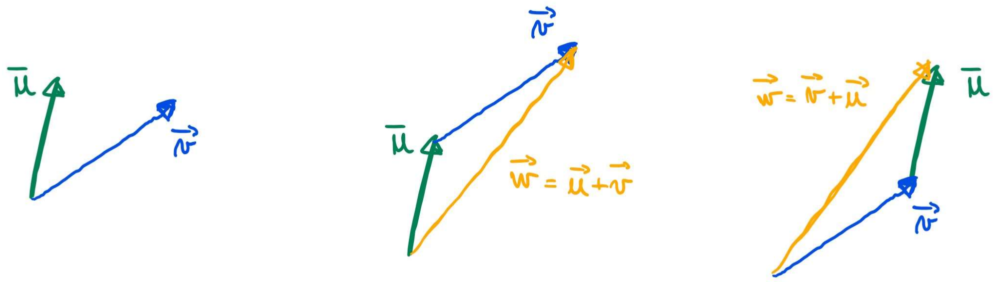
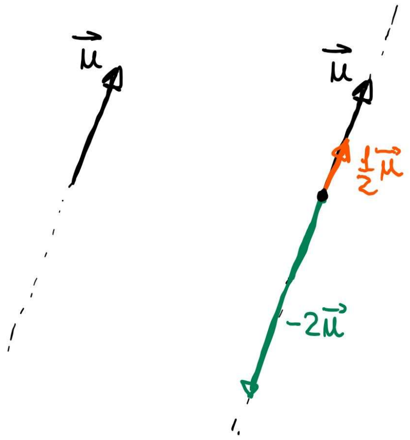
LINEAR ALGEBRA AND VECTOR CALCULUS BASIS

VECTORS

- ADDITION



- MULTIPLICATION BY A SCALAR $\alpha \in \mathbb{R}$



PROPERTIES

- ASSOCIATIVITY $u + (v + w) = (u + v) + w$
- COMMUTATIVITY $u + v = v + u$
- ADDITION IDENTITY $\exists\, 0 \in V$ s.t. $v + 0 = 0$
- ADDITIVE INVERSE $\forall v \in V\ \exists\, w$ s.t. $v + w = 0$
- MULT IDENTITY $1 \cdot v = v$
- DISTRIBUTIVITY $\alpha(v + w) = \alpha v + \alpha w$

  $(\alpha + \beta) v = \alpha v + \beta v$

LINEAR ALGEBRA STUDIES VECTOR SPACES AND $\boxed{\text{LINEAR MAPS}}$ BETWEEN THEM

$\downarrow$

TRANSFORMATION THAT DOES NOT CHANGE THE ORIGIN AND TURNS LINES INTO LINES

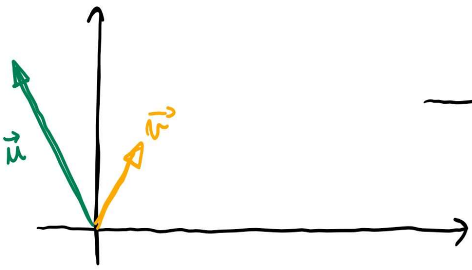
LINEAR INDEPENDENCE, SPAN

I CAN WRITE ANY VECTOR $v \in \mathbb{R}^m$ AS A LINEAR COMBINATION (WEIGHTED SUM) OF m LINEARLY INDEPENDENT VECTORS.

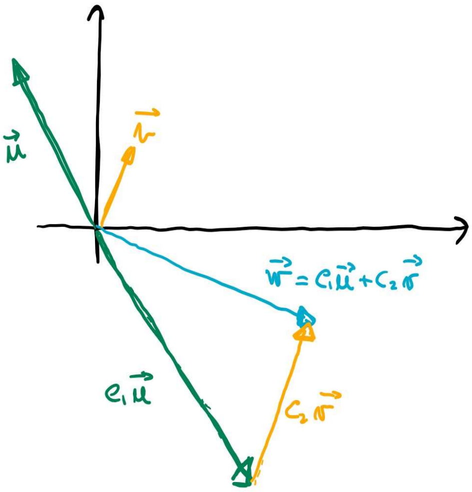
A SET OF VECTORS ARE LINEARLY INDEPENDENT IF NO VECTOR IN THE SET CAN BE WRITTEN AS A LINEAR COMBINATION OF THE OTHERS.

IN $\mathbb{R}^m$ I CAN HAVE AT MOST m INDEPENDENT VECTORS. A SET OF m INDEPENDENT VECTORS IN $\mathbb{R}^m$ IS CALLED A BASIS OF $\mathbb{R}^m$, BECAUSE I CAN OBTAIN EVERY OTHER VECTOR IN $\mathbb{R}^m$ BY ADDING SCALED VERSIONS OF THE VECTORS IN THE BASIS

EXAMPLE $\mathbb{R}^2$

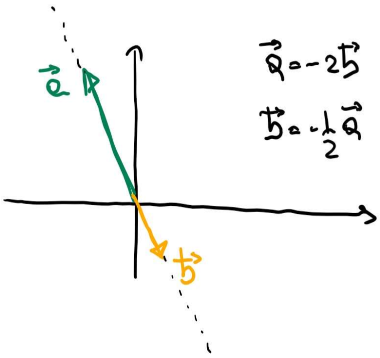


ARE 2 INDEPENDENT VECTORS IN $\mathbb{R}^2$, INDEED I CAN OBTAIN EVERY VECTOR IN $\mathbb{R}^2$ AS A WEIGHTED SUM OF $\vec{u}$ and $\vec{v}$



EXAMPLE $\mathbb{R}^2$

THE TWO VECTORS $\vec{a}$ and $\vec{b}$ are NOT INDEPENDENT.



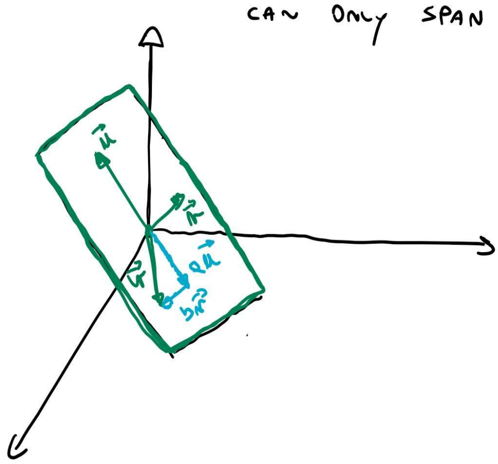
$\vec{a} = -2\vec{b}$

$\vec{b} = -\frac{1}{2}\vec{a}$

IF TWO VECTORS ARE ON THE SAME LINE THEY'RE <u>NOT</u> INDEPENDENT

EXAMPLE $\mathbb{R}^3$

In $\mathbb{R}^3$ IF THREE VECTORS ARE ON THE SAME PLANE THEY ARE <u>NOT</u> INDEPENDENT, THEY CAN ONLY SPAN VECTORS ON THE PLANE (A 2D SPACE)



IN GENERAL

IN $\mathbb{R}^m$, m VECTORS ARE NOT INDEPENDENT IF THEY LAY ON A m-k DIMENSIONAL SPACE

VECTORS AND COORDINATES

GIVEN A VECTOR SPACE V WITH BASIS $B=\{b_0,\dots b_m\}$ WE CAN REPRESENT A VECTOR $v\in V$ IN COORDINATES WITH RESPECT TO THE BASIS B

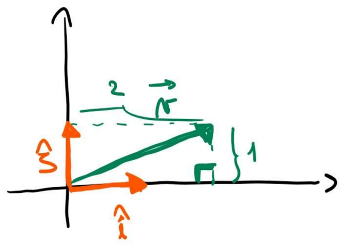
$$\vec{v} = a_1\vec{b}_1 + a_2\vec{b}_2 + \dots + a_m\vec{b}_m \Rightarrow b = \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}$$

USUALLY CHOOSE AN <u>ORTHONORMAL BASIS</u> → (BASIS VECTOR ARE ORTHOGONAL AND HAVE NORM 1)

IN WHICH WE CAN USE PYTAGORA'S THEOREM TO WRITE THE LENGTH OF A VECTOR

$$\vec{v} = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \text{ AS } \|v\| = \sqrt{c_1^2 + \dots + c_m^2}$$

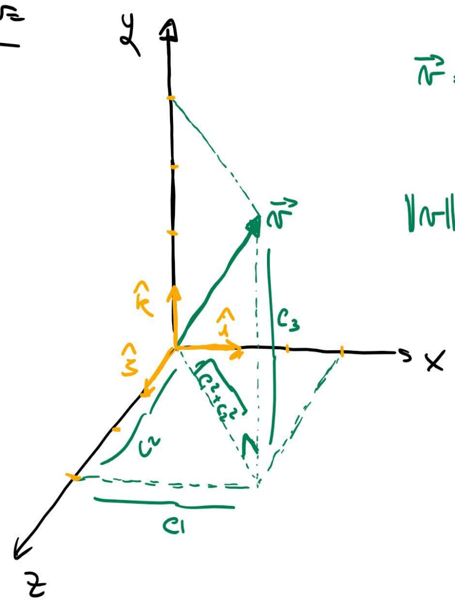
2D EXAMPLE



$$\vec{v} = 2\hat{i} + 1\hat{j} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\|v\| = \sqrt{2^2+1^2} = \sqrt{5}$$

3D EXAMPLE

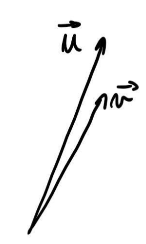


$$\vec{v} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}$$

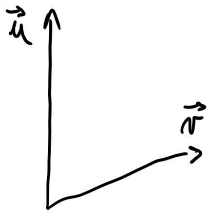
$$\|v\| = \sqrt{\left(\sqrt{c_1^2+c_2^2}\right)^2 + c_3^2} = \sqrt{c_1^2+c_2^2+c_3^2} = \sqrt{9+9+16} = \sqrt{34}$$

INNER PRODUCT

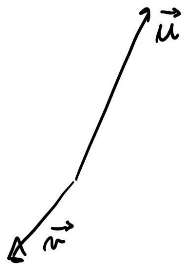
MEASURES HOW MUCH TWO VECTORS ARE "ALIGNED"



NEARLY ALIGNED



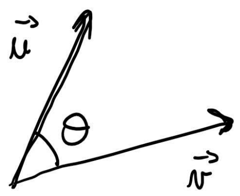
NOT ALIGNED



NEARLY ALIGNED BUT WITH OPPOSITE ORIENTATION

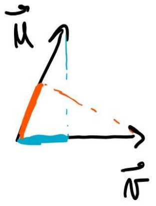
THE EUCLIDEAN INNER PRODUCT IS DEFINED AS

$$\langle \vec{u}, \vec{v} \rangle = \|u\| \cdot \|v\| \cdot \cos\theta$$



$\Rightarrow$ IF $\vec{u}$ AND $\vec{v}$ HAVE UNIT NORM, THEN

$$\langle \vec{u}, \vec{v} \rangle = \cos\theta = \text{proj}_{\vec{u}} \vec{v} = \text{proj}_{\vec{v}} \vec{u}$$



LINEAR MAP

A LINEAR MAP IS A FUNCTIONS BETWEEN VECTOR SPACES THAT TRANSFORMS LINES INTO LINES AND DOES NOT MOVE THE ORIGIN

$$L: V \to V \qquad \text{S.T.} \quad L(c\vec{u}) = c\,L(\vec{u})$$

$$L(\vec{u} + \vec{v}) = L(\vec{u}) + L(\vec{v})$$

A LINEAR MAP CAN BE WRITTEN IN MATRIX FORM (AND EVERY MATRIX CAN BE INTERPRETED AS A LINEAR OPERATOR).

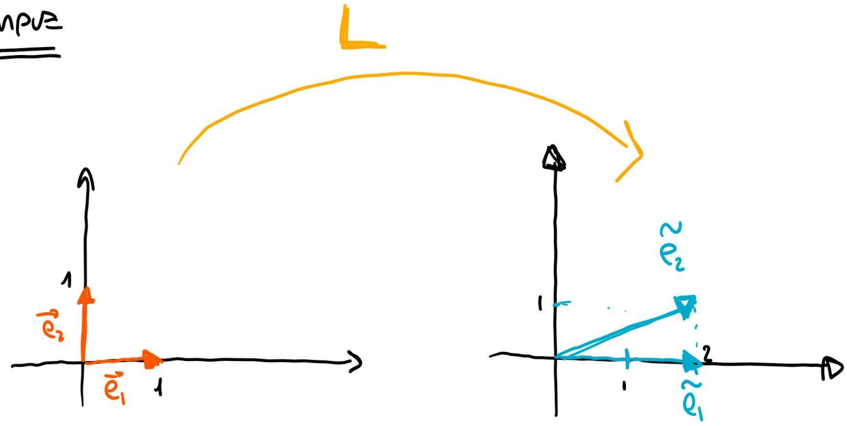
TO WRITE THE LINEAR MAP IN MATRIX FORM WE NEED TO CONSIDER ONLY HOW THE LINEAR OPERATOR TRANSFORMS THE BASIS VECTORS.

GIVEN THE STANDARD BASIS $B = \{e_1, \ldots e_m\}$ AND $\vec{v} \in \mathbb{R}^n = \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix}$

$$L(\vec{v}) = L(c_1\vec{e}_1 + \cdots + c_m\vec{e}_m) = c_1 L(\vec{e}_1) + \cdots + c_m L(\vec{e}_m)$$

BY LINEARITY

EXAMPLE



$$\tilde{e}_1 = L(\vec{e}_1) = \begin{pmatrix}2\\0\end{pmatrix}$$

$$\tilde{e}_2 = L(\vec{e}_2) = \begin{pmatrix}2\\1\end{pmatrix}$$

$$\Rightarrow L = \begin{pmatrix}2 & 2\\0 & 1\end{pmatrix}$$

MATRIX-VECTOR PRODUCT

LINEAR MAP IS WHAT DEFINES MATRIX-VECTOR PRODUCT

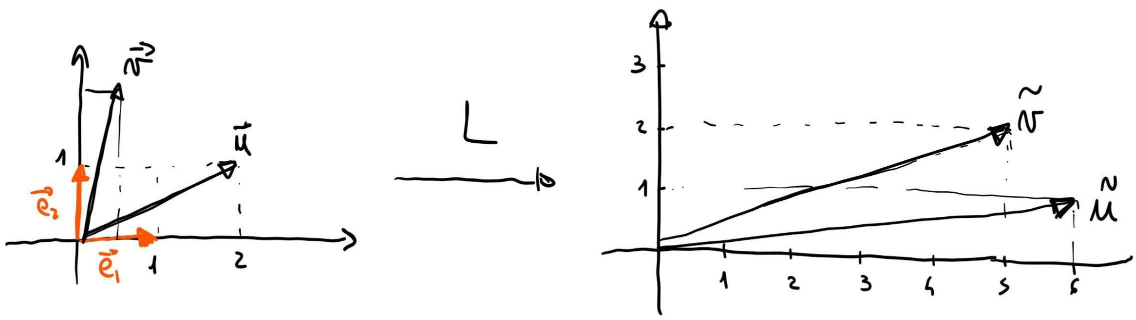
$$\begin{pmatrix}a & b\\e & d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = x\begin{pmatrix}a\\c\end{pmatrix} + y\begin{pmatrix}b\\d\end{pmatrix} = \begin{pmatrix}ax+by\\ex+dy\end{pmatrix}$$

LET'S CHECK

$$L(\vec{e}_1) = \begin{pmatrix}2 & 2\\0 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix}$$

$$L(\vec{e}_2) = \begin{pmatrix}2 & 2\\0 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}2\\1\end{pmatrix}$$

NOW WE CAN TRANSFORM ANY VECTOR



$$L(\vec{u}) = \begin{pmatrix}2 & 2\\0 & 1\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}4+2\\0+1\end{pmatrix} = \begin{pmatrix}6\\1\end{pmatrix} = \tilde{u}$$

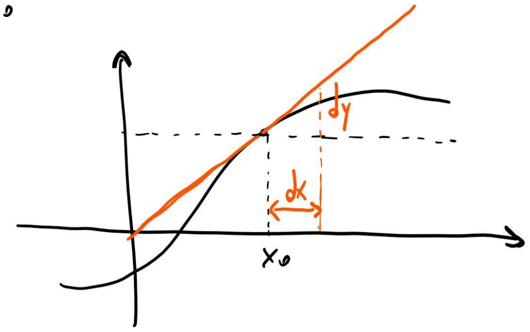
$$L(\vec{v}) = \begin{pmatrix}2 & 2\\0 & 1\end{pmatrix}\begin{pmatrix}1/2\\2\end{pmatrix} = \begin{pmatrix}1+4\\0+2\end{pmatrix} = \begin{pmatrix}5\\2\end{pmatrix} = \tilde{v}$$

VECTOR CALCULUS

VECTOR CALCULUS STUDIES HOW VECTOR FUNCTION CHANGES

DERIVATIVE IN 2D

THE DERIVATIVE OF A FUNCTION f IS A FUNCTION THAT RETURNS THE RATE OF CHANGE OF f AT EVERY POINT IN THE DOMAIN



$$f'(x) = \frac{dy}{dx}$$

- AT EACH POINT x_0 THE TANGENT LINE OF f at x_0 IS GIVEN BY

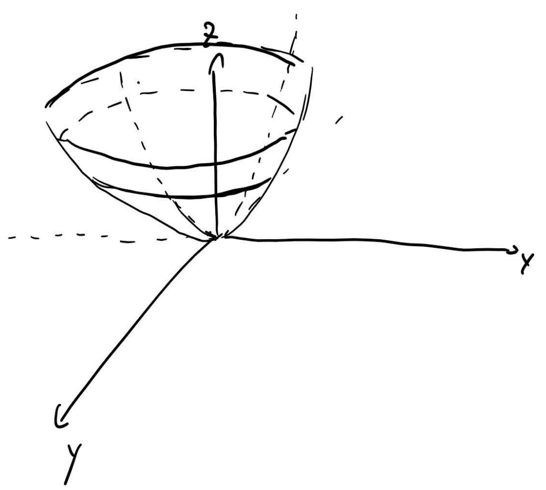
$$l(x) = f(x_0) + f'(x_0)(x - x_0)$$

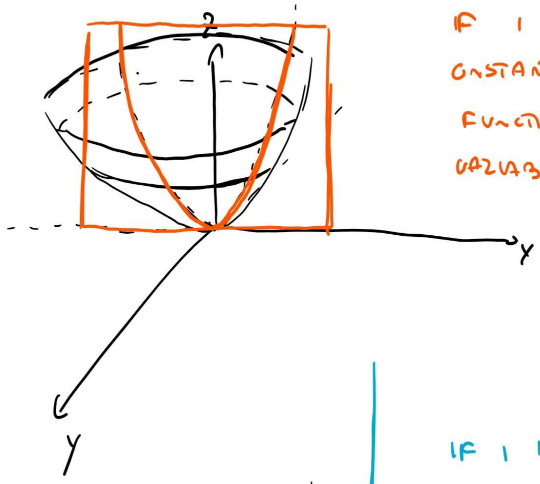
PARTIAL DERIVATIVES

WHEN DIMENSIONS INCREASE THERE ARE MULTIPLE DIRECTIONS IN WHICH I CAN COMPUTE THE RATE OF CHANGE.

HENCE, I NEED TO INTRODUCE THE CONCEPT OF PARTIAL DERIVATIVES

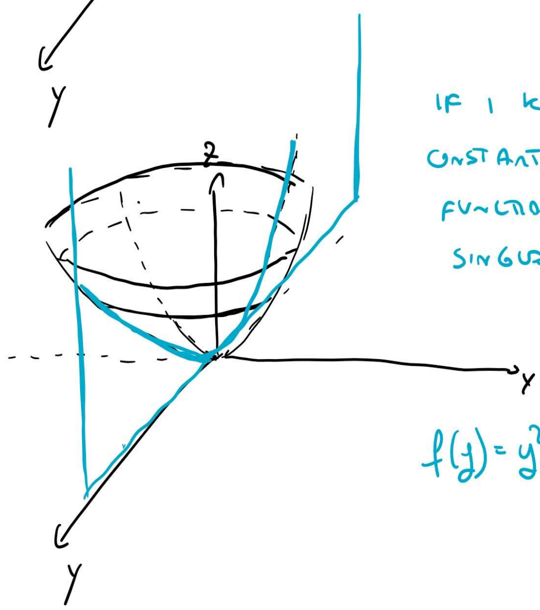
CONSIDER $f(x,y) = x^2 + y^2 + 2y$





IF I KEEP THE y CONSTANT I HAVE A FUNCTION OF A SINGLE VARIABLE x

$$f(x) = x^2 + c$$

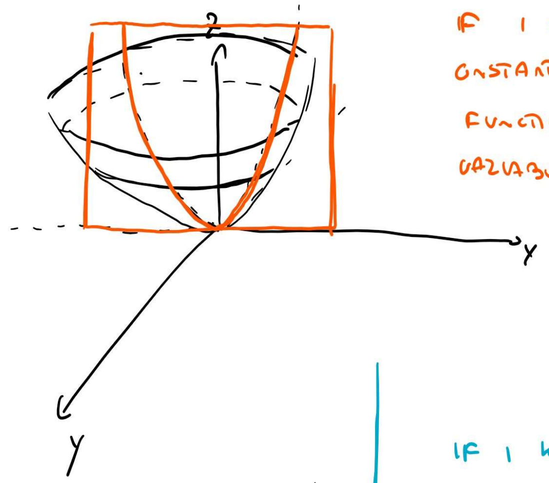


IF I KEEP THE x CONSTANT I HAVE A FUNCTION OF A SINGLE VARIABLE y

$$f(y) = y^2 + 2y + c$$

THE PARTIAL DERIVATIVE

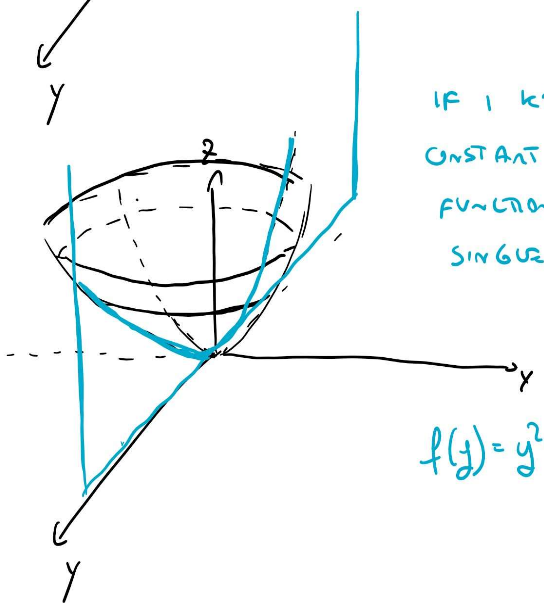
$\frac{\partial f}{\partial x_i}$ IS THE DERIVATIVE OF f WITH RESPECT TO x_i WHEN THE OTHER VARIABLES ARE CONSIDERED CONSTANTS



IF I KEEP THE y CONSTANT I HAVE A FUNCTION OF A SINGLE VARIABLE x

$$\Longrightarrow \frac{\partial f}{\partial x} = 2x$$

$$f(x) = x^2 + c$$



IF I KEEP THE x CONSTANT I HAVE A FUNCTION OF A SINGLE VARIABLE y

$$\Longrightarrow \frac{\partial f}{\partial y} = 2y + 2$$

$$f(y) = y^2 + 2y + c$$

$\boxed{\text{THE GRADIENT VECTOR}}$ IS THE VECTOR OF PARTIAL DERIVATIVES $\Big] \rightarrow$ IT IS A VECTOR FIELD, IT ASSOCIATES A VECTOR TO EACH POINT IN THE DOMAIN OF f

$$\nabla f(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}$$

HOW CAN I COMPUTE THE RATE OF CHANGE IN AN ARBITRARY DIRECTION $\vec{u}$?

JUST TAKE THE DOT PRODUCT BETWEEN $\vec{u}$ AND THE GRADIENT VECTOR

$$D_u f(x,y) = u \cdot \nabla f(x,y)$$

(THE PROOF IS A SIMPLE APPLICATION OF THE CHAIN RULE TO THE DEFINITION OF THE DIRECTIONAL DERIVATIVE)

AS IN 2D I DEFINED THE TANGENT LINE OF f AT x_0 THAT IS THE BEST LINEAR APPROXIMATION OF f AROUND x_0 IN 3D (OR ND) I CAN DEFINE THE TANGENT PLANE (OR TANGENT HYPERPLANE) THAT IS THE BEST LOCAL APPROXIMATION OF f AROUND $\vec{x}_0$

$$L(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)\cdot(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)\cdot(y-y_0)$$

IF YOU ARE INTERESTED, THIS EQUATION COMES FROM

EQN OF THE PLANE $\quad A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$

DIVIDE BY C AND RENAME CONSTANTS $\Rightarrow \quad z = z_0 + a(x-x_0) + b(y-y_0)$

NOW THE INTERSECTION WITH THE PLANE $y=y_0$ IS THE PARTIAL DERIVATIVE ALONG X

$$z - z_0 = a(x-x_0) \Rightarrow a = \frac{\partial f}{\partial x}(x_0,y_0)$$

I CAN DO THE SAME WITH THE PLANE $x-x_0$

$$\Rightarrow z - z_0 = b(y-y_0) \Rightarrow b = \frac{\partial f}{\partial y}(x_0,y_0)$$

$$\Rightarrow \boxed{z = z_0 + \frac{\partial f}{\partial x}(x-x_0) + \frac{\partial f}{\partial y}(y-y_0)}$$