EXTREME VALUES CHARACTERIZATION

A FUNCTION $f: D \subset \mathbb{R}^2 \to \mathbb{R}$ CAN BE APPROXIMATED AROUND A POINT $P(x_0, y_0) \in D$ AS

$$f(x_0+h, y_0+k) \simeq f(x_0,y_0) + \underbrace{\frac{\partial f}{\partial x}(x_0,y_0)\cdot h + \frac{\partial f}{\partial y}(x_0,y_0)\cdot k}_{\text{LINEAR APPROXIMATION}} + \underbrace{\frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}(x_0,y_0)\cdot h^2 + 2\frac{\partial^2 f}{\partial x \partial y}(x_0,y_0)\cdot h \cdot k + \frac{\partial^2 f}{\partial y^2}(x_0,y_0)\cdot k^2\right)}_{\text{QUADRATIC APPROXIMATION}}$$

TO WRITE THIS APPROXIMATION IN MATRIX FORM, WE NEED TO DEFINE THE

HESSIAN MATRIX

$$H_{\vec{x}} = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x_0,y_0) & \frac{\partial^2 f}{\partial x \partial y}(x_0,y_0) \\ \frac{\partial f^2}{\partial y \partial x}(x_0,y_0) & \frac{\partial^2 f}{\partial y^2}(x_0,y_0) \end{pmatrix} \longrightarrow \text{SYMMETRIC MATRIX}$$

$$\Rightarrow f(\vec{x}_0 + \vec{h}) \simeq \underbrace{\nabla f(\vec{x}_0)\cdot \vec{h}}_{\text{LINEAR TERM}} + \underbrace{\frac{1}{2}\vec{h}^T H_{\vec{x}_0} \vec{h}}_{\text{QUADRATIC TERM}} \qquad \text{WHERE } \vec{x}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \text{ AND } \vec{h} = \begin{pmatrix} h \\ k \end{pmatrix}$$

REMEMBER THAT WHEN $\vec{x}_0$ IS AN EXTREME VALUE (MAX, MIN, SADDLE) THEN THE GRADIENT VANISHES

$$\nabla f(\vec{x}_0) = \vec{0}$$

SO THE FUNCTION IS APPROXIMATED ONLY BY ITS SECOND ORDER TERM

$$f(\vec{x}_0 + \vec{h}) \simeq \frac{1}{2}\vec{h}^T H_{\vec{x}} \vec{h}$$ AND IN PARTICULAR

H IS POSITIVE SEMIDEFINITE

- $\vec{x}_0$ IS MAX IF $f(\vec{x}_0 + \vec{h}) \geq f(\vec{x}_0) \; \forall \vec{h}$ HENCE IF $\boxed{\vec{h}^T H_{\vec{x}_0} \vec{h} \geq 0 \; \forall h}$

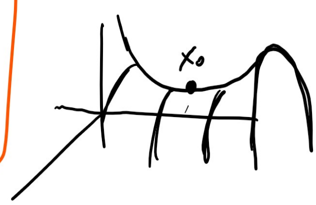
- $\vec{x}_0$ IS MIN IF $f(\vec{x}_0 + \vec{h}) \leq f(\vec{x}_0) \; \forall \vec{h}$ HENCE IF $\boxed{\vec{h}^T H_{\vec{x}_0} \vec{h} \leq 0 \; \forall h}$

H IS NEGATIVE SEMIDEFINITE

- OTHERWISE $\vec{x}_0$ IS NEITHER MAX NOR MIN (SADDLE)

H IS INDEFINITE

$$\boxed{\vec{h}^T H_{\vec{x}_0} \vec{h} \begin{cases} \leq 0 & \text{FOR SOME } h \\ \geq 0 & \text{FOR SOME } h \end{cases}}$$



GIVEN A SYMMETRIC MATRIX H IT IS

→ POSITIVE SEMIDEFINITE IF AND ONLY IF ALL EIGENVALUES ARE POSITIVE OR ZERO (NON NEGATIVE)

→ NEGATIVE SEMIDEFINITE IF AND ONLY IF ALL EIGENVALUES ARE NEGATIVE OR ZERO (NON POSITIVE)