F}}_t) \\ &\qquad\qquad + (\boldsymbol{u}_\perp\cdot\boldsymbol{\nabla})\boldsymbol{U} + \boldsymbol{f}_{E_\parallel} + \boldsymbol{f}_{EX} - \boldsymbol{g} = \boldsymbol{s}. \end{aligned} \tag{46}$$

Multiplying the first equation by the tensor test function $\boldsymbol{\mathcal{G}}$ and the second by a vector test function $\boldsymbol{v}$ and integrating in each element, we obtain

$$\begin{aligned} &\Big(\boldsymbol{\mathcal{G}},\boldsymbol{\mathcal{Q}}\Big)_K - \Big(\boldsymbol{\mathcal{G}},\boldsymbol{\nabla}\boldsymbol{U}\Big)_K = 0 \\ &\Big(\boldsymbol{v},\partial_t\boldsymbol{U}\Big)_K + \Big(\boldsymbol{v},\boldsymbol{\nabla}\cdot(\boldsymbol{\mathcal{F}} - D_f\boldsymbol{\mathcal{Q}} + D_f\boldsymbol{\mathcal{Q}}\boldsymbol{b}\otimes\boldsymbol{b} - \boldsymbol{\mathcal{F}}_t)\Big)_K \\ &\qquad + \Big(\boldsymbol{v},(\boldsymbol{u}_\perp\cdot\boldsymbol{\nabla})\boldsymbol{U}\Big)_K + \Big(\boldsymbol{v},\boldsymbol{f}_{E_\parallel}\Big)_K + \Big(\boldsymbol{v},\boldsymbol{f}_{EX}\Big)_K - \Big(\boldsymbol{v},\boldsymbol{g}\Big)_K = \Big(\boldsymbol{v},\boldsymbol{s}\Big)_K, \end{aligned}$$

which gives, after integration by parts,

$$\begin{aligned} &\Big(\boldsymbol{\mathcal{G}},\boldsymbol{\mathcal{Q}}\Big)_K + \Big(\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{G}},\boldsymbol{U}\Big)_K - \Big\langle\boldsymbol{\mathcal{G}}\boldsymbol{n},\widehat{\boldsymbol{U}}\Big\rangle_{\partial K} = 0 \\ &\Big(\boldsymbol{v},\partial_t\boldsymbol{U}\Big)_K - \Big(\boldsymbol{\nabla}\boldsymbol{v},\boldsymbol{\mathcal{F}} - D_f\boldsymbol{\mathcal{Q}} + D_f\boldsymbol{\mathcal{Q}}\boldsymbol{b}\otimes\boldsymbol{b} - \boldsymbol{\mathcal{F}}_t\Big)_K + \Big\langle\boldsymbol{v},(\widehat{\boldsymbol{\mathcal{F}}} - D_f\widehat{\boldsymbol{\mathcal{Q}}} + D_f\widehat{\boldsymbol{\mathcal{Q}}}\boldsymbol{b}\otimes\boldsymbol{b} - \widehat{\boldsymbol{\mathcal{F}}_t})\boldsymbol{n}\Big\rangle_{\partial K} \\ &\qquad + \Big(\boldsymbol{v},(\boldsymbol{u}_\perp\cdot\boldsymbol{\nabla})\boldsymbol{U}\Big)_K + \Big(\boldsymbol{v},\boldsymbol{f}_{E_\parallel}\Big)_K + \Big(\boldsymbol{v},\boldsymbol{f}_{EX}\Big)_K - \Big(\boldsymbol{v},\boldsymbol{g}\Big)_K = \Big(\boldsymbol{v},\boldsymbol{s}\Big)_K. \end{aligned}$$

The definition of the numerical traces is

$$\begin{aligned}\widehat{\boldsymbol{\mathcal{F}}}(\widehat{\boldsymbol{U}}) &= \boldsymbol{\mathcal{F}}(\widehat{\boldsymbol{U}}) + \boldsymbol{\tau}(\boldsymbol{U} - \widehat{\boldsymbol{U}}) \otimes \boldsymbol{n},\\ \widehat{\boldsymbol{\mathcal{Q}}} &= \boldsymbol{\mathcal{Q}},\\ \widehat{\boldsymbol{\mathcal{F}}_t}(\widehat{\boldsymbol{U}}) &= \boldsymbol{\mathcal{F}}_t(\widehat{\boldsymbol{U}}),\end{aligned}$$

which gives

$$\begin{aligned}&\Big(\boldsymbol{\mathcal{G}}, \boldsymbol{\mathcal{Q}}\Big)_K + \Big(\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{G}}, \boldsymbol{U}\Big)_K - \Big\langle \boldsymbol{\mathcal{G}}\boldsymbol{n}, \widehat{\boldsymbol{U}}\Big\rangle_{\partial K} = 0\\ &\Big(\boldsymbol{v}, \partial_t \boldsymbol{U}\Big)_K - \Big(\boldsymbol{\nabla v}, \boldsymbol{\mathcal{F}}(\boldsymbol{U}) - D_f\boldsymbol{\mathcal{Q}} + D_f\boldsymbol{\mathcal{Q}}\boldsymbol{b}\otimes\boldsymbol{b} - \boldsymbol{\mathcal{F}}_t(\boldsymbol{U})\Big)_K + \Big\langle \boldsymbol{v}, (\boldsymbol{\mathcal{F}}(\widehat{\boldsymbol{U}}) - D_f\boldsymbol{\mathcal{Q}} + D_f\boldsymbol{\mathcal{Q}}\boldsymbol{b}\otimes\boldsymbol{b} - \boldsymbol{\mathcal{F}}_t(\widehat{\boldsymbol{U}}))\boldsymbol{n}\Big\rangle_{\partial K}\\ &\qquad + \Big\langle \boldsymbol{v}, \boldsymbol{\tau}(\boldsymbol{U} - \widehat{\boldsymbol{U}})\Big\rangle_{\partial K} + \Big(\boldsymbol{v}, (\boldsymbol{u}_\perp\cdot\boldsymbol{\nabla})\boldsymbol{U}\Big)_K + \Big(\boldsymbol{v}, \boldsymbol{f}_{E_\parallel}\Big)_K + \Big(\boldsymbol{v}, \boldsymbol{f}_{EX}\Big)_K - \Big(\boldsymbol{v}, \boldsymbol{g}\Big)_K = \Big(\boldsymbol{v}, \boldsymbol{s}\Big)_K.\end{aligned}$$

The time derivative is discretized using an implicit scheme of the form

$$\partial_t \boldsymbol{U} \approx \delta\frac{\boldsymbol{U}}{\Delta t} - \boldsymbol{f}_0$$

where δ is a constant parameter that depends of the time integration scheme, and $\boldsymbol{f}_0$ is a vector that takes into account the previous time steps.

Using the linearization techniques introduced before, and rearranging the terms with reference to the three variables of the local problems $\boldsymbol{\mathcal{Q}}, \boldsymbol{U}, \widehat{\boldsymbol{U}}$, we obtain

$$\begin{aligned}&\Big(\boldsymbol{\nabla v}, D_f\boldsymbol{\mathcal{Q}} - D_f\boldsymbol{b}\otimes\boldsymbol{\mathcal{Q}}\boldsymbol{b} + \boldsymbol{\mathcal{F}}_t^{\boldsymbol{\mathcal{Q}}}\Big)_K + \Big\langle \boldsymbol{v}, (-D_f\boldsymbol{\mathcal{Q}} + D_f\boldsymbol{b}\otimes\boldsymbol{\mathcal{Q}}\boldsymbol{b} - \boldsymbol{\mathcal{F}}_t^{\boldsymbol{\mathcal{Q}}})\boldsymbol{n}\Big\rangle_{\partial K} + \Big(\boldsymbol{v}, \boldsymbol{f}_{E_\parallel}^{\boldsymbol{\mathcal{Q}}}\Big)_K\\ &+ \Big(\boldsymbol{v}, \frac{\delta}{\Delta t}\boldsymbol{U}\Big)_K - \Big(\boldsymbol{\nabla v}, \mathbb{A}_{(\boldsymbol{U})}^{k-1}\boldsymbol{U} - \boldsymbol{\mathcal{F}}_t^{\boldsymbol{U}}\Big)_K + \Big\langle \boldsymbol{v}, \boldsymbol{\tau}\boldsymbol{U}\Big\rangle_{\partial K}\\ &+ \Big(\boldsymbol{v}, (\boldsymbol{u}_\perp\cdot\boldsymbol{\nabla})\boldsymbol{U}\Big)_K + \Big(\boldsymbol{v}, \boldsymbol{f}_{E_\parallel}^{\boldsymbol{U}}\Big)_K + \Big(\boldsymbol{v}, \boldsymbol{f}_{EX}^{\boldsymbol{U}}\Big)_K - \Big(\boldsymbol{v}, \frac{d\boldsymbol{g}}{d\boldsymbol{U}}\Big|_{k-1}\boldsymbol{U}\Big)_K\\ &+ \Big\langle \boldsymbol{v}, (\mathbb{A}_{(\widehat{\boldsymbol{U}})}^{k-1}\widehat{\boldsymbol{U}} - \boldsymbol{\mathcal{F}}_t^{\widehat{\boldsymbol{U}}})\boldsymbol{n}\Big\rangle_{\partial K} - \Big\langle \boldsymbol{v}, \boldsymbol{\tau}\widehat{\boldsymbol{U}}\Big\rangle_{\partial K}\\ &= \Big(\boldsymbol{v}, \boldsymbol{f}_0\Big)_K + \Big(\boldsymbol{v}, \boldsymbol{s}\Big)_K - \Big(\boldsymbol{\nabla v}, \boldsymbol{\mathcal{F}}_t^0\Big)_K + \Big\langle \boldsymbol{v}, \boldsymbol{\mathcal{F}}_t^0\boldsymbol{n}\Big\rangle_{\partial K} - \Big(\boldsymbol{v}, \boldsymbol{f}_{EX}^0\Big)_K,\\ &\Big(\boldsymbol{\mathcal{G}}, \boldsymbol{\mathcal{Q}}\Big)_K + \Big(\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{G}}, \boldsymbol{U}\Big)_K - \Big\langle \boldsymbol{\mathcal{G}}\boldsymbol{n}, \widehat{\boldsymbol{U}}\Big\rangle_{\partial K} = 0.\end{aligned} \tag{47}$$

System (47) can be rewritten as

$$\begin{aligned}A_{uq}\boldsymbol{\mathcal{Q}} + A_{uu}\boldsymbol{U} + A_{ul}\widehat{\boldsymbol{U}} &= \boldsymbol{S}\\ A_{qq}\boldsymbol{\mathcal{Q}} + A_{qu}\boldsymbol{U} + A_{ql}\widehat{\boldsymbol{U}} &= \boldsymbol{0}\end{aligned}$$

where the bilinear forms are

$$\begin{aligned}A_{uq} = &\Big(\boldsymbol{\nabla v}, D_f\boldsymbol{\mathcal{Q}}\Big)_K - \Big\langle \boldsymbol{v}, D_f\boldsymbol{\mathcal{Q}}\boldsymbol{n}\Big\rangle_{\partial K}\\ &- \Big(\boldsymbol{\nabla v}, D_f\boldsymbol{\mathcal{Q}}\boldsymbol{b}\otimes\boldsymbol{b}\Big)_K + \Big\langle \boldsymbol{v}, D_f(\boldsymbol{\mathcal{Q}}\boldsymbol{b}\otimes\boldsymbol{b})\boldsymbol{n}\Big\rangle_{\partial K}\\ &+ \Big(\boldsymbol{\nabla v}, \boldsymbol{\mathcal{F}}_t^{\boldsymbol{\mathcal{Q}}}\Big)_K - \Big\langle \boldsymbol{v}, \boldsymbol{\mathcal{F}}_t^{\boldsymbol{\mathcal{Q}}}\boldsymbol{n}\Big\rangle_{\partial K} + \Big(\boldsymbol{v}, \boldsymbol{f}_{E_\parallel}^{\boldsymbol{\mathcal{Q}}}\Big)_K,\end{aligned}$$

$$
\begin{aligned}
A_{uu} = &\left(v, \frac{\delta}{\Delta t}\boldsymbol{U}\right)_K + \left\langle v, \boldsymbol{\tau}\boldsymbol{U}\right\rangle_{\partial K} + \left(v, (\boldsymbol{u}_\perp \cdot \boldsymbol{\nabla})\boldsymbol{U}\right)_K \\
&- \left(\boldsymbol{\nabla} v, \mathbb{A}^{k-1}\boldsymbol{U}\right)_K + \left(\boldsymbol{\nabla} v, \boldsymbol{\mathcal{F}}_t^{\boldsymbol{U}}\right)_K \\
&+ \left(v, \boldsymbol{f}_{E_\parallel}^{\boldsymbol{U}}\right)_K + \left(v, \boldsymbol{f}_{EX}^{\boldsymbol{U}}\right)_K - \left(v, \frac{d\boldsymbol{g}}{d\boldsymbol{U}}\Big|_{k-1}\boldsymbol{U}\right)_K,
\end{aligned}
$$

$$
A_{ul} = +\left\langle v, (\mathbb{A}^{k-1}\widehat{\boldsymbol{U}})\boldsymbol{n}\right\rangle_{\partial K} - \left\langle v, \boldsymbol{\mathcal{F}}_t^{\widehat{\boldsymbol{U}}}\boldsymbol{n}\right\rangle_{\partial K} - \left\langle v, \boldsymbol{\tau}\widehat{\boldsymbol{U}}\right\rangle_{\partial K},
$$

$$
A_{qq} = \left(\boldsymbol{\mathcal{G}}, \boldsymbol{\mathcal{Q}}\right)_K,
$$

$$
A_{qu} = \left(\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{G}}, \boldsymbol{U}\right)_K,
$$

$$
A_{qu} = -\left\langle \boldsymbol{\mathcal{G}}\boldsymbol{n}, \widehat{\boldsymbol{U}}\right\rangle_{\partial K}
$$

$$
\boldsymbol{S} = \left(v, \boldsymbol{f}_0\right)_K + \left(v, \boldsymbol{s}\right)_K - \left(\boldsymbol{\nabla} v, \boldsymbol{\mathcal{F}}_t^0\right)_K + \left\langle v, \boldsymbol{\mathcal{F}}_t^0\boldsymbol{n}\right\rangle_{\partial K} - \left(v, \boldsymbol{f}_{EX}^0\right)_K.
$$

6.2 The global problem

The global problem derives from the imposition of the continuity of the fluxes in the normal direction of the interior faces, and the boundary conditions, that is

$$
\left\langle \boldsymbol{\mu}, (\widehat{\boldsymbol{\mathcal{F}}} - D_f\widehat{\boldsymbol{\mathcal{Q}}} + D_f\widehat{\boldsymbol{\mathcal{Q}}}\boldsymbol{b}\otimes\boldsymbol{b} - \widehat{\boldsymbol{\mathcal{F}}}_t)\boldsymbol{n}\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega} + \left\langle \boldsymbol{\mu}, \boldsymbol{B}_{\text{BC}}\right\rangle_{\partial\Omega} = 0,
$$

where $\mathcal{T}_h$ represents the skeleton of the triangulation and $\boldsymbol{B}_{\text{BC}}$ is a vector that defines the boundary conditions on $\partial\Omega$. In particular, for the Bohm boundary condition $\boldsymbol{B}_{\text{BC}} = \boldsymbol{B}$ on $\partial\Omega_{\text{Bohm}}$, as defined in 5.6.

Substituting the definition of the fluxes and $\boldsymbol{B}$ we obtain

$$
\left\langle \boldsymbol{\mu}, (\boldsymbol{\mathcal{F}} - D_f\boldsymbol{\mathcal{Q}} + D_f\boldsymbol{\mathcal{Q}}\boldsymbol{b}\otimes\boldsymbol{b} - \boldsymbol{\mathcal{F}}_t)\boldsymbol{n} + \boldsymbol{\tau}(\boldsymbol{U} - \widehat{\boldsymbol{U}})\right\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega} + \left\langle \boldsymbol{\mu}, \boldsymbol{B}_{\text{BC}}\right\rangle_{\partial\Omega} = 0,
$$

7 Discrete form in 2D Cartesian/axisymmetric configuration

A planar configuration is considered. The x and y axis define the coordinate plane. The derivative rules used in the following for Cartesian or axisymmetric configuration are, considering a generic scalar function f and a generic vector function $\boldsymbol{v}$

$$\text{Cartesian: } \boldsymbol{\nabla} f = \frac{\partial f}{\partial x}\hat{\boldsymbol{e}}_x + \frac{\partial f}{\partial y}\hat{\boldsymbol{e}}_y, \quad \boldsymbol{\nabla}\cdot\boldsymbol{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y},$$

$$\text{Axisymmetric: } \boldsymbol{\nabla} f = \frac{\partial f}{\partial x}\hat{\boldsymbol{e}}_x + \frac{\partial f}{\partial y}\hat{\boldsymbol{e}}_y, \quad \boldsymbol{\nabla}\cdot\boldsymbol{v} = \frac{\partial v_x}{\partial x} + \frac{1}{x}v_x + \frac{\partial v_y}{\partial y}.$$

In order to develop a high-order finite-element scheme, high-order polynomial interpolation is considered in each element to represent the unknowns. Defining a set of basis function, a generic scalar function can be represented in a generic point $\boldsymbol{x}$ of the element K as

$$f|_K(\boldsymbol{x}) = \sum_{j=1}^{N_p} N_j(\boldsymbol{x}) f_j,$$

where N_p is the number of nodes in each element, N_j is the $j-th$ basis and f_j is the nodal value of the function f in the $j-th$ node.

Similarly, the vector of nodal values for the vector unknown $\boldsymbol{U}$ in the element K can be represented as (dropping some symbols to easy the notation)

$$\boldsymbol{U} = \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \sum_{j=1}^{N_p} \begin{bmatrix} N_j & 0 & 0 & 0 \\ 0 & N_j & 0 & 0 \\ 0 & 0 & N_j & 0 \\ 0 & 0 & 0 & N_j \end{bmatrix} \begin{Bmatrix} U_1^j \\ U_2^j \\ U_3^j \\ U_4^j \end{Bmatrix} = \sum_{j=1}^{N_p} N_j \boldsymbol{\mathcal{I}}_4 \boldsymbol{U}^j,$$

while the vector of nodal values for the tensor unknown $\boldsymbol{\mathcal{Q}}$ in the same element can be written as

$$\boldsymbol{\mathcal{Q}} = \begin{Bmatrix} \mathcal{Q}_{11} \\ \mathcal{Q}_{12} \\ \mathcal{Q}_{21} \\ \mathcal{Q}_{22} \\ \mathcal{Q}_{31} \\ \mathcal{Q}_{32} \\ \mathcal{Q}_{41} \\ \mathcal{Q}_{42} \end{Bmatrix} = \sum_{j=1}^{N_p} \begin{bmatrix} N_j & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & N_j & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & N_j & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_j & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & N_j & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & N_j & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & N_j & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & N_j \end{bmatrix} \begin{Bmatrix} \mathcal{Q}_{11}^j \\ \mathcal{Q}_{12}^j \\ \mathcal{Q}_{21}^j \\ \mathcal{Q}_{22}^j \\ \mathcal{Q}_{31}^j \\ \mathcal{Q}_{32}^j \\ \mathcal{Q}_{41}^j \\ \mathcal{Q}_{42}^j \end{Bmatrix} = \sum_{j=1}^{N_p} N_j \boldsymbol{\mathcal{I}}_8 \boldsymbol{\mathcal{Q}}^j,$$

where $\boldsymbol{\mathcal{I}}_n$ is the identity matrix of rank n, and $\boldsymbol{U}^j$ and $\boldsymbol{\mathcal{Q}}^j$ are the vectors of the nodal values for the unknowns $\boldsymbol{U}$ and $\boldsymbol{\mathcal{Q}}$ for the node j.

It is useful to define in this framework also some operations applied to the unknowns. For example, the gradient of $\boldsymbol{U}$ is

$$\boldsymbol{\nabla U} = \begin{bmatrix} U_{1,x} & U_{1,y} \\ U_{2,x} & U_{2,y} \\ U_{3,x} & U_{3,y} \\ U_{4,x} & U_{4,y} \end{bmatrix},$$

and can be written in vector-nodal notation as

$$\boldsymbol{\nabla U} = \begin{Bmatrix} U_{1,x} \\ U_{1,y} \\ U_{2,x} \\ U_{2,y} \\ U_{3,x} \\ U_{3,y} \\ U_{4,x} \\ U_{4,y} \end{Bmatrix} = \sum_{j=1}^{N_p} \begin{bmatrix} N_{j,x} & 0 & 0 & 0 \\ N_{j,y} & 0 & 0 & 0 \\ 0 & N_{j,x} & 0 & 0 \\ 0 & N_{j,y} & 0 & 0 \\ 0 & 0 & N_{j,x} & 0 \\ 0 & 0 & N_{j,y} & 0 \\ 0 & 0 & 0 & N_{j,x} \\ 0 & 0 & 0 & N_{j,y} \end{bmatrix} \begin{Bmatrix} U_1^j \\ U_2^j \\ U_3^j \\ U_4^j \end{Bmatrix}.$$

The product of $\boldsymbol{U}$ with the Jacobian matrix $\mathbb{A}$ is

$$\mathbb{A}\boldsymbol{U} = \boldsymbol{\mathcal{A}U} \otimes \boldsymbol{b}^T = \boldsymbol{\mathcal{A}} \begin{bmatrix} N_j & 0 & 0 & 0 \\ 0 & N_j & 0 & 0 \\ 0 & 0 & N_j & 0 \\ 0 & 0 & 0 & N_j \end{bmatrix} \begin{Bmatrix} U_1^j \\ U_2^j \\ U_3^j \\ U_4^j \end{Bmatrix} \otimes \boldsymbol{b}^T.$$

Hence, in vector-nodal notation it becomes

$$\mathbb{A}\boldsymbol{U} = \sum_{j=1}^{N_p} \begin{bmatrix} \mathcal{A}_{11}N_jb_x & \mathcal{A}_{12}N_jb_x & \mathcal{A}_{13}N_jb_x & \mathcal{A}_{14}N_jb_x \\ \mathcal{A}_{11}N_jb_y & \mathcal{A}_{12}N_jb_y & \mathcal{A}_{13}N_jb_y & \mathcal{A}_{14}N_jb_y \\ \mathcal{A}_{21}N_jb_x & \mathcal{A}_{22}N_jb_x & \mathcal{A}_{23}N_jb_x & \mathcal{A}_{24}N_jb_x \\ \mathcal{A}_{21}N_jb_y & \mathcal{A}_{22}N_jb_y & \mathcal{A}_{23}N_jb_y & \mathcal{A}_{24}N_jb_y \\ \mathcal{A}_{31}N_jb_x & \mathcal{A}_{32}N_jb_x & \mathcal{A}_{33}N_jb_x & \mathcal{A}_{34}N_jb_x \\ \mathcal{A}_{31}N_jb_y & \mathcal{A}_{32}N_jb_y & \mathcal{A}_{33}N_jb_y & \mathcal{A}_{34}N_jb_y \\ \mathcal{A}_{41}N_jb_x & \mathcal{A}_{42}N_jb_x & \mathcal{A}_{43}N_jb_x & \mathcal{A}_{44}N_jb_x \\ \mathcal{A}_{41}N_jb_y & \mathcal{A}_{42}N_jb_y & \mathcal{A}_{43}N_jb_y & \mathcal{A}_{44}N_jb_y \end{bmatrix} \begin{Bmatrix} U_1^j \\ U_2^j \\ U_3^j \\ U_4^j \end{Bmatrix}.$$

The divergence of $\boldsymbol{\mathcal{Q}}$ can be written as

$$\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{Q}} = \begin{bmatrix} \mathcal{Q}_{11,x} + \mathcal{Q}_{12,y} \\ \mathcal{Q}_{21,x} + \mathcal{Q}_{22,y} \\ \mathcal{Q}_{31,x} + \mathcal{Q}_{32,y} \\ \mathcal{Q}_{41,x} + \mathcal{Q}_{42,y} \end{bmatrix} = \sum_{j=1}^{N_p} \begin{bmatrix} \tilde{N}_{j,x} & N_{j,y} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{N}_{j,x} & N_{j,y} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tilde{N}_{j,x} & N_{j,y} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \tilde{N}_{j,x} & N_{j,y} \end{bmatrix} \begin{Bmatrix} \mathcal{Q}_{11}^j \\ \mathcal{Q}_{12}^j \\ \mathcal{Q}_{21}^j \\ \mathcal{Q}_{22}^j \\ \mathcal{Q}_{31}^j \\ \mathcal{Q}_{32}^j \\ \mathcal{Q}_{41}^j \\ \mathcal{Q}_{42}^j \end{Bmatrix},$$

where $\tilde{N}_{j,x}$ corresponds to $N_{j,x}$ for Cartesian computations and to $N_{j,x} + \frac{1}{x}N_j$ for axisymmetric computations.

The product of $\boldsymbol{\mathcal{Q}}$ for a generic 2D vector $\boldsymbol{q} = \{q_x, q_y\}^T$ is

$$\boldsymbol{\mathcal{Q}q} = \boldsymbol{\mathcal{Q}} \begin{Bmatrix} q_x \\ q_y \end{Bmatrix} = \begin{bmatrix} \mathcal{Q}_{11}q_x + \mathcal{Q}_{12}q_y \\ \mathcal{Q}_{21}q_x + \mathcal{Q}_{22}q_y \\ \mathcal{Q}_{31}q_x + \mathcal{Q}_{32}q_y \\ \mathcal{Q}_{41}q_x + \mathcal{Q}_{42}q_y \end{bmatrix} = \sum_{j=1}^{N_p} \begin{bmatrix} N_jq_x & N_jq_y & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & N_jq_x & N_jq_y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & N_jq_x & N_jq_y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & N_jq_x & N_jq_y \end{bmatrix} \begin{Bmatrix} \mathcal{Q}_{11}^j \\ \mathcal{Q}_{12}^j \\ \mathcal{Q}_{21}^j \\ \mathcal{Q}_{22}^j \\ \mathcal{Q}_{31}^j \\ \mathcal{Q}_{32}^j \\ \mathcal{Q}_{41}^j \\ \mathcal{Q}_{42}^j \end{Bmatrix}.$$

Finally, the tensor term $\boldsymbol{\mathcal{Q}b}\otimes\boldsymbol{b}$ is

$$\boldsymbol{\mathcal{Q}b}\otimes\boldsymbol{b} = \begin{bmatrix} b_x(\mathcal{Q}_{11}b_x+\mathcal{Q}_{12}b_y) & b_y(\mathcal{Q}_{11}b_x+\mathcal{Q}_{12}b_y) \\ b_x(\mathcal{Q}_{21}b_x+\mathcal{Q}_{22}b_y) & b_y(\mathcal{Q}_{21}b_x+\mathcal{Q}_{22}b_y) \\ b_x(\mathcal{Q}_{31}b_x+\mathcal{Q}_{32}b_y) & b_y(\mathcal{Q}_{31}b_x+\mathcal{Q}_{32}b_y) \\ b_x(\mathcal{Q}_{41}b_x+\mathcal{Q}_{42}b_y) & b_y(\mathcal{Q}_{41}b_x+\mathcal{Q}_{42}b_y) \end{bmatrix},$$

and can be written in vector-nodal notation as

$$\boldsymbol{\mathcal{Q}b}\otimes\boldsymbol{b} = \begin{Bmatrix} b_x(\mathcal{Q}_{11}b_x+\mathcal{Q}_{12}b_y) \\ b_y(\mathcal{Q}_{11}b_x+\mathcal{Q}_{12}b_y) \\ b_x(\mathcal{Q}_{21}b_x+\mathcal{Q}_{22}b_y) \\ b_y(\mathcal{Q}_{21}b_x+\mathcal{Q}_{22}b_y) \\ b_x(\mathcal{Q}_{31}b_x+\mathcal{Q}_{32}b_y) \\ b_y(\mathcal{Q}_{31}b_x+\mathcal{Q}_{32}b_y) \\ b_x(\mathcal{Q}_{41}b_x+\mathcal{Q}_{42}b_y) \\ b_y(\mathcal{Q}_{41}b_x+\mathcal{Q}_{42}b_y) \end{Bmatrix} =$$

$$\sum_{j=1}^{N_p} \begin{bmatrix} N_jb_xb_x & N_jb_xb_y & 0 & 0 & 0 & 0 & 0 & 0 \\ N_jb_xb_y & N_jb_yb_y & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & N_jb_xb_x & N_jb_xb_y & 0 & 0 & 0 & 0 \\ 0 & 0 & N_jb_xb_x & N_jb_xb_y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & N_jb_xb_x & N_jb_xb_y & 0 & 0 \\ 0 & 0 & 0 & 0 & N_jb_xb_x & N_jb_xb_y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & N_jb_xb_x & N_jb_xb_y \\ 0 & 0 & 0 & 0 & 0 & 0 & N_jb_xb_x & N_jb_xb_y \end{bmatrix} \begin{Bmatrix} \mathcal{Q}_{11}^j \\ \mathcal{Q}_{12}^j \\ \mathcal{Q}_{21}^j \\ \mathcal{Q}_{22}^j \\ \mathcal{Q}_{31}^j \\ \mathcal{Q}_{32}^j \\ \mathcal{Q}_{41}^j \\ \mathcal{Q}_{42}^j \end{Bmatrix}.$$

To define a Galerkin method, the test functions are chosen in the same space of the basis functions. Hence, the functions $\boldsymbol{v}_i(v_i)$ and $\boldsymbol{\mathcal{G}}(\mathcal{G}_i)$ are defined as

$$\boldsymbol{v} = N_i\boldsymbol{\mathcal{I}}_4\boldsymbol{v}, \quad \boldsymbol{\mathcal{G}} = N_i\boldsymbol{\mathcal{I}}_8\boldsymbol{\mathcal{G}}$$

where the vector $\boldsymbol{v}$ takes the values

$$\boldsymbol{v} = \begin{Bmatrix}1\\0\\0\\0\end{Bmatrix} \text{ for the first equation,} \quad \boldsymbol{v} = \begin{Bmatrix}0\\1\\0\\0\end{Bmatrix} \text{ for the second equation,}$$

$$\boldsymbol{v} = \begin{Bmatrix}0\\0\\1\\0\end{Bmatrix} \text{ for the third equation,} \quad \boldsymbol{v} = \begin{Bmatrix}0\\0\\0\\1\end{Bmatrix} \text{ for the fourth equation.}$$

The vector $\boldsymbol{\mathcal{G}}$ is constructed in a similar way.

Similarly to the basis functions, some operations on the test functions can be defined as

$$\boldsymbol{\nabla v} = \begin{bmatrix} N_{i,x} & 0 & 0 & 0 \\ N_{i,y} & 0 & 0 & 0 \\ 0 & N_{i,x} & 0 & 0 \\ 0 & N_{i,y} & 0 & 0 \\ 0 & 0 & N_{i,x} & 0 \\ 0 & 0 & N_{i,y} & 0 \\ 0 & 0 & 0 & N_{i,x} \\ 0 & 0 & 0 & N_{i,y} \end{bmatrix}\boldsymbol{v},$$

$$\boldsymbol{\nabla}\cdot\mathcal{G} = \begin{bmatrix} \tilde{N}_{i,x} & N_{i,y} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \tilde{N}_{i,x} & N_{i,y} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tilde{N}_{i,x} & N_{i,y} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \tilde{N}_{i,x} & N_{i,y} \end{bmatrix} \boldsymbol{\mathcal{G}},$$

$$\mathcal{G}\boldsymbol{n} = \begin{bmatrix} N_i b_x & N_i b_y & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & N_i b_x & N_i b_y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & N_i b_x & N_i b_y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & N_i b_x & N_i b_y \end{bmatrix} \boldsymbol{\mathcal{G}}.$$

7.1 Discretization of the bilinear forms

The discretization introduced before allows to discretize the bilinear forms introduced in 6. As an example, the term $\left(\boldsymbol{\nabla v}, D_f \boldsymbol{\mathcal{Q}}\right)_K$ becomes

$$\left(\boldsymbol{\nabla v}, D_f \boldsymbol{\mathcal{Q}}\right)_K = D_f \int_K (\boldsymbol{\nabla v})^T : \boldsymbol{\mathcal{Q}} dK =$$

$$\sum_{j=1}^{N_p} \boldsymbol{v}^t D_f \int_K \boldsymbol{\mathcal{M}} dK \begin{Bmatrix} \mathcal{Q}_{11}^j \\ \mathcal{Q}_{12}^j \\ \mathcal{Q}_{21}^j \\ \mathcal{Q}_{22}^j \\ \mathcal{Q}_{31}^j \\ \mathcal{Q}_{32}^j \\ \mathcal{Q}_{41}^j \\ \mathcal{Q}_{42}^j \end{Bmatrix},$$

where the matrix is $\boldsymbol{\mathcal{M}}$

$$\boldsymbol{\mathcal{M}} = \begin{bmatrix} N_{i,x} & N_{i,y} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & N_{i,x} & N_{i,y} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & N_{i,x} & N_{i,y} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & N_{i,x} & N_{i,y} \end{bmatrix} \begin{bmatrix} N_j & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & N_j & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & N_j & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_j & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & N_j & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & N_j & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & N_j & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & N_j \end{bmatrix},$$

Hence,

$$\boldsymbol{\mathcal{M}} = \begin{bmatrix} N_{i,x}N_j & N_{i,y}N_j & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & N_{i,x}N_j & N_{i,y}N_j & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & N_{i,x}N_j & N_{i,y}N_j & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & N_{i,x}N_j & N_{i,y}N_j \end{bmatrix}$$

In the following part are shown the matrices related to different bilinear forms.

$$\left(\boldsymbol{\nabla v}, \boldsymbol{\mathcal{Q}}\right)_K \rightarrow N_j \begin{bmatrix} N_{i,x} & N_{i,y} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & N_{i,x} & N_{i,y} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & N_{i,x} & N_{i,y} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & N_{i,x} & N_{i,y} \end{bmatrix}$$

$$\left\langle \boldsymbol{v}, \boldsymbol{\mathcal{Q} n}\right\rangle_{\partial K} \rightarrow N_i N_j \begin{bmatrix} n_x & n_y & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & n_x & n_y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & n_x & n_y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & n_x & n_y \end{bmatrix}$$

$$\left(\boldsymbol{\nabla v}, \boldsymbol{\mathcal{Q} b} \otimes \boldsymbol{b}\right)_K \rightarrow (N_{i,x}b_x + N_{i,y}b_y)N_j \begin{bmatrix} b_x & b_y & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_x & b_y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_x & b_y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_x & b_y \end{bmatrix}$$

$$\left\langle \boldsymbol{v}, (\boldsymbol{\mathcal{Q}}\boldsymbol{b} \otimes \boldsymbol{b})\boldsymbol{n} \right\rangle_{\partial K} \to \boldsymbol{n} \cdot \boldsymbol{b} N_i N_j \begin{bmatrix} b_x & b_y & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_x & b_y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_x & b_y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & b_x & b_y \end{bmatrix}$$

$$\left(\boldsymbol{\nabla} \boldsymbol{v}, \mathbb{A}^{k-1} \boldsymbol{U} \right)_K \to (N_{i,x} b_x + N_{i,y} b_y) N_j \boldsymbol{\mathcal{A}}^{k-1}$$

$$\left\langle \boldsymbol{v}, (\mathbb{A}^{k-1} \widehat{\boldsymbol{U}})\boldsymbol{n} \right\rangle_{\partial K} \to \boldsymbol{n} \cdot \boldsymbol{b} N_i N_j \boldsymbol{\mathcal{A}}^{k-1}$$

8 Discrete form in 3D with Fourier series in the toroidal direction

In this section, the HDG scheme is the 2D poloidal plane is coupled with a finite-element scheme based on Fourier expansions in the toroidal direction ϕ. To this aim, some notation is introduced. The gradient of a generic function f is written as

$$\boldsymbol{\nabla} f = \boldsymbol{\nabla} f_{pol} + \frac{1}{x} \frac{\partial f}{\partial \phi} \hat{\boldsymbol{e}}_\phi,$$

where $\boldsymbol{\nabla} f_{pol}$ is the projection of the gradient in the poloidal plane. Therefore the gradient of the vector variable $\boldsymbol{U}$ is

$$\boldsymbol{\nabla} \boldsymbol{U} = \boldsymbol{\nabla} \boldsymbol{U}_{pol} + \frac{1}{x} \frac{\partial \boldsymbol{U}}{\partial \phi} \otimes \hat{\boldsymbol{e}}_\phi = \boldsymbol{\mathcal{Q}} + \frac{1}{x} \frac{\partial \boldsymbol{U}}{\partial \phi} \otimes \hat{\boldsymbol{e}}_\phi,$$

Hence, 46 is rewritten as

$$\begin{aligned} &\boldsymbol{\mathcal{Q}} - \boldsymbol{\nabla} \boldsymbol{U}_{pol} = 0 \\ &\partial_t \boldsymbol{U} + \boldsymbol{\nabla} \cdot \Big(\boldsymbol{\mathcal{F}} - D_f \boldsymbol{\mathcal{Q}} - D_f \frac{1}{x} \frac{\partial \boldsymbol{U}}{\partial \phi} \otimes \hat{\boldsymbol{e}}_\phi + D_f \boldsymbol{\mathcal{Q}} \boldsymbol{b} \otimes \boldsymbol{b} + D_f \frac{1}{x} \frac{\partial \boldsymbol{U}}{\partial \phi} b_\phi \otimes \boldsymbol{b} - \boldsymbol{\mathcal{F}}_t \Big) \\ &\qquad + (\boldsymbol{u}_\perp \cdot \boldsymbol{\nabla}) \boldsymbol{U} + \boldsymbol{f}_{E_\parallel} + \boldsymbol{f}_{EX} - \boldsymbol{g} = \boldsymbol{s}, \end{aligned}$$

that is

$$\begin{aligned} &\boldsymbol{\mathcal{Q}} - \boldsymbol{\nabla} \boldsymbol{U}_{pol} = 0 \\ &\partial_t \boldsymbol{U} + \boldsymbol{\nabla} \cdot (\boldsymbol{\mathcal{F}} - D_f \boldsymbol{\mathcal{Q}} + D_f \boldsymbol{\mathcal{Q}} \boldsymbol{b} \otimes \boldsymbol{b} - \boldsymbol{\mathcal{F}}_t) + \boldsymbol{\nabla} \cdot \Big(-D_f \frac{1}{x} \frac{\partial \boldsymbol{U}}{\partial \phi} \otimes \hat{\boldsymbol{e}}_\phi + D_f \frac{1}{x} \frac{\partial \boldsymbol{U}}{\partial \phi} b_\phi \otimes \boldsymbol{b} \Big) \\ &\qquad + (\boldsymbol{u}_\perp \cdot \boldsymbol{\nabla}) \boldsymbol{U} + \boldsymbol{f}_{E_\parallel} + \boldsymbol{f}_{EX} - \boldsymbol{g} = \boldsymbol{s}. \end{aligned}$$

In order to develop a 3D scheme using Fourier series in the toroidal direction, the following approximation of a generic periodic function is considered

$$f(\phi) \approx a_0 + \sum_{m=1}^{N_m} a_m \cos(m\phi) + b_m \sin(m\phi),$$

where N_m is the number of modes and the a_m and b_m are the coefficients of the modal expansion.

A 3D function in the toroidal space will be defined by the poloidal position $\boldsymbol{x}$ and the toroidal position ϕ. Using high-order polynomials in the poloidal plane and the Fourier expansion in the toroidal direction, a generic function can be approximated as

$$f|_K(\boldsymbol{x}, \phi) = \sum_{j=1}^{N_p} N_j(\boldsymbol{x}) f_j (a_0 + \sum_{m=1}^{N_m} a_m \cos(m\phi) + b_m \sin(m\phi)).$$

Hence, the unknowns $\boldsymbol{U}$ and $\boldsymbol{\mathcal{Q}}$ are written as

$$\boldsymbol{U}(\boldsymbol{x},\phi) = \sum_{j=1}^{N_p} N_j(\boldsymbol{x})\boldsymbol{\mathcal{I}}_4\boldsymbol{U}^j(a_0 + \sum_{m=1}^{N_m} a_m \cos(m\phi) + b_m \sin(m\phi)),$$

$$\boldsymbol{\mathcal{Q}}(\boldsymbol{x},\phi) = \sum_{j=1}^{N_p} N_j(\boldsymbol{x})\boldsymbol{\mathcal{I}}_8\boldsymbol{\mathcal{Q}}^j(a_0 + \sum_{m=1}^{N_m} a_m \cos(m\phi) + b_m \sin(m\phi)),$$

which provides,

$$\boldsymbol{U}(\boldsymbol{x},\phi) = \sum_{j=1}^{N_p} N_j(\boldsymbol{x})\boldsymbol{\mathcal{I}}_4(\boldsymbol{U}_0^j + \sum_{m=1}^{N_m} \boldsymbol{U}_{cm}^j \cos(m\phi) + \boldsymbol{U}_{sm}^j \sin(m\phi)),$$

$$\boldsymbol{\mathcal{Q}}(\boldsymbol{x},\phi) = \sum_{j=1}^{N_p} N_j(\boldsymbol{x})\boldsymbol{\mathcal{I}}_8(\boldsymbol{\mathcal{Q}}_0^j + \sum_{m=1}^{N_m} \boldsymbol{\mathcal{Q}}_{cm}^j \cos(m\phi) + \boldsymbol{\mathcal{Q}}_{sm}^j \sin(m\phi)),$$

where $\boldsymbol{U}_0^j$, $\boldsymbol{U}_{cm}^j$ and $\boldsymbol{U}_{sm}^j$ and $\boldsymbol{\mathcal{Q}}_0^j$, $\boldsymbol{\mathcal{Q}}_{cm}^j$ and $\boldsymbol{\mathcal{Q}}_{sm}^j$ are the nodal values for the modal expansions respectively for the variable $\boldsymbol{U}$ and $\boldsymbol{\mathcal{Q}}$. Note that the problem presents $2N_m - 1$ vector unknowns for each variable.

The test functions are defined in this case as

$$\boldsymbol{v} = N_i\boldsymbol{\mathcal{I}}_4\boldsymbol{v}\phi, \quad \boldsymbol{\mathcal{G}} = N_i\boldsymbol{\mathcal{I}}_8\boldsymbol{\mathcal{G}}\phi,$$

where ϕ are the test functions in the toroidal directions, that is

$$\phi = 1, \cos(n\phi), \sin(n\phi), \text{ for } n = 1, \cdots, N_m,$$

that is, there are $2N_m - 1$ test functions in the toroidal direction.

$$\begin{aligned}
&\left(\boldsymbol{\mathcal{G}},\boldsymbol{\mathcal{Q}}\right)_K + \left(\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{G}},\boldsymbol{U}\right)_K - \left\langle\boldsymbol{\mathcal{G}}\boldsymbol{n},\widehat{\boldsymbol{U}}\right\rangle_{\partial K} = 0\\
&\left(\boldsymbol{v},\partial_t\boldsymbol{U}\right)_K - \left(\boldsymbol{\nabla}\boldsymbol{v},\boldsymbol{\mathcal{F}}(\boldsymbol{U}) - D_f\boldsymbol{\mathcal{Q}} + D_f\boldsymbol{\mathcal{Q}}\boldsymbol{b}\otimes\boldsymbol{b} - \boldsymbol{\mathcal{F}}_t(\boldsymbol{U})\right)_K + \left\langle\boldsymbol{v},(\boldsymbol{\mathcal{F}}(\widehat{\boldsymbol{U}}) - D_f\boldsymbol{\mathcal{Q}} + D_f\boldsymbol{\mathcal{Q}}\boldsymbol{b}\otimes\boldsymbol{b} - \boldsymbol{\mathcal{F}}_t(\widehat{\boldsymbol{U}}))\boldsymbol{n}\right\rangle_{\partial K}\\
&\qquad + \left(\boldsymbol{\nabla}\boldsymbol{v}, D_f\frac{1}{x}\frac{\partial\boldsymbol{U}}{\partial\phi}\otimes\hat{\boldsymbol{e}}_\phi - D_f\frac{1}{x}\frac{\partial\boldsymbol{U}}{\partial\phi}b_\phi\otimes\boldsymbol{b}\right)_K - \left\langle\boldsymbol{v},(D_f\frac{1}{x}\frac{\partial\boldsymbol{U}}{\partial\phi}\otimes\hat{\boldsymbol{e}}_\phi - D_f\frac{1}{x}\frac{\partial\boldsymbol{U}}{\partial\phi}b_\phi\otimes\boldsymbol{b})\boldsymbol{n}\right\rangle_{\partial K}\\
&\qquad + \left\langle\boldsymbol{v},\boldsymbol{\tau}(\boldsymbol{U}-\widehat{\boldsymbol{U}})\right\rangle_{\partial K} + \left(\boldsymbol{v},(\boldsymbol{u}_\perp\cdot\boldsymbol{\nabla})\boldsymbol{U}\right)_K + \left(\boldsymbol{v},\boldsymbol{f}_{E_\parallel}\right)_K + \left(\boldsymbol{v},\boldsymbol{f}_{EX}\right)_K - \left(\boldsymbol{v},\boldsymbol{g}\right)_K = \left(\boldsymbol{v},\boldsymbol{s}\right)_K.
\end{aligned}$$