Time Crystals for Students

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Abstract

Abstract: This review explores time crystals, a novel concept proposed by physicist Frank Wilczek in 2012. Time crystals, exhibiting a self-organized, time-periodic structure, deviate from equilibrium states and hold promise for applications like atomic clocks and quantum computers. The discussion focuses on the breaking of continuous time invariance, analogous to spatial crystals breaking spatial invariance. Starting with Spontaneous Symmetry Breaking, the review navigates Wilczek's theories, including challenges and refutations by physicists like Bruno. The proposed realization through Floquet states and discrete time crystals concludes the exploration, offering insights into the intricate interplay of symmetry and the temporal dimension of matter.

1. Introduction

In this presentation we will discuss an interesting concept postulated for the first time in 2012 by the physicist Frank Wilczek and still the subject of discussion and study among the scientific community: time crystals. A time crystal consists in a self-organised time-periodic structure, as a space crystal does but in space, that in its ground state displays a time-dependent behavior. It represents the first example of a non-equilibrium state of matter and, thanks to its stability, its realization would be useful for many applications like atomic clocks, quantum computers or high precision magnetic field's sensors.
The key point of the discussion lies in the concept of symmetry and its breaking; as for space crystals, where the creation of a spatial self-organised structure breaks spontaneously continuous spacial invariance, time crystals are supposed to break spontaneously the continuous time invariance. Before proceeding with the developed theories on time crystals it is therefore necessary to clarify some topics related to symmetry. For this reason the discussion will follow this guideline: firstly, a brief insight into the theory of Spontaneous Symmetry Breaking will be given. Next, the Wilczek's classic and quantum theory for the time crystals will be presented, emphasizing the critical aspects which led Bruno to confute that model, proving that they are impossible to realize. In conclusion, the concept of Floquet states and discrete time crystals for the experimental realization will be proposed.

2. Spontaneous Symmetry Breaking

In this section several concepts about symmetry and its breaking will be introduced. The purpose of this discussion is to provide the reader with some useful tools for a better understanding of how Wilczek's paper were evaluated and how the modern concept of time crystals has been built.

2.1. Recall on symmetry formalism

We already know that a state $|\psi\rangle$ is said to be symmetric under a unitary transformation U if the transformed state is identical to the original state, up to a phase factor:

$$U|\psi\rangle = e^{i\varphi}|\psi\rangle \tag{2.1}$$

Since the dynamics of quantum states follow directly from the Hamiltonian, symmetries of the equation of motion correspond to symmetries of the Hamiltonian operator. Of course, symmetries can be defined whitin the Lagrangian formulation just as well as whitin the Hamiltonian formalism. It is known from the Noether's theorem that for any unitary and continuous symmetry transformation U is associated an observable Q such that $U = e^{iQ}$, and that Q is a conserved quantity.
In general, any quantum operator A is called invariant under the unitary transformation U if $U^\dagger AU = A$, or equivalently, if $[U, A] = 0$. If the Hamiltonian H is invariant under some unitary transformation U, then U is called a symmetry of the Hamiltonian and their commutator vanishes:

$$[U, H] = 0 \tag{2.2}$$

2.2. Basic notion of SSB

As seen in the previous subsection, having the commutator between H and U that vanishes means that if H has an eigenstate $|\psi\rangle$ with eigenvalue E_ψ, then the transformed state $U|\psi\rangle$ is also an eigenstate of H with the same eigenvalue:

$$H\left(U\left|\psi\right\rangle\right) = UH\left|\psi\right\rangle = UE_\psi\left|\psi\right\rangle = E_\psi\left(U\left|\psi\right\rangle\right) \tag{2.3}$$

In case the state $|\psi\rangle$ is itself symmetric under U, once the transformation is applied the resulting state is the same original state with the same energy. However, if the state $|\psi\rangle$ is not symmetric under U what we obtain is a new set of states $U|\psi\rangle$ that are degenerate between them with energy E_ψ. If these states describe the ground state of a system, then this non-trivial phenomenon represents the so-called Spontaneous Symmetry Breaking (SSB) and the state is said to have spontaneously broken symmetry.
Since it's impossible, in terms of energy, to distinguish symmetry-broken states one from another, let's introduce a new operator, which we'll call *Order Parameter Operator* $\hat{\mathcal{O}}$, characterized by the following properties:

- Its eigenstates are the inequivalent set of symmetry-broken states;
- Each of its eigenvalues is different from zero;
- The expectation value of the order parameter operator is zero in the case of symmetric states (the one we are not interested).

One non-trivial property of $\hat{\mathcal{O}}$ is that it does not generally commute with the Hamiltonian, meaning that they do not share the same eigenvalues ($\Rightarrow$the symmetry-broken states are not eigenstates of the Hamiltonian). The last result we have found is very important, since it expresses a fundamental paradox. Despite that, these states exist and are stable: this is due to the **singularity** of the *themodynamic limit*, an important aspect of SSB that will recurr frequently througout the Lecture. It is defined, for a system of N particles and volume V, by taking the limit $N \to \infty$ and $V \to \infty$, while kepping the ratio N/V fixed. In this limit, the symmetry broken states become orthogonal to one another, and they become degenerate with the symmetric exact eigenstates of the Hamiltonian.
As a practical result, in the thermdynamic limit it's possible to actually detect spontaneous symmetry breaking, that shows itself in the *singularity of the limit*: shortly the limits of two physical quantities, generally the size of the system and a parameter describing the perturbation we use to test the stability of our state, do not commute. A rapid example of this phenomenon can be seen if we try intuitively to balance a pencil on the table: if we start fro a cylinder perfectly standing and then we sharpen the edge until we get a very sharp tip (in the limit, thickness to 0) we are able to keep the pencil balanced. Doing the opposite, that means balancing an already sharpened pencil is impossible.
It's important to keep in mind, though, that the thermodynamic limit represent a mathematical idealisation physicists use as a guide to what happen in the real world: aside from this, we are far from describing how finite-sized object can be observed in symmetry-broken configurations.

2.3. Tower of states, stable and unstable states, ergodicity breaking

There is a way to approach the description of systems with finite size. First things first, we observe that the spectra of symmetric Hamiltonians have some common properties. Through a Fourier transform, it can always be separated into a center-of-mass part (corresponding to $\vec{\mathbf{k}} = \mathbf{0}$), also called *collective part*, and a part for finite wave nummbers, which describes the internal degress of freedom of the system. Let's consider the free space case: here, the collective Hamiltonian is just that of a free particle of mass mN, and its lowst energy levels are spaced by an amount of energy $\Delta E \sim 1/N$. These states form the so-called *tower of states*. Their peculiarity is to be non-local states (they cannot be written as a product of single-particle states), and consequently, unstable to the dimension of the system. Stable states instead are local: fruthermore they are superpositions of the unstable states in the tower. That's why the concept of tower of states represent a crucial point in SSB treatment. For finite-sized system, stable states are not orthogonal, but their overlap drops as e^N; once the system ends up in one of them, the probability of tunneling is exponentially suppressed. Because of this overlap, a system which goes through SSB can be treated as if it has only a symmetry-broken state. All the other (quasi-)degenerate states are inaccessible to the system. This represent an alternative way of saying that the **ergodicity theorem is not valid**. In practical approaches,we are far more interested in the concept of *stability against local measurements*. Indeed, it can be demonstrated that unstable states of this kind are defined so that a single measurement influences the outcome of many subsequent measurements.
Using *cluster decomposition* it's possible to quantify the meaning of "many". A state is said to satisfy this property if and only if for all local observables $a(\vec{x})$ and $b(\vec{x})$ we have:

$$C_{ab}(\vec{x}, \vec{y}) = \langle a(\vec{x}) | b(\vec{y}) \rangle - \langle a(\vec{x}) \rangle \langle b(\vec{y}) \rangle \to 0 \quad \text{when } |\vec{x} - \vec{y}| \to \infty \tag{2.4}$$

States that do not satisfy this property are called sometimes *cat states*. As an almost general rule, the exact ground states of Hamiltonians that undergo SSB are cat states.

2.4. Correlation function and LRO

Aside from the analysis for stable and unstable states, we are interested, in a more practical approach, in having a way to detect SSB. Clearly, in uniform states, checking the breaking of symmetries via order parameter is the best way, but in the real world, many systems exist in condition that are not well uniform. In those cases it's better to use a differnt method to quantify the occurrence of SSB, that is checking the behavior of the **two-point correlation function**:

$$C(\vec{x}, \vec{x}') = \langle \psi | \, \mathcal{O}^\dagger(\vec{x}) \mathcal{O}'(\vec{x}') \, | \psi \rangle \tag{2.5}$$

The behavior of this function can be divided into two distinct classes:

- Long-Range Ordered (LRO), if it is constant as the two-point distant diverges, as well as the correlation length.
- Disordered where the correlation function behaves like an exponential

3. Wilczek hypothesis: Classical and Quantum Time Crystals

Time crystals are systems that repeat their state periodically over time. The idea of this kind of systems arises from the consideration of the temporal translation spontaneous symmetry breaking. Suppose to have a system described by a time-independent Hamiltonian, therefore symmetrical respect to time translation. There are states of the system, solutions of this Hamiltonian, which are at the minimum energy level, but are not symmetric with respect to the time translation. So, we can deduce that there must be a system motion which over time passes from one state to another, always remaining at minimum energy. Furthermore, if the system time-dependent physical variable was an angular variable φ of 2π period, the system would present temporal discrete symmetry, and no more continuous . Our system would act like a time crystal since the system would return to the same state periodically over time.

3.1. Classical Time Crystals

In classical physics the system motion equations are derived either from the Lagrangian $\mathcal{L}$ or from the Hamiltonian $\mathcal{H}$f unctions. Suppose to have a system whose motion equations depend on the angular coordinate φ, $\dot{\varphi}$: Wilczek [1] proposes a Lagrangian specific form:

$$\mathcal{L}(\varphi, \dot{\varphi}) = -\frac{\kappa}{2}\dot{\varphi}^2 + \frac{\lambda}{4}\dot{\varphi}^4 \tag{3.1}$$

A physical system so described has minimum energy when the angular velocity is different from zero: therefore, there is continuous time translation symmetry spontaneous broken.
However, the system Hamiltonian (which for each value of p and q assumes a value equal to energy) under the minimization energy condition (null gradient of the Hamiltonian) implies, from Hamilton's equations, that $\frac{\partial q}{\partial t}$is equal to zero: that is, the system cannot exhibit motion in the state of minimum energy. This contradiction is solved by considering that the Hamiltonian of the system described by 3.1 has a cusp at minimum energy point, therefore the consideration that the gradient must vanish at minimum point is no longer valid.
A second apparent contradiction is observed if a potential energy $V(\phi)$ is added to 3.1, since the kinetic term would be minimized with non-zero angular velocity as shown above, while the potential energy would be minimized for a specific value ϕ_0. So, the two conditions are in conflict because the total energy would appear to be minimized in a simultaneous state of motion and rest. Carrying out tedious calculations Wilczek demonstrates that also this contradiction is resolved since the minimum energy state corresponds to a sort of oscillatory motion around the point ϕ_0 where the potential energy is minimum, without ever canceling its angular velocity $\frac{\partial \phi}{\partial t}$, since the inversion of motion occurs instantaneously.
This last observation pose the problem of a diverging angular acceleration; nevertheless it can be demonstrated that the energy remains finite under suitable conditions and that, passing to a quantum system, this problem can be solved because the wave function has a certain spatial extension and therefore it would not reverse the motion all together, so that without an infinite angular acceleration a motion inversion would happen without vanishing of the expectation value $\left\langle \frac{\partial \phi}{\partial t} \right\rangle$.

3.2. Quantum Time Crystals

Therefore, we pass to a system adequately described by quantum physics. A quantum time crystal example was formulated by Wilczek with the ring particle model [2]. It consists of a unitary mass q-charged particle forced to rotate in a superconducting ring with a unitary fixed radius, crossed by a magnetic field flux. We start by noting that the time-dependent variable for the particle is the angular coordinate ϕ. In this case, one has to solve the quantum problem described by the Hamiltonian operator:

$$\hat{H} = \frac{1}{2}\left(\hat{\pi} - \alpha\right)^2 \tag{3.2}$$

where $\hat{\pi}$ is the kinetic momentum operator ($i\hbar\frac{d}{d\phi}$ with eigenvalue equation $\hat{\pi}\,|l\rangle = l\,|l\rangle$ and l is an integer) and α is the quantum number associated to the magnetic field flux. It is observed that the angular velocity expectation value in the system ground state (denoted by $|l_0\rangle$):

$$\langle l_0|\,\hat{\dot{\phi}}\,|l_0\rangle = l_0 - \alpha \tag{3.3}$$

is non-zero for suitable values of the magnetic field flux. Being the system Hamiltonian independent of time, the states $|l_0, t\rangle$ spontaneously break the temporal symmetry; moreover, since each state is associated with a particle angular position (within the delocalization limits imposed by the uncertainty principle), after a certain period the state in time must repeat itself, so this example can be considered a quantum time crystal. To be rigorous, to speak of spontaneous symmetry breaking, we must be in the thermodynamic limit for which we must consider a model with N particles where N tends to infinity. We can consider many superconducting rings lying on the same axis, in which particles, identical to those above described, are forced to rotate. In this case, they are subject not only to the magnetic field flux but also to their attractive interaction, which

makes them move as a lump with the same angular coordinate. In this case, the Hamiltonian operator is corrected as follows:

$$\hat{\mathcal{H}} = \sum_{j=1}^{N} \frac{1}{2} \left(\hat{\pi}_j - \alpha\right)^2 + \frac{\lambda}{N-1} \sum_{j \neq k,1}^{N} \delta\left(\varphi_j - \varphi_k\right) \tag{3.4}$$

where the second term on the right represents the attractive interaction between the particles. In the mean-field approximation we obtain the nonlinear Schrödinger equation:

$$i\frac{\partial \psi}{\partial t} = \frac{1}{2}\left(\hat{\pi} - \alpha\right)^2 \psi - \lambda \left|\psi\right|^2 \psi \tag{3.5}$$

We can demonstrate that again, under flux particular values, this lump of particles has an angular velocity expectation value not zero. This corresponds to a situation of time translation symmetry spontaneous breaking and to a such a cluster behavior as a time crystal.

4. Discrediting Wilczek Theory

It is now interesting to look at the comments that followed Wilczek hypothesis, particularly the ones by Patrick Bruno, right after in 2013, and by Ohsikawa and Watanabe, in 2015.

4.1. Patrick Bruno's comment on Quantum Time Crystals

Bruno stated that Wilczek did not actually prove there is not, after the AB flux is turned on, a lower-energy state than the one he labeled as "quantum time crystal". In particular, he considered that the rotating lump, when coupled to some external environment, would radiate energy while being in its ground state, violating the principle of energy conservation. Explicit solutions of nonlinear Schrodinger equations lead Patrick Bruno to assert that the system has a lower energy level than that of Wilczek.

$$i\partial_t \psi = \left[\tfrac{1}{2}(-i\partial_\phi - \alpha)^2 - \lambda|\psi|^2\right]\psi, \tag{4.1}$$

Another objection was demonstrated by Philippe Nozières [6], in his note he present this example: a superfluid ring threaded by a magnetic field which develops a charge density wave (CDW). A naive calculation shows that diamagnetic currents cannot drive rotation of the CDW.

4.2. No-go theorem: Oshikawa and Watanabe definition of time crystals

These mistakes arise to the fact that a spontaneous break of symmetry is being sought, take place only in the thermodynamic limit, and cannot be represented by a calculation for finite N. Bruno and Nozières articles strictly exclude the possibility of a spontaneous rotational motion in the ground state, for a wide class of systems. These arguments, however, not generally preclude the existence of time crystals. Other models are, in fact, proposed successively as possible different realizations of temporal crystals, systematically correcting the emerged inaccuracies.
Watanabe and Oshikawa [4] proposed two definitions of "time crystal":

- The first one consists in applying a symmetry breaking field. Computing the expectation value of the Order Parameter, and then take the limit in the thermodynamic limit. If this quantity is non-vanishing, this is a clear sign of spontaneous symmetry breaking.
- The second one consists in the analysis of the equal-time correlation function of the local order parameter. If it is non-vanishing for large enough distances and exhibits a nontrivial periodic oscillation in time.

  $$\lim_{V\to\infty} \langle \phi(x,t) | \phi(0,0) \rangle = f(t) \tag{4.2}$$

  where $\phi(x,t)$, represent the local order parameter.

Thus,a system is a time crystal if the correlation function does not cancel as the distance increases and has a periodic oscillation over time.
In addition, Watanabe and Oshikawa showed that no orders on long distance dependent on time is possible, in other words the existence of time crystals to equilibrium is definitely prohibited by this "No-Go theorem".

$$\lim_{V\to\infty} \frac{\langle\phi(x,t)|\phi(0,0)\rangle}{V^2} = c \tag{4.3}$$

where c is a constant.
So, the initial model of Wilczek crystal that takes in consideration the fundamental state, and therefore the equilibrium, is physically impossible.
If we wonder why it is this way, even though there surely exist crystals with a spatial LRO (long-range order), we should recall that Lorentz invariance does not mean the complete equivalence between space and time. While the eigenvalues of the Hamiltonian are bounded from below, the eigenvalues of the momentum are unbounded. Moreover, the equilibrium is determined by the Hamiltonian, and generally it is not Lorentz invariant in thermodynamic equilibrium.

5. Discrete Time Crystals

5.1. Floquet States

Let's now describe the problem in an alternative way. According to the non-go theorem, preparing a time crystal in the equilibrium state is impossible, so let's look at what happens when we are out of equilibrium. This kind of situation can be described by an explicit time-dependent Hamiltonian $H(t)$ that, in general, does not guarantee the conservation of the energy. Nevertheless, in the particular case in which the Hamiltonian is time-periodic, thus $H(t+T) = H(t)$, there exist a kind of time-periodic stationary states so-called Floquet states $|u_n(t+T)\rangle = |u_n(t)\rangle$. Also, such states form a complete basis and they can be used to write the time evolution of any quantum state:

$$|\psi(t)\rangle = \sum_n c_n e^{-iE_n t} |u_n(t)\rangle \tag{5.1}$$

The consequences of this Floquet Theorem are in full analogy with the Bloch Theorem that we should know very well. As one can see by watching at equation (5.1), a general quantum eigenstate of a time-periodic Hamiltonian is described by the product of an exponential part and a time-periodic function $u_n(t)$ as well as the eigenstates of a space periodic Hamiltonian are described by the product between planes waves and space periodic functions. Also, substituting $e^{-iE_n t}|u_n(t)\rangle$ in the time-dependent Schrodinger equation one has,

$$i\hbar\partial_t \left(e^{-iE_n t}|u_n(t)\rangle\right) = \hat{H}(t)e^{-iE_n t}|u_n(t)\rangle \tag{5.2}$$

then, by rearranging the terms of the equations and simplifying the exponential part one obtain an eigenvalue problem for the so-called Floquet Hamiltonian $\hat{H}_F$:

$$E_n|u_n(t)\rangle = \left(\hat{H}(t) - i\hbar\partial_t\right)|u_n(t)\rangle \equiv \hat{H}_F|u_n(t)\rangle \tag{5.3}$$

The eigenvalues E_n are real and they are called quasi-energies of the system. Quasi-energy spectrum is not bounded from below, it is periodic with a period $\frac{2\pi}{T}$ and it is sufficient to consider only a single Floquet zone to fully describe the system, again in full analogy with the Brillouin zone in the case of space crystals. The periodicity of the Hamiltonian implies that it commutes with the discrete-time translation operator $\mathcal{T}_T$ related to the evolution of a system by period T and Floquet eigenstates are also eigenstates of $\mathcal{T}_T$ (because $[\hat{H}(t);\mathcal{T}_T] = 0$). In other words, this means that if we choose a point in the configuration space and ask how the probability density for the detection of a single or many particles at this point changes in time, the answer is it is periodic with a period T if a system is prepared in a Floquet eigenstate. This interesting conclusion makes a question arises: can a many-body periodically driven system prepared in a Floquet eigenstate spontaneously self-organize in time and start evolving with a period that is not equal to T? The answer is yes and this phenomenon, which leads to the spontaneous breaking of discrete time translation symmetry, is called a discrete or Floquet time crystal, and it is the one that has been recently realized experimentally.

5.2. Discrete Time crystals for atoms bouncing on an oscillating mirror

Let's start considering a single particle with unitary mass bouncing on a T-periodic oscillating mirror in 1D approximation under the action of the gravitational field. Let's consider the frame oscillating with the mirror in which the position of the mirror is fixed at $z = 0$. In this case the quantum Hamiltonian is:

$$\hat{H} = -\frac{\hbar^2}{2}\partial_z^2 + V(z) + \lambda z \cos(\omega t) \tag{5.4}$$

where $V(z) = gz$ for $z \geq 0$ and $V(z) = \infty$ for $z < 0$ is the gravitation potential energy and the sinusoidal term is due to the inertial periodic forces acting on the non-inertial system that we are considering. Lastly, λ and ω are the mirror oscillations' wavelength and angular frequency, respectively. In the absence of mirror oscillations ($\lambda = 0$), all classical trajectories of a particle are periodic. When mirror oscillations are turned on, classical motion becomes irregular but some of the periodic orbits survive. In particular, there is a 1:1 resonant orbit where a particle moves periodically with a period equal to the period of the mirror oscillations $T = \frac{2\pi}{\omega}$ and, more important for us, there exist 2:1 resonant orbits where particle bounces on a mirror with a period twice longer than that of the mirror oscillations.

Switching to the quantum description, a periodic motion of a particle is described by Floquet eigenstates. Several descriptions can be used to describe our particle bouncing on the oscillating mirror along the 2:1 resonant orbit. The most intuitive one, but also trivial, is to use a single localized wave-packet moving along the 2:1 resonant orbit. Such a packet would have a period equal to $2T$, i.e. twice longer than the period of the Hamiltonian, and it wouldn't be a Floquet Eigenstate. However, in general, it's possible to demonstrate, that superpositions of s wave-packets each with period equal to sT but after T exchange their position can form a proper Floquet eigenstate for the system. Using s wave-packets we can build s superpositions that are linearly independent, hence, there are s Floquet eigenstates made by localized wave-packets moving along an s-resonant orbit. For what concerns us, we would like to describe the 2:1 resonant orbits and so for s=2, called $v_1(t)$ and $v_2(t)$ the wave-packet moving along a 2:1 resonant orbit, thus with a period equal to 2T, there are two orthonormal Floquet eigenstates:

$$u_1(t) = v_1(t) + v_2(t)$$
$$u_2(t) = v_1(t) - v_2(t)$$

To convince you that such eigenstates have a period equal to T, despite the two wave-packet having a 2T-period, one can watch the graphs reported in Fig. 1 representing the time evolution of the probability density for the bouncing particle described as a superposition of $v_1(t)$ and $v_2(t)$. Initially ($t = 0$) two localized wave-packets overlap but because they propagate in opposite directions one can see interference fringes. In the course of the time evolution, one wave-packet moves towards the mirror (located at $z = 0$) bounces off the mirror and returns. The other wave-packet moves towards the classical turning point in the gravitational field and also returns. Even though each of the wave-packets evolves with a period 2T, at $t = T$ we end up with the initial situation because at this moment the indistinguishable wave-packets exchange their roles. In the end, we have been able to create two different Floquet eigenstates, i.e. $u_1(t+T) = u_1(t)$ and $u_2(t+T) = u_2(t)$, made of two wave-packets with period 2T.

Now we want to consider the generic case of N particles bouncing on an oscillating mirror. Let's consider a system made of N bosons interacting with each other via δ-contact potential. In this case, more and more issues arise due to the form of the Hamiltonian and the strength of the interactions. Long story short, it's possible to demonstrate that if the interactions between bosons are strong enough is energetically favourable for the system to collect all bosons in a single localized wave-packet similar to the one that we described before, in the single-particle case. Nevertheless, also in this case, to form a Floquet state T-periodic we have to use two wave-packet with period 2T: this means that the many-body ground state of the system $|\psi_0\rangle$ would be made of the superpositions of this two wave-packet. Indeed, because, as told, is energetically favourable to collect all the bosons in the same packet but, at the same time, not knowing which of these two packets is the

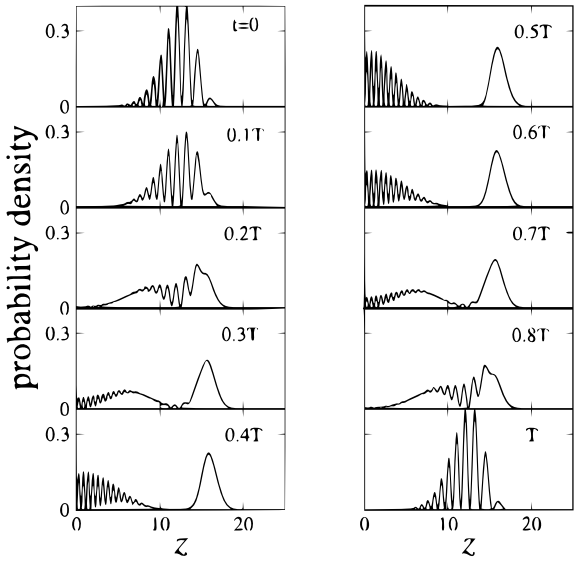

Figure 1. Time evolution of the probability density for a particle bouncing on an oscillating mirror and prepared in a Floquet eigenstate consisting of two localized wave-packets moving along 2:1 resonant trajectory.

proper one, the many-body ground state must be a superposition of the configurations in which all the boson are either in the packet one or in the packet two. Such a quantum state, made of two diametrically opposite conditions is called Schrödinger cat-like state. Calling n_1 the occupation number of the first packet and $n_2 = N - n_1$ the occupation number of the second one and using the Fock basis one has:

$$|\psi_0\rangle = \frac{1}{\sqrt{2}} \left(|N, 0\rangle + |0, N\rangle\right) \tag{5.5}$$

The corresponding single particle probability density is depicted in the left panel of Fig. 2. The discrete-time translation symmetry is preserved in the time evolution of the many-body Floquet eigenstate $|\psi_0\rangle$ but this state is extremely vulnerable to any perturbation. For example, after a measurement of a position x_1 of one of the N particles, the symmetry is gone because the ground state will collapse on one of the two terms in (5.5). Indeed, if the N particles are in the same packets the detection of one of these particles in a particular packet, immediately define the position of the other $N - 1$ particles, thus the ground state wavefunction collapse on the state $|N - 1, 0\rangle$ or $|0, N - 1\rangle$ depending on the result of the measurement of the position of x_1. So now, as one can see in the right panel of Fig. 2, our system is made by only one packet evolving with period 2T and this means that the system broke the discrete-time translation symmetry of period T to form a system with still a discrete time translation symmetry but with a higher-order parameter, i.e. with a lower symmetry. Is possible to demonstrate that the resulting state ($|N - 1, 0\rangle$ or $|0, N - 1\rangle$) is robust against any further perturbation one can perform many measurements and still the period of the time evolution remains 2T. A discrete (o Floquet) time-crystal is born.

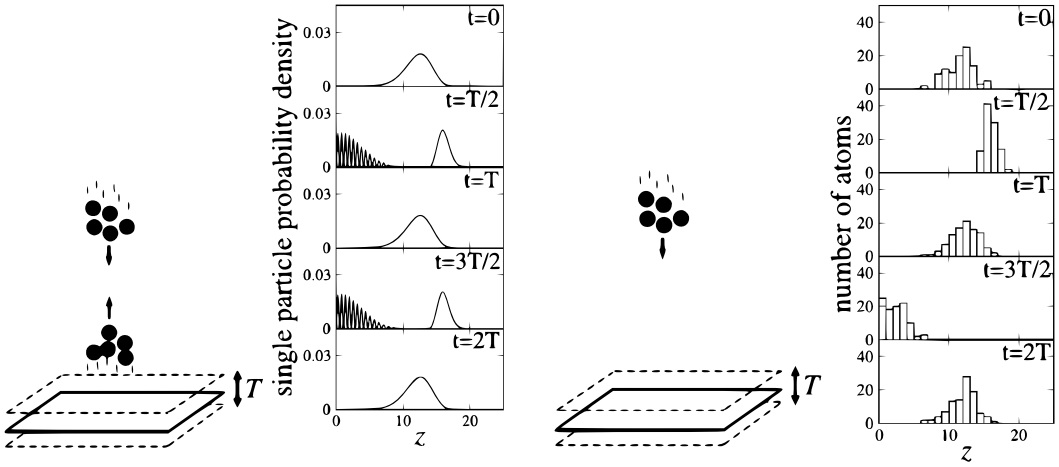

Figure 2. *Left panel*: schematic plot of a system of N atoms bouncing on an oscillating mirror and prepared in a many-body Floquet state. Each of the two atomic clouds moves with a period 2T but they exchange their positions after time T so that the entire Floquet state is periodic with a period T. *Right panel*: schematic plot of a system of N atoms bouncing on an oscillating mirror as in the left panel but after the measurement of the position of a single atom; the atomic cloud visible in the plot moves with a period 2T. The histograms show the results of the measurements of positions of 100 atoms, i.e. at $t = 0$ one measures the positions of 100 atoms, lets the remaining atoms evolve and after T/2 one again measures the positions of 100 atoms and so on.

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