7.5 Convergence order

When a sequence converges, it may be interesting also to define a mathematical concept that represents the *speed* at which it does so.

To this end the following definition is important.

Definition 1 *Let z_0 be a zero of a function f. If $\{x_k\}$ is a sequence such that*

$$\lim_{k\to\infty} x_k = z_0\,,$$

and if there exist a real number p and a real constant $C \neq 0$ such that

$$\lim_{k\to\infty} \frac{|x_{k+1}-z_0|}{|x_k - z_0|^p} = C\,, \tag{7.3}$$

then we say that the sequence converges to z_0 ***with convergence order*** *p, and C is called* ***asymptotic error constant*** *or* ***convergence rate****. When $p = 1, 2, 3$ the convergence order is said to be linear, quadratic, cubic, respectively.*

7.5.1 Convergence orders

It can be shown that, under standard assumptions, the convergence order of the following methods is:

- Bisection: $p = 1$
- Fixed point: $p = 1$
- Newton: $p = 2$
- Secant: $p = \frac{1+\sqrt{5}}{2} \approx 1.618$

These numbers may happen to be bigger for certain functions, but in general that's what you have.

7.5.2 Approximation of the convergence order

It turns out that another way to compute the asymptotic error constant is

$$\lim_{k\to\infty} \frac{|x_{k+1}-x_k|}{|x_k - x_{k-1}|^p} = C\,.$$

Notice that this way doesn't require the knowledge of the true root z_0, which is what in general we don't know and want to approximate.

One thing that can be approximately computed with this is the convergence order p. From before we have

$$\frac{|x_{k+1}-x_k|}{|x_k - x_{k-1}|^p} \approx C$$

This estimate involves three indices $k+1, k, k-1$ and it can be used for all such triplets, so we can write it for $k+2, k+1, k$ as well. We then have

$$|x_{k+1}-x_k| \approx C|x_k - x_{k-1}|^p$$
$$|x_{k+2}-x_{k+1}| \approx C|x_{k+1} - x_k|^p$$

The advantage of using these two is that we can now take the ratio of these estimates and we get rid of the constant C, which we don't know a priori. So,

$$\frac{|x_{k+2}-x_{k+1}|}{|x_{k+1}-x_k|} \approx (\frac{|x_{k+1}-x_k|}{|x_k-x_{k-1}|})^p$$

take the log of both sides to bring the p down

$$\log(\frac{|x_{k+2}-x_{k+1}|}{|x_{k+1}-x_k|}) \approx p\log(\frac{|x_{k+1}-x_k|}{|x_k-x_{k-1}|})$$

Therefore,

$$p \approx \frac{\log(\frac{|x_{k+2}-x_{k+1}|}{|x_{k+1}-x_k|})}{\log(\frac{|x_{k+1}-x_k|}{|x_k-x_{k-1}|})}$$

This is the estimate of the convergence order p that we can use in our code.

Notice that this formula involves 4 indices $k+2, k+1, k, k-1$.

7.5.3 Approximation of the asymptotic error constant

The convergence order p can be known either theoretically or through numerical computation with the above procedure.

Once we know p or its numerical approximation, another thing we can do in our code to verify its correctness is to compute the quantities

$$\frac{|x_{k+1}-x_k|}{|x_k-x_{k-1}|^p}\,.$$

We should observe that these tend to some value, which will be an approximation of the asymptotic error constant.