Green hyperbolic complexes on Lorentzian manifolds and quantizations of gauge field theories

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In memory of my grandfather, Giuseppe

Abstract

Gauge field theories have a central role in modern mathematical physics. The space of solutions of a gauge field theory is described by the moduli problem associated with the derived critical locus of its action functional. As such it is expected to carry a (-1)-shifted Poisson structure. On the other hand, its quantization on Lorentzian manifolds, especially in the algebraic approach, requires an unshifted Poisson structure. In this thesis we develop a framework, covering many examples of derived critical loci of gauge-theoretic quadratic action functionals, including Abelian Chern-Simons theory and Maxwell p-forms, in which these (-1)-shifted Poisson structures can be unshifted. The key ingredient of our approach is a homological generalization of Green hyperbolic operators, called Green hyperbolic complexes. We define the latter through a generalization of retarded and advanced Green's operators, called retarded and advanced Green's homotopies. We show that these generalized notions admit homological variants of the main features of their ordinary counterparts. Namely, retarded and advanced Green's homotopies are unique up to contractible spaces of choices, they induce a retarded-minus-advanced quasi-isomorphism, replacing the causal propagator, and, together with a differential pairing generalizing fiber metrics, they lead to unshifted covariant and fixed-time Poisson structures, which are compatible (up to homotopy) with the retarded-minus-advanced quasi-isomorphism. Furthermore, we exploit these constructions to quantize Green hyperbolic complexes in two alternative approaches: as time-orderable prefactorization algebras via BV quantization and as algebraic quantum field theories via canonical quantization. Finally, we compare the two approaches by constructing an explicit isomorphism of time-orderable prefactorization algebras from the BV quantization to the time-orderable prefactorization algebra canonically associated to the algebraic quantum field theory.

Contents

Introduction

State of the art. Gauge field theory plays a central role in modern theoretical and mathematical physics. The two principal theories that together provide the cutting-edge description of physical phenomena from the infinitesimal scales of particle physics to cosmological distances, Standard Model and General Relativity, respectively, are based on the profound idea that the laws of Nature should be invariant under the action of suitable groups of *gauge symmetries*. Yang-Mills theory, which encompasses the fundamental forces in the Standard Model, is a theory of connections on principal bundles with a compact Lie structure group, whose gauge symmetries are principal bundle automorphisms. General Relativity, that is the theory of gravitation, describes the dynamics of Lorentzian metrics over smooth manifolds with gauge symmetries encoded by diffeomorphisms.

The presence of these gauge symmetries, far from being just redundancies of the theories, encodes richer information that deserves to be treated carefully. They encode information about higher structures of the moduli space of gauge field configurations. To be more precise, we first observe that the configuration space of a (higher) gauge field theory is properly described by an ∞-groupoid. Gauge fields are the points of the ∞-groupoid, while the n-morphisms describe the gauge transformations between gauge fields (for $n = 1$) and the higher gauge transformations between other (higher) gauge transformations (for $n > 1$). Passing from the ∞-groupoid to the quotient space of configurations modulo gauge transformations corresponds to the decategorification of the ∞-groupoid, which computes just its 0-th homotopy group, forgetting all the higher homotopy groups. This, for example, forgets the information about the automorphism groups of the objects of the ∞-groupoid. Moreover, ∞-groupoids are ∞-categorical objects, whose equivalences are the ∞-categorical equivalences [Lur09]. This hints to the origins of the higher structures behind gauge theories. The ∞-groupoid of gauge field configurations is a *kinematic* object, no information about the field dynamics is encoded in it, moreover, it lacks also of a 'smooth structure.' In order to endow the ∞-groupoid with a smooth structure one can employ a sheaf theoretic point of view and define all the smooth maps from all finite-dimensional manifolds valued into the ∞-groupoid. Intuitively, this is tantamount to declaring smooth parameterizations of ∞-groupoids (the smooth curves, surfaces, etc. . .). More precisely, this is achieved by describing the smooth configuration space of gauge fields as an ∞-stack [Lur09; TV05; TV08]. In a (gauge) field theoretic setting one starts with a space (an ∞-stack) of fields and an action functional over it. The dynamics is then implemented by taking the critical locus of the action. In general, the result of such construction may be badly behaved and, moreover, it neglects more refined information, like multiplicities of multiple intersections. A way to overcome these problems is offered by ideas coming from derived algebraic geometry [PTVV13; Cal+17]. It consists in replacing the naive critical locus with the *derived critical locus* of the action functional [Vez11; PTVV13]. It follows that the space of the gauge fields that are solutions of the equation of motion is described by the more refined notion of *derived ∞-stacks*. See [TV04] and [Toë09] for an overview. If ∞-stacks are higher analogs of sheaves that, instead of sets, assign ∞-groupoids (to capture, for instance, gauge symmetries), derived ∞-stacks (informally) correspond to enlarging the category of test spaces that parametrize the ∞-groupoids from manifolds to cosimplicial manifolds. Intuitively, the 'stacky' simplicial degrees in the target category (∞-groupoids may be realized as simplicial sets) encode refined information about gauge symmetries, while the 'derived' cosimplicial degrees in the source category encode refined information about the critical loci. In a more physical jargon, the former are related to *ghost fields* while the latter to *antifields*. The structure of the derived ∞-stack of the solutions to the equation of motion 'infinitesimally close' to an arbitrary solution is then described by a *formal moduli problem*. See [CG21a] for a survey paper. A renown result due independently to Lurie and Pridham [Lur11; Pri10] shows that there is an equivalence of ∞-categories between the ∞-category of formal moduli problems and that of differential graded Lie algebras. Hence, dg-Lie algebras provide an equivalent description of a formal neighborhood of a point in the space of solutions to equation of motion of a gauge theory. From a more physical point of view, the kinematic ∞-groupoid of gauge field configurations is related to the Becchi-Rouet-Stora-Tyutin (BRST) formalism, which introduces ghost fields to resolve gauge symmetries in cohomology. When also the dynamics is considered, BRST formalism gets refined to the Batalin-Vilkovisky (BV) formalism, which is then related to the formal moduli problem of the solutions to equation of motion.

Working directly with derived ∞-stacks is very hard, and it is not clear yet how to describe examples of solution spaces of physically realistic gauge field theories within this approach, except for some toy-models [PTVV13; Cal+17]. For this reason, in order to tackle the problem of studying the quantum theory of gauge fields, one can first try to consider directly a formal moduli problem, in order to get an approximation of the space of solutions to equation of motion, and then to quantize it adopting a perturbative approach. Problems related to these are currently investigated both in the setting of *algebraic quantum field theory* (AQFT) [FR12a; FR12b; Hol08; BSW21; DR23] and of *factorization algebra* (FA) [Cos07; CG17; CG21b]. These are two rigorous axiomatizations of quantum field theory which focus on different algebraic structures of the space of quantum observables, adopting a point of view which generalizes the Heisenberg picture of quantum mechanics. The AQFT formalism is particularly suited to the Lorentzian setting. According to it an algebraic quantum field theory is defined as a functor $\mathcal{A} : \mathbf{Loc}_m \to \mathbf{Alg}$ from the category of globally hyperbolic Lorentzian manifolds, see Definition 1.1.6, taking values into a suitable category $\mathbf{Alg}$ of algebras. It associates with each spacetime $M \in \mathbf{Loc}_m$ the algebra $\mathcal{A}(M) \in \mathbf{Alg}$ of the quantum observables over it. The assignment $\mathcal{A}$ has to satisfy certain physically motivated axioms: the *Einstein causality axiom* imposes that observables pertaining to spacetime regions that are causally disjoint must commute; the *time-slice axiom* imposes a notion of well-posed dynamics at the level of the algebra of observables by requiring that the algebra assigned by $\mathcal{A}$ to any time-slab around a Cauchy surface is equivalent to that assigned to the whole spacetime. The FA approach focuses instead on the products between observables with disjoint support. For this, it is a formalism more suitable for the description of topological or Riemannian quantum field theories. A factorization algebra $\mathcal{F}$ assigns to each manifold M a space $\mathcal{F}(M)$ of quantum observables over it, and to each n-tuple of morphisms $f_i : M_i \to N$, $i = 1, \dots, n$, whose images $f_i(M_i)$ are disjoint in N, a n-ary product, called *factorization product*, $\mathcal{F}(\underline{f}) : \bigotimes_i \mathcal{F}(M_i) \to \mathcal{F}(N)$ on the spaces of observables pertaining to the disjoint regions. The factorization products have to fulfill suitable composition and permutation equivariance axioms. Moreover, factorization algebras must satisfy a suitable descent condition with respect to Weiss covers, otherwise one talks of *prefactorization algebras*. Note in particular that a factorization algebra, in general, does not allow one to define an algebra structure on the space of observables, in contrast to the axiomatization provided by algebraic quantum field theory.

To describe perturbatively the solution space of a gauge field theory, and hence to quantize its associated space of observables, both in the AQFT and in the FA approaches, the first step consists in considering the free gauge theory underlying the interacting one, which corresponds to the 0-th order in the perturbative expansion. Afterwards one regards the non-linear contributions to the equation of motion and gauge transformations as perturbations over the linear theory. Therefore, it is of paramount importance to have fully under control linear gauge field theories before addressing the perturbative problem. For a linear gauge field theory the description of the configuration space may be significantly simplified passing from ∞-groupoids to cochain complexes. In fact, the linear assumption yields that there is a cochain complex $\mathfrak{C}$ canonically associated with the ∞-groupoid (a simplicial vector space in the linear case) of gauge field configurations via the Dold-Kan correspondence. The cochain complex $\mathfrak{C}$ is such that its degree 0 is the vector space of gauge fields and its negative degrees are the vector spaces of ghosts and higher ghosts. The differential encodes the action of (higher) gauge transformations. Hence, it is related to the BRST complex from the more physically-oriented approaches. The higher structures described by the ∞-groupoid of gauge field configurations are now encoded at the level of cohomology of the cochain complex and, hence, by the ∞-category $\mathbf{Ch}_{\mathbb{K}}$ of cochain complexes, whose equivalences are the quasi-isomorphisms. When also the dynamics is ruled by a linear differential operator, namely it is induced by a quadratic gauge invariant action functional, we can also implement it at the level of cochain complexes. This amounts to consider the variation of the action, $\mathrm{d}S : \mathfrak{C} \to T^*\mathfrak{C}$ in $\mathbf{Ch}_{\mathbb{K}}$, which provides a section of a suitably defined 'smooth' cotangent bundle $T^*\mathfrak{C}$ over the cochain complex $\mathfrak{C}$ of field configurations, and to take its homotopy pullback with the zero section. This computes a linear analog of the derived critical locus of the action S and, in a sense, it provides a linear approximation of the derived ∞-stack considered above. See [BS19] for more details. The cochain complex $\mathfrak{F}$ which is obtained from this procedure encodes the dynamical features of the linear gauge field theory in cohomology, and it is related to the BV complex. In addition to the gauge fields in degree 0 and the (higher) ghosts in the negative degrees, the cochain complex $\mathfrak{F}$ includes also the vector spaces of the antifields (of both the gauge and ghost fields) in its positive cohomological degrees. Its differential encodes simultaneously the action of the gauge transformations and the dynamics of the theory. In particular, the 0-th cohomology $\mathsf{H}^0\mathfrak{F}$ is the usual vector space of equivalence classes of on-shell gauge fields modulo gauge transformations.

General results of derived algebraic geometry [PTVV13; Cal+17; Pri18] entail that the cochain complex $\mathfrak{F}$ comes endowed with a (-1)-shifted Poisson structure $\tau^{(-1)}$ as a consequence of being a derived critical locus. The (-1)-shifted Poisson structure $\tau^{(-1)}$ pairs the fields with the associated antifields. To fix the ideas, in the case of the free scalar field theory over a manifold M the (-1)-shifted Poisson structure explicitly reads as $\tau^{(-1)}(\phi \otimes \phi^\ddagger) = \int_M \phi\phi^\ddagger \,\mathrm{vol}_M$, where ϕ is a field and $\phi^\ddagger$ an antifield. This structure is related to the *antibracket* from the BV formalism. Such (-1)-shifted Poisson structure plays a crucial role in the factorization algebra approach to quantum field theory by Costello and Gwilliam [CG17; CG21b] to quantize a classical field theory according to the BV quantization. Therefore, it is particularly useful in the context of topological and Riemannian quantum field theories.

Our main results. The constructions outlined above are reminiscent of the usual characterization of the space of solutions to equation of motion in classical mechanics as the critical locus of an action functional. The more refined language of derived geometry is required to handle the higher structures encoded by gauge symmetries in a coherent way. Also the origin of the

(-1)-shifted Poisson structure $\tau^{(-1)}$ may be understood in a simpler way by making a comparison with the more familiar setting of analytical mechanics. In fact, the cotangent bundle is endowed with a symplectic structure, which, roughly speaking, pairs the positions (the 'q-s') with the momenta (the 'p-s'). When the field configurations are described by a cochain complex, we may think of the fields as the upgrade of the q-s and of the antifields as the upgrade of the p-s. Computing the derived critical locus in this setting shifts the fibers of the cotangent bundle, where the antifields live, by -1. As a consequence also the Poisson structure that pairs the fields with the antifields gets shifted accordingly to the (-1)-shifted Poisson structure $\tau^{(-1)}$. As already said $\tau^{(-1)}$ may be used to quantize a field theory as a factorization algebra, especially in a topological or Riemannian setting. Unfortunately, it cannot be directly exploited to quantize it as an AQFT. This may be unsatisfactory since the AQFT formalism provides a robust axiomatization of quantum field theories on Lorentzian manifolds, i.e. on physical spacetimes. Indeed, in the construction of the algebra $\mathcal{A}(M)$ assigned by an AQFT $\mathcal{A}$ to a spacetime M a crucial role is typically played by an *unshifted* Poisson structure τ_M. This is the starting point of the deformation arguments used to construct the quantum product of $\mathcal{A}(M)$ by imposing canonical commutation relations. In this thesis we will develop a framework, which covers many examples of derived critical loci for gauge-theoretic quadratic action functionals in Lorentzian signature, see Chapter 6, in which the (-1)-shifted Poisson structure $\tau^{(-1)}$ can be unshifted to a Poisson structure τ in a homotopy coherent fashion. This will allow us to provide both quantizations as factorization algebras and as AQFTs in a Lorentzian setting, and to compare the results of such constructions, by showing that they induce the same time-ordered products. The key concept we will need to get this results is a homological generalization of wave operators, called *Green hyperbolic complexes*, which covers many examples of gauge theories in the above formalism. As wave operators play an important role in the study of ordinary field theories (e.g. of Klein-Gordon field), our Green hyperbolic complexes have a similar importance for linear gauge field theories.

Green hyperbolic linear differential operators [BG12; Bär15], generalizing wave operators [BGP07], are one of the fundamental building blocks of mathematical field theory on (globally hyperbolic) Lorentzian manifolds. They are defined as linear differential operators $P : \Gamma(E) \to \Gamma(E)$, acting on smooth sections of a vector bundle $E \to M$ of finite rank, which admit retarded and advanced Green's operators $G^{\pm} : \Gamma_{\mathrm{c}}(E) \to \Gamma(E)$, namely linear maps satisfying $G^{\pm} P \psi = \psi = P G^{\pm} \psi$ which propagate any compactly supported section $\psi \in \Gamma_{\mathrm{c}}(E)$ to the causal future/past $\mathrm{supp}(G^{\pm}\psi) \subseteq J^{\pm}_M(\mathrm{supp}\,\psi)$ of its support. The importance of Green hyperbolic operators in the mathematical description of Lorentzian field theory [BFV03; FV15; HW15] is related to the following key results. First of all, when they exist, retarded/advanced Green's operators $G^{\pm}$ are unique. Second, given the retarded and advanced Green's operators $G^{\pm}$ one defines the *retarded-minus-advanced propagator* $G := G^{+} - G^{-}$. It descends to an isomorphism of vector spaces

$$G : \mathrm{coker_c}(P) \xrightarrow{\cong} \mathrm{ker_{sc}}(P) \tag{1}$$

between the vector space of compactly supported sections modulo the equation of motion, $\mathrm{coker_c}(P) := \mathrm{coker}(P : \Gamma_{\mathrm{c}}(E) \to \Gamma_{\mathrm{c}}(E)) = \Gamma_{\mathrm{c}}(E)/P\Gamma_{\mathrm{c}}(E)$, and the space of solutions with spacelike compact support, $\mathrm{ker_{sc}}(P) := \mathrm{ker}(P : \Gamma_{\mathrm{sc}}(E) \to \Gamma_{\mathrm{sc}}(E))$. Furthermore, when P is formally self-adjoint with respect to a fiber metric $\langle -, - \rangle$ on E, it is possible to endow the vector space $\mathrm{coker_c}(P)$, which admits an interpretation as the space of linear observables of the classical field theory ruled by P, with an unshifted Poisson structure

$$\tau_M([\psi_1] \otimes [\psi_2]) := \int_M \langle \psi_1, G\psi_2 \rangle \,\mathrm{vol}_M \,, \tag{2}$$

for all $[\psi_1], [\psi_2] \in \mathrm{coker_c}(P)$, where vol_M denotes the volume form on M. This structure plays a central role in the canonical quantization of the free ordinary field theory associated with P in the AQFT formalism. Finally, assume the form $P = \Box^{\nabla} + B$ for the Green hyperbolic operator P, where $\Box^{\nabla}$ is the d'Alembert operator of the metric connection ∇ on $(E, \langle -, - \rangle)$ and B is a symmetric endomorphism. Then, it is possible to endow the vector space $\mathrm{ker_{sc}}(P)$ of spacelike solutions with the *fixed-time* unshifted Poisson structure

$$\sigma_\Sigma(\varphi_1 \otimes \varphi_2) := \int_\Sigma \big(\langle \varphi_2, \nabla_{\mathfrak{n}}\varphi_1 \rangle - \langle \varphi_1, \nabla_{\mathfrak{n}}\varphi_2 \rangle\big)\big|_\Sigma \,\mathrm{vol}_\Sigma \tag{3}$$

for all $\varphi_1, \varphi_2 \in \mathrm{ker_{sc}}(P)$, which relies on the choice of a spacelike Cauchy surface $\Sigma \subseteq M$. Then G upgrades to an isomorphism of Poisson vector spaces since it is compatible with the Poisson structures, i.e. $\sigma_\Sigma \circ G^{\otimes 2} = \tau_M$.

The main goal of this work is to develop a homological generalization of Green hyperbolic operators, and in particular of retarded/advanced Green's operators, which upgrades these results, in a homotopy coherent fashion, to the case of derived critical loci of gauge invariant quadratic action functionals. This generalization is important because Green hyperbolic operators cannot rule the dynamics of a gauge theory since the latter is necessarily degenerate. (The differential operator ruling the dynamics of a gauge theory must vanish on gauge transformations while a Green hyperbolic operator cannot vanish on any non-zero compactly supported section.)

The setting we shall consider is the following. Abstracting away from the cochain complexes one gets as the derived critical locus of gauge-theoretic quadratic action functionals, one sees that they are generally associated with a *complex of linear differential operators* (F, Q). This consists of a $\mathbb{Z}$-graded vector bundle $F \to M$, degree-wise of finite rank, over a globally hyperbolic Lorentzian manifold M and of a family Q of degree increasing linear differential operators which square to zero. See [Tar95] for a related concept. The cochain complex $\mathfrak{F}(M) \in \mathbf{Ch}_{\mathbb{K}}$ whose underlying graded vector space is obtained by taking smooth sections of the graded vector bundle F and whose differential is given by Q is then designed to model a BV complex.

Similarly to the ordinary definition of Green hyperbolic operators, in Definition 3.1.5 we define a *Green hyperbolic complex* as a complex of linear differential operators which admits a *retarded* and an *advanced Green's homotopy* $\Lambda^{\pm}$. The latter are homological generalizations of the familiar retarded and advanced Green's operators.

Retarded and advanced Green's homotopies are formalized in Definition 3.1.4 as homotopies connecting the identity and 0 in a suitable complex of homotopy coherent natural transformations. They reduce to retarded and advanced Green's operators in the classical case of a Green hyperbolic operator, see Example 3.1.7. Most importantly, Green hyperbolic complexes admit homotopy coherent variants of the main features Green hyperbolic operators have. First of all, Proposition 3.2.2 shows that retarded and advanced Green's homotopies are unique (in a homotopically meaningful way, namely up to a contractible space of choices). Second, given a Green hyperbolic complex (F, Q), any choice of a retarded and an advanced Green's homotopies $\Lambda^{\pm}$ induces a *retarded-minus-advanced quasi-isomorphism*

$$\Lambda := \Lambda^{+} - \Lambda^{-} : \mathfrak{F}_{\mathrm{c}}(M)[1] \xrightarrow{\sim} \mathfrak{F}_{\mathrm{sc}}(M) \tag{1'}$$

in $\mathbf{Ch}_{\mathbb{K}}$ from the 1-shift of the cochain complex $\mathfrak{F}_{\mathrm{c}}(M)$ of compactly supported sections (interpreted as the complex of linear observables) to the cochain complex $\mathfrak{F}_{\mathrm{sc}}(M)$ of sections with spacelike compact support of (F, Q) (interpreted as the complex of solutions with spacelike compact support). Furthermore, introducing the concept of a *differential pairing* $(-,-)$ (Definition 3.3.1), replacing that of a fiber metric, we endow both the source and the target of the quasi-isomorphism Λ with an unshifted Poisson structure. The covariant Poisson structure τ_M on $\mathfrak{F}_{\mathrm{c}}(M)[1]$ is defined as

$$\tau_M(\psi_1 \otimes \psi_2) := \int_M (\psi_1, \Lambda\psi_2)\,, \tag{2'}$$

for all homogeneous $\psi_1, \psi_2 \in \mathfrak{F}_{\mathrm{c}}(M)[1]$, generalizing the classical Poisson structure (2). After fixing a spacelike Cauchy surface $\Sigma \subseteq M$, one defines also the fixed-time Poisson structure on the cochain complex $\mathfrak{F}_{\mathrm{sc}}(M)$

$$\sigma_\Sigma(\varphi_1 \otimes \varphi_2) := (-1)^{m-1} \int_\Sigma (\varphi_1, \varphi_2)\,, \tag{3'}$$

for all $\varphi_1, \varphi_2 \in \mathfrak{F}_{\mathrm{sc}}(M)$, which upgrades (3). It is shown in Theorem 3.3.7 that the quasi-isomorphism Λ is compatible with the Poisson structures $\sigma_\Sigma \circ \Lambda^{\otimes 2} = \tau_M + \partial\lambda$ up to an explicit homotopy λ, thus generalizing the analogous classical result.

The results above shall be used to quantize Green hyperbolic complexes on globally hyperbolic Lorentzian manifolds both as AQFTs and as *time-orderable* prefactorization algebras. The latter is a Lorentzian variant of prefactorization algebras, proposed in [BPS20], whose factorization products are assigned only to tuples of time-orderable spacetime regions. Moreover, we shall also compare the results of these two constructions, generalizing the model-based comparison due to Gwilliam and Rejzner [GR20] to the context of gauge field theories. The key ingredient for this purpose is the concept of a *Green's witness* W, introduced in Chapter 4 as a tool that witnesses the Green hyperbolicity of a complex of differential operators (F, Q). This is an auxiliary structure, slightly reminiscent of the gauge fixing operator of the BV formalism [CG17; CG21b], which may be informally thought of as witnessing the existence of a gauge fix which makes the equation of motion Green hyperbolic. In fact, a Green's witness W gives rise to the Green hyperbolic operator $P := QW + WQ$, which allows us to find particularly simple retarded and advanced Green's homotopies $\Lambda^{\pm} := WG^{\pm}$, for $G^{\pm}$ the retarded/advanced Green's operator of P. The ancillary nature of Green's witnesses is made more evident by the uniqueness of retarded and advanced Green's homotopies (Proposition 3.2.2). Proposition 3.3.5, in particular, entails that each possible choice of a Green's witness gives rise to equivalent models. Moreover, there exist interesting examples of complexes of differential operators which are Green hyperbolic but do not admit a Green's witness. See for example [BGS23], where a gauge-theoretic model that possesses (a variant of) Green's homotopies, but does not seem to admit a Green's witness, is illustrated. When both Q and W are natural (with respect to spacetime embeddings), it follows that these simplified Green's homotopies behave as ordinary retarded and advanced Green's operators. This is crucial for constructing the AQFT quantization of the associated complex of differential operators since, in particular, makes the covariant Poisson structure τ_M compatible with spacetime embeddings. Let us note that the examples of free ordinary field theory, Abelian Chern-Simons theory and Maxwell p-forms, all admit a Green's witness, see Chapter 6.

Coming back to the constructions of AQFTs and time-orderable prefactorization algebras quantizing Green hyperbolic complexes, the input datum is a *covariant free BV theory* $(\mathsf{F}, Q, (-,-), W)$, which consists of a collection of complexes of differential operators $(\mathsf{F}(M), Q_M)$, differential pairings $(-,-)_M$ and Green's witnesses W_M, labeled by globally hyperbolic Lorentzian manifolds M and natural with respect to spacetime embeddings. Both the quantization procedures happen at the level of the symmetric dg-algebra $\mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}(M)[1])$ generated by the cochain complex $\mathfrak{F}_{\mathrm{c}}(M)[1]$. The time-orderable prefactorization algebra $\mathcal{F}$ is constructed via BV quantization. This consists in deforming the differential $\mathcal{Q}_M$ of $\mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}(M)[1])$ to the quantized differential $\mathcal{Q}^{\hbar}_M := \mathcal{Q}_M + \mathrm{i}\,\hbar\Delta_{\mathrm{BV}}$, where Δ_{BV} is the *BV*

Laplacian built out of the (-1)-shifted Poisson structure $\tau_M^{(-1)}$ on $\mathfrak{F}_{\mathrm{c}}(M)[1]$, which is obtained by integrating over M the differential pairing $(-,-)_M$. The time-ordered factorization products are the ones induced by the commutative multiplication μ of the symmetric dg-algebra. The Green's witness W does not play any role in this construction. On the contrary, the latter is crucial to define the AQFT $\mathcal{A}$ quantizing the same covariant free BV theory $(\mathsf{F}, Q, (-,-), W)$. Here the deformation is moved from the differential $\mathcal{Q}_M$ to the multiplication μ_M of the symmetric dg-algebra. Indeed, we exploit the unshifted Poisson structure τ_M to deform μ_M into the, in general non-commutative, *Moyal-Weyl star product* $\mu_M^\hbar$. Finally, we construct an isomorphism $T : \mathcal{F} \to \mathcal{F}_{\mathcal{A}}$ of time-orderable prefactorization algebras that compares the BV quantization $\mathcal{F}$ and the time-orderable prefactorization algebra $\mathcal{F}_{\mathcal{A}}$ associated with the AQFT $\mathcal{A}$. (See [BPS20] and also Section 1.4 for the construction of the time-orderable prefactorization algebra $\mathcal{F}_{\mathcal{A}}$ associated with an AQFT $\mathcal{A}$.)

Future outlook. We conclude this introduction by pointing out some future outlooks and possible continuations of this work. In this thesis we focused in particular on free (i.e. linear) classical and quantum field theories for which we developed the theory of Green's homotopies in a way which takes due care of the higher structures that are encoded by gauge symmetries. It is then natural to ask what happens when interactions, hence non-linear contributions, are considered. First of all, since in the classical scenario the theory of Green hyperbolic operators does not extend to non-linear differential operators, we do not expect that our constructions translate *verbatim* to interacting field theories. However, they can definitely be applied to the linear theory underlying a perturbatively interacting one. For this reason, as already noticed, the study of linear gauge field theories performed in this thesis may be seen as propaedeutic to the study of arbitrary gauge field theories within a perturbative approach. Importantly, this encompasses non-Abelian gauge field theories. Including interactions in our framework is not an easy task, not even in a perturbative fashion. The issues trace back to the fact that interaction terms are typically described by distributions whose support is concentrated along the diagonal. Since interactions should be in the algebra of observables, the latter needs to be extended to include more singular objects. This brings us to consider as the cochain complex of linear classical observables not the compactly supported smooth sections of the graded vector bundle $F \to M$, but rather its compactly supported distributional sections. Working with distributional sections introduces several difficulties. A first source of problems is that products and compositions of distributions are, in general, ill-defined because of the bad interplay of their singularities. The theory of microlocal analysis [Hör90] provides sufficient conditions to keep this problem under control. These techniques are exploited in the perturbative AQFT (pAQFT) literature [BF00; DF01; HW01; HW03; BDF09; Rej16] to first extend the free algebra to include distributional observables and then to implement interactions. In fact, the Moyal-Weyl star product defined by taking contractions against the retarded-minus-advanced propagator G does not extend to arbitrary distributional sections because of the propagation of singularities property of G. In pAQFT this problem is circumvented by restricting the observables to those satisfying the technical condition of having a *microcausal wavefront set* [Rad96; BFK96] and replacing the retarded-minus-advanced propagator G with the so-called *Wightman propagator* H. The latter is a bidistribution whose anti-symmetric part coincides with G and whose wavefront set satisfies the *microlocal spectrum condition*. Contracting microcausal distributions with H allows one to get a well-defined quantum star product [HW01; BDF09]. The Wightman propagator H is not unique and its choice is related to that of a *Hadamard state* of the free theory [Rad96]. Following these suggestions, we may try to introduce a regularized Moyal-Weyl star product on the dg-algebra of polynomials with coefficients given by microcausal distributional sections of (F, Q) by replacing the retarded-minus-advanced cochain map Λ with a *Wightman cochain map* Γ satisfying suitable regularity conditions. Note that simple choices of such Γ may come from a suitably refined Green's witness W in concrete examples. This procedure constructs the free field algebra encompassing distributional observables. The long-term goal would be to build the perturbatively interacting algebra on the free one. In the pAQFT approach this is achieved by relying on the so-called *Bogoliubov map* and resorting to *renormalization* techniques. We currently do not fully understand how to import these techniques into our formalism. Furthermore, it is even unclear if Bogoliubov maps exist for the examples of gauge theories of interest, since we and our collaborators have recently found obstructions to the existence of their classical counterparts [BGMS23].

Let us recall that in the BV formalism the space of solutions to the (in general, non-linear) equation of motion of a gauge field theory is perturbatively modeled by a formal moduli problem, hence equivalently by a L_∞-algebra, whose higher arity operations encode the perturbative interactions. In this framework, it would be possible to exploit the homotopy techniques employed in this work to deal with perturbatively interacting gauge field theory. This would require us to consider a model structure on the category of L_∞-algebras and to perform derived constructions in it. In particular, one would be interested in the derived hom space between two L_∞-algebras, which would be the natural space where to search for a generalization of our retarded and advanced Green's homotopies.

A second open problem concerns the construction of *homotopy coherent quantizations* of an arbitrary covariant Green hyperbolic complex (F, Q). Indeed, in this work we restrict ourselves to those covariant complexes of differential operators whose Green hyperbolicity is witnessed by a natural

Green's witness W. This is a fundamental assumption to ensure that the unshifted Poisson structure τ_M is natural and, hence, that the symmetric dg-algebra can be quantized as a (strict) AQFT $\mathcal{A}$ by means of the Moyal-Weyl star product. In absence of a natural Green's witness, there are no preferred choices of retarded and advanced Green's homotopies, hence there is no reason to expect that choices may be performed in a natural way. Since retarded and advanced Green's homotopies are in general only natural up to higher homotopies, the unshifted Poisson structures they induce fail to be natural on the nose with respect to spacetime embeddings. However, the higher homotopies witnessing the failure of $\Lambda^{\pm}$ to be natural may be used to control this failure. The challenge here is to understand how to organize these structures in order to get a well-behaved object that may replace the functor $(\mathfrak{F}_{\mathrm{c}}[1], \tau)$ valued in Poisson complexes, and that may serve as the basis for a homotopy coherent quantization. The outcome of this construction would be a *homotopy coherent AQFT*, where both the functorial and algebraic structures may be homotopically relaxed.

Summary. The outline of the remainder of the thesis is the following. In Chapter 1 we recollect the required preliminary material. More specifically, in Section 1.1 we recall some basic aspects of the theory of globally hyperbolic Lorentzian manifolds. Section 1.2 formalizes some useful notions of natural geometric structures with respect to spacetime embeddings, such as natural vector bundles, natural linear differential operators and natural fiber metrics. In Section 1.3 we recall the notion of Green hyperbolic linear differential operators and we collect some facts about them. Section 1.4 reviews the concepts of algebraic quantum field theories and of time-orderable prefactorization algebras valued in a bicomplete closed symmetric monoidal category $\mathbf{M}$. We also include the time-slice axiom, both in its strict and homotopy variant, when $\mathbf{M}$ is also a homotopical category.

Chapter 2 is devoted to introducing some useful tools related to homological algebra. In particular, Section 2.1 recalls the basics of the theory of cochain complexes over a field $\mathbb{K}$ of characteristics zero. In Section 2.2 we extend graded symmetric and anti-symmetric $(-p)$-shifted pairings τ, for $p \in \mathbb{Z}$, over a cochain complex $V \in \mathbf{Ch}_{\mathbb{K}}$ to suitable biderivations $\{\{-,-\}\}_{\tau}$ and, when τ is symmetric, to suitable Laplacians Δ_{τ} over the symmetric dg-algebra $\operatorname{Sym} V \in \mathbf{dgCAlg}_{\mathbb{K}}$. In Section 2.3 we consider functors valued in cochain complexes. In particular, we regard the category $\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ of functors from a category $\mathbf{C}$ to the category of cochain complexes $\mathbf{Ch}_{\mathbb{K}}$ as a dg-category by introducing the mapping complex $\underline{\mathrm{map}}$. This formalizes a concept of homotopy coherent natural transformations which is compatible with weak equivalences of functors in $\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$. We also introduce a concrete model to compute homotopy colimits of $\mathbf{Ch}_{\mathbb{K}}$-valued diagrams. Section 2.4 recalls the notion of Kan complexes as models of ∞-groupoids and reviews a standard construction of a Kan complex describing the space of elements of a cochain complex V having an assigned differential and (higher) homotopies between them.

The core of the thesis are Chapters 3, 4 and 5. In Section 3.1 we introduce the central concepts of retarded/advanced Green's homotopies and Green hyperbolic complexes and we show that they generalize the ordinary retarded/advanced Green's operators and Green hyperbolic differential operators to the context of complexes of differential operators. In Section 3.2 we prove the main results concerning the properties of retarded and advanced Green's homotopies. In particular, Proposition 3.2.1 provides a characterization of Green hyperbolic complexes. Proposition 3.2.2 shows that retarded/advanced Green's homotopies (when they exist) are unique, in the sense that their spaces are contractible. Finally, Theorem 3.2.4 proves that the retarded-minus-advanced cochain map Λ constructed in Section 3.1 is a quasi-isomorphism. Section 3.3 introduces the notion of a differential pairing. This is exploited to define two types of Poisson structures, $\tau_M, \tau_M^{\pm}$ on the complex of linear observables, and σ_Σ on the complex of spacelike solutions. The first type relies on the choice of a retarded and an advanced Green's homotopy while the second one on that of a spacelike Cauchy surface $\Sigma \subseteq M$. Proposition 3.3.5 and Corollary 3.3.8 show that different choices lead to Poisson structures that coincide up to homotopy. Theorem 3.3.7 shows that Λ is compatible with τ_M and σ_Σ up to an explicit homotopy.

In Chapter 4 we cover the useful concept of a Green's witness W of a complex of differential operators (F, Q). This is introduced in Section 4.1, where Theorem 4.1.5 shows that W witnesses the Green hyperbolicity of (F, Q) by picking out special retarded and advanced Green's homotopies. Furthermore, we provide the notion of a natural Green's witness which allows one to construct natural retarded-minus-advanced quasi-isomorphisms. In Section 4.2 we specialize to formally self-adjoint Green's witnesses which simplify the construction of the Poisson structure τ_M and its comparison to σ_Σ through Λ. Finally, Section 4.3 introduces the concept of a covariant free BV theory as a natural family of complexes of differential operators endowed with differential pairings and Green's witnesses. To each of them, we associate a natural Poisson structure τ, a natural (-1)-shifted Poisson structure $\tau^{(-1)}$ and a natural symmetric Dirac pairing τ^D. In particular, Theorem 4.3.3 shows that the functor $(\mathfrak{F}_\mathrm{c}[1], \tau) : \mathbf{Loc}_m \to \mathbf{PoCh}_\mathbb{R}$ which assigns to each $M \in \mathbf{Loc}_m$ the Poisson complex of the linear observables $\mathfrak{F}_\mathrm{c}(M)[1]$ with the Poisson structure τ_M satisfies classical analogs of the Einstein causality and homotopy time-slice axioms.

These structures and results are exploited in Chapter 5, which is devoted to the construction and comparison of two different quantization schemes of a covariant free BV theory. Section 5.1 quantizes the covariant free BV theory $(\mathsf{F}, Q, (-,-), W)$ on $\mathbf{Loc}_m$ as a time-orderable prefactorization algebra $\mathcal{F} \in \mathbf{tPFA}_m$, according to the BV quantization approach, by deforming the differential $\mathcal{Q}$ of the symmetric dg-algebra $\mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}[1])$ by means of the BV Laplacian Δ_{BV} associated with the natural (-1)-shifted Poisson structure $\tau^{(-1)}$. Proposition 5.1.3 proves that $\mathcal{F}$ satisfies the homotopy time-slice axiom. Section 5.2 quantizes a covariant free BV theory as an algebraic quantum field theory $\mathcal{A} \in \mathbf{AQFT}_m$ by deforming the graded commutative multiplication μ of $\mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}[1])$ to the Moyal-Weyl star product $\mu^{\hbar}$ built out of the natural unshifted Poisson structure τ. Proposition 5.2.2 shows, in particular, that $\mathcal{A}$ fulfills the homotopy time-slice axiom. The natural Dirac pairing τ^D, via its associated Dirac Laplacian Δ_D, is exploited in Section 5.3 to build the isomorphism $T : \mathcal{F} \to \mathcal{F}_{\mathcal{A}}$ in $\mathbf{tPFA}_m$ which compares $\mathcal{F}$ with the time-orderable prefactorization algebra $\mathcal{F}_{\mathcal{A}}$, canonically associated with the AQFT $\mathcal{A}$, whose time-ordered products are built out of the Moyal-Weyl star product $\mu^{\hbar}$.

The thesis is concluded by Chapter 6 which contains some examples of Green hyperbolic complexes motivated by classical and quantum field theory. Section 6.1 discusses the example of ordinary free field theories, Section 6.2 describes the Abelian Chern-Simons theory and Section 6.3 considers the theory of Maxwell p-forms. Appendix A provides detailed proofs of some results stated in Section 2.3.

Chapter 1

Preliminaries

We start this thesis by collecting some known results pertaining to various areas of mathematics that we will need in the rest of the work. They allow us to establish the language we shall use in the following and, in particular Section 1.3, they give us the opportunity to introduce the notions we aim to generalize in the main part of this thesis.

The structure of the chapter is as follows. In Section 1.1 we will introduce the notion of Lorentzian manifolds and some of the main concepts related to it. In particular, we will recall the definition of the category $\mathbf{Loc}_m$ of globally hyperbolic orientable and time-orientable m-dimensional Lorentzian manifolds. We will also introduce the notion of time-orderable n-tuples of $\mathbf{Loc}_m$ morphisms which we will need to define time-orderable prefactorization algebras (tPFAs) later on in Section 1.4. In Section 1.2 we will provide the reader with a notion of natural vector bundles, natural differential operators and natural fiber metrics with respect to morphisms in $\mathbf{Loc}_m$. In Section 1.3 we will recall the definition of Green hyperbolic linear differential operators on globally hyperbolic Lorentzian manifolds and we will recall their main features. Section 1.4 will be devoted to the description of the mathematical formalism behind two commonly used frameworks that formalize the algebraic structure of classical and quantum field theories on Lorentzian manifolds: algebraic quantum field theory (AQFT) and time-orderable prefactorization algebra (tPFA).

1.1 Lorentzian geometry

In this section we will recollect some important facts from Lorentzian geometry. The definitions and results presented here are all well-known, for this reason we will keep the narrative as streamlined as possible, without going into too much detail. We refer to the books [HE73; ONe83; Wal84; BGP07] for an in-depth introduction to these topics and for the proofs of the facts stated below. In the following, our *manifolds* will be always assumed to be second-countable, Hausdorff, paracompact and smooth.

Definition 1.1.1. A *Lorentzian manifold* (M, g) is the datum of a manifold M and of a smooth metric g such that the symmetric bilinear form $g_x : T_xM^{\times 2} \to \mathbb{R}$ on the tangent space T_xM at any point $x \in M$ is of signature $(-, +, \dots, +)$.

The semi-definite metric g of a Lorentzian manifold (M, g) allows us to endow the latter with a *causal structure* which originates from the subdivision of the (non-zero) tangent vectors to M into three disjoint subsets. Let $v \in T_xM\backslash\{0\}$ be a non-zero tangent vector at a point $x \in M$. We say that v is *spacelike* when $g_x(v, v) > 0$, *lightlike* when $g_x(v, v) = 0$ and *timelike* when $g_x(v, v) < 0$. Furthermore, we call it *causal* a v such that $g_x(v, v) \leq 0$, that is v is either timelike or lightlike. The same classification extends to piecewise C^1 curves on M by considering the tangent vectors to it. More precisely, let $I \subseteq \mathbb{R}$ be an open interval and $c : I \to M$ a piecewise C^1 curve. It is called *spacelike* (*lightlike*, *timelike* or *causal*) if its tangent vector $\dot{c}(t) \in T_{c(t)}M$ is spacelike (lightlike, timelike or causal, respectively) for every $t \in I$. We denote by $V_{(M,g)}(x) := \{ v \in T_xM \mid g_x(v, v) < 0 \}$ the *open light cone* stemmed by a point $x \in M$ and by $\overline{V}_{(M,g)}(x) := \{ v \in T_xM \mid g_x(v, v) \leq 0 \}$ the *closed light cone* centered in $x \in M$.

In the following, we shall restrict our attention to those Lorentzian manifolds for which it is possible to continuously assign a 'time direction', that is the ones which admit an everywhere timelike vector field $\mathfrak{t} \in \Gamma(TM)$, $g_x(\mathfrak{t}(x), \mathfrak{t}(x)) < 0$ for any $x \in M$. A Lorentzian manifold which satisfies this condition is said to be *time-orientable* and such a timelike vector field $\mathfrak{t}$ is called a *time-orientation* on M. Whenever a time-orientation is assigned it is possible to further classify the causal curves $c : I \to M$. We say that c is *future directed* if the inner product $g_{c(t)}(\dot{c}(t), \mathfrak{t}(c(t))) < 0$ for every $t \in I$, and *past directed* if $g_{c(t)}(\dot{c}(t), \mathfrak{t}(c(t))) > 0$ for every $t \in I$. Finally, we define causal relations between points $x, y \in M$ in a time-orientable Lorentzian manifold (M, g) endowed with a time-orientation $\mathfrak{t}$. We say that a point $x \in M$ *chronologically precedes* $y \in M$, and we write $x \ll y$, if there exists a future directed timelike curve starting from x and reaching y. A point $x \in M$ is said to *strictly causally precede* a point $y \in M$, $x < y$, if there is a future directed causal curve from x to y and to *causally precede* it, and we write $x \prec y$, if either $x < y$ or $x = y$.

Given a point $x \in M$ we define its *chronological future* as the subset $I^+_M(x) := \{ y \in M \mid x \ll y \} \subseteq M$ of all points that can be reached by a future directed timelike curve stemming from x. The *chronological past* of $x \in M$ is the subset $I^-_M(x) := \{ y \in M \mid y \ll x \} \subseteq M$ of all points that can be reached by a past directed timelike curve stemming from x. Analogously, the *causal future* $J^+_M(x) := \{ y \in M \mid x \prec y \} \subseteq M$ of $x \in M$ is given by x itself and by all the points that can be reached by a future directed causal curve stemming from it, while the *causal past* $J^-_M(x) := \{ y \in M \mid y \prec x \} \subseteq M$ contains the point x and all points that can be reached by a past directed causal curve stemming from it. Given a subset $S \subseteq M$, we define its chronological future/past simply as the union

$$I_M^{\pm}(S) := \bigcup_{x \in S} I_M^{\pm}(x) \subseteq M \tag{1.1}$$

of the chronological future/past of all its points. Similarly, we denote by

$$J_M^{\pm}(S) := \bigcup_{x \in S} J_M^{\pm}(x) \subseteq M \tag{1.2}$$

its causal future/past. We introduce the following shorthand notation: for a subset $S \subseteq M$ we write $J_M(S) := J_M^+(S) \cup J_M^-(S) \subseteq M$. From the definition, it directly follows that $I_M^{\pm}(S) \subseteq J_M^{\pm}(S)$, moreover the chronological future/past $I_M^{\pm}(S)$ of any subset $S \subseteq M$ coincides with the interior of the causal future/past $J_M^{\pm}(S)$, hence it is always an open subset. On the other hand, the causal past/future $J_M^{\pm}(S)$ is not closed in general, even when the subset $S \subseteq M$ is closed. One has only the inclusion $J_M^{\pm}(S) \subseteq \overline{I_M^{\pm}(S)}$ of the causal future/past in the closure of the chronological future/past. Nonetheless, it is true, see [BGP07, Appx. 5], that $J_M^{\pm}(K)$ is closed when the subset $K \subseteq M$ is compact.

A subset $S \subseteq M$ is called *causally convex* if

$$J_M^+(S) \cap J_M^-(S) \subseteq S\,, \tag{1.3}$$

i.e. when all causal curves with endpoints in S are entirely contained in S. It readily follows that given a subset $S \subseteq M$ the smallest causally convex subset of M that contains S is

$$J_M^{+\cap -}(S) := J_M^+(S) \cap J_M^-(S) \subseteq M\,, \tag{1.4}$$

which is called the *causally convex hull* of S.

Definition 1.1.2. Let (M, g) be a time-orientable Lorentzian manifold. A subset $S \subseteq M$ is called *achronal* (or *acausal*) if each timelike (respectively causal) curve meets S at most once. Moreover, a subset $\Sigma \subseteq M$ is called a *Cauchy surface* if every inextendible timelike curve meets Σ exactly once.

Obviously, each achronal subset is causal, while the converse is not true. A Cauchy surface Σ is in particular an achronal subset. Moreover, it is possible to prove that it is a closed subset and that it is met exactly once by each inextendible causal curve as well.

The following definitions formalize notions of Lorentzian manifolds with a causal well-behavior.

Definition 1.1.3. A Lorentzian manifold (M, g) satisfies the *causality condition* if it does not contain any closed causal curve. It is said to satisfy the *strong causality condition* if for any $x \in M$ and any open neighborhood $U \subseteq M$ of x there exists an open neighborhood $V \subseteq U$ of x such that each causal curve in M with endpoints in V is entirely contained in U.

Note that the strong causality condition implies the causality condition.

Definition 1.1.4. A *globally hyperbolic* Lorentzian manifold $(M, g, \mathfrak{o}, \mathfrak{t})$ is the datum of an orientable and time-orientable Lorentzian manifold (M, g) endowed with a choice of orientation $\mathfrak{o}$ and time-orientation $\mathfrak{t}$ such that it satisfies the strong causality condition and for all $x, y \in M$ the intersection $J^-_M(x) \cap J^+_M(y) \subseteq M$ is compact.

The following result provides a useful characterization of globally hyperbolic Lorentzian manifolds. It gathers a paramount result by Bernal and Sánchez [BS05] (the equivalence of items i and iii) with previous well-known results [HE73; ONe83].

Theorem 1.1.5. *Given an orientable and time-orientable Lorentzian manifold $(M, g, \mathfrak{o}, \mathfrak{t})$, endowed with a choice of orientation and time-orientation, the following are equivalent:*

i. $(M, g, \mathfrak{o}, \mathfrak{t})$ *is globally hyperbolic;*

ii. *There exists a Cauchy surface* $\Sigma \subseteq M$ *for* $(M, g, \mathfrak{o}, \mathfrak{t})$*;*

iii. (M, g) *is isometrically diffeomorphic to* $(\mathbb{R} \times \Sigma, -\beta \mathrm{d}t^2 + h_t)$ *where* $\beta : \mathbb{R} \times \Sigma \to \mathbb{R}$ *is a smooth positive function,* $(h_t)_{t \in \mathbb{R}}$ *is a family of Riemannian metrics on* Σ*, smoothly parameterized by* $t \in \mathbb{R}$*, and* $\{t\} \times \Sigma$ *is a spacelike smooth Cauchy surface in* $\mathbb{R} \times \Sigma$ *for each* $t \in \mathbb{R}$*.*

In the following we will write just M to denote a globally hyperbolic Lorentzian manifold, leaving the metric and the choice of orientation and time-orientation implicit when confusion should not arise.

Globally hyperbolic Lorentzian manifolds assemble into the following category, which will have an ubiquitous role in the rest of this work.

Definition 1.1.6. Let $m \geq 2$ be an integer. $\mathbf{Loc}_m$ is the category whose objects are the m-dimensional globally hyperbolic Lorentzian manifolds M and whose morphisms $f : M \to N$ between objects $M, N \in \mathbf{Loc}_m$ are the orientation and time-orientation preserving isometric embeddings whose image $f(M) \subseteq N$ is open and causally convex, see Equation (1.3).

Among all morphisms in $\mathbf{Loc}_m$, there is a subset of special morphisms which will play an important role in Section 1.4 to introduce the (homotopy) time-slice axiom for AQFTs and tPFAs. These are the *Cauchy morphisms*.

Definition 1.1.7. Let $M, N \in \mathbf{Loc}_m$ be two globally hyperbolic Lorentzian manifolds of dimension $m \geq 2$ and $f : M \to N$ be a morphism in $\mathbf{Loc}_m$. We say that f is a *Cauchy morphism* if its image $f(M) \subseteq N$ contains a Cauchy surface $\Sigma \subseteq f(M)$ of N.

Let $M \in \mathbf{Loc}_m$ be a globally hyperbolic Lorentzian manifold. We give the following useful definitions. A closed subset $S \subseteq M$ is called *past compact* if the intersection $S \cap J_M^-(x)$ is compact for all $x \in M$. $S \subseteq M$ is called *strictly past compact* if there exists a compact subset $K \subseteq M$ such that $S \subseteq J_M^+(K)$. Note that strictly past compact subsets S are also past compact since $S \cap J_M^-(x) \subseteq J_M^+(K) \cap J_M^-(x)$ which is compact because of global hyperbolicity. Moreover, it is possible to prove, see e.g. [Bär15], that a closed subset $S \subseteq M$ is past compact if and only if there exists a smooth spacelike Cauchy surface Σ for M such that $S \subseteq J_M^+(\Sigma)$ and it is strictly past compact if and only if there is a smooth spacelike Cauchy surface $\Sigma \subseteq M$ and a compact $K_\Sigma \subseteq \Sigma$ such that $S \subseteq J_M^+(K_\Sigma)$. Interchanging the role of past and future, we define *future compact* and *strictly future compact* subsets and we get similar results. A subset which is both past and future compact is called *temporally compact*. Finally, a closed $S \subseteq M$ is said to be *spacelike compact* if there exists a compact $K \subseteq M$ such that $S \subseteq J_M(K)$, equivalently $S \subseteq J_M(K_\Sigma)$ for some compact $K_\Sigma \subseteq \Sigma$ of some smooth spacelike Cauchy surface $\Sigma \subseteq M$. From a result by Sanders [San13], a closed $S \subseteq M$ is spacelike compact if and only if the intersection $S \subseteq \Sigma$ is compact for all smooth spacelike Cauchy surfaces $\Sigma \subseteq M$ for M.

In Section 1.4, where we will provide the reader with the definitions of AQFTs and tPFAs, we will need to compare (tuple of) morphisms in $\mathbf{Loc}_m$. The following definitions give us the required language to do it.

Definition 1.1.8. Let $M_1, M_2, N \in \mathbf{Loc}_m$ be m-dimensional globally hyperbolic Lorentzian manifolds. Let then $f_1 : M_1 \to N \leftarrow M_2 : f_2$ be two morphisms in $\mathbf{Loc}_m$ with the same target. We say that f_1 and f_2 are *causally disjoint* if there are no causal curves in N connecting their images. Equivalently, the intersection $J_N(f_1(M_1)) \cap f_2(M_2) = \emptyset$ is empty.

Definition 1.1.9. Let $M_1, \dots, M_n, N \in \mathbf{Loc}_m$ be m-dimensional globally hyperbolic Lorentzian manifolds. An n-tuple $(f_1 : M_1 \to N, \dots, f_n : M_n \to N)$ of morphisms in $\mathbf{Loc}_m$ to a common target N is *time-ordered* if the intersection $J_N^+(f_i(M_i)) \cap f_j(M_j) = \emptyset$ is empty for all $i < j$. The n-tuple $(f_1, \dots, f_n)$ is *time-orderable* if there exists a permutation $\rho \in \Sigma_n$ of n objects, called *time-ordering* permutation, such that the ρ-permuted n-tuple $(f_{\rho(1)}, \dots, f_{\rho(n)})$ is time-ordered.

In order to lighten the notation we will write $\underline{f} : \underline{M} \to N$ to denote the tuple $(f_1 : M_1 \to N, \dots, f_n : M \to N)$ and $\underline{f}\rho : \underline{M}\rho \to N$ for the ρ-permuted tuple $(f_{\rho(1)} : M_{\rho(1)} \to N, \dots, f_{\rho(n)} : M_{\rho(n)} \to N)$, with $\rho \in \Sigma_n$ a permutation. A time-orderable 1-tuple $(f) : \underline{M} \to N$ will be simply denoted as the morphism $f : M \to N$ in $\mathbf{Loc}_m$. For later convenience, we also define a unique empty tuple $\emptyset \to N$ for each $N \in \mathbf{Loc}_m$.

Remark 1.1.10. Note that the time-ordering permutation ρ for a time-orderable tuple $\underline{f}$ may be not unique. Consider for example a pair $(f_1 : M_1 \to N, f_2 : M_2 \to N)$ of morphisms in $\mathbf{Loc}_m$. Then, they are causally disjoint, $J_N(f_1(M_1)) \cap f_2(M_2) = \emptyset$, if and only if both the intersections $J_N^+(f_1(M_1)) \cap f_2(M_2) = \emptyset$ and $J_N^-(f_1(M_1)) \cap f_2(M_2) = \emptyset$ are empty. Hence, precisely if and only if both (f_1, f_2) and (f_2, f_1) are time-ordered pairs. Moreover, note that not all n-tuples of morphisms in $\mathbf{Loc}_m$, for $n \geq 2$, may be time-orderable. See [BPS20] for an explicit counterexample where $N = (\mathbb{R} \times \mathbb{S}^1, g = -\mathrm{d}t^2 + \mathrm{d}\theta^2)$ is the Lorentzian cylinder. $\triangledown$

The following lemma collects some results from [BPS20] that concern the interplay between permutations and time-ordering permutations and show that an internal composition of time-orderable tuples is well-defined. We refer to [BPS20, Lemma 4.3] for the proof of these facts.

Lemma 1.1.11.

i. Let $\underline{f} : \underline{M} \to N$ be a time-orderable n-tuple of morphisms in $\mathbf{Loc}_m$ and $\sigma \in \Sigma_n$ a permutation. Then the σ-permuted tuple $\underline{f}\sigma : \underline{M}\sigma \to N$ is time-orderable.

ii. Let $\underline{f} = (f_1, \dots, f_n) : \underline{M} \to N$ and $\underline{g_i} = (g_{i1}, \dots, g_{ik_i}) : \underline{L_i} \to M_i$, for $1 \leq i \leq n$, be time-orderable tuples. Then, the $\big(\sum_i k_i\big)$-tuple of morphisms in $\mathbf{Loc}_m$

$$\underline{f}(\underline{g}_1, \dots, \underline{g}_n) := (f_1 g_{11}, \dots, f_n g_{nk_n}) : (\underline{L}_1, \dots, \underline{L}_n) \longrightarrow N \tag{1.5}$$

is time-orderable.

iii. Let $\underline{f} : \underline{M} \to N$ be a time-orderable n-tuple of morphisms in $\mathbf{Loc}_m$ and let $\rho, \rho' \in \Sigma_n$ be two time-ordering permutations for $\underline{f}$. Then, the right permutation $\rho^{-1}\rho' : \underline{f}\rho \to \underline{f}\rho'$ acts by the transpositions of adjacent causally disjoint pairs of morphisms.

The results above will be crucial in Section 1.4 to prove that time-orderable prefactorization algebras on $\mathbf{Loc}_m$ are well-defined (items i and ii) and that it is possible to construct a functor that assigns a ((homotopy) Cauchy constant) time-orderable prefactorization algebra to each algebraic quantum field theory (satisfying the (homotopy) time-slice axiom) (item iii). See Remark 1.4.6 and Theorem 1.4.11, respectively.

1.2 Natural geometric structures

In the main part of this work, particularly in Section 4.3 and in Chapter 5, we will have to deal with vector bundles on manifolds $M \in \mathbf{Loc}_m$, linear differential operators between them and other structures, like fiber metrics, which are required to be assigned in a natural fashion with respect to morphisms in $\mathbf{Loc}_m$. In this section we shall introduce the categorical language needed to correctly formalize these kinds of naturality. Let us first introduce a suitably designed category of vector bundles over globally hyperbolic m-dimensional Lorentzian manifolds. In the following let $\mathbb{K}$ be a fixed field of characteristic zero.

Definition 1.2.1. We denote by $\mathbb{K}$-$\mathbf{VBnd}_m$, for $m \geq 2$, the category whose objects are the $\mathbb{K}$-vector bundles of finite rank $E \to M$ with a globally hyperbolic m-dimensional Lorentzian manifold $M \in \mathbf{Loc}_m$ as the base and whose morphisms $(\overline{f}, f) : (E \to M) \to (E' \to M')$ are the $\mathbb{K}$-vector bundle maps

$$\begin{tikzcd} E \arrow[r, "\overline{f}"] \arrow[d] & E' \arrow[d] \\ M \arrow[r, "f"'] & M' \end{tikzcd} \tag{1.6}$$

covering morphisms $f : M \to M'$ in $\mathbf{Loc}_m$, such that they are fiberwise isomorphisms, namely the restriction $\overline{f}_x : E_x \xrightarrow{\cong} E'_{f(x)}$ to the fibers over a point is an isomorphism of vector spaces for each $x \in M$.

Note that there is an obvious functor $\mathsf{\Pi} : \mathbb{K}\text{-}\mathbf{VBnd}_m \to \mathbf{Loc}_m$ which projects down to the base manifold, i.e. it acts on objects as $\mathsf{\Pi}(E \to M) := M$ and on morphisms as $\mathsf{\Pi}(\overline{f}, f) := f$.

Definition 1.2.2. A *natural* $\mathbb{K}$*-vector bundle* is a functor $\mathsf{E} : \mathbf{Loc}_m \to \mathbb{K}\text{-}\mathbf{VBnd}_m$ which is a section of $\mathsf{\Pi}$, that is $\mathsf{\Pi} \circ \mathsf{E} = \mathrm{id}$, where $\mathrm{id} : \mathbf{Loc}_m \to \mathbf{Loc}_m$ is the identity functor.

It means that a natural $\mathbb{K}$-vector bundle E is a functor that assigns to each globally hyperbolic Lorentzian manifold $M \in \mathbf{Loc}_m$ an object $\mathsf{E}(M) = (E_M \to M) \in \mathbb{K}\text{-}\mathbf{VBnd}_m$ and to each morphism $f : M \to N$ in $\mathbf{Loc}_m$ a morphism $\mathsf{E}(f) = (\mathsf{E}(f), f) : (E_M \to M) \to (E_N \to N)$ in $\mathbb{K}\text{-}\mathbf{VBnd}_m$ covering f. Note that, from the definition of the category $\mathbb{K}\text{-}\mathbf{VBnd}_m$, the restriction to the fibers $\mathsf{E}(f)_x : E_{Mx} \cong E_{Nf(x)}$ of the $\mathbb{K}$-vector bundle map $\mathsf{E}(f)$ is an isomorphism for all $x \in M$.

Let us consider smooth global sections of $\mathbb{K}$-vector bundles. There is a functor

$$\Gamma : \mathbb{K}\text{-}\mathbf{VBnd}_m^{\mathrm{op}} \longrightarrow \mathbf{Vec}_{\mathbb{K}}\,, \tag{1.7}$$

where $\mathbf{Vec}_{\mathbb{K}}$ is the category of $\mathbb{K}$-vector spaces and $\mathbb{K}$-linear maps, which to any $\mathbb{K}$-vector bundle $E \to M$ over a globally hyperbolic Lorentzian manifold $M \in \mathbf{Loc}_m$ assigns the vector space of its global smooth sections $\Gamma(E) \in$

$\mathbf{Vec}_{\mathbb{K}}$ and to any morphism $(\overline{f}, f) : (E \to M) \to (E' \to M')$ in $\mathbb{K}$-$\mathbf{VBnd}_m$ it assigns the *pullback* of smooth sections along f,

$$\Gamma(\overline{f}, f) := f^* : \Gamma(E') \longrightarrow \Gamma(E)\,, \tag{1.8a}$$

which is defined on a section $\varphi \in \Gamma(E')$ by

$$(f^*\varphi)(x) := \overline{f}_x^{-1}(\varphi \circ f(x)) \in E_x\,, \tag{1.8b}$$

for any $x \in M$. Note that the inverse of $\overline{f}_x$ exists since only fibers on $f(M)$ are involved, $\varphi \circ f(x) \in E'_{f(x)}$ for all $x \in M$. See that the support $\operatorname{supp}(f^*\varphi)$ of the pullback of a section is related to the support $\operatorname{supp}\varphi$ of the original section by $\operatorname{supp}(f^*\varphi) = f^{-1}(\operatorname{supp}\varphi)$. In fact, one readily sees that

$$\begin{aligned}
\operatorname{supp}(f^*\varphi) &= \overline{\{\, x \in M \,|\, \varphi(f(x)) \neq 0 \,\}} \\
&= \overline{f^{-1}\{\, y \in f(M) \,|\, \varphi(y) \neq 0 \,\}} \\
&\subseteq f^{-1}\overline{\{\, y \in f(M) \,|\, \varphi(y) \neq 0 \,\}} \\
&= f^{-1}\left(\overline{\{\, y \in M' \,|\, \varphi(y) \neq 0 \,\} \cap f(M)}\right) \\
&\subseteq f^{-1}\left(\operatorname{supp}\varphi \cap \overline{f(M)}\right) \subseteq f^{-1}(\operatorname{supp}\varphi)\,,
\end{aligned} \tag{1.9a}$$

where in the first step we used that $\overline{f}_x^{-1}$ is an isomorphism and in the third that f is continuous. The opposite inclusion can be shown as follows. Let $x \in f^{-1}(\operatorname{supp}\varphi)$, then there exists a sequence $M' \ni y_n \to f(x)$ such that $\varphi(y_n) \neq 0$ for all $n \in \mathbb{N}$, because of the definition of support. Since f is a morphism in $\mathbf{Loc}_m$, $f(M)$ is open and there exists a neighborhood $f(x) \ni U \subseteq f(M)$. Therefore, we can assume that $y_n \in U \subseteq f(M)$ for all $n \in \mathbb{N}$. Since $f^{-1} : f(M) \to M$ is continuous, we have a converging sequence $x_n = f^{-1}(y_n) \to x$ in M. Note that $f^*\varphi(x_n) = \overline{f}_{x_n}^{-1}\varphi(y_n) \neq 0$ since $\overline{f}_{x_n}^{-1}$ is an isomorphism. Hence, $x_n \in \operatorname{supp}(f^*\varphi)$ for all $n \in \mathbb{N}$ and, since $\operatorname{supp}(f^*\varphi)$ is closed, $x \in \operatorname{supp}(f^*\varphi)$ as well. This yields that

$$f^{-1}(\operatorname{supp}\varphi) \subseteq \operatorname{supp}(f^*\varphi)\,. \tag{1.9b}$$

When only sections with compact support are considered one defines the functor

$$\Gamma_{\mathrm{c}} : \mathbb{K}\text{-}\mathbf{VBnd}_m \longrightarrow \mathbf{Vec}_{\mathbb{K}}\,, \tag{1.10}$$

which assigns to any object $(E \to M) \in \mathbb{K}$-$\mathbf{VBnd}_m$ the vector space of compactly supported smooth sections $\Gamma_{\mathrm{c}}(E) \in \mathbf{Vec}_{\mathbb{K}}$ and to any morphism $(\overline{f}, f) : (E \to M) \to (E' \to M')$ in $\mathbb{K}$-$\mathbf{VBnd}_m$ the *pushforward* of compactly support sections

$$\Gamma_{\mathrm{c}}(\overline{f}, f) := f_* : \Gamma_{\mathrm{c}}(E) \longrightarrow \Gamma_{\mathrm{c}}(E')\,, \tag{1.11a}$$

which is given on any $\psi \in \Gamma_{\mathrm{c}}(E)$ by the extension to zero along the open embedding $f(M) \subseteq M'$ of the compactly supported section

$$\widetilde{f_*\psi} : f(M) \longrightarrow E'|_{f(M)} \tag{1.11b}$$

$$\widetilde{f_*\psi}(y) := (\overline{f} \circ \psi)(x)\,, \tag{1.11c}$$

where $x \in M$ is the unique point in M such that $f(x) = y$. (Note that the uniqueness follows from the fact that f is a diffeomorphism when restricted to its image.) From the definition, it follows that $\mathrm{supp}(f_*\psi) = f(\mathrm{supp}\,\psi)$.

The pullback and pushforward maps satisfy some useful compatibility conditions. Let $(E \to M) \in \mathbb{K}\text{-}\mathbf{VBnd}_m$ be a $\mathbb{K}$-vector bundle over a globally hyperbolic m-dimensional Lorentzian manifold $M \in \mathbf{Loc}_m$, then there exists an obvious linear map $\Gamma_{\mathrm{c}}(E) \subseteq \Gamma(E)$ which regards compactly supported sections simply as smooth sections, forgetting the condition on the support. Let $(\overline{f}, f) : (E \to M) \to (E' \to M')$ be a morphism in $\mathbb{K}\text{-}\mathbf{VBnd}_m$, then the composition

$$\Gamma_{\mathrm{c}}(E) \xrightarrow{\ f_*\ } \Gamma_{\mathrm{c}}(E') \subseteq \Gamma(E') \xrightarrow{\ f^*\ } \Gamma(E) \tag{1.12}$$

factors through the inclusion $\Gamma_{\mathrm{c}}(E) \subseteq \Gamma(E)$ as $f^* f_* = \subseteq \circ\, \mathrm{id}$. This follows directly from the definitions (1.8) and (1.11): For a compactly supported section $\psi \in \Gamma_{\mathrm{c}}(E)$ one has

$$(f^*(f_*\psi))(x) = \overline{f}_x^{-1}(f_*\psi \circ f(x)) = \overline{f}_x^{-1}(\overline{f} \circ \psi(x)) = \psi(x) \tag{1.13}$$

for all $x \in M$, where the first step uses (1.8a) and the second step follows from the fact that $f_*\psi \circ f = \widetilde{f_*\psi} \circ f$ because of the definition of the pushforward map. Equation (1.13) proves our claim and in particular shows that the support $\mathrm{supp}(f^* f_*\psi) = \mathrm{supp}\,\psi \subseteq M$ is compact. Similarly, denoting by $\Gamma_{f(M)}(E') \subseteq \Gamma_{\mathrm{c}}(E')$ the linear subspace of the sections $\psi \in \Gamma_{\mathrm{c}}(E')$ with support $\mathrm{supp}\,\psi \subseteq f(M)$ which is compact in $f(M)$, one has that the composition

$$\begin{tikzcd} \Gamma_{f(M)}(E') \subseteq \Gamma(E') \arrow[r, "f^*"] \arrow[dr, "f^*"'] & \Gamma(E) \\ & \Gamma_{\mathrm{c}}(E) \arrow[u, phantom, "\cup"] \end{tikzcd} \tag{1.14}$$

factors through the inclusion $\Gamma_{\mathrm{c}}(E) \subseteq \Gamma(E)$. Indeed, for any section $\psi \in \Gamma_{f(M)}(E')$, one has that the support $\mathrm{supp}(f^*\psi) = f^{-1}(\mathrm{supp}\,\psi)$, see Equation (1.9), is compact, since $\mathrm{supp}\,\psi \subseteq f(M)$ is compact by hypothesis and $f^{-1} : f(M) \to M$ is continuous. Finally, we have that the composition

$$\Gamma_{f(M)}(E') \xrightarrow{\ f^*\ } \Gamma_{\mathrm{c}}(E) \xrightarrow{\ f_*\ } \Gamma_{\mathrm{c}}(E') \tag{1.15}$$

factors through the inclusion $\Gamma_{f(M)}(E') \subseteq \Gamma_{\mathrm{c}}(E')$ as $f_* f^* = \subseteq \circ \,\mathrm{id}$. Let $\psi \in \Gamma_{f(M)}(E')$, exploiting definitions (1.8) and (1.11) one computes

$$(f_*(f^*\psi))(y) = (\overline{f} \circ f^*\psi)(f^{-1}(y)) = \overline{f}(\overline{f}_x^{-1}(\psi(y))) = \psi(y)\,, \tag{1.16}$$

for $y \in f(M)$. Note that for $y \in M' \setminus f(M)$ both sides of Equation (1.16) vanish, the left-hand side because of the definition of the pushforward map and the right-hand side since $y \notin \operatorname{supp} \psi$. This concludes our proof.

Let $\mathsf{E} : \mathbf{Loc}_m \to \mathbb{K}\text{-}\mathbf{VBnd}_m$ be a natural $\mathbb{K}$-vector bundle, composing with the functors of smooth sections (1.7) and of compactly supported sections (1.10) we get functors

$$\mathfrak{E} := \Gamma \circ \mathsf{E}^{\mathrm{op}} : \mathbf{Loc}_m^{\mathrm{op}} \longrightarrow \mathbf{Vec}_{\mathbb{K}}, \qquad \mathfrak{E}_{\mathrm{c}} := \Gamma_{\mathrm{c}} \circ \mathsf{E} : \mathbf{Loc}_m \longrightarrow \mathbf{Vec}_{\mathbb{K}} \tag{1.17}$$

which take values into the category of $\mathbb{K}$-vector spaces and linear maps.

Definition 1.2.3. Let E and E' be two natural $\mathbb{K}$-vector bundles. A *natural linear differential operator* P between them is a natural transformation

$$P : \mathfrak{E} \longrightarrow \mathfrak{E}' \tag{1.18}$$

between the associated functors of smooth sections (1.17), such that its component

$$P_M : \Gamma(E_M) \longrightarrow \Gamma(E'_M) \tag{1.19}$$

is a linear differential operator, for all $M \in \mathbf{Loc}_m$.

Since P is a natural transformation, it means that the diagram

$$\begin{tikzcd} \Gamma(E_{M'}) \arrow[r, "P_{M'}"] \arrow[d, "f^*"'] & \Gamma(E'_{M'}) \arrow[d, "f^*"] \\ \Gamma(E_M) \arrow[r, "P_M"'] & \Gamma(E'_M) \end{tikzcd} \tag{1.20}$$

in $\mathbf{Vec}_{\mathbb{K}}$ commutes for any morphism $f : M \to M'$ in $\mathbf{Loc}_m$. (By Definition 1.2.2 both $\mathsf{E}(f)$ and $\mathsf{E}'(f)$ cover the same $\mathbf{Loc}_m$-morphism f.)

Differential operators P_M do not enlarge supports, $\operatorname{supp}(P_M\varphi) \subseteq \operatorname{supp}\varphi$ for all smooth sections $\varphi \in \Gamma(E_M)$, hence in particular they restrict to compactly supported sections, $P_M : \Gamma_{\mathrm{c}}(E_M) \to \Gamma_{\mathrm{c}}(E'_M)$. By exploiting this and the identities (1.13) and (1.16), one finds that the diagram

$$\begin{tikzcd} \Gamma_{\mathrm{c}}(E_M) \arrow[r, "P_M"] \arrow[d, "f_*"'] & \Gamma_{\mathrm{c}}(E'_M) \arrow[d, "f_*"] \\ \Gamma_{\mathrm{c}}(E_{M'}) \arrow[r, "P_{M'}"'] & \Gamma_{\mathrm{c}}(E'_{M'}) \end{tikzcd} \tag{1.21}$$

in $\mathbf{Vec}_{\mathbb{K}}$ commutes for any morphism $f : M \to M'$ in $\mathbf{Loc}_m$. In fact, for any compactly supported section $\psi \in \Gamma_c(E_M)$ one computes

$$f_* P_M \psi = f_* P_M f^* f_* \psi = f_* f^* P_{M'} f_* \psi = P_{M'} f_* \psi \,, \tag{1.22}$$

where in the first step we used that $f^* f_* = \mathrm{id}$ on compactly supported sections, in the second one the naturality (1.20) of P and in the last step that $\mathrm{supp}(P_{M'} f_* \psi) \subseteq \mathrm{supp}(f_* \psi) \subseteq f(M)$, because of the definition (1.11) of the pushforward map and the support properties of differential operators, and that $f_* f^* = \mathrm{id}$ on sections whose support is compact in $f(M)$, see Equation (1.16).

The commutativity of the diagram (1.21) entails that any natural differential operator $P : \mathfrak{E} \to \mathfrak{E}'$ induces also a natural transformation

$$P : \mathfrak{E}_c \longrightarrow \mathfrak{E}'_c \tag{1.23}$$

between the functors of compactly supported sections, which we still denote by P, slightly abusing notation.

The last type of natural structure on globally hyperbolic Lorentzian manifolds we will need to introduce are natural fiber metrics. To do so we will restrict ourselves to the case of the field $\mathbb{K} = \mathbb{R}$ of real numbers.

Definition 1.2.4. Let E be a natural $\mathbb{R}$-vector bundle on $\mathbf{Loc}_m$. A *natural fiber metric* on E is a natural transformation

$$\langle -, - \rangle : \mathsf{E} \otimes \mathsf{E} \longrightarrow \underline{\mathbb{R}} \tag{1.24}$$

from the natural $\mathbb{R}$-vector bundle $\mathsf{E} \otimes \mathsf{E}$, which assigns to $M \in \mathbf{Loc}_m$ the tensor product of vector bundles $E_M \otimes E_M$, to the natural trivial line bundle $\underline{\mathbb{R}} : \mathbf{Loc}_m \to \mathbb{R}\text{-}\mathbf{VBnd}_m$, $M \mapsto M \times \mathbb{R}$, whose components $\langle -, - \rangle_M : E_M \otimes E_M \to \mathbb{R}$, for all $M \in \mathbf{Loc}_m$, are fiber metrics, that is fiberwise non-degenerate, symmetric linear maps.

Given a natural $\mathbb{R}$-vector bundle E endowed with a natural fiber metric $\langle -, - \rangle$, one introduces the *integration pairing*

$$\langle\!\langle -, - \rangle\!\rangle_M : \Gamma(E_M) \otimes \Gamma_c(E_M) \longrightarrow \mathbb{R} \,, \tag{1.25a}$$

for all $M \in \mathbf{Loc}_m$, defined by the integral

$$\langle\!\langle \varphi, \psi \rangle\!\rangle_M := \int_M \langle \varphi, \psi \rangle_M \, \mathrm{vol}_M \tag{1.25b}$$

for all sections $\varphi \in \Gamma(E_M)$ and $\psi \in \Gamma_c(E_M)$, where vol_M denotes the volume form on M. Note that the properties of fiber metric imply that the support $\mathrm{supp}\langle \varphi, \psi \rangle \subseteq \mathrm{supp}\, \varphi \cap \mathrm{supp}\, \psi \subseteq M$ is compact, hence the integral (1.25b) is well-defined.

Since all the ingredients in the definition (1.25) of the integral pairings $\langle\langle -,-\rangle\rangle_M$ for $M \in \mathbf{Loc}_m$ are natural, they also inherit some naturality properties with respect to morphisms in $\mathbf{Loc}_m$. A rigorous formalization of these naturality properties is given by the notion of *dinatural transformations*: A *dinatural transformation* $\alpha : F \to G$, between functors $F, G : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{D}$, consists of morphisms $\alpha_c : F(c,c) \to G(c,c)$ in $\mathbf{D}$ for every object $c \in \mathbf{C}$ such that the hexagon

$$
\begin{tikzcd}
 & F(c,c) \arrow[r, "\alpha_c"] & G(c,c) \arrow[dr, "G(\mathrm{id},f)"] & \\
F(c',c) \arrow[ur, "F(f,\mathrm{id})"] \arrow[dr, "F(\mathrm{id},f)"'] & & & G(c,c') \\
 & F(c',c') \arrow[r, "\alpha_{c'}"'] & G(c',c') \arrow[ur, "G(f,\mathrm{id})"'] &
\end{tikzcd}
\tag{1.26}
$$

in $\mathbf{D}$ commutes for each morphism $f : c \to c'$ in $\mathbf{C}$.

Let then consider the integration pairing from Equation (1.25), and let $f : M \to M'$ be a morphism in $\mathbf{Loc}_m$. For all sections $\varphi \in \Gamma(E_{M'})$ and $\psi \in \Gamma_{\mathrm{c}}(E_M)$ one computes

$$
\begin{aligned}
\langle\langle f^*\varphi, \psi\rangle\rangle_M = \langle\langle f^*\varphi, f^* f_*\psi\rangle\rangle_M &= \int_M \left(\langle \varphi, f_*\psi\rangle_{M'} \circ f\right) \mathrm{vol}_M \\
&= \int_{f(M)} \langle \varphi, f_*\psi\rangle_{M'} \,\mathrm{vol}_{M'} = \langle\langle \varphi, f_*\psi\rangle\rangle_{M'} \,,
\end{aligned}
\tag{1.27}
$$

where in the first step we used Equation (1.13), in the second step the naturality of the fiber metric $\langle -,-\rangle$ and finally that $\mathrm{supp}(f_*\psi) = f(\mathrm{supp}\,\psi) \subseteq f(M)$ by construction. Identity (1.27) is equivalent to the fact that the integration pairings as per Equation (1.25) are the components of a dinatural transformation

$$
\langle\langle -,-\rangle\rangle : \mathfrak{E} \otimes \mathfrak{E}_{\mathrm{c}} \longrightarrow \Delta\mathbb{R} \,,
\tag{1.28}
$$

from the functor $\mathfrak{E} \otimes \mathfrak{E}_{\mathrm{c}} : \mathbf{Loc}_m^{\mathrm{op}} \times \mathbf{Loc}_m \to \mathbf{Vec}_{\mathbb{R}}$, $(M, M') \mapsto \mathfrak{E}(M) \otimes \mathfrak{E}(M') = \Gamma(E_M) \otimes \Gamma_{\mathrm{c}}(E_{M'})$ to the constant functor $\Delta\mathbb{R} : \mathbf{Loc}_m^{\mathrm{op}} \times \mathbf{Loc}_m \to \mathbf{Vec}_{\mathbb{R}}, (M, M') \mapsto \mathbb{R}$. Indeed, since the target functor is constant, the hexagon (1.26) collapses to the square

$$
\begin{tikzcd}
\Gamma(E_{M'}) \otimes \Gamma_{\mathrm{c}}(E_M) \arrow[r, "f^*\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes f_*"'] & \Gamma(E_M) \otimes \Gamma_{\mathrm{c}}(E_M) \arrow[d, "\langle\langle -,-\rangle\rangle_M"] \\
\Gamma(E_{M'}) \otimes \Gamma_{\mathrm{c}}(E_{M'}) \arrow[r, "\langle\langle -,-\rangle\rangle_{M'}"'] & \mathbb{R}
\end{tikzcd}
\tag{1.29}
$$

for any $f : M \to M'$ in $\mathbf{Loc}_m$, which commutes because of Equation (1.27).

1.3 Green hyperbolic operators

This section recollects the notions of Green hyperbolic operators on globally hyperbolic Lorentzian manifolds and of retarded and advanced Green's operators. We will also state some important and well-known results about them. We refer the reader to the classic books and papers on the topic [BGP07; BG12; Bär15] for further details and for the proofs of the results mentioned below. In the following let $M \in \mathbf{Loc}_m$ be a globally hyperbolic Lorentzian manifold and $(E \to M) \in \mathbb{K}\text{-}\mathbf{VBnd}_m$ a finite rank $\mathbb{K}$-vector bundle over M. In this section, the field $\mathbb{K}$ will be either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers.

Let us start by fixing some notations. Recall from Section 1.1 the definitions of (strict) past/future compactness and spacelike compactness for a closed subset $S \subseteq M$. Note that all these notions give (non-empty) directed subsets $\mathscr{D} \subseteq \mathrm{cl}$ of the directed set cl of closed subsets of M (partially) ordered by subset inclusion. Remember that a *directed set* is a non-empty set $\mathscr{D}$ together with a reflexive and transitive binary relation $\subseteq$ such that any pair of elements of $\mathscr{D}$ has an upper bound, i.e. for any $a, b \in \mathscr{D}$ there is a $c \in \mathscr{D}$ such that $a \subseteq c \supseteq b$. We will denote by $\mathscr{D} = \mathrm{c}, \mathrm{(s)pc}, \mathrm{(s)fc}, \mathrm{sc}$ the directed sets of compact, (strictly) past compact, (strictly) future compact and, respectively, spacelike compact closed subsets of M. To fix ideas, let us see that past compact closed subsets of M form a directed set. Let $S_1, S_2 \in \mathrm{pc}$ be past compact sets in M, then we show that their union $S_1 \cup S_2$ is past compact. It is clearly a closed set since both S_1 and S_2 are closed, moreover, there are smooth spacelike Cauchy surfaces $\Sigma_1, \Sigma_2 \subseteq M$ such that $S_i \subseteq J^+_M(\Sigma_i)$, $i = 1, 2$. Therefore, $S_1 \cup S_2 \subseteq J^+_M(\Sigma_1) \cup J^+_M(\Sigma_2) \subseteq J^+_M(\Sigma)$, where $\Sigma \subseteq M$ is some spacelike smooth Cauchy surface of M in the past of both Σ_1 and Σ_2. This proves that $S_1 \cup S_2$ is past compact, therefore pc is a directed set. With similar arguments one sees that also the other sets mentioned above are actually directed sets.

Let $C \in \mathrm{cl}$ be a closed subset of M, we write $\Gamma_C(E)$ for the vector space of the smooth sections of the $\mathbb{K}$-vector bundle $(E \to M) \in \mathbb{K}\text{-}\mathbf{VBnd}_m$ with support contained in C. When $C = M$ is the full manifold we write $\Gamma(E) := \Gamma_M(E)$, following the standard convention and coherently with our notation in Section 1.2. Consider a directed subset $\mathscr{D} \subseteq \mathrm{cl}$ of the directed set of closed subsets of M, then $\Gamma_C(E)$ upgrades to a functor

$$\Gamma_{(-)}(E) : \mathscr{D} \longrightarrow \mathbf{Vec}_{\mathbb{K}}\, , \; C \longmapsto \Gamma_C(E)\, , \tag{1.30}$$

since $\Gamma_C(E) \subseteq \Gamma_{C'}(E)$ for $C \subseteq C'$, where the directed set $\mathscr{D}$ is regarded as a category. We define the vector space of sections of $(E \to M) \in \mathbb{K}\text{-}\mathbf{VBnd}_m$ with support prescribed by $\mathscr{D}$ as the colimit

$$\Gamma_{\mathscr{D}}(E) := \operatorname*{colim}_{C \in \mathscr{D}} \Gamma_C(E) \in \mathbf{Vec}_{\mathbb{K}} \tag{1.31}$$

over the directed set $\mathscr{D}$. For instance, for $\mathscr{D} = \mathrm{c}$ the directed set of compact subsets of M one has that $\mathrm{colim}_{K\in\mathrm{c}}\,\Gamma_K(E) = \Gamma_{\mathrm{c}}(E)$ is exactly the usual vector space of compactly supported sections, consistently with notation used in Section 1.2.

Definition 1.3.1. Let $P : \Gamma(E) \to \Gamma(E)$ be a linear differential operator. A *retarded/advanced Green's operator* of P is a linear map $G^\pm : \Gamma_{\mathrm{c}}(E) \to \Gamma(E)$ which satisfies

1. $PG^\pm\psi = \psi$,
2. $G^\pm P\psi = \psi$,
3. $\mathrm{supp}(G^\pm\psi) \subseteq J^\pm_M(\mathrm{supp}\,\psi)$,

for all compactly supported sections $\psi \in \Gamma_{\mathrm{c}}(E)$.

Definition 1.3.2. A linear differential operator $P : \Gamma(E) \to \Gamma(E)$ is called *Green hyperbolic* if it admits a retarded and an advanced Green's operator G^+ and G^-.

Remark 1.3.3. From item 3 of Definition 1.3.1, and recalling the characterization of strict past/future compactness from Section 1.1, one realizes that retarded and advanced Green's operators restrict to linear maps

$$G^+ : \Gamma_{\mathrm{c}}(E) \longrightarrow \Gamma_{\mathrm{spc}}(E) \subseteq \Gamma_{\mathrm{sc}}(E)\,, \tag{1.32a}$$
$$G^- : \Gamma_{\mathrm{c}}(E) \longrightarrow \Gamma_{\mathrm{sfc}}(E) \subseteq \Gamma_{\mathrm{sc}}(E)\,, \tag{1.32b}$$

that take values in the vector spaces of strictly past compact and strictly future compact sections, respectively. ▽

The following result, due to Bär [Bär15], shows that retarded and advanced Green's operators admit an extension on a domain larger than compactly supported sections. These extended retarded and advanced Green's operators will inspire our definition of retarded and advanced Green's homotopies in Section 3.1.

Theorem 1.3.4. *Let $P : \Gamma(E) \to \Gamma(E)$ be a Green hyperbolic operator and let $G^\pm : \Gamma_{\mathrm{c}}(E) \to \Gamma(E)$ be retarded/advanced Green's operators of it. Then, there exists a unique linear extension*

$$\widetilde{G}^\pm : \Gamma_{\substack{\mathrm{(s)pc}\\ \mathrm{(s)fc}}}(E) \longrightarrow \Gamma_{\substack{\mathrm{(s)pc}\\ \mathrm{(s)fc}}}(E) \tag{1.33}$$

of $G^\pm$ such that

1. *$P\widetilde{G}^\pm\psi = \psi$,*
2. *$\widetilde{G}^\pm P\psi = \psi$,*

3. $\operatorname{supp}(\widetilde{G}^{\pm}\psi) \subseteq J^{\pm}_M(\operatorname{supp}\psi)$,

for all $\psi \in \Gamma_{\substack{\mathrm{(s)pc}\\ \mathrm{(s)fc}}}(E)$.

Remark 1.3.5. Items 1 and 2 of Theorem 1.3.4 imply that the restriction to sections with (strictly) past/future compact support of a Green hyperbolic operator $P : \Gamma_{\substack{\mathrm{(s)pc}\\ \mathrm{(s)fc}}}(E) \to \Gamma_{\substack{\mathrm{(s)pc}\\ \mathrm{(s)fc}}}(E)$ is an isomorphism of vector spaces with $\widetilde{G}^{\pm}$ as its unique inverse. In particular, this means that no non-trivial solutions with past/future compact support exist for the homogeneous equation $P\psi = 0$ and that for any (strictly) past/future compactly supported source $\psi \in \Gamma_{\substack{\mathrm{(s)pc}\\ \mathrm{(s)fc}}}(E)$ there exists a unique solution $\varphi \in \Gamma(E)$ of the differential equation $P\varphi = \psi$ with $\operatorname{supp}\varphi \subseteq J^{\pm}_M(\operatorname{supp}\psi)$. ▽

Remark 1.3.6. From Theorem 1.3.4, it also follows that retarded and advanced Green's operators $G^{\pm}$ of a Green hyperbolic operator P are unique. Indeed, such a $G^{\pm} : \Gamma_{\mathrm{c}}(E) \to \Gamma_{\substack{\mathrm{(s)pc}\\ \mathrm{(s)fc}}}(E)$ must be the restriction of the operator $\widetilde{G}^{\pm} : \Gamma_{\substack{\mathrm{(s)pc}\\ \mathrm{(s)fc}}}(E) \to \Gamma_{\substack{\mathrm{(s)pc}\\ \mathrm{(s)fc}}}(E)$ which is uniquely determined as the inverse of the isomorphism $P : \Gamma_{\substack{\mathrm{(s)pc}\\ \mathrm{(s)fc}}}(E) \to \Gamma_{\substack{\mathrm{(s)pc}\\ \mathrm{(s)fc}}}(E)$. ▽

In order to simplify the notation, and since no confusion would arise, from now on we will simply write $G^{\pm}$, in place of $\widetilde{G}^{\pm}$, to denote the extended retarded and advanced Green's operators as well.

Definition 1.3.7. Let $P : \Gamma(E) \to \Gamma(E)$ be a Green hyperbolic operator with retarded/advanced Green's operator $G^{\pm} : \Gamma_{\mathrm{c}}(E) \to \Gamma_{\mathrm{sc}}(E)$, see (1.32). Then, the difference

$$G := G^{+} - G^{-} : \Gamma_{\mathrm{c}}(E) \longrightarrow \Gamma_{\mathrm{sc}}(E) \tag{1.34}$$

is called the *retarded-minus-advanced* (or *causal*) *propagator* of P.

The following theorem concerns the exactness of a renown sequence involving the retarded-minus-advanced propagator G. That gives us some insights about the solutions of partial differential equations involving Green hyperbolic operators.

Theorem 1.3.8. *Let $P : \Gamma(E) \to \Gamma(E)$ be a Green hyperbolic operator and $G : \Gamma_{\mathrm{c}}(E) \to \Gamma_{\mathrm{sc}}(E)$ be its retarded-minus-advanced propagator. Then,*

$$0 \longrightarrow \Gamma_{\mathrm{c}}(E) \xrightarrow{\;P\;} \Gamma_{\mathrm{c}}(E) \xrightarrow{\;G\;} \Gamma_{\mathrm{sc}}(E) \xrightarrow{\;P\;} \Gamma_{\mathrm{sc}}(E) \longrightarrow 0 \tag{1.35}$$

is an exact sequence.

This result is proved for example in [BG12] and it relies on the defining properties of (extended) retarded and advanced Green's operators.

Remark 1.3.9. Recall that a sequence in an Abelian category $\mathbf{C}$

$$\dots \longrightarrow c_{i-1} \xrightarrow{f_{i-1}} c_i \xrightarrow{f_i} c_{i+1} \longrightarrow \dots, \tag{1.36}$$

$c_i \in \mathbf{C}$ and $f_i : c_i \to c_{i+1}$ in $\mathbf{C}$, $\mathrm{im}(f_{i-1}) \subseteq \ker(f_i)$, for all $i \in \mathbb{Z}$, is called *exact* when $\mathrm{im}(f_{i-1}) = \ker(f_i)$ for all $i \in \mathbb{Z}$. (In other words it is an acyclic cochain complex in $\mathbf{Ch}(\mathbf{C})$, cf. Section 2.1.) Therefore, Theorem 1.3.8 yields that (1) there are no non-trivial compactly supported solutions to $P\psi = 0$, (2) $G\psi = 0$ if and only if there exists a unique $\psi' \in \Gamma_{\mathrm{c}}(E)$ such that $\psi = P\psi'$, (3) any spacelike compact solution $\varphi \in \Gamma_{\mathrm{sc}}(E)$ to the homogeneous equation $P\varphi = 0$ is of the type $\varphi = G\psi$ for some compactly supported $\psi \in \Gamma_{\mathrm{c}}(E)$ and (4) the inhomogeneous equation $P\varphi = \varphi'$ admits a spacelike compact solution $\varphi \in \Gamma_{\mathrm{sc}}(E)$ for any spacelike compact source $\varphi' \in \Gamma_{\mathrm{sc}}(E)$. ▽

Let us provide the reader with some examples of Green hyperbolic operators. All of them have applications in mathematical physics, especially in classical and quantum field theory.

Example 1.3.10. The first, and arguably most important, class of examples is offered by *wave operators*, or *normally hyperbolic operators*. A linear second-order differential operator $P : \Gamma(E) \to \Gamma(E)$ is called *normally hyperbolic* if its principal symbol $\sigma_P(\xi) = -g_x^{-1}(\xi, \xi)\,\mathrm{id}$ is given by the (inverse of the) Lorentzian metric g for all $x \in M$ and $\xi \in T_x^*M$, where id denotes the identity on the fiber E_x. In other words, for a choice of local coordinates $x^1, \dots, x^m$ on M, and of a local trivialization of E, one has

$$P = -\sum_{i,j=1}^{m} g_x^{ij} \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^{m} A_i(x) \frac{\partial}{\partial x^i} + B(x)\,, \tag{1.37}$$

where A_i, B are matrix-valued smooth functions, for $i = 1, \dots, m$, and (g_x^{ij}) is the inverse of $(g_{x,ij} = g_x(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}))$. Bär, Ginoux and Päffle in [BGP07, Prop. 3.4.2] proved that normally hyperbolic operators admit retarded and advanced Green's operators, hence they are Green hyperbolic. This class includes many important examples. Indeed, they are normally hyperbolic the *Klein-Gordon operator* $P := \Box + \mathbf{m}^2$ of mass $\mathbf{m} \geq 0$, which acts on functions, namely $E := M \times \mathbb{K}$ is the trivial line bundle, and the *Laplace-de Rham operator* $P := \mathrm{d}_{\mathrm{dR}}\delta_{\mathrm{dR}} + \delta_{\mathrm{dR}}\mathrm{d}_{\mathrm{dR}}$ on p-forms, i.e. $E := \Lambda^p(M)$, $p = 0, \dots, m$, where d_{dR} is the de Rham differential and δ_{dR} the codifferential built out of the orientation and metric of M. In all generality, all normally hyperbolic operators are *connection-d'Alembert operators* $P = \Box^\nabla + B$ for a unique connection ∇ on E, up to a (unique) endomorphism B of E. See [BGP07, Lemma 1.5.5]. △

Example 1.3.11. A first class of Green hyperbolic operators that are not normally hyperbolic is obtained by suitably changing the metric of the background manifold. More precisely, let $(M, g) \in \mathbf{Loc}_m$ be a globally hyperbolic

Lorentzian manifold and P a normally hyperbolic operator on M. Consider another Lorentzian metric g' on the same manifold such that $(M, g') \in \mathbf{Loc}_m$ is globally hyperbolic. If the inclusion $V_{(M,g)}(x) \subseteq V_{(M,g')}(x)$ of the open light cones (see Section 1.1 for their definition) holds for all $x \in M$, then P is Green hyperbolic on (M, g'). See [MMV23, Thm. 3.14] for the proof of this fact. $\triangle$

Example 1.3.12. Another example of a Green hyperbolic operator is offered by the *Proca operator* on p-forms on a m-dimensional globally hyperbolic Lorentzian manifold M, $P := \delta_{\mathrm{dR}}\mathrm{d}_{\mathrm{dR}} + \mathbf{m}^2 : \Omega^p(M) \to \Omega^p(M)$ of mass $\mathbf{m} > 0$ (which is not normally hyperbolic for $p \geq 1$). To see this, consider the differential operator $W := \mathbf{m}^{-2}\mathrm{d}_{\mathrm{dR}}\delta_{\mathrm{dR}} + \mathrm{id} : \Omega^p(M) \to \Omega^p(M)$, then $WP = PW = \mathrm{d}_{\mathrm{dR}}\delta_{\mathrm{dR}} + \delta_{\mathrm{dR}}\mathrm{d}_{\mathrm{dR}} + \mathbf{m}^2$ is normally hyperbolic, see Example 1.3.10. Denoting by $G^\pm$ the retarded/advanced Green's operator of WP, we set $\overline{G}^\pm := WG^\pm = G^\pm W$. It is easy to see that $\overline{G}^\pm$ is the retarded/advanced Green's operator of P. Compare the role of W with that of our Green's witnesses from Chapter 4. $\triangle$

Example 1.3.13. Bär proved in [Bär15] that a differential operator P is Green hyperbolic whenever its square P^2 is Green hyperbolic. Therefore, a class of examples of Green hyperbolic operators is given by those P whose square is normally hyperbolic. They are called operators of *Dirac-type* and a primary example of them is the *classical Dirac operator* $P = \gamma \circ \nabla : \Gamma(SM) \to \Gamma(SM)$, where $E = SM$ is the spinor bundle over $M \in \mathbf{Loc}_m$, ∇ a spin connection and $\gamma : TM \to \mathrm{End}(SM)$ the Clifford multiplication. See [BG12] for further details. Note that the Dirac-type operators are first-order differential operators, in contrast to the normally hyperbolic ones and the other examples presented above. $\triangle$

Let us borrow the language and notations of Section 1.2 and let us consider a natural linear differential operator $P : \mathfrak{E} \to \mathfrak{E}$ on a natural $\mathbb{K}$-vector bundle E, whose components $P_M : \Gamma(E_M) \to \Gamma(E_M)$ are Green hyperbolic operators for all $M \in \mathbf{Loc}_m$. We denote by $G^\pm_M : \Gamma_{\mathrm{c}}(E_M) \to \Gamma(E_M)$ the associated retarded/advanced Green's operator for $M \in \mathbf{Loc}_m$. A straightforward computation shows that the diagram

$$\begin{tikzcd}
\Gamma_{\mathrm{c}}(E_{M_1}) \arrow[r, "G^\pm_{M_1}"] \arrow[d, "f_*"'] & \Gamma(E_{M_1}) \\
\Gamma_{\mathrm{c}}(E_{M_2}) \arrow[r, "G^\pm_{M_2}"'] & \Gamma(E_{M_2}) \arrow[u, "f^*"']
\end{tikzcd} \tag{1.38}$$

in $\mathbf{Vec}_{\mathbb{K}}$ commutes for all morphisms $f : M_1 \to M_2$ in $\mathbf{Loc}_m$. This follows from the fact that $f^* G^\pm_{M_2} f_*$ is a retarded/advanced Green's operator of

P_{M_1}, thanks to the uniqueness of retarded/advanced Green's operator, see Remark 1.3.6. Indeed, for all section $\psi \in \Gamma_c(E_{M_1})$ with compact support, we compute

$$1. \qquad P_{M_1}(f^* G^{\pm}_{M_2} f_*)\psi = f^* P_{M_2} G^{\pm}_{M_2} f_* \psi = f^* f_* \psi = \psi \,, \tag{1.39a}$$

where we used in the first step the naturality (1.20) of P, in the second step that $G^{\pm}_{M_2}$ is the retarded/advanced Green's operator of P_{M_2} and, finally, in the last step Equation (1.13). Similarly,

$$2. \qquad (f^* G^{\pm}_{M_2} f_*) P_{M_1} \psi = f^* G^{\pm}_{M_2} P_{M_2} f_* \psi = f^* f_* \psi = \psi \,, \tag{1.39b}$$

now exploiting (1.21) in the first step. Finally, let us check the support propagation property:

$$\begin{aligned} 3. \quad \operatorname{supp}(f^* G^{\pm}_{M_2} f_* \psi) &= f^{-1}(\operatorname{supp}(G^{\pm}_{M_2} f_* \psi)) \subseteq f^{-1} J^{\pm}_{M_2}(\operatorname{supp}(f_* \psi)) \\ &= f^{-1} J^{\pm}_{M_2} f(\operatorname{supp} \psi) = J^{\pm}_{M_1}(\operatorname{supp} \psi) \,, \end{aligned} \tag{1.39c}$$

where we used (1.9) in the first step, the support propagation property of the retarded/advanced Green's operator $G^{\pm}_{M_2}$ in the second one, the fact that $\operatorname{supp}(f_* \psi) = f(\operatorname{supp} \psi)$ by construction in the third step and, finally, that $f^{-1} J^{\pm}_{M_2} f(S) = J^{\pm}_{M_1}(S)$ for all subsets $S \subseteq M_1$. To show the last fact, note that for $x \in J^{\pm}_{M_1}(S)$ there is a future/past directed causal curve $c : [0,1] \to M_1$ such that $c(0) \in S$ and $c(1) = x$. Then, $f \circ c : [0,1] \to M_2$ is again a future/past directed causal curve because f is a time-orientation preserving isometry. Since $f \circ c(0) \in f(S)$ and $f \circ c(1) = f(x)$, $f(x) \in J^{\pm}_{M_2}(f(S))$, hence $J^{\pm}_{M_1}(S) \subseteq f^{-1} J^{\pm}_{M_2}(f(S))$. For the opposite inclusion, notice that for an $x \in f^{-1} J^{\pm}_{M_2} f(S)$, i.e. $f(x) \in J^{\pm}_{M_2}(f(S))$, there exists a future/past directed causal curve $c' : [0,1] \to M_2$ such that $c'(0) \in f(S) \subseteq f(M_1)$ and $c'(1) = f(x)$. In particular, both the endpoints of c' are in $f(M_1)$ which is causally convex by hypothesis, therefore $c'([0,1]) \subseteq f(M_1)$ and $f^{-1} \circ c'$ defines a past/future directed causal curve in M_1. Since $f^{-1} \circ c'(0) \in S$ and $f^{-1} \circ c'(1) = x$ we conclude.

The commutativity of (1.38) is equivalent to that of the hexagon (1.26) between functors $\Delta_1 \mathfrak{E}_c : \mathbf{Loc}_m^{\mathrm{op}} \times \mathbf{Loc}_m \to \mathbf{Vec}_{\mathbb{K}}$, $(M_2, M_1) \mapsto \mathfrak{E}_c(M_1) = \Gamma_c(E_{M_1})$, and $\Delta_2 \mathfrak{E} : \mathbf{Loc}_m^{\mathrm{op}} \times \mathbf{Loc}_m \to \mathbf{Vec}_{\mathbb{K}}$, $(M_2, M_1) \mapsto \mathfrak{E}(M_2) = \Gamma(E_{M_2})$, which are constant in one of their entries. To sum up, the naturality of the retarded/advanced Green's operators of a natural Green hyperbolic operator is correctly captured by saying that they arrange themselves as the components of a dinatural transformation

$$G^{\pm} : \Delta_1 \mathfrak{E}_c \longrightarrow \Delta_2 \mathfrak{E} \,. \tag{1.40}$$

The same naturality features are inherited by the retaded-minus-advanced propagators $G_M = G^+_M - G^-_M$, $M \in \mathbf{Loc}_m$, which in turn are the components of the dinatural transformation $G : \Delta_1 \mathfrak{E}_c \to \Delta_2 \mathfrak{E}$.

Given two $\mathbb{R}$-vector bundles E_1, E_2 over a globally hyperbolic Lorentzian manifold $M \in \mathbf{Loc}_m$ endowed with fiber metrics $\langle -,-\rangle_1$ and $\langle -,-\rangle_2$, one defines the integration pairings $\langle\!\langle -,-\rangle\!\rangle_1, \langle\!\langle -,-\rangle\!\rangle_2$ according to (1.25). Then, let $P : \Gamma(E_1) \to \Gamma(E_2)$ be a linear differential operator. Its *formal adjoint* is the linear differential operator $P^* : \Gamma(E_2) \to \Gamma(E_1)$ uniquely determined by the condition

$$\langle\!\langle P^*\varphi_2, \varphi_1\rangle\!\rangle_1 := \langle\!\langle \varphi_2, P\varphi_1\rangle\!\rangle_2 \,, \tag{1.41}$$

for all sections $\varphi_1 \in \Gamma(E_1), \varphi_2 \in \Gamma(E_2)$ with compactly overlapping supports, namely such that $\operatorname{supp}\varphi_1 \cap \operatorname{supp}\varphi_2 \subseteq M$ is compact. Note that the Equation (1.25b) defining the integration pairing makes sense as soon as $\langle\!\langle -,-\rangle\!\rangle$ is evaluated upon sections with compactly overlapping supports. A linear differential operator $P : \Gamma(E) \to \Gamma(E)$ acting on sections of the $\mathbb{R}$-vector bundle $E \to M$ endowed with a fiber metric $\langle -.-\rangle$ is called *formally self-adjoint* when $P^* = P$. Consider a formally self-adjoint operator P which is also Green hyperbolic and let $G^\pm$ be the associated retarded and advanced Green's operators. One has that they are one the formal adjoint of the other, namely

$$\langle\!\langle G^\pm\psi_1, \psi_2\rangle\!\rangle = \langle\!\langle \psi_1, G^\mp\psi_2\rangle\!\rangle \,, \tag{1.42}$$

for all compactly supported sections $\psi_1, \psi_2 \in \Gamma_c(E)$. This follows from the following straightforward computation,

$$\langle\!\langle G^\pm\psi_1, \psi_2\rangle\!\rangle = \langle\!\langle G^\pm\psi_1, PG^\mp\psi_2\rangle\!\rangle = \langle\!\langle PG^\pm\psi_1, G^\mp\psi_2\rangle\!\rangle = \langle\!\langle \psi_1, G^\mp\psi_2\rangle\!\rangle \,, \tag{1.43}$$

which exploits the properties of retarded/advanced Green's operators, see Definition 1.3.1, in the first and last steps and the formal self-adjointness of P in the second one. Note that $\operatorname{supp}(G^\pm\psi_1)\cap\operatorname{supp}(G^\mp\psi_2) \subseteq J^\pm_M(\operatorname{supp}\psi_1)\cap J^\mp_M(\operatorname{supp}\psi_2)$ is compact since both the supports are compact and M is globally hyperbolic, i.e. $G^\pm\psi_1$ and $G^\mp\psi_2$ have compactly overlapping supports and Equation (1.42) applies. As a consequence, the retarded-minus-advanced propagator $G := G^+ - G^-$ is *formally skew-adjoint*, that is

$$\langle\!\langle G\psi_1, \psi_2\rangle\!\rangle = -\langle\!\langle \psi_1, G\psi_2\rangle\!\rangle \,, \tag{1.44}$$

for all compactly supported sections $\psi_1, \psi_2 \in \Gamma_c(E)$. This implies that the linear map

$$\begin{aligned} \tau : \Gamma_c(E) \otimes \Gamma_c(E) &\longrightarrow \mathbb{R} \\ \psi_1 \otimes \psi_2 &\longmapsto \tau(\psi_1, \psi_2) := \langle\!\langle \psi_1, G\psi_2\rangle\!\rangle \end{aligned} \tag{1.45}$$

is antisymmetric, $\tau(\psi_1,\psi_2) = -\tau(\psi_2,\psi_1)$. (Recall that a fiber metric $\langle -,-\rangle$ is symmetric by definition.) Therefore, it descends to a *Poisson structure*[1]

[1]We will provide the reader with a precise definition of what we mean by a (possibly shifted) Poisson structure in Section 2.2.

$\tau : \Gamma_c(E)^{\wedge 2} \to \mathbb{R}$ on the space of compactly supported sections $\Gamma_c(E)$. By exploiting the exactness of sequence (1.35), this Poisson structure descends to the cokernel $\mathrm{coker}_c(P) := \mathrm{coker}(P : \Gamma_c(E) \to \Gamma_c(E)) = \Gamma_c(E)/P\Gamma_c(E)$ as a non-degenerate Poisson structure. Indeed, one has that $\tau(\psi_1, P\psi_2) = \langle\langle \psi_1, GP\psi_2 \rangle\rangle = 0$ for all $\psi_1, \psi_2 \in \Gamma_c(E)$ and that $\tau(\psi_1, \psi_2) = 0$ for all $\psi_1 \in \Gamma_c(E)$ implies $G\psi_2 = 0$, which in turn yields $\psi_2 = P\widetilde{\psi}$ for some $\widetilde{\psi} \in \Gamma_c(E)$, see Remark 1.3.9, i.e. $[\psi_2] = 0$ in $\Gamma_c(E)/P\Gamma_c(E)$.

Remark 1.3.14. In mathematical physics, in the context of classical and quantum field theory, the smooth sections of the vector bundle $E \to M$ are interpreted as the *fields* of the theory and the (linear) differential operator P describes their (free) dynamics. Therefore, one may be interested in the *on-shell observables* of the theory, that is functions on the space of fields that are solutions of the equation of motion. When the theory is free (i.e. linear), and by identifying the dual bundle E^* with E itself by means of a fiber metric $\langle -, - \rangle$, we may model the vector space of linear on-shell observables as the quotient $\Gamma_c(E)/P\Gamma_c(E)$. Results above show that if the dynamical operator P is Green hyperbolic, then the space of linear on-shell observables can be endowed with a (non-degenerate) Poisson structure τ. This plays a crucial role in the quantization of the classical field theory. $\triangledown$

Remark 1.3.15. Let us just note that when all data are naturally assigned, namely a natural $\mathbb{R}$-vector bundle E on $\mathbf{Loc}_m$ endowed with a natural fiber metric $\langle -, - \rangle$, and a natural differential operator P, whose components P_M are all formally self-adjoint with respect to $\langle -, - \rangle_M$, are given, the Poisson structures $\tau_M := \langle\langle -, G_M - \rangle\rangle_M$ assemble into a natural transformation $\tau : \mathfrak{E}_c \otimes \mathfrak{E}_c \to \Delta\mathbb{R}$ to the constant functor $\Delta\mathbb{R}$, as one can easily see by exploiting the naturality of G and that of the integral pairing, Equations (1.38) and (1.29), respectively. $\triangledown$

In the remainder of this section we shall restrict ourselves to the case of a normally hyperbolic linear differential operator P, see Example 1.3.10. Recall that $P = \Box^\nabla + B$ is the connection-d'Alembert operator for a connection ∇ on the vector bundle $E \to M$ plus an endomorphism field of E. An important feature of normally hyperbolic operators is that they have a well-posed Cauchy problem on globally hyperbolic Lorentzian manifolds. The following theorem, whose proof takes up most of Section 3.2 in [BGP07], states precisely this fact.

Theorem 1.3.16. *Let $M \in \mathbf{Loc}_m$ be a globally hyperbolic Lorentzian manifold and $\iota : \Sigma \hookrightarrow M$ a spacelike Cauchy surface of it. Denote by $\mathfrak{n}$ the future directed timelike unit vector field normal to Σ. Let then $E \to M$ be a vector bundle on M and $P = \Box^\nabla + B : \Gamma(E) \to \Gamma(E)$ be a normally hyperbolic operator. Then, for any compactly supported sections $\psi \in \Gamma_c(E)$ and $\varphi_0, \varphi_1 \in \Gamma_c(\iota^* E)$ there exists a unique solution $\varphi \in \Gamma(E)$ of the Cauchy* *problem*

$$\begin{cases} P\varphi = \psi \\ \varphi|_\Sigma = \varphi_0 \\ \nabla_{\mathfrak{n}}\varphi|_\Sigma = \varphi_1 \,. \end{cases} \tag{1.46}$$

Moreover, $\operatorname{supp}\varphi \subseteq J_M(\operatorname{supp}\psi \cup \operatorname{supp}\varphi_0 \cup \operatorname{supp}\varphi_1)$*.*

Note that, since they can be of different order, see Example 1.3.13, it is not clear how to generalize this result to Green hyperbolic operators.

Suppose that the vector bundle E comes endowed with a fiber metric $\langle -,-\rangle$ and the P-compatible connection ∇ is *metric* with respect to it, that is $\langle \nabla_V\varphi_1, \varphi_2\rangle + \langle \varphi_1, \nabla_V\varphi_2\rangle = \partial_V\langle\varphi_1,\varphi_2\rangle$ for all sections $\varphi_1, \varphi_2 \in \Gamma(E)$ and vector fields $V \in \Gamma(TM)$. Moreover, assume that the endomorphism B is self-adjoint with respect to $\langle -,-\rangle$. Then, the normally hyperbolic operator $P = \Box^\nabla + B$ is formally self-adjoint. In this setting, we pick a spacelike smooth Cauchy surface $\Sigma \subseteq M$ with future directed timelike unit vector field $\mathfrak{n}$, and define the *fixed-time Poisson structure*

$$\begin{aligned} \sigma_\Sigma : \ker_{\mathrm{sc}}(P) \wedge \ker_{\mathrm{sc}}(P) &\longrightarrow \mathbb{R} \\ \varphi_1 \wedge \varphi_2 &\longmapsto \sigma_\Sigma(\varphi_1,\varphi_2) := -\int_\Sigma \big(\langle\varphi_1, \nabla_{\mathfrak{n}}\varphi_2\rangle - \langle\varphi_2, \nabla_{\mathfrak{n}}\varphi_1\rangle\big)\big|_\Sigma \operatorname{vol}_\Sigma \end{aligned} \tag{1.47}$$

on the kernel $\ker_{\mathrm{sc}}(P) := \ker(P : \Gamma_{\mathrm{sc}}(E) \to \Gamma_{\mathrm{sc}}(E))$ of the restriction of P to spacelike compact sections, where $\operatorname{vol}_\Sigma$ denotes the induced volume form on Σ. Equation (1.47) is explicitly antisymmetric, therefore σ_Σ is a well-defined Poisson structure. Note also that the integral is finite since sections φ_1, φ_2 are spacelike compact, hence compactly supported when restricted to a spacelike Cauchy surface.

The exact sequence (1.35) yields that the retarded-minus-advanced propagator G descends to an isomorphism

$$G : \operatorname{coker}_{\mathrm{c}}(P) \longrightarrow \ker_{\mathrm{sc}}(P) \tag{1.48}$$

of vector spaces. Furthermore, it is compatible with Poisson structures, in that $\sigma_\Sigma \circ G^{\wedge 2} = \tau$, as one can see by integrating by parts twice, exploiting that $P = \Box^\nabla + B$ with ∇ a metric connection and B a self-adjoint endomorphism of E, and recalling that $PG^{\pm}\psi = \psi$ for all $\psi \in \Gamma_c(E)$:

$$\begin{aligned}
\tau(\psi_1,\psi_2) &= \int_M \langle \psi_1, G\psi_2\rangle \,\mathrm{vol}_M \\
&= \int_{J_M^+(\Sigma)} \langle PG^-\psi_1, G\psi_2\rangle \,\mathrm{vol}_M + \int_{J_M^-(\Sigma)} \langle PG^+\psi_1, G\psi_2\rangle \,\mathrm{vol}_M \\
&= -\int_\Sigma \langle \nabla_{\mathfrak{n}} G^-\psi_1, G\psi_2\rangle\big|_\Sigma \,\mathrm{vol}_\Sigma + \int_\Sigma \langle G^-\psi_1, \nabla_{\mathfrak{n}} G\psi_2\rangle\big|_\Sigma \,\mathrm{vol}_\Sigma \\
&\quad + \int_\Sigma \langle \nabla_{\mathfrak{n}} G^+\psi_1, G\psi_2\rangle\big|_\Sigma \,\mathrm{vol}_\Sigma - \int_\Sigma \langle G^+\psi_1, \nabla_{\mathfrak{n}} G\psi_2\rangle\big|_\Sigma \,\mathrm{vol}_\Sigma \\
&= -\int_\Sigma \langle G\psi_1, \nabla_{\mathfrak{n}} G\psi_2\rangle\big|_\Sigma \,\mathrm{vol}_\Sigma + \int_\Sigma \langle G\psi_2, \nabla_{\mathfrak{n}} G\psi_1\rangle\big|_\Sigma \,\mathrm{vol}_\Sigma \\
&= \sigma_\Sigma(G\psi_1, G\psi_2)\,,
\end{aligned} \tag{1.49}$$

where we have picked a Cauchy surface $\Sigma \subseteq M$ and used that $M = J_M^+(\Sigma) \cup J_M^-(\Sigma)$ in the second step. This makes evident that the fixed-time Poisson structure σ_Σ from Equation (1.47) does not depend on the choice of the spacelike Cauchy surface Σ, $\sigma_\Sigma = \sigma_{\Sigma'}$ for all $\Sigma, \Sigma' \subseteq M$ smooth spacelike Cauchy surfaces, as one can also see by a direct computation exploiting that only solutions to equation of motion $P\varphi = 0$ are considered. The map G then upgrades to an isomorphism in the category $\mathbf{PoVec}_{\mathbb{R}}$ of Poisson vector spaces, i.e. $\mathbb{R}$-vector spaces endowed with a Poisson structure, and linear maps compatible with the Poisson structures between them.

Remark 1.3.17. Since the isomorphism (1.48) is also compatible with the Poisson structures one realizes that the Poisson vector space $(\ker_{\mathrm{sc}}(P), \sigma_\Sigma)$ of the spacelike compact solutions to the equation of motion with the fixed-time Poisson structure is, when available, a model of the linear on-shell observables equivalent to $(\mathrm{coker}_c(P), \tau)$ from Remark 1.3.14. We would like to mention that in this representation the space of linear observables is 'localized' on a Cauchy surface and the initial data of fields on it are considered, cf. Theorem 1.3.16. Hence, the model it provides is closer in spirit to Hamiltonian formalism. ▽

1.4 AQFTs and tPFAs

Algebraic quantum field theory (AQFT) [HK64; BDFY15; FV15; BSW21; BF23] and factorization algebras [CG17; BPS20; CG21b] are axiomatic frameworks which describe the algebraic structure on the observables of classical and quantum field theories in several geometric settings. In this section we will recall some basic concepts about these in the Lorentzian case.

First, we shall consider $\mathbf{M}$-valued AQFTs and (time-orderable) prefactorization algebras, for an arbitrary bicomplete closed symmetric monoidal category $\mathbf{M}$. In this setting, we shall recall the *(strict) time-slice axiom* for both frameworks. Afterwards, we will specialize to the case of a homotopical category $\mathbf{M}$. Indeed, in the main part of this thesis we will be interested in the case of the category $\mathbf{M} = \mathbf{Ch}_{\mathbb{K}}$ of (unbounded) cochain complexes on a field $\mathbb{K}$ of characteristic zero, endowed with the tensor product monoidal structure. The category $\mathbf{Ch}_{\mathbb{K}}$ is a homotopical category when endowed with the weak equivalences given by the quasi-isomorphisms. This choice is motivated by recent developments [Yau19; BS19; BBS20; BSW21; Car23] in the study of higher structures in quantum field theory. From a physical point of view the presence of higher structures in quantum field theory, in particular in gauge field theory, has been known for a long time in the cohomological approach of BV/BRST formalism [HT92; BBH95]. A consistent treatment of these higher structures requires us to work in a higher categorical framework. For this reason, we will also provide a relaxation of the time-slice axiom, which besides the fact that it is conceptually more consistent with the homotopical category approach, it is also the one that is satisfied by the field theoretical examples of linear gauge theories, see e.g. [BBS20; BGS23; BMS24] and Chapter 6.

Let us start our review with the definition of an algebraic quantum field theory on globally hyperbolic Lorentzian manifolds. Recall from Definition 1.1.8 the notion of causally disjoint pair of morphisms in $\mathbf{Loc}_m$.

Definition 1.4.1. Let $\mathbf{M} = (\mathbf{M}, \otimes, I, \gamma)$ be a bicomplete closed symmetric monoidal category. A $\mathbf{M}$*-valued AQFT on globally hyperbolic Lorentzian manifolds* is a functor $\mathcal{A} : \mathbf{Loc}_m \to \mathbf{Mon}(\mathbf{M})$ to the category of monoids in $\mathbf{M}$ such that the *Einstein causality axiom* is satisfied, that is the diagram

$$\begin{tikzcd} \mathcal{A}(M_1) \otimes \mathcal{A}(M_2) \arrow[r, "\mathcal{A}(f_1)\otimes\mathcal{A}(f_2)"] \arrow[d, "\mathcal{A}(f_1)\otimes\mathcal{A}(f_2)"'] & \mathcal{A}(N) \otimes \mathcal{A}(N) \arrow[d, "\mu_N"] \\ \mathcal{A}(N) \otimes \mathcal{A}(N) \arrow[r, "\mu_N^{\mathrm{op}}"'] & \mathcal{A}(N) \end{tikzcd} \tag{1.50}$$

in $\mathbf{M}$ commutes for any pair $f_1 : M_1 \to N \leftarrow M_2 : f_2$ of causally disjoint morphisms in $\mathbf{Loc}_m$. Here, μ_N denotes the multiplication of the monoid $\mathcal{A}(N)$ while $\mu_N^{\mathrm{op}} := \mu_N \circ \gamma$ is the opposite multiplication.

Remark 1.4.2. For the convenience of the reader, we briefly recall that a bicomplete symmetric monoidal category $(\mathbf{M}, \otimes, I, \gamma)$ is the datum of a category $\mathbf{M}$ which admits all small limits and colimits together with a functor $\otimes : \mathbf{M}\times\mathbf{M} \to \mathbf{M}$, called the *tensor product*, a *unit object* $I \in \mathbf{M}$, and natural isomorphisms $\alpha_{m_1,m_2,m_3} : (m_1 \otimes m_2) \otimes m_3 \to m_1 \otimes (m_2 \otimes m_3)$ (*associator*), $\rho_{m_1} : m_1 \otimes I \to m_1$ (*right unitor*), $\lambda_{m_1} : I \otimes m_1 \to m_1$ (*left unitor*) and $\gamma_{m_1,m_2} : m_1 \otimes m_2 \to m_2 \otimes m_1$ (*braiding*), subject to suitable compatibility axioms and such that the braiding is *symmetric*, i.e. $\gamma_{m_1,m_2} \circ \gamma_{m_2,m_1} = \mathrm{id}$. A monoidal category is said to be *closed* if, for all $m_1 \in \mathbf{M}$, the functor $- \otimes m_1 : \mathbf{M} \to \mathbf{M}$ which takes tensor products with $m_1 \in \mathbf{M}$ has a right adjoint functor $[m_1, -] : \mathbf{M} \to \mathbf{M}$, which takes *internal homs* out of m_1. A *monoid* (a, μ, η) in the monoidal category $\mathbf{M}$ is given by an object $a \in \mathbf{M}$ and morphisms $\mu : a \otimes a \to a$, called *multiplication*, and $\eta : I \to a$, called *unit* in $\mathbf{M}$, subject to the *associative* and *unit laws*. $\triangledown$

Remark 1.4.3. Note that in the definition of an AQFT $\mathcal{A}$ we did not mention any particular feature of the background category $\mathbf{Loc}_m$ but the existence of the distinguished class of causally disjoint morphisms. Indeed, it is possible to abstract away from this choice introducing the notion of orthogonal categories $(\mathbf{C}, \perp)$, see [BSW21; BS23], that are (small) categories $\mathbf{C}$ with a distinguished set $\perp \subseteq \mathrm{Mor}(\mathbf{C})\,{}_{\mathrm{t}}{\times}_{\mathrm{t}}\,\mathrm{Mor}(\mathbf{C})$ of pairs of morphisms in $\mathbf{C}$ with the same target $f_1 : c_1 \to c \leftarrow c_2 : f_2$, subject to suitable conditions. Then, an AQFT on $\mathbf{C}$ is a functor $\mathcal{A} : \mathbf{C} \to \mathbf{Mon}(\mathbf{M})$ which is asked to satisfy the condition (1.50) for all pairs $(f_1, f_2) \in \perp$. In the following we will not need such a level of generality since we will only consider AQFTs on $\mathbf{Loc}_m$, nevertheless the flexibility offered by orthogonal categories allows several flavors of quantum field theory to be treated along the same lines, see also [Car23]. Let us just mention that when $(\mathbf{C}, \perp)$ is the category of oriented Riemannian m-manifold with orientation preserving isometric open embeddings as morphisms and the orthogonality relation given by having disjoint image, one gets Euclidean quantum field theory, while when $\mathbf{C} = \mathbf{Loc}_m \downarrow M$ is the slice category for a $M \in \mathbf{Loc}_m$ and $\perp$ is given again by causally disjoint subregions, one gets the nets of algebras à la Haag and Kastler [HK64] on the globally hyperbolic Lorentzian manifold M. $\triangledown$

Given two $\mathbf{M}$-valued AQFTs $\mathcal{A}_1$ and $\mathcal{A}_2$, a morphism $\kappa : \mathcal{A}_1 \to \mathcal{A}_2$ between them is a natural transformation. Hence, we define the category of $\mathbf{M}$-valued AQFTs on $\mathbf{Loc}_m$ as the full subcategory

$$\mathbf{AQFT}_m(\mathbf{M}) \subseteq \mathbf{Mon}(\mathbf{M})^{\mathbf{Loc}_m} \tag{1.51}$$

of the category of functors from $\mathbf{Loc}_m$ to $\mathbf{Mon}(\mathbf{M})$ and natural transformations, given by all the objects which fulfill the Einstein causality axiom.

The interpretation of the AQFT $\mathcal{A}$ is immediate. It gives a law to assign to each globally hyperbolic Lorentzian manifold $M \in \mathbf{Loc}_m$ the algebra of the observables of the field theory on it, $\mathcal{A}(M) \in \mathbf{Mon}(\mathbf{M})$. The Einstein causality axiom imposes that this assignment is compatible with a notion of causality, in that observables assigned to regions that are causally disjoint must commute. Up to this point, no dynamical information is encoded in the definition of an AQFT. The task of implementing a well-behaved dynamics at the level of observables is carried out by the *time-slice axiom*, which formalizes the idea that a field theory that admits a well-posed evolution problem is such that it assigns equivalent algebras of observables to regions containing the same spacelike Cauchy surface.

As previously said, this idea can be formalized both in a strict and, when $\mathbf{M}$ is a homotopical category, in a weak sense. The two formulations are, in general, not equivalent. Here, we start with the strict version, while later on in this section we will introduce the weak one, which is the one we will actually need in the main part of this work.

Definition 1.4.4. Let $\mathbf{M}$ be a bicomplete closed symmetric monoidal category and $\mathcal{A} \in \mathbf{AQFT}_m(\mathbf{M})$ a $\mathbf{M}$-valued AQFT on $\mathbf{Loc}_m$. We say that $\mathcal{A}$ satisfies the *time-slice axiom* if for any Cauchy morphism $f : M \to N$ in $\mathbf{Loc}_m$, see Definition 1.1.7, the associated morphism

$$\mathcal{A}(f) : \mathcal{A}(M) \xrightarrow{\cong} \mathcal{A}(N) \tag{1.52}$$

in $\mathbf{Mon}(\mathbf{M})$ is an isomorphism.

Let us now consider prefactorization algebras. First, let us note that the standard setting in which (pre)factorization algebras are considered [CG17; CG21b] is that of topological or Riemannian manifolds. While this is totally satisfactory for studying topological or Euclidean field theories, it is not so for our purposes since we work in a Lorentzian setting. In particular, one of our goals is to produce a comparison between an AQFT and a prefactorization algebra description of a certain class of field theories on $\mathbf{Loc}_m$. Therefore, we need a Lorentzian variant of prefactorization algebras, namely the *time-orderable prefactorization algebras* from [BPS20].

Definition 1.4.5. Let $\mathbf{M}$ be a bicomplete closed symmetric monoidal category. A $\mathbf{M}$*-valued time-orderable prefactorization algebra (tPFA)* $\mathcal{F}$ on $\mathbf{Loc}_m$ is given by the following data:

i. For each $M \in \mathbf{Loc}_m$ the assignment of an object $\mathcal{F}(M) \in \mathbf{M}$;

ii. For any n-tuple, $n \geq 1$, $\underline{f} = (f_1, \dots, f_n) : \underline{M} \to N$ of time-orderable morphisms in $\mathbf{Loc}_m$, see Definition 1.1.9, the assignment of a morphism, called *time-ordered product*, $\mathcal{F}(\underline{f}) : \mathcal{F}(\underline{M}) := \bigotimes_{i=1}^n \mathcal{F}(M_i) \to \mathcal{F}(N)$. To the empty tuple $\emptyset \to N$ it is associate a morphism $I \to \mathcal{F}(N)$ in $\mathbf{M}$ from the monoidal unit;

They are subject to the axioms listed below:

1. For all time-orderable tuples $\underline{f} = (f_1, \dots, f_n) : \underline{M} \to N$ and $\underline{f'}_i = (f'_{i1}, \dots, f'_{ik_i}) : \underline{L}_i \to M_i$, $i = 1, \dots, n$, in $\mathbf{Loc}_m$ the diagram

$$\begin{tikzcd} \bigotimes_{i=1}^n \mathcal{F}(\underline{L}_i) \arrow[r, "\bigotimes_{i=1}^n \mathcal{F}(\underline{f'}_i)"] \arrow[dr, "\mathcal{F}(\underline{f}(\underline{f'}_1, \dots, \underline{f'}_n))"'] & \mathcal{F}(\underline{M}) \arrow[d, "\mathcal{F}(\underline{f})"] \\ & \mathcal{F}(N) \end{tikzcd} \tag{1.53}$$

in $\mathbf{M}$ commutes, where the notation for the composite tuple $\underline{f}(\underline{f}'_1, \dots, \underline{f}'_n)$ is the one from Lemma 1.1.11;

2. For all $M \in \mathbf{Loc}_m$, $\mathcal{F}(\mathrm{id}_M) = \mathrm{id}_{\mathcal{F}(M)} : \mathcal{F}(M) \to \mathcal{F}(M)$ in $\mathbf{M}$ is the identity;

3. For all time-orderable n-tuples $\underline{f} : \underline{M} \to N$ in $\mathbf{Loc}_m$ and permutations $\sigma \in \Sigma_n$, the diagram

$$\begin{tikzcd} \mathcal{F}(\underline{M}) \arrow[r, "\mathcal{F}(\underline{f})"] \arrow[d, "\gamma_\sigma"'] & \mathcal{F}(N) \\ \mathcal{F}(\underline{M}\sigma) \arrow[ur, "\mathcal{F}(\underline{f}\sigma)"'] & \end{tikzcd} \tag{1.54}$$

in $\mathbf{M}$ commutes, where γ_σ denotes the action of the permutation σ on tensor powers $\bigotimes_{i=1}^n \mathcal{F}(M_i)$ via the symmetric braiding γ.

Remark 1.4.6. In the definition of a tPFA we implicitly used the facts stated in Lemma 1.1.11. The crucial role played by these is to guarantee that time-orderable tuples can be composed producing another time-orderable tuple, item ii of the lemma, and that they carry permutation actions, item i, which, respectively, are required to make sense of axioms 1 and 3 of Definition 1.4.5. $\triangledown$

Given two $\mathbf{M}$-valued tPFAs $\mathcal{F}_1, \mathcal{F}_2$ on $\mathbf{Loc}_m$, a morphism $\zeta : \mathcal{F}_1 \to \mathcal{F}_2$ between them is given by a collection $\zeta = (\zeta_M)_{M \in \mathbf{Loc}_m}$ of morphisms $\zeta_M : \mathcal{F}_1(M) \to \mathcal{F}_2(M)$ in $\mathbf{M}$ for $M \in \mathbf{Loc}_m$ which are compatible with the time-ordered products, namely for any time-orderable tuple $\underline{f} : \underline{M} \to N$ in $\mathbf{Loc}_m$ the diagram

$$\begin{tikzcd} \mathcal{F}_1(\underline{M}) \arrow[r, "\mathcal{F}_1(\underline{f})"] \arrow[d, "\zeta_{\underline{M}}"'] & \mathcal{F}_1(N) \arrow[d, "\zeta_N"] \\ \mathcal{F}_2(\underline{M}) \arrow[r, "\mathcal{F}_2(\underline{f})"'] & \mathcal{F}_2(N) \end{tikzcd} \tag{1.55}$$

in $\mathbf{M}$ commutes, where $\zeta_{\underline{M}} := \bigotimes_i \zeta_{M_i}$. We denote by $\mathbf{tPFA}_m(\mathbf{M})$ the category whose objects are all $\mathbf{M}$-valued time-orderable prefactorization algebras on $\mathbf{Loc}_m$ and whose morphisms are the ones described above.

We introduce also a time-slice axiom, or a *Cauchy constancy axiom* by using a terminology borrowed from factorization algebra literature [CG17; CG21b], for $\mathbf{M}$-valued prefactorization algebras on $\mathbf{Loc}_m$. The strict version is defined as follows.

Definition 1.4.7. Let $\mathbf{M}$ be a bicomplete closed symmetric monoidal category and $\mathcal{F} \in \mathbf{tPFA}_m(\mathbf{M})$ be a $\mathbf{M}$-valued time-orderable prefactorization algebra on $\mathbf{Loc}_m$. We say that $\mathcal{F}$ satisfies the *time-slice axiom* (or, equivalently, that is *Cauchy constant*) when $\mathcal{F}(f) : \mathcal{F}(M) \xrightarrow{\cong} \mathcal{F}(N)$ is an isomorphism in $\mathbf{M}$ for all Cauchy morphisms $f : M \to N$ in $\mathbf{Loc}_m$, see Definition 1.1.7.

In applications to gauge field theory in the BV formalism, the strict time-slice axiom, given in terms of isomorphisms, may be too strong a requirement, and it may not be satisfied even by simple models, see e.g. [BBS20; BGS23; BMS24]. Moreover, the same applications suggest that isomorphisms are not the right way to compare models. Indeed, there are models which are not isomorphic but still must be considered as equivalent since they describe the same physical theory. For instance, this is the case of the model for linear Yang-Mills theory provided by Section 6.3 and models which include the so-called *auxiliary fields*. Both these issues may be overcome at once by relaxing the way we compare objects: instead of using isomorphisms we will need to use something weaker. A possible way to make this idea rigorous is to use an ∞-categorical language, regarding our target categories as suitably presenting ∞-categories. For our purposes, it will be enough to require that the category $\mathbf{M}$ is a *homotopical category* in the sense of [DHKS04]. This means to endow $\mathbf{M}$ with the choice of a wide subcategory, whose morphisms are called *weak equivalences*, such that all isomorphisms are included and that the weak equivalences satisfy the 2-out-of-6 property (i.e. for any triple of composable morphisms f, g, h such that fg and gh are weak equivalences, also f, g, h and fgh are so). Then, weak equivalences replace, and extend, isomorphisms as the right way to compare objects of $\mathbf{M}$.

Going back to applications to gauge field theory, BV formalism requires us to work in the category $\mathbf{M} = \mathbf{Ch}_{\mathbb{K}}$ of cochain complexes, see Section 2.1, which is a homotopical category (even more, it is a model category, see Section 2.1, and [Hov99] for a deeper analysis) whose weak equivalences are the quasi-isomorphisms. In this setting, by replacing isomorphisms with quasi-isomorphisms the issues above are solved. This is why we introduce the notion of *homotopy time-slice axiom*. While in the following we will stick with the category $\mathbf{Ch}_{\mathbb{K}}$ of cochain complexes, here we will still remain a bit more general.

First, let us note that given a homotopical category $\mathbf{M}$, there is an induced choice of weak equivalences for the category $\mathbf{Mon}(\mathbf{M})$ of its monoids. Indeed, one says that a morphism $f : a_1 \to a_2$ in $\mathbf{Mon}(\mathbf{M})$ is a weak equivalence if and only if it is so when regarded as a morphism in $\mathbf{M}$. With this choice of weak equivalences $\mathbf{Mon}(\mathbf{M})$ becomes a homotopical category. (Note that the homotopical category axioms, i.e. isomorphisms being weak equivalences and the 2-out-of-6 property, can be both checked down at the level of the underlying category $\mathbf{M}$.)

Definition 1.4.8. Let $\mathbf{M}$ be a closed symmetric monoidal homotopical category. An $\mathbf{M}$-valued AQFT $\mathcal{A} \in \mathbf{AQFT}_m(\mathbf{M})$ on $\mathbf{Loc}_m$ is said to satisfy the *homotopy time-slice axiom* if for any Cauchy morphism $f : M \to N$ in $\mathbf{Loc}_m$, see Definition 1.1.7, the morphism

$$\mathcal{A}(f) : \mathcal{A}(M) \xrightarrow{\sim} \mathcal{A}(N) \,, \tag{1.56}$$

in $\mathbf{Mon}(\mathbf{M})$ is a weak equivalence.

Similarly, for a tPFA we give the following definition.

Definition 1.4.9. Let $\mathbf{M}$ be a closed symmetric monoidal homotopical category. A $\mathbf{M}$-valued tPFA $\mathcal{F} \in \mathbf{tPFA}_m(\mathbf{M})$ on $\mathbf{Loc}_m$ satisfies the *homotopy time-slice axiom* (it is *homotopy Cauchy constant*) if

$$\mathcal{F}(f) : \mathcal{F}(M) \xrightarrow{\sim} \mathcal{F}(N) \tag{1.57}$$

is a weak equivalence in $\mathbf{M}$ for all Cauchy morphisms $f : M \to N$ in $\mathbf{Loc}_m$, see Definition 1.1.7.

Remark 1.4.10. In the previous definitions we exploited the very flexible notion of homotopical category in order to introduce the homotopy time-slice axiom. The flexibility of this framework has as its drawback a lack of control on performing constructions and controlling notions up to weak equivalences. A more powerful framework is offered by model category theory, where in addition to weak equivalences, other two compatible subclasses of morphisms, called fibrations and cofibrations, have to be declared. These two auxiliary classes of morphisms allows for more controlled constructions. Note, in particular, that a model category is also a homotopical category when fibrations and cofibrations are forgotten and only the weak equivalences are kept. The issue is that, even if $\mathbf{M}$ is a (closed symmetric) monoidal model category, it is not, in general, possible to endow the category $\mathbf{Mon}(\mathbf{M})$ of the monoids of it with a model structure inherited from the one of $\mathbf{M}$. Therefore, requiring that $\mathbf{M}$ is a model category is not enough to give sense to the Definition 1.4.8. Hence the necessity to invoke the weaker homotopical categories.

Nevertheless, in the main part of this thesis we will work with the category $\mathbf{M} = \mathbf{Ch}_{\mathbb{K}}$ of cochain complexes which is a model category when endowed with the projective model structure, see Section 2.1 and [Hov99]. This will allow us to enjoy the more controlled constructions made available by model category theory when dealing with cochain complexes. Since $\mathbf{Ch}_{\mathbb{K}}$ is a model category, it is in particular a homotopical category, hence the category $\mathbf{Mon}(\mathbf{Ch}_{\mathbb{K}}) = \mathbf{dgAlg}_{\mathbb{K}}$ of differential graded algebras is a homotopical category as well. Moreover, the category $\mathbf{dgAlg}_{\mathbb{K}}$ is a model category with the model structure inherited from the projective one on $\mathbf{Ch}_{\mathbb{K}}$, see [Hin97, Thm. 4.1.1]. In the following we will not need this extra structure layer, for our purposes it will be enough to regard $\mathbf{dgAlg}_{\mathbb{K}}$ as a homotopical category. ▽

Recall that the time-slice axiom formalizes a notion of well-posed dynamics for a field theory by requiring that its observables are determined everywhere just by their behavior around a spacelike Cauchy surface. The homotopy time-slice axiom makes this idea consistent with the slogan that weak equivalent objects in a homotopical category should be considered as 'being the same', since it imposes that there is a weak equivalence between the spaces of observables assigned to regions which share a Cauchy surface.

Let us now review a construction from [BPS20] which produces a ((homotopy) Cauchy constant) tPFA from an AQFT (satisfying the (homotopy) time-slice axiom). We will need this construction in Section 5.3 to compare our $\mathbf{Ch}_{\mathbb{K}}$-valued AQFTs and tPFAs. Let us note that AQFTs and tPFAs encode quite different algebraic structures on the space of observables of a field theory. An AQFT assigns to each $M \in \mathbf{Loc}_m$ the algebra of observables on it, while a tPFA assigns just a space (an object in $\mathbf{M}$) of observables on M, without an algebra structure. The operations between observables that the tPFA implements are the time-ordered products, which however are assigned only between observables pertaining to regions that are time-orderable. It follows that it is not clear whether it is possible to define an algebra structure on the space of observables out of the data provided by a tPFA, or under what conditions. (See [BPS20] for a positive answer when the strict time-slice axiom and a suitable descent condition, therein called *additivity*, are assumed.) The other way around, however, is easier and it corresponds to the well-known construction of the time-ordered products associated with an AQFT.

Theorem 1.4.11. *Let $\mathbf{M}$ be a bicomplete closed symmetric monoidal category and let $\mathcal{A} \in \mathbf{AQFT}_m(\mathbf{M})$ be a $\mathbf{M}$-valued AQFT on $\mathbf{Loc}_m$. Let us assign the object*

$$\mathcal{F}_{\mathcal{A}}(M) := \mathcal{A}(M) \in \mathbf{M}\,, \tag{1.58a}$$

underlying the monoid $\mathcal{A}(M)$, to all $M \in \mathbf{Loc}_m$, and the morphism

$$\begin{tikzcd}
\mathcal{F}_{\mathcal{A}}(\underline{M}) = \bigotimes_{i=1}^n \mathcal{A}(M_i) \arrow[r, dashed, "\mathcal{F}_{\mathcal{A}}(\underline{f})"] \arrow[d, "\gamma_\rho"'] & \mathcal{F}_{\mathcal{A}}(N) = \mathcal{A}(N) \\
\bigotimes_{i=1}^n \mathcal{A}(M_{\rho(i)}) \arrow[r, "\bigotimes_i^n \mathcal{A}(f_{\rho(i)})"'] & \mathcal{A}(N)^{\otimes n} \arrow[u, "\mu_N^{(n)}"']
\end{tikzcd} \tag{1.58b}$$

in $\mathbf{M}$, to any time-orderable n-tuple $\underline{f} : \underline{M} \to N$ in $\mathbf{Loc}_m$, where ρ is a time-ordering permutation for the tuple $\underline{f}$ and $\mu_N^{(n)}$ is the n-ary multiplication in the associative and unital monoid $\mathcal{A}(N)$ in the given order. To the empty tuple $\emptyset \to N$ it is assigned the unit $\eta_N : I \to \mathcal{F}_{\mathcal{A}}(N) = \mathcal{A}(N)$. Then, $\mathcal{F}_{\mathcal{A}} \in \mathbf{tPFA}_m(\mathbf{M})$ is a $\mathbf{M}$-valued tPFA on $\mathbf{Loc}_m$. This assignment extends to a functor

$$\mathcal{F}_{(-)} : \mathbf{AQFT}_m(\mathbf{M}) \longrightarrow \mathbf{tPFA}_m(\mathbf{M})\,, \tag{1.59}$$

from the category of $\mathbf{M}$*-valued algebraic quantum field theories, valued in the category of time-orderable prefactorization algebras. Furthermore, it preserves the property of satisfying the (homotopy) time-slice axiom.*

The proof of the theorem can be found in [BPS20] and it is based on the facts recalled in Lemma 1.1.11: More precisely, item i implies that the assigned $\mathcal{F}_{\mathcal{A}}(\underline{f})$ are equivariant, see Equation (1.54), and item ii that they compose in the correct way, see Equation (1.53). Item iii, together with the Einstein causality axiom for $\mathcal{A} \in \mathbf{AQFT}_m(\mathbf{M})$, yields that the morphism $\mathcal{F}_{\mathcal{A}}(\underline{f})$ in $\mathbf{M}$ is well-defined for all time-orderable tuples $\underline{f}$ since it does not depend on the choice of the time-ordering permutation ρ. In [BPS20] it is only proved that AQFTs which satisfy the time-slice axiom are sent to Cauchy constant tPFAs, since there the homotopy time-slice axiom was not considered at all. Nevertheless, it is immediate to see that the claim extends to homotopy time-slice axiom since $\mathcal{F}_{\mathcal{A}}(f) = \mathcal{A}(f) : \mathcal{A}(M) \to \mathcal{A}(N)$ and the weak equivalences in $\mathbf{Mon}(\mathbf{M})$ are detected on the underlying homotopical category $\mathbf{M}$.

Chapter 2

On homological algebra

In this chapter we shall discuss some notions and develop constructions that will play a paramount role in the main part of this thesis, particularly in Chapter 3 to introduce *retarded and advanced Green's homotopies* (Section 3.1) and to prove their main features, as uniqueness up to a contractible space of choices (Section 3.2). The topics covered by this chapter, although in general not totally new, have different degrees of technicality. Section 2.1 will deal with the basics of the theory of cochain complexes over a field $\mathbb{K}$ of characteristic zero. This is a well-known topic and it is widely covered by the literature, see the textbook [Wei94]. We will use this section to fix our notation and conventions and, in particular, we will focus our attention on the (projective) model structure on the category $\mathbf{Ch}_{\mathbb{K}}$ of cochain complexes over $\mathbb{K}$, see [Hov99]. In Section 2.2 we shall review the notion of a *shifted Poisson structure* on a cochain complex and we will extend shifted (anti)symmetric pairings on cochain complexes to the free symmetric dg-algebra. In Section 2.3 we will consider the category $\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ of cochain complex-valued functors from a (small) category $\mathbf{C}$. We will show that it may be enriched over the category $\mathbf{Ch}_{\mathbb{K}}$ and we will provide the reader with models to compute homotopy colimits on it. Moreover, we will show that our model for homotopy colimits extends to a dg-functor and that our dg-enrichment is such that an enriched adjunction $\mathrm{hocolim} \dashv \Delta$ between the homotopy colimit dg-functor hocolim and the diagonal dg-functor Δ holds true. Finally, in Section 2.4 we will review the notion of Kan complexes and we will provide a construction to associate a Kan complex with any cochain complex. Furthermore, we will provide an argument which shows, under the condition that the cochain complex is acyclic, that the associated Kan complex is contractible. While we are sure that the content of the last sections is well-known to experts in the field, we are not aware, at the best of our knowledge, of any reference where it is spelled out in details. For this reason, we will include the proofs of all statements of these sections, directly in this chapter or in Appendix A.

2.1 Cochain complexes

In this section we review the basics of the theory of cochain complexes over a field $\mathbb{K}$ of characteristics zero. We use this section mainly to fix our notation and conventions, since the topics presented here are all well covered by the literature. We refer the reader in particular to the books by Weibel [Wei94] for the homological algebra part and by Hovey [Hov99] for the topics concerning model category theory.

Definition 2.1.1. A *cochain complex* $V = (V, Q)$ over $\mathbb{K}$ is given by a $\mathbb{Z}$-graded $\mathbb{K}$-vector space, that is a family of $\mathbb{K}$-vector spaces $(V^n)_{n\in\mathbb{Z}}$ labeled by integers $n \in \mathbb{Z}$, together with a *differential* $Q = (Q^n)_{n\in\mathbb{Z}}$, that is a family of degree-increasing $\mathbb{K}$-linear maps $Q^n : V^n \to V^{n+1}$ such that the composition $Q^{n+1}Q^n = 0$ vanishes for all $n \in \mathbb{Z}$.

In the following, we will use the diagrammatic picture

$$\cdots \longrightarrow V^{n-1} \xrightarrow{Q^{n-1}} V^n \xrightarrow{Q^n} V^{n+1} \longrightarrow \cdots \tag{2.1}$$

to represent the cochain complex (V, Q), where the differential is represented by right pointing horizontal arrows.

Definition 2.1.2. Let $V = (V, Q_V)$ and $W = (W, Q_W)$ be cochain complexes over $\mathbb{K}$, a *cochain map* $f : V \to W$ between them consists of a family $f = (f^n : V^n \to W^n)_{n\in\mathbb{Z}}$ of degree preserving linear maps, labeled by integers $n \in \mathbb{Z}$, which commute with the differentials, $Q^n_W f^n = f^{n+1}Q^n_V$ for all $n \in \mathbb{Z}$. We denote by $\mathbf{Ch}_{\mathbb{K}}$ the category whose objects are the cochain complexes over the field $\mathbb{K}$ and whose morphisms are the cochain maps.

Diagrammatically, cochain maps are represented by vertical arrows.

$$\begin{tikzcd}
\cdots \arrow[r] & V^{n-1} \arrow[r, "Q_V^{n-1}"] \arrow[d, "f^{n-1}"] & V^n \arrow[r, "Q_V^n"] \arrow[d, "f^n"] & V^{n+1} \arrow[r, "Q_V^{n+1}"] \arrow[d, "f^{n+1}"] & \cdots \\
\cdots \arrow[r] & W^{n-1} \arrow[r, "Q_W^{n-1}"'] & W^n \arrow[r, "Q_W^n"'] & W^{n+1} \arrow[r, "Q_W^{n+1}"'] & \cdots
\end{tikzcd} \tag{2.2}$$

Compatibility with differentials is equivalent to the commutativity of all the squares above.

The category $\mathbf{Ch}_{\mathbb{K}}$ of cochain complexes is a closed symmetric monoidal category when endowed with the tensor product, unit, symmetric braiding and internal hom described below. Given cochain complexes $V, W \in \mathbf{Ch}_{\mathbb{K}}$, their tensor product $V \otimes W \in \mathbf{Ch}_{\mathbb{K}}$ is the cochain complex whose underlying graded vector space is given by

$$(V \otimes W)^n := \bigoplus_{i\in\mathbb{Z}} (V^i \otimes W^{n-i}) \tag{2.3a}$$

for all $n \in \mathbb{Z}$, where on the right-hand side $\otimes$ denotes the standard tensor product of vector spaces, and whose differential $Q_\otimes = (Q_\otimes^n)_{n\in\mathbb{Z}}$ is defined by the *(graded) Leibniz rule*

$$Q_\otimes^n(v\otimes w) := Q_V^i v\otimes w + (-1)^i v\otimes Q_W^{n-i}w\,, \tag{2.3b}$$

for all $v \in V^i$ and $w \in W^{n-i}$. The monoidal unit $(\mathbb{K},0) \in \mathbf{Ch}_\mathbb{K}$ is obtained by regarding the ground field as a cochain complex concentrated in degree 0 with, necessarily, vanishing differential. The symmetric braiding is given by the cochain isomorphisms $\gamma : V\otimes W \to W\otimes V$, determined by the *Koszul sign rule*,

$$\gamma(v\otimes w) := (-1)^{ij} w\otimes v\,, \tag{2.4}$$

for all homogeneous $v \in V$ and $w \in W$ of degree $|v| = i, |w| = j$. Finally, given cochain complexes $V,W \in \mathbf{Ch}_\mathbb{K}$, the internal hom object $[V,W] \in \mathbf{Ch}_\mathbb{K}$ is the cochain complex which consists of the graded vector space defined degree-wise by

$$[V,W]^n := \prod_{i\in\mathbb{Z}} \mathrm{Hom}_\mathbb{K}(V^i, W^{i+n})\,, \tag{2.5a}$$

for all $n \in \mathbb{Z}$, where on the right-hand side $\mathrm{Hom}_\mathbb{K}(V^i, W^{i+n})$ denotes the vector space of $\mathbb{K}$-linear maps from V^i to W^{i+n}, and of the 'adjoint' differential $\partial = (\partial^n)_{n\in\mathbb{Z}}$ defined by

$$\partial^n f := Q_W \circ f - (-1)^n f\circ Q_V\,, \tag{2.5b}$$

for all $f \in [V,W]^n$, $n \in \mathbb{Z}$. It is just a matter of straightforward computations to check that, with the choices above, $(\mathbf{Ch}_\mathbb{K}, \otimes, \mathbb{K}, \gamma)$ is a symmetric monoidal category. Moreover, it is closed since it is possible to show that there are natural isomorphisms

$$\mathbf{Ch}_\mathbb{K}(V\otimes W, Z) \cong \mathbf{Ch}_\mathbb{K}(V, [W,Z]) \tag{2.6}$$

between the sets of cochain maps, for all $V,W,Z \in \mathbf{Ch}_\mathbb{K}$. As a general result for closed monoidal categories, natural isomorphisms (2.6) 'internalize' to isomorphisms

$$[V\otimes W, Z] \cong [V,[W,Z]] \tag{2.7}$$

in $\mathbf{Ch}_\mathbb{K}$ by exploiting the Yoneda lemma.

Definition 2.1.3. Let $V \in \mathbf{Ch}_\mathbb{K}$ be a cochain complex. Its *cohomology* is the graded vector space $\mathsf{H}^\bullet V$ defined degree-wise by the quotient of vector spaces

$$\mathsf{H}^n V := \frac{\mathsf{Z}^n V}{\mathsf{B}^n V} := \frac{\ker(Q^n : V^n \to V^{n+1})}{\mathrm{im}(Q^{n-1} : V^{n-1} \to V^n)}\,, \tag{2.8}$$

for all $n \in \mathbb{Z}$. Elements $v \in \mathsf{Z}^n V$ are called *n-cocycles*, while the vectors in $\mathsf{B}^n V$ are called *n-coboundaries*.

By regarding $\mathsf{H}^\bullet V = (\mathsf{H}^\bullet V, 0) \in \mathbf{Ch}_\mathbb{K}$ as a cochain complex with trivial differential, cohomology extends to a functor $\mathsf{H}^\bullet : \mathbf{Ch}_\mathbb{K} \to \mathbf{Ch}_\mathbb{K}$ on cochain complexes in an obvious way. (A cochain map f sends cocycles $Q_V v = 0$ to cocycles $Q_W(fv) = f(Q_V v) = 0$ and coboundaries $Q_V v$ to coboundaries $f(Q_V v) = Q_W(fv)$ as a consequence of its compatibility with differentials. Hence, it descends to cohomology classes.)

Definition 2.1.4. Let $V, W \in \mathbf{Ch}_\mathbb{K}$ be two cochain complexes. A cochain map $f : V \to W$ in $\mathbf{Ch}_\mathbb{K}$ is called a *quasi-isomorphism* if it descends to an isomorphism

$$\mathsf{H}^\bullet f : \mathsf{H}^\bullet V \xrightarrow{\cong} \mathsf{H}^\bullet W \tag{2.9}$$

on the cohomologies.

Given cochain complexes $V, W \in \mathbf{Ch}_\mathbb{K}$, their internal hom $[V, W] \in \mathbf{Ch}_\mathbb{K}$ can be interpreted in terms of (higher) homotopies. First, note that a cochain map $f : V \to W$ is just a 0-cocycle $f \in \mathsf{Z}^0[V, W]$ in the internal hom, namely the condition $0 = \partial f = Q_W f - f Q_V$ of having a vanishing differential is equivalent to the compatibility with differentials from Definition 2.1.1. Then, given two cochain maps $f_1, f_2 \in \mathsf{Z}^0[V, W]$, a *cochain homotopy* between them is a (-1)-cochain $h \in [V, W]^{-1}$ such that $\partial h = f_2 - f_1$. Since $\partial h \in \mathsf{B}^0[V, W]$ is a 0-coboundary, a homotopy from f_1 to f_2 exists if and only if the cohomology classes $[f_1] = [f_2] \in \mathsf{H}^0[V, W]$ coincide. This pattern extends even to higher cohomological degrees: a (higher) homotopy between two n-cocycles $f_1, f_2 \in \mathsf{Z}^n[V, W]$, $\partial f_1 = 0 = \partial f_2$, is a $(n-1)$-cochain $h \in [V, W]^{n-1}$ such that $\partial h = f_2 - f_1$. With the same argument as above, one realizes that a necessary and sufficient condition for the existence of a homotopy between two n-cocycles f_1, f_2 is that their cohomology classes $[f_1] = [f_2] \in \mathsf{H}^n[V, W]$ in the internal hom coincide.

A cochain map $f : V \to W$ is called a *homotopy equivalence* if there exists a cochain map $g : W \to V$ going in the opposite direction such that both the compositions fg and gf are homotopic to the identity maps. It is clear, by passing to cohomologies, that any homotopy equivalence is a quasi-isomorphism. Moreover, exploiting that we are working over a field $\mathbb{K}$, it is possible to show that each cochain complex $V \in \mathbf{Ch}_\mathbb{K}$ is homotopy equivalent to its cohomology,

$$\begin{tikzcd} \mathsf{H}^\bullet V \arrow[r, shift right, "i"'] & V \arrow[l, shift right, "p"'] \arrow[loop right, "h"] \end{tikzcd} , \tag{2.10}$$

$pi = \mathrm{id}$, $\mathrm{id} - ip = \partial h$, see [Wei94]. By composing these homotopy equivalences with the isomorphism induced on cohomologies, one realizes that a quasi-isomorphism $f : V \to W$ is also a homotopy equivalence,

$$\begin{tikzcd} V \arrow[loop left, "h_V"] \arrow[r, shift right, "p_V"'] & \mathsf{H}^\bullet V \arrow[l, shift right, "i_V"'] \arrow[r, shift right, "\mathsf{H}^\bullet f"'] & \mathsf{H}^\bullet W \arrow[l, shift right, "(\mathsf{H}^\bullet f)^{-1}"'] \arrow[r, shift right, "i_W"'] & W \arrow[l, shift right, "p_W"'] \arrow[loop right, "h_W"] \end{tikzcd} , \tag{2.11}$$

hence there always exist a *quasi-inverse* $g : W \to V$ in $\mathbf{Ch}_{\mathbb{K}}$ and cochain homotopies $h \in [V,V]^{-1}, k \in [W,W]^{-1}$ witnessing that the compositions $gf = \mathrm{id} - \partial h$ and $fg = \mathrm{id} - \partial k$ are homotopic to the identities. It follows that the relation given by the existence of a quasi-isomorphism from V to W is symmetric. Therefore, we can safely say that two cochain complexes $V, W \in \mathbf{Ch}_{\mathbb{K}}$ are quasi-isomorphic when there is a quasi-isomorphism between them. We say that a cochain complex $V \in \mathbf{Ch}_{\mathbb{K}}$ is *acyclic* if it is quasi-isomorphic $V \xrightarrow{\sim} 0$ to the zero cochain complex. The above argument entails that V is acyclic if and only if there exists a *contracting homotopy* $h \in [V,V]^{-1}$ such that $\partial h = \mathrm{id}$.

Exploiting the internal hom (2.5), it is possible to regard the category $\mathbf{Ch}_{\mathbb{K}}$ as enriched over itself, i.e. as a *dg-category*. Let us recall that a (small) category $\mathbf{C}$ enriched over the monoidal category $(\mathbf{M}, \otimes, I)$ is given by defining a *hom-object* $\mathrm{hom}(c_1, c_2) \in \mathbf{M}$ for all $c_1, c_2 \in \mathbf{C}$, together with a *composition morphism* $\circ_{c_1,c_2,c_3} : \mathrm{hom}(c_2, c_3) \otimes \mathrm{hom}(c_1, c_2) \to \mathrm{hom}(c_1, c_3)$ in $\mathbf{M}$, for all $c_1, c_2, c_3 \in \mathbf{C}$ and an *identity morphism* $j_c : I \to \mathrm{hom}(c, c)$ in $\mathbf{M}$, $c \in \mathbf{C}$, such that the composition is associative and unital. In the case of $\mathbf{Ch}_{\mathbb{K}}$, we simply set $\mathrm{hom}(V, W) := [V, W] \in \mathbf{Ch}_{\mathbb{K}}$, for $V, W \in \mathbf{Ch}_{\mathbb{K}}$ any two cochain complexes. Then, the composition morphism $\circ : [W, Z] \otimes [V, W] \to [V, Z]$ in $\mathbf{Ch}_{\mathbb{K}}$ is given by the obvious sequential composition of graded maps, $\circ : f_2 \otimes f_1 \mapsto f_2 \circ f_1 := (f_2^{i+p} f_1^i)_{i\in\mathbb{Z}} \in [V, Z]^n$, where $f_1 \in [V, W]^p$ and $f_2 \in [W, Z]^{n-p}$. (Compatibility with differentials follows easily from the definitions of the 'adjoint' differential (2.5b) and of the tensor product differential (2.3b).) The identity morphism $j : \mathbb{K} \to [V, V]$ is given by sending the unit $1 \mapsto \mathrm{id} \in \mathsf{Z}^0[V, V]$ to the identity. (Compatibility with differentials is trivial since the identity $\partial(\mathrm{id}) = 0$ is a cochain map.)

Isomorphisms between cochain complexes may be a too strong condition, not satisfied in many concrete situations, see for example the discussion in Section 1.4 about the homotopy time-slice axiom in AQFTs and tPFAs. In those cases what seems to be more useful is to weaken this condition and to regard as 'being the same' the cochain complexes which are only quasi-isomorphic. An approach meant to formalize this idea is offered by *model category theory* [Hov99]. A *model structure* on a category $\mathbf{C}$ consists of three distinguished classes of morphisms in $\mathbf{C}$, called *weak equivalences*, *fibrations* and *cofibrations*, subject to a suitable list of axioms, see e.g. [Hov99, Sec. 1.1]. Then, a *model category* is a bicomplete category together with a model structure on it. From a conceptual point of view, weak equivalences is the most important class since they describe the more relaxed notion of 'being the same' compared to isomorphism. Fibrations and cofibrations have a more technical role as they allow one to construct *derived functors*, i.e. functors that preserve weak equivalences, in a controlled way. In other words, they allow one to perform constructions that work up to weak equivalences.

This is a stronger framework than that of homotopical categories in the sense of [DHKS04] we had to use in Section 1.4 to give the definition of homotopy time-slice axiom (especially in the AQFT context, Definition 1.4.8). Note in particular that a model category has also the structure of a homotopical category, with the same choice of weak equivalences. The facts that weak equivalences include all isomorphisms and that they satisfy the 2-out-of-6 property follow directly from the model category axioms.

The category $\mathbf{Ch}_{\mathbb{K}}$ of cochain complexes admits, in particular, the projective model structure, which has

- quasi-isomorphisms as its weak equivalences,
- degree-wise surjective cochain maps as its fibrations,

while cofibrations are detected by the left lifting property with respect to acyclic fibrations[1], namely morphisms that are simultaneously weak equivalences and fibrations. This means that a cochain map $f : V \to W$ in $\mathbf{Ch}_{\mathbb{K}}$ is a cofibration if and only if for all commutative squares

$$\begin{tikzcd} V \arrow[r, "j"] \arrow[d, "f"'] & V' \arrow[d, "f'"] \\ W \arrow[r, "k"'] \arrow[ur, dashed, "h"] & W' \end{tikzcd} \tag{2.12}$$

in $\mathbf{Ch}_{\mathbb{K}}$, where $f' : V' \to W'$ is an acyclic fibration, there is a lift $h : W \to V'$ in $\mathbf{Ch}_{\mathbb{K}}$, such that $hf = j$ and $f'h = k$. It is shown in [Hov99, Ch. 2] that this choice of weak equivalences and fibrations defines a model structure on $\mathbf{Ch}_{\mathbb{K}}$. Let us recall that any model category, since it is bicomplete, admits a initial object 0 and a terminal object 1. In $\mathbf{Ch}_{\mathbb{K}}$ they are both isomorphic to the zero cochain complex $0 \in \mathbf{Ch}_{\mathbb{K}}$. An object in a model category is called *fibrant* when the (unique) morphism from it to the terminal object is a fibration, while it is called *cofibrant* when the (unique) morphism it gets from the initial object is a cofibration. It is possible to show that every cochain complex $V \in \mathbf{Ch}_{\mathbb{K}}$ is both fibrant and cofibrant with the model structure above. See e.g. [BMR14, Rem. 1.8]. Moreover, the projective model structure and the monoidal structure on $\mathbf{Ch}_{\mathbb{K}}$ are compatible in the sense that the axioms of a *closed symmetric monoidal model category*, see [Hov99, Sec. 4.2], are fulfilled. This, together with the previous fact that all objects are both fibrant and cofibrant, yields that the tensor product functor $\otimes : \mathbf{Ch}_{\mathbb{K}} \times \mathbf{Ch}_{\mathbb{K}} \to \mathbf{Ch}_{\mathbb{K}}$ and the internal hom functor $[-,-] : \mathbf{Ch}_{\mathbb{K}}^{\mathrm{op}} \times \mathbf{Ch}_{\mathbb{K}} \to \mathbf{Ch}_{\mathbb{K}}$ are already derived, namely they preserve quasi-isomorphisms.

[1] It is a general fact that the data of a model category are overdetermined. Indeed, in a model category, the class of cofibrations/fibrations is uniquely determined by the classes of weak equivalences and that of fibrations/cofibrations via the left/right lifting property, see [Hov99, Lemma 1.1.10].

Finally, let us fix our convention for shifts of cochain complexes. Let $V = (V, Q) \in \mathbf{Ch}_{\mathbb{K}}$ be a cochain complex and $p \in \mathbb{Z}$ an integer, the *p-shift* of V is the cochain complex $V[p] \in \mathbf{Ch}_{\mathbb{K}}$ which consists of the graded vector space defined degree-wise for all $n \in \mathbb{Z}$ by

$$V[p]^n := V^{n+p}\,, \tag{2.13a}$$

and of the differential $Q_{[p]}$ defined by

$$Q^n_{[p]} := (-1)^p Q^{n+p}\,, \tag{2.13b}$$

for all $n \in \mathbb{Z}$. It is immediate to see that $V[p][q] = V[p+q]$ for all $p, q \in \mathbb{Z}$, and $V[0] = V$. The definition (2.3) of the tensor product yields natural isomorphisms $\mathbb{K}[p] \otimes V \cong V[p]$, for all $p \in \mathbb{Z}$. Moreover, considering the internal hom, one gets natural cochain isomorphisms

$$[V, W[p]] \cong [V, W][p]\,, \tag{2.14a}$$

for all $p \in \mathbb{Z}$, determined on components by

$$\begin{aligned} [V, W[p]]^n &\longrightarrow [V, W][p]^n = [V, W]^{n+p} \\ (f^i : V^i \to W[p]^{i+n})_{i \in \mathbb{Z}} &\longmapsto (f^i : V^i \to W^{i+n+p})_{i\in\mathbb{Z}}\,, \end{aligned} \tag{2.14b}$$

for all $n \in \mathbb{Z}$, and

$$[V[p], W] \cong [V, W[-p]] \cong [V, W][-p]\,, \tag{2.15a}$$

for all $p \in \mathbb{Z}$, where the first isomorphism is given on components by

$$\begin{aligned} [V[p], W]^n &\longrightarrow [V, W[-p]]^n \\ (f^i : V[p]^i \to W^{i+n})_{i\in\mathbb{Z}} &\longmapsto (\widetilde{f^i} := (-1)^{np} f^{i-p} : V^i \to W[-p]^{i+n})_{i\in\mathbb{Z}}\,, \end{aligned} \tag{2.15b}$$

for all $n \in \mathbb{Z}$. Compatibilities of maps (2.14) and (2.15) with the 'adjoint' differentials follows from direct computations. Note that the sign $(-1)^{np}$ in the definition of the isomorphism (2.15), given by pulling the shift out of the internal hom, is necessary to get a cochain map.

2.2 Extension of pairings to symmetric algebras

In the core part of this thesis, especially in Sections 3.3 and 4.3, we shall consider several $(-p)$-shifted and unshifted (i.e. 0-shifted) pairings on a cochain complex $V \in \mathbf{Ch}_{\mathbb{K}}$, that is graded maps $\tau \in [V \otimes V, \mathbb{K}]^p$ of degree $p \in \mathbb{Z}$. Our pairings will always be either symmetric $\tau = \tau \circ \gamma$ or antisymmetric $\tau = -\tau \circ \gamma$, where γ is the symmetric braiding of $\mathbf{Ch}_{\mathbb{K}}$. In view of their importance in mathematical physics, in particular for the quantization of linear field theories, we will be especially interested in $(-p)$-*shifted* and also *unshifted* (i.e. 0-shifted) *Poisson structures* on V.

Definition 2.2.1. A $(-p)$-*shifted Poisson structure* on a cochain complex $V \in \mathbf{Ch}_{\mathbb{K}}$, $p \in \mathbb{Z}$, is a p-cocycle $\tau \in [V \otimes V, \mathbb{K}]^p$, $\partial\tau = 0$, in the internal hom, which is symmetric for p odd and antisymmetric for p even, $\tau = -(-1)^p \tau \circ \gamma$, γ being the symmetric braiding of $\mathbf{Ch}_{\mathbb{K}}$.

Remark 2.2.2. We agree that there may be some ambiguity in the terminology since *'Poisson structure'* is used by some authors as a synonym of *'Poisson bracket,'* while a (shifted) Poisson structure τ, in the sense of Definition 2.2.1 above, is **not** a (shifted) Poisson bracket on V. Furthermore, (anti)symmetric bilinear forms are sometimes called *presymplectic structures*. However, (pre)symplectic forms are typically understood as acting on vector fields, while our τ coincides with the restriction to generators V of a honest (shifted) Poisson bracket acting on the algebra of functions modeled by the polynomial dg-algebra $\operatorname{Sym} V$, see Remark 2.2.5 and 2.2.6. For this reason we adopted the term 'Poisson structure,' hoping this does not cause any confusion. $\triangledown$

Shifted and also unshifted Poisson structures will appear in Sections 3.3 and 4.3. They will also play a crucial role in Chapter 5, where two different quantization schemes, as AQFTs and as tPFAs, of a suitable class of field theories will be presented. In the latter chapter, we will consider structures on the *free symmetric algebra* $\operatorname{Sym} V \in \mathbf{dgCAlg}_{\mathbb{K}}$ generated by the cochain complex $V \in \mathbf{Ch}_{\mathbb{K}}$. For this reason, this section introduces extensions of shifted and unshifted pairings to the symmetric dg-algebra $\operatorname{Sym} V$ as suitable bi-derivations and, in the symmetric case, as Laplacians. The constructions below are well-known and they appeared frequently in the previous literature. Here, we follow the same approach of [BMS24] and go into a little more detail.

First, let us fix some notations. The category $\mathbf{dgAlg}_{\mathbb{K}} := \mathbf{Mon}(\mathbf{Ch}_{\mathbb{K}})$ of *differential graded (associative and unital) algebras over* $\mathbb{K}$ (*dg-algebras* for short) is just the category of monoids in $\mathbf{Ch}_{\mathbb{K}}$, while with $\mathbf{dgCAlg}_{\mathbb{K}} \subseteq \mathbf{dgAlg}_{\mathbb{K}}$ we denote the full subcategory of differential graded *commutative* algebras over $\mathbb{K}$, namely dg-algebras $A \in \mathbf{dgAlg}_{\mathbb{K}}$ whose multiplication $\mu = \mu \circ \gamma$ is (graded) commutative.

Given a cochain complex $V \in \mathbf{Ch}_{\mathbb{K}}$, its associated *free symmetric dg-algebra* $\operatorname{Sym} V = (\operatorname{Sym} V, \mu, \eta) \in \mathbf{dgCAlg}_{\mathbb{K}}$ is the differential graded commutative algebra whose underlying cochain complex is given by the sum

$$\operatorname{Sym} V := \bigoplus_{n \geq 0} \operatorname{Sym}^n V := \bigoplus_{n \geq 0} (V^{\otimes n})_{\Sigma_n} \in \mathbf{Ch}_{\mathbb{K}}\,, \tag{2.16}$$

of the symmetric tensor powers $\operatorname{Sym}^n V := (V^{\otimes n})_{\Sigma_n}$, defined as the coinvariants (i.e. quotients) of the action of the permutation group Σ_n on tensor powers $V^{\otimes n}$ via the symmetric braiding γ, with the convention that $\operatorname{Sym}^0 V := \mathbb{K} \in \mathbf{Ch}_{\mathbb{K}}$. The unit $\eta : \mathbb{K} \to \operatorname{Sym} V$ is given by the inclusion

$\mathbb{K} = \mathrm{Sym}^0 V \hookrightarrow \mathrm{Sym}\, V$ and the multiplication $\mu : \mathrm{Sym}\, V \otimes \mathrm{Sym}\, V \to \mathrm{Sym}\, V$ reads explicitly as $\mu([v_1 \otimes \cdots \otimes v_i] \otimes [v_{i+1} \otimes \cdots \otimes v_{i+j}]) := [v_1 \otimes \cdots \otimes v_{i+j}]$. In the following we will simply write $vw := \mu(v \otimes w)$ to lighten the notation. Observe that the differential of $\mathrm{Sym}\, V$ is uniquely determined by its action on generators in $\mathrm{Sym}^1 V = V$, where it coincides with the differential Q of V, thanks to the Leibniz rule, $Q(ab) = (Qa)b + (-1)^{|a|} a(Qb)$, where $|a|$ denotes the cohomological degree of the homogeneous element $a \in \mathrm{Sym}\, V$.

Let us consider a (anti)symmetric $(-p)$-shifted pairing $\tau \in [V \otimes V, \mathbb{K}]^p$ on $V \in \mathbf{Ch}_{\mathbb{K}}$, not necessarily a $(-p)$-shifted Poisson structure. We define the following extension of the pairing τ to the symmetric dg-algebra $\mathrm{Sym}\, V \in \mathbf{dgCAlg}_{\mathbb{K}}$ as a graded bi-derivation.

Definition 2.2.3. Given a (anti)symmetric pairing $\tau \in [V \otimes V, \mathbb{K}]^p$ on $V \in \mathbf{Ch}_{\mathbb{K}}$ of degree p, we define the p-cochain

$$\{\{-,-\}\}_\tau \in [\mathrm{Sym}\, V \otimes \mathrm{Sym}\, V, \mathrm{Sym}\, V \otimes \mathrm{Sym}\, V]^p \tag{2.17}$$

as the unique linear map which satisfies the following conditions:

1. $\{\{-,-\}\}_\tau$ is (anti)symmetric, $\{\{-,-\}\}_\tau = s\gamma \circ \{\{-,-\}\}_\tau \circ \gamma$, with $s = +1$ in the symmetric and $s = -1$ in the antisymmetric case;

2. for all $v, w \in V$, $\{\{v, w\}\}_\tau := \tau(v \otimes w)\, 1 \otimes 1 \in \mathrm{Sym}\, V \otimes \mathrm{Sym}\, V$, where $1 \in \mathbb{K} = \mathrm{Sym}^0 V \hookrightarrow \mathrm{Sym}\, V$ is the unit element of the symmetric algebra $\mathrm{Sym}\, V$;

3. for all homogeneous $a \in \mathrm{Sym}\, V$, $\{\{a, -\}\}_\tau : \mathrm{Sym}\, V \to \mathrm{Sym}\, V \otimes \mathrm{Sym}\, V$ is a graded derivation of degree $|a| + p$. Here, $\mathrm{Sym}\, V \otimes \mathrm{Sym}\, V$ is regarded as a $(\mathrm{Sym}\, V)$-module with the multiplication $\mathrm{Sym}\, V \otimes (\mathrm{Sym}\, V \otimes \mathrm{Sym}\, V) \to \mathrm{Sym}\, V \otimes \mathrm{Sym}\, V, r \otimes (a \otimes b) \mapsto \mu((1 \otimes r) \otimes (a \otimes b))$ given by the $(\mathrm{Sym}\, V)$-multiplication on the second tensor factor. More explicitly, one has

$$\{\{a, bc\}\}_\tau = \{\{a, b\}\}_\tau\, (1 \otimes c) + (-1)^{(|a|+p)|b|} (1 \otimes b)\, \{\{a, c\}\}_\tau\,, \tag{2.18}$$

for all homogeneous $b, c \in \mathrm{Sym}\, V$.

Note that condition 3, together with symmetry (or antisymmetry) from condition 1, implies that the linear map $\{\{-,-\}\}_\tau$ is known on all $(\mathrm{Sym}\, V)^{\otimes 2}$ as soon as its action on generators in V is assigned. Hence, condition 2 in Definition 2.2.3 is enough to uniquely fix it.

For an arbitrary (anti)symmetric pairing $\tau \in [V \otimes V, \mathbb{K}]^p$, its extension $\{\{-,-\}\}_\tau \in [(\mathrm{Sym}\, V)^{\otimes 2}, (\mathrm{Sym}\, V)^{\otimes 2}]^p$ on the symmetric dg-algebra $\mathrm{Sym}\, V$ is just a p-cochain, i.e. it is, in general, *not* closed $\partial \{\{-,-\}\}_\tau \neq 0$ in the internal hom. The following proposition computes the 'adjoint' differential of the bi-derivation extension in terms of the 'adjoint' differential of the original pairing.

Proposition 2.2.4. *Let $\tau \in [V \otimes V, \mathbb{K}]^p$ be a (anti)symmetric pairing of degree p on the cochain complex $V \in \mathbf{Ch}_{\mathbb{K}}$. Consider the graded endomorphism $\{\{-,-\}\}_\tau \in [(\mathrm{Sym}\, V)^{\otimes 2}, (\mathrm{Sym}\, V)^{\otimes 2}]^p$, as per Definition 2.2.3. Then,*

$$\partial \{\{-,-\}\}_\tau = \{\{-,-\}\}_{\partial\tau} \,. \tag{2.19}$$

Proof. We argue by induction on the polynomial weight of the arguments. First, note that if at least one of the arguments is of degree 0, i.e. it is a multiple of the algebra unit element $1 \in \mathbb{K} = \mathrm{Sym}^0 V$, identity (2.19) holds trivially since both sides vanish. Therefore, we can start by checking the claim on generators $v, w \in V = \mathrm{Sym}^1 V \hookrightarrow \mathrm{Sym}\, V$:

$$\begin{aligned}
\partial \{\{-,-\}\}_\tau (v \otimes w) &= Q_\otimes (\tau(v \otimes w)\, 1 \otimes 1) - (-1)^p \{\{-,-\}\}_\tau Q_\otimes (v \otimes w) \\
&= -(-1)^p (\tau \circ Q_\otimes)(v \otimes w)\, 1 \otimes 1 = \partial\tau(v \otimes w)\, 1 \otimes 1 \\
&= \{\{-,-\}\}_{\partial\tau}(v \otimes w) \,,
\end{aligned} \tag{2.20}$$

where in the first step we used the definition (2.5b) of the 'adjoint' differential and item 1 of Definition 2.2.3. In the second step we used that the unit element $Q\, 1 = 0$ is closed and, again, item 1 of Definition 2.2.3. The third step used the definition of the 'adjoint' differential and the last one is just the definition of the bracket $\{\{-,-\}\}_{\partial\tau}$. Let us now assume as inductive hypothesis that (2.19) holds true when evaluated on $a \otimes \underline{v}$, for any homogeneous $a \in \mathrm{Sym}\, V$ and any $\underline{v} := v_1 \cdots v_n$, for $v_1, \dots, v_n \in V$, $n \geq 1$. Let us check that the claim is still true when the polynomial weight of the second argument is increased by 1. Therefore, let $w \in V$ and compute

$$\begin{aligned}
&\partial \{\{-,-\}\}_\tau (a \otimes w\underline{v}) = Q_\otimes \{\{a, w\underline{v}\}\}_\tau - (-1)^p \{\{Qa, w\underline{v}\}\}_\tau \\
&\quad - (-1)^{p+|a|} (\{\{a, (Qw)\underline{v})\}\}_\tau + (-1)^{|w|} \{\{a, w\, Q\underline{v}\}\}_\tau) \\
&= Q_\otimes (\{\{a, w\}\}_\tau\, 1 \otimes \underline{v} + (-1)^{(|a|+p)|w|} 1 \otimes w\, \{\{a, \underline{v}\}\}_\tau) \\
&\quad - (-1)^p (\{\{Qa, w\}\}_\tau\, 1 \otimes \underline{v} + (-1)^{(|a|+1+p)|w|} 1 \otimes w\, \{\{Qa, \underline{v}\}\}_\tau) \\
&\quad - (-1)^{p+|a|} \{\{a, Qw\}\}_\tau\, 1 \otimes \underline{v} - (-1)^{(|a|+p)|w|} 1 \otimes Qw\, \{\{a, \underline{v}\}\}_\tau \\
&\quad - (-1)^{p+|a|+|w|} (\{\{a, w\}\}_\tau\, 1 \otimes Q\underline{v} + (-1)^{(|a|+p)|w|} 1 \otimes w\, \{\{a, Q\underline{v}\}\}_\tau) \\
&= (Q\{\{a, w\}\}_\tau - (-1)^p \{\{Qa, w\}\}_\tau - (-1)^{p+|a|} \{\{a, Qw\}\}_\tau) 1 \otimes \underline{v} \\
&\quad + (-1)^{(|a|+(p+1))|w|} 1 \otimes w (Q\{\{a, \underline{v}\}\}_\tau - (-1)^p \{\{Qa, \underline{v}\}\}_\tau - (-1)^{p+|a|} \{\{a, Q\underline{v}\}\}_\tau) \\
&= (\partial \{\{-,-\}\}_\tau (a \otimes w))\, 1 \otimes \underline{v} + (-1)^{(|a|+(p+1))|w|} 1 \otimes w (\partial \{\{-,-\}\}_\tau (a \otimes \underline{v})) \\
&= \{\{a, w\}\}_{\partial\tau}\, 1 \otimes \underline{v} + (-1)^{(|a|+(p+1))|w|} 1 \otimes w\, \{\{a, \underline{v}\}\}_{\partial\tau} \\
&= \{\{a, w\underline{v}\}\}_{\partial\tau}
\end{aligned} \tag{2.21}$$

where in the first step we used the definition (2.5b) of the 'adjoint' differential ∂, the definition (2.3b) of $Q_\otimes$ and the Leibniz rule for the differential Q on $\mathrm{Sym}\, V$. In the second step we exploited item 3 of the Definition 2.2.3

repeatedly. The third step used the Leibniz rule for Q with respect to the product μ of $\operatorname{Sym} V$. Step four simply follows from the definition of ∂. The inductive hypothesis has been used in step five since the second argument of $\partial\{\{-,-\}\}_\tau$ is of weight 1 in the first contribution and n in the second one. Finally, in the last step we used the fact that $\{\{a,-\}\}_{\partial\tau}$ is a graded derivation of degree $|a|+(p+1)$ by definition. Claim (2.19) then follows by exploiting also the (anti)symmetry of the endomorphisms $\{\{-,-\}\}_\tau$ and $\{\{-,-\}\}_{\partial\tau}$, see item 1 in Definition 2.2.3. $\square$

Remark 2.2.5. Let us notice that a $(-p)$-shifted Poisson structure $\tau \in [V\otimes V, \mathbb{K}]^p$ on V, see Definition 2.2.1, induces a $(-p)$-*shifted Poisson bracket* on the symmetric dg-algebra $\operatorname{Sym} V$. The latter is given by the composition

$$\{-,-\}_\tau := \mu \circ \{\{-,-\}\}_\tau \in [\operatorname{Sym} V \otimes \operatorname{Sym} V, \operatorname{Sym} V]^p\,, \tag{2.22}$$

where μ is the commutative product of $\operatorname{Sym} V$ and $\{\{-,-\}\}_\tau$ is given by Definition 2.2.3. Definition 2.2.1 and Proposition 2.2.4 yield that the bracket $\partial\{-,-\}_\tau = \mu\circ\{\{-,-\}\}_{\partial\tau} = 0$ is compatible with differentials. Moreover, it satisfies a graded Jacobi identity, $\{a,\{b,c\}_\tau\}_\tau = (-1)^{(|a|+p)p}\{\{a,b\}_\tau,c\}_\tau + (-1)^{(|a|+p)(|b|+p)}\{b,\{a,c\}_\tau\}_\tau$, for all homogeneous $a,b,c\in\operatorname{Sym} V$. The signs in the formula can be understood by regarding the degree p of the bracket $\{-,-\}_\tau$ as concentrated on its left end. Jacobi identity can be proved by induction on the polynomial weight of its arguments, exploiting Definition 2.2.3 and arguing similarly to the proof of Proposition 2.2.4. Furthermore, the bracket $\{-,-\}_\tau$ is symmetric for p odd and antisymmetric for p even, $\{-,-\}_\tau\circ\gamma = -(-1)^p\mu\circ\gamma\circ\{\{-,-\}\}_\tau = -(-1)^p\{-,-\}_\tau$ as a consequence of Definitions 2.2.1 and 2.2.3 and of the commutativity of the multiplication μ. Finally, Definition 2.2.3, item 3, entails that $\{a,-\}_\tau$ is a graded derivation of degree $|a|+p$ on $\operatorname{Sym} V$, for all $a\in\operatorname{Sym} V$. $\triangledown$

Remark 2.2.6. It is worth noticing that other conventions for the definition of shifted Poisson brackets can be found in the literature. Arguably the most important one, see e.g. [CFL06], defines a $(-p)$-*Poisson algebra* as a triple $(A,\mu,\widetilde{\{-,-\}})$ consisting of a dg-commutative algebra $(A,\mu)\in\mathbf{dgCAlg}_\mathbb{K}$ and of a p-cochain $\widetilde{\{-,-\}}\in[A\otimes A,A]^p$ which induces a dg-Lie algebra structure $(A[-p],\widetilde{\{-,-\}})$ on the $(-p)$-shift of the cochain complex A, and such that $\widetilde{\{a,-\}}$ is a degree $|a|+p$ derivation with respect to μ, for all $a\in A$. According to this definition, a $(-p)$-shifted Poisson structure $\tau\in[V\otimes V,\mathbb{K}]^p$ on V endows the symmetric dg-algebra $\operatorname{Sym} V\in\mathbf{dgCAlg}_\mathbb{K}$ with the structure of a $(-p)$-Poisson algebra by introducing the bracket

$$\widetilde{\{a,b\}}_\tau := (-1)^{|a|}\mu(\{\{a,b\}\}_\tau) = (-1)^{|a|}\{a,b\}_\tau\,, \tag{2.23}$$

for all $a,b\in\operatorname{Sym} V$. Note that it differs from (2.22) simply by a sign. The disadvantage of working with this notion of $(-p)$-Poisson algebras in a differential graded setting is mainly that the bracket $\widetilde{\{-,-\}}$ does not have a natural compatibility with the differential of $A \in \mathbf{Ch}_{\mathbb{K}}$, since the former originates from a dg-Lie structure on the $(-p)$-shift $A[-p]$. ▽

Proposition 2.2.7. *Given (anti)symmetric pairings $\tau \in [V \otimes V, \mathbb{K}]^p$ and $\omega \in [W \otimes W, \mathbb{K}]^p$ of degree p on cochain complexes V and $W \in \mathbf{Ch}_{\mathbb{K}}$, respectively, and a cochain map $f : V \to W$ which preserves the pairings, i.e. $\omega \circ (f \otimes f) = \tau$, one has that the morphism of dg-algebras $\operatorname{Sym} f : \operatorname{Sym} V \to \operatorname{Sym} W$ preserves the graded endomorphisms from Definition 2.2.3, namely*

$$\{\{-,-\}\}_\omega \circ (\operatorname{Sym} f \otimes \operatorname{Sym} f) = (\operatorname{Sym} f \otimes \operatorname{Sym} f) \circ \{\{-,-\}\}_\tau \tag{2.24}$$

Proof. The proof of this fact goes along the same lines of the proof of Proposition 2.2.4. Since f preserves the pairings it is immediate to see that Equation (2.24) holds when evaluated on generators in $V = \operatorname{Sym}^1 V \hookrightarrow \operatorname{Sym} V$. Then, exploiting the (anti)symmetry of the pairings it is enough to argue inductively on the weight of the second of their arguments. Assuming that the identity is fulfilled upon evaluation on $a \otimes \underline{v}$, for arbitrary homogeneous $a \in \operatorname{Sym} V$ and $\underline{v} := v_1 \cdots v_n$, for any $v_1, \dots, v_n \in V$, $n \geq 1$, a straightforward calculation, exploiting also item 3 of Definition 2.2.3, shows that the claim is also true upon evaluation on $a \otimes w\underline{v}$, for any $w \in W$, thus concluding the proof. □

We now restrict ourselves to the case of a symmetric $(-p)$-shifted pairing $\tau \in [V \otimes V, \mathbb{K}]^p$. One can extend it as a *Laplacian* over $\operatorname{Sym} V \in \mathbf{dgCAlg}_{\mathbb{K}}$.

Definition 2.2.8. Let $\tau \in [V \otimes V, \mathbb{K}]^p$ be a symmetric pairing of degree p on the cochain complex $V \in \mathbf{Ch}_{\mathbb{K}}$. We define its associated *Laplacian* as the unique p-cochain

$$\Delta_\tau \in [\operatorname{Sym} V, \operatorname{Sym} V]^p \tag{2.25}$$

such that:

1. $\Delta_\tau(1) = 0$, for $1 \in \operatorname{Sym} V$ the unit element;
2. $\Delta_\tau(v) = 0$, for all $v \in V$;
3. It satisfies a modified Leibniz rule, the extended pairing $\{\{-,-\}\}_\tau$ from Definition 2.2.3 witnessing the violation of the graded Leibniz rule. More explicitly, for all homogeneous $a, b \in \operatorname{Sym} V$, one has

$$\Delta_\tau(ab) = \Delta_\tau(a)b + (-1)^{p|a|} a\Delta_\tau(b) + \mu(\{\{a,b\}\}_\tau)\,. \tag{2.26}$$

Note that Definition 2.2.8 uniquely fixes the Laplacian Δ_τ on the whole $\operatorname{Sym} V$. Indeed, the modified Leibniz rule (2.26) allows one to iteratively reduce the polynomial weight of the arguments up to weight 1. Therefore, it is enough to assign Δ_τ on generators in V (item 2 of Definition 2.2.8) to fix it everywhere. By exploiting this strategy, one comes to the explicit formula

$$\Delta_\tau(v_1 \cdots v_n) = \sum_{i<j} (-1)^{\sigma_{ij}} \tau(v_i \otimes v_j)\, v_1 \cdots \check{v}_i \cdots \check{v}_j \cdots v_n \tag{2.27}$$

for any homogeneous $v_1 \ldots v_n \in \operatorname{Sym} V$, where $\check{\cdot}$ denotes the omission of the corresponding factor and the sign, $\sigma_{ij} := p \sum_{k=1}^{i-1} |v_k| + |v_j| \sum_{k=i+1}^{j-1} |v_k|$, may be understood by pulling v_i and v_j to the left. In particular, for any $v, w \in V$, one has

$$\Delta_\tau(vw) = \mu(\{\{v, w\}\}_\tau) = \tau(v \otimes w)\, 1\,. \tag{2.28}$$

Remark 2.2.9. We restricted ourselves to symmetric pairings because, otherwise, the modified Leibniz rule (2.26) would not have made sense for $\tau \neq 0$, since it does not descend to the quotients defining $\operatorname{Sym} V$. For simplicity, consider the monomial $vw = (-1)^{|v|\,|w|} wv \in \operatorname{Sym} V$ of weight 2. One sees that

$$\tau(v \otimes w)\, 1 = \Delta_\tau(vw) = (-1)^{|v|\,|w|} \Delta_\tau(wv) = (-1)^{|v|\,|w|} \tau(w \otimes v)\, 1\,, \tag{2.29}$$

which, for an antisymmetric τ, would imply $\tau = 0$. $\triangledown$

Proposition 2.2.10. *Let $\tau \in [V \otimes V, \mathbb{K}]^p$ be a $(-p)$-shifted symmetric pairing on the cochain complex $V \in \mathbf{Ch}_\mathbb{K}$. Then, the 'adjoint' differential of the associated Laplacian,*

$$\partial \Delta_\tau = \Delta_{\partial \tau}\,, \tag{2.30}$$

equals the Laplacian associated to $\partial\tau$.

Proof. The strategy is again to exploit the defining properties of the Laplacian, Definition 2.2.8, and to argue by induction on the weight of the argument. Since the idea of the proof is not different from that of Proposition 2.2.4 and not much insight will be gained from it, we just sketch it. Note that the claim is trivially true on generators $v \in V$, since both sides vanish, $\partial \Delta_\tau v = 0 = \Delta_{\partial\tau} v$. Assuming as inductive hypothesis that the claim holds true on monomials $\underline{v} := v_1 \cdots v_n \in \operatorname{Sym} V$ of weight n, one shows that it is also true on monomials of the form $w\underline{v} \in \operatorname{Sym} V$ for any homogeneous $w \in V$, by exploiting Equation (2.26), the Leibniz rule for the differential Q of $\operatorname{Sym} V$ and Proposition 2.2.4. $\square$

The following proposition exports the result of Proposition 2.2.7 to the case of Laplacians.

Proposition 2.2.11. *Let $\tau \in [V \otimes V, \mathbb{K}]^p, \omega \in [W \otimes W, \mathbb{K}]^p$ be symmetric pairings of degree p on cochain complexes $V, W \in \mathbf{Ch}_\mathbb{K}$. It holds that*

$$\Delta_\omega \circ \operatorname{Sym} f = \operatorname{Sym} f \circ \Delta_\tau \tag{2.31}$$

for all morphisms $f : V \to W$ in $\mathbf{Ch}_\mathbb{K}$ compatible with the pairings, i.e. such that $\omega \circ (f \otimes f) = \tau$.

Proof. The proof follows the same ideas of the previous ones, using defining properties of the Laplacians and Equation (2.24), along with an induction on the polynomial weight of the argument of the maps. □

A useful observation is that the Laplacian and the bi-derivation extending the same symmetric $(-p)$-shifted pairing graded commute.

Proposition 2.2.12. *Let $\tau \in [V \otimes V, \mathbb{K}]^p$ be a symmetric $(-p)$-shifted pairing. The associated Laplacian $\Delta_\tau \in [\operatorname{Sym} V, \operatorname{Sym} V]^p$ and graded endomorphism $\{\{-,-\}\}_\tau \in [(\operatorname{Sym} V)^{\otimes 2}, (\operatorname{Sym} V)^{\otimes 2}]^p$ graded commute, namely*

$$(\mathrm{id} \otimes \Delta_\tau) \circ \{\{-,-\}\}_\tau = (-1)^p \{\{-,-\}\}_\tau \circ (\mathrm{id} \otimes \Delta_\tau) \tag{2.32a}$$

and

$$(\Delta_\tau \otimes \mathrm{id}) \circ \{\{-,-\}\}_\tau = (-1)^p \{\{-,-\}\}_\tau \circ (\Delta_\tau \otimes \mathrm{id})\,. \tag{2.32b}$$

Proof. First, note that it is enough to show only one of the claimed identities. Indeed, suppose that Equation (2.32a) holds true, one computes

$$\begin{aligned}(\Delta_\tau \otimes \mathrm{id}) \circ \{\{-,-\}\}_\tau &= \gamma \circ (\mathrm{id} \otimes \Delta_\tau) \circ \{\{-,-\}\}_\tau \circ \gamma \\ &= (-1)^p \gamma \circ \{\{-,-\}\}_\tau \circ (\mathrm{id} \otimes \Delta_\tau) \circ \gamma \\ &= (-1)^p \{\{-,-\}\}_\tau \circ (\Delta_\tau \otimes \mathrm{id})\,,\end{aligned} \tag{2.33}$$

where in the first and last steps we used symmetry of $\{\{-,-\}\}_\tau$ and that $\Delta_\tau \otimes \mathrm{id} = \gamma \circ (\mathrm{id} \otimes \Delta_\tau) \circ \gamma$. The latter follows by recalling that the evaluation of the tensor product of two graded linear maps gets a sign from the Koszul rule, i.e. $(f \otimes g)(v \otimes w) := (-1)^{|g|\,|v|} f(v) \otimes g(w)$. The second step used Equation (2.32a).

Therefore, let us consider just the first identity. One can prove that it holds true by arguing by induction on the polynomial weight of the arguments. One immediately realizes that Equation (2.32a) is trivially true upon evaluation on elements of the type $a \otimes v$, for arbitrary $a \in \operatorname{Sym} V$ and $v \in V$, since both the sides vanish. Recall, in particular, that the Laplacian $\Delta_\tau(v) = 0$ is identically zero on generators. Afterwards, it is possible to show, with a tedious yet straightforward calculation, that the identity is satisfied also on elements $a \otimes \underline{v} w$ for homogeneous $a \in \operatorname{Sym} V$ and $\underline{v} := v_1 \cdots v_n$, for $v_1, \dots, v_n, w \in V$, with the second tensor factor of weight $n+1$, once it is assumed to be true upon evaluation on any element of the type $a \otimes \underline{v} \in \operatorname{Sym} V$. □

An immediate consequence of Proposition 2.2.12 is the following formula.

Proposition 2.2.13. *Let $\tau \in [V \otimes V, \mathbb{K}]^p$ be a symmetric $(-p)$-shifted pairing on $V \in \mathbf{Ch}_\mathbb{K}$. Then, for p even, one has*

$$\Delta_\tau^n \circ \mu = \sum_{k=0}^{n} \binom{n}{k} \mu \circ \{\{-,-\}\}_\tau^k \circ \Delta_{\tau\otimes}^{n-k}\,, \tag{2.34}$$

for all $n \geq 1$. While, for p odd, $\Delta_\tau^n = 0$ for all $n \geq 2$.

Proof. From item 3 of Definition 2.2.8 of the Laplacian one has

$$\Delta_\tau \circ \mu = \mu \circ (\Delta_{\tau\otimes} + \{\{-,-\}\}_\tau)\,, \tag{2.35}$$

where $\Delta_{\tau\otimes} := \Delta_\tau \otimes \mathrm{id} + \mathrm{id} \otimes \Delta_\tau$. By iterating this formula, one gets

$$\begin{aligned}\Delta_\tau^n \circ \mu &= \Delta_\tau^{n-1} \circ \mu \circ (\Delta_{\tau\otimes} + \{\{-,-\}\}_\tau) = \mu \circ (\Delta_{\tau\otimes} + \{\{-,-\}\}_\tau)^n \\ &= \sum_{k=0}^{n} \binom{n}{k} \mu \circ \{\{-,-\}\}_\tau^k \circ \Delta_{\tau\otimes}^{n-k}\,,\end{aligned} \tag{2.36}$$

where the last step follows from the fact that both $\Delta_\tau \otimes \mathrm{id}$ and $\mathrm{id} \otimes \Delta_\tau$ commute with $\{\{-,-\}\}_\tau$ for p even, see Proposition 2.2.12. For the second part of the proposition, let p be odd and let us argue by induction. First note that $\Delta_\tau^2(v) = 0$ for all $v \in V$ by definition of the Laplacian. Assume now that $\Delta_\tau^2 \underline{v} = 0$, where $\underline{v} := v_1 \cdots v_n$ for any $v_1, \ldots, v_n \in V$, $n \geq 1$, and let us show that Δ_τ^2 vanishes also on monomials of weight $n+1$. Let $w \in V$ and compute

$$\begin{aligned}\Delta_\tau^2(\underline{v}w) &= \Delta_\tau((\Delta_\tau \underline{v})w + \mu(\{\{\underline{v}, w\}\}_\tau)) \\ &= \Delta_\tau^2(\underline{v})w + \mu \circ \{\{-,-\}\}_\tau \circ (\Delta_\tau \otimes \mathrm{id})(\underline{v} \otimes w) \\ &\quad + \mu \circ (\Delta_\tau \otimes \mathrm{id}) \circ \{\{-,-\}\}_\tau(\underline{v} \otimes w) = 0\,,\end{aligned} \tag{2.37}$$

where we used the defining properties of Δ_τ in the first and second steps. Last step used the inductive hypothesis and the fact that $\Delta_\tau \otimes \mathrm{id}$ and $\{\{-,-\}\}_\tau$ anticommute for p even because of Proposition 2.2.12. □

We conclude this section with a last technical result.

Proposition 2.2.14. *Given symmetric pairings $\tau \in [V \otimes V, \mathbb{K}]^p$ and $\omega \in [V \otimes V, \mathbb{K}]^q$ on $V \in \mathbf{Ch}_\mathbb{K}$ of degree p and q, respectively, one has*

$$\Delta_\tau \circ \Delta_\omega = (-1)^{pq} \Delta_\omega \circ \Delta_\tau\,, \tag{2.38}$$

where Δ_τ and Δ_ω are the associated Laplacians from Definition 2.2.8.

Proof. The identity (2.38) is proven by a direct computation exploiting the explicit formula (2.27) for the Laplacian associated to a symmetric pairing. Since the computation is pretty lengthy and not particularly interesting, whose most painful part is just carefully taking account of all the signs coming from the Koszul rule, we will omit it. □

Remark 2.2.15. Let $V \in \mathbf{Ch}_\mathbb{K}$ be a cochain complex endowed with a $(-p)$-shifted Poisson structure $\tau \in [V \otimes V, \mathbb{K}]^p$, as per Definition 2.2.1, for p odd. The triple $(\mathrm{Sym}\, V, \widetilde{\{-,-\}}_\tau, \Delta_\tau)$ consisting of the free symmetric dg-algebra

$\operatorname{Sym} V \in \mathbf{dgAlg}_{\mathbb{K}}$, the degree p bracket $\widetilde{\{-,-\}}_\tau \in [(\operatorname{Sym} V)^{\otimes 2}, \operatorname{Sym} V]^p$ from (2.23) and the Laplacian $\Delta_\tau \in [\operatorname{Sym} V, \operatorname{Sym} V]^p$ from Definition 2.2.8 is a *dg* $(-p)$*-BV algebra*. See [CFL06] for arbitrary p in the graded case and [DV13] for $p = -1$ in the homological convention. Briefly, a dg $(-p)$-BV algebra $(A, \widetilde{\{-,-\}}, \Delta)$ is the datum of a $(-p)$-Poisson algebra $(A, \widetilde{\{-,-\}})$, see Remark 2.2.6, and of a p-cocycle $\Delta \in [A, A]^p$, $\partial\Delta = 0$, which is an *exact generator* for the bracket $\widetilde{\{-,-\}}$, namely $\Delta^2 = 0$ and the bracket is the obstruction to Δ being a derivation with respect to the product μ of A,

$$\widetilde{\{a,b\}} = (-1)^{|a|}\Delta(\mu(a \otimes b)) - (-1)^{|a|}\mu((\Delta a) \otimes b) - \mu(a \otimes \Delta b)\,, \tag{2.39}$$

for all $a, b \in A$. Note that these two conditions immediately imply that Δ is a derivation with respect to the bracket $\widetilde{\{-,-\}}$. More explicitly, one has

$$\Delta\widetilde{\{a,b\}} = \widetilde{\{\Delta a, b\}} - (-1)^{|a|}\widetilde{\{a, \Delta b\}}\,, \tag{2.40}$$

for any $a, b \in A$. The fact that $(\operatorname{Sym} V, \widetilde{\{-,-\}}_\tau, \Delta_\tau)$ is a dg $(-p)$-BV algebra readily follows from Remark 2.2.6 (see also Remark 2.2.5) together with item 3 of Definition 2.2.8 which yields (2.39), Proposition 2.2.13 which proves that Δ_τ squares to zero and, finally, Proposition 2.2.10 which shows the compatibility $\partial\Delta_\tau = \Delta_{\partial\tau} = 0$ of the Laplacian Δ_τ with the differential of $\operatorname{Sym} V$. Recall that $\partial\tau = 0$ as a part of Definition 2.2.1 of $(-p)$-shifted Poisson structure. ▽

2.3 Homotopy colimits of $\mathbf{Ch}_{\mathbb{K}}$-valued functors

In the main part of the thesis, especially in Section 3, we will extensively consider functors $\mathcal{V} : \mathbf{C} \to \mathbf{Ch}_{\mathbb{K}}$ from a (small) category $\mathbf{C}$ taking values into the category $\mathbf{Ch}_{\mathbb{K}}$ of cochain complexes over a field $\mathbb{K}$ of characteristic zero. (Typically, $\mathbb{K}$ will be either the field $\mathbb{K} = \mathbb{R}$ of real or $\mathbb{K} = \mathbb{C}$ of complex numbers.) In particular, we will have to deal with homotopy colimits of such functors over directed sets. In this section we will introduce a (non-standard) dg-enrichment of the category $\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ of $\mathbf{Ch}_{\mathbb{K}}$-valued functors and natural transformations between them and we will present a model for the homotopy colimit, which we will extend to an enriched functor that is the dg-left adjoint of the diagonal dg-functor Δ.

Consider the category $\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ whose objects are functors $\mathcal{V} : \mathbf{C} \to \mathbf{Ch}_{\mathbb{K}}$ from a (small) category $\mathbf{C}$ to the category $\mathbf{Ch}_{\mathbb{K}}$ of cochain complexes and whose morphisms $\eta : \mathcal{V} \to \mathcal{W}$ are natural transformations between functors $\mathcal{V}, \mathcal{W} : \mathbf{C} \to \mathbf{Ch}_{\mathbb{K}}$, namely, for all $c \in \mathbf{C}$ the component $\eta_c : \mathcal{V}(c) \to \mathcal{W}(c)$ in $\mathbf{Ch}_{\mathbb{K}}$ is a cochain map and for each morphism $f : c_1 \to c_2$ in $\mathbf{C}$ one has the naturality condition $\mathcal{W}(f) \circ \eta_{c_1} = \eta_{c_2} \circ \mathcal{V}(f)$. The category $\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ inherits some structures from its target category $\mathbf{Ch}_{\mathbb{K}}$. First, it admits a straightforward enrichment over $\mathbf{Ch}_{\mathbb{K}}$. Given functors $\mathcal{V}, \mathcal{W} \in \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ one defines the hom-object $\underline{\hom}(\mathcal{V}, \mathcal{W}) \in \mathbf{Ch}_{\mathbb{K}}$ as the limit

$$\underline{\hom}(\mathcal{V}, \mathcal{W}) := \lim \left(\prod_{c_1 \in \mathbf{C}} [\mathcal{V}(c_1), \mathcal{W}(c_1)] \overset{\mathcal{V}^*}{\underset{\mathcal{W}_*}{\rightrightarrows}} \prod_{f : c_1 \to c_2} [\mathcal{V}(c_1), \mathcal{W}(c_2)] \right), \tag{2.41}$$

in $\mathbf{Ch}_{\mathbb{K}}$, where $\mathcal{V}^*$ is defined, on the f-component of the codomain, by 'pulling back' along $f : c_1 \to c_2$ the c_2-component of the domain, i.e. $\mathrm{pr}_f(\mathcal{V}^* \eta) := (\mathrm{pr}_{c_2} \eta) \circ \mathcal{V}(f)$, while $\mathcal{W}_*$ is given by 'pushing forward' the c_1-component of the domain along $f : c_1 \to c_2$, i.e. $\mathrm{pr}_f(\mathcal{W}_* \eta) := \mathcal{W}(f) \circ \mathrm{pr}_{c_1} \eta$, for $\eta = (\eta_c) \in \prod_{c_1 \in \mathbf{C}} [\mathcal{V}(c_1), \mathcal{W}(c_1)]$ and pr denoting the projection onto the components of the product. Making explicit the equalizer (2.41), one finds that a model for the hom-object $\underline{\hom}(\mathcal{V}, \mathcal{W}) \in \mathbf{Ch}_{\mathbb{K}}$ is given by the cochain complex whose underlying graded vector space is defined degree-wise by

$$\begin{aligned} &\underline{\hom}(\mathcal{V}, \mathcal{W})^n \\ &= \big\{ (\eta_c \in [\mathcal{V}(c), \mathcal{W}(c)]^n)_{c \in \mathbf{C}} \,\big|\, \eta_{c_2} \circ \mathcal{V}(f) = \mathcal{W}(f) \circ \eta_{c_1}, \forall f : c_1 \to c_2 \text{ in } \mathbf{C} \big\} , \end{aligned} \tag{2.42a}$$

for all $n \in \mathbb{Z}$, and whose differential is given by the (restriction of the) differential on the product cochain complex, namely

$$\mathrm{pr}_c(\partial \eta) = \partial (\mathrm{pr}_c \eta) , \tag{2.42b}$$

for all $c \in \mathbf{C}$. Note that $\partial \eta \in \underline{\hom}(\mathcal{V}, \mathcal{W})$ is again an element in the hom complex since $\mathcal{V}(f)$ and $\mathcal{W}(f)$ are both cochain maps for all $f : c_1 \to c_2$ in $\mathbf{C}$. Equation (2.42a) makes clear that the degree n of the hom complex $\underline{\hom}(\mathcal{V}, \mathcal{W})$ describes graded natural transformations of degree n between $\mathcal{V}$ and $\mathcal{W}$, regarded as functors valued in graded vector spaces, i.e. forgetting the differentials. In particular, a 0-cocycle $\eta \in \mathsf{Z}^0 \underline{\hom}(\mathcal{V}, \mathcal{W})$ is just a natural transformation from $\mathcal{V}$ to $\mathcal{W}$, since all its c-components $\partial(\eta_c) = (\partial \eta)_c = 0$ are cochain maps. Compare this with the fact that $f \in \mathsf{Z}^0[V, W]$ is a 0-cocycle in the internal hom if and only if $f : V \to W$ is a cochain map, for $V, W \in \mathbf{Ch}_{\mathbb{K}}$. The remaining data of the $\mathbf{Ch}_{\mathbb{K}}$-enrichment of $\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ are the obvious identity cochain map $j : \mathbb{K} \to \underline{\hom}(\mathcal{V}, \mathcal{V})$ in $\mathbf{Ch}_{\mathbb{K}}$, which sends $1 \mapsto \mathrm{id} \in \mathsf{Z}^0 \underline{\hom}(\mathcal{V}, \mathcal{V})$ to the identity natural transformation, and the composition morphism $\circ : \underline{\hom}(\mathcal{W}, \mathcal{Z}) \otimes \underline{\hom}(\mathcal{V}, \mathcal{W}) \to \underline{\hom}(\mathcal{V}, \mathcal{Z})$ in $\mathbf{Ch}_{\mathbb{K}}$ given component-wise by $\mathrm{pr}_c(\zeta \circ \eta) := \mathrm{pr}_c \zeta \circ \mathrm{pr}_c \eta$, for any $\eta \in \underline{\hom}(\mathcal{V}, \mathcal{W})$ and $\zeta \in \underline{\hom}(\mathcal{W}, \mathcal{Z})$, where the composition $\circ$ on the right-hand side is the one in the dg-category $\mathbf{Ch}_{\mathbb{K}}$. It is easy to check that these data fulfill the associativity and unitality laws, hence they define a dg-enrichment of $\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$.

The dg-category $\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ is also *tensored* and *cotensored* over $\mathbf{Ch}_{\mathbb{K}}$. In few words, see e.g. [Kel05] for more details, it means that there is a *tensor product* functor

$$- \otimes - : \mathbf{Ch}_{\mathbb{K}} \times \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}} \longrightarrow \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}} , \tag{2.43}$$

and a *cotensor product* functor

$$- \pitchfork - : \mathbf{Ch}_{\mathbb{K}}^{\mathrm{op}} \times \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}} \longrightarrow \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}} , \tag{2.44}$$

such that there are natural isomorphisms

$$\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}(V \otimes \mathcal{V}, \mathcal{W}) \cong \mathbf{Ch}_{\mathbb{K}}(V, \underline{\hom}(\mathcal{V}, \mathcal{W})) \cong \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}(\mathcal{V}, V \pitchfork \mathcal{W}) \tag{2.45}$$

of the hom sets. Notice that the first isomorphism recalls an enriched variant of the closed monoidal category adjunction (2.6). In the case of $\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ we define the tensor product

$$V \otimes \mathcal{V} \in \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}} \tag{2.46}$$

as the functor which assigns the tensor product of cochain complexes $(V \otimes \mathcal{V})(c) := V \otimes \mathcal{V}(c) \in \mathbf{Ch}_{\mathbb{K}}$ to any $c \in \mathbf{C}$ and the cochain map $\mathrm{id} \otimes \mathcal{V}(f) : V \otimes \mathcal{V}(c_1) \to V \otimes \mathcal{V}(c_2)$ to any morphism $f : c_1 \to c_2$ in $\mathbf{C}$. Given a cochain complex $V \in \mathbf{Ch}_{\mathbb{K}}$ and a functor $\mathcal{V} \in \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$, their cotensor product

$$V \pitchfork \mathcal{V} \in \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}} \tag{2.47}$$

is the functor which associates the internal hom $(V \pitchfork \mathcal{V})(c) := [V, \mathcal{V}(c)] \in \mathbf{Ch}_{\mathbb{K}}$ to any $c \in \mathbf{C}$ and the cochain map $[\mathrm{id}, \mathcal{V}(f)] : [V, \mathcal{V}(c_1)] \to [V, \mathcal{V}(c_2)]$ in $\mathbf{Ch}_{\mathbb{K}}$ to each morphism $f : c_1 \to c_2$ in $\mathbf{C}$.

The category $\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ of $\mathbf{Ch}_{\mathbb{K}}$-valued functors is a model category when endowed with the model structure transferred from the one on $\mathbf{Ch}_{\mathbb{K}}$. This is called the projective model structure on the functor category, and it exists because of general results, see for example [Hir03, Thm. 11.6.1], since the target model category $\mathbf{Ch}_{\mathbb{K}}$ endowed with the projective model structure is *cofibrantly generated*. Being cofibrantly generated is a technical condition which, roughly speaking, means that the model structure is determined by certain nice (in a precise way) sets of generating cofibrations and generating trivial cofibrations. See [Hov99, Ch. 2] for the precise definition and for the proof that the projective model structure on $\mathbf{Ch}_{\mathbb{K}}$ is cofibrantly generated. The model structure on the functor category $\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ is determined as follows.

- The weak equivalences are the natural transformations $\eta : \mathcal{V} \to \mathcal{W}$ which are component-wise weak equivalences $\eta_c : \mathcal{V}(c) \to \mathcal{W}(c)$ in $\mathbf{Ch}_{\mathbb{K}}$, i.e. quasi-isomorphisms, for all $c \in \mathbf{C}$;

- The fibrations are the natural transformations $\eta : \mathcal{V} \to \mathcal{W}$ which are component-wise fibrations $\eta_c : \mathcal{V}(c) \to \mathcal{W}(c)$ in $\mathbf{Ch}_{\mathbb{K}}$, i.e. degree-wise surjective cochain maps, for all $c \in \mathbf{C}$.

Similarly to $\mathbf{Ch}_{\mathbb{K}}$, the cofibrations are detected by the left lifting property with respect to the morphisms which are simultaneously fibrations and weak equivalences.

The projective model structure and the (co)tensoring of the dg-category $\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ are compatible, in the sense that they make $\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ a $\mathbf{Ch}_{\mathbb{K}}$-*model category*. This means that tensoring $-\otimes- : \mathbf{Ch}_{\mathbb{K}} \times \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}} \to \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ satisfies the technical condition of being a *Quillen bifunctor*. See [Hov99, Sec. 4.2] for the details. However, the model structure on $\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ loses some of the nice features of the projective model structure of $\mathbf{Ch}_{\mathbb{K}}$. For instance, while all objects $V \in \mathbf{Ch}_{\mathbb{K}}$ are both fibrant and cofibrant, objects $\mathcal{V} \in \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ are in general only fibrant. ($\mathcal{V} \to 0$ being a fibration follows directly from the fact that fibrations are checked component-wise down in $\mathbf{Ch}_{\mathbb{K}}$, where all objects are fibrant.)

Example 2.3.1. It is not hard to show that objects in $\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ are in general not cofibrant, even for a simple shape category $\mathbf{C}$. As an example, consider as shape the horn $\mathbf{C} := \Lambda := \{1 \leftarrow 0 \to 2\}$ seen as a category of three objects and two arrows (in addition to identities). An object $\mathcal{W} \in \mathbf{Ch}_{\mathbb{K}}^{\Lambda}$ is a span $\mathcal{W}_1 \xleftarrow{w_1} \mathcal{W}_0 \xrightarrow{w_2} \mathcal{W}_2$ in $\mathbf{Ch}_{\mathbb{K}}$. We show that if $\mathcal{W} \in \mathbf{Ch}_{\mathbb{K}}^{\Lambda}$ is cofibrant then the cochain maps w_1, w_2 are cofibrations in $\mathbf{Ch}_{\mathbb{K}}$. Hence, the spans $\mathcal{W}$ with (at least) one arrow which is not a cofibration are not cofibrant in $\mathbf{Ch}_{\mathbb{K}}^{\Lambda}$. Let us prove that w_1 is a cofibration by showing the left lifting property against arbitrary acyclic fibration $f : A \to B$ in $\mathbf{Ch}_{\mathbb{K}}$. (A similar argument works for w_2 as well.) Consider the commutative square

$$\begin{tikzcd} \mathcal{W}_0 \arrow[r, "g"] \arrow[d, "w_1"'] & A \arrow[d, "f"] \\ \mathcal{W}_1 \arrow[r, "h"'] & B \end{tikzcd} \tag{2.48}$$

in $\mathbf{Ch}_{\mathbb{K}}$, $f \circ g = h \circ w_1$, and let us thicken it to the diagram

$$\begin{tikzcd} \mathcal{W}_2 \arrow[r] & 0 & 0 \arrow[l] \\ \mathcal{W}_0 \arrow[u, "w_2"] \arrow[r, "g"] \arrow[d, "w_1"'] & A \arrow[u] \arrow[d, "f"'] & A \arrow[l, "\mathrm{id}"'] \arrow[u] \arrow[d, "\mathrm{id}"] \\ \mathcal{W}_1 \arrow[r, "h"'] & B & A \arrow[l, "f"] \end{tikzcd} \tag{2.49}$$

in $\mathbf{Ch}_{\mathbb{K}}$. This is the exploded picture of the cospan $\mathcal{W} \xrightarrow{\zeta} \mathcal{B} \xleftarrow{\eta} \mathcal{A}$ in $\mathbf{Ch}_{\mathbb{K}}^{\Lambda}$ where $\mathcal{A} := \{0 \leftarrow A \xrightarrow{\mathrm{id}} A\}$ and $\mathcal{B} := \{0 \leftarrow A \xrightarrow{f} B\}$ in $\mathbf{Ch}_{\mathbb{K}}^{\Lambda}$. Commutativity of (2.49) implies that η and ζ are natural transformations. Moreover, η is an acyclic fibration in $\mathbf{Ch}_{\mathbb{K}}^{\Lambda}$ since both id and f (by hypothesis) are so in $\mathbf{Ch}_{\mathbb{K}}$. Since we have assumed $\mathcal{W}$ to be cofibrant, there is a lift $\xi : \mathcal{W} \to \mathcal{A}$ in $\mathbf{Ch}_{\mathbb{K}}^{\Lambda}$, $\eta \circ \xi = \zeta$. In particular, $\xi_0 = \zeta_0 = g$ and $f \circ \xi_1 = \eta_1 \circ \xi_1 = \zeta_1 = h$. Naturality of ξ yields $\xi_1 \circ w_1 = \xi_0 = g$. The last two equations show that $\xi_1 : \mathcal{W}_1 \to A$ is a lift of (2.48), hence $w_1 : \mathcal{W}_0 \to \mathcal{W}_1$ is a cofibration as claimed. △

The fact that not all objects of the functor category $\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ are cofibrant has profound implications. In particular, the hom complex $\underline{\mathrm{hom}} : (\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}})^{\mathrm{op}} \times \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}} \to \mathbf{Ch}_{\mathbb{K}}$ may not be a homotopical functor, hence it may fail to preserve weak equivalences. This would be a very unpleasant feature since it contrasts the idea that weak equivalent objects in a model category should be considered as 'being the same'. As a side remark, recall from Section 2.1 that this issue does not arise for the enriched (internal) hom $[-,-] : \mathbf{Ch}_{\mathbb{K}}^{\mathrm{op}} \times \mathbf{Ch}_{\mathbb{K}} \to \mathbf{Ch}_{\mathbb{K}}$ in $\mathbf{Ch}_{\mathbb{K}}$. Therefore, we need to find a replacement for the enriched hom $\underline{\mathrm{hom}}$ which is compliant with the idea that weak equivalent objects in $\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ should be considered as 'being the same', in other words we have to construct a *mapping complex* $\underline{\mathrm{map}} : (\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}})^{\mathrm{op}} \times \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}} \to \mathbf{Ch}_{\mathbb{K}}$ which sends weak equivalent $\mathbf{Ch}_{\mathbb{K}}$-functors in both entries to quasi-isomorphic cochain complexes. Moreover, we ask that the original hom complex $\underline{\mathrm{hom}}(\mathcal{V},\mathcal{W}) \to \underline{\mathrm{map}}(\mathcal{V},\mathcal{W})$ may be realized as a subcomplex of the new, weaker, mapping complex. In the following, we will provide a concrete model for a mapping complex $\underline{\mathrm{map}}$ which fulfills all these requirements and whose construction is motivated and inspired by classical constructions modelling derived functors through (co)simplicial resolutions [Dug01; Fre09; Fre16]. This model was first exploited by us in [BMS23], while a similar construction previously appeared in [Tam07]. Let us start our construction by introducing a simplicial[2] resolution $R(\mathcal{V})_\bullet \in (\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}})^{\Delta^{\mathrm{op}}}$ for the functor $\mathcal{V} \in \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ coming from a cotriple resolution in the sense of [Fre09, Sec. 13.3]. This stems from the free-forgetful adjunction

$$F : \prod_{c\in\mathbf{C}} \mathbf{Ch}_{\mathbb{K}} \rightleftarrows \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}} : U\,, \tag{2.50}$$

where the forgetful functor $U(\mathcal{V}) := (\mathcal{V}(c))_{c\in\mathbf{C}}$ remembers only the family of values of the functor $\mathcal{V} \in \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ on the objects. The left adjoint F defines a notion of free objects in $\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$. Given a family $\underline{V} := (V_c)_{c\in\mathbf{C}}$ of cochain complexes, the associated free $\mathbf{Ch}_{\mathbb{K}}$-valued functor $F(\underline{V}) \in \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ is defined as the coproduct

$$F(\underline{V}) := \coprod_{c_0\in\mathbf{C}} V_{c_0} \otimes \mathbf{C}(c_0,-)\,, \tag{2.51}$$

where $\otimes$ denotes the tensoring of $\mathbf{Ch}_{\mathbb{K}}$ over the category $\mathbf{Set}$ of sets.

Lemma 2.3.2. *The free functor $F : \prod_{c\in\mathbf{C}} \mathbf{Ch}_{\mathbb{K}} \to \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ from (2.51) and the forgetful functor $U : \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}} \to \prod_{c\in\mathbf{C}} \mathbf{Ch}_{\mathbb{K}}$ defined above form a pair of adjoint functors, $F \dashv U$.*

[2] Recall that the *simplex category* Δ is the category whose objects are finite ordinals $[n] = \{0 \to 1 \to \cdots \to n\}$, $n \geq 0$, and morphisms are order-preserving functions between them.

Proof. We prove the adjunction by directly showing the natural isomorphisms of hom sets

$$\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}(F(\underline{V}),\mathcal{W}) \cong \prod_{c\in\mathbf{C}} \mathbf{Ch}_{\mathbb{K}}(V_c,\mathcal{W}(c))\,, \tag{2.52}$$

for $\underline{V} \in \prod_{c\in\mathbf{C}} \mathbf{Ch}_{\mathbb{K}}$ and $\mathcal{W} \in \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$. Let $\eta : F(\underline{V}) \to \mathcal{W}$ in $\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ be a natural transformation. For any object $c \in \mathbf{C}$, one has the cochain map

$$\eta_c : F(\underline{V})(c) = \coprod_{c_0\in\mathbf{C}} V_{c_0} \otimes \mathbf{C}(c_0,c) \longrightarrow \mathcal{W}(c) \tag{2.53}$$

in $\mathbf{Ch}_{\mathbb{K}}$. Denoting by $\iota^c_{c_0} : V_{c_0} \otimes \mathbf{C}(c_0,c) \to F(\underline{V})(c)$ the insertion into the c_0-factor of the coproduct, naturality of η entails that

$$\mathcal{W}(f)\eta_c\iota^c_{c_0}(v\otimes g) = \eta_{c'}F(\underline{V})(f)\iota^c_{c_0}(v\otimes g) = \eta_{c'}\iota^{c'}_{c_0}(v\otimes fg) \tag{2.54}$$

for any $f : c \to c'$ in $\mathbf{C}$ and $v \otimes g \in V_{c_0} \otimes \mathbf{C}(c_0,c)$. Therefore, the c-component of the natural transformation η is uniquely determined by its evaluation upon the objects of type $\iota^{c_0}_{c_0}(v\otimes \mathrm{id})$ for $v \in V_{c_0}$ by the identity

$$\eta_c\iota^c_{c_0}(v\otimes f) = \mathcal{W}(f)\eta_{c_0}\iota^{c_0}_{c_0}(v\otimes \mathrm{id})\,. \tag{2.55}$$

Therefore, the function which sends $\eta \mapsto (\eta_c\iota^c_c(-\otimes\mathrm{id}) : V_c \to \mathcal{W}(c))$ realizes the sought isomorphism of hom sets, whose inverse is provided by the formula (2.55). □

The adjunction $F \dashv U$ yields a cotriple (i.e. a comonad) $C := FU : \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}} \to \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ with comultiplication $C = FU \to F(UF)U = C^2$ defined out of the unit of the adjunction, and counit $C = FU \to \mathrm{id}$ given by the adjunction counit. To be more explicit, on an object $\mathcal{V} \in \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ the counit assigns the natural transformation $FU(\mathcal{V}) \to \mathcal{V}$ in $\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ given componentwise by the cochain map

$$\begin{aligned} FU(\mathcal{V})(c) = \coprod_{c_0\in\mathbf{C}} \mathcal{V}(c_0)\otimes\mathbf{C}(c_0,c) &\longrightarrow \mathcal{V}(c)\\ \iota^c_{c_0}(v\otimes f) &\longmapsto \mathcal{V}(f)v\,, \end{aligned} \tag{2.56}$$

for $v \in \mathcal{V}(c_0)$ and $f : c_0 \to c$ in $\mathbf{C}$. On the other hand, the adjunction unit is given on an object $\underline{V} \in \prod_{c\in\mathbf{C}} \mathbf{Ch}_{\mathbb{K}}$ by the morphism $\underline{V} \to UF(\underline{V})$ in $\prod_{c\in\mathbf{C}} \mathbf{Ch}_{\mathbb{K}}$ whose c-component is the cochain map

$$\begin{aligned} V_c \longrightarrow UF(\underline{V})_c = \coprod_{c_0\in\mathbf{C}} V_{c_0}\otimes\mathbf{C}(c_0,c)&\\ v \longmapsto \iota^c_c(v\otimes\mathrm{id})\,,& \end{aligned} \tag{2.57}$$

in $\mathbf{Ch}_{\mathbb{K}}$. Then, the comultiplication, for $\mathcal{V} \in \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$, returns the morphism in $\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ whose c-component explicitly reads as

$$\begin{aligned} \coprod_{c_0 \in \mathbf{C}} \mathcal{V}(c_0) \otimes \mathbf{C}(c_0, c) &\longrightarrow \coprod_{c_1 \in \mathbf{C}} \Big(\coprod_{c_0 \in \mathbf{C}} \mathcal{V}(c_0) \otimes \mathbf{C}(c_0, c_1) \Big) \otimes \mathbf{C}(c_1, c) \\ \iota_{c_0}^{c}(v \otimes f) &\longmapsto \iota_{c_0}^{c}(\iota_{c_0}^{c_0}(v \otimes \mathrm{id}) \otimes f) \end{aligned} \tag{2.58}$$

where $v \in \mathcal{V}(c_0)$ and $f : c_0 \to c$ in $\mathbf{C}$. The cotriple C allows us to define the *simplicial resolution* of an object $\mathcal{V} \in \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ as the simplicial $\mathbf{Ch}_{\mathbb{K}}$-valued functor

$$R(\mathcal{V})_\bullet := \left(C(\mathcal{V}) \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; C^2(\mathcal{V}) \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \cdots \right) \in (\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}})^{\Delta^{\mathrm{op}}}, \tag{2.59}$$

where the faces are built out of the counit and the degeneracies out of the comultiplication of the cotriple. Given the objects $\mathcal{V}, \mathcal{W} \in \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$, one gets a cosimplicial cochain complex by plugging the simplicial resolution $R(\mathcal{V})_\bullet$ into the enriched hom $\underline{\mathrm{hom}}(-, \mathcal{W})$:

$$\begin{aligned} &\underline{\mathrm{hom}}(R(\mathcal{V})_\bullet, \mathcal{W}) \\ &= \left(\underline{\mathrm{hom}}(C(\mathcal{V}), \mathcal{W}) \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}\; \underline{\mathrm{hom}}(C^2(\mathcal{V}), \mathcal{W}) \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}\; \cdots \right) \in \mathbf{Ch}_{\mathbb{K}}^{\Delta}. \end{aligned} \tag{2.60}$$

Note that we only need to resolve the $\mathbf{Ch}_{\mathbb{K}}$-functor $\underline{\mathrm{hom}}$ in its first entry as a consequence of the fact that objects $\mathcal{W} \in \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ are all fibrant. Let us compute the first cosimplicial degrees of the cosimplicial cochain complex (2.60) in order to get a more manageable and enlightening presentation of it. Let us start by considering the 0-cosimplex. Since $\underline{\mathrm{hom}}(-, \mathcal{W}) : (\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}})^{\mathrm{op}} \to \mathbf{Ch}_{\mathbb{K}}$ is a right adjoint, one has

$$\underline{\mathrm{hom}}(C(\mathcal{V}), \mathcal{W}) \cong \prod_{c_0 \in \mathbf{C}} \underline{\mathrm{hom}}(\mathcal{V}(c_0) \otimes \mathbf{C}(c_0, -), \mathcal{W}) \in \mathbf{Ch}_{\mathbb{K}}. \tag{2.61}$$

Each component of the n-cochain $\eta = (\eta_c) \in \underline{\mathrm{hom}}(C(\mathcal{V}), \mathcal{W})^n$ in it is a degree n natural transformation $\eta_{c_0} : \mathcal{V}(c_0) \otimes \mathbf{C}(c_0, -) \to \mathcal{W}$ of the underlying graded vector space-valued functors (i.e. forgetting the differentials). Naturality condition implies that for all $f : c \to c'$ in $\mathbf{C}$ the identity

$$\mathcal{W}(f)(\eta_{c_0})_c(v \otimes g) = (\eta_{c_0})_{c'}(v \otimes fg) \tag{2.62}$$

holds true for all objects $v \in \mathcal{V}(c_0)$ and morphisms $g : c_0 \to c$ in $\mathbf{C}$. Arguing as in the proof of Lemma 2.3.2 above, one realizes that the graded linear maps $(\eta_{c_0})_c$ are uniquely determined by the graded linear map $\overline{\eta}_{c_0} := (\eta_{c_0})_{c_0}(- \otimes \mathrm{id}) \in [\mathcal{V}(c_0), \mathcal{W}(c_0)]^n$. Hence, there is an isomorphism

$$\underline{\mathrm{hom}}(C(\mathcal{V}), \mathcal{W}) \cong \prod_{c_0 \in \mathbf{C}} [\mathcal{V}(c_0), \mathcal{W}(c_0)] \tag{2.63}$$

of graded vector spaces. It is also compatible with the differentials, namely

$$\partial\overline{\eta} = (\partial\overline{\eta}_{c_0}) = (\partial(\eta_{c_0})_{c_0}(-\otimes \mathrm{id})) = ((\partial\eta_{c_0})_{c_0}(-\otimes \mathrm{id})) = \overline{\partial\eta}\,, \tag{2.64}$$

hence the isomorphism (2.63) upgrades to an isomorphism in $\mathbf{Ch}_{\mathbb{K}}$. Since no confusion would arise, we will drop the notation for the isomorphism and simply write η instead. Let us move to the 1-cosimplex,

$$\underline{\mathrm{hom}}(C^2(\mathcal{V}),\mathcal{W}) \cong \prod_{c_1\in\mathbf{C}}\prod_{c_0\in\mathbf{C}} \underline{\mathrm{hom}}((\mathcal{V}(c_0)\otimes\mathbf{C}(c_0,c_1))\otimes\mathbf{C}(c_1,-),\mathcal{W})\,, \tag{2.65}$$

where we used the definition of the comonad C and that $\underline{\mathrm{hom}}(-,\mathcal{W})$ is a right adjoint. Let $\zeta = (\zeta_{c_0,c_1}) \in \underline{\mathrm{hom}}(C^2(\mathcal{V}),\mathcal{W})$ be a homogeneous element in the enriched hom, namely

$$\zeta_{c_0,c_1} : \mathcal{V}(c_0)\otimes\mathbf{C}(c_0,c_1)\otimes\mathbf{C}(c_1,-) \longrightarrow \mathcal{W} \tag{2.66}$$

is a graded natural transformation between the underlying graded vector space-valued functors, for all $c_0, c_1 \in \mathbf{C}$. Let then $f : c \to c'$ be a morphism in $\mathbf{C}$, naturality of ζ_{c_0,c_1} entails that

$$\mathcal{W}(f)(\zeta_{c_0,c_1})_c(v\otimes g\otimes h) = (\zeta_{c_0,c_1})_{c'}(v\otimes g\otimes fh), \tag{2.67}$$

for all $v \in \mathcal{V}(c_0)$ and $g : c_0 \to c_1, h : c_1 \to c$ in $\mathbf{C}$. Once again, one finds that ζ_{c_0,c_1} is uniquely determined by the graded linear map given by its evaluation $(\zeta_{c_0,c_1})_{c_1}(-\otimes-\otimes \mathrm{id})$ on the identity map $\mathrm{id} : c_1 \to c_1$ on its last entry. Therefore, we have isomorphisms

$$\begin{aligned}
\underline{\mathrm{hom}}(C^2(\mathcal{V}),\mathcal{W}) &\cong \prod_{c_1\in\mathbf{C}}\prod_{c_0\in\mathbf{C}} [\mathcal{V}(c_0)\otimes\mathbf{C}(c_0,c_1),\mathcal{W}(c_1)]\\
&\cong \prod_{c_1\in\mathbf{C}}\prod_{c_0\in\mathbf{C}} [\mathcal{V}(c_0),\prod_{f:c_0\to c_1}\mathcal{W}(c_1)]\\
&\cong \prod_{\underline{c}:[1]\to\mathbf{C}} [\mathcal{V}(c_0),\mathcal{W}(c_1)]
\end{aligned} \tag{2.68}$$

of graded vector spaces, where in the second step we used the isomorphism $[\mathbf{C}(c_0,c_1),\mathcal{W}(c_1)] \cong \prod_{f:c_0\to c_1}\mathcal{W}(c_1)$ in $\mathbf{Ch}_{\mathbb{K}}$. The last product runs over all functors $\underline{c} : [1] \to \mathbf{C}$ from the totally ordered set $[1] = \{0 \to 1\}$, regarded as a category. By a straightforward computation, as in Equation (2.64), one checks that the isomorphism (2.68) is compatible with differentials, hence it upgrades to an isomorphism of cochain complexes. Exploiting arguments similar to the ones above, one soon realizes that the pattern goes on also for higher cosimplices, namely there are isomorphisms

$$\underline{\mathrm{hom}}(C^n(\mathcal{V}),\mathcal{W}) \cong \prod_{\underline{c}:[n]\to\mathbf{C}} [\mathcal{V}(c_0),\mathcal{W}(c_n)] \tag{2.69}$$

in $\mathbf{Ch}_{\mathbb{K}}$ for all $n \geq 0$. The product in (2.69) runs over all functors $\underline{c} : [n] \to \mathbf{C}$, where the totally ordered set $[n] = \{0 \to 1 \to \cdots \to n\}$ is regarded as a category. In other words, a functor $\underline{c}$ is given by a set of n composable morphisms $c_0 \xrightarrow{\gamma_0} c_1 \xrightarrow{\gamma_1} \cdots \xrightarrow{\gamma_{n-1}} c_n$ in $\mathbf{C}$.

These isomorphisms give us a more concrete presentation of the cosimplicial cochain complex (2.60). It just remains to explicitly write out the coface and codegeneracy cochain maps. We will proceed as above and perform the explicit computations for the lower degrees only, since it turns to be sufficient to get to the pattern which is followed also on higher cosimplicial degrees. Let us start with the codegeneracy map $s^0 : \prod_{\underline{c}:[1]\to\mathbf{C}} [\mathcal{V}(c_0), \mathcal{W}(c_1)] \to \prod_{c_0\in\mathbf{C}} [\mathcal{V}(c_0), \mathcal{W}(c_0)]$. To write it out explicitly we have to intertwine the degeneracy map in (2.59), given by the comultiplication (2.58), with the isomorphisms from Equations (2.63) and (2.68). Consider a homogeneous element $\zeta = (\zeta_f) \in \prod_{\underline{c}:[1]\to\mathbf{C}} [\mathcal{V}(c_0), \mathcal{W}(c_1)]$, under the isomorphism (2.68) it provides the graded natural transformation $\zeta \in \underline{\hom}(C^2(\mathcal{V}), \mathcal{W})$ whose c-component is

$$\begin{aligned} \zeta_c : \coprod_{c_1\in\mathbf{C}} \coprod_{c_0\in\mathbf{C}} \mathcal{V}(c_0) \otimes \mathbf{C}(c_0, c_1) \otimes \mathbf{C}(c_1, c) &\longrightarrow \mathcal{W}(c) \\ \iota^c_{c_0,c_1}(v \otimes f \otimes g) &\longmapsto \mathcal{W}(g)\eta_f(v)\,, \end{aligned} \tag{2.70}$$

for any $c \in \mathbf{C}$. The pullback along the comultiplication (2.58) returns the graded natural transformation $s^0\zeta \in \underline{\hom}(C(\mathcal{V}), \mathcal{W})$ given by components

$$\begin{aligned} (s^0\zeta)_c : \coprod_{c_0\in\mathbf{C}} \mathcal{V}(c_0) \otimes \mathbf{C}(c_0, c) &\longrightarrow \mathcal{W}(c) \\ \iota^c_{c_0}(v \otimes f) &\longmapsto \zeta_c \iota^c_{c_0,c_0}(v \otimes \mathrm{id} \otimes f) = \mathcal{W}(f)\zeta_{\mathrm{id}}(v)\,, \end{aligned} \tag{2.71}$$

for all $c \in \mathbf{C}$. Finally, using the isomorphism (2.63) one finds the codegeneracy map

$$\begin{aligned} s^0 : \prod_{\underline{c}:[1]\to\mathbf{C}} [\mathcal{V}(c_0), \mathcal{W}(c_1)] &\longrightarrow \prod_{c_0\in\mathbf{C}} [\mathcal{V}(c_0), \mathcal{W}(c_0)] \\ \zeta = (\zeta_f) &\longmapsto s^0\zeta = (\zeta_{\mathrm{id}})\,, \end{aligned} \tag{2.72}$$

which acts selecting the components associated to the identity morphisms $\mathrm{id} : c_0 \to c_0$ in $\mathbf{C}$. Since the degeneracy maps in (2.59) are built out of the comultiplication, by acting with it on different stages of the C power, e.g. $C^2(\mathcal{V}) \to (CC)C(\mathcal{V}) = C^3(\mathcal{V})$ and $C^2(\mathcal{V}) \to C(CC)(\mathcal{V}) = C^3(\mathcal{V})$, the higher codegeneracy maps s^k of (2.60) are given by selecting the components associated to the identity morphisms stemming from the k-position of the chain of composable arrows $\underline{c}$. In other words, one produces out of $\underline{c}$ a chain of composable maps of length increased by 1 by inserting the arrow $c_k \to c_k$. See Equation (2.78) below for a precise definition.

We conclude the construction of our cosimplicial resolution with the description of its coface maps. As it will be clear in a moment, there are three different types of coface maps $d^k : \prod_{\underline{c}:[n]\to\mathbf{C}}[\mathcal{V}(c_0),\mathcal{W}(c_n)] \to \prod_{\underline{c}:[n+1]\to\mathbf{C}}[\mathcal{V}(c_0),\mathcal{W}(c_{n+1})]$, depending if $k = 0, n+1$ is an extreme index or it is $1 \leq k \leq n$ intermediate. Therefore, we have to go up to degree $n = 1$ in order to see all of them. In this case one has to compute the cochain maps $d^0, d^1, d^2 : \prod_{\underline{c}:[1]\to\mathbf{C}}[\mathcal{V}(c_0),\mathcal{W}(c_1)] \to \prod_{\underline{c}:[2]\to\mathbf{C}}[\mathcal{V}(c_0),\mathcal{W}(c_2)]$. They are obtained by intertwining the face maps of (2.59), built out of the counit (2.56), with the isomorphisms from (2.69). Given a family $\zeta = (\zeta_f) \in \prod_{\underline{c}:[1]\to\mathbf{C}}[\mathcal{V}(c_0),\mathcal{W}(c_1)]$ of graded linear maps, it identifies a graded natural transformation $\zeta \in \underline{\mathrm{hom}}(C^2(\mathcal{V}),\mathcal{W})$ according to the isomorphism (2.63) as per Equation (2.70). We pullback it along the enriched hom $\underline{\mathrm{hom}}(-,\mathcal{W})$ of the face maps of the simplicial resolution (2.59) . Recall that the latter are obtained from the counit (2.56), hence they explicitly reads as

$$\partial_0 := C^2(C(\mathcal{V}) \longrightarrow \mathcal{V})$$
$$\partial_0 : \iota^c_{c_0,c_1,c_2}(v \otimes f \otimes g \otimes h) \longmapsto \iota^c_{c_1,c_2}(\mathcal{V}(f)v \otimes g \otimes h)\,, \tag{2.73a}$$

$$\partial_1 := C(C(C(\mathcal{V})) \longrightarrow C(\mathcal{V}))$$
$$\partial_1 : \iota^c_{c_0,c_1,c_2}(v \otimes f \otimes g \otimes h) \longmapsto \iota^c_{c_0,c_2}(v \otimes gf \otimes h)\,, \tag{2.73b}$$

$$\partial_2 := C(C^2(\mathcal{V})) \longrightarrow C^2(\mathcal{V})$$
$$\partial_2 : \iota^c_{c_0,c_1,c_2}(v \otimes f \otimes g \otimes h) \longmapsto \iota^c_{c_0,c_1}(v \otimes f \otimes gh)\,, \tag{2.73c}$$

where $f : c_0 \to c_1, g : c_1 \to c_2,\ h : c_2 \to c$ in $\mathbf{C}$ and $v \in \mathcal{V}(c_0)$. After a straightforward computation, one gets the graded natural transformations $d^k\zeta$, $0 \leq k \leq 2$, whose c-components are

$$(d^k\zeta)_c : \coprod_{(c_0,c_1,c_2)\in\mathbf{C}^{\times 3}} \mathcal{V}(c_0) \otimes \mathbf{C}(c_0,c_1) \otimes \mathbf{C}(c_1,c_2) \otimes \mathbf{C}(c_2,c) \longrightarrow \mathcal{W}(c)$$
$$(d^0\zeta)_c : \iota^c_{c_0,c_1,c_2}(v \otimes f \otimes g \otimes h) \longmapsto \mathcal{W}(h)\zeta_g(\mathcal{V}(f)v)\,, \tag{2.74a}$$
$$(d^1\zeta)_c : \iota^c_{c_0,c_1,c_2}(v \otimes f \otimes g \otimes h) \longmapsto \mathcal{W}(h)\zeta_{gf}(v)\,, \tag{2.74b}$$
$$(d^2\zeta)_c : \iota^c_{c_0,c_1,c_2}(v \otimes f \otimes g \otimes h) \longmapsto \mathcal{W}(hg)\zeta_f(v)\,. \tag{2.74c}$$

Finally, exploiting the isomorphism (2.69), we identify the natural transformations above with the coface cochain maps

$$d^k : \prod_{\underline{c}:[1]\to\mathbf{C}}[\mathcal{V}(c_0),\mathcal{W}(c_1)] \longrightarrow \prod_{\underline{c}:[2]\to\mathbf{C}}[\mathcal{V}(c_0),\mathcal{W}(c_2)]$$
$$d^0 : \zeta = (\zeta_f) \longmapsto d^0\zeta = ((d^0\zeta)_{f,g} = \zeta_g \circ \mathcal{V}(f))\,, \tag{2.75a}$$
$$d^1 : \zeta = (\zeta_f) \longmapsto d^1\zeta = ((d^1\zeta)_{f,g} = \zeta_{gf})\,, \tag{2.75b}$$
$$d^2 : \zeta = (\zeta_f) \longmapsto d^2\zeta = ((d^2\zeta)_{f,g} = \mathcal{W}(g) \circ \zeta_f)\,. \tag{2.75c}$$

The first and last coface maps, $k = 0, 2$, act by taking the pullback $[\mathcal{V}(f), \mathrm{id}]$ along the first map and the pushforward $[\mathrm{id}, \mathcal{W}(g)]$ along the last map of the chain of composable arrows $\underline{c}$, respectively. The intermediate coface map d^1, instead, picks the component of ζ associated to the composition gf of the two arrows. This behavior continues even on higher cosimplicial degrees and it always corresponds to pick the component associated to the chain of composable arrows obtained from $\underline{c}$ by reducing by 1 its length by dropping the element c_k. If c_k is intermediate, $1 \le k \le n$, the new chain is obtained by composing arrows γ_{k-1} and γ_k. If c_k, $k = 0, n+1$, is extremal the dropped arrow goes, respectively, in the pullback $[\mathcal{V}(\gamma_0), \mathrm{id}]$ or in the pushforward $[\mathrm{id}, \mathcal{W}(\gamma_n)]$. See Equations (2.79) for a precise definition.

We sum up the results of our computations in the following lemma.

Lemma 2.3.3. *Given $\mathbf{Ch}_{\mathbb{K}}$-valued functors $\mathcal{V}, \mathcal{W} \in \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$, the cosimplicial cochain complex $\underline{\mathrm{hom}}(R(\mathcal{V})_\bullet, \mathcal{W}) \in \mathbf{Ch}_{\mathbb{K}}^{\Delta}$ from (2.60) is isomorphic to the cosimplicial cochain complexes*

$$C(\mathcal{V}, \mathcal{W})^\bullet := \Big(C(\mathcal{V}, \mathcal{W})^0 \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}\; C(\mathcal{V}, \mathcal{W})^1 \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}\; \cdots \Big) \in \mathbf{Ch}_{\mathbb{K}}^{\Delta}\,, \tag{2.76}$$

consisting of the cochain complex

$$C(\mathcal{V}, \mathcal{W})^n := \prod_{\underline{c}:[n]\to\mathbf{C}} [\mathcal{V}(c_0), \mathcal{W}(c_n)] \in \mathbf{Ch}_{\mathbb{K}}\,, \tag{2.77}$$

for all $n \ge 0$, of the codegeneracy maps $s^k : C(\mathcal{V}, \mathcal{W})^{n+1} \to C(\mathcal{V}, \mathcal{W})^n$ in $\mathbf{Ch}_{\mathbb{K}}$ defined, for all $n \ge 0$ and $0 \le k \le n$, by Equation (2.78), and of the coface maps $d^k : C(\mathcal{V}, \mathcal{W})^n \to C(\mathcal{V}, \mathcal{W})^{n+1}$ in $\mathbf{Ch}_{\mathbb{K}}$ given, for all $n \ge 0$ and $0 \le k \le n+1$, as per Equations (2.79) below. Introducing the order preserving map $\check{k} : [n+1] \to [n]$ which hits the element $k \in [n]$ twice, the codegeneracy maps s^k, $0 \le k \le n$, are given by

$$\begin{tikzcd}
C(\mathcal{V}, \mathcal{W})^{n+1} \arrow[r, dashed, "s^k"] \arrow[d, "\mathrm{pr}_{\underline{c}\circ\check{k}}"'] & C(\mathcal{V}, \mathcal{W})^n \arrow[d, "\mathrm{pr}_{\underline{c}}"] \\
{[\mathcal{V}(c_0), \mathcal{W}(c_n)]} \arrow[r, "\mathrm{id}"'] & {[\mathcal{V}(c_0), \mathcal{W}(c_n)]}
\end{tikzcd} \tag{2.78}$$

Introducing also the order preserving map $\widehat{k} : [n] \to [n+1]$ which skips the element $k \in [n+1]$, one writes the coface map d^0 as

$$\begin{tikzcd}
C(\mathcal{V}, \mathcal{W})^n \arrow[r, dashed, "d^0"] \arrow[d, "\mathrm{pr}_{\underline{c}\circ\widehat{0}}"'] & C(\mathcal{V}, \mathcal{W})^{n+1} \arrow[d, "\mathrm{pr}_{\underline{c}}"] \\
{[\mathcal{V}(c_1), \mathcal{W}(c_{n+1})]} \arrow[r, "{[\mathcal{V}(\gamma_0), \mathrm{id}]}"'] & {[\mathcal{V}(c_0), \mathcal{W}(c_{n+1})]}
\end{tikzcd} \tag{2.79a}$$

the coface maps d^k, $1 \leq k \leq n$, as

$$\begin{tikzcd}
C(\mathcal{V},\mathcal{W})^n \arrow[r, dashed, "d^k"] \arrow[d, "\mathrm{pr}_{\underline{c}\circ\widehat{k}}"'] & C(\mathcal{V},\mathcal{W})^{n+1} \arrow[d, "\mathrm{pr}_{\underline{c}}"] \\
{[\mathcal{V}(c_0),\mathcal{W}(c_{n+1})]} \arrow[r, "\mathrm{id}"'] & {[\mathcal{V}(c_0),\mathcal{W}(c_{n+1})]}
\end{tikzcd} \tag{2.79b}$$

and the coface map d^{n+1} as

$$\begin{tikzcd}
C(\mathcal{V},\mathcal{W})^n \arrow[r, dashed, "d^{n+1}"] \arrow[d, "\mathrm{pr}_{\underline{c}\circ\widehat{n+1}}"'] & C(\mathcal{V},\mathcal{W})^{n+1} \arrow[d, "\mathrm{pr}_{\underline{c}}"] \\
{[\mathcal{V}(c_0),\mathcal{W}(c_n)]} \arrow[r, "{[\mathrm{id},\mathcal{W}(\gamma_n)]}"'] & {[\mathcal{V}(c_0),\mathcal{W}(c_{n+1})]}
\end{tikzcd} \tag{2.79c}$$

We conclude the construction of our model for the homotopical replacement of the enriched hom $\underline{\mathrm{hom}}$ by taking the $\prod$-total complex associated to the cosimplicial cochain complex (2.76). To be more precise, to the cosimplicial cochain complex $C(\mathcal{V},\mathcal{W})^\bullet \in \mathbf{Ch}_{\mathbb{K}}^{\Delta}$ it is associated the double complex $C(\mathcal{V},\mathcal{W})^{\bullet,\bullet} \in \mathbf{Ch}^{\geq 0}(\mathbf{Ch}_{\mathbb{K}})$ by taking the alternating sum of the coface maps as its horizontal differential. Then, we define the *mapping complex* between $\mathbf{Ch}_{\mathbb{K}}$-valued functors $\mathcal{V},\mathcal{W} \in \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ as the $\prod$-total complex of the double complex $C(\mathcal{V},\mathcal{W})^{\bullet,\bullet}$. Therefore, $\underline{\mathrm{map}}(\mathcal{V},\mathcal{W}) \in \mathbf{Ch}_{\mathbb{K}}$ is the cochain complex consisting of the graded vector space defined degree-wise for all $n \in \mathbb{Z}$ as

$$\underline{\mathrm{map}}(\mathcal{V},\mathcal{W})^n := \prod_{q\geq 0} C(\mathcal{V},\mathcal{W})^{q,n-q} = \prod_{q\geq 0}\prod_{\underline{c}:[q]\to\mathbf{C}} [\mathcal{V}(c_0),\mathcal{W}(c_q)]^{n-q}\,, \tag{2.80}$$

where $q \geq 0$ is the cosimplicial degree while $n-q \in \mathbb{Z}$ is the cochain degree, and of the differential

$$\delta := \delta_{\mathrm{h}} + \delta_{\mathrm{v}} \tag{2.81a}$$

given by the sum of the *horizontal differential* δ_{h} defined component-wise by

$$\mathrm{pr}_0 \circ \delta_{\mathrm{h}} := 0\,, \qquad \mathrm{pr}_q \circ \delta_{\mathrm{h}} := \sum_{k=0}^{q} (-1)^k d^{q-k} \circ \mathrm{pr}_{q-1}\,, \tag{2.81b}$$

for all $q \geq 1$, and of the *vertical differential* δ_{v} defined component-wise by

$$\mathrm{pr}_{q,\underline{c}} \circ \delta_{\mathrm{v}} := (-1)^q\, \partial \circ \mathrm{pr}_{q,\underline{c}}\,, \tag{2.81c}$$

for all $q \geq 0$ and $\underline{c} : [q] \to \mathbf{C}$, where $\mathrm{pr}_{q,\underline{c}} := \mathrm{pr}_{\underline{c}} \circ \mathrm{pr}_q$ denotes the projection onto the $(q,\underline{c})$-component $[\mathcal{V}(c_0),\mathcal{W}(c_q)]^{n-q}$ of $\underline{\mathrm{map}}(\mathcal{V},\mathcal{W})^n$ and ∂ is the 'adjoint' differential of the internal hom $[\mathcal{V}(c_0),\mathcal{W}(c_q)] \in \mathbf{Ch}_{\mathbb{K}}$. The signs in δ_{h} can be understood projecting onto $\underline{c}$-components. Let us think of the

q-tuple $\underline{c}$ of composable morphisms in $\mathbf{C}$ as concentrated on the projection map $\mathrm{pr}_{\underline{c}}$. Since the k-th summand of the right-hand side of (2.81b) acts with d^{q-k}, it has to be pulled, from right to left, through the last k arrows of $\underline{c}$ in order to reach the object c_{q-k} it has to drop. Since each morphism in $\underline{c}$ contributes 1 to the total degree and δ_{h} has degree 1, this gives rise to the sign $(-1)^k$. Similarly, the sign in (2.81c) arises from the fact that δ_{v} has to be pulled through all the q morphisms of $\underline{c}$ for ∂ to act. Since $\underline{c}$ contributes q to the total degree and δ_{v} has degree 1, this gives rise to the sign $(-1)^q$.

Remark 2.3.4. It is common folklore that a model which computes the homotopy limit $\mathrm{holim}\, V^\bullet \in \mathbf{Ch}_{\mathbb{K}}$ of a cosimplicial cochain complex $V^\bullet \in \mathbf{Ch}_{\mathbb{K}}^{\Delta}$ is provided by the totalization of the double complex $V^{\bullet,\bullet} \in \mathbf{Ch}^{\geq 0}(\mathbf{Ch}_{\mathbb{K}})$ associated to it. See e.g. [Bun13, Problem 4.23], [Dug17, Sec. 5][AØ23] for related statements. This entails that the mapping complex $\underline{\mathrm{map}}(\mathcal{V},\mathcal{W})$ from (2.80) may be regarded as the homotopy limit $\mathrm{holim}\, C(\mathcal{V},\mathcal{W})^\bullet$ of the cosimplicial resolution from Lemma 2.3.3. ▽

Proposition 2.3.5. *The mapping complex* $\underline{\mathrm{map}}(-,-) : (\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}})^{\mathrm{op}} \times \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}} \to \mathbf{Ch}_{\mathbb{K}}$ *is a homotopical functor, namely it preserves weak equivalences in both entries.*

Proof. Let $\eta : \mathcal{V}' \xrightarrow{\sim} \mathcal{V}$ and $\zeta : \mathcal{W} \xrightarrow{\sim} \mathcal{W}'$ in $\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ be weak equivalences. For all integers $n \geq 0$ and functors $\underline{c} : [n] \to \mathbf{C}$ one has that the cochain map $[\eta_{c_0}, \zeta_{c_n}] : [\mathcal{V}(c_0), \mathcal{W}(c_n)] \xrightarrow{\sim} [\mathcal{V}'(c_0), \mathcal{W}'(c_n)]$ in $\mathbf{Ch}_{\mathbb{K}}$ is a quasi-isomorphism, since the internal hom $[-,-]$ is homotopical and the components $\eta_c : \mathcal{V}'(c) \xrightarrow{\sim} \mathcal{V}(c)$ and $\zeta_c : \mathcal{W}(c) \xrightarrow{\sim} \mathcal{W}'(c)$ in $\mathbf{Ch}_{\mathbb{K}}$ of the weak equivalences η, ζ in $\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ are quasi-isomorphisms for all $c \in \mathbf{C}$ by definition. Since cohomology commutes with taking products in $\mathbf{Ch}_{\mathbb{K}}$ we get that $C(\eta,\zeta)^n : C(\mathcal{V},\mathcal{W})^n \xrightarrow{\sim} C(\mathcal{V}',\mathcal{W}')^n$ in $\mathbf{Ch}_{\mathbb{K}}$ is a quasi-isomorphism for all $n \geq 0$, hence $C(\eta,\zeta)^\bullet$ in $\mathbf{Ch}_{\mathbb{K}}^{\Delta}$ is a natural quasi-isomorphism, that is a natural transformation which is component-wise a quasi-isomorphism. We conclude by exploiting that the homotopy limit functor $\mathrm{holim} : \mathbf{Ch}_{\mathbb{K}}^{\Delta} \to \mathbf{Ch}_{\mathbb{K}}$ is homotopical by definition, hence $\mathrm{holim}\, C(\eta,\zeta)^\bullet : \underline{\mathrm{map}}(\mathcal{V},\mathcal{W}) \xrightarrow{\sim} \underline{\mathrm{map}}(\mathcal{V}',\mathcal{W}')$ in $\mathbf{Ch}_{\mathbb{K}}$ is a weak equivalence, see Remark 2.3.4. □

Remark 2.3.6. Let us comment on how to interpret the mapping complex $\underline{\mathrm{map}}$. While the n-cocycles of the enriched hom complex $\underline{\mathrm{hom}}$ describe graded natural transformations between functors $\mathcal{V},\mathcal{W} \in \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$, the mapping complex $\underline{\mathrm{map}}$ encodes a notion of 'homotopy coherent' graded natural transformations. Indeed, let $\eta \in \underline{\mathrm{map}}(\mathcal{V},\mathcal{W})^n$ be a n-cochain in the mapping complex. It is given by a family of $(n-q)$-cochains $\mathrm{pr}_{q,\underline{c}}\, \eta \in [\mathcal{V}(c_0), \mathcal{W}(c_q)]^{n-q}$ in the internal hom labeled by integers $q \geq 0$ and functors $\underline{c} : [q] \to \mathbf{C}$. The cocycle condition $\delta\eta = 0$ has the following implications: First, the $q = 0$ components $\mathrm{pr}_{0,c}\, \eta \in [\mathcal{V}(c), \mathcal{W}(c)]^n$ of η, for each $c \in \mathbf{C}$, are the components of a degree n transformation between the functors $\mathcal{V}$ and $\mathcal{W}$ which is compatible with differentials. In fact, they are all cocycles in the internal hom,

$$\partial(\mathrm{pr}_{0,c}\,\eta) = \mathrm{pr}_{0,c}(\delta_{\mathrm{v}}\eta) = \mathrm{pr}_{0,c}(\delta\eta) = 0\,, \tag{2.82}$$

as a consequence of the definition (2.81) of the differential δ of $\underline{\mathrm{map}}$. However, these may fail to be the components of a natural transformation. In this case the higher cosimplicial components $\mathrm{pr}_q\,\eta$, for $q > 0$, of the cocycle η play the role of (higher) homotopies which provide η as a natural transformation up to homotopy. To fix ideas, consider a functor $\underline{c} : [1] \to \mathbf{C}$, i.e. a morphism $\gamma_0 : c_0 \to c_1$ in $\mathbf{C}$, then compute

$$\begin{aligned}\partial(\mathrm{pr}_{1,\gamma_0}\,\eta) &= -\,\mathrm{pr}_{1,\gamma_0}(\delta_{\mathrm{v}}\eta) = \mathrm{pr}_{1,\gamma_0}(\delta_{\mathrm{h}}\eta)\\ &= \mathcal{W}(\gamma_0)\circ(\mathrm{pr}_{0,c_0}\,\eta) - (\mathrm{pr}_{0,c_1}\,\eta)\circ\mathcal{V}(\gamma_0)\end{aligned} \tag{2.83}$$

using (2.81), (2.79) and that $\delta\eta = 0$. It follows that, in general, $\mathrm{pr}_0\,\eta$ is not a natural transformation, yet its failure is witnessed by the homotopy $\mathrm{pr}_1\,\eta$, as explicitly shown by Equation (2.83). The data $\mathrm{pr}_{1,\gamma_0}\,\eta$, for all $\gamma_0 : c_0 \to c_1$, may again fail to be natural in the appropriate sense, but the components of η of cosimplicial degree 2 provide the higher homotopies $\mathrm{pr}_{2,\underline{c}}\,\eta$, for all $\underline{c} = (c_0 \xrightarrow{\gamma_0} c_1 \xrightarrow{\gamma_1} c_2)$ in $\mathbf{C}$, which witness this failure:

$$\partial(\mathrm{pr}_{2,\underline{c}}\,\eta) = -\mathcal{W}(\gamma_1)\circ(\mathrm{pr}_{1,\gamma_0}\,\eta) + \mathrm{pr}_{1,\gamma_1\gamma_0}\,\eta - (\mathrm{pr}_{1,\gamma_1}\,\eta)\circ\mathcal{V}(\gamma_0)\,. \tag{2.84}$$

This pattern goes on with higher and higher homotopies. More precisely, one has

$$\begin{aligned}\partial(\mathrm{pr}_{q,\underline{c}}\,\eta) &= (-1)^{q+1}\mathcal{W}(\gamma_{q-1})\circ(\mathrm{pr}_{q-1,\underline{c}\circ\widehat{q}}\,\eta)\\ &\quad - \sum_{k=1}^{q-1}(-1)^{q+k}\,\mathrm{pr}_{q-1,\underline{c}\circ\widehat{q-k}}\,\eta + (\mathrm{pr}_{q-1,\underline{c}\circ\widehat{0}})\circ\mathcal{V}(\gamma_0)\,.\end{aligned} \tag{2.85}$$

for any chain $\underline{c} = (c_0 \xrightarrow{\gamma_0} \cdots \xrightarrow{\gamma_{n-1}} c_q)$ of morphisms in $\mathbf{C}$. $\triangledown$

Observe that given functors $\mathcal{V},\mathcal{W} \in \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ one has the inclusion

$$\underline{\mathrm{hom}}(\mathcal{V},\mathcal{W}) \xrightarrow{\subseteq} \underline{\mathrm{map}}(\mathcal{V},\mathcal{W}) \tag{2.86}$$

in $\mathbf{Ch}_{\mathbb{K}}$ of the hom complex into the mapping complex. This is obtained by assigning to any n-cochain $\eta \in \underline{\mathrm{hom}}(\mathcal{V},\mathcal{W})^n$ in the hom complex the n-cochain $\widetilde{\eta} \in \underline{\mathrm{map}}(\mathcal{V},\mathcal{W})^n$ in the mapping complex defined by the components $\mathrm{pr}_0\,\widetilde{\eta} := \eta \in \underline{C}(\mathcal{V},\mathcal{W})^{0,n}$ and $\mathrm{pr}_q\,\widetilde{\eta} := 0 \in C(\mathcal{V},\mathcal{W})^{q,n-q}$ for all $q \geq 1$. This assignment is clearly compatible with differentials, as a direct check shows: In cosimplicial degree 0 one has

$$\mathrm{pr}_0(\delta\widetilde{\eta}) = \partial(\mathrm{pr}_0\,\widetilde{\eta}) = \partial\eta = \mathrm{pr}_0\,\widetilde{\partial\eta}\,, \tag{2.87}$$

while for $q \geq 1$ and any $\underline{c} : [q] \to \mathbf{C}$,

$$\mathrm{pr}_{q,\underline{c}}(\delta\widetilde{\eta}) = (-1)^q \partial(\mathrm{pr}_{q,\underline{c}}\,\widetilde{\eta}) + \sum_{k=0}^{q} (-1)^k\, \mathrm{pr}_{\underline{c}} \circ d^{q-k} \circ \mathrm{pr}_{q-1}\,\widetilde{\eta}\,. \tag{2.88}$$

Since by definition all components $\mathrm{pr}_p\,\widetilde{\eta} = 0$, vanish for $p \geq 1$, the right-hand side of (2.88) vanishes for all $q \geq 2$. The only possibly non vanishing component of $\delta\widetilde{\eta}$ is then for $q = 1$. Yet, $\mathrm{pr}_{1,\gamma_0}(\delta\widetilde{\eta}) = \mathcal{W}(\gamma_0)\circ\eta_{c_1} - \mathcal{V}(\gamma_0)\circ\eta_{c_0} = 0$ for all $\gamma_0 : c_0 \to c_1$ in $\mathbf{C}$ because of naturality of $\eta \in \underline{\mathrm{hom}}(\mathcal{V},\mathcal{W})^n$. Hence, $\delta\widetilde{\eta} = \widetilde{\partial\eta}$ and the 0 degree map in (2.86) is actually a cochain map. This inclusion may be interpreted by saying that the strict naturality (encoded by $\underline{\mathrm{hom}}$) is a particular instance of the homotopy coherent naturality (encoded by $\underline{\mathrm{map}}$). In the rest of this work we will implicitly identify n-cochains $\eta \in \underline{\mathrm{hom}}(\mathcal{V},\mathcal{W})^n$ with their counterpart $\widetilde{\eta} \in \underline{\mathrm{map}}(\mathcal{V},\mathcal{W})^n$ in the mapping complex without making explicit use of $\widetilde{\cdot}$ in our notation.

2.3.1 $\underline{\mathrm{map}}(-,-)$ as an enriched hom

The mapping complex $\underline{\mathrm{map}}(\mathcal{V},\mathcal{W})$ may be used to provide a cochain complex of maps from $\mathcal{V} \in \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ to $\mathcal{W} \in \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$, endowing the functor category $\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ with a $\mathbf{Ch}_{\mathbb{K}}$-enrichment alternative to the one introduced at the beginning of Section 2.3 based on the non-homotopical hom complex $\underline{\mathrm{hom}}(\mathcal{V},\mathcal{W})$. This subsection shall be devoted to the description of the former $\mathbf{Ch}_{\mathbb{K}}$-enrichment. In order to regard the functor category $\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ as a dg-category (see Section 2.1, or [Kel05] for more details) we have to provide, in addition to the hom-object $\underline{\mathrm{map}} : (\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}})^{\mathrm{op}} \times \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}} \to \mathbf{Ch}_{\mathbb{K}}$, a composition cochain map

$$\circ_{\mathcal{V},\mathcal{W},\mathcal{Z}} : \underline{\mathrm{map}}(\mathcal{W},\mathcal{Z}) \otimes \underline{\mathrm{map}}(\mathcal{V},\mathcal{W}) \longrightarrow \underline{\mathrm{map}}(\mathcal{V},\mathcal{Z}) \tag{2.89}$$

in $\mathbf{Ch}_{\mathbb{K}}$, for all $\mathcal{V},\mathcal{W}$ and $\mathcal{Z} \in \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$, and an identity cochain map

$$j_{\mathcal{V}} : \mathbb{K} \longrightarrow \underline{\mathrm{map}}(\mathcal{V},\mathcal{V}) \tag{2.90}$$

in $\mathbf{Ch}_{\mathbb{K}}$, for all $\mathcal{V} \in \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$, where $\mathbb{K}$ is the monoidal unit of $\mathbf{Ch}_{\mathbb{K}}$. The composition has to be associative and unital with respect to the identity cochain map. Let us define such maps, starting with the identity cochain map. Given any object $\mathcal{V} \in \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$, we define $j_{\mathcal{V}}$ as the graded linear map

$$\mathbb{K} \ni 1 \longmapsto \mathrm{id} \in \mathsf{Z}^0\underline{\mathrm{hom}}(\mathcal{V},\mathcal{V}) \subseteq \mathsf{Z}^0\underline{\mathrm{map}}(\mathcal{V},\mathcal{V}) \tag{2.91}$$

which assigns the identity natural transformation id to 1. It is a cochain map because id is closed, $\delta\mathrm{id} = 0$. Here we used the inclusion (2.86) in $\mathbf{Ch}_{\mathbb{K}}$. For all $\mathcal{V},\mathcal{W},\mathcal{Z} \in \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ we define the composition map $\circ_{\mathcal{V},\mathcal{W},\mathcal{Z}}$ as the degree 0 graded linear map whose n-component

$$\circ^n_{\mathcal{V},\mathcal{W},\mathcal{Z}} : \bigoplus_{i\in\mathbb{Z}} \underline{\mathrm{map}}(\mathcal{W},\mathcal{Z})^i \otimes \underline{\mathrm{map}}(\mathcal{V},\mathcal{W})^{n-i} \longrightarrow \underline{\mathrm{map}}(\mathcal{V},\mathcal{Z})^n \tag{2.92a}$$

is the linear map that sends all homogeneous $\zeta \in \underline{\mathrm{map}}(\mathcal{W},\mathcal{Z})^i$ and $\eta \in \underline{\mathrm{map}}(\mathcal{V},\mathcal{W})^{n-i}$ to the n-cochain $\zeta \circ \eta \in \underline{\mathrm{map}}(\mathcal{V},\mathcal{Z})^n$, defined componentwise by

$$\mathrm{pr}_{q,\underline{c}}\,(\zeta \circ \eta) := \sum_{k=0}^{q} (-1)^{k(q-k+i)} (\mathrm{pr}_{q-k,\underline{c}^{\geq k}}\,\zeta) \circ (\mathrm{pr}_{k,\underline{c}^{\leq k}}\,\eta)\,, \tag{2.92b}$$

for all integers $q \geq 0$ and functors $\underline{c} : [q] \to \mathbf{C}$. In the equation displayed above we used the notation $\underline{c}^{\leq k} := (c_0 \xrightarrow{\gamma_0} \cdots \xrightarrow{\gamma_{k-1}} c_k)$ for the k-uple of composable morphisms in $\mathbf{C}$ obtained from $\underline{c}$ by keeping only the first k arrows and, similarly, $\underline{c}^{\geq k} := (c_k \xrightarrow{\gamma_k} \cdots \xrightarrow{\gamma_{q-1}} c_q)$ for the $(q-k)$-uple built out of $\underline{c}$ by discarding its first k arrows. As for the signs appearing in the definition (2.81) of the differential δ of the mapping complex, we can provide you with an intuitive justification for the signs appearing in the definition of the composition morphism (2.92). Recall that each arrow in $\underline{c}$ contributes 1 to the total degree and that they are thought of as concentrated on the projection map $\mathrm{pr}_{\underline{c}}$. In the k-th summand of Equation (2.92b), the first k arrows of $\underline{c}$ (i.e. $\underline{c}^{\leq k}$) are pushed through the last $q-k$ (i.e. $\underline{c}^{\geq k}$) and ζ, which is of degree i, thus accounting for the sign $(-1)^{k((q-k)+i)}$. The following lemma states that Equation (2.92) defines a cochain map, i.e. the composition map $\circ_{\mathcal{V},\mathcal{W},\mathcal{Z}}$ is compatible with differentials. The proof of this fact, that involves quite long and unpleasant computations, is postponed to Appendix A.

Lemma 2.3.7. *The composition map $\circ_{\mathcal{V},\mathcal{W},\mathcal{Z}}$ defined by Equation (2.92) is closed in the internal hom, namely*

$$\partial(\circ_{\mathcal{V},\mathcal{W},\mathcal{Z}}) = 0 \in \big[\underline{\mathrm{map}}(\mathcal{W},\mathcal{Z}) \otimes \underline{\mathrm{map}}(\mathcal{V},\mathcal{W}), \underline{\mathrm{map}}(\mathcal{V},\mathcal{Z})\big]^1\,, \tag{2.93}$$

hence it is a cochain map in $\mathbf{Ch}_{\mathbb{K}}$, for all $\mathcal{V},\mathcal{W},\mathcal{Z} \in \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$.

We conclude this subsection by claiming that the mapping complex $\underline{\mathrm{map}}$ from (2.80), the identity cochain maps $j_{\mathcal{V}}$ from (2.90) and the composition cochain maps $\circ_{\mathcal{V},\mathcal{W},\mathcal{Z}}$ from (2.92) define a dg-enrichment of the functor category $\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$. The proof of the Proposition below is given in Appendix A.

Proposition 2.3.8. *The composition cochain map $\circ_{\mathcal{V},\mathcal{W},\mathcal{Z}}$, defined in (2.92) above, is associative and unital with respect to the identity cochain map $j_{\mathcal{V}}$ from (2.90). Hence, together with the mapping complex $\underline{\mathrm{map}}$ from (2.80), they provide an enrichment over $\mathbf{Ch}_{\mathbb{K}}$ of the functor category $\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$.*

2.3.2 The homotopy colimit as a dg-left adjoint

In this subsection we shall produce a model for the homotopy colimit of a $\mathbf{Ch}_{\mathbb{K}}$-valued diagram $\mathcal{V} \in \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$, which is obtained from the standard procedure, related to the bar construction, of taking the totalization of a simplicial replacement of $\mathcal{V}$. See for example [Shu09; Rie14], [CG17, Appx. C] and [Dug17, Sec. 4] for similar discussions. Moreover, we shall extend the homotopy colimit functor hocolim : $\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}} \to \mathbf{Ch}_{\mathbb{K}}$ to a dg-functor, where $\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ is regarded as a dg-category endowed with the enrichment introduced in Section 2.3.1, while $\mathbf{Ch}_{\mathbb{K}}$ is enriched over itself in the standard way (see e.g. Section 2.1). Finally, we will conclude this subsection with Proposition 2.3.12 which shows that the hocolim dg-functor is a dg-adjoint of the diagonal dg-functor Δ.

Let $\mathcal{V} \in \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ be a $\mathbf{Ch}_{\mathbb{K}}$-valued functor. We introduce the simplicial cochain complex $S(\mathcal{V})_\bullet \in \mathbf{Ch}_{\mathbb{K}}^{\Delta^{\mathrm{op}}}$, coming through a bar construction,

$$S(\mathcal{V})_\bullet := \Big(S(\mathcal{V})_0 \;\substack{\longleftarrow\\ \Longrightarrow}\; S(\mathcal{V})_1 \;\substack{\longleftarrow\\ \Longrightarrow\\ \longleftarrow\\ \longrightarrow}\; \cdots \Big) \in \mathbf{Ch}_{\mathbb{K}}^{\Delta^{\mathrm{op}}}\,, \tag{2.94a}$$

consisting of the cochain complexes

$$S(\mathcal{V})_n := \bigoplus_{\underline{c}:[n]\to\mathbf{C}} \mathcal{V}(c_0) \in \mathbf{Ch}_{\mathbb{K}}\,, \tag{2.94b}$$

for all integers $n \geq 0$, of the face maps

$$d_k : S(\mathcal{V})_{n+1} \longrightarrow S(\mathcal{V})_n \tag{2.94c}$$

in $\mathbf{Ch}_{\mathbb{K}}$, defined in (2.95) below for all integers $n \geq 0$ and $0 \leq k \leq n+1$, and of the degeneracy maps

$$s_k : S(\mathcal{V})_n \longrightarrow S(\mathcal{V})_{n+1} \tag{2.94d}$$

in $\mathbf{Ch}_{\mathbb{K}}$, defined in (2.96) for all integers $n \geq 0$ and $0 \leq k \leq n$. The d_0 face map is defined by

$$\begin{tikzcd} S(\mathcal{V})_{n+1} \arrow[r, dashed, "d_0"] & S(\mathcal{V})_n \\ \mathcal{V}(c_0) \arrow[u, "\iota_{\underline{c}}"] \arrow[r, "\mathcal{V}(\gamma_0)"'] & \mathcal{V}(c_1) \arrow[u, "\iota_{\underline{c}\circ\hat{0}}"'] \end{tikzcd} \tag{2.95a}$$

for $1 \leq k \leq n+1$, one defines the face map d_k by

$$\begin{tikzcd} S(\mathcal{V})_{n+1} \arrow[r, dashed, "d_k"] & S(\mathcal{V})_n \\ \mathcal{V}(c_0) \arrow[u, "\iota_{\underline{c}}"] \arrow[r, "\mathrm{id}"'] & \mathcal{V}(c_0) \arrow[u, "\iota_{\underline{c}\circ\hat{k}}"'] \end{tikzcd} \tag{2.95b}$$

Finally, the degeneracy map s_k, for $0 \leq k \leq n$, is defined by

$$\begin{tikzcd} S(\mathcal{V})_{n} \arrow[r, dashed, "s_k"] & S(\mathcal{V})_{n+1} \\ \mathcal{V}(c_0) \arrow[u, "\iota_{\underline{c}}"] \arrow[r, "\mathrm{id}"'] & \mathcal{V}(c_0) \arrow[u, "\iota_{\underline{c}\circ\check{k}}"'] \end{tikzcd} \tag{2.96}$$

where $\iota_{\underline{c}} : \mathcal{V}(c_0) \to S(\mathcal{V})_n$ in $\mathbf{Ch}_{\mathbb{K}}$, for $\underline{c} : [n] \to \mathbf{C}$, denotes the injection into the $\underline{c}$-summand of the direct sum $S(\mathcal{V})_n = \bigoplus_{\underline{c}:[n]\to\mathbf{C}} \mathcal{V}(c_0)$. The *homotopy colimit* of $\mathcal{V} \in \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ may be modeled as the $\bigoplus$-totalization of the double complex $S(\mathcal{V})^{\bullet,\bullet} \in \mathbf{Ch}^{\leq 0}(\mathbf{Ch}_{\mathbb{K}})$ associated to the simplicial replacement $S(\mathcal{V})_{\bullet} \in \mathbf{Ch}_{\mathbb{K}}^{\Delta^{\mathrm{op}}}$. More precisely, the double complex consists of the (bi)graded vector space defined degree-wise for all integers $p \geq 0$ and $q \in \mathbb{Z}$ by

$$S(\mathcal{V})^{-p,q} := (S(\mathcal{V})_p)^q \in \mathbf{Vec}_{\mathbb{K}}\,, \tag{2.97}$$

where the sign in front of p is due to the fact that we are turning a simplicial degree into a cohomological degree, of the horizontal differential

$$\mathrm{d}_{\mathrm{h}} : S(\mathcal{V})^{-p,q} \longrightarrow S(\mathcal{V})^{-p+1,q} \tag{2.98a}$$

which necessarily vanishes, $\mathrm{d}_{\mathrm{h}} = 0$, for $p = 0$, and it is given by the alternating sum of the face cochain maps (2.95),

$$\mathrm{d}_{\mathrm{h}} := -\sum_{k=0}^{p} (-1)^{-k} d_k\,, \tag{2.98b}$$

for all integers $p \geq 1$, where the global sign is purely conventional and it has been chosen so that the dg-adjunction from Proposition 2.3.12 holds true, and of the vertical differential

$$\begin{aligned} \mathrm{d}_{\mathrm{v}} : S(\mathcal{V})^{-p,q} &\longrightarrow S(\mathcal{V})^{-p,q+1}\,,\\ \mathrm{d}_{\mathrm{v}} \circ \iota_{\underline{c}} &:= (-1)^{-p} \iota_{\underline{c}} \circ Q\,, \end{aligned} \tag{2.99}$$

for all functors $\underline{c} : [p] \to \mathbf{C}$, where Q denotes the differential of the cochain complex $\mathcal{V}(c_0) \in \mathbf{Ch}_{\mathbb{K}}$. The signs in the definition of both the horizontal and the vertical differential admit an interpretation similar to the one we gave for the differential δ of the mapping complex. They may be interpreted by considering the degree of $\underline{c}$ as located on the inclusion maps $\iota_{\underline{c}}$. Keeping this in mind, the signs in d_{h} are due to the fact that its k-th summand acts with d_k, which needs to be pulled through the first k morphisms of $\underline{c}$. Since each arrow in $\underline{c}$ contributes -1 to the total degree and d_{h} has degree 1, this gives rise to the sign $(-1)^{-k}$. Similarly, the sign in d_{v} arises from the fact that it acts with Q after being pulled through the whole $\underline{c}$. Since the latter contributes $-p$ to the total degree and d_{v} has degree 1, this gives rise to the sign $(-1)^{-p}$. The homotopy colimit $\mathrm{hocolim}(\mathcal{V}) \in \mathbf{Ch}_{\mathbb{K}}$ is given by the cochain complex consisting of the graded vector space defined degree-wise for all $n \in \mathbb{Z}$ by

$$\mathrm{hocolim}(\mathcal{V})^n := \bigoplus_{p\geq 0} S(\mathcal{V})^{-p,n+p} = \bigoplus_{p\geq 0} \bigoplus_{\underline{c}:[p]\to\mathbf{C}} \mathcal{V}(c_0)^{n+p}\,, \tag{2.100}$$

and of the differential $\mathrm{d} : \mathrm{hocolim}(\mathcal{V})^n \to \mathrm{hocolim}(\mathcal{V})^{n+1}$ defined by

$$\mathrm{d} \circ \iota_0 := \iota_0 \circ \mathrm{d}_{\mathrm{v}} \tag{2.101a}$$

and

$$\mathrm{d} \circ \iota_p := \iota_{p-1} \circ \mathrm{d}_{\mathrm{h}} + \iota_p \circ \mathrm{d}_{\mathrm{v}} \tag{2.101b}$$

for all integers $p \geq 1$.

Remark 2.3.9. It is immediate to see that the hocolim functor preserves weak equivalences. Indeed, given weak equivalent functors $\mathcal{V} \sim \mathcal{W} \in \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$, one has that the cochain complexes $\mathcal{V}(c_0) \sim \mathcal{W}(c_0) \in \mathbf{Ch}_{\mathbb{K}}$ are quasi-isomorphic for all $p \geq 0$ and $\underline{c} : [p] \to \mathbf{C}$, by definition. Since quasi-isomorphisms are preserved under arbitrary direct sums, one concludes that the homotopy colimits $\mathrm{hocolim}(\mathcal{V}) \sim \mathrm{hocolim}(\mathcal{W})$ are quasi-isomorphic. ▽

Remark 2.3.10. As usual, the homotopy colimit functor (2.100) comes with a canonical natural transformation hocolim → colim to the ordinary colimit. The $\mathcal{V}$-component of this natural transformation is given by

$$\begin{tikzcd} \mathrm{hocolim}(\mathcal{V}) \arrow[r, dashed] & \mathrm{colim}(\mathcal{V}) \\ \mathcal{V}(c_0) \arrow[u, "\iota_{0,c_0}"] \arrow[r, "\mathrm{id}"'] & \mathcal{V}(c_0) \arrow[u, "\iota_{c_0}"'] \end{tikzcd} \tag{2.102a}$$

for all $c_0 \in \mathbf{C}$, where the vertical arrow on the right, $\iota_{c_0} : \mathcal{V}(c_0) \to \mathrm{colim}(\mathcal{V})$ in $\mathbf{Ch}_{\mathbb{K}}$, is the canonical cochain map from a node of a diagram to its colimit, and by

$$\begin{tikzcd} \mathrm{hocolim}(\mathcal{V}) \arrow[r, dashed] & \mathrm{colim}(\mathcal{V}) \\ \mathcal{V}(c_0) \arrow[u, "\iota_{p,\underline{c}}"] \arrow[r] & 0 \arrow[u] \end{tikzcd} \tag{2.102b}$$

for all $p \geq 1$ and $\underline{c} : [p] \to \mathbf{C}$, where $\iota_{p,\underline{c}} := \iota_p \circ \iota_{\underline{c}}$. Let us note that, when the shape category $\mathbf{C}$ is filtered[3], the natural transformation (2.102) is a natural quasi-isomorphism. Indeed, the Grothendieck AB5 axiom implies that the filtered (ordinary) colimits are already homotopical, hence they provide a model for the homotopy colimits. This situation will be common in Chapters 3 and 4. ▽

Enriched functors are the natural generalization of the notion of functors in the context of enriched category theory. Given categories $\mathbf{C}, \mathbf{D}$ enriched over a monoidal category $\mathbf{M}$, an enriched functor $F : \mathbf{C} \to \mathbf{D}$ sends objects

[3] A category $\mathbf{C}$ is called *filtered* if (1) it is not empty, (2) for every pair of objects $c_1, c_2 \in \mathbf{C}$ there exist an object $c \in \mathbf{C}$ and morphisms $f_1 : c_1 \to c$ and $f_2 : c_2 \to c$ in $\mathbf{C}$ and (3) for every two parallel morphisms $f, g : c_1 \to c_2$ in $\mathbf{C}$ there is an object $c \in \mathbf{C}$ and a morphism $j : c \to c_1$ such that $f \circ j = g \circ j$.

$c \in \mathbf{C}$ to the objects $F(c) \in \mathbf{D}$, just as a regular functor, but instead of mapping $\mathbf{C}$-morphisms to $\mathbf{D}$-morphisms, it assigns morphisms $F_{c_1,c_2} : \mathrm{hom}_{\mathbf{C}}(c_1, c_2) \to \mathrm{hom}_{\mathbf{D}}(F(c_1), F(c_2))$ in $\mathbf{M}$ between the hom-objects, that are compatible, in the obvious way, with the enriched compositions and identities. Let us promote the homotopy colimit functor $\mathrm{hocolim} : \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}} \to \mathbf{Ch}_{\mathbb{K}}$ from (2.100) to a $\mathbf{Ch}_{\mathbb{K}}$-enriched functor (a *dg-functor* for short). This amounts to define cochain maps

$$\mathrm{hocolim} : \underline{\mathrm{map}}(\mathcal{V}, \mathcal{W}) \longrightarrow [\mathrm{hocolim}(\mathcal{V}), \mathrm{hocolim}(\mathcal{W})] \tag{2.103}$$

in $\mathbf{Ch}_{\mathbb{K}}$, for all objects $\mathcal{V}, \mathcal{W} \in \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$, and to check their compatibility with compositions and identities. Recalling that $\mathbf{Ch}_{\mathbb{K}}$ is a closed monoidal category, we define them by giving the adjunct cochain maps

$$\mathrm{hocolim} : \underline{\mathrm{map}}(\mathcal{V}, \mathcal{W}) \otimes \mathrm{hocolim}(\mathcal{V}) \longrightarrow \mathrm{hocolim}(\mathcal{W}) \tag{2.104}$$

in $\mathbf{Ch}_{\mathbb{K}}$. We denote the latter with the same symbol with a slight abuse of notation. Let $\eta \in \underline{\mathrm{map}}(\mathcal{V}, \mathcal{W})^i$ and $\iota_{p,\underline{c}} v \in \mathrm{hocolim}(\mathcal{V})^{n-i}$, with $p \geq 0$, $\underline{c} : [p] \to \mathbf{C}$ and $v \in \mathcal{V}(c_0)^{n-i+p}$. The degree $n \in \mathbb{Z}$ of (2.104) is defined by the linear extension of

$$\mathrm{hocolim}^n(\eta \otimes \iota_{p,\underline{c}} v) := \sum_{k=0}^{p} (-1)^{-pi+k(p-k)} \iota_{p-k,\underline{c}^{\geq k}}((\mathrm{pr}_{k,\underline{c}^{\leq k}}\, \eta)\, v)\,. \tag{2.105}$$

As usual, we motivate the signs in the equation displayed above. Since each arrow in $\underline{c}$ contributes -1 to the total degree of the homotopy colimit, the sign in each summand of (2.105) may be understood by observing that η, which is of degree i, is pulled through $\underline{c}$ and $\underline{c}^{\leq k} : [k] \to \mathbf{C}$ is pulled through $\underline{c}^{\geq k} : [q-k] \to \mathbf{C}$. The following proposition states that the above definition of the action on the hom-objects allows us to upgrade the homotopy colimit functor hocolim to a dg-functor. To avoid annoying the reader with some lengthy, and not particularly enlightening, computations, we move the proof to Appendix A.

Proposition 2.3.11. *Consider the dg-categories $\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$, endowed with the $\underline{\mathrm{map}}$ enrichment from Subsection 2.3.1, and $\mathbf{Ch}_{\mathbb{K}}$. Then, the homotopy colimit* $\mathrm{hocolim} : \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}} \to \mathbf{Ch}_{\mathbb{K}}$ *from* (2.100)*, together with the action* (2.105) $\mathrm{hocolim} : \underline{\mathrm{map}}(\mathcal{V}, \mathcal{W}) \to [\mathrm{hocolim}(\mathcal{V}), \mathrm{hocolim}(\mathcal{W})]$ *on the hom-objects, defines a dg-functor.*

Consider also the diagonal functor

$$\Delta : \mathbf{Ch}_{\mathbb{K}} \longrightarrow \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}\,, \tag{2.106}$$

which sends the cochain complex $V \in \mathbf{Ch}_{\mathbb{K}}$ to the constant functor $\Delta(V) \in \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$. It is a dg-functor whose action $\Delta : [V, W] \to \underline{\mathrm{map}}(\Delta(V), \Delta(W))$ on the hom complexes, for all $V, W \in \mathbf{Ch}_{\mathbb{K}}$, is defined as follows: Given any n-cochain $f \in [V, W]^n$ in the internal hom, $\Delta(f) \in \underline{\mathrm{map}}(\Delta(V), \Delta(W))^n$ is defined as the n-cochain in the mapping complex whose components are $\mathrm{pr}_{0,c_0}(\Delta(f)) := f$, for all $c_0 \in \mathbf{C}$, and $\mathrm{pr}_{q,\underline{c}}(\Delta(f)) := 0$, for all integers $q \geq 1$ and functors $\underline{c} : [q] \to \mathbf{C}$. It is clear from the definition that $\Delta(f)$ is strictly natural for all $f : V \to W$ in $\mathbf{Ch}_{\mathbb{K}}$. Hence, Δ factors through the enriched hom, $\Delta : [V, W] \to \underline{\mathrm{hom}}(\Delta(V), \Delta(W)) \subseteq \underline{\mathrm{map}}(\Delta(V), \Delta(W))$. This makes clear that Δ is a cochain map, indeed, for all $f \in [V, W]^n$, we have $(\partial(\Delta(f)))_c = \partial(\Delta(f))_c = \partial f = (\Delta(\partial f))_c$, for all $c \in \mathbf{C}$. Compatibilities with the enriched identities and compositions are immediate as well. Let us just comment briefly on the compatibility with compositions: It is enough to note that the only non vanishing $(q, \underline{c})$-components of $\Delta(g) \circ \Delta(f)$, for all homogeneous $f \in [V, W]$ and $g \in [W, Z]$, are for $q = 0$, and they are given by $\mathrm{pr}_{0,c_0}(\Delta(g) \circ \Delta(f)) = g \circ f$, for all $c_0 \in \mathbf{C}$.

We conclude this subsection by showing that the homotopy colimit dg-functor hocolim is dg-left adjoint to the diagonal dg-functor Δ. This extends to the enriched context the standard adjunction colim $\dashv \Delta$ between the (ordinary) colimit and the diagonal functor.

Recall that, given categories $\mathbf{C}_1, \mathbf{C}_2$ enriched over the closed monoidal category $\mathbf{M}$, a pair of enriched functors $L : \mathbf{C}_1 \rightleftarrows \mathbf{C}_2 : R$ is an enriched adjunction if there is an enriched natural isomorphism bewteen the hom-objects functors $\mathbf{C}_2(L(-), -) \cong \mathbf{C}_1(-, R(-)) : \mathbf{C}_1^{\mathrm{op}} \otimes \mathbf{C}_2 \to \mathbf{M}$, where $\mathbf{C}_1^{\mathrm{op}} \otimes \mathbf{C}_2$ denotes the product category endowed with the tensor product enrichment, while the closed monoidal category $\mathbf{M}$ is canonically enriched over itself. See for example [Kel69; Kel05] for more details. We are now ready to state and prove our proposition.

Proposition 2.3.12. *The homotopy colimit dg-functor* $\mathrm{hocolim} : \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}} \to \mathbf{Ch}_{\mathbb{K}}$ *from Equation* (2.100) *is dg-left adjoint to the diagonal dg-functor* $\Delta : \mathbf{Ch}_{\mathbb{K}} \to \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ *from* (2.106).

Proof. To prove that hocolim $\dashv \Delta$ is a dg-adjunction of dg-functors, we have to provide a dg-natural isomorphism $\underline{\mathrm{map}}(-, \Delta(-)) \cong [\mathrm{hocolim}(-), -]$ between the hom complexes. Recall, e.g. [Kel05], that a dg-natural transformation is given by a family of cochain maps

$$\Psi_{(\mathcal{V},W)} : \mathbb{K} \longrightarrow [\underline{\mathrm{map}}(\mathcal{V}, \Delta(W)), [\mathrm{hocolim}(\mathcal{V}), W]] \tag{2.107}$$

in $\mathbf{Ch}_{\mathbb{K}}$ for all objects $(\mathcal{V}, W) \in (\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}})^{\mathrm{op}} \otimes \mathbf{Ch}_{\mathbb{K}}$, subject to a suitable naturality condition. Note that the family (2.107) is equivalent to the family of cochain maps

$$\psi_{(\mathcal{V},W)} := \Psi_{(\mathcal{V},W)}(1) : \underline{\mathrm{map}}(\mathcal{V}, \Delta(W)) \longrightarrow [\mathrm{hocolim}(\mathcal{V}), W] \tag{2.108}$$

in $\mathbf{Ch}_{\mathbb{K}}$, for all $(\mathcal{V}, W) \in (\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}})^{\mathrm{op}} \otimes \mathbf{Ch}_{\mathbb{K}}$. Let us define our candidates $\psi_{(\mathcal{V},W)}$ in $\mathbf{Ch}_{\mathbb{K}}$ and then prove that they are all isomorphisms and that they assemble in a dg-natural transformation. For $\mathcal{V} \in \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ and $W \in \mathbf{Ch}_{\mathbb{K}}$, we define $\psi_{(\mathcal{V},W)}$ as the adjunct of the map (denoted by the same symbol abusing the notation)

$$\psi_{(\mathcal{V},W)} : \underline{\mathrm{map}}(\mathcal{V}, \Delta(W)) \otimes \mathrm{hocolim}(\mathcal{V}) \longrightarrow W \tag{2.109a}$$

which assigns the $(n+i)$-cochain

$$\psi_{(\mathcal{V},W)}(\eta \otimes \iota_{p,\underline{c}} v) := (-1)^{-pi} (\mathrm{pr}_{p,\underline{c}}\, \eta)^{n+p} v \in W^{n+i} \tag{2.109b}$$

to each homogeneous $\eta \in \underline{\mathrm{map}}(\mathcal{V}, \Delta(W))^i$ and $\iota_{p,\underline{c}} v \in \mathrm{hocolim}(\mathcal{V})^n$, for all $p \geq 0$, $\underline{c} : [p] \to \mathbf{C}$ and $v \in \mathcal{V}(c_0)^{n+p}$. First, note that Equation (2.109) defines a cochain map. Indeed, a direct computation, using the definitions of all ingredients, shows that

$$\begin{aligned}
&\psi_{(\mathcal{V},W)}(\delta_{\mathrm{v}}\eta \otimes \iota_{p,\underline{c}}) + (-1)^i \psi_{(\mathcal{V},W)}(\eta \otimes \mathrm{d}_{\mathrm{v}} \iota_{p,\underline{c}} v)\\
&\quad = (-1)^{-pi} \partial(\mathrm{pr}_{p,\underline{c}}\, \eta)\, v + (-1)^{i-p-pi} (\mathrm{pr}_{p,\underline{c}}\, \eta)\, Q_{\mathcal{V}(c_0)} v\\
&\quad = (-1)^{-pi} Q_W (\mathrm{pr}_{p,\underline{c}}\, \eta) v\\
&\quad = Q_W \psi_{(\mathcal{V},W)}(\eta \otimes \iota_{p,\underline{c}} v)\,.
\end{aligned} \tag{2.110}$$

Similarly, one shows that $\psi_{(\mathcal{V},W)}(\delta_{\mathrm{h}}\eta \otimes \iota_{p,\underline{c}}) + (-1)^i \psi_{(\mathcal{V},W)}(\eta \otimes \mathrm{d}_{\mathrm{h}} \iota_{p,\underline{c}} v) = 0$. Hence, the graded linear map $\psi_{(\mathcal{V},W)}$ is a cochain map, $\partial \psi_{(\mathcal{V},W)} = 0$, for all $(\mathcal{V}, W) \in (\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}})^{\mathrm{op}} \otimes \mathbf{Ch}_{\mathbb{K}}$. Furthermore, it is an isomorphism in $\mathbf{Ch}_{\mathbb{K}}$. Consider the degree 0 graded linear map $\phi_{(\mathcal{V},W)} : [\mathrm{hocolim}(\mathcal{V}), W] \to \underline{\mathrm{map}}(\mathcal{V}, \Delta(W))$ which sends a i-cochain $f \in [\mathrm{hocolim}(\mathcal{V}), W]^i$ to the i-cochain $\phi_{(\mathcal{V},W)}(f) \in \underline{\mathrm{map}}(\mathcal{V}, \Delta(W))^i$ defined by

$$\begin{gathered}
\mathrm{pr}_{q,\underline{c}}(\phi_{(\mathcal{V},W)}(f)) \in [\mathcal{V}(c_0), W]^{i-q}\,,\\
\mathcal{V}(c_0)^n \ni v \longmapsto \mathrm{pr}_{q,\underline{c}}(\phi_{(\mathcal{V},W)}(f))^n\, v := (-1)^{qi} f^{n-q}(\iota_{q,\underline{c}} v) \in W^{n+i-q}\,,
\end{gathered} \tag{2.111}$$

for all $q \geq 0$ and $\underline{c} : [q] \to \mathbf{C}$. It is immediate to check that $\phi^i_{(\mathcal{V},W)}$ is the inverse of $\psi^i_{(\mathcal{V},W)}$, for all $i \in \mathbb{Z}$. In order to conclude the proof, it just remains to check the naturality of the cochain maps $\Psi_{(\mathcal{V},W)}$ from (2.107). In terms of the cochain maps $\psi_{(\mathcal{V},W)}$ the naturality hexagon for dg-natural transformations, see e.g. [Kel05, Sec. 1.2], reduces to the more familiar square

$$\begin{tikzcd}
\underline{\mathrm{map}}(\mathcal{V}_2, \Delta(W_1)) \arrow[r, "\psi_{(\mathcal{V}_2,W_1)}"] \arrow[d, "\underline{\mathrm{map}}(\eta,\Delta(f))"'] & {[\mathrm{hocolim}(\mathcal{V}_2), W_1]} \arrow[d, "{[\mathrm{hocolim}(\eta), f]}"] \\
\underline{\mathrm{map}}(\mathcal{V}_1, \Delta(W_2)) \arrow[r, "\psi_{(\mathcal{V}_1,W_2)}"'] & {[\mathrm{hocolim}(\mathcal{V}_1), W_2]}
\end{tikzcd} \tag{2.112}$$

of graded linear maps, for any homogeneous $\eta \in \underline{\mathrm{map}}(\mathcal{V}_1, \mathcal{V}_2)^i$ and $f \in [W_1, W_2]^q$. The vertical arrows in the diagram displayed above are given by the action of the enriched hom-functors on the hom complexes of morphisms. They are defined by the obvious pullbacks and pushforwards. Let then $\zeta \in \underline{\mathrm{map}}(\mathcal{V}_2, \Delta(W_1))^r$ be any r-cochain in the mapping complex. For all $p \geq 0$, $\underline{c} : [p] \to \mathbf{C}$ and $v \in \mathcal{V}_1(c_0)^n$, one has

$$
\begin{aligned}
&([\mathrm{hocolim}(\eta), f](\psi_{(\mathcal{V}_2,W_1)}(\zeta)))(\iota_{p,\underline{c}} v) \\
&\quad = (-1)^{i(q+r)} (f \circ \psi_{(\mathcal{V}_2,W_1)}(\zeta) \circ \mathrm{hocolim}(\eta))(\iota_{p,\underline{c}} v) \\
&\quad = (-1)^{i(q+r)} \sum_{k=0}^{p} (-1)^{-pi+k(p-k)} f(\psi_{(\mathcal{V}_2,W_1)}(\zeta)(\iota_{p-k,\underline{c}^{\geq k}}((\mathrm{pr}_{k,\underline{c}^{\leq k}}\, \eta)\, v))) \\
&\quad = (-1)^{i(q+r)} \sum_{k=0}^{p} (-1)^{-p(i+r)+k(p-k+r)} f(\mathrm{pr}_{p-k,\underline{c}^{\geq k}}\, \zeta)(\mathrm{pr}_{k,\underline{c}^{\leq k}}\, \eta)\, v
\end{aligned} \tag{2.113}
$$

where the first step exploits the definition of the dg-functor $[\mathrm{hocolim}(-), -]$. Second step uses the definition (2.105) of the homotopy colimit dg-functor and third step the definition (2.109) of the isomorphism $\psi_{(\mathcal{V}_2,W_1)}$. Similarly,

$$
\begin{aligned}
&(\psi_{(\mathcal{V}_1,W_2)}(\underline{\mathrm{map}}(\eta, \Delta(f)))(\zeta))(\iota_{p,\underline{c}} v) \\
&\quad = (-1)^{i(q+r)} (\psi_{(\mathcal{V}_1,W_2)}(\Delta(f) \circ \zeta \circ \eta))(\iota_{p,\underline{c}} v) \\
&\quad = (-1)^{i(q+r)} (-1)^{-p(q+r+i)} (\mathrm{pr}_{p,\underline{c}}(\Delta(f) \circ \zeta \circ \eta))\, v \\
&\quad = (-1)^{i(q+r)} \sum_{k=0}^{p} (-1)^{-p(q+r+i)+k(p-k+q+r)}\, \mathrm{pr}_{p-k,\underline{c}^{\geq k}}(\Delta(f) \circ \zeta)(\mathrm{pr}_{k,\underline{c}^{\leq k}}\, \eta)\, v \\
&\quad = (-1)^{i(q+r)} \sum_{k=0}^{p} (-1)^{-p(r+i)+k(p-k+r)} f(\mathrm{pr}_{p-k,\underline{c}^{\geq k}}\, \zeta)(\mathrm{pr}_{k,\underline{c}^{\leq k}}\, \eta)\, v\,,
\end{aligned} \tag{2.114}
$$

where we used the definition of the dg-functor $\underline{\mathrm{map}}(-, \Delta(-))$ in the first step and the definition of $\psi_{(\mathcal{V}_1,W_2)}$ in the second. The enriched composition in $\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ from (2.92) is used in the last two steps and the definition of the diagonal dg-functor Δ in the last one. Since Equations (2.113) and (2.114) agree, we conclude our proof. □

2.4 Kan complexes from cochain complexes

In Section 3.2 we shall need Kan complexes to correctly describe the space of retarded/advanced Green's homotopies (Definition 3.1.4) and, importantly, to prove that the latter are unique in a suitable sense, see Proposition 3.2.2. For this reason, in this subsection we recall the notion of Kan complexes as a 'geometric model' for ∞-groupoids and we introduce constructions and review results that will come in handy later on in Section 3.2.

We need some terminology from simplicial homotopy theory. For any integer $n \geq 0$, the *standard simplicial n-simplex* $\Delta^n := \Delta(-, [n]) \in \mathbf{sSet} :=$

$\mathbf{Set}^{\Delta^{\mathrm{op}}}$ is the simplicial set represented, as a presheaf, by the object $[n] \in \Delta$. The standard n-simplex may be thought of as a combinatorial incarnation of the n-ball. It is in particular 'filled' in the sense that it has a (unique) non-degenerate (i.e. not image of a degeneracy map) cell in dimension n. (All cells in dimension strictly greater than n are degenerate.) Therefore, a combinatorial model for the $(n-1)$-sphere is given by the *boundary* $\partial\Delta^n \in \mathbf{sSet}$, which is the simplicial set generated from Δ^n minus its non-degenerate cell of dimension n. Equivalently, it is given degree-wise by the coequalizer

$$\coprod_{0\le i<j\le n} \Delta^{n-2} \rightrightarrows \coprod_{0\le i\le n} \Delta^{n-1} \longrightarrow \partial\Delta^n \tag{2.115}$$

defined by the simplicial identities $d_i d_j = d_{j-1} d_i$. Another shape that can be built out of the simplicial n-simplex, which is the one we are more interested in, is the (n,k)-*horn* $\Lambda^n_k \in \mathbf{sSet}$, for integers $n \ge 1$ and $0 \le k \le n$. It is obtained by removing the k-th face from the boundary $\partial\Delta^n$. As a functor, $\Lambda^n_k : \Delta^{\mathrm{op}} \to \mathbf{Set}$ sends the object $[m] \in \Delta$ to the set of order preserving maps $f \in \Lambda^n_k[m] \subseteq \Delta([m],[n])$ which factor through some face map $d_i : [n-1] \to [n]$ with $i \ne k$. The (n,k)-horns with $k = 0$ or $k = n$ are called *outer horns*.

Definition 2.4.1. A *Kan complex* is a simplicial set $K \in \mathbf{sSet}$ that satisfies the *Kan condition*, i.e. it is such that all its horns have fillers. In other words, all (solid) diagrams

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & K \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd} \tag{2.116}$$

admit a diagonal (dashed) morphism which makes them commutative.

Remark 2.4.2. An ∞-groupoid is a structure which contains n-morphisms, for all $n \ge 0$. For $n = 0$ one recovers the objects of the ∞-groupoid, while for $n \ge 1$ one gets n-morphisms going between $(n-1)$-morphisms. Adjacent n-morphisms are composable, up to $(n+1)$-morphisms, and all n-morphisms are invertible. A model for ∞-groupoids is offered by Kan complexes $K \in \mathbf{sSet}$, in the following sense. The n-morphisms of the ∞-groupoid are organized by the n-cells K_n. The 0-cells K_0 are the objects in the groupoid, the 1-cells K_1 describe the 1-morphisms between them. The 2-cells K_2 are 2-morphisms between 1-morphisms, and this pattern goes on with increasingly higher morphisms. The idea that adjacent n-morphisms have a (not unique) composite n-morphism and that each n-morphism is invertible with respect to this composition is encoded by the Kan condition. Let us give an heuristic idea of this fact by considering just the 1-morphisms. Two *composable* 1-morphisms $f : x_0 \to x_1$, $g : x_1 \to x_2$ correspond to the image in K of the inner horn $\Lambda^2_1[2]$. Kan condition (2.116) implies that there exist a 1-morphism $h : x_0 \to x_2$, which may be thought of as a composition of g and f, and a 2-morphism filling the triangle, comparing f, g and h.

$$\begin{tikzcd} & x_1 \arrow[dr, "g"] & \\ x_0 \arrow[ur, "f"] & & x_2 \end{tikzcd} \longrightarrow \begin{tikzcd} & x_1 \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow"] & \\ x_0 \arrow[ur, "f"] \arrow[rr, "h"'] & {} & x_2 \end{tikzcd} \tag{2.117}$$

The invertibility condition is captured by filling outer horns. Consider, to fix the ideas, the image in K of $\Lambda^2_2[2]$, that is 1-morphisms $h : x_0 \to x_2$ and $g : x_1 \to x_2$ with the same target. Then, if g were invertible one could find a morphism $f : x_0 \to x_1$ such that the composition of g and f is h, up to a 2-morphism.

$$\begin{tikzcd} & x_1 \arrow[dr, "g"] & \\ x_0 \arrow[rr, "h"'] & & x_2 \end{tikzcd} \longrightarrow \begin{tikzcd} & x_1 \arrow[dr, "g"] \arrow[d, phantom, "\Downarrow"] & \\ x_0 \arrow[ur, "f"] \arrow[rr, "h"'] & {} & x_2 \end{tikzcd} \tag{2.118}$$

This is exactly the kind of phenomena implemented by the Kan condition. $\triangledown$

In Section 3.2 we shall define the space of retarded/advanced Green's homotopies as a simplicial set, which turns out to be a Kan complex, built out of a suitable evaluation of the mapping complex functor $\underline{\mathrm{map}}$ from Section 2.3. In order to streamline the story later on in Section 3.2, we anticipate here the discussion of a standard procedure which constructs a special Kan complex out of a given cochain complex. Let $V \in \mathbf{Ch}_{\mathbb{K}}$ be a cochain complex. Suppose to be interested in the set $\mathcal{H}_u$ of the $(k-1)$-cochains whose differential coincides with a fixed k-cocycle $u \in \mathsf{Z}^k V$. This may be obtained as the pullback

$$\begin{tikzcd} \mathcal{H}_u \arrow[r, dashed] \arrow[d, dashed] & \{*\} \arrow[d, "u"] \\ V^{k-1} \arrow[r, "Q"'] & \mathsf{Z}^k V \end{tikzcd} \tag{2.119}$$

in $\mathbf{Set}$, where $\{*\} \in \mathbf{Set}$ is the singleton, the vertical solid arrow denotes the map which assigns the k-cocycle $u \in \mathsf{Z}^k V$ and the horizontal solid arrow is given by the differential Q of V acting on $(k-1)$-cochains. An interesting example may be provided by choosing $k = 0$, $V = [W, W] \in \mathbf{Ch}_{\mathbb{K}}$ the internal hom from $W \in \mathbf{Ch}_{\mathbb{K}}$ to itself and $u = \mathrm{id}$ the identity cochain map. In this case $\mathcal{H}_{\mathrm{id}}$ is the set of contracting homotopies $\partial\eta = \mathrm{id}$ for the complex W. It is clear that, in general, the set $\mathcal{H}_u$ may be empty. For instance, in the case of contracting homotopies, $\mathcal{H}_{\mathrm{id}}$ is not empty if and only if the cochain complex $W \xrightarrow{\sim} 0$ is acyclic.

However, the information encoded by the set $\mathcal{H}_u$ may not be rich enough in some circumstances. For example, let $v \in \mathcal{H}_u$, then the point $v + Qw$, for all $w \in V^{k-2}$, is always in $\mathcal{H}_u$ because the differential of a cochain complex squares to zero. Hence, it would be more significant to consider a 'space' $\underline{\mathcal{H}}_u$ where points $v \in \mathcal{H}_u$ and $v + Qw \in \mathcal{H}_u$ are regarded as 'the same point' and where two points $v_1, v_2 \in \mathcal{H}_u$ are considered as distinct only if it does not exist a $w \in V^{k-2}$ such that $Qw = v_1 - v_2$. In a sense we would like to organize points in $\mathcal{H}_u$ 'up to homotopies'. This idea is made more evident by the example $\mathcal{H}_{\mathrm{id}}$ of contracting homotopies. In that case, we are saying that we may want to identify those contracting homotopies which differ by an higher homotopy. Clearly, the pattern may go on with even 'higher homotopies'. For instance, we may be interested also in classifying in how many different ways the identification of the points in $\mathcal{H}_u$ may be achieved. In general, for $v_1, v_2, v_3 \in \mathcal{H}_u$ and $w_1, w_2, w_3 \in V^{k-2}$ such that $Qw_1 = v_1 - v_2$, $Qw_2 = v_2 - v_3$ and $Qw_3 = v_1 - v_3$, we may want to compare the 'direct path' w_3 from v_1 to v_3 with the 'composition' of w_1 and w_2, which instead passes through v_2. We may consider them as being equivalent if there is a $h \in V^{k-3}$ such that $Qh = (w_1 - w_2) - w_3$. A way to capture these homotopical phenomena is to upgrade the set $\mathcal{H}_u \in \mathbf{Set}$ given by the pullback (2.119) to the simplicial set $\underline{\mathcal{H}}_u \in \mathbf{sSet}$ given by the pullback

$$
\begin{tikzcd}
\underline{\mathcal{H}}_u \arrow[r, dashed] \arrow[d, dashed] & \Delta^0 \arrow[d, "u"] \\
{[N(\Delta^\bullet), V]^{k-1}} \arrow[r, "\partial"'] & \mathsf{Z}^k[N(\Delta^\bullet), V]
\end{tikzcd}
\tag{2.120}
$$

in $\mathbf{sSet}$, where now $\Delta^0 = \{*\} \in \mathbf{sSet}$ is the constant simplicial set with only one point, $N : \mathbf{sSet} \to \mathbf{Ch}_{\mathbb{K}}$ is the normalized cochains functor and $\Delta^n \in \mathbf{sSet}$ is the simplicial n-simplex, $n \geq 0$. The vertical solid arrow denotes the map of simplicial sets which in each simplicial degree n assigns the k-cocycle $u \in \mathsf{Z}^k[N(\Delta^n), V]$ whose only non-vanishing component is the map in cohomological degree 0, $N(\Delta^n)^0 \to V^k$, which constantly assigns $u \in \mathsf{Z}^k V$. Finally, the horizontal solid arrow is given by the 'adjoint' differential ∂ from (2.5b) acting on $(k-1)$-cochains of the internal hom.

Recall that the normalized cochains functor $N : \mathbf{sSet} \to \mathbf{Ch}_{\mathbb{K}}$ assigns to a simplicial set $S_\bullet \in \mathbf{sSet}$ the cochain complex $N(S_\bullet) \in \mathbf{Ch}_{\mathbb{K}}$, concentrated in the non-positive degrees, defined as $N(S_\bullet)^0 := S_0 \otimes \mathbb{K}$ in degree 0 and $N(S_\bullet)^{-k} := S_k \otimes \mathbb{K}/s(S_{k-1})$, where $s(S_{k-1})$ denotes the image of the degeneracy maps, in degree $-k \leq -1$. The differential of $N(S_\bullet)$ is given by the alternating sum of the face maps, $\sum_{i=0}^k (-1)^i d_i : N(S_\bullet)^{-k} \to N(S_\bullet)^{-k+1}$. It is immediate to see that $u \in [N(\Delta^n), V]^k$ is coclosed. Indeed, the only components of ∂u which are not manifestly vanishing are the map $(\partial u)^0 : N(S_\bullet)^0 \to V^{k+1}$, which however vanishes since $Qu = 0$ by hypothesis, and the map $(\partial u)^{-1} : N(S_\bullet)^{-1} \to V^k$, which evaluated on any $[x] \in N(S_\bullet)^{-1}$ returns $(\partial u)^{-1}[x] = -(-1)^k u(d_0(x) - d_1(x)) = -(-1)^k (u - u) = 0$.

Let us consider the simplicial set $\underline{\mathcal{H}}_u \in \mathbf{sSet}$ defined by (2.120) and let us explore how it encodes the desired homotopical phenomena mentioned above. First, note that the set of its 0-simplices coincides with the naive set $\mathcal{H}_u = (\underline{\mathcal{H}}_u)_0$ given by (2.119). This follows immediately once one notes that the normalized cochain $N(\Delta^0) \cong \mathbb{K}$ coincides with the field concentrated in degree 0. Then, let us consider the set of 1-simplices. The normalized cochains functor evaluated on the simplicial 1-simplex Δ^1 gives the cochain complex $N(\Delta^1) \cong \big(\overset{(-1)}{\mathbb{K}} \xrightarrow{\left(\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right)} \overset{(0)}{\mathbb{K}^2} \big)$, where the numbers in round brackets denote the cohomological degrees and the differential is written in matrix notation. Therefore, the graded maps in the internal hom $[N(\Delta^1), V]^{k-1}$ are specified by two $(k-1)$-cochains $v_1, v_2 \in V^{k-1}$ and by a $(k-2)$-cochain $w \in V^{k-2}$. Acting with the 'adjoint' differential ∂ and intersecting with the k-cocycle $u \in \mathsf{Z}^k[N(\Delta^1), V]$ to compute the pullback (2.120), we get the following system of constraints

$$\begin{cases} Qv_1 = u \\ Qv_2 = u \\ Qw - v_1 + v_2 = 0 \end{cases} \tag{2.121}$$

This shows that the 1-simplices in $(\underline{\mathcal{H}}_u)_1$ may be represented by arrows labeled by $w \in V^{k-2}$ which link two 0-simplices $v_1, v_2 \in (\underline{\mathcal{H}}_u)_0$. Analog computations reveal a similar pattern in higher dimensions: In dimension n the set $(\underline{\mathcal{H}}_u)_n$ contains an 'higher homotopy' $h \in V^{k-1-n}$, corresponding to the inner of the n-simplex, between a hierarchy of lower homotopies corresponding to the faces of the n-simplex, up to points $v_i \in \mathcal{H}_u$, $0 \leq i \leq n$, which are associated to the 0-simplices. This is exactly the kind of structure we want to capture.

From the definition as the pullback (2.120), it readily follows that the simplicial set $\underline{\mathcal{H}}_u \in \mathbf{sSet}$, when non-empty, is affine over the simplicial vector space

$$\underline{H}_u := \mathsf{Z}^{k-1}[N(\Delta^\bullet), V] \in \mathbf{sVec}_{\mathbb{K}}\,, \tag{2.122}$$

with the affine action

$$\underline{\mathcal{H}}_u \times \underline{H}_u \longrightarrow \underline{\mathcal{H}}_u \tag{2.123}$$

in $\mathbf{sSet}$ that sends n-simplices $\upsilon \in (\underline{\mathcal{H}}_u)_n$ and $\nu \in (\underline{H}_u)_n$ to their sum $\upsilon + \nu \in (\underline{\mathcal{H}}_u)_n$ in the vector space $[N(\Delta^n), V]^{k-1}$. The affine action is well defined since $\partial\nu = 0$, hence it does not contribute to the computation of the pullback defining $(\underline{\mathcal{H}}_u)_n$. The affine structure of the simplicial set $\underline{\mathcal{H}}_u \in \mathbf{sSet}$ allows us to prove that, when non-empty, it is actually a Kan complex. This is a consequence of the following standard result, see for example [Wei94, Lemma 8.2.8] for a proof.

Proposition 2.4.3. *If $G_\bullet \in \mathbf{Grp}^{\Delta^{\mathrm{op}}}$ is a simplicial object in the category* **Grp** *of groups and group homomorphisms, then its underlying simplicial set* *is a Kan complex. In particular, any simplicial vector space* $V_\bullet \in \mathbf{sVec}_{\mathbb{K}}$ *is a Kan complex when regarded as a simplicial set.*

Let us make our argument explicit.

Proposition 2.4.4. *The simplicial set* $\underline{\mathcal{H}}_u \in \mathbf{sSet}$ *defined by* (2.120) *is a Kan complex, for all* $u \in \mathsf{Z}^k V$.

Proof. If $\underline{\mathcal{H}}_u = \emptyset$ is the empty simplicial set there is nothing to show. Assume $\underline{\mathcal{H}}_u \neq \emptyset$ is not empty, then the affine structure (2.122) induces an isomorphism of simplicial sets $\underline{\mathcal{H}}_u \xrightarrow{\cong} \underline{H}_u$ in $\mathbf{sSet}$, where $\underline{H}_u$ is regarded as a simplicial set by forgetting its linear structure. To realize such isomorphism, we fix an arbitrary 0-simplex $\rho_0 \in (\underline{\mathcal{H}}_u)_0$ and set (with a little abuse of notation) $\rho_n := (s_0)^n(\rho_0) \in (\underline{\mathcal{H}}_u)_n$, where s_i are the degeneracy maps of $\underline{\mathcal{H}}_u$. Then, for $n \geq 0$ we define the map

$$\begin{aligned} (\underline{\mathcal{H}}_u)_n &\longrightarrow (\underline{H}_u)_n \\ \upsilon &\longmapsto \upsilon - \rho_n \,, \end{aligned} \tag{2.124}$$

exploiting the inverse of the affine action $(\underline{H}_u)_n \to (\underline{\mathcal{H}}_u)_n, \nu \mapsto \upsilon + \nu$. This clearly defines degree-wise isomorphisms in $\mathbf{Set}$. In order to show that it is actually an isomorphism of simplicial sets we have to check that the maps (2.124) are compatible with faces and degeneracies. This boils down to check that $d_i(\rho_n) = \rho_{n-1}$ and $s_i(\rho_n) = \rho_{n+1}$ for all $0 \leq i \leq n$. Both are consequence of the fact that $\rho_n = s_{i_1} \cdots s_{i_n}(\rho_0)$ may be obtained by any combination of n degeneracy maps, as a simple computation exploiting the simplicial identity $s_i s_j = s_j s_{i-1}$, for $i > j$, reveals. The second identity, $s_i(\rho_n) = \rho_{n+1}$, follows immediately from this observation, while the first one uses also the simplicial identity $d_i s_j = \mathrm{id}$ for $i = j, j+1$. We conclude by observing that $\underline{H}_u \in \mathbf{sVec}_{\mathbb{K}}$ is a Kan complex as a consequence of Proposition 2.4.3 and that the property of being a Kan complex is preserved under isomorphism of simplicial sets. □

We conclude the section with a useful result providing a sufficient condition for the contractibility of the Kan complex $\underline{\mathcal{H}}_u \in \mathbf{sSet}$. First, let us recall, see for example [GJ09], that the category $\mathbf{sSet}$ of simplicial sets admits a model structure with

- as weak equivalences the morphisms whose geometric realization $|\cdot| : \mathbf{sSet} \to \mathbf{Top}$ is a weak homotopy equivalence of topological spaces (i.e. if it induces isomorphisms of the homotopy groups),
- as cofibrations the morphisms which are degree-wise injections and
- as fibrations the *Kan fibrations*, that is morphisms which have the right lifting property with respect to all horn inclusions $\Lambda^n_k \to \Delta^n$.

It is clear that all simplicial sets $S \in \mathbf{sSet}$ are cofibrant objects. Moreover, Definition 2.4.1 directly implies that Kan complexes $K \in \mathbf{sSet}$ are exactly the fibrant objects in $\mathbf{sSet}$ when endowed with the above model structure. Then, any object $S \in \mathbf{sSet}$ admits a *cylinder object* $\Delta^1 \times S \in \mathbf{sSet}$. Moreover, if $K \in \mathbf{sSet}$ is fibrant, namely a Kan complex, its *path object* is given by $\mathbf{sSet}(\Delta^\bullet \times \Delta^1, S) \in \mathbf{sSet}$. Therefore, there is a well behaved notion of *homotopic* morphisms with a fibrant target, see [GJ09, Cor. 1.9]. This may be spelled out as in the following definition.

Definition 2.4.5. Let $S, K \in \mathbf{sSet}$ be two simplicial sets. Moreover, let K be a Kan complex. Given a pair of morphisms $f, g : S \to K$ in $\mathbf{sSet}$, they are said to be *homotopic* if there exists a morphism $h : \Delta^1 \times S \to K$ in $\mathbf{sSet}$ such that $h|_{\{0\}\times S} = f$ and $h|_{\{1\}\times K} = g$.

Moreover, when both the source and the target are fibrant one has the following.

Definition 2.4.6. Let $K_1, K_2 \in \mathbf{sSet}$ be Kan complexes. A morphism $f : K_1 \to K_2$ in $\mathbf{sSet}$ is a *homotopy equivalence* if there is a morphism $g : K_2 \to K_1$ in $\mathbf{sSet}$ such that the compositions $f \circ g \sim \mathrm{id}$ and $g \circ f \sim \mathrm{id}$ are homotopic to the identities.

The Whitehead theorem for model categories, see [GJ09, Thm 1.10] or [Hov99, Prop. 1.2.8], implies that weak equivalences between Kan complexes are all homotopy equivalences. Therefore, we say that a Kan complex $K \in \mathbf{sSet}$ is *contractible* if it is homotopy equivalent $K \xrightarrow{\sim} \Delta^0$ to the point.

Proposition 2.4.7. *Let $V \xrightarrow{\sim} 0 \in \mathbf{Ch}_{\mathbb{K}}$ be an acyclic cochain complex and $u \in \mathsf{Z}^k V$ a k-cocycle in it. The Kan complex $\underline{\mathcal{H}}_u \in \mathbf{sSet}$ is either empty or contractible.*

Proof. Suppose $\underline{\mathcal{H}}_u \neq \emptyset$ is not empty. We have already noted that it is affine over the simplicial vector space $\underline{H}_u \in \mathbf{sVec}_{\mathbb{K}}$ from (2.122). From the definition of the latter, $\underline{H}_u \cong \Gamma(\tau^{\leq 0}(V[k-1])) \in \mathbf{sVec}_{\mathbb{K}}$ is isomorphic to the simplicial vector space assigned by the Dold-Kan correspondence $\Gamma : \mathbf{Ch}_{\mathbb{K}}^{\leq 0} \to \mathbf{sVec}_{\mathbb{K}}$, $\Gamma(-)_\bullet := \mathsf{Z}^0[N(\Delta^\bullet), -]$, see [SS03] for a brief overview, to the good truncation $\tau^{\leq 0} : \mathbf{Ch}_{\mathbb{K}} \to \mathbf{Ch}_{\mathbb{K}}^{\leq 0}$, which truncates a cochain complex at the degree 0 replacing the 0-th vector space with the kernel $\ker(Q^0)$ of the differential, of the $(k-1)$-shifted cochain complex $V[k-1] \in \mathbf{Ch}_{\mathbb{K}}$. Since both Γ and $\tau^{\leq 0}$ preserve weak equivalences, see [Qui67; SS03], it follows from the hypothesis that $V \xrightarrow{\sim} 0$ is acyclic that $\underline{H}_u \in \mathbf{sVec}_{\mathbb{K}}$ is contractible. Since $\underline{\mathcal{H}}_u \xrightarrow{\cong} \underline{H}_u$ are isomorphic as objects of $\mathbf{sSet}$, as we showed in the proof of Proposition 2.4.4, we conclude that also $\underline{\mathcal{H}}_u \in \mathbf{sSet}$ is contractible. $\square$

Chapter 3

Green hyperbolic complexes

In this chapter we build on the preliminary work done in Chapter 2 to develop a *homotopy coherent* generalization of the Green hyperbolic linear differential operators recalled in Section 1.3, which we call *Green hyperbolic complexes*. This notion plays a central role in the study of the derived critical loci of gauge-theoretic quadratic action functionals on globally hyperbolic Lorentzian manifolds, see [BS19; BBS20]. These provide a rigorous description of the field content and dynamical features of linear gauge field theories, in a language related to that of BRST/BV formalism [HT92; BBH95; Cos11; FR12a; FR12b; Gwi13]. Our extension of the concept of Green hyperbolic operators makes available even in this differential graded context the whole arsenal of useful tools and constructions that are already widely exploited in the study of ordinary linear field theories on globally hyperbolic Lorentzian manifolds.

The content of this chapter is organized as follows. In Section 3.1 we introduce our setting by abstracting away from examples coming from derived critical loci of gauge-theoretic quadratic action functionals. This leads us to the Definition 3.1.1 of *complexes of linear differential operators*. Definition 3.1.4 introduces the central notion of *retarded/advanced Green's homotopies* which are meant to generalize the usual retarded/advanced Green's operators from Definition 1.3.1 to complexes of linear differential operators. The notion of Green hyperbolic complexes is provided by Definition 3.1.5, which mimics the one of Green hyperbolic operators. We shall introduce also the *retarded-minus-advanced cochain map* Λ and the *Dirac homotopy* Λ^D which are built out of the retarded and advanced Green's homotopies, and replace similar constructions performed in the ordinary setting. The main properties of Green's homotopies and Green hyperbolic complexes are discussed in Section 3.2. Generalizing the uniqueness result (see Remark 1.3.6) for retarded/advanced Green's operators, our Proposition 3.2.2 states that retarded and advanced Green's homotopies, when they exist, are unique (in the appropriate sense). Furthermore, Theorem 3.2.4 provides a non-trivial generalization of the exact sequence (1.35) associated to Green hyperbolic operators. The latter is interpreted as witnessing the fact that the retarded-minus-advanced propagator extends to a quasi-isomorphism between the (suitably shifted) cochain complex of compactly supported sections and that with spacelike compact support, whose differential is just the Green hyperbolic operator. See Remark 3.2.6. Finally, in Section 3.3 we specialize to the case $\mathbb{K} = \mathbb{R}$ and introduce the concept of a *differential pairing*, see Definition 3.3.1. This replaces the usual fiber metrics on vector bundles encoding at the same time enough structure to make available the Stokes' theorem upon integration. Similarly to the case of formally self-adjoint Green hyperbolic operators, see Section 1.3 or [BGP07], we shall show that endowing a Green hyperbolic complex with a differential pairing makes available two types of Poisson structures. The first type consists of three Poisson structures $\tau_M^{\pm}, \tau_M : \mathfrak{F}_{\mathrm{hc}}(M)[1]^{\wedge 2} \to \mathbb{R}$ in $\mathbf{Ch}_{\mathbb{R}}$ on the 1-shift of the complex of compactly supported sections, which are the counterpart of the covariant Poisson structure (1.45). All these Poisson structures heavily rely on the fact that we are considering Green hyperbolic complexes and, in general, they are related by suitable homotopies. The second type of Poisson structures $\sigma_\Sigma : \mathfrak{F}_{\mathrm{hsc}}(M)^{\wedge 2} \to \mathbb{R}$ in $\mathbf{Ch}_{\mathbb{R}}$ replaces the fixed-time Poisson structure (1.47) and, just like in that case, it relies on the choice of a spacelike Cauchy surface $\Sigma \subseteq M$ and it is defined on the complex of sections with spacelike compact support. Finally, Theorem 3.3.7 shows that the retarded-minus-advanced quasi-isomorphism Λ is compatible with these Poisson structures up to an explicit homotopy.

This chapter is mainly based on the results presented in [BMS23].

3.1 Green's homotopies

From now on, for the rest of this chapter, we fix a globally hyperbolic Lorentzian manifold $M \in \mathbf{Loc}_m$ of dimension $m \geq 2$. Whenever not explicitly stated differently, we assume we are working over the field $\mathbb{K} = \mathbb{R}$ of real or $\mathbb{K} = \mathbb{C}$ of complex numbers. We stick with the notations of Sections 1.2 and 1.3 for smooth sections of vector bundles with prescriptions on their supports. Moreover, given a ($\mathbb{Z}$-)graded ($\mathbb{K}$-)vector bundle $F := (F^n) \to M$, that is a family of ($\mathbb{K}$-)vector bundles $(F^n \to M) \in \mathbb{K}\text{-}\mathbf{VBnd}_m$ labeled by integers $n \in \mathbb{Z}$, in accordance with the previously adopted notation, we denote by

$$\mathfrak{F}^n(M) := \Gamma(F^n) \in \mathbf{Vec}_{\mathbb{K}} \tag{3.1}$$

the vector space of degree n smooth sections, namely smooth sections of the degree n vector bundle $F^n \to M$. Similarly, for $C \in \mathrm{cl}$ a closed subset of M, $\mathfrak{F}^n_C(M) := \Gamma_C(F^n) \in \mathbf{Vec}_{\mathbb{K}}$ denotes the vector space of degree n smooth sections with support contained in C and, for any directed subset $\mathscr{D} \subseteq \mathrm{cl}$,

$\mathfrak{F}^n_{\mathscr{D}}(M) := \Gamma_{\mathscr{D}}(F^n) \in \mathbf{Vec}_{\mathbb{K}}$ denotes the vector space of degree n smooth sections with support prescribed by $\mathscr{D}$, see Equation (1.31).

According to the BV formalism, and coherently with the computation of derived critical loci of gauge invariant quadratic action functionals, the field content of a linear gauge field theory over the globally hyperbolic Lorentzian manifold $M \in \mathbf{Loc}_m$ is correctly described by a cochain complex whose underlying graded vector space is given by smooth sections of a graded vector bundle and whose differential is degree-wise a linear differential operator. Therefore, we may abstract away from the concrete examples and introduce the following definition.

Definition 3.1.1. A *complex of (linear) differential operators* on $M \in \mathbf{Loc}_m$ is a pair (F, Q) consisting of a graded vector bundle $F \to M$ and of a collection $Q = (Q^n : \mathfrak{F}^n(M) \to \mathfrak{F}^{n+1}(M))_{n\in\mathbb{Z}}$ of degree increasing linear differential operators such that $Q^{n+1} \circ Q^n = 0$, for all $n \in \mathbb{Z}$.

With a complex of differential operators (F, Q) over M, we associate the cochain complex $\mathfrak{F}(M) = (\mathfrak{F}(M), Q) \in \mathbf{Ch}_{\mathbb{K}}$, whose underlying graded vector space is the space of smooth sections of the graded vector bundle $F \to M$. We shall collect some examples of complexes of differential operators related to applications to classical and quantum field theory in Chapter 6. They include ordinary free field theories, Abelian Chern-Simons theory and Maxwell p-forms, which comprehends, for $p = 1$, the linear Yang-Mills theory. According to the BV formalism, the degree 0 of the cochain complex of sections, $\mathfrak{F}^0(M)$, describes the fields of the theory. The negative degrees, $\mathfrak{F}^i(M)$ for $i < 0$, are the spaces of ghost and ghost of ghosts fields. The positive degrees, $\mathfrak{F}^i(M)$ for $i > 0$, are the spaces of the antifields. The differential Q encodes simultaneously the dynamics and the action of the gauge symmetries of the theory.

Here we introduce a couple of examples which will play a distinguished role in the forthcoming discussions.

Example 3.1.2. A linear differential operator $P : \Gamma(E) \to \Gamma(E)$, acting on the smooth sections of the vector bundle $(E \to M) \in \mathbb{K}\text{-}\mathbf{VBnd}_m$, identifies a complex of differential operators $(F_{(E,P)}, Q_{(E,P)})$ consisting of the graded vector bundle $F_{(E,P)} \to M$ concentrated in degrees 0 and 1, given by $F^n_{(E,P)} := E$, for $n = 0, 1$, and of the differential $Q_{(E,P)}$ whose only non-vanishing component is $Q^0_{(E,P)} := P : \mathfrak{F}^0_{(E,P)}(M) \to \mathfrak{F}^1_{(E,P)}(M)$. This class of examples includes ordinary free field theories, see Section 6.1 for more details. △

Example 3.1.3. A second paramount example of a complex of differential operators is provided by the de Rham complex $(\Lambda^\bullet M, \mathrm{d_{dR}})$. It consists of the graded vector bundle $\Lambda^\bullet M \to M$ of the exterior powers of the cotangent bundle of M, $\Lambda^n M := \Lambda^n T^* M$, and of the usual de Rham differential $\mathrm{d_{dR}}$ acting on differential forms. △

Since the differential Q is a degree-wise differential operator, hence it does not enlarge supports, it admits a restriction to the sections with support contained in a closed subset $C \in \mathrm{cl}$ of M. We denote by $\mathfrak{F}_C(M) \in \mathbf{Ch}_{\mathbb{K}}$ the subcomplex of $\mathfrak{F}(M)$ given by the sections with support in C. Given a directed subset $\mathscr{D} \subseteq \mathrm{cl}$ of the closed subsets of M, one defines the $\mathbf{Ch}_{\mathbb{K}}$-valued functor

$$\mathfrak{F}_{(-)}(M) \in \mathbf{Ch}_{\mathbb{K}}^{\mathscr{D}} \tag{3.2}$$

from the directed set $\mathscr{D}$ regarded as a category. The functor $\mathfrak{F}_{(-)}(M)$ assigns to $D \in \mathscr{D} \subseteq \mathrm{cl}$ the cochain complex $\mathfrak{F}_D(M) \in \mathbf{Ch}_{\mathbb{K}}$ and to the subset inclusion $D \subseteq D'$ the obvious inclusion cochain map $\mathfrak{F}_D(M) \to \mathfrak{F}_{D'}(M)$ in $\mathbf{Ch}_{\mathbb{K}}$. We obtain the complex of sections with support prescribed by $\mathscr{D}$ by passing to (homotopy) colimit

$$\mathfrak{F}_{(\mathrm{h})\mathscr{D}}(M) := (\mathrm{ho})\mathrm{colim}\,\big(\mathfrak{F}_{(-)}(M) : \mathscr{D} \to \mathbf{Ch}_{\mathbb{K}}\big) \in \mathbf{Ch}_{\mathbb{K}}\,. \tag{3.3}$$

A primary example is provided by the directed set $\mathscr{D} = \mathrm{c} \subseteq \mathrm{cl}$ of the compact subsets of M. In this case the (homotopy) colimit (3.3) produces the cochain complex $\mathfrak{F}_{(\mathrm{h})\mathrm{c}}(M) \in \mathbf{Ch}_{\mathbb{K}}$ of the smooth sections of $F \to M$ with compact support. When the ordinary colimit is considered, the cochain complex (3.3) is isomorphic to the obvious subcomplex of $\mathfrak{F}(M) \in \mathbf{Ch}_{\mathbb{K}}$ given by restricting to compactly supported sections. The homotopy colimit version, instead, presents the cochain complex of compactly supported sections in a slightly different manner. Recalling the definition (2.100) of the dg-functor hocolim, we find that its degree n component, $n \in \mathbb{Z}$, is

$$\mathfrak{F}^n_{\mathrm{hc}}(M) = \bigoplus_{p \geq 0} \bigoplus_{\underline{K}:[p]\to \mathrm{c}} \mathfrak{F}^{n+p}_{K_0}(M)\,, \tag{3.4}$$

where the functor $\underline{K} : [p] \to \mathrm{c}$ can be equivalently seen as a chain of p composable inclusions of compact subsets, $\underline{K} = (K_0 \subseteq K_1 \subseteq \cdots \subseteq K_p)$. The differential of $\mathfrak{F}_{\mathrm{hc}}(M)$ is given by (2.101) as the sum of the horizontal differential coming from the simplicial resolution of $\mathfrak{F}_{(-)}(M) \in \mathbf{Ch}_{\mathbb{K}}^{\mathrm{c}}$ plus the vertical differential built out of the differential Q of the complex of differential operators. However, the content of the homotopy colimit $\mathfrak{F}_{\mathrm{hc}}(M)$ is the same of that of the ordinary colimit, in the sense that there is a quasi-isomorphism $\mathfrak{F}_{\mathrm{hc}}(M) \xrightarrow{\sim} \mathfrak{F}_{\mathrm{c}}(M)$. This is actually a general fact, since a directed set $\mathscr{D}$ is in particular a filtered category, the canonical cochain map

$$\mathfrak{F}_{\mathrm{h}\mathscr{D}}(M) \xrightarrow{\ \sim\ } \mathfrak{F}_{\mathscr{D}}(M) \tag{3.5}$$

in $\mathbf{Ch}_{\mathbb{K}}$ is always a quasi-isomorphism, see Remark 2.3.10. Recall from Section 1.1 that the causal future/past $J^{\pm}_M(K) \in \mathrm{cl}$ is closed for all compact subsets $K \in \mathrm{c}$ in M and that there is an obvious inclusion $J^{\pm}_M(K) \subseteq J^{\pm}_M(K')$ of closed subset for all inclusions $K \subseteq K'$ of compact subsets of M. Then, taking the causal future/past of compact subsets defines a functor

$$J^{\pm}_M : \mathrm{c} \longrightarrow \mathrm{cl} \tag{3.6}$$

into the directed set of closed subsets of M. Composing it with the functor $\mathfrak{F}_{(-)}(M) \in \mathbf{Ch}_{\mathbb{K}}^{\mathrm{cl}}$ we define the functor

$$\mathfrak{F}_{J_M^{\pm}(-)}(M) := \mathfrak{F}_{(-)}(M) \circ J_M^{\pm} \in \mathbf{Ch}_{\mathbb{K}}^{\mathrm{c}} , \tag{3.7}$$

which assigns to any compact set $K \in \mathrm{c}$ the cochain complex $\mathfrak{F}_{J_M^{\pm}(K)}(M) \in \mathbf{Ch}_{\mathbb{K}}$ of smooth sections of (F, Q) with support contained in the causal past/future $J_M^{\pm}(K)$ of K. All the ingredients are ready to cook up the central definition of *retarded/advanced Green's homotopies*.

Definition 3.1.4. Given a complex of differential operators (F, Q) on $M \in \mathbf{Loc}_m$, we define a *retarded/advanced Green's homotopy* of it as a (-1)-cochain

$$\Lambda^{\pm} \in \underline{\mathrm{map}}(\mathfrak{F}_{J_M^{\pm}(-)}(M), \mathfrak{F}_{J_M^{\pm}(-)}(M))^{-1} \tag{3.8}$$

in the mapping complex (2.80) from the $\mathbf{Ch}_{\mathbb{K}}$-valued functor $\mathfrak{F}_{J_M^{\pm}(-)}(M) \in \mathbf{Ch}_{\mathbb{K}}^{\mathrm{c}}$ to itself, whose differential

$$\delta\Lambda^{\pm} = \mathrm{id} \in \underline{\mathrm{map}}(\mathfrak{F}_{J_M^{\pm}(-)}(M), \mathfrak{F}_{J_M^{\pm}(-)}(M))^{0} \tag{3.9}$$

is the identity natural transformation of the functor $\mathfrak{F}_{J_M^{\pm}(-)}(M) \in \mathbf{Ch}_{\mathbb{K}}^{\mathrm{c}}$.

We define *Green hyperbolic complexes* in analogy with the Green hyperbolic operators from Section 1.3.

Definition 3.1.5. A complex of differential operators (F, Q) on M is called a *Green hyperbolic complex* if it admits a retarded and an advanced Green's homotopy $\Lambda^{\pm}$.

Remark 3.1.6. The Definitions 3.1.4 and 3.1.5 generalize the notions of retarded/advanced Green's operators $G^{\pm}$, Definition 1.3.1, and of Green hyperbolic operators P, Definition 1.3.2, respectively. While it is clear that the definition of a Green hyperbolic complex is analogous to that of Green hyperbolic operators, it is perhaps less evident that the retarded/advanced Green's homotopies $\Lambda^{\pm}$ are a homotopy coherent generalization of the retarded/advanced Green's operators $G^{\pm}$, or more precisely of their extensions from Theorem 1.3.4. Let us first provide the reader with a heuristic argument, we shall make all claims more precise in few moments. The (extension of) retarded/advanced Green's operators are characterized by two types of conditions. Items 1-2 in Theorem 1.3.4 tell that the compositions $PG^{\pm} = G^{\pm}P = \mathrm{id}$ are the identity on the vector space $\Gamma_{\substack{\mathrm{spc}\\ \mathrm{sfc}}}(E)$ of sections of strictly past/future compact support. This is a 'contractibility condition' similar in nature to Equation (3.9). Item 3 of the same theorem encodes a 'support propagation' property which is now replaced by the fact that retarded/advanced Green's homotopies $\Lambda^{\pm}$ are enriched morphisms (of degree -1) from $\mathfrak{F}_{J_M^{\pm}(-)}(M) \in \mathbf{Ch}_{\mathbb{K}}^{\mathrm{c}}$ to itself, in the homotopy coherent $\underline{\mathrm{map}}$-enrichment of the category $\mathbf{Ch}_{\mathbb{K}}^{\mathrm{c}}$. Hence, the correct support propagation is now encoded by the functor $\mathfrak{F}_{J_M^{\pm}(-)}(M)$. ▽

The following example shows that each Green hyperbolic operator P generates a Green hyperbolic complex, thus making clear that our Definition 3.1.5 of Green hyperbolic complexes provides a generalization of the usual Green hyperbolic operators.

Example 3.1.7. Let $P : \Gamma(E) \to \Gamma(E)$ be a linear differential operator acting on sections of the vector bundle $E \to M$. We have already observed with Example 3.1.2 that a complex of differential operators $(F_{(E,P)}, Q_{(E,P)})$ is associated with it. We show that the latter is Green hyperbolic if and only if P is a Green hyperbolic operator in the usual sense.

In order to do so, we first have to unravel Definition 3.1.4 and specialize it to this simple situation. Since the cochain complex

$$\mathfrak{F}_{J^\pm_M(K)}(M) = \Big(\overset{(0)}{\Gamma_{J^\pm_M(K)}(E)} \xrightarrow{\;P\;} \overset{(1)}{\Gamma_{J^\pm_M(K)}(E)}\Big) \in \mathbf{Ch}_\mathbb{K} \tag{3.10}$$

assigned by the functor $\mathfrak{F}_{J^\pm_M(-)}(M)$ to the compact subset $K \subseteq M$ is concentrated in degrees 0 and 1, the product appearing in the definition of the mapping complex (2.80) immediately stops, namely

$$\underline{\mathrm{map}}(\mathfrak{F}_{J^\pm_M(-)}(M), \mathfrak{F}_{J^\pm_M(-)}(M))^{-1} \cong \prod_{K_0 \in \mathrm{c}} [\mathfrak{F}_{J^\pm_M(K_0)}(M), \mathfrak{F}_{J^\pm_M(K_0)}(M)]^{-1} \,. \tag{3.11}$$

It follows that the only non-vanishing components of a (-1)-cochain $\Lambda_\pm = (\mathrm{pr}_{0,K_0} \Lambda^\pm)_{K_0 \in \mathrm{c}}$ in the mapping complex are the degree -1 linear maps $\mathrm{pr}_{0,K_0} \Lambda^\pm \in [\mathfrak{F}_{J^\pm_M(K_0)}(M), \mathfrak{F}_{J^\pm_M(K_0)}(M)]^{-1}$ labeled by compact subsets $K_0 \subseteq M$. For degree reasons, they only have a non-vanishing component, that is the morphism

$$(\mathrm{pr}_{0,K_0} \Lambda^\pm)^1 : \mathfrak{F}^1_{J^\pm_M(K_0)}(M) \longrightarrow \mathfrak{F}^0_{J^\pm_M(K_0)}(M) \tag{3.12}$$

in $\mathbf{Vec}_\mathbb{K}$, for all $K_0 \in \mathrm{c}$. Moreover, any 0-cochain η in the mapping complex can only have non-vanishing projections $\mathrm{pr}_q \eta$ for $q = 0, 1$. From the definition (2.81) of the differential δ, it follows that the identity $\delta\Lambda^\pm = \mathrm{id}$ involves two different kinds of conditions: The first comes from the vertical differential,

$$\mathrm{pr}_{0,K_0} \delta\Lambda^\pm = \mathrm{pr}_{0,K_0} \delta_\mathrm{v}\Lambda^\pm = \partial(\mathrm{pr}_{0,K_0} \Lambda^\pm) = \mathrm{id} \,, \tag{3.13a}$$

which, making also the 'adjoint' differential ∂ explicit, gives rise to the following conditions on the maps $(\mathrm{pr}_{0,K_0} \Lambda^\pm)^1$ for all $K_0 \in \mathrm{c}$,

$$(\partial(\mathrm{pr}_{0,K_0} \Lambda^\pm))^0 = (\mathrm{pr}_{0,K_0} \Lambda^\pm)^1 \circ Q^0_{(E,P)} = (\mathrm{pr}_{0,K_0} \Lambda^\pm)^1 \circ P = \mathrm{id} \,, \tag{3.13b}$$

$$(\partial(\mathrm{pr}_{0,K_0} \Lambda^\pm))^1 = Q^0_{(E,P)} \circ (\mathrm{pr}_{0,K_0} \Lambda^\pm)^1 = P \circ (\mathrm{pr}_{0,K_0} \Lambda^\pm)^1 = \mathrm{id} \,. \tag{3.13c}$$

The second type involves only the horizontal differential and comes from projecting onto the $q = 1$ components for all inclusions $K_0 \subseteq K_1$ of compact subsets of M. More explicitly one has

$$\begin{aligned}(\mathrm{pr}_{1,K_0\subseteq K_1}\, \delta\Lambda^{\pm})^1 &= (\mathrm{pr}_{1,K_0\subseteq K_1}\, \delta_{\mathrm{h}}\Lambda^{\pm})^1 \\ &= \mathfrak{F}^0_{J^{\pm}_M(K_0\subseteq K_1)}(M) \circ (\mathrm{pr}_{0,K_0}\, \Lambda^{\pm})^1 - (\mathrm{pr}_{0,K_1}\, \Lambda^{\pm})^1 \circ \mathfrak{F}^0_{J^{\pm}_M(K_0\subseteq K_1)}(M) = 0\,.\end{aligned} \tag{3.14}$$

Summing up, a retarded/advanced Green's homotopy of the complex of differential operators $(F_{(E,P)}, Q_{(E,P)})$ associated with the linear differential operator P is equivalently given by

a. a family of linear maps $(\mathrm{pr}_{0,K_0}\, \Lambda^{\pm})^1 : \mathfrak{F}^1_{J^{\pm}_M(K_0)}(M) \to \mathfrak{F}^0_{J^{\pm}_M(K_0)}(M)$ labeled by compact subsets $K_0 \subseteq M$, which are compatible with subsets inclusions, in that they arrange themselves as the components of a natural transformation $(\mathrm{pr}_{0,(-)}\, \Lambda^{\pm})^1 : \mathfrak{F}^1_{J^{\pm}_M(-)}(M) \to \mathfrak{F}^0_{J^{\pm}_M(-)}(M)$, such that

b. the linear maps $(\mathrm{pr}_{0,K_0}\, \Lambda^{\pm})^1 \circ P = \mathrm{id}$ and $(\mathrm{pr}_{0,K_0}\, \Lambda^{\pm})^1 \circ P = \mathrm{id}$ coincide with the appropriate identities, for all compact subsets $K_0 \subseteq M$.

It follows that, when P is a Green hyperbolic operator, its extended retarded/advanced Green's operator $G^{\pm} : \Gamma_{\substack{\mathrm{spc}\\\mathrm{sfc}}}(E) \to \Gamma_{\substack{\mathrm{spc}\\\mathrm{sfc}}}(E)$ restricts to the linear map $G^{\pm}_{K_0} : \mathfrak{F}^1_{J^{\pm}_M(K_0)}(M) \to \mathfrak{F}^0_{J^{\pm}_M(K_0)}(M)$ for all $K_0 \subseteq M$ compact, because of item 3 of Theorem 1.3.4. Therefore, they are the components $(\mathrm{pr}_{0,K_0}\, \Lambda^{\pm}_P)^1 := G^{\pm}_{K_0}$ of a candidate retarded/advanced Green's homotopy $\Lambda^{\pm}_P$. Since they are obtained by restricting $G^{\pm}$, they are obviously compatible with subset inclusions (item a). Moreover, items 1 and 2 in Theorem 1.3.4 imply that also item b above is fulfilled by $G^{\pm}_{K_0}$ for all $K_0 \in \mathrm{c}$. Hence, $\Lambda^{\pm}_P$ is an explicit retarded/advanced Green's homotopy and it witnesses that the complex $(F_{(E,P)}, Q_{(E,P)})$ is Green hyperbolic. To go in the opposite direction, assume that the complex $(F_{(E,P)}, Q_{(E,P)})$ is Green hyperbolic, then the natural transformation $(\mathrm{pr}_{0,(-)}\, \Lambda^{\pm})^1$ from item a defines a linear map $G^{\pm}_{\Lambda} : \Gamma_{\mathrm{c}}(E) \to \Gamma(E)$ by setting, for any compactly supported section $\psi \in \Gamma_{\mathrm{c}}(E)$, $G^{\pm}_{\Lambda}\psi := (\mathrm{pr}_{0,K_0}\, \Lambda^{\pm})^1\psi$, for a compact subset $K_0 \supseteq \mathrm{supp}\,\psi$. The definition of $G^{\pm}_{\Lambda}$ is well-posed since $(\mathrm{pr}_{0,(-)}\, \Lambda^{\pm})^1$ is natural with respect to subset inclusions. Moreover, $\mathrm{supp}(G^{\pm}_{\Lambda}\psi) \subseteq J^{\pm}_M(\mathrm{supp}\,\psi)$ as one realizes from item a by taking $K_0 = \mathrm{supp}\,\psi$. This shows that $G^{\pm}_{\Lambda}$ fulfills item 3 of Definition 1.3.1. Item b above implies that also items 1 and 2 of Definition 1.3.1 are met by $G^{\pm}_{\Lambda}$, thus proving that they are retarded/advanced Green's operators of P. △

The example above shows that a primary class of Green hyperbolic complexes is given by the complexes associated to Green hyperbolic operators. In Chapter 4 we will provide you with the useful concept of *Green's witnesses*, which will allow us to find a sufficient condition for the Green hyperbolicity of a given complex of differential operators. In Chapter 6, this will prove useful to show that the complexes therein, inspired by applications in physics, are actually Green hyperbolic.

Remark 3.1.8. Let us stress that we considered the mapping complex $\underline{\mathrm{map}}$ in Definition 3.1.4 in order to get a notion compatible with weak equivalences $\mathfrak{F}_{J^{\pm}_M(-)}(M) \xrightarrow{\sim} \mathfrak{F}'_{J^{\pm}_M(-)}(M)$ in $\mathbf{Ch}_{\mathbb{K}}^{\mathrm{c}}$ of the $\mathbf{Ch}_{\mathbb{K}}$-valued functors of sections (with prescribed support). This forced us to choose the homotopical mapping complex $\underline{\mathrm{map}}$ instead of the naive enriched hom $\underline{\mathrm{hom}}$. By recalling the Remark 2.3.6, we observe that this forced choice has as its direct consequence that our retarded/advanced Green's homotopies $\Lambda^{\pm}$ are a more 'relaxed' object than the usual retarded/advanced Green's operators $G^{\pm}$. In fact, as noted in the previous remark, Green's operators $G^{\pm}$ are compatible with inclusions of compact subsets of M, hence they define strict natural transformations, $G^{\pm} \in \underline{\mathrm{hom}}(\mathfrak{F}_{J^{\pm}_M(-)}(M), \mathfrak{F}_{J^{\pm}_M(-)}(M))^{-1}$. On the other hand, Green's homotopies $\Lambda^{\pm}$, laying in the homotopical mapping complex, are only 'homotopy coherent' natural with respect to inclusions $K_0 \subseteq K_1$ in c. This means that, in general, the 0-projections of a retarded/advanced Green's homotopy, while satisfying the standard support propagation property, are not natural on the nose but only up to a homotopy provided by the appropriate 1-projection,

$$\begin{aligned}&\partial(\mathrm{pr}_{1,K_0\subseteq K_1}\Lambda^{\pm})\\ &\quad= \mathfrak{F}_{J^{\pm}_M(K_0\subseteq K_1)}(M)\circ \mathrm{pr}_{0,K_0}\Lambda^{\pm} - \mathrm{pr}_{0,K_1}\Lambda^{\pm}\circ \mathfrak{F}_{J^{\pm}_M(K_0\subseteq K_1)}(M)\,.\end{aligned} \tag{3.15}$$

Note that the homotopies $\mathrm{pr}_{1,K_0\subseteq K_1}\Lambda^{\pm} \in [\mathfrak{F}_{J^{\pm}_M(K_0)}(M), \mathfrak{F}_{J^{\pm}_M(K_1)}(M)]^{-2}$ satisfy a more relaxed support propagation property in that they may enlarge the supports a bit more, $J^{\pm}_M(K_1) \supseteq J^{\pm}_M(K_0)$. Finally, notice that $\Lambda^{\pm}$ contains, with its higher projections, an infinite tower of higher homotopies each of which witnesses the lack of naturality of the lower ones and it is allowed to enlarge the supports always a bit more. ▽

As in the classic, non-graded, context one defines the retarded-minus-advanced propagator $G : \Gamma_{\mathrm{c}}(E) \to \Gamma_{\mathrm{sc}}(E)$, see Definition 1.3.7, out of the retarded and advanced Green's operators $G^{\pm}$, in our framework we are able to define a *retarded-minus-advanced cochain map* Λ out of the retarded and advanced Green's homotopies $\Lambda^{\pm}$ of a Green hyperbolic complex. Let us introduce some notation. Let $F_1, F_2 : \mathrm{c} \to \mathrm{cl}$ be functors and $\eta : F_1 \to F_2$ a natural transformation between them. We denote the natural transformation obtained by whiskering the functor $\mathfrak{F}_{(-)}(M) \in \mathbf{Ch}_{\mathbb{K}}^{\mathrm{cl}}$ and the natural transformation η by

$$j^{F_2}_{F_1} := \mathfrak{F}_{(-)}(M)\circ\eta : \mathfrak{F}_{F_1}(M) \longrightarrow \mathfrak{F}_{F_2}(M)\,. \tag{3.16}$$

This is the natural transformation whose $K \in \mathrm{c}$ component is the cochain map $\mathfrak{F}_{\eta_K}(M)$ in $\mathbf{Ch}_{\mathbb{K}}$. The typical situation we shall encounter is that of the natural transformations associated to the inclusions $K \subseteq J^{\pm}_M(K) \subseteq J_M(K) \subseteq M$, for all compact sets $K \subseteq M$.

Definition 3.1.9. Let (F, Q) be a Green hyperbolic complex and $\Lambda^{\pm}$ be a choice of a retarded and an advanced Green's homotopies of it. Consider the (-1)-cocycle

$$\Lambda := j^{J_M(-)}_{J^+_M(-)} \circ \Lambda^+ \circ j^{J^+_M(-)}_{(-)} - j^{J_M(-)}_{J^-_M(-)} \circ \Lambda^- \circ j^{J^-_M(-)}_{(-)} \tag{3.17}$$

in the mapping complex $\underline{\mathrm{map}}(\mathfrak{F}_{(-)}(M), \mathfrak{F}_{J_M(-)}(M))$. (Note that $\delta\Lambda = 0$ follows immediately from $\delta\Lambda^{\pm} = \mathrm{id}$ since $j^{F_2}_{F_1}$ is a strict natural transformation between $\mathbf{Ch}_{\mathbb{K}}$-valued functors, hence $\delta j^{F_2}_{F_1} = 0$.) We define the *retarded-minus-advanced cochain map*

$$\boldsymbol{\Lambda} := \mathrm{hocolim}\, \Lambda : \mathfrak{F}_{\mathrm{hc}}(M)[1] \longrightarrow \mathfrak{F}_{\mathrm{hsc}}(M) \tag{3.18}$$

in $\mathbf{Ch}_{\mathbb{K}}$ by applying the dg-functor hocolim from (2.103). Here we used the isomorphism (2.15a) to get $\mathsf{Z}^{-1}[\mathfrak{F}_{\mathrm{hc}}(M), \mathfrak{F}_{\mathrm{hsc}}(M)] \cong \mathsf{Z}^0[\mathfrak{F}_{\mathrm{hc}}(M)[1], \mathfrak{F}_{\mathrm{hsc}}(M)]$.

Another useful map built out of retarded and advanced Green's homotopies is the *Dirac homotopy*. It is defined by mimicking the construction of the ordinary Dirac propagator.

Definition 3.1.10. Given a Green hyperbolic complex (F, Q) and a choice of retarded and advanced Green's homotopies $\Lambda^{\pm}$ of it, the associated *Dirac homotopy* is the 0-cochain defined as the homotopy colimit

$$\boldsymbol{\Lambda}^D := \mathrm{hocolim}\, \Lambda^D \in [\mathfrak{F}_{\mathrm{hc}}(M)[1], \mathfrak{F}_{\mathrm{hsc}}(M)]^0 \tag{3.19}$$

of the (-1)-cochain

$$\Lambda^D := \frac{1}{2}\Big(j^{J_M(-)}_{J^+_M(-)} \circ \Lambda^+ \circ j^{J^+_M(-)}_{(-)} + j^{J_M(-)}_{J^-_M(-)} \circ \Lambda^- \circ j^{J^-_M(-)}_{(-)} \Big) \tag{3.20}$$

in the mapping complex $\underline{\mathrm{map}}(\mathfrak{F}_{(-)}(M), \mathfrak{F}_{J_M(-)}(M))$. We exploited the isomorphism (2.15a) to identify $[\mathfrak{F}_{\mathrm{hc}}(M), \mathfrak{F}_{\mathrm{hsc}}(M)]^{-1} \cong [\mathfrak{F}_{\mathrm{hc}}(M)[1], \mathfrak{F}_{\mathrm{hsc}}(M)]^0$.

Dirac homotopy will be exploited in Chapter 5 to characterize and compare the time-orderable prefactorization algebra associated to the Moyal-Weyl quantization of a *covariant free BV theory*, see Definition 4.3.2.

Remark 3.1.11. The cochain complex $\mathfrak{F}_{\mathrm{(h)sc}}(M) \in \mathbf{Ch}_{\mathbb{K}}$ of sections with spacelike compact support is defined, according to Equation (3.3), as the (homotopy) colimit of the functor $\mathfrak{F}_{(-)}(M) : \mathrm{sc} \to \mathbf{Ch}_{\mathbb{K}}$. Nonetheless, in Equations (3.18) and (3.19), we implicitly assumed the cochain complex $\mathfrak{F}_{\mathrm{(h)sc}}(M)$

to be (quasi-)isomorphic to the (homotopy) colimit (ho)colim$(\mathfrak{F}_{J_M(-)}(M) : \mathrm{c} \to \mathbf{Ch}_{\mathbb{K}})$ of the functor $\mathfrak{F}_{(-)} \circ J_M \in \mathbf{Ch}_{\mathbb{K}}^{\mathrm{c}}$. This follows from the fact that the functor $J_M : \mathrm{c} \to \mathrm{sc}$ is (homotopy) final, i.e. for each spacelike compact $C \in \mathrm{sc}$ subset of M the (nerve of the) *comma category* $C \downarrow J_M$ is (contractible) connected. Since the category c is filtered, J_M is homotopy final if and only if it is final, see e.g. [Rie14, Lemma 8.5.5]. To show that J_M is final observe that the category $C \downarrow J_M$ is isomorphic to the full subcategory of c given by the compact subsets $K \subseteq M$ such that $C \subseteq J_M(K)$. It is clearly connected since given any pair of objects $K, K' \in C \downarrow J_M$, the union $K \cup K' \in C \downarrow J_M$ is again an object of the comma category and there are morphisms $K \subseteq K \cup K' \supseteq K'$ in $C \downarrow J_M$. It follows that the claimed (quasi-)isomorphism holds true. $\triangledown$

Proving that the retarded-minus-advanced cochain map Λ enjoys homotopy coherent variants of the main features of G recalled in Section 1.3 will be the goal of Sections 3.2 and 3.3.

3.2 Main results

This section is devoted to state and prove the main results concerning the properties of retarded and advanced Green's homotopies. They offer homotopy coherent generalizations of the main features of the ordinary retarded and advanced Green's operators. We shall focus in particular on a suitable uniqueness result (Proposition 3.2.2) generalizing the content of Remark 1.3.6 and on providing a new interpretation of the exactness of the sequence (1.35) through Theorem 3.2.4.

We start with the following proposition which characterizes Green hyperbolic complexes in terms of the, generally easier to verify, acyclicity of suitable cochain complexes.

Proposition 3.2.1. *Let (F, Q) be a complex of differential operators on M. It is a Green hyperbolic complex if and only if both the cochain complexes $\mathfrak{F}_{J_M^+(K)}(M)$ and $\mathfrak{F}_{J_M^-(K)}(M)$ are acyclic for all compact subsets $K \subseteq M$.*

Proof. Let us first assume that (F, Q) is Green hyperbolic. By Definition 3.1.5, there exist a retarded and an advanced Green's homotopies $\Lambda^\pm \in \underline{\mathrm{map}}(\mathfrak{F}_{J_M^\pm(-)}(M), \mathfrak{F}_{J_M^\pm(-)}(M))^{-1}$. For each $K \in \mathrm{c}$, compact subset of M, the $(0, K)$-projection of $\Lambda^\pm$ returns a (-1)-cochain $\mathrm{pr}_{0,K}\, \Lambda^\pm \in [\mathfrak{F}_{J_M^\pm(K)}(M), \mathfrak{F}_{J_M^\pm(K)}(M)]^{-1}$ in the internal hom. The contractibility condition $\delta\Lambda^\pm = \mathrm{id}$ yields that

$$\partial(\mathrm{pr}_{0,K}\, \Lambda^\pm) = \mathrm{pr}_{0,K}\, \delta\Lambda^\pm = \mathrm{id} \in [\mathfrak{F}_{J_M^\pm(K)}(M), \mathfrak{F}_{J_M^\pm(K)}(M)]^0 \,, \qquad (3.21)$$

by exploiting the definition (2.81) of the differential δ of the mapping complex. Therefore, $\mathrm{pr}_{0,K}\, \Lambda^\pm$ is a contracting homotopy witnessing that the cochain complex $\mathfrak{F}_{J_M^\pm(K)}(M)$ is acyclic.

Concerning the opposite implication, note that since $\mathfrak{F}_{J^{\pm}_M(K)}(M) \xrightarrow{\sim} 0$ in $\mathbf{Ch}_{\mathbb{K}}$ is a quasi-isomorphism for all compact subsets $K \subseteq M$ by hypothesis, we have also that the natural transformation $\mathfrak{F}_{J^{\pm}_M(-)}(M) \xrightarrow{\sim} 0$ in $\mathbf{Ch}^{\mathsf{c}}_{\mathbb{K}}$ is a weak equivalence. (Recall from Section 2.3 that weak equivalences in $\mathbf{Ch}^{\mathbf{C}}_{\mathbb{K}}$, for any shape category $\mathbf{C}$, are detected component-wise.) From Proposition 2.3.5 we know that taking the mapping complex $\underline{\mathrm{map}}(\mathfrak{F}_{J^{\pm}_M(-)}(M), -) : \mathbf{Ch}^{\mathsf{c}}_{\mathbb{K}} \to \mathbf{Ch}_{\mathbb{K}}$ preserves weak equivalences. Then, there is a weak-equivalence

$$\underline{\mathrm{map}}(\mathfrak{F}_{J^{\pm}_M(-)}(M), \mathfrak{F}_{J^{\pm}_M(-)}(M)) \xrightarrow{\sim} \underline{\mathrm{map}}(\mathfrak{F}_{J^{\pm}_M(-)}(M), 0) \cong 0 \tag{3.22}$$

in $\mathbf{Ch}_{\mathbb{K}}$. It follows that the identity $\mathrm{id} \in \mathsf{Z}^0\underline{\mathrm{map}}(\mathfrak{F}_{J^{\pm}_M(-)}(M), \mathfrak{F}_{J^{\pm}_M(-)}(M))$ must be exact, i.e. there exists a $\Lambda^{\pm} \in \underline{\mathrm{map}}(\mathfrak{F}_{J^{\pm}_M(-)}(M), \mathfrak{F}_{J^{\pm}_M(-)}(M))^{-1}$ such that $\delta\Lambda^{\pm} = \mathrm{id}$. According to Definition 3.1.4 these provide a retarded and an advanced Green's homotopy, hence the complex (F, Q) is Green hyperbolic. □

One of the most important properties of ordinary retarded/advanced Green's operators $G^{\pm}$ of a Green hyperbolic differential operator P is that they are unique. This is proved in [Bär15] as a consequence of the existence of unique extensions to (strictly) past/future compactly supported sections of the retarded/advanced Green's operators $G^{\pm}$ from Definition 1.3.1. Note that a similar result of strict uniqueness cannot be available for our retarded/advanced Green's homotopies $\Lambda^{\pm}$ from Definition 3.1.4. In fact, if $\Lambda^{\pm} \in \underline{\mathrm{map}}(\mathfrak{F}_{J^{\pm}_M(-)}(M), \mathfrak{F}_{J^{\pm}_M(-)}(M))^{-1}$ is a retarded/advanced Green's homotopy for a complex (F, Q), then also the (-1)-cochain $\Lambda^{\pm} + \Gamma^{\pm} \in \underline{\mathrm{map}}(\mathfrak{F}_{J^{\pm}_M(-)}(M), \mathfrak{F}_{J^{\pm}_M(-)}(M))^{-1}$ is a retarded/advanced Green's homotopy for all (-1)-cocycles $\Gamma^{\pm} \in \mathsf{Z}^{-1}\underline{\mathrm{map}}(\mathfrak{F}_{J^{\pm}_M(-)}(M), \mathfrak{F}_{J^{\pm}_M(-)}(M))$. However, a strict uniqueness result involving just the *set* of retarded/advanced Green's homotopies is not the kind of result one would expect to hold in this context. Indeed, it would neglect the homotopical phenomena occurring in the *space* of retarded/advanced Green's homotopies. For example, two distinct (as points of a set) retarded/advanced Green's homotopies $\Lambda^{\pm}_1, \Lambda^{\pm}_2$ which differ by an exact term $\Lambda^{\pm}_1 - \Lambda^{\pm}_2 = \delta\lambda^{\pm}$, for $\lambda^{\pm} \in \underline{\mathrm{map}}(\mathfrak{F}_{J^{\pm}_M(-)}(M), \mathfrak{F}_{J^{\pm}_M(-)}(M))^{-2}$, should be considered as 'being the same' since they define the same cohomology class $[\Lambda^{\pm}_1] = [\Lambda^{\pm}_2]$. These homotopical phenomena are the ones we considered in Section 2.4, hence we may introduce the Kan complex $\underline{\mathcal{GH}}^{\pm} \in \mathbf{sSet}$ which upgrades the set of retarded/advanced Green's homotopies by describing also the homotopies between them. We recall from Section 2.4, by setting $V = \underline{\mathrm{map}}(\mathfrak{F}_{J^{\pm}_M(-)}(M), \mathfrak{F}_{J^{\pm}_M(-)}(M)) \in \mathbf{Ch}_{\mathbb{K}}$ and $u = \mathrm{id} \in \mathsf{Z}^0\underline{\mathrm{map}}(\mathfrak{F}_{J^{\pm}_M(-)}(M), \mathfrak{F}_{J^{\pm}_M(-)}(M))$ in Equation (2.120), that the Kan complex

$\underline{\mathcal{GH}}^{\pm}$ of the retarded/advanced Green's homotopies is given by the pullback

$$\begin{tikzcd} \underline{\mathcal{GH}}^{\pm} \arrow[r, dashed] \arrow[d, dashed] & \Delta^0 \arrow[d, "\mathrm{id}"] \\ {[N(\Delta^\bullet), \underline{\mathrm{map}}(\mathfrak{F}_{J^\pm_M(-)}(M), \mathfrak{F}_{J^\pm_M(-)}(M))]^{-1}} \arrow[r, "\partial"'] & \mathsf{Z}^0[N(\Delta^\bullet), \underline{\mathrm{map}}(\mathfrak{F}_{J^\pm_M(-)}(M), \mathfrak{F}_{J^\pm_M(-)}(M))] \end{tikzcd} \tag{3.23}$$

in **sSet**. Recall that $\underline{\mathcal{GH}}^{\pm} \in \mathbf{sSet}$ is a Kan complex as proved by Proposition 2.4.4. We refer the reader to Section 2.4 for a thorough analysis on the content of this Kan complex, here we would like to mention just that the set of 0-simplices of $\underline{\mathcal{GH}}^{\pm}$ is exactly the set of retarded/advanced Green's homotopies, as presented by the pullback

$$\begin{tikzcd} \underline{\mathcal{GH}}^{\pm}_0 \arrow[r, dashed] \arrow[d, dashed] & \{*\} \arrow[d, "\mathrm{id}"] \\ \underline{\mathrm{map}}(\mathfrak{F}_{J^\pm_M(-)}(M), \mathfrak{F}_{J^\pm_M(-)}(M))^{-1} \arrow[r, "\delta"'] & \mathsf{Z}^0\underline{\mathrm{map}}(\mathfrak{F}_{J^\pm_M(-)}(M), \mathfrak{F}_{J^\pm_M(-)}(M)) \end{tikzcd} \tag{3.24}$$

in **Set**. By exploiting the construction (3.23) of the space $\underline{\mathcal{GH}}^{\pm}$ and Proposition 2.4.7, we prove a suitable uniqueness result for retarded and advanced Green's homotopies.

Proposition 3.2.2. *The Kan complex $\underline{\mathcal{GH}}^{\pm} \in \mathbf{sSet}$ of retarded/advanced Green's homotopies given by (3.23) is either empty or contractible. In particular, a Green hyperbolic complex (F, Q) has unique retarded and advanced Green's homotopies up to the contractible space of choices $\underline{\mathcal{GH}}^{\pm}$.*

Proof. Assume $\underline{\mathcal{GH}}^{\pm}$ is not empty. Then, there exists a retarded/advanced Green's homotopy $\Lambda^{\pm} \in \underline{\mathcal{GH}}^{\pm}_0$. From the proof of Proposition 3.2.1 we know that the existence of a retarded/advanced Green's homotopy entails the acyclicity of the mapping complex $\underline{\mathrm{map}}(\mathfrak{F}_{J^\pm_M(-)}(M), \mathfrak{F}_{J^\pm_M(-)}(M)) \sim 0$. Therefore, Proposition 2.4.7 yields that $\underline{\mathcal{GH}}^{\pm}$ is contractible as claimed. □

Remark 3.2.3. Everything stated so far in this chapter, including the Definitions 3.1.4 and 3.1.5 of retarded/advanced Green's homotopies and Green hyperbolic complexes, the recognition principle from Proposition 3.2.1 and the uniqueness result of Proposition 3.2.2, admits a straightforward generalization to the category $\mathbf{Loc}_m$ of m-dimensional globally hyperbolic Lorentzian manifolds. Instead of a single complex of differential operators (F, Q) on $M \in \mathbf{Loc}_m$ we need to consider a *covariant complex of natural differential operators* (F, Q) consisting of a natural graded vector bundle $\mathsf{F} = (\mathsf{F}^n)_{n\in\mathbb{Z}}$, see Definition 1.2.2, and of a family of degree increasing natural linear differential operators $Q = (Q^n : \mathfrak{F}^n \to \mathfrak{F}^{n+1})_{n\in\mathbb{Z}}$, see Definition 1.2.3, such that $(\mathsf{F}(M), Q_M)$ is a complex of differential operators, as per Definition 3.1.1, for all $M \in \mathbf{Loc}_m$. Since the background manifold $M \in \mathbf{Loc}_m$ is let free to vary we need to replace the functors $\mathfrak{F}_{J^{\pm}_M(-)}(M) \in \mathbf{Ch}^{\mathrm{c}}_{\mathbb{K}}$ appearing in Definition 3.1.4 of retarded and advanced Green's homotopies. For this reason, we consider the category $\mathbf{LocC}_m$ whose objects are the pair (M, K) consisting of a m-dimensional globally hyperbolic Lorentzian manifold $M \in \mathbf{Loc}_m$ and of a compact subset $K \subseteq M$. A morphism $f : (M_1, K_1) \to (M_2, K_2)$ in $\mathbf{LocC}_m$ consists of a morphism $f : M_1 \to M_2$ in $\mathbf{Loc}_m$ such that $K_2 \subseteq f(K_1)$. The opposite category $\mathbf{LocC}^{\mathrm{op}}_m$ is a generalization to varying background manifolds of the directed set c of compact subsets of a fixed $M \in \mathbf{Loc}_m$, in that there is an obvious functor $\mathrm{c} \to \mathbf{LocC}^{\mathrm{op}}_m$ sending $K \subseteq M$ to the pair (M, K). Note also that there is an obvious forgetful functor $\mathbf{LocC}_m \to \mathbf{Loc}_m$ which forgets the second entry of the pairs. Recalling from (1.8) the pullback f^* of smooth sections of a natural vector bundle along the morphism f in $\mathbf{Loc}_m$ and exploiting that Equation (1.9) provides a control on the support of the image under f^* of any section, one defines the functor $\mathfrak{F}_{J^{\pm}} : \mathbf{LocC}^{\mathrm{op}}_m \to \mathbf{Ch}_{\mathbb{K}}$ that assigns to the object $(M, K) \in \mathbf{LocC}^{\mathrm{op}}_m$ the cochain complex $\mathfrak{F}_{J^{\pm}_M(K)}(M) \in \mathbf{Ch}_{\mathbb{K}}$ of the sections of the graded vector bundle $\mathsf{F}(M) = (F_M \to M)$ with support contained in the closed subset $J^{\pm}_M(K) \subseteq M$. Given a morphism $f : (M_1, K_1) \to (M_2, K_2)$ in $\mathbf{LocC}_m$, $\mathfrak{F}_{J^{\pm}}$ assigns to it the pullback $\mathfrak{F}_{J^{\pm}}(f) := f^* : \mathfrak{F}_{J^{\pm}_{M_2}(K_2)}(M_2) \to \mathfrak{F}_{J^{\pm}_{M1}(K_1)}(M_1)$ in $\mathbf{Ch}_{\mathbb{K}}$ of smooth sections along the underlying morphism $f : M_1 \to M_2$ in $\mathbf{Loc}_m$. Note that it is well defined because $\operatorname{supp}(f^*\psi) = f^{-1}(\operatorname{supp}\psi) \subseteq f^{-1}(J^{\pm}_{M_2}(K_2)) \subseteq f^{-1}(J^{\pm}_{M_2}(f(K_1))) = J^{\pm}_{M_1}(K_1)$, for all $\psi \in \mathfrak{F}_{J^{\pm}_{M_2}(K_2)}(M_2)$. First identity is Equation (1.9), while the last one relies on f being a $\mathbf{Loc}_m$-morphism. Indeed, let $x \in f^{-1}(J^{\pm}_{M_2}(f(K_1)))$, then from the definition of causal future/past there is a causal curve $c : [0, 1] \to M_2$ pointing in the future/past such that $c(0) \in f(K_1)$ and $c(1) = f(x)$. Since $f(M_1)$ is causally convex, the curve c is entirely supported in $f(M_1)$, then taking its preimage $f^{-1} \circ c : [0, 1] \to M_1$ along f defines a causal curve pointing in the future/past (since f is a time-ordering preserving isometry) such that $f^{-1} \circ c(0) \in K_1$ and $f^{-1} \circ c(1) = x$, thus proving that $x \in J^{\pm}_{M_1}(K_1)$. The opposite inclusion may be proved with a similar argument. Moreover, the pullback f^* is compatible with differentials, $Q_{M_1} f^* = f^* Q_{M_2}$, since Q is degree-wise a natural linear differential operator. If we replace the complex (F, Q) with its covariant counterpart (F, Q), the category c with $\mathbf{LocC}^{\mathrm{op}}_m$ and the functor $\mathfrak{F}_{J^{\pm}_M(-)}(M) \in \mathbf{Ch}^{\mathrm{c}}_{\mathbb{K}}$ with $\mathfrak{F}_{J^{\pm}} \in \mathbf{Ch}_{\mathbb{K}}^{\mathbf{LocC}^{\mathrm{op}}_m}$ in the Definition 3.1.4, we get the definition of *homotopy coherently* $\mathbf{Loc}_m$*-natural* Green's homotopies. We would like to notice that, as a result of their definition, these generalized Green's homotopies $\Lambda^{\pm}$ are weakly natural even with respect to morphisms in $\mathbf{Loc}_m$. To be more precise, following the same line of reasoning of Remark 3.1.8, one sees that the $(0, (M_0, K_0))$-

projections of $\Lambda^{\pm}$, $\mathrm{pr}_{0,(M_0,K_0)}\,\Lambda^{\pm} \in [\mathfrak{F}_{J^{\pm}_{M_0}(K_0)}(M_0), \mathfrak{F}_{J^{\pm}_{M_0}(K_0)}(M_0)]^{-1}$, are *not* natural on the nose but only up to the explicit homotopies $\mathrm{pr}_{1,f^{\mathrm{op}}}\,\Lambda^{\pm} \in [\mathfrak{F}_{J^{\pm}_{M_0}(K_0)}(M_0), \mathfrak{F}_{J^{\pm}_{M_1}(K_1)}(M_1)]^{-2}$, labeled by morphisms $f^{\mathrm{op}} : (M_0, K_0) \to (M_1, K_1)$ in $\mathbf{LocC}^{\mathrm{op}}_m$. If $f : M_1 \to M_0$ denotes the underlying morphism in $\mathbf{Loc}_m$, one explicitly has

$$f^* \circ \mathrm{pr}_{0,(M_0,K_0)}\,\Lambda^{\pm} - \mathrm{pr}_{0,(M_1,K_1)}\,\Lambda^{\pm} \circ f^* = \partial(\mathrm{pr}_{1,f^{\mathrm{op}}}\,\Lambda^{\pm})\,. \tag{3.25}$$

Analog weak naturality conditions hold also for the higher homotopies. Similarly to Definition 3.1.5 a covariant complex (F, Q) is said to be Green hyperbolic if it admits homotopy coherently $\mathbf{Loc}_m$-natural retarded and advanced Green's homotopies. With the same substitutions as above the analog of Propositions 3.2.1 and 3.2.2 hold true. They in fact depend only on the structure of the mapping complex, in particular on the fact that it preserves the weak equivalences. In fact, their proofs do not exploit the shape of the category indexing the $\mathbf{Ch}_{\mathbb{K}}$-valued functors that appear. However, the next result does not admit a similarly straightforward generalization to the covariant case since our proof will explicitly make use of the fact that we have a fixed background manifold $M \in \mathbf{Loc}_m$. $\triangledown$

Theorem 3.2.4. *Let (F, Q) be a Green hyperbolic complex on M. The retarded-minus-advanced cochain map $\Lambda : \mathfrak{F}_{\mathrm{hc}}(M)[1] \to \mathfrak{F}_{\mathrm{hsc}}(M)$ in $\mathbf{Ch}_{\mathbb{K}}$ from Definition 3.1.9 is a quasi-isomorphism.*

Proof. The proof is constructive in that we shall show that Λ is a quasi-isomorphism by providing you with an explicit quasi-inverse $\Theta : \mathfrak{F}_{\mathrm{hsc}}(M) \to \mathfrak{F}_{\mathrm{hc}}(M)[1]$ in $\mathbf{Ch}_{\mathbb{K}}$. For reader's convenience, we will split the proof in four steps. The first step develops a geometric construction which is used in the following steps. The second step defines the candidate quasi-inverse Θ. The third and the fourth steps show that Θ is actually a quasi-inverse of Λ by constructing the homotopies $\Xi \in [\mathfrak{F}_{\mathrm{hc}}(M)[1], \mathfrak{F}_{\mathrm{hc}}(M)[1]]^{-1}$ and $\Upsilon \in [\mathfrak{F}_{\mathrm{hsc}}(M), \mathfrak{F}_{\mathrm{hsc}}(M)]^{-1}$ which witness that $\Theta \circ \Lambda \sim \mathrm{id}$ and $\Lambda \circ \Theta \sim \mathrm{id}$, respectively.

Geometric construction. Let us fix two arbitrary spacelike Cauchy surfaces $\Sigma_{\pm} \subseteq M$ of M such that $\Sigma_+ \subseteq I^+_M(\Sigma_-)$ is contained in the chronological future of the Cauchy surface Σ_-. Consider a functor

$$\Sigma : \mathrm{c} \longrightarrow \mathrm{c}\,, \tag{3.26}$$

i.e. a order preserving map on the directed set of compact subsets of M, which satisfies the condition

$$J^{\pm}_M(\Sigma_{\mp}) \cap J^{\mp}_M(K) \subseteq \Sigma(K)\,, \tag{3.27}$$

for all compact subsets $K \subseteq M$. Note in particular that this implies that $K \subseteq \Sigma(K)$ for each compact subset $K \subseteq M$. Indeed, by construction, $M = I^+_M(\Sigma_-) \cup I^-_M(\Sigma_+) \subseteq J^+_M(\Sigma_-) \cup J^-_M(\Sigma_+)$ and $K \subseteq J^\pm_M(K)$ by definition. Such a $\Sigma : \mathrm{c} \to \mathrm{c}$ always exists. For example, a (minimal) model for it is given by the order preserving map which assigns to $K \in \mathrm{c}$ the subset $\Sigma(K) := (J^+_M(\Sigma_-) \cap J^-_M(K)) \cup (J^-_M(\Sigma_+) \cap J^+_M(K)) \subseteq M$, which is compact because M is globally hyperbolic.

Quasi-inverse Θ of Λ. Given the previous choice of Cauchy surfaces $\Sigma_\pm \subseteq M$, we have an open cover $\{I^+_M(\Sigma_-), I^-_M(\Sigma_+)\}$ of M. We choose in addition a partition of unity $\{\chi_+, \chi_-\}$ subordinate to it. Taking advantage of the choices made, one builds a candidate quasi-inverse Θ to the retarded-minus-advanced cochain map Λ. Explicitly, it is the cochain map

$$\Theta : \mathfrak{F}_{\mathrm{hsc}}(M) \longrightarrow \mathfrak{F}_{\mathrm{hc}}(M)[1] \tag{3.28a}$$

in $\mathbf{Ch}_\mathbb{K}$ which is uniquely determined by

$$\mathrm{hocolim}(j_{(-)}^{J_M(-)}) \circ \Theta := \pm \partial \theta_\pm \in [\mathfrak{F}_{\mathrm{hsc}}(M), \mathfrak{F}_{\mathrm{hsc}}(M)]^1 \tag{3.28b}$$

where $\theta_\pm$ is defined in (3.30) below. Note that in (3.28b) it appears the dg-functor hocolim from (2.103) and that for $n \in \mathbb{Z}$, $p \geq 0$, $\underline{K} : [p] \to \mathrm{c}$ and $\psi \in \mathfrak{F}^{n+p}_{K_0}(M)$, one has

$$\mathrm{hocolim}(j_{(-)}^{J_M(-)})(\iota_{p,\underline{K}}\psi) = \iota_{p,\underline{K}}(j_{(-)}^{J_M(-)}(\psi))\,. \tag{3.29}$$

It readily follows that $\mathrm{hocolim}(j_{(-)}^{J_M(-)}) : \mathfrak{F}_{\mathrm{hc}}(M) \to \mathfrak{F}_{\mathrm{hsc}}(M)$ is degree-wise injective, hence (3.28b) defines Θ uniquely, provided the latter exists. Moreover, degree-wise injectivity of $\mathrm{hocolim}(j_{(-)}^{J_M(-)}) : \mathfrak{F}_{\mathrm{hc}}(M) \to \mathfrak{F}_{\mathrm{hsc}}(M)$ implies also that Θ is a cochain map, in fact $\mathrm{hocolim}(j_{(-)}^{J_M(-)}) \circ \partial\Theta = \partial\big(\mathrm{hocolim}(j_{(-)}^{J_M(-)}) \circ \Theta\big) = 0$. Recalling the dg-adjunction $\mathrm{hocolim} \dashv \Delta$ from Proposition 2.3.12, we define the 0-cochain

$$\theta_\pm \in [\mathfrak{F}_{\mathrm{hsc}}(M), \mathfrak{F}_{\mathrm{hsc}}(M)]^0 \cong \underline{\mathrm{map}}(\mathfrak{F}_{J_M(-)}(M), \Delta\mathfrak{F}_{\mathrm{hsc}}(M))^0 \tag{3.30a}$$

in the mapping complex, by assigning its $(q,\underline{K})$-projections, $\mathrm{pr}_{q,\underline{K}}\,\theta_\pm \in [\mathfrak{F}_{J_M(K_0)}(M), \mathfrak{F}_{\mathrm{hsc}}(M)]^{-q}$, for $q \geq 0$ and $\underline{K} : [q] \to \mathrm{c}$, according to

$$(\mathrm{pr}_{q,\underline{K}}\,\theta_\pm)^n \psi := \iota_{q,\Sigma(\underline{K})}(\chi_\pm \psi) \in \mathfrak{F}^{n-q}_{\mathrm{hsc}}(M) \tag{3.30b}$$

for any $n \in \mathbb{Z}$ and $\psi \in \mathfrak{F}^n_{J_M(K_0)}(M)$. Note that the equation above involves the functor Σ from (3.26). Since $K_0 \subseteq \Sigma(K_0)$, as a consequence of the property (3.27), we also have that $\mathrm{supp}\,\psi \subseteq J_M(K_0) \subseteq J_M(\Sigma(K_0))$. It follows that (3.30b) is well-defined. Exploiting the differentials δ of the mapping complex from (2.81) and d of the homotopy colimit from (2.101), we compute

$$\begin{aligned}
\mathrm{pr}_{q,\underline{K}}(\delta\theta_\pm)\,\psi &= \mathrm{pr}_{q,\underline{K}}(\delta_{\mathrm{v}}\theta_\pm)\,\psi + \mathrm{pr}_{q,\underline{K}}(\delta_{\mathrm{h}}\theta_\pm)\,\psi \\
&= (-1)^q\partial(\mathrm{pr}_{q,\underline{K}}\,\theta_\pm)\,\psi + \sum_{j=0}^{q}(-1)^j\,\mathrm{pr}_{\underline{K}}\circ d^{q-j}\circ \mathrm{pr}_{q-1}(\theta_\pm)\,\psi \\
&= (-1)^q(\mathrm{d}_{\mathrm{v}}+\mathrm{d}_{\mathrm{h}})(\mathrm{pr}_{q,\underline{K}}\,\theta_\pm\,\psi) - (\mathrm{pr}_{q,\underline{K}}\,\theta_\pm)(Q\psi) \\
&\quad + \sum_{j=0}^{q}(-1)^j\,\mathrm{pr}_{q-1,\underline{K}\circ\widehat{q-j}}\,\theta_\pm\,\psi \\
&= (-1)^q(\mathrm{d}_{\mathrm{v}}+\mathrm{d}_{\mathrm{h}})\iota_{q,\Sigma(\underline{K})}(\chi_\pm\psi) - \iota_{q,\Sigma(\underline{K})}(\chi_\pm Q\psi) \\
&\quad + \sum_{j=0}^{q}(-1)^{q-j}\iota_{q-1,\Sigma(\underline{K}\circ\widehat{j})}(\chi_\pm\psi) \\
&= \iota_{q,\Sigma(\underline{K})}(Q(\chi_\pm\psi) - \chi_\pm Q\psi)\,,
\end{aligned}\tag{3.31}$$

for all $q \geq 0$, $\underline{K} : [q] \to \mathrm{c}$ and $\psi \in \mathfrak{F}^n_{J_M(K_0)}(M)$, where we dropped from the notation the action of the functor $\mathfrak{F}_{J_M(-)}(M)$ on the subset inclusion $K_0 \subseteq K_1$. In the second step we just used the definition of δ, in the third step we exploited the definitions of the 'adjoint' differential ∂ and of the coface maps (2.79). The fourth step uses the definition (3.30) of $\theta_\pm$ and the last one follows from the definitions of the vertical d_{v} and horizontal d_{h} differentials. Using that $\chi_+ + \chi_- = 1$ on M and that the differential Q consists of differential operators in each degree, it follows that $Q(\chi_+\psi) - \chi_+ Q\psi = -(Q(\chi_-\psi) - \chi_- Q\psi)$ is a section supported in $\mathrm{supp}\,\psi \cap \mathrm{supp}\,\chi_+ \cap \mathrm{supp}\,\chi_- \subseteq J_M(K_0) \cap J_M^+(\Sigma_-) \cap J_M^-(\Sigma_+) \subseteq \Sigma(K_0)$. The last inclusion is a consequence of property (3.27). Hence, the map $\partial\theta_+ = -\partial\theta_- \in [\mathfrak{F}_{\mathrm{hsc}}(M), \mathfrak{F}_{\mathrm{hsc}}(M)]^1$ factors through the inclusion $\mathrm{hocolim}(j_{(-)}^{J_M(-)})$, ensuring that the cochain map Θ defined by (3.28) exists.

Homotopy Ξ witnessing $\Theta\circ\Lambda \sim \mathrm{id}$. We now build a homotopy Ξ showing that Θ is a left quasi-inverse of Λ, namely $\partial\Xi = \mathrm{id} - \Theta\circ\Lambda$. We uses the isomorphisms (2.14) and (2.15) to pull out the shifts from the internal hom and the dg-adjunction hocolim $\dashv \Delta$ from Proposition 2.3.12 to regard

$$\Xi \in [\mathfrak{F}_{\mathrm{hc}}(M)[1], \mathfrak{F}_{\mathrm{hc}}(M)[1]]^{-1} \cong \underline{\mathrm{map}}(\mathfrak{F}_{(-)}(M), \Delta\mathfrak{F}_{\mathrm{hc}}(M))^{-1} \tag{3.32a}$$

as a (-1)-cochain in the mapping complex. We explicitly define it as

$$\Xi := \xi_- \circ \Lambda^+ \circ j_{(-)}^{J_M^+(-)} + \xi_+ \circ \Lambda^- \circ j_{(-)}^{J_M^-(-)} + \xi\,, \tag{3.32b}$$

where $\Lambda^\pm$ are the retarded and advanced Green's homotopies of (F, Q) chosen in Definition 3.1.9 and $\xi_\mp$ and ξ are defined, respectively, in Equations (3.33) and (3.35) below. The 0-cochains in the mapping complex

$$\xi_\mp \in \underline{\mathrm{map}}(\mathfrak{F}_{J_M^\pm(-)}(M), \Delta\mathfrak{F}_{\mathrm{hc}}(M))^0 \tag{3.33a}$$

are given by assigning their components $\mathrm{pr}_{q,\underline{K}}\,\xi_\mp \in [\mathfrak{F}_{J^\pm_M(K_0)}(M), \mathfrak{F}_{\mathrm{hc}}(M)]^{-q}$, for all $q \geq 0$ and $\underline{K} : [q] \to \mathrm{c}$, according to

$$(\mathrm{pr}_{q,\underline{K}}\,\xi_\mp)^n \psi := \iota_{q,\Sigma(\underline{K})}(\chi_\mp \psi) \in \mathfrak{F}^{n-q}_{\mathrm{hc}}(M)\,, \tag{3.33b}$$

for all $n \in \mathbb{Z}$ and $\psi \in \mathfrak{F}^n_{J^\pm_M(K_0)}(M)$. Note that formula (3.33b) is well-defined since the section $\chi_\mp \psi$ is supported in $J^\pm_M(K_0) \cap J^\mp_M(\Sigma_\pm) \subseteq \Sigma(K_0)$ because of property (3.27). Direct inspection, formally through the same computations as in (3.31), shows that

$$\delta \xi_\mp = \mp \Theta \circ j^{J_M(-)}_{J^\pm_M(-)} \in \underline{\mathrm{map}}(\mathfrak{F}_{J^\pm_M(-)}(M), \Delta \mathfrak{F}_{\mathrm{hc}}(M))^1\,, \tag{3.34}$$

where Θ is regarded as a 1-cocycle $\Theta \in \underline{\mathrm{map}}(\mathfrak{F}_{J_M(-)}(M), \Delta \mathfrak{F}_{\mathrm{hc}}(M))^1$ in the mapping complex exploiting the isomorphism (2.14) and the dg-adjunction $\mathrm{hocolim} \dashv \Delta$. The (-1)-cochain

$$\xi \in \underline{\mathrm{map}}(\mathfrak{F}_{(-)}(M), \Delta \mathfrak{F}_{\mathrm{hc}}(M))^{-1} \tag{3.35a}$$

in the mapping complex is defined by assigning the components $\mathrm{pr}_{q,\underline{K}}\,\xi \in [\mathfrak{F}_{K_0}(M), \mathfrak{F}_{\mathrm{hc}}(M)]^{-1-q}$, for all $q \geq 0$ and $\underline{K} : [q] \to \mathrm{c}$. For all integers $n \in \mathbb{Z}$ and sections $\psi \in \mathfrak{F}^n_{K_0}(M)$, we define

$$(\mathrm{pr}_{q,\underline{K}}\,\xi)^n \psi := \sum_{k=0}^{q} (-1)^{q-k} \iota_{q+1,\underline{K}^{\leq k} \subseteq \Sigma(\underline{K}^{\geq k})} \psi \in \mathfrak{F}^{n-q-1}_{\mathrm{hc}}(M)\,, \tag{3.35b}$$

which may be thought of as a sum over all possible paths of length $q+1$ in the diagram $\underline{K} \subseteq \Sigma(\underline{K})$ starting in K_0 and ending in $\Sigma(K_q)$. The sign in the k-th summand may be interpreted as due to the fact that the morphism $K_k \subseteq \Sigma(K_k)$ in c is pulled through $\underline{K}^{\geq k} : [q-k] \to \mathrm{c}$ and each morphism in c contributes -1 to the total degree in the homotopy colimit. A direct computation shows that

$$\delta \xi = \eta - \xi_- \circ j^{J^+_M(-)}_{(-)} - \xi_+ \circ j^{J^-_M(-)}_{(-)} \in \underline{\mathrm{map}}(\mathfrak{F}_{(-)}(M), \Delta \mathfrak{F}_{\mathrm{hc}}(M))^0\,, \tag{3.36}$$

where $\eta \in \underline{\mathrm{map}}(\mathfrak{F}_{(-)}(M), (\Delta \circ \mathrm{hocolim})\mathfrak{F}_{(-)}(M))^0$ denotes the unit of the dg-adjunction $\mathrm{hocolim} \dashv \Delta$, $(\mathrm{pr}_{q,\underline{K}}\,\eta)^n \psi = \iota_{q,\underline{K}} \psi$ for all $n \in \mathbb{Z}$, $q \geq 0$, $\underline{K} : [q] \to \mathrm{c}$, and $\psi \in \mathfrak{F}^n_{K_0}(M)$. To prove that Equation (3.36) holds true one argues by projecting onto the $(q, \underline{K})$-components. Let us make the computation explicit for the simplest case, $q = 0$, when there is not any contribution coming from the horizontal differential δ_{h} of the mapping complex:

$$\begin{aligned}
(\mathrm{pr}_{0,K_0}\,\delta\xi)\psi &= (\mathrm{pr}_{0,K_0}\,\delta_{\mathrm{v}}\xi)\psi = \partial(\mathrm{pr}_{0,K_0}\,\xi)\psi \\
&= (\mathrm{d_h}+\mathrm{d_v})\iota_{1,K_0\subseteq\Sigma(K_0)}\psi + \iota_{1,K_0\subseteq\Sigma(K_0)}Q\psi \\
&= -\iota_{0,\Sigma(K_0)}\psi + \iota_{0,K_0}\psi \\
&= -\iota_{0,\Sigma(K_0)}(\chi_+\psi) - \iota_{0,\Sigma(K_0)}(\chi_-\psi) + \iota_{0,K_0}\psi \\
&= \big(-\mathrm{pr}_{0,K_0}\,\xi_+\circ j_{(-)}^{J_M^-(-)} - \mathrm{pr}_{0,K_0}\,\xi_-\circ j_{(-)}^{J_M^+(-)} + \mathrm{pr}_{0,K_0}\,\eta\big)\psi\,,
\end{aligned} \tag{3.37}$$

where we dropped from the notation the action of $\mathfrak{F}_{(-)}(M)$ on the subset inclusion $K_0\subseteq\Sigma(K_0)$. The second step is the definition (2.81c) of the vertical differential δ_{v} of the mapping complex, the third step uses the definition (2.5b) of the 'adjoint' differential ∂ and Equation (3.35b) defining ξ. Fourth step follows from the definitions (2.98) and (2.99) of, respectively, the horizontal $\mathrm{d_h}$ and vertical $\mathrm{d_v}$ differentials of the homotopy colimit. Finally, in the fifth step we used that $\chi_+ + \chi_- = 1$ on M. The computations for $q\geq 1$ go on in a similar fashion, yet they are longer and more painful since also the contributions due to δ_{h} are present and the combinatorics becomes more involved. Combining Equation (3.34) and (3.36), one finds

$$\begin{aligned}
\delta\Xi &= \delta\xi_-\circ\Lambda^+\circ j_{(-)}^{J_M^+(-)} + \xi_-\circ\delta\Lambda^+\circ j_{(-)}^{J_M^+(-)} \\
&\qquad + \delta\xi_+\circ\Lambda^-\circ j_{(-)}^{J_M^-(-)} + \xi_+\circ\delta\Lambda^-\circ j_{(-)}^{J-+_M(-)} + \delta\xi \\
&= -\Theta\circ j_{J_M^+(-)}^{J_M(-)}\circ\Lambda^+\circ j_{(-)}^{J_M^+(-)} + \Theta\circ j_{J_M^-(-)}^{J_M(-)}\circ\Lambda^-\circ j_{(-)}^{J_M^-(-)} + \eta \\
&= -\Theta\circ\Lambda + \eta\,,
\end{aligned} \tag{3.38}$$

where in the last step we used Equation (3.17). This yields, under the dg-adjunction hocolim $\dashv\Delta$ from Proposition 2.3.12, the desired $\partial\Xi = \mathrm{id}-\Theta\circ\Lambda$.

Homotopy Υ witnessing $\Lambda\circ\Theta\sim\mathrm{id}$. We shall now construct the homotopy $\Upsilon\in[\mathfrak{F}_{\mathrm{hsc}}(M),\mathfrak{F}_{\mathrm{hsc}}(M)]^{-1}$ witnessing that Θ is a right quasi-inverse of Λ, i.e. $\partial\Upsilon = \mathrm{id}-\Lambda\circ\Theta$. Explicitly, we consider the (-1)-cochain

$$\Upsilon := \mathrm{hocolim}(j_{J_M^+(-)}^{J_M(-)}\circ\Lambda^+)\circ\upsilon_+ + \mathrm{hocolim}(j_{J_M^-(-)}^{J_M(-)}\circ\Lambda^-)\circ\upsilon_- + \upsilon \tag{3.39}$$

in the internal hom, where $\upsilon_\pm$ and υ are defined, respectively, in (3.40) and (3.42) below. Recall that spc/sfc denotes the directed set of the strictly past/future compact subsets of M, see Section 1.1. The 0-cochain

$$\upsilon_\pm \in [\mathfrak{F}_{\mathrm{hsc}}(M),\mathfrak{F}_{\mathrm{h}\,{}^{\mathrm{spc}}_{\mathrm{sfc}}}(M)]^0 \tag{3.40a}$$

in the internal hom is uniquely determined by

$$\mathrm{hocolim}(j_{J_M^\pm(-)}^{J_M(-)})\circ\upsilon_\pm := \theta_\pm \in [\mathfrak{F}_{\mathrm{hsc}}(M),\mathfrak{F}_{\mathrm{hsc}}(M)]^0\,, \tag{3.40b}$$

where $\theta_\pm$ is defined in (3.30). The equation displayed above defines uniquely $\upsilon_\pm$, provided it exists, since $\mathrm{hocolim}(j^{J_M(-)}_{J^\pm_M(-)}) : \mathfrak{F}_{\mathrm{h}^{\mathrm{spc}}_{\mathrm{sfc}}}(M) \to \mathfrak{F}_{\mathrm{hsc}}(M)$ in $\mathbf{Ch}_\mathbb{K}$ is degree-wise injective. See the analog situation in Equation (3.29). Moreover, a $\upsilon_\pm$ exists since for all $\psi \in \mathfrak{F}^n_{J_M(K_0)}(M)$, the section $\chi_\pm\psi$ is supported in $J^\pm_M(\Sigma_\mp) \cap J_M(K_0) \subseteq \Sigma(K_0) \cup J^\pm_M(K_0) \subseteq J^\pm_M(\Sigma(K_0))$ because of property (3.27) of the order preserving map Σ. Therefore, $\theta_\pm$ factors through $\mathrm{hocolim}(j^{J_M(-)}_{J^\pm_M(-)})$. Factoring out $\mathrm{hocolim}(j^{J_M(-)}_{J^\pm_M(-)})$, Equation (3.28b) yields

$$\partial\upsilon_\pm = \pm\,\mathrm{hocolim}(j^{J^\pm_M(-)}_{(-)}) \circ \Theta \in [\mathfrak{F}_{\mathrm{hsc}}(M), \mathfrak{F}_{\mathrm{h}^{\mathrm{spc}}_{\mathrm{sfc}}}(M)]^1\,. \tag{3.41}$$

The (-1)-cochain

$$\upsilon \in [\mathfrak{F}_{\mathrm{hsc}}(M), \mathfrak{F}_{\mathrm{hsc}}(M)]^{-1} \cong \underline{\mathrm{map}}(\mathfrak{F}_{J_M(-)}(M), \Delta\mathfrak{F}_{\mathrm{hsc}}(M))^{-1}\,, \tag{3.42}$$

that is regarded as in the mapping complex by means of the dg-adjunction $\mathrm{hocolim} \dashv \Delta$ from Proposition 2.3.12, is formally defined by the same formula as (3.35). (Note that source and target changed, in particular $\psi \in \mathfrak{F}^n_{J_M(K_0)}(M)$.) Similar computations to those above yield

$$\partial\upsilon = \mathrm{id} - \mathrm{hocolim}(j^{J_M(-)}_{J^-_M(-)}) \circ \upsilon_- - \mathrm{hocolim}(j^{J_M(-)}_{J^+_M(-)}) \circ \upsilon_+\,. \tag{3.43}$$

Finally, combining (3.41) and (3.43), and exploiting definition (3.39) of the (-1)-cochain Υ, one computes

$$\begin{aligned}
\partial\Upsilon &= \mathrm{hocolim}(j^{J_M(-)}_{J^+_M(-)}) \circ \mathrm{hocolim}(\delta\Lambda^+) \circ \upsilon_+ - \mathrm{hocolim}(j^{J_M(-)}_{J^+_M(-)} \circ \Lambda^+) \circ \partial\upsilon_+ \\
&\quad + \mathrm{hocolim}(j^{J_M(-)}_{J^-_M(-)}) \circ \mathrm{hocolim}(\delta\Lambda^-) \circ \upsilon_- \\
&\quad - \mathrm{hocolim}(j^{J_M(-)}_{J^-_M(-)} \circ \Lambda^-) \circ \partial\upsilon_- + \partial\upsilon \\
&= -\,\mathrm{hocolim}(j^{J_M(-)}_{J^+_M(-)} \circ \Lambda^+ \circ j^{J^+_M(-)}_{(-)} - j^{J_M(-)}_{J^-_M(-)} \circ \Lambda^- \circ j^{J^-_M(-)}_{(-)}) \circ \Theta + \mathrm{id} \\
&= -\Lambda \circ \Theta + \mathrm{id}\,,
\end{aligned} \tag{3.44}$$

exploiting also in the first and second step that hocolim is a dg-functor. This proves that Θ is also a right quasi-inverse of Λ, and concludes the proof. □

Remark 3.2.5. Notice that the cochain map $\Theta : \mathfrak{F}_{\mathrm{hsc}}(M) \to \mathfrak{F}_{\mathrm{hc}}(M)[1]$ in $\mathbf{Ch}_\mathbb{K}$ exists independently on the fact that (F, Q) is Green hyperbolic. This should be clear since its construction in the proof of Theorem 3.2.4 relies only on geometric properties of M, via the choices of the spacelike Cauchy surfaces $\Sigma_\pm$, of the order preserving map Σ, and of the partition of unity $\{\chi_+, \chi_-\}$. What Theorem 3.2.4 guarantees is that when (F, Q) is Green hyperbolic the cochain map Θ is, in addition, a quasi-isomorphism. ▽

Remark 3.2.6. Theorem 3.2.4 offers at the same time a new interpretation and a generalization of the well-known exact sequence (1.35) associated with a Green hyperbolic operator $P : \Gamma(E) \to \Gamma(E)$. Let us explain the argument. Recall that with the linear differential operator P is associated the Green hyperbolic complex $(F_{(E,P)}, Q_{(E,P)})$ concentrated in degrees 0 and 1 and whose differential has only one non-vanishing component $Q^0_{(E,P)} = P$. See e.g. Example 3.1.2 and 3.1.7. Denoting by $G^\pm$ the retarded and advanced Green's operators of P, a choice of retarded and advanced Green's homotopies for $(F_{(E,P)}, Q_{(E,P)})$ is given by $\Lambda^\pm \in \underline{\mathrm{hom}}(\mathfrak{F}_{J^\pm_M(-)}(M), \mathfrak{F}_{J^\pm_M(-)}(M))^{-1} \subseteq \underline{\mathrm{map}}(\mathfrak{F}_{J^\pm_M(-)}(M), \mathfrak{F}_{J^\pm_M(-)}(M))^{-1}$ of components $\mathrm{pr}_{0,K_0} \Lambda^\pm = G^\pm : \mathfrak{F}^1_{J^\pm_M(K_0)}(M) \to \mathfrak{F}^0_{J^\pm_M(K_0)}(M)$, for all compact subsets $K_0 \subseteq M$. The associated retarded-minus-advanced cochain map is given, according to Definition 3.1.9, by the homotopy colimit of $\Lambda \in \mathsf{Z}^0\underline{\mathrm{hom}}(\mathfrak{F}_{(-)}(M)[1], \mathfrak{F}_{J_M(-)}(M)) \subseteq \mathsf{Z}^0\underline{\mathrm{map}}(\mathfrak{F}_{(-)}(M)[1], \mathfrak{F}_{J_M(-)}(M))$, defined by $\mathrm{pr}_{0,K_0} \Lambda = G := G^+ - G^- : \mathfrak{F}^0_{K_0}(M)[1] \to \mathfrak{F}^0_{J_M(K_0)}(M)$, for any $K_0 \subseteq M$ compact. We dropped from the notation the explicit reference to the inclusions of sections associated to the subset inclusions $K_0 \subseteq J^\pm_M(K_0) \subseteq J_M(K_0)$. As we have already mentioned, see Remark 2.3.10, since the category c indexing the homotopy colimits is filtered (it is actually a directed set) there are quasi-isomorphisms $\mathfrak{F}_{\mathrm{hc}}(M) \xrightarrow{\sim} \mathfrak{F}_{\mathrm{c}}(M)$ and $\mathfrak{F}_{\mathrm{hsc}}(M) \xrightarrow{\sim} \mathfrak{F}_{\mathrm{sc}}(M)$ to ordinary colimits. Since $\Lambda : \mathfrak{F}_{(-)}(M)[1] \to \mathfrak{F}_{J_M(-)}(M)$ in $\mathbf{Ch}^{\mathrm{c}}_{\mathbb{K}}$ is a natural transformation, taking its colimit

$$\Lambda := \mathrm{colim}(\Lambda) : \mathfrak{F}_{\mathrm{c}}(M)[1] \longrightarrow \mathfrak{F}_{\mathrm{sc}}(M) \tag{3.45}$$

returns a cochain map, which we denote by the same symbol with a slight abuse of notation. Putting together these two facts, we get the commutative diagram

$$\begin{tikzcd} \mathfrak{F}_{\mathrm{hc}}(M)[1] \arrow[r, "\Lambda"] \arrow[d, "\sim"'] & \mathfrak{F}_{\mathrm{hsc}}(M) \arrow[d, "\sim"] \\ \mathfrak{F}_{\mathrm{c}}(M)[1] \arrow[r, "\Lambda"'] & \mathfrak{F}_{\mathrm{sc}}(M) \end{tikzcd} \tag{3.46}$$

in $\mathbf{Ch}_{\mathbb{K}}$. From Theorem 3.2.4 we have that $\Lambda : \mathfrak{F}_{\mathrm{hc}}(M)[1] \xrightarrow{\sim} \mathfrak{F}_{\mathrm{hsc}}(M)$ in $\mathbf{Ch}_{\mathbb{K}}$ is a quasi-isomorphism, hence the diagram displayed above, together with the 2-out-of-3 property for the weak equivalences of a model category, yields that $\Lambda : \mathfrak{F}_{\mathrm{c}}(M)[1] \xrightarrow{\sim} \mathfrak{F}_{\mathrm{sc}}(M)$ is a quasi-isomorphism too. Recalling that a cochain map is a quasi-isomorphism if and only if its *cone complex* is acyclic, see e.g. [Wei94, Cor. 1.5.4], one gets that

$$\begin{aligned} \mathrm{cone}(\Lambda) &:= \left(\mathfrak{F}_{\mathrm{c}}(M)[2] \oplus \mathfrak{F}_{\mathrm{sc}}(M), \begin{pmatrix} Q_{[2]} & 0 \\ -\Lambda & Q \end{pmatrix}\right) \\ &= \Big(\cdots \longrightarrow 0 \longrightarrow \overset{(-2)}{\Gamma_{\mathrm{c}}(E)} \xrightarrow{P} \overset{(-1)}{\Gamma_{\mathrm{c}}(E)} \xrightarrow{-G} \overset{(0)}{\Gamma_{\mathrm{sc}}(E)} \xrightarrow{P} \overset{(1)}{\Gamma_{\mathrm{sc}}(E)} \longrightarrow 0 \longrightarrow \cdots \Big), \end{aligned} \tag{3.47}$$

where in the first line the differential is written in matrix notation, is an acyclic cochain complex. Since it is clear that $\mathrm{cone}(\Lambda)$ is isomorphic to the sequence (1.35) regarded as a cochain complex concentrated between degrees -2 and 1, we conclude that the exactness of (1.35) is equivalent to the acyclicity of $\mathrm{cone}(\Lambda) \in \mathbf{Ch}_{\mathbb{K}}$. Therefore, Theorem 1.3.8 is recovered by our Theorem 3.2.4. ▽

3.3 Differential pairings and Poisson structures

Working over $\mathbb{K} = \mathbb{R}$, for a formally self-adjoint Green hyperbolic operator P acting on smooth sections of a vector bundle $E \to M$ endowed with a fiber metric $\langle -, - \rangle$, the vector space of the linear observables $\mathrm{coker}_{\mathrm{c}}(P)$ comes endowed with a Poisson structure $\tau_M : \mathrm{coker}_{\mathrm{c}}(P)^{\wedge 2} \to \mathbb{R}$, see Equation (1.45) and Remark 1.3.14, which may be written in the following three equivalent ways

$$\tau_M([\psi_1],[\psi_2]) = \langle\!\langle \psi_1, G\psi_2 \rangle\!\rangle = \pm \langle\!\langle \psi_1, G^{\pm}\psi_2 \rangle\!\rangle \mp \langle\!\langle \psi_2, G^{\pm}\psi_1 \rangle\!\rangle \,, \tag{3.48}$$

for all $[\psi_1],[\psi_2] \in \mathrm{coker}_{\mathrm{c}}(P)$, where $\langle\!\langle -, - \rangle\!\rangle = \int_M \langle -, - \rangle \, \mathrm{vol}_M$ denotes the integration pairing built out of the fiber metric. Moreover, if P is normally hyperbolic (Example 1.3.10), also the vector space of spacelike solutions $\ker_{\mathrm{sc}}(P)$, see Remark 1.3.17, admits a Poisson structure $\sigma_\Sigma : \ker_{\mathrm{sc}}(P)^{\wedge 2} \to \mathbb{R}$, which relies on the additional choice of a spacelike Cauchy surface $\Sigma \subseteq M$. In this section we shall show that similar structures are available also for the complexes of differential operators that are Green hyperbolic. First, we need to introduce the concept of a *differential pairing* for a complex of differential operators. This generalizes the usual fiber metric $\langle -, - \rangle$ on a vector bundle $E \to M$, required to construct the Poisson structures above, and, at the same time, it encodes enough structure to make Stokes' theorem available upon integration.

Definition 3.3.1. Let (F, Q) be a complex of differential operators on $M \in \mathbf{Loc}_m$. A *differential pairing* $(-, -)$ on (F, Q) is a graded anti-symmetric cochain map

$$(-,-) : \mathfrak{F}(M) \otimes \mathfrak{F}(M) \longrightarrow \Omega^\bullet(M)[m-1] \tag{3.49}$$

in $\mathbf{Ch}_{\mathbb{K}}$, which is degree-wise a bi-differential operator. Here $\mathfrak{F}(M) \in \mathbf{Ch}_{\mathbb{K}}$ is the cochain complex of smooth sections of (F, Q) and $\Omega^\bullet(M) \in \mathbf{Ch}_{\mathbb{K}}$ denotes the cochain complex of smooth sections of the de Rham complex, see Example 3.1.3, i.e. it is the cochain complex consisting of differential forms on M and of the de Rham differential.

In Chapter 6, we will show that all our physically motivated examples of complexes of differential operators admit differential pairings on them.

In particular, in the case of the complex associated to an ordinary field theory ruled by a normally hyperbolic operator, one recovers both the pairings needed to define, upon integration, the Poisson structures τ_M and σ_Σ mentioned above and introduced in Section 1.3.

Note that a differential pairing (3.49) encodes the information of a fiber metric (actually its action on smooth sections) in cohomological degree 1 since $(-,-)^1 : \bigoplus_{p\in\mathbb{Z}} \mathfrak{F}^p(M) \otimes \mathfrak{F}^{1-p}(M) \to \Omega^m(M)$ takes values into the linear space of top-dimensional differential forms on M. When everything is well-defined, e.g. if the arguments of $(-,-)^1$ have compactly overlapping supports, its integration over M defines an integration pairing analogous to the one obtained by integrating a fiber metric. Nevertheless, a differential pairing contains more information thanks to the lower cohomological degrees. For $k \leq 0$, $(-,-)^k : \bigoplus_{p\in\mathbb{Z}} \mathfrak{F}^p(M) \otimes \mathfrak{F}^{k-p}(M) \to \Omega^{m-1+k}(M)$ yields differential forms which have to be integrated over submanifolds of codimension $1 + |k|$. The compatibility of $(-,-)$ with respect to the differentials of the tensor product and of the ($(m-1)$-shifted) de Rham complex allows us to relate the structures induced on submanifolds of different codimension. In practice, it makes available the Stokes' theorem after integration.

Let us be more precise and introduce some key ingredients for the construction of our Poisson structures. Given a complex of differential operators (F, Q) and a differential pairing $(-,-)$ on it, we define an *evaluation pairing* by the composition

$$\mathrm{ev}_M : \mathfrak{F}_\mathrm{c}(M)[1] \otimes \mathfrak{F}(M) \xrightarrow{\mathrm{id}_{\mathbb{R}[1]}\otimes(-,-)} \Omega^\bullet_\mathrm{c}(M)[m] \xrightarrow{\int_M} \mathbb{R} \tag{3.50}$$

in $\mathbf{Ch}_\mathbb{R}$. In the construction above we implicitly used the isomorphisms $\mathfrak{F}_\mathrm{c}(M)[1] \cong \mathbb{R}[1]\otimes\mathfrak{F}_\mathrm{c}(M)$ and $\mathbb{R}[1]\otimes\Omega^\bullet_\mathrm{c}(M)[m-1] \cong \Omega^\bullet_\mathrm{c}(M)[m]$ in $\mathbf{Ch}_\mathbb{R}$. Furthermore, we exploited that, being a bi-differential operator by hypothesis, $(-,-)$ preserves supports and that integrating over M defines a cochain map $\int_M : \Omega^\bullet_\mathrm{c}(M)[m] \to \mathbb{R}$ in $\mathbf{Ch}_\mathbb{R}$ because of Stokes' theorem, $(-1)^m \int_M \mathrm{d}_\mathrm{dR}\omega = 0$ for all $\omega \in \Omega^m_\mathrm{c}(M)$.

We construct now the Poisson structures $\tau^\pm_M$ on the 'complex of linear observables' $\mathfrak{F}_\mathrm{hc}(M)[1] \in \mathbf{Ch}_\mathbb{R}$ which generalize the two equivalent presentations of the usual covariant Poisson structure appearing in the second identity of (3.48). Let the complex (F, Q) be Green hyperbolic and let $\Lambda^\pm$ be a choice of retarded/advanced Green's homotopy for it. Exploiting the natural transformations (3.16) obtained by whiskering $\mathfrak{F}_{(-)}(M)$ with the order preserving maps associated to the inclusions $K \subseteq J^\pm_M(K) \subseteq M$, for all compact subsets $K \subseteq M$, we define the 0-cochain

$$\Lambda^\pm \in [\mathfrak{F}_\mathrm{hc}(M)[1], \mathfrak{F}(M)]^0 \cong [\mathfrak{F}_\mathrm{hc}(M), \mathfrak{F}(M)]^{-1} \tag{3.51a}$$

in the internal hom, as the adjunct of the (-1)-cochain

$$j^M_{J^\pm_M(-)} \circ \Lambda^\pm \circ j^{J^\pm_M(-)}_{(-)} \in \underline{\mathrm{map}}(\mathfrak{F}_{(-)}, \Delta\mathfrak{F}(M))^{-1} \tag{3.51b}$$

with respect to the dg-adjunction $\mathrm{hocolim} \dashv \Delta$ from Proposition 2.3.12. Combining the evaluation cochain map (3.50) and the 0-cochain (3.51), we define the cochain map

$$\tau_M^{\pm} := \pm\, \mathrm{ev}_M \circ (\mathrm{id} \otimes \Lambda^{\pm}) \mp \mathrm{ev}_M \circ \gamma \circ (\Lambda^{\pm} \otimes \mathrm{id}) : \mathfrak{F}_{\mathrm{hc}}(M)[1]^{\otimes 2} \longrightarrow \mathbb{R} \tag{3.52}$$

where γ denotes the symmetric braiding on $\mathbf{Ch}_{\mathbb{R}}$ and the 1-shift of the $\mathfrak{F}_{(-)}$-component of the natural transformation $\mathrm{hocolim} \to \mathrm{colim}$ from (2.102) is suppressed from our notation for the sake of simplicity. Let us check that the equation displayed above defines a cochain map by computing, for all homogeneous $\psi_1, \psi_2 \in \mathfrak{F}_{\mathrm{hc}}(M)[1]$,

$$\begin{aligned}(\partial\tau_M^{\pm})(\psi_1 \otimes \psi_2) &= (\pm \mathrm{ev}_M \circ (\mathrm{id} \otimes \partial\Lambda^{\pm}) \mp \mathrm{ev}_M \circ \gamma \circ (\partial\Lambda^{\pm} \otimes \mathrm{id}))(\psi_1 \otimes \psi_2)\\ &= \pm(-1)^{|\psi_1|} \int_M (\psi_1, \psi_2) \mp (-1)^{|\psi_2|(|\psi_1|+1)} \int_M (\psi_2, \psi_1)\\ &= 0\,,\end{aligned} \tag{3.53}$$

where $|\psi_i|$, $i = 1, 2$, denotes the degree of $\psi_i \in \mathfrak{F}_{\mathrm{hc}}(M)[1]$. The first step follows since both γ and ev_M are cochain maps, in the second step we used that the 1-cochain $\partial\Lambda^{\pm} \in [\mathfrak{F}_{\mathrm{hc}}(M)[1], \mathfrak{F}(M)]^1 \cong [\mathfrak{F}_{\mathrm{hc}}(M), \mathfrak{F}(M)]^0$ is the adjunct with respect to the dg-adjunction $\mathrm{hocolim} \dashv \Delta$ from Proposition 2.3.12 of the inclusion $j_{(-)}^M : \mathfrak{F}_{(-)}(M) \to \Delta\mathfrak{F}(M)$ in $\mathbf{Ch}_{\mathbb{R}}^{\mathrm{c}}$ since $\delta\Lambda^{\pm} = \mathrm{id}$. Finally, in the last step we used that the differential pairing $(-,-)$ is graded anti-symmetric when evaluated on the two-fold tensor product of the unshifted complex $\mathfrak{F}(M)$, then taking into account the 1-shift one has $(\psi_2, \psi_1) = -(-1)^{(|\psi_1|+1)(|\psi_2|+1)}(\psi_1, \psi_2)$. Since $\tau_M^{\pm}$ is manifestly graded anti-symmetric, $\tau_M^{\pm} \circ \gamma = -\tau_M^{\pm}$, it descends to a (unshifted) Poisson structure $\tau_M^{\pm} : \mathfrak{F}_{\mathrm{hc}}(M)[1]^{\wedge 2} \to \mathbb{R}$, recall Definition 2.2.1.

The Poisson structure τ_M generalizing the first equivalent presentation of the usual covariant Poisson structure (3.48) is built as follows. Given a Green hyperbolic complex (F, Q) and a choice of retarded and advanced Green's homotopies $\Lambda^{\pm}$ of it, we first define the cochain map

$$\widetilde{\tau}_M := \mathrm{ev}_M \circ (\mathrm{id} \otimes \Lambda) : \mathfrak{F}_{\mathrm{hc}}(M)[1]^{\otimes 2} \longrightarrow \mathbb{R} \tag{3.54}$$

in $\mathbf{Ch}_{\mathbb{R}}$, where $\Lambda : \mathfrak{F}_{\mathrm{hc}}(M)[1] \to \mathfrak{F}_{\mathrm{hsc}}(M)$ is the retarded-minus-advanced cochain map associated with the choice of $\Lambda^{\pm}$. In Equation (3.54) we suppressed from our notation both the 1-shift of the $\mathfrak{F}_{(-)}$-component of the natural transformation $\mathrm{hocolim} \to \mathrm{colim}$ and the adjunct of the natural transformation $\mathfrak{F}_{J_M(-)}(M) \to \Delta\mathfrak{F}(M)$ in $\mathbf{Ch}_{\mathbb{R}}^{\mathrm{c}}$ with respect to the dg-adjunction $\mathrm{hocolim} \dashv \Delta$. Notice that $\widetilde{\tau}_M$ is always a cochain map, since it is a composition of cochain maps, but it is, in general, not graded anti-symmetric. For this reason, in order to get a Poisson structure on the complex $\mathfrak{F}_{\mathrm{hc}}(M)[1] \in \mathbf{Ch}_{\mathbb{R}}$ we have to take its (graded) anti-symmetrization

$$\tau_M := \mathrm{asym}\,(\widetilde{\tau}_M) : \mathfrak{F}_{\mathrm{hc}}(M)[1]^{\otimes 2} \longrightarrow \mathbb{R} \tag{3.55}$$

in $\mathbf{Ch}_{\mathbb{R}}$, which by construction descends to a (unshifted) Poisson structure $\tau_M : \mathfrak{F}_{\mathrm{hc}}(M)[1]^{\wedge 2} \to \mathbb{R}$ on $\mathfrak{F}_{\mathrm{hc}}(M)[1]$.

Remark 3.3.2. As mentioned above, the anti-symmetrization is required because we did not impose any compatibility condition between the chosen retarded and advanced Green's homotopies and the differential pairing. Hence, in general the cochain map $\widetilde{\tau}_M$ may fail to be graded anti-symmetric. In the standard, non-graded, context this issue is mitigated by the fact that retarded and advanced Green's operators are unique. Therefore, it is sufficient to ask the Green hyperbolic operator P to be formally self-adjoint with respect to the fiber metric to automatically get formally skew-adjoint retarded/advanced Green's operators $G^{\pm}$, see (1.42), and hence a Poisson structure τ_M without the need of the anti-symmetrization, see (1.44). Something similar can be obtained in our framework as well. Indeed, for a large class of examples, including all our examples from Chapter 6, there exist special choices of retarded and advanced Green's homotopies which are compatible with the differential pairing and make the cochain map $\widetilde{\tau}_M$ graded anti-symmetric, thus making superfluous taking the anti-symmetrization in (3.55). We will go into more detail in Section 4.2. $\triangledown$

In the ordinary case of a Green hyperbolic linear differential operator the Poisson structures $\tau_M^{\pm}, \tau_M$ coincide on the nose, as shown by Equation (3.48). For an arbitrary Green hyperbolic complex this is not the case and the three presentations may actually be different Poisson structures. To fix ideas, since the choice of a Λ^+ and a Λ^- is not required to be performed in a compatible way with respect to the differential pairing, it follows that there is no reason for τ_M^+ and τ_M^- to coincide. Nevertheless, the three are always equivalent in the sense that they differ by a homotopy.

Proposition 3.3.3. *Given a Green hyperbolic complex (F, Q) on $M \in \mathbf{Loc}_m$ endowed with a differential pairing $(-,-)$ and a choice of retarded and advanced Green's homotopies $\Lambda^{\pm}$ for it, let $\tau_M^{\pm}, \tau_M : \mathfrak{F}_{\mathrm{hc}}(M)[1]^{\wedge 2} \to \mathbb{R}$ be the Poisson structures defined by Equations (3.52) and (3.55), respectively. The Poisson structures coincide $\tau_M^{\pm} = \tau_M + \partial \lambda_M^{\pm}$ up to a homotopy $\lambda_M^{\pm} \in [\mathfrak{F}_{\mathrm{hc}}(M)[1]^{\wedge 2}, \mathbb{R}]^{-1}$.*

Proof. First, let us observe that the anti-symmetrization (3.55) is computed as $\tau_M = 1/2(\widetilde{\tau}_M - \widetilde{\tau}_M \circ \gamma)$. Exploiting the formula (3.54) defining $\widetilde{\tau}_M$, one gets

$$\begin{aligned} \tau_M &= \frac{1}{2}\big(\mathrm{ev}_M \circ (\mathrm{id} \otimes \Lambda) - \mathrm{ev}_M \circ \gamma \circ (\Lambda \otimes \mathrm{id})\big) \\ &= \frac{1}{2}(\tau_M^+ + \tau_M^-)\,, \end{aligned} \tag{3.56}$$

where the last step follows from Equation (3.52) and the Definition 3.1.9 of the retarded-minus-advanced cochain map Λ. It follows that $\tau_M^+ - \tau_M =$

$-(\tau_M^- - \tau_M)$. Hence, it is sufficient to prove that there is a homotopy $\lambda_M^+ \in [\mathfrak{F}_{\mathrm{hc}}(M)[1]^{\wedge 2}, \mathbb{R}]^{-1}$ such that $\tau_M^+ - \tau_M = \partial \lambda_M^+$. Then, $\lambda_M^- := -\lambda_M^+$ is a homotopy comparing the Poisson structures τ_M^- and τ_M. We construct the homotopy λ_M^+ as the (graded) anti-symmetrization

$$\lambda_M^+ := \operatorname{asym}(\widetilde{\lambda}_M^+) \in [\mathfrak{F}_{\mathrm{hc}}(M)[1]^{\otimes 2}, \mathbb{R}]^{-1} \tag{3.57a}$$

of the (-1)-cochain

$$\widetilde{\lambda}_M^+ \in [\mathfrak{F}_{\mathrm{hc}}(M)[1]^{\otimes 2}, \mathbb{R}]^{-1} \tag{3.57b}$$

defined below. Let us make some preparations. Abusing a bit the notation, we denote by $\Lambda^\pm := \operatorname{hocolim}(\Lambda^\pm) \in [\mathfrak{F}_{\mathrm{h}^{\mathrm{spc}}_{\mathrm{sfc}}}(M)[1], \mathfrak{F}_{\mathrm{h}^{\mathrm{spc}}_{\mathrm{sfc}}}(M)]^0$ also the homotopy colimit of the chosen retarded/advanced Green's homotopy, regarded as a 0-cochain in the internal hom by exploiting the isomorphism (2.15). As we have already done for the definitions of the Poisson structures, we will drop from our notation the components of the natural transformation $\operatorname{hocolim} \to \operatorname{colim}$, the 1-shift of the homotopy colimit $\operatorname{hocolim}(j_{(-)}^{J_M^\pm(-)}) \in [\mathfrak{F}_{\mathrm{hc}}(M), \mathfrak{F}_{\mathrm{h}^{\mathrm{spc}}_{\mathrm{sfc}}}(M)]^0$ associated to the inclusion $\mathfrak{F}_{(-)}(M) \to \mathfrak{F}_{J_M^\pm(-)}(M)$ in $\mathbf{Ch}_{\mathbb{R}}^{\mathrm{c}}$ and the adjuncts of the inclusions $\mathfrak{F}_{(-)}(M) \to \Delta\mathfrak{F}(M)$, $\mathfrak{F}_{J_M^\pm(-)}(M) \to \Delta\mathfrak{F}(M)$, $\mathfrak{F}_{J_M(-)}(M) \to \Delta\mathfrak{F}(M)$ in $\mathbf{Ch}_{\mathbb{R}}^{\mathrm{c}}$ with respect to the dg-adjunction $\operatorname{hocolim} \dashv \Delta$ from Proposition 2.3.12. Keeping also in mind that the intersection of a strictly past compact subset of M with a strictly future compact one is compact and that the differential pairing $(-,-)$ preserves the supports, we define

$$\widetilde{\lambda}_M^+(\psi_1 \otimes \psi_2) := -\int_M (\Lambda^+\psi_1, \Lambda^-\psi_2)\,, \tag{3.57c}$$

for all homogeneous $\psi_1, \psi_2 \in \mathfrak{F}_{\mathrm{hc}}(M)[1]$. For degree reasons, the integration cochain map $\int_M \in \mathsf{Z}^0[\Omega_{\mathrm{c}}^\bullet(M)[m], \mathbb{R}] \cong \mathsf{Z}^{-1}[\Omega_{\mathrm{c}}^\bullet(M)[m-1], \mathbb{R}]$ is regarded as a (-1)-cocycle in the internal hom. A straightforward computation shows that

$$\partial \widetilde{\lambda}_M^+ = \tau_M^+ - \widetilde{\tau}_M\,. \tag{3.58}$$

Let us explicitly compute, for all homogeneous $\psi_1, \psi_2 \in \mathfrak{F}_{\mathrm{hc}}(M)[1]$, that

$$\begin{aligned}
(\partial\widetilde{\lambda}_M^+)(\psi_1 \otimes \psi_2) &= -\int_M (\Lambda^+ \mathrm{d}_{[1]}\psi_1, \Lambda^-\psi_2) - (-1)^{|\psi_1|}\int_M (\Lambda^+\psi_1, \Lambda^- \mathrm{d}_{[1]}\psi_2) \\
&= -\int_M (\mathrm{d}\Lambda^+\psi_1, \Lambda^-\psi_2) - (-1)^{|\psi_1|}\int_M (\Lambda^+\psi_1, \mathrm{d}\Lambda^-\psi_2) \\
&\quad + \int_M (\psi_1, \Lambda^-\psi_2) + (-1)^{|\psi_1|}\int_M (\Lambda^+\psi_1, \psi_2) \\
&= -\int_M \mathrm{d}_{\mathrm{dR}[m-1]}(\Lambda^+\psi_1, \Lambda^-\psi_2) + \int_M (\psi_1, \Lambda^-\psi_2) \\
&\quad - (-1)^{|\psi_1||\psi_2|}\int_M (\psi_2, \Lambda^+\psi_1) \\
&= (\tau_M^+ - \widetilde{\tau}_M)(\psi_1 \otimes \psi_2)\,.
\end{aligned} \tag{3.59}$$

In the first step we used that $\partial\widetilde{\lambda}^+_M = \widetilde{\lambda}^+_M \circ \mathrm{d}_{[1]\otimes}$, because of the definition of the 'adjoint' differential ∂. The second step follows from $\mathrm{d}\circ\Lambda^\pm - \Lambda^\pm\circ\mathrm{d}_{[1]} = \partial(\Lambda^\pm) = \mathrm{id}$ (suppressing the 1-shift of the adjunct of $\mathfrak{F}_{(-)}(M) \to \Delta\mathfrak{F}(M)$ in $\mathbf{Ch}^{\mathrm{c}}_{\mathbb{R}}$, with respect to hocolim $\dashv \Delta$). In the third step we used that the differential pairing $(-,-)$ is graded anti-symmetric and it is compatible with differentials $(-,-)\circ Q_\otimes = \mathrm{d}_{\mathrm{dR}[m-1]}\circ(-,-)$. (Here the passage from d to Q is left implicit since the components of the natural transformation hocolim $\to$ colim have been suppressed from our notation.) Last step uses the Stokes' theorem and the definitions (3.54) of $\widetilde{\tau}_M$ and (3.52) of τ^+_M, together with $\Lambda = \Lambda^+ - \Lambda^-$. Since λ^+_M is graded anti-symmetric by construction, it descends to a homotopy $\lambda^+_M \in [\mathfrak{F}_{\mathrm{hc}}(M)[1]^{\wedge 2}, \mathbb{R}]^{-1}$ such that

$$\partial\lambda^+_M = \mathrm{asym}(\partial\widetilde{\lambda}^+_M) = \tau^+_M - \tau_M\,, \tag{3.60}$$

which follows from (3.55) and the graded anti-symmetry of τ^+_M. □

Remark 3.3.4. Even though for generic Green hyperbolic complex (F,Q) and choices of retarded/advanced Green's homotopies $\Lambda^\pm$ of it the Poisson structures $\tau_M, \tau^\pm_M$ on the complex of linear observables $\mathfrak{F}_{\mathrm{hc}}(M)[1]$ agree only up to the explicit homotopies from Proposition 3.3.3, there exists a large enough class of examples, which includes all our examples from Chapter 6, for which there are particular choices of the retarded and advanced Green's homotopies that are compatible with the differential pairing. With these choices the Poisson structures introduced above $\tau_M = \tau^+_M = \tau^-_M$ all agree on the nose, similarly to the more traditional case. We will deal with this topic in Section 4.2. ▽

In general, different choices of retarded/advanced Green's homotopies for a Green hyperbolic complex are possible, since they are not strictly unique. For each choice, we get *a priori* different Poisson structures $\tau^\pm_M, \tau_M$ according to formulae (3.52) and (3.55). One may then ask if there is a 'true' dependence on these choices. The following proposition, taking advantage of our contractibility result (Proposition 3.2.2), shows that the Poisson structures one gets by choosing different retarded/advanced Green's homotopies $\Lambda^\pm$ all coincide up to homotopy.

Proposition 3.3.5. *Let (F,Q) be a Green hyperbolic complex endowed with a differential pairing $(-,-)$ and let $\Lambda^\pm, \Lambda'^\pm$ be two different choices of retarded/advanced Green's homotopies for it. Denote by τ_M and τ'_M the Poisson structures on $\mathfrak{F}_{\mathrm{hc}}(M)[1]$ built out of $\Lambda^\pm$ and $\Lambda'^\pm$, respectively. Then, there exists a homotopy $\lambda' \in [\mathfrak{F}_{\mathrm{hc}}(M)[1]^{\wedge 2}, \mathbb{R}]^{-1}$ such that $\tau'_M = \tau_M + \partial\lambda'$.*

Proof. Since (F,Q) is Green hyperbolic, the Kan complex $\underline{\mathcal{GH}}^\pm \in \mathbf{sSet}$ of retarded/advanced Green's homotopies, defined by the pullback (3.23), is contractible by Proposition 3.2.2. It follows, in particular, that for any pair of retarded/advanced Green's homotopies $\Lambda^\pm, \Lambda'^\pm \in \underline{\mathcal{GH}}^\pm_0$ there is a

1-simplex $H^{\pm} \in \underline{\mathcal{GH}}^{\pm}_1$ such that $d_0 H^{\pm} = \Lambda^{\pm}$ and $d_1 H^{\pm} = \Lambda'^{\pm}$. Recalling Section 2.4, in particular Equation (2.121), this means that there exists a homotopy $\eta^{\pm} \in \underline{\mathrm{map}}(\mathfrak{F}_{J^{\pm}_M(-)}(M), \mathfrak{F}_{J^{\pm}_M(-)}(M))^{-2}$ such that $\delta\eta^{\pm} = \Lambda'^{\pm} - \Lambda^{\pm}$. It follows that

$$
\begin{aligned}
\Lambda' &= \mathrm{hocolim}\left(j^{J_M(-)}_{J^+_M(-)} \circ \Lambda'^{+} \circ j^{J^+_M(-)}_{(-)} - j^{J_M(-)}_{J^-_M(-)} \circ \Lambda'^{-} \circ j^{J^-_M(-)}_{(-)} \right) \\
&= \mathrm{hocolim}\left(j^{J_M(-)}_{J^+_M(-)} \circ \Lambda^{+} \circ j^{J^+_M(-)}_{(-)} - j^{J_M(-)}_{J^-_M(-)} \circ \Lambda^{-} \circ j^{J^-_M(-)}_{(-)} \right) \\
&\quad + \mathrm{hocolim}\left(j^{J_M(-)}_{J^+_M(-)} \circ \delta\eta^{+} \circ j^{J^+_M(-)}_{(-)} - j^{J_M(-)}_{J^-_M(-)} \circ \delta\eta^{-} \circ j^{J^-_M(-)}_{(-)} \right) \\
&= \Lambda + \partial \upeta \,,
\end{aligned} \tag{3.61}
$$

where in the first and last step we used the Definition 3.1.9 of the retarded-minus-advanced cochain maps Λ' and Λ associated, respectively, to $\Lambda'^{\pm}$ and $\Lambda^{\pm}$. The second step uses the homotopies $\eta^{\pm}$ to relate the primed retarded and advanced Green's homotopies to the unprimed ones. Finally, we used that the homotopy colimit and the functors $j^{J^{\pm}_M(-)}_{(-)}$ and $j^{J_M(-)}_{J^{\pm}_M(-)}$ are compatible with differentials and we set

$$
\upeta := \mathrm{hocolim}\left(j^{J_M(-)}_{J^+_M(-)} \circ \eta^{+} \circ j^{J^+_M(-)}_{(-)} - j^{J_M(-)}_{J^-_M(-)} \circ \eta^{-} \circ j^{J^-_M(-)}_{(-)} \right), \tag{3.62}
$$

regarded as a (-1)-cochain in the internal hom $[\mathfrak{F}_{\mathrm{hc}}(M)[1], \mathfrak{F}_{\mathrm{hsc}}(M)]$. From the definition (3.55) of the Poisson structure, one has

$$
\begin{aligned}
\tau'_M &= \mathrm{asym}(\mathrm{ev}_M \circ (\mathrm{id} \otimes \Lambda')) \\
&= \mathrm{asym}(\mathrm{ev}_M \circ (\mathrm{id} \otimes \Lambda)) + \mathrm{asym}(\mathrm{ev}_M \circ (\mathrm{id} \otimes \partial\upeta)) && (3.63) \\
&= \tau_M + \partial \lambda' \,, && (3.64)
\end{aligned}
$$

where

$$
\lambda' := \mathrm{asym}(\mathrm{ev}_M \circ (\mathrm{id} \otimes \upeta)) \in [\mathfrak{F}_{\mathrm{hc}}(M)[1]^{\otimes 2}, \mathbb{R}]^{-1} \tag{3.65}
$$

descends to a homotopy $\lambda' \in [\mathfrak{F}_{\mathrm{hc}}(M)[1]^{\wedge 2}, \mathbb{R}]^{-1}$ since it is graded anti-symmetric by construction. □

The second type of Poisson structure on a complex of differential operators (F, Q) endowed with a differential pairing $(-,-)$ is a structure on the 'complex of spacelike solutions' $\mathfrak{F}_{\mathrm{hsc}}(M)$ which generalizes the fixed-time Poisson structure σ_Σ from (1.47). As the latter, also our generalization relies on the choice of a spacelike Cauchy surface $\Sigma \overset{\iota}{\hookrightarrow} M$. Given such a choice, we define a second type of evaluation pairing ev_Σ given by the composition

$$
\mathrm{ev}_\Sigma : \mathfrak{F}_{\mathrm{sc}}(M) \otimes \mathfrak{F}(M) \xrightarrow{(-,-)} \Omega^{\bullet}_{\mathrm{sc}}(M)[m-1] \xrightarrow{\iota^*} \Omega^{\bullet}_{\mathrm{c}}(\Sigma)[m-1] \xrightarrow{\int_\Sigma} \mathbb{R} \tag{3.66}
$$

in $\mathbf{Ch}_{\mathbb{R}}$. Note that in the construction above we exploited that $(-,-)$ preserves supports since it is a bi-differential operator and that the intersection of a spacelike compact subset of M with any spacelike Cauchy surface Σ is a compact subset of Σ (as we recalled in Section 1.1). Recalling that $\dim(\Sigma) = \dim(M) - 1 = m - 1$, Stokes' theorem guarantees that integrating over Σ defines a cochain map $\int_\Sigma : \Omega^\bullet_{\mathrm{c}}(\Sigma)[m-1] \to \mathbb{R}$ in $\mathbf{Ch}_{\mathbb{R}}$. The *fixed-time Poisson structure* σ_Σ on $\mathfrak{F}_{\mathrm{hsc}}(M)$ is constructed in the next proposition, which is an immediate consequence of the graded anti-symmetry of the differential pairing $(-,-)$.

Proposition 3.3.6. *Let (F, Q) be a complex of differential operators on $M \in \mathbf{Loc}_m$ endowed with a differential pairing $(-,-)$. Given a spacelike Cauchy surface $\Sigma \subseteq M$, the composition*

$$\sigma_\Sigma : \mathfrak{F}_{\mathrm{hsc}}(M)^{\otimes 2} \longrightarrow \mathfrak{F}_{\mathrm{sc}}(M) \otimes \mathfrak{F}(M) \xrightarrow{(-1)^{m-1}\,\mathrm{ev}_\Sigma} \mathbb{R} \tag{3.67}$$

in $\mathbf{Ch}_{\mathbb{R}}$ is graded anti-symmetric, hence it descends to a Poisson structure on $\mathfrak{F}_{\mathrm{hsc}}(M)$. The first arrow in the diagram above consists in its first factor of the $\mathfrak{F}_{J_M(-)}$-component of the natural transformation hocolim $\to$ colim *from* (2.3.10) *and in its second factor of the adjunct of the inclusion $\mathfrak{F}_{J_M(-)}(M) \to \Delta\mathfrak{F}(M)$ with respect to the dg-adjunction* hocolim $\dashv \Delta$ *from Proposition 2.3.12.*

Note that this second type of Poisson structures is defined regardless of whether the complex (F, Q) is Green hyperbolic, since it relies entirely on the differential pairing $(-,-)$.

Classically, the Poisson vector spaces $(\mathrm{coker}_{\mathrm{c}}(P), \tau_M)$ of linear observables and covariant Poisson structure and $(\ker_{\mathrm{sc}}(P), \sigma_\Sigma)$ of spacelike solutions and fixed-time Poisson structure are isomorphic as objects in the category $\mathbf{PoVec}_{\mathbb{R}}$ of Poisson vector spaces since the retarded-minus-advanced propagator is compatible with the Poisson structures, see (1.49). In our context, we would be interested in the category $\mathbf{PoCh}_{\mathbb{R}}$ of *Poisson complexes*, whose objects are pairs (V, τ) consisting of a cochain complex $V \in \mathbf{Ch}_{\mathbb{R}}$ and of an unshifted Poisson structure $\tau : V^{\wedge 2} \to \mathbb{R}$ in $\mathbf{Ch}_{\mathbb{R}}$. A morphism $f : (V, \tau) \to (V', \tau')$ in $\mathbf{PoCh}_{\mathbb{R}}$ between Poisson complexes is a cochain map $f : V \to V'$ in $\mathbf{Ch}_{\mathbb{R}}$ between the underlying cochain complexes, which is compatible with the Poisson structures, i.e. $\tau' \circ f^{\wedge 2} = \tau$. The analog of the usual Poisson vector spaces above are the Poisson complexes $(\mathfrak{F}_{\mathrm{hc}}(M)[1], \tau_M)$ (or $(\mathfrak{F}_{\mathrm{hc}}(M)[1], \tau_M^{\pm})$) and $(\mathfrak{F}_{\mathrm{hsc}}(M), \sigma_\Sigma)$. While the retarded-minus-advanced cochain map $\Lambda : \mathfrak{F}_{\mathrm{hc}}(M)[1] \to \mathfrak{F}_{\mathrm{hsc}}(M)$ in $\mathbf{Ch}_{\mathbb{R}}$, for any choice of retarded and advanced Green's homotopies, is a quasi-isomorphism, as proved by Theorem 3.2.4, it does *not* upgrade to a morphism of Poisson complexes. However, as we are going to prove in Theorem 3.3.7, the failure of Λ to be a morphism of Poisson complexes is witnessed by prescribed homotopies.

Hence, it may be regarded as a morphism in a suitably constructed simplicial category of Poisson complexes, see [GH18, Sec. 3.1]

The following theorem shows that Λ is compatible with the Poisson structures τ_M from (3.55) and σ_Σ from (3.67) up to an explicit homotopy. By combining this result with Proposition 3.3.3, one gets also homotopies relating $\sigma_\Sigma \circ \Lambda^{\wedge 2}$ with $\tau_M^\pm$.

Theorem 3.3.7. *Let (F, Q) be a Green hyperbolic complex on $M \in \mathbf{Loc}_m$ endowed with a differential pairing $(-,-)$. Consider a choice of retarded and advanced Green's homotopies $\Lambda^\pm$ of it and a spacelike Cauchy surface $\Sigma \subseteq M$. Then, the retarded-minus-advanced quasi-isomorphism $\Lambda : \mathfrak{F}_{\mathrm{hc}}(M)[1] \to \mathfrak{F}_{\mathrm{hsc}}(M)$ from Theorem 3.2.4 is compatible with the Poisson structures τ_M from Proposition 3.3.3 and σ_Σ from Proposition 3.3.6 up to a homotopy $\lambda \in [\mathfrak{F}_{\mathrm{hc}}(M)[1]^{\wedge 2}, \mathbb{R}]^{-1}$, i.e. $\sigma_\Sigma \circ \Lambda^{\wedge 2} = \tau_M + \partial \lambda$.*

Proof. Let us construct the candidate homotopy

$$\lambda := \mathrm{asym}(\widetilde{\lambda}) \in [\mathfrak{F}_{\mathrm{hc}}(M)[1]^{\wedge 2}, \mathbb{R}]^{-1} \tag{3.68a}$$

as the graded anti-symmetrization of the (-1)-cochain

$$\widetilde{\lambda} \in [\mathfrak{F}_{\mathrm{hc}}(M)[1]^{\otimes 2}, \mathbb{R}]^{-1} \tag{3.68b}$$

defined for all homogeneous $\psi_1, \psi_2 \in \mathfrak{F}_{\mathrm{hc}}(M)[1]$ by

$$\widetilde{\lambda}(\psi_1 \otimes \psi_2) := \int_{\Sigma^+} (\Lambda^- \psi_1, \Lambda \psi_2) + \int_{\Sigma^-} (\Lambda^+ \psi_1, \Lambda \psi_2)\,, \tag{3.68c}$$

where $\Sigma^\pm := J_M^\pm(\Sigma)$, $\Lambda^\pm := \mathrm{hocolim}(\Lambda^\pm \circ j_{(-)}^{J_M^\pm(-)}) \in [\mathfrak{F}_{\mathrm{hc}}(M)[1], \mathfrak{F}_{\mathrm{h}\,{}^{\mathrm{spc}}_{\mathrm{sfc}}}(M)]^0$ denotes the homotopy colimit of the chosen retarded/advanced Green's homotopy, regarded as a 0-cochain in the internal hom, and the relevant components of the natural transformation hocolim $\to$ colim on the right-hand side are suppressed from our notation. Note that the integrals above are finite since $(-,-)$ preserves supports and the intersection $S^\pm \cap J_M^\mp(\Sigma) \subseteq M$ is compact for all strictly past/future compact subset $S^\pm \in$ spc/sfc. Moreover, the integration over $\Sigma^\pm$ defines a (-1)-cochain $\int_{\Sigma^\pm} \in [\Omega^\bullet_{\substack{\mathrm{sfc}\\ \mathrm{spc}}}(M)[m-1], \mathbb{R}]^{-1}$ in the internal hom. By direct inspection, one finds that, for all homogeneous $\psi_1, \psi_2 \in \mathfrak{F}_{\mathrm{hc}}(M)[1]$,

$$
\begin{aligned}
(\partial\widetilde{\lambda})(\psi_1 \otimes \psi_2) &= \int_{\Sigma^+} (\Lambda^- \mathrm{d}_{[1]}\psi_1, \Lambda\psi_2) + (-1)^{|\psi_1|} \int_{\Sigma^+} (\Lambda^- \psi_1, \Lambda \mathrm{d}_{[1]}\psi_2) \\
&\quad + \int_{\Sigma^-} (\Lambda^+ \mathrm{d}_{[1]}\psi_1, \Lambda\psi_2) + (-1)^{|\psi_1|} \int_{\Sigma^-} (\Lambda^+ \psi_1, \Lambda \mathrm{d}_{[1]}\psi_2) \\
&= \int_{\Sigma^+} (\mathrm{d}\Lambda^- \psi_1, \Lambda\psi_2) - \int_{\Sigma^+} (\psi_1, \Lambda\psi_2) \\
&\quad + (-1)^{|\psi_1|} \int_{\Sigma^+} (\Lambda^- \psi_1, \mathrm{d}\Lambda\psi_2) + \int_{\Sigma^-} (\mathrm{d}\Lambda^+ \psi_1, \Lambda\psi_2) \\
&\quad - \int_{\Sigma^-} (\psi_1, \Lambda\psi_2) + (-1)^{|\psi_1|} \int_{\Sigma^-} (\Lambda^+ \psi_1, \mathrm{d}\Lambda\psi_2)
\end{aligned}
$$

where in the first step we used that $\partial\widetilde{\lambda} = \widetilde{\lambda} \circ \mathrm{d}_{[1]\otimes}$ and in the second one that $\mathrm{d} \circ \Lambda^\pm - \Lambda^\pm \circ \mathrm{d}_{[1]} = \partial\Lambda^\pm = \mathrm{id}$ and $\mathrm{d} \circ \Lambda = \Lambda \circ \mathrm{d}_{[1]}$. Exploiting also that the differential pairing $(-,-)$ is compatible with differentials and that $\Sigma^+ \cup \Sigma^- = M$, one continues the chain of equalities with

$$
\begin{aligned}
&= \int_{\Sigma^+} \mathrm{d}_{\mathrm{dR}[m-1]} (\Lambda^- \psi_1, \Lambda\psi_2) + \int_{\Sigma^-} \mathrm{d}_{\mathrm{dR}[m-1]} (\Lambda^+ \psi_1, \Lambda\psi_2) - \int_M (\psi_1, \Lambda\psi_2) \\
&= (-1)^{m-1} \int_\Sigma \iota^* (\Lambda\psi_1, \Lambda\psi_2) - \int_M (\psi_1, \Lambda\psi_2) \\
&= \sigma_\Sigma(\Lambda\psi_1, \Lambda\psi_2) - \widetilde{\tau}_M(\psi_1, \psi_2)\,,
\end{aligned} \tag{3.69}
$$

where in the first step we used Stokes' theorem and in the last one just the definitions (3.67) of σ_Σ and (3.54) of $\widetilde{\tau}_M$. Taking the graded anti-symmetrization of both sides, and recalling also the definition (3.55) of τ_M, one finally finds the identity

$$
\partial\lambda = \mathrm{asym}(\partial\widetilde{\lambda}) = \sigma_\Sigma \circ \Lambda^{\wedge 2} - \tau_M\,, \tag{3.70}
$$

which concludes our proof. □

In the classical scenario, the choice of the Cauchy surface Σ in the definition of the fixed-time Poisson structure σ_Σ does not really matter since different choices give rise to the same Poisson structure. A similar result holds true also for our fixed-time Poisson structure. Now different choices of the Cauchy surface Σ do *not* give rise to the same Poisson structure but to Poisson structures that coincide up to homotopy.

Corollary 3.3.8. *Let (F, Q) be a Green hyperbolic complex on $M \in \mathbf{Loc}_m$ endowed with a differential pairing $(-,-)$. Moreover, let $\Sigma, \Sigma' \subseteq M$ be two spacelike Cauchy surfaces of M. Denote by σ_Σ and $\sigma_{\Sigma'}$ the Poisson structures on $\mathfrak{F}_{\mathrm{hsc}}(M)$ from Proposition 3.3.6 obtained by choosing Σ and, respectively, Σ'. Then, there exists a homotopy $\lambda_{\Sigma\Sigma'} \in [\mathfrak{F}_{\mathrm{hsc}}(M)^{\wedge 2}, \mathbb{R}]^{-1}$ such that $\sigma_{\Sigma'} = \sigma_\Sigma + \partial\lambda_{\Sigma\Sigma'}$.*

Proof. Given a choice of retarded and advanced Green's homotopies for the complex (F,Q), Theorem 3.3.7 provides us with homotopies $\lambda,\lambda' \in [\mathfrak{F}_{\mathrm{hc}}(M)[1]^{\wedge 2},\mathbb{R}]^{-1}$ such that

$$\sigma_\Sigma \circ \Lambda^{\wedge 2} - \partial\lambda = \tau_M = \sigma_{\Sigma'} \circ \Lambda^{\wedge 2} - \partial\lambda' \,, \tag{3.71}$$

where $\Lambda : \mathfrak{F}_{\mathrm{hc}}(M)[1] \to \mathfrak{F}_{\mathrm{hsc}}(M)$ in $\mathbf{Ch}_\mathbb{R}$ is the retarded-minus-advanced quasi-isomorphism associated to that choice of Green's homotopies. Since Λ is a quasi-isomorphism, it admits a quasi-inverse $\Theta : \mathfrak{F}_{\mathrm{hsc}}(M) \to \mathfrak{F}_{\mathrm{hc}}(M)[1]$ in $\mathbf{Ch}_\mathbb{R}$, such that, in particular, $\Lambda \circ \Theta = \mathrm{id} - \partial\Upsilon$, for a homotopy $\Upsilon \in [\mathfrak{F}_{\mathrm{hsc}}(M),\mathfrak{F}_{\mathrm{hsc}}(M)]^{-1}$. (In the proof of Theorem 3.2.4 we constructed an explicit choice of such Θ and Υ.) By exploiting this, we compute

$$\begin{aligned}
\sigma_{\Sigma'} - \sigma_\Sigma &= (\sigma_{\Sigma'} - \sigma_\Sigma) \circ (\Lambda \circ \Theta + \partial\Upsilon)^{\wedge 2} \\
&= (\partial(\lambda' - \lambda)) \circ \Theta^{\wedge 2} + (\sigma_{\Sigma'} - \sigma_\Sigma) \circ (\partial\Upsilon \wedge \partial\Upsilon + 2\, \Lambda \circ \Theta \wedge \partial\Upsilon) \\
&= \partial\big((\lambda' - \lambda) \circ \Theta^{\wedge 2} + (\sigma_{\Sigma'} - \sigma_\Sigma) \circ ((\mathrm{id} + \Lambda \circ \Theta) \wedge \Upsilon)\big) =: \partial\lambda_{\Sigma\Sigma'} \,,
\end{aligned} \tag{3.72}$$

where in the second step we used Equation (3.71) and in the third one that all the degree 0 maps appearing in the equation are cochain maps. $\square$

Chapter 4

Green's witnesses

In the previous chapter we introduced the notion of Green hyperbolic complexes which generalizes to complexes of differential operators the standard notion of Green hyperbolic operators. In analogy with the ordinary situation, we defined them as those complexes of differential operators that admit a retarded and an advanced Green's homotopy. For a general complex of differential operators on a globally hyperbolic Lorentzian manifold $M \in \mathbf{Loc}_m$, it may not be easy to prove whether it is Green hyperbolic since it would be equivalent to prove the contractibility of a family of cochain complexes labeled by the compact subsets $K \subseteq M$ (Proposition 3.2.1). Furthermore, even assuming we have proven that a complex is Green hyperbolic, there is no reason, in general, to expect to find a retarded/advanced Green's homotopy (namely, a point in the contractible space $\underline{\mathcal{GH}}^{\pm}$ from (3.23)) $\Lambda^{\pm} \in \underline{\mathrm{hom}}(\mathfrak{F}_{J^{\pm}_M(-)}(M), \mathfrak{F}_{J^{\pm}_M(-)}(M))^{-1} \subseteq \underline{\mathrm{map}}(\mathfrak{F}_{J^{\pm}_M(-)}(M), \mathfrak{F}_{J^{\pm}_M(-)}(M))^{-1}$ which is strictly natural with respect to inclusions of compact subsets. Although conceptually this is not an issue, and indeed we needed to define retarded and advanced Green's homotopies as (-1)-cochains in the mapping complex of homotopy coherent natural transformations in order to get the contractibility of their space (Proposition 3.2.2), for some applications, in particular to quantum field theory, see Chapter 5, it would be preferable to work with strictly natural transformations. Indeed, in this case the tower of compatibilities encoded by the projections $\mathrm{pr}_q \Lambda^{\pm}$, for $q \geq 1$, collapses and constructions become more manageable.

To solve both these issues, in Section 4.1 we shall introduce the notion of a *Green's witness*, that is a family of degree decreasing linear differential operators which fulfill suitable conditions. The role of a Green's witness is, as the name suggests, to witness the Green hyperbolicity of a complex of differential operators, by providing us with a special choice of retarded and advanced Green's homotopies of it. The Green's homotopies built out of the Green's witness are strictly natural, i.e. they are (-1)-cochains in the enriched hom $\underline{\mathrm{hom}}(\mathfrak{F}_{J^{\pm}_M(-)}(M), \mathfrak{F}_{J^{\pm}_M(-)}(M))$, hence they allow us to simplify some constructions and results from Chapter 3. Indeed, the retarded-minus-advanced quasi-isomorphism associated with them 'descends to ordinary colimits', making superfluous to consider homotopy colimits for the complexes of linear observables and of spacelike solutions. This, in particular, allows us to simplify the statement and the proof of Theorem 3.2.4. Moreover, when the data of complexes of differential operators are assigned in a natural fashion with respect to morphisms in $\mathbf{Loc}_m$, in the sense of Section 1.2 and Remark 3.2.3, it is possible to refine the notion of Green's witnesses to that of natural Green's witnesses. In this case, the retarded/advanced Green's homotopy and the retarded-minus-advanced quasi-isomorphism become natural with respect to morphisms in $\mathbf{Loc}_m$ in the sense of dinatural transformations.

In Section 4.2 we further specify the notion of a Green's witness to a *formally self-adjoint* Green's witness with respect to a given differential pairing. The additional conditions a formally self-adjoint Green's witness is asked to satisfy are partially reminiscent of the condition of formal self-adjointness in the ordinary sense. Formally self-adjoint Green's witnesses are such that the retarded and advanced Green's homotopies they select allow for simpler constructions and comparison of the Poisson structures τ_M and σ_Σ from Proposition 3.3.3 and 3.3.6.

Finally, in Section 4.3, we shall introduce the notion of *(covariant) free BV theories* as the datum of a (covariant) complex of (natural) differential operators endowed with a (natural) differential pairing and a (natural) formally self-adjoint Green's witness. We would like to mention that all our examples are such, as we will see in Chapter 6. For a natural free BV theory, we shall show that there is a functor $(\mathfrak{F}_{\mathrm{c}}[1], \tau) : \mathbf{Loc}_m \to \mathbf{PoCh}_{\mathbb{R}}$, taking values into the category of Poisson complexes, assigning the complex of linear observables endowed with the covariant Poisson structure from Proposition 3.3.3. Moreover, we shall construct a (-1)-shifted Poisson structure $\tau^{(-1)}$ on the linear observables $\mathfrak{F}_{\mathrm{c}}[1] \in \mathbf{Ch}_{\mathbb{R}}^{\mathbf{Loc}_m}$. We shall also prove that the functor $(\mathfrak{F}_{\mathrm{c}}[1], \tau) \in \mathbf{PoCh}_{\mathbb{R}}^{\mathbf{Loc}_m}$ fulfills classical analogs of the Einstein causality and (homotopy) time-slice axioms. These unshifted and shifted Poisson structures will prove fundamental in Chapter 5 to construct, respectively, a strict AQFT and a strict tPFA quantizing the observables of the natural free BV theory.

The content of this chapter is based on [BMS23] and [BMS24].

4.1 Definition and first results

We start with the definition of a Green's witness for a complex (F, Q) of differential operators.

Definition 4.1.1. Given a complex of differential operators (F, Q) on a globally hyperbolic Lorentzian manifold $M \in \mathbf{Loc}_m$, a *Green's witness* $W =$

$(W^n)_{n\in\mathbb{Z}}$ for it consists of a collection of degree-decreasing linear differential operators $W^n : \mathfrak{F}^n(M) \to \mathfrak{F}^{n-1}(M)$, labeled by integers $n \in \mathbb{Z}$, such that the linear differential operator

$$P^n := Q^{n-1}W^n + W^{n+1}Q^n : \mathfrak{F}^n(M) \longrightarrow \mathfrak{F}^n(M) \tag{4.1}$$

is Green hyperbolic (in the ordinary sense, see Section 1.3), for all $n \in \mathbb{Z}$.

Remark 4.1.2. Given a complex of differential operators (F, Q) on a globally hyperbolic Lorentzian manifold $M \in \mathbf{Loc}_m$, we recall that $\mathfrak{F}(M) \in \mathbf{Ch}_{\mathbb{K}}$ denotes the complex of its smooth sections, whose degree $n \in \mathbb{Z}$ is the vector space $\mathfrak{F}^n(M)$ from Equation (3.1) and whose differential is Q. Keeping this in mind, a Green's witness W for (F, Q) is in particular a (-1)-cochain $W \in [\mathfrak{F}(M), \mathfrak{F}(M)]^{-1}$ in the internal hom, whose differential $\partial W =: P$ is a 0-coboundary whose n-th component P^n is an ordinary Green hyperbolic linear differential operator for any $n \in \mathbb{Z}$. Since Q, W and P are all made up of linear differential operators, which in particular preserve supports, they admit restrictions to the complex $\mathfrak{F}_{\mathscr{D}}(M) \in \mathbf{Ch}_{\mathbb{K}}$ of sections with support prescribed by any directed subset $\mathscr{D} \subseteq \mathrm{cl}$ of the directed set of closed subsets of M. ▽

Remark 4.1.3. Our concept of Green's witness has its roots in the theory of elliptic complexes. A more recent and closely related concept, which inspired us, is that of the gauge fixing operator appearing in the works of Costello and Gwilliam [CG17; CG21b]. However, we would like to point out the following differences. First, the analogs of the differential operators P^n in [CG17; CG21b] are assumed to be elliptic instead of Green hyperbolic. This is obviously related to the fact that Costello and Gwilliam consider Riemannian instead of Lorentzian manifolds M. In a similar setting, see [SS24] for a generalization of gauge fixing operators where even the ellipticity hypothesis is dropped. However, in this case regularization techniques are then required to ensure that the heat kernels involved are microlocally well-behaved. Second, [CG17; CG21b] include also a self-adjointness condition and a square-zero condition for their gauge fixing operators. While our general Definition 4.1.1 of a Green's witness does not require such conditions, in Section 4.2 the stronger Definition 4.2.1 of formally self-adjoint Green's witness shall introduce a relaxation of the square-zero condition, see item 1, and a certain self-adjointness condition, see item 2. These additional requirements are all met by our examples from Chapter 6. Let us mention that the square-zero condition, $W^2 = 0$, for a Green's witness is not only unnecessary for our purposes, since the weaker condition 1 of Definition 4.2.1 is enough to prove all our results, but would also rule out important examples that we would like to study, such as the Maxwell p-forms from Section 6.3. ▽

Example 4.1.4. We refer the reader to Chapter 6 for some explicit examples of Green's witnesses for the complexes of differential operators associated with ordinary free field theories, Abelian Chern-Simons theory and Maxwell p-forms, which includes linear Yang-Mills theory for $p = 1$. Let us just comment here on the simple case of the complex $(F_{(E,P)}, Q_{(E,P)})$ associated with a Green hyperbolic linear differential operator $P : \Gamma(E) \to \Gamma(E)$ acting on the sections of a vector bundle $E \to M$, see Examples 3.1.2 and 3.1.7. A Green's witness for it has a unique non-vanishing component, $W^1 : \mathfrak{F}^1(M) \to \mathfrak{F}^0(M)$, such that $P \circ W^1$ and $W^1 \circ P$ are both Green hyperbolic operators. It is clear that, since P is Green hyperbolic, both conditions are met by choosing $W^1 := \mathrm{id} : \Gamma(E) \to \Gamma(E)$. △

As previously mentioned, the main role of a Green's witness W of a complex of differential operators is to witness that the latter is Green hyperbolic. This is proved by the following theorem.

Theorem 4.1.5. *Let (F, Q) be a complex of differential operators and W be a Green's witness of it. Then, (F, Q) is a Green hyperbolic complex. Furthermore, denote by $(G^\pm)^n : \mathfrak{F}^n_{\substack{\mathrm{spc}\\ \mathrm{sfc}}}(M) \to \mathfrak{F}^n_{\substack{\mathrm{spc}\\ \mathrm{sfc}}}(M)$ the extended Green's operators of the Green hyperbolic operators P^n from Equation (4.1). Then, the (-1)-cochain*

$$\Lambda^\pm \in \underline{\mathrm{hom}}(\mathfrak{F}_{J^\pm_M(-)}(M), \mathfrak{F}_{J^\pm_M(-)}(M))^{-1} \subseteq \underline{\mathrm{map}}(\mathfrak{F}_{J^\pm_M(-)}(M), \mathfrak{F}_{J^\pm_M(-)}(M))^{-1} \tag{4.2a}$$

defined, for all integers $n \in \mathbb{Z}$ and compact subsets $K \subseteq M$, by

$$(\mathrm{pr}_K \Lambda^\pm)^n := W^n (G^\pm)^n : \mathfrak{F}^n_{J^\pm_M(K)}(M) \longrightarrow \mathfrak{F}^{n-1}_{J^\pm_M(K)}(M) \tag{4.2b}$$

is a retarded/advanced Green's homotopy of (F, Q).

Proof. We recall from Proposition 3.2.1 that a necessary and sufficient condition for the Green hyperbolicity of the complex (F, Q) is that the cochain complex $\mathfrak{F}_{J^\pm_M(K)}(M) \in \mathbf{Ch}_\mathbb{K}$ of sections supported in the causal future/past of K is acyclic, for all compact subsets $K \subseteq M$. Then, let us provide you with a contracting homotopy

$$\Lambda^\pm_K \in [\mathfrak{F}_{J^\pm_M(K)}(M), \mathfrak{F}_{J^\pm_M(K)}(M)]^{-1}\,, \tag{4.3a}$$

for any compact subset $K \in \mathrm{c}$, defined degree-wise for all $n \in \mathbb{Z}$ by

$$(\Lambda^\pm_K)^n := W^n (G^\pm)^n\,. \tag{4.3b}$$

By direct inspection, one sees that $\partial \Lambda^\pm_K = \mathrm{id}$, hence $\Lambda^\pm_K$ is a contracting homotopy witnessing that $\mathfrak{F}_{J^\pm_M(K)}(M) \in \mathbf{Ch}_\mathbb{K}$ is an acyclic complex, for all compact subset $K \subseteq M$. More explicitly, one computes

$$(\partial \Lambda^\pm_K)^n = Q^{n-1} W^n (G^\pm)^n + W^{n+1} (G^\pm)^{n+1} Q^n = P^n (G^\pm)^n = \mathrm{id}\,, \tag{4.4}$$

for all $n \in \mathbb{Z}$, where in the first step we used the definition of the 'adjoint' differential from (2.5b), in the second step the identity

$$(G^{\pm})^{n+1} Q^n = Q^n (G^{\pm})^n \tag{4.5}$$

which follows from $P^{n+1} Q^n = Q^n P^n$, since $P \in \mathsf{B}^0[\mathfrak{F}(M), \mathfrak{F}(M)]$ is a 0-coboundary, see Remark 4.1.2. The last step follows from the properties of ordinary retarded/advanced Green's operators, see Theorem 1.3.4. This proves the first part of the theorem. Concerning the second part of the statement, there remains to show that the (-1)-cochains $\Lambda^{\pm}_K \in [\mathfrak{F}_{J^{\pm}_M(K)}(M), \mathfrak{F}_{J^{\pm}_M(K)}(M)]^{-1}$, for all compact subsets $K \subseteq M$, assemble into a (-1)-cochain $\Lambda^{\pm} \in \underline{\mathrm{hom}}(\mathfrak{F}_{J^{\pm}_M(-)}(M), \mathfrak{F}_{J^{\pm}_M(-)}(M))^{-1}$ in the enriched hom, according to $\mathrm{pr}_K \Lambda^{\pm} := \Lambda^{\pm}_K$. In other words, we have to show that they are natural $\mathfrak{F}_{J^{\pm}_M(K \subseteq K')}(M) \circ \Lambda^{\pm}_K = \Lambda^{\pm}_{K'} \circ \mathfrak{F}_{J^{\pm}_M(K \subseteq K')}(M)$ with respect to the inclusion $K \subseteq K'$ of compact sets. Since $\Lambda^{\pm}_K$ is degree-wise given by the composition of a retarded/advanced Green's operator with a linear differential operator, this is an immediate consequence of the support properties of Green's operators, see Section 1.3. □

Remark 4.1.6. Note that a Green's witness W not only offers a convenient way to check the Green hyperbolicity of certain complexes of differential operators, but it also points out particularly simple retarded and advanced Green's homotopies for the said complex. In fact, the retarded and advanced Green's homotopies (4.2) built out of W are strictly natural as they lie in the enriched hom $\underline{\mathrm{hom}}$. This should be compared to the case of generic retarded and advanced Green's homotopies which instead are (-1)-cochains in the mapping complex $\underline{\mathrm{map}}$, hence they are required to be natural only in a homotopy coherent fashion. While the latter consists of a more involved tower of consistencies controlling naturality up to homotopies, the retarded/advanced Green's homotopies $\Lambda^{\pm}$ picked by W have vanishing components $\mathrm{pr}_q \Lambda^{\pm}$ for all $q \geq 1$. From a practical point of view, not having to deal with these additional data brings a lot of simplifications and, in particular, it will allow us to construct concrete examples of strict AQFTs and tPFAs from Green hyperbolic covariant complexes endowed with natural Green's witnesses, see Chapter 5. ▽

Remark 4.1.7. From the proof of Theorem 4.1.5, one may realize that the retarded and advanced Green's homotopies (4.3) are not the only choice that one can cook up using the Green's witness W. For example, one may consider the (-1)-cochain

$$\widetilde{\Lambda}^{\pm} \in \underline{\mathrm{hom}}(\mathfrak{F}_{J^{\pm}_M(-)}(M), \mathfrak{F}_{J^{\pm}_M(-)}(M))^{-1} \subseteq \underline{\mathrm{map}}(\mathfrak{F}_{J^{\pm}_M(-)}(M), \mathfrak{F}_{J^{\pm}_M(-)}(M))^{-1}\,, \tag{4.6a}$$

in the enriched hom subcomplex, which is defined, for all compact $K \subseteq M$ and $n \in \mathbb{Z}$, by

$$(\mathrm{pr}_K\, \widetilde{\Lambda}^{\pm})^n := (G^{\pm})^{n-1} W^n : \mathfrak{F}^n_{J^{\pm}_M(K)}(M) \longrightarrow \mathfrak{F}^{n-1}_{J^{\pm}_M(K)}(M)\,, \tag{4.6b}$$

where the order of the composition is reversed with respect to that appearing in Theorem 4.1.5. The same argument as in the proof of the theorem shows that (4.6b) are the components of a strictly natural transformation, hence they assemble into a (-1)-cochain $\widetilde{\Lambda}^{\pm}$ in the enriched hom. Furthermore, a straightforward computation allows us to prove that they are actually retarded and advanced Green's homotopies. Indeed, one has

$$(\partial \widetilde{\Lambda}^{\pm})^n = Q^{n-1}(G^{\pm})^{n-1}W^n + (G^{\pm})^n W^{n+1} Q^n = (G^{\pm})^n P^n = \mathrm{id}\,, \tag{4.7}$$

where in the first step we used the definition (2.5b) of ∂ and (4.6) of $\widetilde{\Lambda}^{\pm}$, in the second one the identity (4.5) and in the last one the properties of retarded/advanced Green's operators, see Theorem 1.3.4. Notice that, without any additional requirement on the Green's witness W, the retarded/advanced Green's homotopies $\widetilde{\Lambda}^{\pm}$ from (4.6) and $\Lambda^{\pm}$ from Theorem 4.1.5 do *not* coincide. This shows, once more, in the concrete example of a complex of differential operators (F, Q) endowed with a Green's witness W, that a strict uniqueness result for retarded/advanced Green's homotopies cannot be achieved. On the contrary, we known from Proposition 3.2.2 that retarded/advanced Green's homotopies (when they exist) are unique up to a contractible space of choices. Informally, this means that they are unique up to higher homotopies, which are themselves unique up to even higher homotopies, and so on. Let us see it concretely in this setting, by exhibiting a higher homotopy

$$\lambda^{\pm} \in \underline{\mathrm{hom}}(\mathfrak{F}_{J^{\pm}_M(-)}(M), \mathfrak{F}_{J^{\pm}_M(-)}(M))^{-2} \subseteq \underline{\mathrm{map}}(\mathfrak{F}_{J^{\pm}_M(-)}(M), \mathfrak{F}_{J^{\pm}_M(-)}(M))^{-2} \tag{4.8a}$$

comparing $\Lambda^{\pm}$ and $\widetilde{\Lambda}^{\pm}$. It is given, for all compact subsets $K \subseteq M$ and integers $n \in \mathbb{Z}$, by

$$(\mathrm{pr}_K\, \lambda^{\pm})^n := W^{n-1}(G^{\pm})^{n-1}(G^{\pm})^{n-1}W^n\,. \tag{4.8b}$$

Strict naturality of $\lambda^{\pm}$ follows as usual from the support properties of retarded/advanced Green's operators $G^{\pm}$, see Theorem 1.3.4. In order to show that it compares the two different retarded/advanced Green's homotopies introduced above, we calculate, for all $n \in \mathbb{Z}$,

$$\begin{aligned}
(\partial\lambda^{\pm})^n &= Q^{n-2}W^{n-1}(G^{\pm})^{n-1}(G^{\pm})^{n-1}W^n - W^n(G^{\pm})^n(G^{\pm})^n W^{n+1}Q^n \\
&= (P^{n-1} - W^n Q^{n-1})(G^{\pm})^{n-1}(G^{\pm})^{n-1}W^n \\
&\quad - W^n(G^{\pm})^n(G^{\pm})^n(P^n - Q^{n-1}W^n) \\
&= -W^n(Q^{n-1}(G^{\pm})^{n-1}(G^{\pm})^{n-1} - (G^{\pm})^n(G^{\pm})^n Q^{n-1})W^n \\
&\qquad + (G^{\pm})^{n-1}W^n - W^n(G^{\pm})^n \\
&= (\widetilde{\Lambda}^{\pm})^n - (\Lambda^{\pm})^n\,, \qquad (4.9)
\end{aligned}$$

where the first step used the definition (2.5b) of the 'adjoint' differential, in the second step we used (4.1), in the third step the properties of the retarded/advanced Green's operators $G^{\pm}$. Finally, the last step follows from applying (4.5) twice. It is definitely possible to come up with other higher homotopies $\widetilde{\lambda}^{\pm} \in \underline{\mathrm{map}}(\mathfrak{F}_{J^{\pm}_M(-)}(M), \mathfrak{F}_{J^{\pm}_M(-)}(M))^{-2}$ relating $\Lambda^{\pm}$ and $\widetilde{\Lambda}^{\pm}$. Our Proposition 3.2.2, then, assures that any two choices agree up to an even higher homotopy, and so on. $\triangledown$

In Chapter 6, we will exploit Theorem 4.1.5 to prove that the complexes of differential operators associated to the concrete examples of the ordinary free field theories, the Abelian Chern-Simons theory and the Maxwell p-forms (including the linear Yang-Mills theory for $p = 1$) are Green hyperbolic and, in particular, to write explicit retarded and advanced Green's homotopies for them. For Maxwell p-forms and linear Yang-Mills, in particular, the retarded/advanced Green's homotopies pointed out by the Green's witnesses considered in Chapter 6 agree with the contracting homotopies presented in [BBS20; AB23].

Remark 4.1.8. We would like to notice that our approach to the definition of retarded/advanced Green's homotopies $\Lambda^{\pm}$ is purely algebraic, in the sense that in Definition 3.1.4 we defined $\Lambda^{\pm}$ as collections of linear maps that satisfy some suitable algebraic conditions. In this, we proceeded in a very similar way to that adopted by [BGP07; BG12; Bär15] to deal with retarded/advanced Green's operators. Indeed, these references prove that retarded/advanced Green's operators are continuous (with respect to suitable topologies) as a by-product of their algebraic definition. (To be more precise their argument is based on the fact that the retarded/advanced Green's operator is the unique inverse of a suitable restriction of a Green hyperbolic linear differential operator and exploits the open mapping theorem for Fréchet spaces.) Continuity of ordinary retarded/advanced Green's operators yields that they admit a presentation in terms of suitable distributional kernels. This regularity result turned out to be very useful and had several applications in the AQFT literature, especially in perturbative theory. On the contrary, our retarded/advanced Green's homotopies are more abstract concepts, therefore we do not expect that their algebraic definition is enough to guarantee their continuity and presentability via distributional kernels.

A way out of this problem might be to endow the relevant complexes of sections of graded vector bundles with suitable additional structures, for instance topologies or bornologies [Hog77; Hog81]. This would force all morphisms to be continuous or, respectively, bounded by construction. However, it is well known that these are *not* Abelian categories, hence they have a bad interplay with homological algebra, while having a homologically well-behaved concept of retarded and advanced Green's homotopies is vital for our constructions and to prove crucial results, as the uniqueness from Proposition 3.2.2. Unfortunately, the homotopy theory of, for instance, quasi-Abelian categories is way more involved than that of the Abelian categories which we are considering in this thesis.

Note that the existence of a Green's witness W allows us to fully circumvent this problem. In fact, the retarded/advanced Green's homotopy $\Lambda^{\pm} = WG^{\pm}$ from (4.2) suggested by the Green's witness is degree-wise given by the composition of a linear differential operator with an ordinary retarded/advanced Green's operator. Since the latter is continuous, see [BGP07; BG12; Bär15], it follows that also $(\Lambda^{\pm})^n$ is a continuous map and hence it admits a useful presentation in terms of distributional kernels, for all $n \in \mathbb{Z}$. In other words, while the algebraic definition of retarded/advanced Green's homotopies $\Lambda^{\pm}$ ensures that they are unique up to a contractible space of choices, when a Green's witness W is given, which is often the case in concrete examples, see Chapter 6, it is guaranteed also that the contractible space of retarded/advanced Green's homotopies from (3.23) contains analitically well-behaved points, e.g. $\Lambda^{\pm} := WG^{\pm}$. ▽

The existence of a Green's witness W considerably simplifies the constructions we performed in Chapter 3. Indeed, in Remark 4.1.9 we will show how the construction of the retarded-minus-advanced cochain map from Definition 3.1.9 turns out to be simplified by a Green's witness W, while in Remark 4.1.10 we will show that the statement and proof of Theorem 3.2.4 can be simplified as well.

Remark 4.1.9. Given a complex of differential operators (F, Q) endowed with a Green's witness W, the construction of the retarded-minus-advanced cochain map from (3.18) is simplified by choosing the retarded and advanced Green's homotopies $\Lambda^{\pm}$ from Theorem 4.1.5. Indeed, the latter are strictly natural, in contrast with the general case, as noted in Remark 4.1.6. This implies that the construction of the retarded-minus-advanced cochain map Λ 'descends to ordinary colimits'. To be more precise, the special choice of retarded/advanced Green's homotopy

$$\Lambda^{\pm} = WG^{\pm} \in \underline{\mathrm{hom}}(\mathfrak{F}_{J^{\pm}_M(-)}(M), \mathfrak{F}_{J^{\pm}_M(-)}(M))^{-1} \tag{4.10}$$

lies in the enriched hom, therefore the (-1)-cocycle in the enriched hom

$$\Lambda = j^{J_M(-)}_{J_M^+(-)} \circ \Lambda^+ \circ j^{J_M^+(-)}_{(-)} - j^{J_M(-)}_{J_M^-(-)} \circ \Lambda^- \circ j^{J_M^-(-)}_{(-)} , \tag{4.11}$$

$\partial\Lambda = 0$, from Equation (3.17), defines a natural transformation

$$\Lambda : \mathfrak{F}_{(-)}(M)[1] \longrightarrow \mathfrak{F}_{J_M(-)}(M) \tag{4.12}$$

in $\mathbf{Ch}_{\mathbb{K}}^{\mathrm{c}}$. Then, by taking its ordinary colimit one gets the cochain map

$$\Lambda := \mathrm{colim}(\Lambda) : \mathfrak{F}_{\mathrm{c}}(M)[1] \longrightarrow \mathfrak{F}_{\mathrm{sc}}(M) , \tag{4.13}$$

in $\mathbf{Ch}_{\mathbb{K}}$, which we denote with abuse of notation by the same symbol. Recalling that the category c shaping the colimit is filtered, it follows that the natural quasi-isomorphism (2.3.10) comparing homotopy and ordinary colimits yields a commutative diagram

$$\begin{tikzcd} \mathfrak{F}_{\mathrm{hc}}(M)[1] \arrow[r, "\Lambda"] \arrow[d, "\sim"'] & \mathfrak{F}_{\mathrm{hsc}}(M) \arrow[d, "\sim"] \\ \mathfrak{F}_{\mathrm{c}}(M)[1] \arrow[r, "\Lambda"'] & \mathfrak{F}_{\mathrm{sc}}(M) \end{tikzcd} \tag{4.14}$$

in $\mathbf{Ch}_{\mathbb{K}}$. This makes precise the statement that the retarded-minus-advanced cochain map descends to ordinary colimits. See also Remark 3.2.6, where the complex of differential operators $(F_{(E,P)}, Q_{(E,P)})$ associated to a Green hyperbolic operator $P : \Gamma(E) \to \Gamma(E)$ is considered. The discussion above includes also that case, since $W = \mathrm{id}$ is a Green's witness for $(F_{(E,P)}, Q_{(E,P)})$, as noted in Remark 4.1.4. $\triangledown$

Remark 4.1.10. For a complex of differential operators (F, Q) endowed with a Green's witness W, the statement and proof of Theorem 3.2.4 can be simplified by passing to ordinary colimits as well. Indeed, exploiting the commutative diagram (4.14) and the 2-out-of-3 property of weak equivalences in a model category, one finds that the retarded-minus-advanced cochain map Λ is a quasi-isomorphism if and only if Λ is so. Therefore, the statement of Theorem 3.2.4 becomes equivalent to $\Lambda : \mathfrak{F}_{\mathrm{c}}(M)[1] \xrightarrow{\sim} \mathfrak{F}_{\mathrm{sc}}(M)$ in $\mathbf{Ch}_{\mathbb{K}}$ being a quasi-isomorphism. In this case it is easier, compared to Section 3.2, to find an explicit quasi-inverse

$$\Theta : \mathfrak{F}_{\mathrm{sc}}(M) \longrightarrow \mathfrak{F}_{\mathrm{c}}(M)[1] , \quad \psi \longmapsto \pm\big(Q(\chi_\pm\psi) - \chi_\pm Q\psi\big) \tag{4.15}$$

in $\mathbf{Ch}_{\mathbb{K}}$ of Λ, where recall that $\{\chi_+, \chi_-\}$ is a partition of unity subordinate to the open cover $\{I_M^+(\Sigma_-), I_M^-(\Sigma_+)\}$ of M for a choice of spacelike Cauchy surfaces $\Sigma_\pm$ for M such that $\Sigma_+ \subseteq I_M^+(\Sigma_-)$ is in the chronological future of Σ_-. It is also easier to write out explicit expressions for the homotopies

$$\Xi \in [\mathfrak{F}_{\mathrm{c}}(M)[1], \mathfrak{F}_{\mathrm{c}}(M)[1]]^{-1} , \quad \psi \longmapsto -\chi_-\Lambda^+\psi - \chi_+\Lambda^-\psi \tag{4.16}$$

witnessing that $\Theta \circ \Lambda \sim \mathrm{id}$, and

$$\Upsilon \in [\mathfrak{F}_{\mathrm{sc}}(M), \mathfrak{F}_{\mathrm{sc}}(M)]^{-1}\,, \quad \psi \longmapsto \Lambda^+ \chi_+ \psi + \Lambda^- \chi_- \psi \tag{4.17}$$

witnessing that $\Lambda \circ \Theta \sim \mathrm{id}$, where we abused the notation by denoting with the same symbol $\Lambda^\pm$ also their ordinary colimits. (The seeming sign discrepancy between Equation (4.16) and (3.32) is explained by the fact that the latter involves pulling the shifts out of the internal hom, which contributes a sign -1 in degree -1, cf. isomorphism (2.15).) $\triangledown$

Let us conclude this section by introducing the concept of a *natural Green's witness* for a covariant complex of natural differential operators (F, Q), see Remark 3.2.3. Recalling from Section 1.2 the notion of natural $\mathbb{K}$-vector bundles $\mathsf{F}^n : \mathbf{Loc}_m \to \mathbb{K}\text{-}\mathbf{VBnd}_m$, for $n \in \mathbb{Z}$, from Definition 1.2.2, the associated functors $\mathfrak{F}^n : \mathbf{Loc}_m^{\mathrm{op}} \to \mathbf{Vec}_\mathbb{K}$ and $\mathfrak{F}^n_{\mathrm{c}} : \mathbf{Loc}_m \to \mathbf{Vec}_\mathbb{K}$ from (1.17) of the smooth and compactly supported sections, respectively, and the notion of natural linear differential operators, Definition 1.2.3, the Definition 4.1.1 of a Green's witness admits a straightforward generalization to natural assigned data.

Definition 4.1.11. Given a covariant complex of natural differential operators (F, Q), a *natural Green's witness* $W = (W^n)_{n\in\mathbb{Z}}$ for it consists of a collection of degree-decreasing natural linear differential operators $W^n : \mathfrak{F}^n \to \mathfrak{F}^{n-1}$, for any integer $n \in \mathbb{Z}$, such that the natural linear differential operators

$$P^n := Q^{n-1} \circ W^n + W^{n+1} \circ Q^n : \mathfrak{F}^n \longrightarrow \mathfrak{F}^n \tag{4.18}$$

are object-wise Green hyperbolic operators $P^n_M : \mathfrak{F}^n(M) \to \mathfrak{F}^n(M)$, for all $M \in \mathbf{Loc}_m$ and $n \in \mathbb{Z}$.

Given a covariant complex of natural linear differential operators (F, Q) endowed with a natural Green's witness W, one has retarded/advanced Green's operators $(G^\pm_M)^n : \mathfrak{F}^n_{\mathrm{c}}(M) \to \mathfrak{F}^n(M)$ of P^n_M, for all $M \in \mathbf{Loc}_m$ and $n \in \mathbb{Z}$, which are the components of the dinatural transformations $(G^\pm)^n : \Delta_1 \mathfrak{F}^n_{\mathrm{c}} \to \Delta_2 \mathfrak{F}^n$, see Equation (1.40), for all $n \in \mathbb{Z}$. This means that for all morphisms $f : M_1 \to M_2$ in $\mathbf{Loc}_m$ we have $(G^\pm_{M_1})^n = f^*(G^\pm_{M_2})^n f_*$, cf. (1.38). Moreover, since from (4.18) it follows that $Q^n \circ P^n = P^{n+1} \circ Q^n$, one finds that $Q^n \circ (G^\pm)^n = (G^\pm)^{n+1} \circ Q^n$ for all $n \in \mathbb{Z}$. In other words, for any $M \in \mathbf{Loc}_m$ the retarded/advanced Green's operators from above are the components of a cochain map

$$G^\pm_M : \mathfrak{F}_{\mathrm{c}}(M) \longrightarrow \mathfrak{F}(M) \tag{4.19}$$

in $\mathbf{Ch}_\mathbb{K}$, which is the M-component of the dinatural transformation

$$G^\pm : \Delta_1 \mathfrak{F}_{\mathrm{c}} \longrightarrow \Delta_2 \mathfrak{F} \tag{4.20}$$

between functors $\Delta_1\mathfrak{F}_{\mathrm{c}} : \mathbf{Loc}_m^{\mathrm{op}} \times \mathbf{Loc}_m \to \mathbf{Ch}_{\mathbb{K}}$, $(M_1, M_2) \mapsto \mathfrak{F}_{\mathrm{c}}(M_1)$ and $\Delta_2\mathfrak{F} : \mathbf{Loc}_m^{\mathrm{op}} \times \mathbf{Loc}_m \to \mathbf{Ch}_{\mathbb{K}}$, $(M_1, M_2) \mapsto \mathfrak{F}(M_2)$. For each fixed $M \in \mathbf{Loc}_m$ one has a complex of differential operators $(\mathsf{F}(M), Q_M)$ endowed with a Green's witness W_M, hence Theorem 4.1.5 applies and guarantees us that every complex $(\mathsf{F}(M), Q_M)$ is Green hyperbolic, for all $M \in \mathbf{Loc}_m$. It also provides us with the component-wise retarded/advanced Green's homotopies

$$\Lambda_M^{\pm} := W_M G_M^{\pm} \in [\mathfrak{F}_{\mathrm{c}}(M), \mathfrak{F}(M)]^{-1}\,, \tag{4.21}$$

where we implicitly descended to ordinary colimits over the index category $\mathrm{c} \subseteq \mathbf{LocC}_m^{\mathrm{op}}$ relying on Remark 4.1.9. Note that the support propagation property of ordinary retarded/advanced Green's operators together with the fact that differential operators preserve supports yield the contracting homotopies $\Lambda_M^{\pm} \in [\mathfrak{F}_{J_M^{\pm}(K)}(M), \mathfrak{F}_{J_M^{\pm}(K)}(M)]^{-1}$ for all $(M, K) \in \mathbf{LocC}_m$. A generalization of Proposition 3.2.1 (see Remark 3.2.3) then implies that the covariant complex (F, Q) is Green hyperbolic. The same constructions as in Section 3.1 allow us to introduce the retarded-minus-advanced dinatural transformation

$$\Lambda : \Delta_1\mathfrak{F}_{\mathrm{c}}[1] \longrightarrow \Delta_2\mathfrak{F} \tag{4.22a}$$

between the 1-shifted functor $\Delta_1\mathfrak{F}_{\mathrm{c}}[1] : \mathbf{Loc}_m^{\mathrm{op}} \times \mathbf{Loc}_m \to \mathbf{Ch}_{\mathbb{K}}$ and the functor $\Delta_2\mathfrak{F}_{\mathrm{sc}} : \mathbf{Loc}_m^{\mathrm{op}} \times \mathbf{Loc}_m \to \mathbf{Ch}_{\mathbb{K}}$, whose M-component is given by the difference

$$\Lambda_M := \Lambda_M^{+} - \Lambda_M^{-} = W_M G_M : \mathfrak{F}_{\mathrm{c}}(M)[1] \longrightarrow \mathfrak{F}(M) \tag{4.22b}$$

in $\mathbf{Ch}_{\mathbb{K}}$, and the dinatural Dirac homotopy

$$\Lambda^D : \Delta_1\mathfrak{F}_{\mathrm{c}}[1] \longrightarrow \Delta_2\mathfrak{F}\,, \tag{4.23a}$$

here the functors $\Delta_1\mathfrak{F}_{\mathrm{c}}[1], \Delta_2\mathfrak{F}$ are regarded as valued on graded vector spaces (i.e. forgetting the differentials), given by the component

$$\Lambda_M^D := \frac{1}{2}(\Lambda_M^{+} + \Lambda_M^{-}) \in [\mathfrak{F}_{\mathrm{c}}(M)[1], \mathfrak{F}(M)]^0 \tag{4.23b}$$

for any $M \in \mathbf{Loc}_m$. Note that the components of the dinatural Dirac homotopy Λ^D are not cochain maps. Indeed, their differential reads as

$$\partial\Lambda_M^D = \frac{1}{2}\partial W_M(G_M^{+} + G_M^{-}) = \frac{1}{2}P_M(G_M^{+} + G_M^{-}) = j\,, \tag{4.24}$$

where $j \in [\mathfrak{F}_{\mathrm{c}}(M)[1], \mathfrak{F}(M)]^1 \cong [\mathfrak{F}_{\mathrm{c}}(M), \mathfrak{F}(M)]^0$ simply regards compactly supported sections as smooth sections (without any condition on their support). Let us just recall from Sections 1.2 and 1.3 that $\Lambda^{(D)}$ being a dinatural transformation means that for all morphism $f : M_1 \to M_2$ in $\mathbf{Loc}_m$ the identities

$$f^*\Lambda_{M_2} f_* = \Lambda_{M_1}\,, \qquad f^*\Lambda_{M_2}^D f_* = \Lambda_{M_1} \tag{4.25}$$

hold true. This immediately follows from $G^{\pm}$ being a dinatural transformation, see (1.38), and W being a degree-wise natural differential operator.

4.2 Formally self-adjoint Green's witnesses

In this section we introduce the notion of a *formally self-adjoint Green's witness* for a complex of differential operators (F, Q) endowed with a differential pairing $(-,-)$. (Recall that in this case $\mathbb{K} = \mathbb{R}$.) That is a Green's witness W for (F, Q) which is compatible with $(-,-)$ in a way that is partially reminiscent of formal self-adjointness in the ordinary sense, see Section 1.3. When a formally self-adjoint Green's witness is available the retarded and advanced Green's homotopies provided by Theorem 4.1.5 have such properties that simplify the construction and the comparison of the Poisson structures τ_M and σ_Σ from Proposition 3.3.3 and 3.3.6, respectively. Let us anticipate that all our examples from Chapter 6 admit formally self-adjoint Green's witnesses, hence they would enjoy from the simplified construction and comparison of the Poisson structures we will discuss later in this section.

Definition 4.2.1. Given a complex of differential operators (F, Q) on $M \in \mathbf{Loc}_m$ endowed with a differential pairing $(-,-)$, a Green's witness W for it is called *formally self-adjoint* with respect to $(-,-)$ if the following compatibility conditions are met:

1. $QWW = WWQ$;
2. $\int_M (W\varphi_1, \varphi_2) = (-1)^{|\varphi_1|} \int_M (\varphi_1, W\varphi_2)$, for all homogeneous sections $\varphi_1, \varphi_2 \in \mathfrak{F}(M)$ with compact overlapping support.

Remark 4.2.2. We list here some direct consequences of Definition 4.2.1 of a formally self-adjoint Green's witness.

i. Recalling from Definition 4.1.1 that $QW + WQ = P$, item 1 of Definition 4.2.1 implies that $PW = WP$. Since P is, by definition, a degree-wise ordinary Green hyperbolic operator, let us denote by $G^\pm$ the degree-wise retarded/advanced Green's operator associated with it. Then, the previous identity yields $G^\pm W = WG^\pm$. In particular, the special choice of retarded/advanced Green's homotopy $\Lambda^\pm$ provided by Theorem 4.1.5 coincides on the nose with the one $\widetilde{\Lambda}^\pm$ from Remark 4.1.7, namely there is not need of an higher homotopy to compare them. As a consequence, the associated retarded-minus-advanced quasi-isomorphism (descended to ordinary colimits) $\Lambda : \mathfrak{F}_\mathrm{c}(M)[1] \to \mathfrak{F}_\mathrm{sc}(M)$ in $\mathbf{Ch}_\mathbb{R}$ from (4.13) can be equivalently expressed as $GW = \Lambda = WG$, where $G := G^+ - G^-$ is the degree-wise retarded-minus-advanced propagator of P;

ii. One finds that the Green hyperbolic operator P is formally self-adjoint, namely $\int_M (P\varphi_1, \varphi_2) = \int_M (\varphi_1, P\varphi_2)$ for all sections $\varphi_1, \varphi_2 \in \mathfrak{F}(M)$ with compact overlapping support. This is a consequence of item 2 of Definition 4.2.1 and of the compatibility of $(-,-)$ with the differentials, see Definition 3.3.1. This implies that $\int_M (G^\pm\psi_1, \psi_2) =$

$\int_M (\psi_1, G^{\mp}\psi_2)$, for all homogeneous sections $\psi_1, \psi_2 \in \mathfrak{F}_{\mathrm{c}}(M)$, see Equation (1.42). Hence, the retarded-minus-advanced propagator $G := G^+ - G^-$ is formally skew-adjoint, $\int_M (G\psi_1, \psi_2) = -\int_M (\psi_1, G\psi_2)$, and the Dirac propagator $G^D := \frac{1}{2}(G^+ + G^-)$, is formally self-adjoint, $\int_M (G^D\psi_1, \psi_2) = \int_M (\psi_1, G^D\psi_2)$.

These observations shall be extensively used in our constructions of the remaining of this work. $\triangledown$

Remark 4.2.3. Let us observe that in Definition 4.2.1 we have not used the whole content of the differential pairing. In fact, the differential pairing $(-,-)$ enters the game only in item 2 after being integrated over the m-dimensional manifold $M \in \mathbf{Loc}_m$. Hence, it is only the component of cohomological degree 1 of $(-,-)$ that matters. (Recall that the differential pairing is in particular a cochain map $(-,-) : \mathfrak{F}(M)^{\otimes 2} \to \Omega^\bullet(M)[m-1]$ in $\mathbf{Ch}_\mathbb{R}$ which takes values into the $(m-1)$-shifted de Rham complex.) For this reason, one may define a formally self-adjoint Green's witness also in a slightly more general context. Indeed, one may 'truncate' the differential pairing at its cohomological degree 1. In other words, one may replace it with a (fiber-wise non degenerate,) graded anti-symmetric, graded vector bundle map $\langle -,-\rangle : F \otimes F \to M \times \mathbb{R}[-1]$ such that the identity

$$\int_M \langle Q\varphi_1, \varphi_2\rangle \,\mathrm{vol}_M + (-1)^{|\varphi_1|} \int_M \langle \varphi_1, Q\varphi_2\rangle \,\mathrm{vol}_M = 0\,, \tag{4.26}$$

where vol_M denotes the volume form on M, holds true for all homogeneous sections $\varphi_1, \varphi_2 \in \mathfrak{F}(M)$ with compact overlapping support. Equation (4.26) replaces in this context the compatibility of the full differential pairing $(-,-)$ with differentials, imposing by hand that the analog of the evaluation map ev_M from (3.50) is a cochain map. This is what is needed for the item ii of Remark 4.2.2 to still hold true. This kind of relaxed data is what the we and our collaborators considered in [BMS24] under the name of *compatible* (-1)-*shifted fiber metric* on (F, Q). Let us just point out that all our examples from Chapter 6 admit a full-fledged differential pairing, so we will never need to consider less structure. $\triangledown$

Recall that the existence of any Green's witness W permits to simplify the construction of the retarded-minus-advanced quasi-isomorphism $\Lambda : \mathfrak{F}_{\mathrm{c}}(M)[1] \to \mathfrak{F}_{\mathrm{sc}}(M)$ passing to ordinary (as opposed to homotopy) colimits, see (4.13) and (4.14). When on top of that W is formally self-adjoint, one proves that also Propositions 3.3.3, 3.3.6 and Theorem 3.3.7 have simplified counterparts.

Proposition 4.2.4. *Let* (F, Q) *be a complex of differential operators endowed with a differential pairing* $(-,-)$ *and a formally self-adjoint Green's* *witness W. Consider the retarded-minus-advanced quasi-isomorphism $\Lambda : \mathfrak{F}_{\mathrm{c}}(M)[1] \to \mathfrak{F}_{\mathrm{sc}}(M)$ in $\mathbf{Ch}_{\mathbb{R}}$ from Remark 4.1.9. Then the composition*

$$\tau_M : \mathfrak{F}_{\mathrm{c}}(M)[1]^{\otimes 2} \xrightarrow{\mathrm{id}\otimes\Lambda} \mathfrak{F}_{\mathrm{c}}(M)[1] \otimes \mathfrak{F}_{\mathrm{sc}}(M) \subseteq \mathfrak{F}_{\mathrm{c}}(M)[1] \otimes \mathfrak{F}(M) \xrightarrow{\mathrm{ev}_M} \mathbb{R} \tag{4.27}$$

in $\mathbf{Ch}_{\mathbb{R}}$ is graded anti-symmetric, hence it descends to a Poisson structure on $\mathfrak{F}_{\mathrm{c}}(M)[1]$. The evaluation cochain map ev_M is defined in (3.50).

Proof. We have to check that $\tau_M \circ \gamma = -\tau_M$, where γ denotes the symmetric braiding (2.4) on $\mathbf{Ch}_{\mathbb{R}}$. Since τ_M takes values in the cochain complex $\mathbb{R} \in \mathbf{Ch}_{\mathbb{R}}$ concentrated in degree 0, it is enough to check the identity upon the evaluation on all homogeneous sections $\psi_1 \in \mathfrak{F}_{\mathrm{c}}^p(M)[1]$ and $\psi_2 \in \mathfrak{F}_{\mathrm{c}}^{-p}(M)[1]$, for all $p \in \mathbb{Z}$. Recalling Remark 4.2.2, we compute

$$\begin{aligned} \tau_M(\gamma(\psi_1 \otimes \psi_2)) = (-1)^p \int_M (\psi_2, WG\psi_1) &= \int_M (GW\psi_2, \psi_1) \\ &= -(-1)^{-p(p+1)} \int_M (\psi_1, GW\psi_2) \\ &= -\tau_M(\psi_1 \otimes \psi_2)\,, \end{aligned} \tag{4.28}$$

where in the first step we used the definition of γ and $\Lambda = WG$. The second step follows from both item i and ii of Remark 4.2.2, and the third step from the graded anti-symmetry of the differential pairing. To get the correct signs, recall that both the graded anti-symmetry of $(-,-)$ and the compatibility of W with respect to the latter are expressed in terms of the degree in the unshifted complex $\mathfrak{F}_{\mathrm{c}}(M)$. The last step follows from $\Lambda = GW$, see item i of Remark 4.2.2. □

Remark 4.2.5. Proposition 4.2.4 yields, in particular, that with the choice of retarded and advanced Green's homotopies $\Lambda^{\pm}$ from Theorem 4.1.5, and up to the quasi-isomorphism $\mathfrak{F}_{\mathrm{hc}}(M) \sim \mathfrak{F}_{\mathrm{c}}(M)$, the cochain map $\widetilde{\tau}_M$ from Equation (3.54) coincides with the Poisson structure τ_M from (4.27). Indeed, the same calculations as in the proof of Proposition 4.2.4 show that $\widetilde{\tau}_M$ is already graded anti-symmetric, hence the anti-symmetrization in (3.55) becomes superfluous as anticipated in Remark 3.3.2. Furthermore, similar computations show that the Poisson structures $\tau_M^{+} = \tau_M^{-} = \tau_M$, from (3.52) and (4.27), coincide on the nose, making superfluous the homotopies from Proposition 3.3.3 comparing them, as anticipated by Remark 3.3.4. To be more explicit, one needs to check only that they coincide upon the evaluation on all homogeneous $\psi_1 \in \mathfrak{F}_{\mathrm{c}}^p(M)[1]$ and $\psi_2 \in \mathfrak{F}_{\mathrm{c}}^{-p}(M)[1]$, for all $p \in \mathbb{Z}$, since they land on $\mathbb{R} \in \mathbf{Ch}_{\mathbb{R}}$ that is concentrated in degree 0. Hence, one has

$$\begin{aligned}
\tau_M^{\pm}(\psi_1 \otimes \psi_2) &= \pm \int_M (\psi_1, WG^{\pm}\psi_2) \mp (-1)^p \int_M (\psi_2, WG^{\pm}\psi_1) \\
&= \pm \int_M (\psi_1, WG^{\pm}\psi_2) \pm \int_M (G^{\mp}W\psi_2, \psi_1) \\
&= \pm \int_M (\psi_1, WG^{\pm}\psi_2) \mp (-1)^{-p(p+1)} \int_M (\psi_1, G^{\mp}W\psi_2) \\
&= \int_M (\psi_1, WG\psi_2) \\
&= \tau_M(\psi_1 \otimes \psi_2)\,.
\end{aligned} \tag{4.29}$$

The first step follows from the definition (3.52) of $\tau_M^{\pm}$, and in particular from the definition of the symmetric braiding γ and from $\Lambda^{\pm} = WG^{\pm}$. The second step is a consequence of item ii of Remark 4.2.2 and of item 2 of Definition 4.2.1. Third step used the graded anti-symmetry of $(-,-)$ and the fourth one follows from item i of Remark 4.2.2 and $G = G^{+} - G^{-}$. Finally, last step is just the definition (4.27) of τ_M. ▽

Proposition 3.3.6 admits a simplification to ordinary, as opposed to homotopy, colimits. The next proposition makes this claim precise and, just as the original proposition, it is a straightforward consequence of the graded anti-symmetry of the differential pairing $(-,-)$.

Proposition 4.2.6. *Let (F, Q) be a complex of differential operators on $M \in \mathbf{Loc}_m$ endowed with a differential pairing $(-,-)$. Given a spacelike Cauchy surface $\Sigma \subseteq M$, the composition*

$$\sigma_\Sigma : \mathfrak{F}_{\mathrm{sc}}(M)^{\otimes 2} \xrightarrow{\mathrm{id}\otimes\subseteq} \mathfrak{F}_{\mathrm{sc}}(M) \otimes \mathfrak{F}(M) \xrightarrow{(-1)^{m-1}\,\mathrm{ev}_\Sigma} \mathbb{R} \tag{4.30}$$

in $\mathbf{Ch}_{\mathbb{R}}$ is graded anti-symmetric. Hence, it descends to a Poisson structure $\sigma_\Sigma : \mathfrak{F}_{\mathrm{sc}}(M)^{\wedge 2} \to \mathbb{R}$ on $\mathfrak{F}_{\mathrm{sc}}(M)$. Recall that the cochain map ev_Σ is defined in (3.66).

When a formally self-adjoint Green's witness is available, one has the Poisson complexes $(\mathfrak{F}_{\mathrm{c}}(M)[1], \tau_M)$ and $(\mathfrak{F}_{\mathrm{sc}}(M), \sigma_\Sigma) \in \mathbf{PoCh}_{\mathbb{R}}$ which are the ordinary colimits counterparts of the ones constructed in Section 3.3. From Remark 4.1.9, also the retarded-minus-advanced quasi-isomorphism $\Lambda : \mathfrak{F}_{\mathrm{c}}(M)[1] \to \mathfrak{F}_{\mathrm{sc}}(M)$ in $\mathbf{Ch}_{\mathbb{R}}$ descends to ordinary colimits. However, it is *not* compatible with the simplified Poisson structures from Propositions 4.2.4 and 4.2.6, hence it does not descend to a morphism of Poisson complexes. Similarly to Theorem 3.3.7, there is a homotopy λ witnessing this failure, that is the simplified counterpart of the homotopy λ from (3.68). This determines the following simplification of Theorem 3.3.7.

Theorem 4.2.7. *Let (F,Q) be a complex of differential operators on $M \in \mathbf{Loc}_m$ endowed with a differential pairing $(-,-)$ and a formally self-adjoint Green's witness W. Denote by $\Sigma \subseteq M$ a spacelike Cauchy surface of M. Recall from Propositions 4.2.4 and 4.2.6 the Poisson structures τ_M and σ_Σ, and from Remark 4.1.9 the retarded-minus-advanced quasi-isomorphism $\Lambda : \mathfrak{F}_{\mathrm{c}}(M)[1] \to \mathfrak{F}_{\mathrm{sc}}(M)$. Then there is a homotopy $\lambda \in [\mathfrak{F}_{\mathrm{c}}(M)[1]^{\wedge 2}, \mathbb{R}]^{-1}$ such that $\sigma_\Sigma \circ \Lambda^{\wedge 2} = \tau_M + \partial\lambda$.*

Proof. We define the candidate homotopy as the graded anti-symmetrization

$$\lambda := \mathrm{asym}(\widetilde{\lambda}) \in [\mathfrak{F}_{\mathrm{c}}(M)[1]^{\wedge 2}, \mathbb{R}]^{-1} \tag{4.31a}$$

of the (-1)-cochain

$$\widetilde{\lambda} \in [\mathfrak{F}_{\mathrm{c}}(M)[1]^{\otimes 2}, \mathbb{R}]^{-1} \tag{4.31b}$$

defined, for all homogeneous sections $\psi_1, \psi_2 \in \mathfrak{F}_{\mathrm{c}}(M)[1]$, by

$$\widetilde{\lambda}(\psi_1 \otimes \psi_2) := \int_{\Sigma^+} (\Lambda^- \psi_1, \Lambda\psi_2) + \int_{\Sigma^-} (\Lambda^+ \psi_1, \Lambda\psi_2)\,, \tag{4.31c}$$

where $\Sigma^\pm := J_M^\pm(\Sigma)$. Similar, yet simpler, computations as in (3.69) allow us to conclude. Explicitly, for all homogeneous $\psi_1, \psi_2 \in \mathfrak{F}_{\mathrm{c}}(M)[1]$, one has

$$\begin{aligned}
(\partial\widetilde{\lambda})(\psi_1 \otimes \psi_2) &= \int_{\Sigma^+} (\Lambda^- Q_{[1]}\psi_1, \Lambda\psi_2) + (-1)^{|\psi_1|} \int_{\Sigma^+} (\Lambda^- \psi_1, \Lambda Q_{[1]}\psi_2) \\
&\quad + \int_{\Sigma^-} (\Lambda^+ Q_{[1]}\psi_1, \Lambda\psi_2) + (-1)^{|\psi_1|} \int_{\Sigma^-} (\Lambda^+ \psi_1, \Lambda Q_{[1]}\psi_2) \\
&= \int_{\Sigma^+} (Q\Lambda^- \psi_1, \Lambda\psi_2) - \int_{\Sigma^+} (\psi_1, \Lambda\psi_2) \\
&\quad + (-1)^{|\psi_1|} \int_{\Sigma^+} (\Lambda^- \psi_1, Q\Lambda\psi_2) + \int_{\Sigma^-} (Q\Lambda^+ \psi_1, \Lambda\psi_2) \\
&\quad - \int_{\Sigma^-} (\psi_1, \Lambda\psi_2) + (-1)^{|\psi_1|} \int_{\Sigma^-} (\Lambda^+ \psi_1, Q\Lambda\psi_2) \\
&= (-1)^{m-1} \Big(\int_{\Sigma^+} \mathrm{d}_{\mathrm{dR}} (\Lambda^- \psi_1, \Lambda\psi_2) + \int_{\Sigma^-} \mathrm{d}_{\mathrm{dR}} (\Lambda^+ \psi_1, \Lambda\psi_2) \Big) \\
&\quad - \int_M (\psi_1, \Lambda\psi_2) \\
&= (-1)^{m-1} \int_\Sigma \iota^* ((\Lambda^+ - \Lambda^-)\psi_1, \Lambda\psi_2) - \int_M (\psi_1, \Lambda\psi_2) \\
&= \sigma_\Sigma(\Lambda\psi_1, \Lambda\psi_2) - \tau_M(\psi_1, \psi_2)\,,
\end{aligned} \tag{4.32}$$

where in the first step we used that $\partial\widetilde{\lambda} = \widetilde{\lambda} \circ Q_{[1]\otimes}$, in the second step that Λ is a cochain map and that $Q\Lambda^\pm + \Lambda^\pm Q = PG^\pm = \mathrm{id}$. The latter follows from $\Lambda^\pm = WG^\pm$, identities (4.5) and (4.1). Third step used the compatibility of $(-,-)$ with the differential $Q_\otimes$ and $M = \Sigma^+ \cup \Sigma^-$. Fourth step consists of applying the Stokes' theorem. Taking the graded anti-symmetrization of the identity above yields

$$\partial\lambda = \partial\,\mathrm{asym}(\widetilde{\lambda}) = \sigma_\Sigma \circ \Lambda^{\wedge 2} - \tau_M\,, \tag{4.33}$$

since both τ_M and σ_Σ are already graded anti-symmetric. □

4.3 Natural shifted and unshifted pairings

From now on we shall always consider complexes of differential operators (F, Q) on a globally hyperbolic Lorentzian manifold $M \in \mathbf{Loc}_m$, endowed with a differential pairing $(-,-)$ and a formally self-adjoint Green's witness W with respect to the latter. In view of the central role they play in the rest of our story, they deserve a name of their own. Following the terminology in the Riemannian setting [CG21b] and in the paper [BMS24] by collaborators and us, we introduce the following definition.

Definition 4.3.1. A *free BV theory* $(F, Q, (-,-), W)$ on $M \in \mathbf{Loc}_m$ consists of a complex of differential operators (F, Q) on M endowed with a differential pairing $(-,-)$ and a formally self-adjoint Green's witness W.

The differential pairing $(-,-)$ induces the evaluation map

$$\mathrm{ev}_M : \mathfrak{F}_\mathrm{c}(M)[1] \otimes \mathfrak{F}(M) \longrightarrow \mathbb{R} \tag{4.34}$$

in $\mathbf{Ch}_\mathbb{R}$ according to Equation (3.50). This allows us to interpret the 1-shift of the complex of compactly supported sections $\mathfrak{F}_\mathrm{c}(M)[1] \in \mathbf{Ch}_\mathbb{R}$ as the cochain complex of (linear) observables of the free BV theory. From Section 4.2 we know that the cochain complex $\mathfrak{F}_\mathrm{c}(M)[1] \in \mathbf{Ch}_\mathbb{R}$ comes endowed with a (unshifted) Poisson structure $\tau_M : \mathfrak{F}_\mathrm{c}(M)[1]^{\wedge 2} \to \mathbb{R}$, which relies on the formally self-adjoint Green's witness W, see Proposition 4.2.4. Therefore, with any free BV theory $(F, Q, (-,-), W)$ one associates the Poisson complex $(\mathfrak{F}_\mathrm{c}(M)[1], \tau_M) \in \mathbf{PoCh}_\mathbb{R}$ describing the (linear) observables of the associated field theory. This is completely analog to the ordinary situation of the Poisson vector space of linear observables of an ordinary field theory ruled by a Green hyperbolic operator, see Remarks 1.3.14 and 1.3.17. As in that case, this Poisson complex will prove to be crucial to quantize the theory as an AQFT, see Section 5.2.

The cochain complex $\mathfrak{F}_\mathrm{c}(M)[1] \in \mathbf{Ch}_\mathbb{R}$ of the observables of a free BV theory $(F, Q, (-,-), W)$ may be also endowed with different structures. In particular, it admits a (-1)-shifted Poisson structure $\tau_M^{(-1)} \in [\mathfrak{F}_\mathrm{c}(M)[1]^{\otimes 2}, \mathbb{R}]^1$, which relies only on the differential pairing $(-,-)$. This plays a crucial role in the BV quantization of the theory and hence in the construction of the tPFA from Section 5.1. Finally, exploiting again the Green's witness W, it can be endowed with a graded symmetric pairing

$\tau_M^D \in [\mathfrak{F}_{\mathrm{c}}(M)[1]^{\otimes 2}, \mathbb{R}]^0$, that we called *Dirac pairing*, trivializing the (-1)-shifted Poisson structure, i.e. $\partial \tau_M^D = \tau_M^{(-1)}$. The role of the Dirac pairing will be clear in Section 5.3 when we will introduce a comparison of the AQFT and tPFA quantizations of a free BV theory.

In this section we shall focus on introducing these additional structures and we shall prove some preliminary results. In view of their applications to quantum field theory in Chapter 5, we shall provide the reader directly with the $\mathbf{Loc}_m$-natural variant of them. Let us start by defining a *covariant free BV theory.*

Definition 4.3.2. A *covariant free BV theory* $(\mathsf{F}, Q, (-,-), W)$ consists of a covariant complex of natural differential operators (F, Q), see Remark 3.2.3, endowed with a natural Green's witness W, see Definition 4.1.11, and a *natural differential pairing* $(-,-) : \mathfrak{F} \otimes \mathfrak{F} \to \Omega^\bullet[m-1]$, that is a natural transformation from the functor $\mathfrak{F} \otimes \mathfrak{F} \in \mathbf{Ch}_{\mathbb{R}}^{\mathbf{Loc}_m^{\mathrm{op}}}$ of the two-fold tensor product of the smooth sections to the functor $\Omega^\bullet[m-1] \in \mathbf{Ch}_{\mathbb{R}}^{\mathbf{Loc}_m^{\mathrm{op}}}$ of the $(m-1)$-shifted de Rham complex, which is component-wise a differential pairing $(-,-)_M : \mathfrak{F}(M)^{\otimes 2} \to \Omega^\bullet(M)[m-1]$ for all $M \in \mathbf{Loc}_m$, such that $(\mathsf{F}(M), Q_M, (-,-)_M, W_M)$ is a free BV theory in the sense of Definition 4.3.1 for all $M \in \mathbf{Loc}_m$.

In Chapter 6 we will show that all our examples give rise to covariant free BV theories in a straightforward manner. This comprehends ordinary free field theories on $\mathbf{Loc}_m$, the Abelian Chern-Simons theory on $\mathbf{Loc}_3$, and the Maxwell p-forms, including linear Yang-Mills theory, on $\mathbf{Loc}_m$, $p \leq m-1$.

Recall from Section 4.1 that the natural Green's witness W allows us to construct the retarded-minus-advanced dinatural transformation Λ of components $\Lambda_M = W_M G_M : \mathfrak{F}_{\mathrm{c}}(M)[1] \to \mathfrak{F}(M)$ in $\mathbf{Ch}_{\mathbb{R}}$ from (4.22), and the dinatural Dirac homotopy Λ^D given by components $\Lambda_M^D = W_M G_M^D \in [\mathfrak{F}_{\mathrm{c}}(M)[1], \mathfrak{F}(M)]^0$, from (4.23), for each $M \in \mathbf{Loc}_m$, where $G_M := G_M^+ - G_M^-$ is the retarded-minus-advanced and $G_M^D := \frac{1}{2}(G_M^+ + G_M^-)$ the Dirac propagators built out of the degree-wise retarded/advanced Green's operators $G_M^\pm$ of the degree-wise Green hyperbolic differential operators $P_M := \partial W_M$. Combining these with the natural differential pairing $(-,-)$ one defines the following natural pairings.

First, we have the natural transformation

$$\tau : \mathfrak{F}_{\mathrm{c}}[1] \otimes \mathfrak{F}_{\mathrm{c}}[1] \longrightarrow \Delta\mathbb{R} \tag{4.35a}$$

in $\mathbf{Ch}_{\mathbb{K}}^{\mathbf{Loc}_m}$, where $\Delta\mathbb{R} \in \mathbf{Ch}_{\mathbb{K}}^{\mathbf{Loc}_m}$ is the constant functor which sends any $M \in \mathbf{Loc}_m$ to $\Delta\mathbb{R}(M) := \mathbb{R} \in \mathbf{Ch}_{\mathbb{R}}$, whose component at $M \in \mathbf{Loc}_m$ is the unshifted Poisson structure

$$\tau_M : \mathfrak{F}_{\mathrm{c}}(M)[1]^{\otimes 2} \xrightarrow{\mathrm{id} \otimes \Lambda_M} \mathfrak{F}_{\mathrm{c}}(M)[1] \otimes \mathfrak{F}(M) \xrightarrow{\mathrm{ev}_M} \mathbb{R} \tag{4.35b}$$

in $\mathbf{Ch}_{\mathbb{R}}$ from Proposition 4.2.4, where ev_M is the evaluation pairing (3.50) built out of the differential pairing $(-,-)_M$. The only thing we have to check is that this assignment is natural. In other words, let $f : M_1 \to M_2$ be a morphism in $\mathbf{Loc}_m$, we have to prove that

$$\tau_{M_2} \circ (f_* \otimes f_*) = \tau_{M_1} \,. \tag{4.36}$$

This readily follows from the naturality of the differential pairing and of the retarded-minus-advanced dinatural transformation, indeed one finds

$$\begin{aligned} \tau_{M_2}(f_*\psi_1 \otimes f_*\psi_2) &= \int_{M_2} (f_*\psi_1, \Lambda_{M_2} f_*\psi_2)_{M_2} = \int_{M_2} (f_*\psi_1, f_*\Lambda_{M_1}\psi_2)_{M_2} \\ &= \int_{M_2} f_*(\psi_1, \Lambda_{M_1}\psi_2)_{M_1} = \int_{M_1} (\psi_1, \Lambda_{M_1}\psi_2)_{M_1} \\ &= \tau_{M_1}(\psi_1 \otimes \psi_2) \,, \end{aligned} \tag{4.37}$$

for all compactly supported sections $\psi_1, \psi_2 \in \mathfrak{F}_{\mathrm{c}}(M_1)[1]$. The second step follows from the dinaturality of Λ, see (4.25), and the third step follows from the fact that $(-,-)$ is degree-wise a natural bi-differential operator by hypothesis.

Second, we define the *natural Dirac pairing* as the 0-cochain

$$\tau^D \in \underline{\hom}(\mathfrak{F}_{\mathrm{c}}[1] \otimes \mathfrak{F}_{\mathrm{c}}[1], \Delta\mathbb{R})^0 \tag{4.38a}$$

in the enriched hom $\underline{\hom}$ given, for all $M \in \mathbf{Loc}_m$, by the composition

$$\tau^D_M : \mathfrak{F}_{\mathrm{c}}(M)[1]^{\otimes 2} \xrightarrow{\mathrm{id}\otimes\Lambda^D_M} \mathfrak{F}_{\mathrm{c}}(M)[1] \otimes \mathfrak{F}(M) \xrightarrow{\mathrm{ev}_M} \mathbb{R} \tag{4.38b}$$

of degree preserving graded linear maps. Recall that the Dirac homotopy $\Lambda^D_M \in [\mathfrak{F}_{\mathrm{c}}(M)[1], \mathfrak{F}(M)]^0$ from (4.23) is a 0-cochain in the internal hom that is *not* a 0-cocycle, i.e. it is not a cochain map, indeed its differential

$$\partial\Lambda^D_M = j \in [\mathfrak{F}_{\mathrm{c}}(M)[1], \mathfrak{F}(M)]^1 \cong [\mathfrak{F}_{\mathrm{c}}(M), \mathfrak{F}(M)]^0 \tag{4.39}$$

coincides with the cochain map $j : \mathfrak{F}_{\mathrm{c}}(M) \to \mathfrak{F}(M)$ which forgets supports, see (4.24). Naturality of τ^D, namely that it is a 0-cochain in the enriched hom $\underline{\hom}$, follows from the naturality of $(-,-)$ and of the dinatural Dirac homotopy (4.25) through a similar computation to (4.37). A straightforward computation exploiting the properties of the Dirac propagators G^D_M shows that the Dirac pairing is graded symmetric, $\tau^D_M \circ \gamma = \tau^D_M$, for any $M \in \mathbf{Loc}_m$. Indeed, similarly to (4.28), it is sufficient to check the identity upon the evaluation on all homogeneous sections $\psi_1 \in \mathfrak{F}^p_{\mathrm{c}}(M)[1], \psi_2 \in \mathfrak{F}^{-p}_{\mathrm{c}}(M)[1]$, for $p \in \mathbb{Z}$. In this case, one finds

$$\begin{aligned} \tau^D_M(\gamma(\psi_1 \otimes \psi_2)) &= (-1)^p \int_M (\psi_2, W_M G^D_M \psi_1)_M = -\int_M (G^D_M W_M \psi_2, \psi_1)_M \\ &= (-1)^{-p(p+1)} \int_M (\psi_1, G^D_M W_M \psi_2)_M \\ &= \tau^D_M(\psi_1 \otimes \psi_2) \,, \end{aligned} \tag{4.40}$$

where the first step is a consequence of the definition (2.4) of the symmetric braiding γ and (4.23) of the Dirac homotopy Λ^D_M. Second step follows from the fact that $(\mathsf{F}(M), Q_M, (-,-)_M, W_M)$ is a free BV theory by definition, hence W_M is a formally self-adjoint Green's witness with respect to $(-,-)_M$. In particular, item 2 of Definition 4.2.1 and item ii of Remark 4.2.2 hold true. The third step used the graded anti-symmetry of the differential pairing $(-,-)_M$ and the last one the definition of the Dirac pairing (4.38) and the fact that $W_M G^D_M = \Lambda^D_M = G^D_M W_M$, see item i of Remark 4.2.2.

Finally, we introduce the *natural* (-1)*-shifted Poisson structure*

$$\tau^{(-1)} : \mathfrak{F}_c[1] \otimes \mathfrak{F}_c[1] \longrightarrow \Delta\mathbb{R}[1] \tag{4.41a}$$

in $\mathbf{Ch}_{\mathbb{R}}^{\mathbf{Loc}_m}$, where $\Delta\mathbb{R}[1] \in \mathbf{Ch}_{\mathbb{R}}^{\mathbf{Loc}_m}$ is the 1-shift of the functor $\Delta\mathbb{R} \in \mathbf{Ch}_{\mathbb{R}}^{\mathbf{Loc}_m}$, $\Delta\mathbb{R}[1](M) := \mathbb{R}[1] \in \mathbf{Ch}_{\mathbb{R}}$ for all $M \in \mathbf{Loc}_m$, which is given by the $M \in \mathbf{Loc}_m$ components defined by the composition

$$\begin{tikzcd}
\mathfrak{F}_c(M)[1] \otimes \mathfrak{F}_c(M)[1] \arrow[d, "\cong"'] \arrow[rr, dashed, "\tau^{(-1)}_M"] & & \mathbb{R}[1] \\
\mathfrak{F}_c(M)[1] \otimes \mathbb{R}[1] \otimes \mathfrak{F}_c(M) \arrow[r, "\gamma\otimes j"'] & \mathbb{R}[1] \otimes \mathfrak{F}_c(M)[1] \otimes \mathfrak{F}(M) \arrow[r, "\mathrm{id}\otimes \mathrm{ev}_M"'] & \mathbb{R}[1] \otimes \mathbb{R} \arrow[u, "\cong"']
\end{tikzcd} \tag{4.41b}$$

in $\mathbf{Ch}_{\mathbb{R}}$, where recall that $j : \mathfrak{F}_c(M) \to \mathfrak{F}(M)$ in $\mathbf{Ch}_{\mathbb{R}}$ denotes the cochain map that forgets the support condition. To confirm that (4.41) defines a (-1)-shifted Poisson structure we have to check it is graded symmetric, $\tau^{(-1)}_M \circ \gamma = \tau^{(-1)}_M$ for any $M \in \mathbf{Loc}_m$. Let us consider homogeneous compactly supported sections $\psi_1, \psi_2 \in \mathfrak{F}_c(M)[1]$. One computes

$$\begin{aligned}
\tau^{(-1)}_M(\gamma(\psi_1 \otimes \psi_2)) &= (-1)^{|\psi_2|(|\psi_1|+1)} \int_M (\psi_2, \psi_1)_M \\
&= (-1)^{|\psi_1|} \int_M (\psi_1, \psi_2)_M \\
&= \tau^{(-1)}_M(\psi_1 \otimes \psi_2)\,,
\end{aligned} \tag{4.42}$$

using the definition of $\tau^{(-1)}_M$ from (4.41) in the first and last step and the graded anti-symmetry of the differential pairing $(-,-)_M$ in the second step. Naturality of $\tau^{(-1)}$ from (4.41) follows from the naturality of the differential pairing $(-,-)$, indeed for all morphism $f : M_1 \to M_2$ in $\mathbf{Loc}_m$ one finds

$$\begin{aligned}
\tau^{(-1)}_{M_2}(f_*\psi_1 \otimes f_*\psi_2) &= (-1)^{|f_*\psi_1|} \int_{M_2} (f_*\psi_1, f_*\psi_2)_{M_2} \\
&= (-1)^{|\psi_1|} \int_{M_2} f_*(\psi_1, \psi_2)_{M_1} \\
&= (-1)^{|\psi_1|} \int_{M_1} (\psi_1, \psi_2)_{M_1} \\
&= \tau^{(-1)}_{M_1}(\psi_1 \otimes \psi_2)\,, \qquad (4.43)
\end{aligned}$$

for all homogeneous compactly supported sections $\psi_1, \psi_2 \in \mathfrak{F}_\mathrm{c}(M_1)[1]$, where in the first and last step we used the definition of $\tau^{(-1)}$, in the second step the naturality of the differential pairing and in the third step the naturality of the integral of compactly supported top-dimensional differential forms.

As previously mentioned, the natural (-1)-shifted Poisson structure $\tau^{(-1)}$ from (4.41) is trivialized by the natural Dirac pairing τ^D from (4.38). This means that we have

$$\partial \tau^D = \tau^{(-1)} \in \underline{\mathrm{hom}}(\mathfrak{F}_\mathrm{c}[1] \otimes \mathfrak{F}_\mathrm{c}[1], \Delta\mathbb{R})^1\,, \qquad (4.44\text{a})$$

where we are regarding $\tau^{(-1)} \in \mathsf{Z}^1\underline{\mathrm{hom}}(\mathfrak{F}_\mathrm{c}[1]^{\otimes 2}, \Delta\mathbb{R})$ as a 1-cocycle in the enriched hom $\underline{\mathrm{hom}}$ by pulling out the shift. Indeed, for all $M \in \mathbf{Loc}_m$ and all homogeneous compactly supported sections $\psi_1, \psi_2 \in \mathfrak{F}_\mathrm{c}(M)[1]$ one has

$$\begin{aligned}
\partial \tau^D_M(\psi_1 \otimes \psi_2) &= \int_M (Q_M\psi_1, \Lambda^D_M\psi_2)_M + (-1)^{|\psi_1|} \int_M (\psi_1, \Lambda^D_M Q_M \psi_2)_M \\
&= (-1)^{|\psi_1|} \int_M (\psi_1, (Q\Lambda^D_M - \Lambda^D_M Q_{[1]})\psi_2)_M \\
&= (-1)^{|\psi_1|} \int_M (\psi_1, \psi_2)_M \\
&= \tau^{(-1)}_M(\psi_1 \otimes \psi_2)\,, \qquad (4.44\text{b})
\end{aligned}$$

where the first step follows from the fact that $\partial\tau^D_M = -\tau^D_M \circ Q_{M[1]}$ and from the definition of the Dirac pairing τ^D_M, the second step follows from the fact that $\mathrm{ev}_M := \int_M (-,-)_M : \mathfrak{F}_\mathrm{c}(M)[1] \otimes \mathfrak{F}(M) \to \mathbb{R}$ is a cochain map, see (3.50), the third step uses (4.24) and the last one the definition (4.41) of $\tau^{(-1)}$.

Recalling the natural Poisson structure τ from Equation (4.35), we define the functor

$$(\mathfrak{F}_\mathrm{c}[1], \tau) : \mathbf{Loc}_m \longrightarrow \mathbf{PoCh}_\mathbb{R} \qquad (4.45)$$

which assigns to each manifold $M \in \mathbf{Loc}_m$ the Poisson cochain complex $(\mathfrak{F}_\mathrm{c}(M)[1], \tau_M) \in \mathbf{PoCh}_\mathbb{R}$ and to each morphism $f : M_1 \to M_2$ in $\mathbf{Loc}_m$ the morphism of Poisson complexes $f_* : (\mathfrak{F}_\mathrm{c}(M_1)[1], \tau_{M_1}) \to (\mathfrak{F}_\mathrm{c}(M_2)[1], \tau_{M_2})$ in $\mathbf{PoCh}_\mathbb{R}$ which is (degree-wise) given by the pushforward (1.11) of sections with compact support. Note that the naturality condition (4.36) tells us that the pushforward cochain map f_* respects the Poisson structures, hence it upgrades to a morphism in $\mathbf{PoCh}_{\mathbb{R}}$. We would like to notice that the fact that the functor (4.45) is well-defined is a consequence of the strict naturality of the unshifted Poisson structure (4.35) which in turn is a consequence of the naturality properties of the retarded-minus-advanced dinatural transformation Λ, see the computation (4.37). Moreover, the latter is a consequence of our particular choice of the retarded/advanced Green's homotopies $\Lambda^{\pm} = WG^{\pm}$ and of the naturality properties of the retarded and advanced Green's operator of a natural Green hyperbolic operator, see Section 1.3. Ultimately, it is consequence of the existence of a natural Green's witness W. Indeed, it is W which allows us to perform the choices of the retarded and advanced Green's homotopies for each $M \in \mathbf{Loc}_m$ in a controlled and natural way. Without a natural Green's witness, there would not be a criterion to pick special points in the space of retarded/advanced Green's homotopies $\underline{\mathcal{GH}}^{\pm}$ from (3.23) and even more there would not be clear how to perform these choices consistently while varying the background manifold $M \in \mathbf{Loc}_m$. This is why we stick to the case of covariant free BV theories $(\mathsf{F}, Q, (-,-), W)$ in this section. Especially in view of Chapter 5, where the naturality of the Poisson structure above will be relevant for the quantization of a covariant free BV theory as a strict AQFT.

The next result shows that classical analogs of the Einstein causality and of the (homotopy) time-slice axioms hold true for the functor $(\mathfrak{F}_{\mathrm{c}}[1], \tau) \in \mathbf{PoCh}_{\mathbb{R}}^{\mathbf{Loc}_m}$ introduced above. (Recall that the quantum version of these axioms are found in Definitions 1.4.1 and 1.4.8.)

Theorem 4.3.3. *Let $(\mathsf{F}, Q, (-,-), W)$ be a covariant free BV theory. Then,*

i. for all causally disjoint morphisms $f_1 : M_1 \to N \leftarrow M_2 : f_2$ in $\mathbf{Loc}_m$, recall Definition 1.1.8, the composition

$$\tau_N \circ (f_{1*} \otimes f_{2*}) = 0 \tag{4.46}$$

in $\mathbf{Ch}_{\mathbb{R}}$ vanishes,

ii. for all Cauchy morphisms $f : M \to N$ in $\mathbf{Loc}_m$, recall Definition 1.1.7, the pushforward cochain map

$$f_* : \mathfrak{F}_{\mathrm{c}}(M)[1] \xrightarrow{\sim} \mathfrak{F}_{\mathrm{c}}(N)[1] \tag{4.47}$$

in $\mathbf{Ch}_{\mathbb{R}}$ is a quasi-isomorphism.

Proof. Item i is a straightforward consequence of the definition (4.35) of the natural Poisson structure τ and of the support propagation properties of retarded/advanced Green's operators. Indeed, for all sections $\psi_1 \in \mathfrak{F}_{\mathrm{c}}(M_1)[1]$ and $\psi_2 \in \mathfrak{F}_{\mathrm{c}}(M_2)[1]$ with compact support, one has

$$\tau_N(f_{1*}\psi_1 \otimes f_{2*}\psi_2) = \int_N (f_{1*}\psi_1, W_N G_N f_{2*}\psi_2)_N = 0\,, \tag{4.48}$$

since $\mathrm{supp}(f_{i*}\psi_i) \subseteq f_i(M_i)$, for $i = 1, 2$, because of the definition of the push-forward, and $\mathrm{supp}(f_{1*}\psi_1) \cap \mathrm{supp}(W_N G_N f_{2*}\psi_2) \subseteq f_1(M_1) \cap J_N(f_2(M_2)) = \emptyset$, where in the first step we used that W_N preserves supports being a differential operators and the properties of retarded/advanced Green's operators $G_N^{\pm}$, see items 2 and 3 of Definition 1.3.1. Second step follows because f_1 and f_2 are causally disjoint by hypothesis.

To prove also item ii, we shall construct a quasi-inverse $g : \mathfrak{F}_{\mathrm{c}}(N)[1] \to \mathfrak{F}_{\mathrm{c}}(M)[1]$ in $\mathbf{Ch}_{\mathbb{R}}$ of f_* and homotopies $\eta \in [\mathfrak{F}_{\mathrm{c}}(N)[1], \mathfrak{F}_{\mathrm{c}}(N)[1]]^{-1}$ witnessing that $f_* g \sim \mathrm{id}$, and $\zeta \in [\mathfrak{F}_{\mathrm{c}}(M)[1], \mathfrak{F}_{\mathrm{c}}(M)[1]]^{-1}$ witnessing that $g f_* \sim \mathrm{id}$. Let us start with a preliminary geometric construction. Since $f : M \to N$ in $\mathbf{Loc}_m$ is a Cauchy morphism by hypothesis, we can choose two spacelike Cauchy surfaces $\Sigma_{\pm} \subseteq f(M) \subseteq N$ entirely contained in the image of f and such that $\Sigma_+ \subseteq I_N^+(\Sigma_-)$ is in the chronological future of Σ_-. Given this choice, we will also consider a partition of unity $\{\chi_+, \chi_-\}$ of N subordinate to the open cover $\{I_N^+(\Sigma_-), I_N^-(\Sigma_+)\}$.

Quasi-inverse g. We define a candidate quasi-inverse as the cochain map

$$g : \mathfrak{F}_{\mathrm{c}}(N)[1] \longrightarrow \mathfrak{F}_{\mathrm{c}}(M)[1] \tag{4.49a}$$

in $\mathbf{Ch}_{\mathbb{R}}$ uniquely determined by

$$j f_* g := \mp \partial(\chi_{\pm} \Lambda_N) : \mathfrak{F}_{\mathrm{c}}(N)[1] \longrightarrow \mathfrak{F}(N)[1] \tag{4.49b}$$

in $\mathbf{Ch}_{\mathbb{R}}$, where $j : \mathfrak{F}_{\mathrm{c}}(N)[1] \to \mathfrak{F}(N)[1]$ in $\mathbf{Ch}_{\mathbb{R}}$ denotes the cochain map which forgets compact supports, the (-1)-cochain $\chi_{\pm} \in [\mathfrak{F}(N), \mathfrak{F}(N)[1]]^{-1}$ is given in each degree by the multiplication by the partition of unity function $\chi_{\pm}$, the cochain map $\Lambda_N = W_N G_N : \mathfrak{F}_{\mathrm{c}}(N)[1] \to \mathfrak{F}(N)$ in $\mathbf{Ch}_{\mathbb{R}}$ is the retarded-minus-advanced cochain map (4.21) associated with the natural Green's witness W and ∂ denotes the 'adjoint' differential of the internal hom $[\mathfrak{F}_{\mathrm{c}}(N)[1], \mathfrak{F}(N)[1]] \in \mathbf{Ch}_{\mathbb{R}}$. In order to show that such a cochain map g exists we have to see that the right-hand side of (4.49b) factors through $j f_* : \mathfrak{F}_{\mathrm{c}}(M)[1] \to \mathfrak{F}(N)[1]$ in $\mathbf{Ch}_{\mathbb{R}}$. Since it is manifestly a 0-coboundary, hence a cochain map, this is proven by showing that for all $\psi \in \mathfrak{F}_{\mathrm{c}}(N)[1]$ the support $\mathrm{supp}(\mp\partial(\chi_{\pm}\Lambda_N))\psi$ is a compact contained in the image $f(M) \subseteq N$. Observing that

$$\partial(\chi_- \Lambda_N)\psi = \partial((\mathrm{id}_{[1]} - \chi_+)\Lambda_N)\psi = -\partial(\chi_+ \Lambda_N)\psi\,, \tag{4.50}$$

where we used that $\{\chi_+, \chi_-\}$ is a partition of unity, $\chi_+ + \chi_- = 1$ on N, and that Λ_N is a cochain map, $\partial \Lambda_N = 0$, we conclude that the support $\mathrm{supp}(\mp\partial(\chi_{\pm}\Lambda_N)\psi) \subseteq J_N(\mathrm{supp}\,\psi) \cap J_N^-(\Sigma_+) \cap J_N^+(\Sigma_-)$. The latter is a compact subsets of N because of properties of strictly past/future compact subsets, see [BGP07, Cor. A.5.4], and it is contained in $f(M)$ because by construction $J_N^-(\Sigma_+) \cap J_N^+(\Sigma_-) \subseteq f(M)$. This proves that g exists. It is also unique because jf_* is degree-wise an injective linear map, since both j and f_* are so.

Homotopy η witnessing $f_*g \sim \mathrm{id}$. Consider the (-1)-cochain

$$\eta \in [\mathfrak{F}_{\mathrm{c}}(N)[1], \mathfrak{F}_{\mathrm{c}}(N)[1]]^{-1} \tag{4.51a}$$

in the internal hom, uniquely defined by

$$j\eta := -\chi_-\Lambda_N^+ - \chi_+\Lambda_N^- \in [\mathfrak{F}_{\mathrm{c}}(N)[1], \mathfrak{F}(N)[1]]^{-1}\,, \tag{4.51b}$$

where the retarded/advanced Green's homotopy $\Lambda_N^\pm = W_N G_N^\pm$ selected by the Green's witness is regarded as a 0-cochain in $[\mathfrak{F}_{\mathrm{c}}(N)[1], \mathfrak{F}(N)] \in \mathbf{Ch}_{\mathbb{R}}$, under the isomorphism (2.15) $[\mathfrak{F}_{\mathrm{c}}(N)[1], \mathfrak{F}(N)] \cong [\mathfrak{F}_{\mathrm{c}}(N), \mathfrak{F}(N)][-1]$ in $\mathbf{Ch}_{\mathbb{R}}$ which consists of a sign $(-1)^n$ in degree n. To prove existence of η we have to show that the right-hand side of (4.51b) factors through the inclusion $j : \mathfrak{F}_{\mathrm{c}}(N)[1] \to \mathfrak{F}(N)[1]$ in $\mathbf{Ch}_{\mathbb{R}}$. Hence, we have to show that for any section $\psi \in \mathfrak{F}_{\mathrm{c}}(N)[1]$ with compact support, the support of the section $(-\chi_-\Lambda_N^+ - \chi_+\Lambda_N^-)\psi \in \mathfrak{F}(N)[1]$ is compact. This follows from $\mathrm{supp}(\chi_\mp\Lambda_N^\pm\psi) \subseteq J_N^\mp(\Sigma_\pm) \cap J_N^\pm(\mathrm{supp}\,\psi) \subseteq N$ which is compact again by [BGP07, Cor. A.5.4]. Uniqueness of η then follows from the degree-wise injectivity of j. Once proven that the (-1)-cochain η from (4.51) exists and it is unique, let us prove that it is a homotopy witnessing that g is a left quasi-inverse of f_*, i.e. $\partial\eta = \mathrm{id} - f_*g$. Since j is degree-wise injective, this follows from

$$\begin{aligned}
j\partial\eta &= \partial(-\chi_-\Lambda_N^+ - \chi_+\Lambda_N^-) \\
&= -\partial(\chi_-)\Lambda_N^+ + \chi_-\partial\Lambda_N^+ - \partial(\chi_+)\Lambda_N^- + \chi_+\partial\Lambda_N^- \\
&= j + \partial(\chi_+)\Lambda_N \\
&= j - jf_*g
\end{aligned} \tag{4.52}$$

where in the first step we used that j is a cochain map and the definition (4.51) of η, in the second step that ∂ is a degree 1 derivation with respect to the composition (recall that $|\chi_\pm| = -1$), the third step follows from $\partial\Lambda_N^\pm = j$, and $\chi_+ + \chi_- = \mathrm{id}$ which implies that $-\partial(\chi_-) = \partial(\chi_+)$. Finally, last step uses that Λ_N is a cochain map and Equation (4.49).

Homotopy ζ witnessing $gf_* \sim \mathrm{id}$. Consider the (-1)-cochain

$$\zeta \in [\mathfrak{F}_{\mathrm{c}}(M)[1], \mathfrak{F}_{\mathrm{c}}(M)[1]]^{-1} \tag{4.53a}$$

in the internal hom, uniquely defined by

$$f_*\zeta := \eta f_* \in [\mathfrak{F}_{\mathrm{c}}(M)[1], \mathfrak{F}_{\mathrm{c}}(N)[1]]^{-1}\,. \tag{4.53b}$$

The existence of ζ follows from the fact that the right-hand side of (4.53b) factors through the cochain map $f_* : \mathfrak{F}_c(M)[1] \to \mathfrak{F}_c(N)[1]$ in $\mathbf{Ch}_\mathbb{R}$, namely that the support of the section $\eta f_* \psi \in \mathfrak{F}_c(N)[1]$ is a compact contained in $f(M)$ for all compactly supported sections $\psi \in \mathfrak{F}_c(M)[1]$. This is a consequence of the support of the section $\chi_\mp \Lambda_N^\pm f_* \psi \in \mathfrak{F}(N)[1]$ being contained in the compact subset $J_N^\mp(\Sigma_\pm) \cap J_N^\pm(f(\operatorname{supp}\psi)) \subseteq f(M)$. Since f_* is degree-wise an injective map, it follows that ζ is unique. Moreover, it is an homotopy witnessing that g is a right quasi-inverse of f_*, i.e. $\partial\zeta = \mathrm{id} - g f_*$, as one can show by computing

$$f_* \partial\zeta = (\partial\eta) f_* = f_*(\mathrm{id} - g f_*) \,, \tag{4.54}$$

where in the first step one uses that f_* is a cochain map and the definition (4.53) of ζ and in the last step one uses that $\partial\eta = \mathrm{id} - f_* g$ as proven by (4.52). Since f_* is degree-wise injective, Equation (4.54) yields that $\partial\zeta = \mathrm{id} - g f_*$. □

We conclude this section by proving a simple result which relates the natural Poisson structure τ from (4.35) and the natural Dirac pairing τ^D from (4.38) via time-ordering.

Proposition 4.3.4. *Let $(\mathsf{F}, Q, (-,-), W)$ be a covariant free BV theory. Then, for all time-ordered pairs $(f_1, f_2) : (M_1, M_2) \to N$ of morphisms in* $\mathbf{Loc}_m$*, see Definition 1.1.9, the identity*

$$\tau_N^D \circ (f_{1*} \otimes f_{2*}) = \frac{1}{2} \tau_N \circ (f_{1*} \otimes f_{2*}) \tag{4.55}$$

holds true.

Proof. Identity (4.55) follows from a straightforward computation exploiting the support properties of retarded and advanced Green's operators from Definition 1.3.1. Indeed, one computes, for all compactly supported sections $\psi_1 \in \mathfrak{F}_c(M_1)[1]$ and $\psi_2 \in \mathfrak{F}_c(M_2)[1]$, that

$$\begin{aligned}
\frac{1}{2}\tau_N(f_{1*}\psi_1 \otimes f_{2*}\psi_2) &= \frac{1}{2}\int_N (f_{1*}\psi_1, W_N G_N f_{2*}\psi_2)_N \\
&= \frac{1}{2}\int_N (f_{1*}\psi_1, W_N G_N^+ f_{2*}\psi_2)_N \\
&= \int_N (f_{1*}\psi_1, W_N G_N^D f_{2*}\psi_2)_N \\
&= \tau_N^D(f_{1*}\psi_1 \otimes f_{2*}\psi_2) \,.
\end{aligned} \tag{4.56}$$

In the first step we used the definition (4.35) of the unshifted Poisson structure τ_N, recalling that $\Lambda_N = W_N G_N$ because of the choice of retarded and advanced Green's homotopies provided by the Green's witness W_N. Second and third steps both use that $(f_{1*}\psi_1, W_N G_N^- f_{2*}\psi_2)_N = 0$, in combination either with $G_N = G_N^+ - G_N^-$ or with $G_N^D = \frac{1}{2}(G_N^+ + G_N^-)$. The first identity follows from $\mathrm{supp}(f_{1*}\psi_1) \cap \mathrm{supp}(W_N G_N^- f_{2*}\psi_2) \subseteq f_1(M_1) \cap J_N^-(f_2(M_2)) = \emptyset$, that in turn is a consequence of the support properties of the linear differential operator W_N and of the advanced Green's operator G_N^- together with the fact that (f_1, f_2) is time-ordered. The last step uses the definition (4.38) of the Dirac pairing τ_N^D. $\square$

Chapter 5

Quantizations

This chapter will be devoted to the quantization of a covariant free BV theory $(\mathsf{F}, Q, (-,-), W)$ on $\mathbf{Loc}_m$. We shall proceed by presenting two *a priori* different approaches to this problem. First, in Section 5.1 we shall construct a $\mathbf{Ch}_{\mathbb{C}}$-valued tPFA $\mathcal{F} \in \mathbf{tPFA}_m(\mathbf{Ch}_{\mathbb{C}})$ by deforming the differential of the free symmetric algebra $\mathrm{Sym}(\mathfrak{F}_\mathrm{c}(M)[1]) \in \mathbf{dgAlg}_{\mathbb{C}}$ generated by the complex $\mathfrak{F}_\mathrm{c}(M)[1]$ of linear observables by the *BV Laplacian*, as required by the BV formalism [CG17; CG21b]. Second, in Section 5.2 we shall construct a $\mathbf{Ch}_{\mathbb{C}}$-valued AQFT $\mathcal{A} \in \mathbf{AQFT}_m(\mathbf{Ch}_{\mathbb{C}})$ by deforming the commutative product of the free symmetric algebra $\mathrm{Sym}(\mathfrak{F}_\mathrm{c}(M)[1]) \in \mathbf{dgAlg}_{\mathbb{C}}$ to the non-commutative *Moyal-Weyl star product*. These two approaches do not differ only in the kind of objects they produce (tPFAs versus AQFTs) and in the place where the deformation is concentrated (the differential versus the product), but also in the input data they require. More precisely, the time-orderable prefactorization algebra $\mathcal{F} \in \mathbf{tPFA}_m(\mathbf{Ch}_{\mathbb{C}})$ relies only on the natural differential pairing $(-,-)$ through the natural (-1)-shifted Poisson structure $\tau^{(-1)}$ from (4.41), while the algebraic quantum field theory $\mathcal{A} \in \mathbf{AQFT}_m(\mathbf{Ch}_{\mathbb{C}})$ relies on the natural unshifted Poisson structure τ from (4.35), and through it on the natural differential pairing $(-,-)$ and also on the natural Green's witness W. (To be precise, W is also needed to prove the homotopy time-slice axiom for $\mathcal{F} \in \mathbf{tPFA}_m(\mathbf{Ch}_{\mathbb{C}})$ in Section 5.1. Hence, without a natural Green's witness the tPFA $\mathcal{F}$ can still be constructed but, in general, it may fail to be homotopy Cauchy constant.) In Section 5.3 we shall see that when both the constructions are available, namely when a natural Green's witness is provided, the two seemingly different approaches can be compared in a nice way. Indeed, the natural Dirac pairing τ^D from (4.38) induces an isomorphism $T : \mathcal{F} \to \mathcal{F}_{\mathcal{A}}$ in $\mathbf{tPFA}_m(\mathbf{Ch}_{\mathbb{C}})$ between the BV quantization of the covariant free BV theory and the time-orderable prefactorization algebra $\mathcal{F}_{\mathcal{A}} \in \mathbf{tPFA}_m(\mathbf{Ch}_{\mathbb{C}})$ which is canonically associated to the AQFT $\mathcal{A} \in \mathbf{AQFT}_m(\mathbf{Ch}_{\mathbb{C}})$, see the Theorem 1.4.11 which we recalled from [BPS20]. Since the tPFA $\mathcal{F}_{\mathcal{A}}$ encodes the information about the time-ordered products built out of an AQFT product, this result makes clear that the physical content encoded by the BV quantization of a free BV theory is equivalent to that of the time-ordered products of the Moyal-Weyl quantization of the same theory. This result generalizes to the case of linear gauge field theories the model-based comparison due to Gwilliam and Rejzner [GR20]. The content of this chapter is mainly based on [BMS24].

In the following we will just write $\mathbf{tPFA}_m$ and $\mathbf{AQFT}_m$ for the categories of $\mathbf{Ch}_{\mathbb{C}}$-valued time-orderable prefactorization algebras and algebraic quantum field theories, respectively, since we will consider only theories valued in the monoidal category $\mathbf{Ch}_{\mathbb{C}}$ of cochain complexes.

Let us start by presenting here a geometric construction that we will need frequently in the rest of this chapter. Recall the Definition 1.1.9 of time-ordered tuples of morphisms in $\mathbf{Loc}_m$.

Lemma 5.0.1. *Let $\underline{f} = (f_1, \dots, f_n) : \underline{M} = (M_1, \dots, M_n) \to N$ be a time-ordered tuple of morphisms of $\mathbf{Loc}_m$ of length $n \geq 2$. Then, there exists a manifold $M \in \mathbf{Loc}_m$, a morphism $f : M \to N$ in $\mathbf{Loc}_m$ and a time-ordered tuple $\underline{f}' = (f'_1, \dots, f'_{n-1}) : (M_1, \dots, M_{n-1}) \to M$ in $\mathbf{Loc}_m$ of length $n-1$, such that the pair $(f, f_n) : (M, M_n) \to N$ of morphisms of $\mathbf{Loc}_m$ is time-ordered and $f \circ f'_i = f_i$ for all $i = 1, \dots, n-1$. In other words, each time-ordered tuple $\underline{f} : \underline{M} \to N$ in $\mathbf{Loc}_m$ of length $n \geq 2$ admits a factorization*

$$
\begin{tikzcd}
\underline{M} \arrow[rr, "\underline{f}"] \arrow[dr, "(\underline{f}'{,}\mathrm{id})"'] & & N \\
& (M, M_n) \arrow[ur, "(f{,}f_n)"'] &
\end{tikzcd}
\tag{5.1}
$$

where $\underline{f}'$ is a time-ordered tuple of length $n-1$ and (f, f_n) is a time-ordered pair.

Proof. Recalling Section 1.1, we define the subset

$$
M := J_N^{+\cap -}\Big(\bigcup_{i=1}^{n-1} f_i(M_i)\Big) \subseteq N \tag{5.2}
$$

as the convex hull (1.4) of the union of the images of f_i for $i = 1, \dots, n-1$. By construction $M \subseteq N$ is causally convex. Moreover, since the images $f_i(M_i) \subseteq N$ are open, because of the definition of morphism in $\mathbf{Loc}_m$, M is open as well. Therefore, M endowed with the restrictions of the orientation, time-orientation and metric of N defines an object of $\mathbf{Loc}_m$. Moreover, the subset inclusion $M \subseteq N$ is promoted to a morphism $f : M \to N$ in $\mathbf{Loc}_m$. Since by construction $f_i(M_i) \subseteq M$ for all $i = 1, \dots, n-1$, the morphism $f_i : M_i \to N$ in $\mathbf{Loc}_m$ factors as $f_i = f \circ f'_i$, where $f'_i : M_i \to M$ in $\mathbf{Loc}_m$ is the corestriction of f_i to its codomain, for each $i = 1, \dots, n-1$. Since $\underline{f}$ is time-ordered by hypothesis, it follows that the tuple $\underline{f}'$ of the corestrictions is time-ordered as well. It remains to show that the pair (f, f_n) in $\mathbf{Loc}_m$ is time-ordered, namely that $J_N^+(f(M)) \cap f_n(M_n) = \emptyset$. Let us argue by contradiction. Assume that the intersection is not empty, then there is a point $x \in f_n(M_n)$ and a future directed causal curve $c : [0,1] \to N$ such that $c(1) = x$ and $c(0) \in f(M)$. Since by the definition (1.4) of the convex hull each point of M is in particular in the causal future $J_N^+(f_{\bar{\imath}}(M_{\bar{\imath}}))$ of $f_{\bar{\imath}}(M_{\bar{\imath}})$, for some $\bar{\imath} \in \{1, \dots, n-1\}$, it follows that there exists a future directed causal curve $c' : [0,1] \to N$ such that $c'(0) \in f_{\bar{\imath}}(M_{\bar{\imath}})$ and $c'(1) \in f_n(M_n)$, extending the curve c. This contradicts the hypothesis that the tuple $\underline{f}$ is time-ordered. $\square$

5.1 BV quantization defines a tPFA

In this section we shall construct a $\mathbf{Ch}_{\mathbb{C}}$-valued time-orderable prefactorization algebra $\mathcal{F} \in \mathbf{tPFA}_m$ on $\mathbf{Loc}_m$ quantizing the classical time-orderable prefactorization algebra $\mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}[1]) \in \mathbf{tPFA}_m$ of the polynomial observables associated with a covariant free BV theory $(\mathsf{F}, Q, (-,-), W)$ on $\mathbf{Loc}_m$. Following the prescription of the BV formalism, the quantization will be achieved by suitably deforming the differential of the complexes assigned by the classical tPFA to a 'quantum' differential, see [CG17; CG21b].

Let us dive deeper into details. Recall from Section 2.1 the free symmetric dg-commutative algebra $\mathrm{Sym}\, V \in \mathbf{dgCAlg}_{\mathbb{K}}$ generated by a cochain complex $V \in \mathbf{Ch}_{\mathbb{K}}$, see Equation (2.16). Given the covariant free BV theory $(\mathsf{F}, Q, (-,-), W)$, we associate with it the time-orderable prefactorization algebra $\mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}[1]) \in \mathbf{tPFA}_m$ which is defined by the assignment

$$\mathbf{Loc}_m \ni M \longmapsto \mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}(M)[1]) := \mathrm{Sym}(\mathfrak{F}_{\mathrm{c}}(M)[1] \otimes \mathbb{C}) \in \mathbf{Ch}_{\mathbb{C}} \tag{5.3a}$$

of the cochain complex underlying the free symmetric dg-algebra generated by the complexification of the cochain complex $\mathfrak{F}_{\mathrm{c}}(M)[1] \in \mathbf{Ch}_{\mathbb{R}}$ of the linear observables on $M \in \mathbf{Loc}_m$, and whose time-ordered products are induced by the algebra structure. To be more precise, given a time-orderable n-tuple $\underline{f} = (f_1, \dots, f_n) : \underline{M} = (M_1, \dots, M_n) \to N$ in $\mathbf{Loc}_m$, we define the time-ordered product $\mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}[1])(\underline{f})$ as the composition

$$\begin{tikzcd}
\bigotimes_i \mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}(M_i)[1]) \arrow[rr, dashed, "\mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}[1])(\underline{f})"] \arrow[dr, "\bigotimes_i f_{i*}"'] & & \mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}(N)[1]) \\
& \mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}(N)[1])^{\otimes n} \arrow[ur, "\mu_N^{(n)}"'] &
\end{tikzcd} \tag{5.3b}$$

in $\mathbf{Ch}_{\mathbb{C}}$. Here f_{i*} denotes the symmetric algebra extension of the pushforward cochain maps $f_{i*} : \mathfrak{F}_{\mathrm{c}}(M_i)[1] \to \mathfrak{F}_{\mathrm{c}}(N)[1]$ in $\mathbf{Ch}_{\mathbb{R}}$ for compactly supported sections, see Section 1.2, and $\mu_N^{(n)}$ denotes the n-ary multiplication on the symmetric algebra $\mathrm{Sym}(\mathfrak{F}_{\mathrm{c}}(N)[1] \otimes \mathbb{C})$. The composition displayed above defines a time-ordered product, satisfying the axioms from the Definition 1.4.5 of a tPFA, as a straightforward consequence of the algebra structure of $\mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}(N)[1])$ and, in particular, of the graded commutativity of its multiplication μ_N. Since the functor $\mathfrak{F}_{\mathrm{c}}[1] \in \mathbf{Ch}_{\mathbb{R}}^{\mathbf{Loc}_m}$ is interpreted as assigning the linear observables of $(\mathsf{F}, Q, (-,-), W)$, the time-orderable prefactorization algebra $\mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}[1]) \in \mathbf{tPFA}_m$ may be interpreted as a description of the classical polynomial observables for the given free BV theory.

To quantize the classical tPFA $\mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}[1]) \in \mathbf{tPFA}_m$ on $\mathbf{Loc}_m$, we proceed by deforming, for all $M \in \mathbf{Loc}_m$, the differential $\mathcal{Q}_M$ of the cochain complex $\mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}(M)[1])$, generated by the differential $Q_{M[1]} = -Q_M$ of $\mathfrak{F}_{\mathrm{c}}(M)[1]$, by means of the *BV Laplacian*

$$\Delta_{\mathrm{BV}} := \Delta_{\tau_M^{(-1)}} \in [\mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}(M)[1]), \mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}(M)[1])]^1 \,, \tag{5.4}$$

which is the Laplacian associated with the (-1)-shifted Poisson structure $\tau_M^{(-1)}$ from (4.41) according to Definition 2.2.8. More explicitly, we define the degree increasing graded linear map

$$\mathcal{Q}_M^\hbar := \mathcal{Q}_M + \mathrm{i}\,\hbar \Delta_{\mathrm{BV}} \in [\mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}(M)[1]), \mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}(M)[1])]^1 \,, \tag{5.5}$$

where $\hbar > 0$ is the Planck's constant and $\mathrm{i} \in \mathbb{C}$ is the imaginary unit. First, let us note that $\mathcal{Q}_M^\hbar$ defines a new differential on the graded vector space underlying the free symmetric algebra $\mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}(M)[1])$. In fact, its square

$$(\mathcal{Q}_M^\hbar)^2 = \mathcal{Q}_M^2 + \mathrm{i}\,\hbar \partial \Delta_{\mathrm{BV}} - \hbar^2 \Delta_{\mathrm{BV}}^2 = 0 \tag{5.6}$$

vanishes. In the equation displayed above we used $\mathcal{Q}_M^2 = 0$ since it is a differential, $\partial \Delta_{\mathrm{BV}} = \Delta_{\partial \tau_M^{(-1)}} = 0$, which follows from Proposition 2.2.10 and from the identity $\partial \tau_M^{(-1)} = 0$, and $\Delta_{\mathrm{BV}}^2 = 0$, that is a consequence of Proposition 2.2.14. Therefore, we can replace the original differential $\mathcal{Q}_M$ with the deformed one $\mathcal{Q}_M^\hbar$ and define a cochain complex

$$\mathcal{F}(M) := (\mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}(M)[1]), \mathcal{Q}_M^\hbar) \in \mathbf{Ch}_{\mathbb{C}} \,, \tag{5.7}$$

which we interpret as the *complex of quantum observables*, for all $M \in \mathbf{Loc}_m$.

Remark 5.1.1. The triple $(\mathcal{F}(M), \mu_M, \{-,-\}_{(-1)})$, consisting of $\mathcal{F}(M) \in \mathbf{Ch}_{\mathbb{C}[[\hbar]]}$, regarded as a cochain complex of modules over the ring $\mathbb{C}[[\hbar]]$ of formal power series in a formal parameter $\hbar$, the graded-commutative multiplication μ_M inherited from the symmetric dg-algebra and the (-1)-shifted Poisson bracket $\{-,-\}_{(-1)} := \{-,-\}_{\tau_M^{(-1)}} = \mu_M \circ \{\{-,-\}\}_{\tau_M^{(-1)}}$ from Remark 2.2.5, has the structure of a *BD-algebra*, see [CG17]. To show this, one has to prove that the failure of the (quantum) differential $\mathcal{Q}_M^\hbar$ to be a derivation with respect to μ_M is witnessed by the Poisson bracket times the formal parameter $\hbar$, namely that the identity

$$\mathcal{Q}^{\hbar}_M \circ \mu_M = \mu_M \circ \mathcal{Q}^{\hbar}_{M\otimes} + \mathrm{i}\,\hbar\{-,-\}_{(-1)} \tag{5.8}$$

holds true. This identity is a consequence of $\mathcal{Q}_M$ being a derivation with respect to μ_M, together with item 3 of Definition 2.2.8 for the BV Laplacian Δ_{BV}. $\triangledown$

We now want to promote the assignment $M \mapsto \mathcal{F}(M)$ to a time-orderable prefactorization algebra $\mathcal{F} \in \mathbf{tPFA}_m$. To do so we need to introduce time-ordered products compatible with the quantized differential $\mathcal{Q}^{\hbar}_M$. The next result shows that we can just take the classical time-ordered products from (5.3b), since they are already compatible also with the new differential.

Proposition 5.1.2. *Let $(\mathsf{F}, Q, (-,-), W)$ be a covariant free BV theory and let $\mathcal{F}(M) \in \mathbf{Ch}_{\mathbb{C}}$ be the cochain complex defined by (5.7) for all $M \in \mathbf{Loc}_m$. Then, for all time-orderable tuples $\underline{f} = (f_1, \dots, f_n) : \underline{M} = (M_1, \dots, M_n) \to N$ in $\mathbf{Loc}_m$, the degree 0 graded linear map*

$$\begin{tikzcd} \bigotimes_i \mathcal{F}(M_i) =: \mathcal{F}(\underline{M}) \arrow[rr, dashed, "\mathcal{F}(\underline{f})"] \arrow[dr, "\bigotimes_i f_{i*}"'] & & \mathcal{F}(N) \\ & \mathcal{F}(N)^{\otimes n} \arrow[ur, "\mu_N^{(n)}"'] & \end{tikzcd} \tag{5.9}$$

is a cochain map, namely it is compatible with the quantum differential, $\mathcal{Q}^{\hbar}_N \mathcal{F}(\underline{f}) = \mathcal{F}(\underline{f}) \mathcal{Q}^{\hbar}_{\otimes}$.

Proof. Since the original differential $\mathcal{Q}$ is natural and compatible with the symmetric algebra multiplication, $\mathcal{Q}\mu = \mu \mathcal{Q}_{\otimes}$, it follows that $\mathcal{Q}_N \mathcal{F}(\underline{f}) = \mathcal{F}(\underline{f}) \mathcal{Q}_{\otimes}$. Since as a graded linear map $\mathcal{F}(\underline{f})$ coincides with the classical time-ordered product (5.3b), this is the same as saying that the latter is compatible with the original, unquantized, differential. To conclude, it is enough to show the analog identity $\Delta_{\mathrm{BV}} \mathcal{F}(\underline{f}) = \mathcal{F}(\underline{f}) \Delta_{\mathrm{BV}\otimes}$ concerning the BV Laplacian Δ_{BV}. Since the multiplication μ of the symmetric algebra is graded commutative, we can further reduce ourselves to the case of time-ordered tuples $\underline{f}$. We then argue by induction on the length n of $\underline{f}$.

For $n = 0$, $\mathcal{F}(\emptyset \to N) = \eta_N : \mathbb{C} \to \mathcal{F}(N)$ coincides with the unit of the symmetric algebra $\mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}(N)[1]) \in \mathbf{dgCAlg}_{\mathbb{C}}$. Then, the claim follows from the defining condition $\Delta_{\mathrm{BV}}(1) = 0$, item 1 of Definition 2.2.8.

For $n = 1$, $\mathcal{F}(f) = f_*$, hence the claim $\Delta_{\mathrm{BV}} f_* = f_* \Delta_{\mathrm{BV}}$ follows from the naturality of the (-1)-shifted Poisson structure $\tau^{(-1)}$, see (4.43), as a consequence of Proposition 2.2.11.

For $n = 2$, let us compute

$$\begin{aligned}\Delta_{\mathrm{BV}} \circ \mathcal{F}(f_1, f_2) &= \Delta_{\mathrm{BV}} \circ \mu_N \circ (f_{1*} \otimes f_{2*}) \\ &= \mu_N \circ (\Delta_{\mathrm{BV}\otimes} + \{\{-,-\}\}_{(-1)}) \circ (f_{1*} \otimes f_{2*}) \\ &= \mu_N \circ (f_{1*} \otimes f_{2*}) \circ \Delta_{\mathrm{BV}\otimes} \\ &= \mathcal{F}(f_1, f_2) \circ \Delta_{\mathrm{BV}\otimes}\,, \end{aligned} \tag{5.10}$$

where in the first and last steps we used the definition (5.9) of the time-ordered product $\mathcal{F}(f_1, f_2)$, the second step uses the item 3 of Definition 2.2.8, see also (2.35), where we set $\{\{-,-\}\}_{(-1)} := \{\{-,-\}\}_{\tau_N^{(-1)}}$. Finally, the third step uses the naturality of the BV Laplacian and the fact that $\{\{-,-\}\}_{(-1)} \circ (f_{1*} \otimes f_{2*}) = 0$ vanishes since $f_1(M_1) \cap f_2(M_2) = \emptyset$, because (f_1, f_2) is time-ordered, and $\tau_N^{(-1)}$ vanishes on sections with disjoint supports.

For $n \geq 3$, let us assume that the claim is satisfied for the tuples of length $n-1$. Given the time-ordered tuple $\underline{f}$ of length n, we consider the object $M \in \mathbf{Loc}_m$, the morphism $f : M \to N$ in $\mathbf{Loc}_m$ and the time-ordered tuple $\underline{f}' : (M_1, \dots, M_{n-1}) \to M$ in $\mathbf{Loc}_m$, provided by Lemma 5.0.1. We compute

$$\begin{aligned}\mathcal{F}(\underline{f}) = \mu_N^{(n)} \circ \bigotimes_{i=1}^{n} f_{i*} &= \mu_N^{(n)} \circ (f_* \otimes f_{n*}) \circ \Big(\Big(\mu_M^{(n-1)} \circ \bigotimes_{i=1}^{n-1} f'_{i*}\Big) \otimes \mathrm{id}\Big) \\ &= \mathcal{F}(f, f_n) \circ (\mathcal{F}(\underline{f}') \otimes \mathrm{id})\,, \end{aligned} \tag{5.11}$$

where in the first and last steps we used the definition (5.9) of the time-ordered products, and in the second step we used $\mu_N^{(n)} = \mu_N \circ (\mu_N^{(n-1)} \otimes \mathrm{id})$, $f_i = f \circ f'_i$ for all $i = 1, \dots n-1$ by construction, and the naturality of the symmetric algebra multiplication μ. Finally, one has

$$\begin{aligned}\Delta_{\mathrm{BV}} \circ \mathcal{F}(\underline{f}) &= \Delta_{\mathrm{BV}} \circ \mathcal{F}(f, f_n) \circ (\mathcal{F}(\underline{f}') \otimes \mathrm{id}) \\ &= \mathcal{F}(f, f_n) \circ (\Delta_{\mathrm{BV}} \mathcal{F}(\underline{f}') \otimes \mathrm{id} + \mathcal{F}(\underline{f}') \otimes \Delta_{\mathrm{BV}}) \\ &= \mathcal{F}(f, f_n) \circ (\mathcal{F}(\underline{f}') \circ \Delta_{\mathrm{BV}\otimes} \otimes \mathrm{id} + \mathcal{F}(\underline{f}') \otimes \Delta_{\mathrm{BV}}) \\ &= \mathcal{F}(f, f_n) \circ (\mathcal{F}(\underline{f}') \otimes \mathrm{id}) \circ \Delta_{\mathrm{BV}\otimes} && (5.12) \\ &= \mathcal{F}(\underline{f}) \circ, \Delta_{\mathrm{BV}} && (5.13) \end{aligned}$$

where in the second step is used the claim for the tuple (f, f_n) of length 2 and in the third one that for the tuple $\underline{f}'$ of length $n-1$ which holds because of the inductive hypothesis. □

Given these preparations, we can define the time-orderable prefactorization algebra $\mathcal{F} \in \mathbf{tPFA}_m$ by the following assignments:

i. To each $M \in \mathbf{Loc}_m$ it assigns the cochain complex $\mathcal{F}(M) \in \mathbf{Ch}_{\mathbb{C}}$ from (5.7) given by replacing the differential $\mathcal{Q}_M$ of the free symmetric algebra $\mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}(M)[1])$ by the quantized differential $\mathcal{Q}_M^{\hbar}$;

ii. To each time-orderable tuple $\underline{f} : \underline{M} \to N$ in $\mathbf{Loc}_m$ it assigns the time-ordered products $\mathcal{F}(\underline{f}) : \mathcal{F}(\underline{M}) \to \mathcal{F}(N)$ in $\mathbf{Ch}_{\mathbb{C}}$ from Proposition 5.1.2.

Note that the time-ordered products $\mathcal{F}(\underline{f})$ coincide, as graded liner maps, with the classical ones from (5.3b). Since the compatibilities (1-3) required by the Definition 1.4.5 of a tPFA boil down to the level of graded linear maps and graded vector spaces, it is immediate to conclude that the assignments above define a time-orderable prefactorization algebra since we have already noted that (5.3b) satisfy those axioms. Recall in particular that the latter is a consequence of the functoriality of $\mathfrak{F}_{\mathrm{c}}[1] : \mathbf{Loc}_m \to \mathbf{Ch}_{\mathbb{R}}$ and of the associativity, unitality, and commutativity of the multiplication μ of the symmetric algebra.

The following proposition shows that the time-orderable prefactorization algebra $\mathcal{F} \in \mathbf{tPFA}_m$, that describes the BV quantization of the covariant free BV theory $(\mathsf{F}, Q, (-,-), W)$, satisfies the homotopy time-slice axiom.

Proposition 5.1.3. *Let $\mathcal{F} \in \mathbf{tPFA}_m$ be the BV quantization time-orderable prefactorization algebra defined above. Then, for any Cauchy morphism $f : M \to N$ in $\mathbf{Loc}_m$, the time-ordered product $\mathcal{F}(f) : \mathcal{F}(M) \to \mathcal{F}(N)$ in $\mathbf{Ch}_{\mathbb{C}}$ is a quasi-isomorphism.*

Proof. Let $L \in \mathbf{Loc}_m$, we consider the filtration of the cochain complex $\mathcal{F}(L) = \mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}(L)[1]) = \bigoplus_{n\geq 0} \mathrm{Sym}^n(\mathfrak{F}_{\mathrm{c}}(L)[1] \otimes \mathbb{C}) \in \mathbf{Ch}_{\mathbb{C}}$ given by symmetric powers. More explicitly, for any integer $p \geq 0$ we denote the subcomplex of $\mathcal{F}(L)$ consisting of the symmetric powers up to the order p by

$$F_p(\mathcal{F}(L)) := \Big(\bigoplus_{n=0}^{p} \mathrm{Sym}^n(\mathfrak{F}_{\mathrm{c}}(L)[1] \otimes \mathbb{C}), \mathcal{Q}_L^{\hbar} \Big) \in \mathbf{Ch}_{\mathbb{C}} \,. \tag{5.14}$$

Note that the filtration is compatible with the deformed differential $\mathcal{Q}_L^{\hbar} = \mathcal{Q}_L + \mathrm{i}\,\hbar \Delta_{\mathrm{BV}}$ because the original differential $\mathcal{Q}_L$ preserves the symmetric power while the BV Laplacian Δ_{BV} lowers it by 2. See Definition 2.2.8 and Equation (2.27). Clearly this filtration is bounded from below, since $F_p(\mathcal{F}(L)) = 0$ vanishes for all $p < 0$. Moreover, the quotient maps $\mathcal{F}(L) \to \mathcal{F}(L)/F_p(\mathcal{F}(L))$ in $\mathbf{Ch}_{\mathbb{C}}$, for all $p \geq 0$, form a universal cone, namely $\mathcal{F}(L) \cong \lim_{p\in\mathbb{Z}} \mathcal{F}(L)/F_p(\mathcal{F}(L))$. This shows that the filtration is complete in the sense of [EM62]. Furthermore, for $p \geq 0$, the p-th component of the associated graded cochain complex

$$E_p^{\circ}(L) := F_p(\mathcal{F}(L))/F_{p-1}(\mathcal{F}(L)) \cong \mathrm{Sym}^p(\mathfrak{F}_{\mathrm{c}}(L)[1] \otimes \mathbb{C}) \in \mathbf{Ch}_{\mathbb{C}} \tag{5.15}$$

is isomorphic to the p-th symmetric power of $\mathfrak{F}_{\mathrm{c}}(L)[1] \otimes \mathbb{C}$, endowed with the original differential $\mathcal{Q}_L$. Indeed, the BV Laplacian Δ_{BV} does not contribute since it lowers the symmetric power by 2. Let $f : M \to N$ be any morphism in $\mathbf{Loc}_m$, the functoriality of the filtration (5.14) and the naturality of the isomorphisms (5.15) entail that the diagram

$$
\begin{tikzcd}
E_p^\circ(M) \arrow[r, "E_p^\circ(f_*)"] \arrow[d, "\cong"'] & E_p^\circ(N) \arrow[d, "\cong"] \\
\mathrm{Sym}^p(\mathfrak{F}_\mathrm{c}(M)[1] \otimes \mathbb{C}) \arrow[r, "f_*"'] & \mathrm{Sym}^p(\mathfrak{F}_\mathrm{c}(N)[1] \otimes \mathbb{C})
\end{tikzcd}
\tag{5.16}
$$

in $\mathbf{Ch}_\mathbb{C}$ commutes. When f is a Cauchy morphism the bottom cochain map is a quasi-isomorphism by Theorem 4.3.3, hence the cochain map $E_p^\circ(f_*)$ is a quasi-isomorphism too. It follows that it induces an isomorphism $E_p^1(f_*) : E_p^1(M) := \mathsf{H}^\bullet(E_p^\circ(M)) \xrightarrow{\cong} E_p^1(N) := \mathsf{H}^\bullet(E_p^\circ(N))$ on the first page of the spectral sequence. The claim then follows from [EM62, Thm. 7.4]. □

Remark 5.1.4. We would like to notice once more that the definition of the time-orderable prefactorization algebra $\mathcal{F} \in \mathbf{tPFA}_m$ from above relies only on the natural differential pairing $(-,-)$, through the natural (-1)-shifted Poisson structure $\tau^{(-1)}$ entering the definition of the BV Laplacian Δ_{BV}. Therefore, the existence of a natural Green's witness W, hence the Green hyperbolicity of the complexes $(\mathsf{F}(M), Q_M)$ for all $M \in \mathbf{Loc}_m$, is not needed to provide the BV quantization. Nevertheless, the existence of W is needed in Proposition 5.1.3 to prove that the BV quantization $\mathcal{F} \in \mathbf{tPFA}_m$ is homotopy Cauchy constant, since W was used in the proof of Theorem 4.3.3 to show that the pushforward cochain map f_* is a quasi-isomorphism. ▽

5.2 Moyal-Weyl star product defines an AQFT

We construct a $\mathbf{Ch}_\mathbb{C}$-valued algebraic quantum field theory $\mathcal{A} \in \mathbf{AQFT}_m$ on $\mathbf{Loc}_m$ that provides a quantization of the classical algebra of polynomial observables given by the free symmetric algebra $\mathrm{Sym}_\mathbb{C}(\mathfrak{F}_\mathrm{c}[1]) \in \mathbf{dgCAlg}_\mathbb{C}^{\mathbf{Loc}_m}$ generated by the complexification of the functor $\mathfrak{F}_\mathrm{c}[1] \in \mathbf{Ch}_\mathbb{R}^{\mathbf{Loc}_m}$ of linear observables of a given covariant free BV theory $(\mathsf{F}, Q, (-,-), W)$. First, let us note that the assignment to each manifold $M \in \mathbf{Loc}_m$ of the free symmetric algebra $\mathrm{Sym}_\mathbb{C}(\mathfrak{F}_\mathrm{c}(M)[1]) \in \mathbf{dgCAlg}_\mathbb{C}$ and to each morphism $f : M \to N$ in $\mathbf{Loc}_m$ of the symmetric algebra extension of the pushforward cochain map $f_* : \mathfrak{F}_\mathrm{c}(M)[1] \to \mathfrak{F}_\mathrm{c}(N)[1]$ in $\mathbf{Ch}_\mathbb{R}$ defines a functor $\mathrm{Sym}_\mathbb{C}(\mathfrak{F}_\mathrm{c}[1]) : \mathbf{Loc}_m \to \mathbf{dgCAlg}_\mathbb{C}$ to the category of dg-commutative algebras. This follows immediately from the functoriality of $\mathfrak{F}_\mathrm{c}[1] : \mathbf{Loc}_m \to \mathbf{Ch}_\mathbb{R}$ and of the free symmetric algebra $\mathrm{Sym}_\mathbb{C} : \mathbf{Ch}_\mathbb{C} \to \mathbf{dgCAlg}_\mathbb{C}$. The canonical quantization of the free classical algebra may be realized by deforming, for all $M \in \mathbf{Loc}_m$, the commutative multiplication μ_M of the symmetric algebra

$\mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}(M)[1]) \in \mathbf{dgCAlg}_{\mathbb{C}}$ to the, in general, non-commutative *Moyal-Weyl star product*. It is given by

$$\begin{tikzcd}
\mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}(M)[1])^{\otimes 2} \arrow[rr, dashed, "\mu_M^\hbar"] \arrow[dr, "\exp(\frac{\mathrm{i}\,\hbar}{2}\{\{-,-\}\})"'] & & \mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}(M)[1]) \\
& \mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}(M)[1])^{\otimes 2} \arrow[ur, "\mu_M"'] &
\end{tikzcd} \tag{5.17}$$

where $\{\{-,-\}\} := \{\{-,-\}\}_{\tau_M}$ denotes the degree preserving graded endomorphism associated with the unshifted Poisson structure τ_M from (4.35), see Definition 2.2.3. Note that all elements $a, b \in \mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}(M)[1])$ are polynomials and that each application of the cochain map $\{\{-,-\}\}$ lowers by 1 the symmetric power of both. This entails that only a finite number of iterations of $\{\{-,-\}\}$ on $a \otimes b$ is not vanishing, hence the exponential series defining $\mu_M^\hbar(a \otimes b)$ actually truncates to a finite sum. In particular, there is no need to regard the Planck's constant $\hbar > 0$ as a formal parameter.

To be sure that replacing the multiplication μ_M with the Moyal-Weyl star product $\mu_M^\hbar$ makes the cochain complex $\mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}(M)[1])$ into a dg-algebra, we have to check that the latter is associative, unital and compatible with the differential $\mathcal{Q}_M$ of $\mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}(M)[1])$. Associativity and unitality with respect to the unit $\eta_M : \mathbb{C} = \mathrm{Sym}^0_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}(M)[1]) \hookrightarrow \mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}(M)[1])$ readily follow from the properties of the graded endomorphism $\{\{-,-\}\}$ and of the exponential and from the associativity and unitality of μ_M. Finally, we check that the Moyal-Weyl star product $\mu_M^\hbar$ from (5.17) is a cochain map. We compute its 'adjoint' differential,

$$\begin{aligned}
\partial \mu_M^\hbar &= \mu_M \circ \partial \exp\Big(\frac{\mathrm{i}\,\hbar}{2}\{\{-,-\}\}\Big) \\
&= \mu_M \circ \sum_{n \geq 1} \frac{1}{n!}\Big(\frac{\mathrm{i}\,\hbar}{2}\Big)^n \sum_{k=0}^{n-1} \{\{-,-\}\}^k \circ \partial\{\{-,-\}\} \circ \{\{-,-\}\}^{n-k-1} \\
&= 0\,,
\end{aligned} \tag{5.18}$$

where in the first step we used the definition of $\mu_M^\hbar$ and that the algebra multiplication μ_M, is compatible with the differential $\mathcal{Q}_M$ of $\mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}(M)[1])$, $\partial \mu_M = 0$, the second step expands the exponential series and applies the Leibniz rule for ∂ with respect to the composition $\circ$. Finally, in the last step we used that $\partial\{\{-,-\}\} = \{\{-,-\}\}_{\partial \tau_M} = 0$ vanishes. Here we applied the Proposition 2.2.4 and recalled that τ_M is a cochain map, $\partial \tau_M = 0$. From these preparations, it follows that

$$\mathcal{A}(M) := (\mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}(M)[1]), \mu_M^\hbar, \eta_M) \in \mathbf{dgAlg}_{\mathbb{C}} \tag{5.19}$$

defines a differential graded algebra, whose multiplication is given by the deformed Moyal-Weyl star product.

Remark 5.2.1. We note that the Moyal-Weyl star product $\mu_M^\hbar$ from (5.17) is a, in general, non-commutative deformation of the commutative multiplication μ_M of the symmetric algebra $\mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}(M)[1])$. To be more precise, we mean that the multiplications

$$\mu_M^\hbar = \mu_M \circ \sum_{n\geq 0} \frac{1}{n!}\Big(\frac{\mathrm{i}\,\hbar}{2}\Big)^n \{\{-,-\}\}^n = \mu_M + \mathcal{O}(\hbar) \tag{5.20}$$

differ by terms of order at least $\hbar$ and that the (graded) commutator associated to $\mu_M^\hbar$

$$[-,-]_{\mu_M^\hbar} := \mu_M^\hbar - \mu_M^\hbar \circ \gamma = \mathrm{i}\,\hbar\{-,-\} + \mathcal{O}(\hbar^2) \tag{5.21}$$

is proportional to the Poisson bracket $\{-,-\} := \mu_M \circ \{\{-,-\}\}$, see Remark 2.2.5, up to terms of order at least $\hbar^2$, as a consequence of (graded) commutativity of the multiplication μ_M and of (graded) antisymmetry of the pairing $\{\{-,-\}\}$, which in turn follows from the (graded) antisymmetry of the unshifted Poisson structure τ_M, see item 2 of Definition 2.2.3. $\triangledown$

In order to define an AQFT out of the algebras of quantum observables (5.19), we first have to upgrade the assignment $\mathbf{Loc}_m \ni M \mapsto \mathcal{A}(M) \in \mathbf{dgAlg}_{\mathbb{C}}$ to a functor. From previous arguments we already know that the assignment of the underlying cochain complexes $\mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}(M)[1]) \in \mathbf{Ch}_{\mathbb{C}}$ is functorial and also that the unit η is compatible with morphisms in $\mathbf{Loc}_m$. It remains to check the compatibility of the Moyal-Weyl star product $\mu^\hbar$ with respect to the morphisms $f : M \to N$ in $\mathbf{Loc}_m$. Recalling Proposition 2.2.7 and the naturality (4.36) of the unshifted Poisson structure τ, it follows that the graded endomorphism $\{\{-,-\}\}$ is compatible with f, i.e.

$$\{\{-,-\}\} \circ (f_* \otimes f_*) = (f_* \otimes f_*) \circ \{\{-,-\}\}\,, \tag{5.22}$$

where f_* is the symmetric algebra extension of the pushforward cochain map $f_* : \mathfrak{F}_{\mathrm{c}}(M)[1] \to \mathfrak{F}_{\mathrm{c}}(N)[1]$ in $\mathbf{Ch}_{\mathbb{R}}$. This, together with the naturality of the symmetric algebra multiplication μ, yields that the Moyal-Weyl star product is compatible with the morphism $f : M \to N$ in $\mathbf{Loc}_m$, namely

$$\mu_N^\hbar \circ (f_* \otimes f_*) = \mu_N \circ (f_* \otimes f_*) \circ \exp\Big(\frac{\mathrm{i}\,\hbar}{2}\{\{-,-\}\}\Big) = f_* \circ \mu_M^\hbar\,. \tag{5.23}$$

We are now allowed to define the functor

$$\mathcal{A} : \mathbf{Loc}_m \longrightarrow \mathbf{dgAlg}_{\mathbb{C}}\,, \tag{5.24}$$

given by the assignment of the differential graded algebra $\mathcal{A}(M) \in \mathbf{dgAlg}_{\mathbb{C}}$ from (5.19) to any object $M \in \mathbf{Loc}_m$ and of the morphism $\mathcal{A}(f) : \mathcal{A}(M) \to \mathcal{A}(N)$ in $\mathbf{dgAlg}_{\mathbb{C}}$, whose underlying cochain map is the symmetric algebra extension of the pushforward cochain map $f_* : \mathfrak{F}_{\mathrm{c}}(M)[1] \to \mathfrak{F}_{\mathrm{c}}(N)[1]$ in $\mathbf{Ch}_{\mathbb{C}}$, to any morphism $f : M \to N$ in $\mathbf{Loc}_m$. With the next proposition we show that the functor $\mathcal{A}$ is an AQFT which, moreover, satisfies the homotopy time-slice axiom.

Proposition 5.2.2. *Let* $(\mathsf{F}, Q, (-,-), W)$ *be a covariant free BV theory. Let* $\mathcal{A} : \mathbf{Loc}_m \to \mathbf{dgAlg}_{\mathbb{C}}$ *denote the functor valued in dg-algebras provided by* (5.24). *Then,* $\mathcal{A}$ *satisfies the Einstein causality axiom, hence* $\mathcal{A} \in \mathbf{AQFT}_m$ *is an AQFT according to Definition 1.4.1. Furthermore, it satisfies the homotopy time-slice axiom from Definition 1.4.8.*

Proof. Let us start by proving the Einstein causality axiom, hence that the functor $\mathcal{A} \in \mathbf{dgAlg}_{\mathbb{C}}^{\mathbf{Loc}_m}$ from (5.24) is an AQFT. Consider a pair $f_1 : M_1 \to N \leftarrow M_2 : f_2$ of causally disjoint morphisms in $\mathbf{Loc}_m$. According to item i of Theorem 4.3.3 we have that the composition

$$\tau_N \circ (f_{1*} \otimes f_{2*}) = 0 \tag{5.25}$$

vanishes. The Definition 2.2.3 of the pairing $\{\{-,-\}\} = \{\{-,-\}\}_{\tau_N}$ associated to the unshifted Poisson structure τ_N from (4.35) yields that also the composition

$$\{\{-,-\}\} \circ (f_{1*} \otimes f_{2*}) = 0 \tag{5.26}$$

vanishes identically. Exploiting the definition (5.17) of the Moyal-Weyl star product, one has that on the image of the map $f_{1*} \otimes f_{2*}$ the deformed product $\mu_N^\hbar$

$$\mu_N^\hbar \circ (f_{1*} \otimes f_{2*}) = \mu_N \circ (f_{1*} \otimes f_{2*}) \tag{5.27}$$

coincides with the symmetric algebra multiplication μ_N. Since the latter is (graded) commutative the Einstein causality axiom (1.50) follows.

Let us now check the homotopy time-slice axiom. Let $f : M \to N$ in $\mathbf{Loc}_m$ be a Cauchy morphism. The item ii of Theorem 4.3.3 yields that the cochain map $f_* : \mathfrak{F}_{\mathrm{c}}(M)[1] \to \mathfrak{F}_{\mathrm{c}}(N)[1]$ in $\mathbf{Ch}_{\mathbb{R}}$ is a quasi-isomorphism. Since the underlying cochain map of the morphism $\mathcal{A}(f) : \mathcal{A}(M) \to \mathcal{A}(N)$ in $\mathbf{dgAlg}_{\mathbb{C}}$ is by definition the symmetric algebra extension of the pushforward cochain map f_*, it follows that it is a quasi-isomorphism, hence the homotopy time-slice axiom is fulfilled. □

Remark 5.2.3. In contrast with what we noticed in Remark 5.1.4 for the tPFA $\mathcal{F} \in \mathbf{tPFA}_m$, the definition of the AQFT $\mathcal{A} \in \mathbf{AQFT}_m$ from (5.24) requires the entire information provided by the covariant free BV theory $(\mathsf{F}, Q, (-,-), W)$. In particular, since the quantized product $\mu_M^\hbar$ of the dg-algebra $\mathcal{A}(M) \in \mathbf{dgAlg}_{\mathbb{C}}$ is built out of the extended pairing $\{\{-,-\}\}$, hence, ultimately, of the unshifted Poisson structure τ_M from (4.35), we need the complexes of differential operators $(\mathsf{F}(M), Q_M)$ to be Green hyperbolic for all $M \in \mathbf{Loc}_m$. Furthermore, for $\mathcal{A}$ to be a strict functor, it is crucial that the Moyal-Weyl star product $\mu^\hbar$ is compatible with the pushforward cochain map f_*, see Equation (5.23). This turns out to be a consequence of the naturality (4.36) of the unshifted Poisson structure τ, which is, in turn, a consequence of the dinaturality (4.25) of the retarded-minus-advanced cochain map Λ. It follows that the existence of a natural Green's witness W, and thus the ability to choose strictly natural retarded and advanced Green's homotopies in a suitably natural fashion with respect to morphisms in $\mathbf{Loc}_m$, plays a crucial role in the definition of the AQFT $\mathcal{A} \in \mathbf{AQFT}_m$. $\triangledown$

5.3 Comparison

In this section we will consider a covariant free BV theory $(\mathsf{F}, Q, (-,-), W)$ and we will provide you with a comparison between the two different quantization schemes available in this setting and presented in Section 5.1 and 5.2. More precisely, we will establish an isomorphism of time-orderable prefactorization algebras between the time-orderable prefactorization algebra $\mathcal{F} \in \mathbf{tPFA}_m$ from Section 5.1 describing the BV quantization of the covariant free BV theory and the time-orderable prefactorization algebra $\mathcal{F}_{\mathcal{A}} \in \mathbf{tPFA}_m$ canonically associated with the algebraic quantum field theory $\mathcal{A} \in \mathbf{AQFT}_m$ from Section 5.2 constructed using the Moyal-Weyl star product. For the convenience of the reader, here we recall from Theorem 1.4.11 and from [BPS20] the functor $\mathcal{F}_{(-)} : \mathbf{AQFT}_m \to \mathbf{tPFA}_m$ which assigns to each AQFT the associated tPFA of its time-ordered products. To be more concrete, let us consider directly its evaluation on the Moyal-Weyl star product quantization $\mathcal{A} \in \mathbf{AQFT}_m$ and let us describe it explicitly. The tPFA $\mathcal{F}_{\mathcal{A}} \in \mathbf{tPFA}_m$ is given by the following data:

i. The assignment to each object $M \in \mathbf{Loc}_m$ of the cochain complex $\mathcal{F}_{\mathcal{A}}(M) := \mathcal{A}(M) = \mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}(M)[1]) \in \mathbf{Ch}_{\mathbb{C}}$ underlying the dg-algebra $\mathcal{A}(M) \in \mathbf{dgAlg}_{\mathbb{C}}$,

ii. The assignment to each time-orderable n-tuple $\underline{f} : \underline{M} \to N$ in $\mathbf{Loc}_m$ of the cochain map given by the composition

$$\begin{tikzcd} \mathcal{F}_{\mathcal{A}}(\underline{M}) \arrow[r, dashed, "\mathcal{F}_{\mathcal{A}}(\underline{f})"] \arrow[d, "\gamma_\rho"'] & \mathcal{F}_{\mathcal{A}}(N) \\ \mathcal{F}_{\mathcal{A}}(\underline{M}\rho) \arrow[r, "\underline{f}_*\rho"'] & \mathcal{F}_{\mathcal{A}}(N)^{\otimes n} \arrow[u, "\mu_N^{\hbar(n)}"'] \end{tikzcd} \tag{5.28}$$

in $\mathbf{Ch}_{\mathbb{C}}$, where ρ is a time-ordering permutation for $\underline{f}$, γ_ρ is the cochain map that permutes the tensor factors of $\mathcal{F}_{\mathcal{A}}(\underline{M})$ according to the time-ordering permutation ρ by means of the symmetric braiding γ in $\mathbf{Ch}_{\mathbb{C}}$, $\underline{f}_*\rho := \bigotimes_{i=1}^n f_{\rho(i)*}$ is given by the tensor product of the pushforward cochain maps $f_{i*} : \mathcal{A}(M_i) \to \mathcal{A}(N)$ for $i = 1, \ldots, n$ in the order assigned by ρ, and $\mu_N^{\hbar(n)}$ denotes the n-ary Moyal-Weyl star product in the given order. Finally, recall that the construction from Theorem 1.4.11 assigns to the empty tuple $\emptyset \to N$ in $\mathbf{Loc}_m$ the unit $\eta_N : \mathbb{C} \to \mathcal{A}(N)$.

Let us point out that the time-ordered products (5.28) of the tPFA $\mathcal{F}_{\mathcal{A}} \in \mathbf{tPFA}_m$ canonically associated with $\mathcal{A} \in \mathbf{AQFT}_m$ coincide with the standard time-ordered products built out of the quantum product of the AQFT $\mathcal{A}$ which are usually considered in the (mathematical) physics literature. Indeed, they are obtained by first ordering their arguments so that the i-th argument 'comes later' than all the subsequent ones (more precisely, it does 'not come earlier' than the subsequent ones, $J_N^+(f_i(M_i)) \cap f_j(M_j) = \emptyset$ for all $j > i$), and then by taking their product in the non-commutative quantized algebra.

The next lemma gives an alternative description of the time-ordered products (5.28) making clear that they capture the usual time-ordered products built out of the *Dirac multiplication* μ_M^D, that is the degree preserving graded linear map defined by the composition

$$
\begin{tikzcd}
\mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}(M)[1])^{\otimes 2} \arrow[rr, dashed, "\mu_M^D"] \arrow[dr, "\exp(\,\mathrm{i}\,\hbar\{\{-,-\}\}_D)"'] & & \mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}(M)[1]) \\
& \mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}(M)[1])^{\otimes 2} \arrow[ur, "\mu_M"'] &
\end{tikzcd}
\tag{5.29}
$$

of degree preserving graded linear maps, where $\{\{-,-\}\}_D := \{\{-,-\}\}_{\tau_M^D}$ denotes the degree preserving graded endomorphism associated with the Dirac pairing τ_M^D from (4.38), according to Definition 2.2.3. First, note that similarly to the case of the Moyal-Weyl star product $\mu_M^\hbar$ from (5.17) the power series defining the exponential actually truncates to a finite sum since both the arguments are polynomials and applying the graded endomorphism $\{\{-,-\}\}_D$ lowers the symmetric powers of both of them by 1. This is consistent with not regarding $\hbar$ as a formal parameter. Let us observe that the Dirac multiplication μ_M^D is *not* compatible with the differential $\mathcal{Q}_M$ of $\mathcal{F}_{\mathcal{A}}(M)$. In fact, recall from Proposition 2.2.4 that the 'adjoint' differential of the graded endomorphism $\{\{-,-\}\}_D$ is computed by

$$
\partial\{\{-,-\}\}_D = \{\{-,-\}\}_{\partial\tau_M^D} = \{\{-,-\}\}_{(-1)}\,, \tag{5.30}
$$

where the second step used that $\partial\tau_M^D = \tau_M^{(-1)}$, see Equation (4.44). Nonetheless, it is an associative product, unital with respect to the unit $\eta_M : \mathbb{C} \to \mathcal{F}_{\mathcal{A}}(M)$, as a consequence of the properties of the graded endomorphism $\{\{-,-\}\}_D$ and of the exponential series. Moreover, it is commutative as a consequence of the commutativity of the symmetric algebra product μ_M and of the symmetry of the Dirac pairing τ_M^D, see (4.42), together with item 1 of Definition 2.2.3. Finally, naturality of the Dirac pairing τ^D from (4.38) and of the symmetric algebra multiplication μ, together with Proposition 2.2.7, imply that the Dirac multiplications μ_M^D from (5.29) for $M \in \mathbf{Loc}_m$ arrange themselves as the components of a 0-cochain

$$
\mu^D \in \underline{\mathrm{hom}}(\mathcal{F}_{\mathcal{A}}^{\otimes 2}, \mathcal{F}_{\mathcal{A}})^0 \tag{5.31}
$$

in the enriched hom of the functor category $\mathbf{Ch}_{\mathbb{C}}^{\mathbf{Loc}_m}$. Note that here we are regarding the tPFA $\mathcal{F}_{\mathcal{A}} \in \mathbf{tPFA}_m$ as a functor $\mathcal{F}_{\mathcal{A}} : \mathbf{Loc}_m \to \mathbf{Ch}_{\mathbb{C}}$ by simply forgetting the time-ordered products of arity greater than 1.

Lemma 5.3.1. *Consider the tPFA $\mathcal{F}_{\mathcal{A}} \in \mathbf{tPFA}_m$ canonically associated to the AQFT $\mathcal{A} \in \mathbf{AQFT}_m$ from Section 5.2. Then, for all time-orderable n-tuple $\underline{f} : \underline{M} \to N$ in $\mathbf{Loc}_m$, the time-ordered product $\mathcal{F}_{\mathcal{A}}(\underline{f})$ can be computed using the Dirac multiplication μ^D from (5.31). Namely, the triangle*

$$\begin{tikzcd} \mathcal{F}_{\mathcal{A}}(\underline{M}) \arrow[rr, "\mathcal{F}_{\mathcal{A}}(\underline{f})"] \arrow[dr, "\underline{f_*}"'] & & \mathcal{F}_{\mathcal{A}}(N) \\ & \mathcal{F}_{\mathcal{A}}(N)^{\otimes n} \arrow[ur, "\mu_N^{D(n)}"'] & \end{tikzcd} \tag{5.32}$$

of degree preserving graded linear maps commutes.

Proof. Let us observe that since the Dirac multiplication μ_N^D is commutative and $\mathcal{F}_{\mathcal{A}} \in \mathbf{tPFA}$ is a time-orderable prefactorization algebra, hence its time-ordered products are equivariant with respect to permutations, item 3 of Definition 1.4.5, we can restrict ourselves to consider only n-tuple $\underline{f}$ that are time-ordered. Assume the claim has been proven for time-ordered n-tuples. Then, if $\underline{f}$ is time-orderable but not time-ordered, it exists a time-ordering permutation ρ such that $\underline{f}\rho$ is time-ordered. Therefore, by exploiting the equivariance of the time-ordered products, one finds

$$\mathcal{F}_{\mathcal{A}}(\underline{f}) = \mathcal{F}_{\mathcal{A}}(\underline{f}\rho) \circ \gamma_\rho = \mu_N^{D(n)} \circ \underline{f_*}\rho \circ \gamma_\rho = \mu_N^{D(n)} \circ \gamma_\rho \circ \underline{f_*} = \mu_N^{D(n)} \circ \underline{f_*}\,, \tag{5.33}$$

where the last step follows from the commutativity of μ_N^D.

Therefore, let us consider a time-ordered n-tuple $\underline{f}$ in $\mathbf{Loc}_m$ and let us argue by induction on its length n. For $n = 0$ and $n = 1$ there is nothing to prove. Indeed, $\mathcal{F}_{\mathcal{A}}(\emptyset \to N) = \eta_N = \mu_N^{D(0)}$ and $\mathcal{F}_{\mathcal{A}}(f) = \mu_N^{\hbar(1)} \circ f_* = f_* = \mu_N^{D(1)} \circ f_*$, for all morphisms f in $\mathbf{Loc}_m$.

For $n = 2$, $\underline{f} = (f_1, f_2) : \underline{M} \to N$, Proposition 4.3.4 yields that

$$\{\{-,-\}\}_D^k \circ (f_{1*} \otimes f_{2*}) = \left(\frac{1}{2}\{\{-,-\}\}\right)^k \circ (f_{1*} \otimes f_{2*})\,, \tag{5.34}$$

for all integers $k \geq 1$, as a direct inspection would show. Then, one computes

$$\begin{aligned} \mu_N^D \circ (f_{1*} \otimes f_{2*}) &= \mu_N \circ \exp(\mathrm{i}\,\hbar \{\{-,-\}\}_D) \circ (f_{1*} \otimes f_{2*}) \\ &= \mu_N \circ \exp\left(\frac{\mathrm{i}\,\hbar}{2}\{\{-,-\}\}\right) \circ (f_{1*} \otimes f_{2*}) \\ &= \mathcal{F}_{\mathcal{A}}(f_1, f_2)\,, \end{aligned} \tag{5.35}$$

where in the first step we used the definition (5.29) of the Dirac multiplication, the second step follows from applying (5.35) in the power series expansion of the exponential and, finally, the last step is the definition of the time-ordered product $\mathcal{F}_{\mathcal{A}}(f_1, f_2)$, see (5.28) and (5.17). Recall that the pair (f_1, f_2) is time-ordered.

For $n \geq 3$, assume that (5.32) holds true for any time-ordered tuple of length $n-1$. Let then $\underline{f} : (M_1, \dots, M_n) \to N$ be a time-ordered tuple in $\mathbf{Loc}_m$ of length n. Then, consider the manifold $M \in \mathbf{Loc}_m$, the morphism $f : M \to N$ in $\mathbf{Loc}_m$ and the time-ordered tuple $\underline{f'} : (M_1, \dots, M_{n-1}) \to M$ in $\mathbf{Loc}_m$ provided by Lemma 5.0.1. One computes

$$\mu_N^{D(n)} \circ \underline{f_*} = \mu_N^D \circ (f_* \otimes f_{n*}) \circ \big((\mu_M^{D(n-1)} \circ \underline{f'_*}) \otimes \mathrm{id} \big) \,, \tag{5.36}$$

by exploiting $\mu_N^{D(n)} = \mu_N^D \circ (\mu_N^{D(n-1)} \otimes \mathrm{id})$, the naturality of the Dirac multiplication (5.31) and that $f_i = f \circ f'_i$, for all $i = 1, \dots, n-1$, by construction. Therefore, the claim for length n follows from the ones for lengths 2 and $n-1$,

$$\mu_N^{D(n)} \circ \underline{f_*} = \mathcal{F}_{\mathcal{A}}(f, f_n) \circ (\mathcal{F}_{\mathcal{A}}(\underline{f'}) \otimes \mathrm{id}) = \mathcal{F}_{\mathcal{A}}(\underline{f}) \,, \tag{5.37}$$

where the last step follows from properties of the time-ordered products of a tPFA, item 1 in Definition 1.4.5. □

We are finally ready to state and prove our comparison result between the Moyal-Weyl and the BV quantization schemes. A crucial role in the proof will be played by the alternative description from Lemma 5.3.1 of the time-ordered products $\mathcal{F}_{\mathcal{A}}$ in terms of the Dirac multiplication μ^D.

Theorem 5.3.2. *Consider a covariant free BV theory* $(\mathsf{F}, Q, (-,-), W)$ *on* $\mathbf{Loc}_m$. *Then, the time-orderable prefactorization algebra* $\mathcal{F} \in \mathbf{tPFA}_m$ *(constructed in Section 5.1 via the BV formalism) and the time-orderable prefactorization algebra* $\mathcal{F}_{\mathcal{A}} \in \mathbf{tPFA}_m$ *canonically associated with the AQFT* $\mathcal{A} \in \mathbf{AQFT}_m$ *(constructed in Section 5.2 through the Moyal-Weyl star product) are isomorphic. The explicit isomorphism is given by the time-ordering map*

$$T := \exp(\mathrm{i}\,\hbar \Delta_D) : \mathcal{F} \longrightarrow \mathcal{F}_{\mathcal{A}} \tag{5.38}$$

in $\mathbf{tPFA}_m$, *where* $\Delta_D := \Delta_{\tau^D}$ *denotes the Laplacian associated with the Dirac pairing* τ^D *from (4.38) according to Definition 2.2.8, which we called the Dirac Laplacian.*

Proof. First, note that if T defined as in Equation (5.38) is a morphism of time-orderable prefactorization algebras, then it is also an isomorphism. In fact, it admits an explicit inverse given by $T^{-1} := \exp(-\mathrm{i}\,\hbar\Delta_D) : \mathcal{F}_{\mathcal{A}} \to \mathcal{F}$ in $\mathbf{tPFA}_m$, as one can easily see exploiting the properties of the exponential. Therefore, all we have to do is to prove that T is a morphism in $\mathbf{tPFA}_m$. This check consists of two parts: First, we have to prove that all components $T_M : \mathcal{F}(M) \to \mathcal{F}_{\mathcal{A}}(M)$, for $M \in \mathbf{Loc}_m$, are morphisms in $\mathbf{Ch}_{\mathbb{C}}$, namely they are compatible with differentials. Second, we have to show that they are compatible with the time-ordered products, namely that, for all time-orderable tuple $\underline{f} : \underline{M} \to N$ in $\mathbf{Loc}_m$, $T_N \circ \mathcal{F}(\underline{f}) = \mathcal{F}_{\mathcal{A}}(\underline{f}) \circ T_{\underline{M}}$.

Compatibility with differentials. Let $M \in \mathbf{Loc}_m$ and consider the degree preserving graded linear map $T_M = \exp(\mathrm{i}\,\hbar\Delta_D) : \mathcal{F}(M) \to \mathcal{F}_{\mathcal{A}}(M)$. Recall that the differential of the cochain complex $\mathcal{F}_{\mathcal{A}}(M)$ is given by the differential $\mathcal{Q}_M$ of the symmetric algebra $\mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}(M)[1])$, while that of the BV quantization $\mathcal{F}(M)$ is given by $\mathcal{Q}^{\hbar}_M := \mathcal{Q}_M + \mathrm{i}\,\hbar\Delta_{\mathrm{BV}}$, where $\Delta_{\mathrm{BV}} := \Delta_{\tau^{(-1)}_M}$ is the BV Laplacian, see Definition 2.2.3 and (4.41). Therefore we have to show that the difference

$$\mathcal{Q}_M \circ T_M - T_M \circ (\mathcal{Q}_M + \mathrm{i}\,\hbar\Delta_{\mathrm{BV}}) = \widetilde{\partial} T_M - \mathrm{i}\,\hbar T_M \circ \Delta_{\mathrm{BV}}\,, \tag{5.39}$$

vanishes, where $\widetilde{\partial}$ is the differential of $[\mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}(M)[1]), \mathrm{Sym}_{\mathbb{C}}(\mathfrak{F}_{\mathrm{c}}(M)[1])] \in \mathbf{Ch}_{\mathbb{C}}$. Recalling also the Dirac Laplacian $\Delta_D := \Delta_{\tau^D_M}$ given by Definition 2.2.3 and (4.38), one computes

$$\begin{aligned}
\widetilde{\partial} T_M &= \sum_{n\geq 1}\sum_{k=0}^{n-1} \frac{(\mathrm{i}\,\hbar)^n}{n!} \Delta_D^k \circ \widetilde{\partial}\Delta_D \circ \Delta_D^{n-k-1} \\
&= \sum_{n\geq 1}\sum_{k=0}^{n-1} \frac{(\mathrm{i}\,\hbar)^n}{n!} \Delta_D^k \circ \Delta_{\mathrm{BV}} \circ \Delta_D^{n-k-1} \\
&= \sum_{n\geq 1} \frac{(\mathrm{i}\,\hbar)^n}{(n-1)!} \Delta_D^{n-1} \circ \Delta_{\mathrm{BV}} \\
&= \mathrm{i}\,\hbar T_M \circ \Delta_{\mathrm{BV}}\,,
\end{aligned} \tag{5.40}$$

where in the first step we expanded the exponential series in the definition of T_M and exploited the Leibniz rule for $\widetilde{\partial}$ with respect to composition. In the second step we used that $\widetilde{\partial}\Delta_D = \Delta_{\partial\tau^D_M} = \Delta_{\mathrm{BV}}$, see Proposition 2.2.10 and (4.44), and in the third one that the BV and the Dirac Laplacians commute, $\Delta_{\mathrm{BV}} \circ \Delta_D = \Delta_D \circ \Delta_{\mathrm{BV}}$, see Proposition 2.2.14. Last step follows from the definition (5.38) of T_M. This shows that Equation (5.39) vanishes, thus proving that $T_M : \mathcal{F}(M) \to \mathcal{F}_{\mathcal{A}}(M)$ is a cochain map.

Compatibility with time-ordered products. Since $\mathcal{F}, \mathcal{F}_{\mathcal{A}} \in \mathbf{tPFA}_m$ are time-orderable prefactorization algebras, their time-ordered products are equivariant with respect to permutations, see item 3 in Definition 1.4.5. It follows that it is enough to check the compatibility of T with respect to the time-ordered products associated to time-ordered tuples $\underline{f} : \underline{M} \to N$ in $\mathbf{Loc}_m$. We argue by induction on the length n of the time-ordered tuple $\underline{f}$.

For $n = 0$, it is trivial since $T_N(1) = 1 + \sum_{k\geq 1} \frac{(\mathrm{i}\,\hbar)^k}{k!} \Delta_D^k(1) = 1$, as a consequence of item 1 in Definition 2.2.8.

For $n = 1$, it is a direct consequence of the naturality of the Dirac pairing τ^D and of Proposition 2.2.11, that yield $f_* \circ \Delta_D = \Delta_D \circ f_*$ for each $f : M \to N$ in $\mathbf{Loc}_m$.

For $n = 2$, consider a time-ordered pair $(f_1, f_2) : (M_1, M_2) \to N$ in $\mathbf{Loc}_m$ and compute

$$
\begin{aligned}
T_N \circ \mathcal{F}(f_1, f_2) &= \sum_{k \geq 0} \frac{(\,\mathrm{i}\,\hbar)^k}{k!} \Delta_D^k \circ \mu_N \circ (f_{1*} \otimes f_{2*}) \\
&= \sum_{k \geq 0} \sum_{j=0}^{k} \frac{(\,\mathrm{i}\,\hbar)^k}{j!(k-j)!} \mu_N \circ \{\{-,-\}\}_D^j \circ \Delta_{D\otimes}^{k-j} \circ (f_{1*} \otimes f_{2*}) \\
&= \mu_N \circ \exp(\,\mathrm{i}\,\hbar \{\{-,-\}\}_D) \circ \exp(\,\mathrm{i}\,\hbar \Delta_{D\otimes}) \circ (f_{1*} \otimes f_{2*}) \\
&= \mu_D \circ (T_N \otimes T_N) \circ (f_{1*} \otimes f_{2*}) \\
&= \mathcal{F}_{\mathcal{A}}(f_1, f_2) \circ (T_{M_1} \otimes T_{M_2}) \,, \qquad (5.41)
\end{aligned}
$$

where in the first step we used the definitions of the time-ordering map T_N and of the time-ordered product $\mathcal{F}(f_1, f_2)$, the second step exploited Proposition 2.2.13. In the third step we reordered the series and in the fourth one we exploited the definitions of the Dirac multiplication μ_D and the trivial fact that $\exp(\,\mathrm{i}\,\hbar \Delta_D) \otimes \mathrm{id}$ and $\mathrm{id} \otimes \exp(\,\mathrm{i}\,\hbar \Delta_D)$ commute. Finally, the last step used that the claim holds for $n = 1$ and Lemma 5.3.1.

For $n \geq 3$, let us assume that the claim hold for all time-ordered tuples of length $n - 1$ and let $\underline{f} : (M_1, \dots, M_n) \to N$ be a time-ordered n-tuple in $\mathbf{Loc}_m$. From Lemma 5.0.1, we get $M \in \mathbf{Loc}_m$, $f : M \to N$ in $\mathbf{Loc}_m$ and a time-ordered tuple $\underline{f}' : (M_1, \dots, M_{n-1}) \to M$ in $\mathbf{Loc}_m$ of length $n - 1$. Therefore, one computes

$$
\begin{aligned}
T_N \circ \mathcal{F}(\underline{f}) &= T_N \circ \mathcal{F}(f, f_n) \circ (\mathcal{F}(\underline{f}') \otimes \mathrm{id}) \\
&= \mathcal{F}_{\mathcal{A}}(f, f_n) \circ (\mathcal{F}_{\mathcal{A}}(\underline{f}') \otimes \mathrm{id}) \circ T_{\underline{M}} \\
&= \mathcal{F}_{\mathcal{A}}(\underline{f}) \circ T_{\underline{M}} \,. \qquad (5.42)
\end{aligned}
$$

The first and last steps used the compatibility of time-ordered products with compositions and that $f \circ f_i' = f_i$ for all $i = 1, \dots, n-1$ by construction. The second step used the claim for the tuples of length 2 and $n - 1$. □

Remark 5.3.3. Let us conclude by observing that although the two approaches require different input data, namely the time-ordered prefactorization algebra $\mathcal{F} \in \mathbf{tPFA}_m$ given by the BV formalism exists for any triple $(\mathsf{F}, Q, (-,-))$ of a covariant complex of natural differential operators and a natural differential pairing, while the AQFT $\mathcal{A} \in \mathbf{AQFT}_m$ requires also a natural Green's witness W, cf. Remarks 5.1.4 and 5.2.3, when a natural Green's witness is provided, hence both approaches are available, the previous result allows one to compare them. In particular, it shows that the information encoded by the BV quantization $\mathcal{F} \in \mathbf{tPFA}_m$ is the same as that encoded by the time-ordered products of the canonical quantization $\mathcal{A} \in \mathbf{AQFT}_m$ of the same covariant free BV theory. ▽

Chapter 6

Examples

In this chapter we shall provide some examples for the general theory developed in the previous chapters. They all describe classical and quantum field theories which are of interest in physics and in mathematical physics. In Section 6.1 we consider the case of an ordinary (i.e. non-gauge) free field theory, in Section 6.2 we consider the Abelian Chern-Simons theory and in Section 6.3 we describe the higher gauge field theory of Maxwell p-forms, which in particular includes linear Yang-Mills theory for $p = 1$. These examples recollect those provided in [BMS23; BMS24], which partially overlap with earlier descriptions appeared in [BBS20; AB23].

In order to simplify the constructions and allow the reader to browse the examples below in parallel with the reading of the main part of the thesis, we will introduce the required input data and the main constructions following the same order we adopted in the previous chapters. More explicitly, we will stick to the following scheme:

- We introduce the complex of linear differential operators (F, Q) for the theory on a fixed manifold $M \in \mathbf{Loc}_m$, following Definition 3.1.1.
- We provide it with a differential pairing $(-, -)$, see Definition 3.3.1.
- We introduce also a Green's witness W for the complex of differential operators at hand, Definition 4.1.1, and we note that it is formally self-adjoint with respect to the previously introduced differential pairing, see Definition 4.2.1.
- This allows us to prove that the complex of differential operators for the example is Green hyperbolic by introducing concrete advanced and retarded Green's homotopies $\Lambda^{\pm}$ of it, see Theorem 4.1.5. Furthermore, we provide you with the simplified retarded-minus-advanced quasi isomorphism Λ from (4.14) and the simplified covariant Poisson structure τ_M from Proposition 4.2.4 and fixed-time Poisson structure σ_Σ from Proposition 4.2.6, for $\Sigma \subseteq M$ a spacelike Cauchy surface.

- We upgrade the previous data from a fixed manifold $M \in \mathbf{Loc}_m$ to a covariant free BV theory $(\mathsf{F}, Q, (-,-), W)$ in $\mathbf{Loc}_m$, see Definition 4.3.2. This allows us to upgrade τ_M to the natural Poisson structure τ from (4.35) and to introduce the natural Dirac pairing τ^D from (4.38) and the natural (-1)-shifted Poisson structure $\tau^{(-1)}$ from (4.41).

6.1 Ordinary free field theory

A first example is provided by ordinary free field theories, as opposed to theories which admit non-trivial gauge symmetries. An ordinary free field theory is described by a pair (E, P) consisting of a vector bundle $(E \to M) \in \mathbb{K}\text{-}\mathbf{VBnd}_m$ over the manifold $M \in \mathbf{Loc}_m$ and of a linear differential operator $P : \Gamma(E) \to \Gamma(E)$ acting on smooth section of the bundle E. As already discussed in Example 3.1.2, we associate with these data the complex of differential operators $(F_{(E,P)}, Q_{(E,P)})$ consisting of the graded vector bundle $F_{(E,P)} \to M$ over M concentrated in degrees 0 and 1, defined by $F^n_{(E,P)} := E$ for $n = 0, 1$, and of the differential $Q_{(E,P)}$, whose only non-vanishing component $Q^0_{(E,P)} := P : \Gamma(F^0_{(E,P)}) \to \Gamma(F^1_{(E,P)})$ coincides with the operator P ruling the dynamics of the theory. Let $\mathscr{D} \subseteq \mathrm{cl}$ be a (non-empty) directed subset of the directed set cl of the closed subset of M. The associated cochain complex of sections with a $\mathscr{D}$-support explicitly reads as

$$\mathfrak{F}_{(E,P)\mathscr{D}}(M) = \Big(\overset{(0)}{\Gamma_{\mathscr{D}}(E)} \xrightarrow{\;P\;} \overset{(1)}{\Gamma_{\mathscr{D}}(E)}\Big) \in \mathbf{Ch}_{\mathbb{K}}\,. \tag{6.1}$$

The 0-cochains $\phi \in \mathfrak{F}^0_{(E,P)}(M) = \Gamma(E)$ are the fields of the theory, while the 1-cochains $\phi^\ddagger \in \mathfrak{F}^1_{(E,P)}(M) = \Gamma(E)$ are interpreted as the corresponding antifields.

Let us restrict to the field $\mathbb{K} = \mathbb{R}$ of real numbers. Assume, moreover, that the vector bundle E is endowed with a fiber metric $\langle -,- \rangle : \mathsf{E} \otimes \mathsf{E} \to \underline{\mathbb{R}}$ and that the differential operator P is of the form $P = \Box^\nabla + B$, where ∇ is a metric connection on E and B is a symmetric endomorphism of E. Note that all formally self-adjoint normally hyperbolic linear differential operators are such, see [BGP07, Lemma 1.5.5]. These additional assumptions allow us to endow the complex $(F_{(E,P)}, Q_{(E,P)})$ with the differential pairing

$$(-,-)_{(E,P)} : \mathfrak{F}_{(E,P)}(M)^{\otimes 2} \longrightarrow \Omega^\bullet(M)[m-1] \tag{6.2a}$$

whose only non-vanishing components are defined by

$$(\phi_1, \phi_2)^0_{(E,P)} := (-1)^m \langle \phi_1 \wedge *\nabla\phi_2 - \phi_2 \wedge *\nabla\phi_1 \rangle\,, \tag{6.2b}$$

$$(\phi^\ddagger, \phi)^1_{(E,P)} := \langle \phi^\ddagger \wedge *\phi \rangle : -(\phi, \phi^\ddagger)^1_{(E,P)}\,, \tag{6.2c}$$

for all $\phi, \phi_1, \phi_2 \in \mathfrak{F}^0_{(E,P)}(M) = \Gamma(E)$ and $\phi^\ddagger \in \mathfrak{F}^1_{(E,P)}(M)$, where $*$ denotes the Hodge star operator built out of the metric of M. Note that $(-,-)_{(E,P)}$ is manifestly a bi-differential operator and it is antisymmetric: $(\phi_1, \phi_2)_{(E,P)} = -(\phi_2, \phi_1)_{(E,P)}$ by direct inspection and $(\phi^\ddagger, \phi)_{(E,P)} = -(\phi, \phi^\ddagger)_{(E,P)}$ by definition. To show the compatibility with differential, compute

$$\begin{aligned}
(-,-)_{(E,P)} \circ Q_{(E,P)\otimes}(\phi_1 \otimes \phi_2) &= (P\phi_1, \phi_2)_{(E,P)} + (\phi_1, P\phi_2)_{(E,P)} \\
&= \langle (\Box^\nabla + B)\phi_1 \wedge *\phi_2 \rangle - \langle (\Box^\nabla + B)\phi_2 \wedge *\phi_1 \rangle \\
&= \mathrm{d}_{\mathrm{dR}} \langle \nabla\phi_1 \wedge *\phi_2 - \nabla\phi_2 \wedge *\phi_1 \rangle \\
&= \mathrm{d}_{\mathrm{dR}[m-1]} (\phi_1, \phi_2)_{(E,P)} \,.
\end{aligned} \tag{6.3}$$

In the first step we used the definition of the differential $Q_{(E,P)\otimes}$ on the tensor product. The second step used our assumption on the form of the differential operator $P = \Box^\nabla + B$ and the definition (6.2c) of the pairing. In the third step we exploited the symmetry of B and that ∇ is a metric connection. Last step used the definition (6.2b) of the pairing $(-,-)_{(E,P)}$. This finally proves that (6.2) defines a differential pairing.

A Green's witness $W_{(E,P)}$ for $(F_{(E,P)}, Q_{(E,P)})$ is given by the degree decreasing linear map whose only non-trivial component

$$W^1_{(E,P)} := \mathrm{id} : \mathfrak{F}^1_{(E,P)}(M) = \Gamma(E) \longrightarrow \mathfrak{F}^0_{(E,P)}(M) = \Gamma(E) \tag{6.4}$$

is the identity on the smooth sections of the vector bundle $E \to M$. This defines a Green's witness for $(F_{(E,P)}, Q_{(E,P)})$ since $W^1_{(E,P)} \circ Q^0_{(E,P)} = \mathrm{id} \circ P = P$ and $Q^0_{(E,P)} \circ W^1_{(E,P)} = P \circ \mathrm{id} = P$ are Green hyperbolic operators by hypothesis. Furthermore, the Green's witness $W_{(E,P)}$ is formally self-adjoint with respect to the differential pairing $(-,-)_{(E,P)}$ above. Indeed, condition 1 of Definition 4.2.1 is trivially satisfied since $(W_{(E,P)})^2 = 0$. Condition 2 of the same definition follows from the following computation: Let $\phi^\ddagger_1, \phi^\ddagger_2 \in \mathfrak{F}^1_{(E,P)}(M) = \Gamma(E)$ be sections with compact overlapping support, then

$$\begin{aligned}
\int_M (W_{(E,P)}\phi^\ddagger_1, \phi^\ddagger_2)_{(E,P)} &= -\int_M \langle \phi^\ddagger_2 \wedge *\phi^\ddagger_1 \rangle = -\int_M \langle \phi^\ddagger_1 \wedge *\phi^\ddagger_2 \rangle \\
&= -\int_M (\phi^\ddagger_1, W_{(E,P)}\phi^\ddagger_2)_{(E,P)} \,,
\end{aligned} \tag{6.5}$$

where in the first and last step we used (6.2c), recall that $W_{(E,P)}\phi^\ddagger_i = \phi^\ddagger_i \in \mathfrak{F}^0_{(E,P)}(M)$ regards $\phi^\ddagger_i$ as a 0-cochain, for $i = 1, 2$. Therefore, Theorem 4.1.5 yields that the complex $(F_{(E,P)}, Q_{(E,P)})$ is Green hyperbolic. (Note that $(F_{(E,P)}, Q_{(E,P)})$ is a Green hyperbolic complex for all Green hyperbolic operators P as shown in Example 3.1.7, regardless of further assumptions.)

Let $G^\pm_P : \Gamma_{\substack{\mathrm{spc}\\\mathrm{sfc}}}(E) \to \Gamma_{\substack{\mathrm{spc}\\\mathrm{sfc}}}(E)$ be the retarded/advanced Green's operator of the Green hyperbolic operator P. According to Theorem 4.1.5, a choice of retarded/advanced Green's homotopies for $(F_{(E,P)}, Q_{(E,P)})$ is given by the (-1)-cochain

$$\Lambda^{\pm}_{(E,P)} \in \underline{\hom}(\mathfrak{F}_{(E,P)J^{\pm}_M(-)}(M), \mathfrak{F}_{(E,P)J^{\pm}_M(-)}(M))^{-1} \tag{6.6a}$$

in the enriched hom, whose only non-vanishing components are given by

$$\begin{aligned} (\Lambda^{\pm}_{(E,P)})^1_K : \Gamma_{J^{\pm}_M(K)}(E) &\longrightarrow \Gamma_{J^{\pm}_M(K)}(E) \\ \phi^{\ddagger} &\longmapsto (\Lambda^{\pm}_{(E,P)})^1_K \phi^{\ddagger} := G^{\pm}_P \phi^{\ddagger} \,, \end{aligned} \tag{6.6b}$$

for each compact subset $K \subseteq M$ and section $\phi^{\ddagger} \in \mathfrak{F}^1_{(E,P)J^{\pm}_M(K)}(M) = \Gamma_{J^{\pm}_M(K)}(E)$. This choice provide us with the simplified retarded-minus-advanced quasi-isomorphism described by the commutative square (4.14). It is given explicitly by the cochain map

$$\Lambda_{(E,P)} : \mathfrak{F}_{(E,P)\mathrm{c}}(M)[1] \longrightarrow \mathfrak{F}_{(E,P)\mathrm{sc}}(M) \tag{6.7a}$$

in $\mathbf{Ch}_{\mathbb{R}}$, whose unique non-vanishing component reads as

$$\begin{aligned} \Lambda^0_{(E,P)} : \Gamma_{\mathrm{c}}(E) &\longrightarrow \Gamma_{\mathrm{sc}}(E) \\ \epsilon &\longmapsto \Lambda^0_{(E,P)} \epsilon := G_P \epsilon \,, \end{aligned} \tag{6.7b}$$

for each compactly supported section $\epsilon \in \mathfrak{F}_{(E,P)\mathrm{c}}(M)[1]^0 = \mathfrak{F}^1_{(E,P)\mathrm{c}}(M) = \Gamma_{\mathrm{c}}(E)$, where $G_P := G^+_P - G^-_P$ denotes the retarded-minus-advanced propagator of P. The associated unshifted Poisson structure

$$\tau_{(E,P)M} : \mathfrak{F}_{(E,P)\mathrm{c}}(M)[1]^{\otimes 2} \longrightarrow \mathbb{R} \,, \tag{6.8a}$$

defined on the ordinary colimit as per Proposition 4.2.4, does not vanish only when evaluated on 0-cochains $\epsilon_1 \otimes \epsilon_2$ in the tensor product, $\epsilon_i \in \mathfrak{F}_{(E,P)\mathrm{c}}(M)[1]^0 = \Gamma_{\mathrm{c}}(E)$, for $i = 1, 2$. More explicitly,

$$\tau_{(E,P)M}(\epsilon_1 \otimes \epsilon_2) := \int_M (\epsilon_1, \Lambda^0_{(E,P)} \epsilon_2)_{(E,P)} = \int_M \langle \epsilon_1 \wedge * G_P \epsilon_2 \rangle \,. \tag{6.8b}$$

This coincides with the standard covariant Poisson structure from (1.45). Furthermore, let $\Sigma \overset{\iota}{\hookrightarrow} M$ be a spacelike Cauchy surface. Then the fixed-time Poisson structure

$$\sigma_{(E,P)\Sigma} : \mathfrak{F}_{(E,P)\mathrm{sc}}(M)^{\otimes 2} \longrightarrow \mathbb{R} \,, \tag{6.9a}$$

defined on the ordinary colimit according to Proposition 4.2.6, does not vanish only on the 0-cochains $\phi_1 \otimes \phi_2$ in the tensor product, $\phi_i \in \mathfrak{F}^0_{(E,P)\mathrm{sc}}(M) = \Gamma_{\mathrm{sc}}(E)$, for $i = 1, 2$, where it evaluates to

$$\begin{aligned} \sigma_{(E,P)\Sigma}(\phi_1 \otimes \phi_2) &:= (-1)^{m-1} \int_\Sigma \iota^* (\phi_1, \phi_2)_{(E,P)} \\ &= - \int_\Sigma \iota^* \langle \phi_1 \wedge * \nabla \phi_2 - \phi_2 \wedge * \nabla \phi_1 \rangle \,. \end{aligned} \tag{6.9b}$$

Let us observe that it recovers the standard fixed-time Poisson structure from (1.47).

We would like to notice that the field cochain complex $\mathfrak{F}_{(E,P)}(M)$ has a non-vanishing cohomology only in degree 0, $\mathsf{H}^0\mathfrak{F}_{(E,P)}(M) = \ker P$. (The 1-st cohomology $\Gamma(E)/P\Gamma(E) \cong 0$ is trivial since P is Green hyperbolic, hence it is surjective.) Since each cochain complex is quasi-isomorphic to its cohomology, see (2.10), we have that the field complex $\mathfrak{F}_{(E,P)}(M)$ is weakly equivalent to the cochain complex given by the kernel $\ker P$ concentrated in degree 0

$$\mathfrak{F}_{(E,P)}(M) = \Big(\overset{(0)}{\Gamma(E)} \xrightarrow{\;P\;} \overset{(1)}{\Gamma(E)}\Big) \sim \Big(\overset{(0)}{\ker P}\Big)\,. \tag{6.10}$$

This shows that $\mathfrak{F}_{(E,P)}(M)$ recovers the usual space $\ker P$ of on-shell fields of the ordinary free field theory ruled by P, up to a quasi-isomorphism.

It is immediate to upgrade the above constructions to a covariant free BV theory. Indeed, consider a natural $\mathbb{R}$-vector bundle E on $\mathbf{Loc}_m$ endowed with a natural fiber metric $\langle -,- \rangle : \mathsf{E} \otimes \mathsf{E} \to \underline{\mathbb{R}}$ and a natural linear differential operator $P : \mathfrak{E} \to \mathfrak{E}$, whose M-component $P_M : \Gamma(E_M) \to \Gamma(E_M)$ is a normally hyperbolic operator, for each $M \in \mathbf{Loc}_m$. The arguments above show, in particular, that the quadruple $(\mathsf{F}_{(\mathsf{E},P)}, Q_{(\mathsf{E},P)}, (-,-)_{(\mathsf{E},P)}, W_{(\mathsf{E},P)})$, given by $\mathsf{F}_{(\mathsf{E},P)}(M) := F_{(E_M,P_M)}$, $Q_{(\mathsf{E},P)M} := Q_{(E_M,P_M)}$, $(-,-)_{(\mathsf{E},P)M} := (-,-)_{(E_M,P_M)}$ and $W_{(\mathsf{E},P)M} := W_{(E_M,P_M)}$, is a covariant free BV theory. The unshifted Poisson structure (6.8) upgrades to the natural unshifted Poisson structure

$$\tau_{(\mathsf{E},P)} : \mathfrak{F}_{(\mathsf{E},P)\mathrm{c}}[1]^{\otimes 2} \longrightarrow \Delta\mathbb{R}\,, \tag{6.11a}$$

that is given, for each $M \in \mathbf{Loc}_m$, by the cochain map

$$\tau_{(\mathsf{E},P)M} := \tau_{(E_M,P_M)M} : \mathfrak{F}_{(E_M,P_M)\mathrm{c}}(M)[1]^{\otimes} \longrightarrow \mathbb{R}\,, \tag{6.11b}$$

in $\mathbf{Ch}_{\mathbb{R}}$, whose only non-vanishing component is obtained upon evaluation on the 0-cochain $\epsilon_1 \otimes \epsilon_2 \in \Gamma_\mathrm{c}(E_M)^{\otimes 2}$,

$$\tau_{(\mathsf{E},P)M}(\epsilon_1 \otimes \epsilon_2) = \int_M \langle \epsilon_1 \wedge *_M G_{P_M} \epsilon_2 \rangle_M\,. \tag{6.11c}$$

Similarly, we have also the natural Dirac pairing from (4.38),

$$\tau^D_{(\mathsf{E},P)} \in \underline{\mathrm{hom}}(\mathfrak{F}_{(\mathsf{E},P)\mathrm{c}}[1]^{\otimes 2}, \Delta\mathbb{R})^0 \tag{6.12a}$$

whose non-vanishing contributions are

$$\tau^D_{(\mathsf{E},P)M}(\epsilon_1 \otimes \epsilon_2) := \int_M \langle \epsilon_1 \wedge *_M G^D_{P_M} \epsilon_2 \rangle_M\,, \tag{6.12b}$$

for all $M \in \mathbf{Loc}_m$ and $\epsilon_i \in \mathfrak{F}_{(E_M,P_M)\mathrm{c}}(M)[1]^0 = \Gamma_\mathrm{c}(E_M)$, for $i = 1, 2$. Here, $G^D_{P_M} := \frac{1}{2}(G^+_{P_M} + G^-_{P_M})$ is the Dirac propagator of the normally hyperbolic operator P_M. Finally, we have the natural (-1)-shifted Poisson structure

$$\tau^{(-1)}_{(\mathsf{E},P)} : \mathfrak{F}_{(\mathsf{E},P)\mathrm{c}}[1]^{\otimes 2} \longrightarrow \Delta\mathbb{R}[1] \tag{6.13a}$$

from (4.41), given for each $M \in \mathbf{Loc}_m$ by the cochain map

$$\tau^{(-1)}_{(\mathsf{E},P)M} : \mathfrak{F}_{(E_M,P_M)\mathrm{c}}(M)[1]^{\otimes 2} \longrightarrow \mathbb{R}[1] \tag{6.13b}$$

in $\mathbf{Ch}_{\mathbb{R}}$, which does not vanish only upon evaluation on the (-1)-cochains because of degree reasons,

$$\tau^{(-1)}_{(\mathsf{E},P)M}(\epsilon^\ddagger \otimes \epsilon) := -\int_M \langle \epsilon^\ddagger \wedge *\epsilon \rangle =: \tau^{(-1)}_{(\mathsf{E},P)M}(\epsilon \otimes \epsilon^\ddagger) \tag{6.13c}$$

for all compactly supported sections $\epsilon^\ddagger \in \mathfrak{F}_{(E_M,P_M)\mathrm{c}}(M)[1]^{-1} = \Gamma_{\mathrm{c}}(E_M)$ and $\epsilon \in \mathfrak{F}_{(E_M,P_M)\mathrm{c}}(M)[1]^{0} = \Gamma_{\mathrm{c}}(E_M)$.

Let us observe that this class of examples includes the free Klein-Gordon theory of mass $\mathbf{m} \geq 0$ by choosing as the natural vector bundle on $\mathbf{Loc}_m$ $\mathsf{E}_{\mathrm{KG}} := \underline{\mathbb{R}}$ the natural trivial line bundle, $\underline{\mathbb{R}}(M) = M \times \mathbb{R} \to M$, as the natural differential operator P the Klein-Gordon operator $P_M := \Box + \mathbf{m}^2$, whose naturality is a consequence of the fact that the morphisms in $\mathbf{Loc}_m$ are isometries, and as the natural fiber metric $\langle -,- \rangle$ the canonical fiber metric given component-wise by the multiplication on $\mathbb{R}$.

6.2 Abelian Chern-Simons theory

In this section let $m = 3$. A first example of a non-trivial gauge field theory is provided by Abelian Chern-Simons theory. Fix a 3-dimensional manifold $M \in \mathbf{Loc}_3$. The Abelian Chern-Simons theory on M is described by the complex of differential operators $(F_{\mathrm{CS}}, Q_{\mathrm{CS}}) := (\Lambda^\bullet M[1], \mathrm{d}_{\mathrm{dR}[1]})$ given by the 1-shift of the de Rham complex on M, see Example 3.1.3. The nihilpotency of the differential Q_{CS} follows from that of the de Rham differential d_{dR}. Given a (non-empty) directed subset $\mathscr{D} \subseteq \mathrm{cl}$ of the directed set cl of closed subsets of M, the cochain complex of $\mathscr{D}$-supported sections associated to the Abelian Chern-Simons theory on M is explicitly given by the cochain complex

$$\mathfrak{F}_{\mathrm{CS}\mathscr{D}}(M) = \Big(\overset{(-1)}{\Omega^0_{\mathscr{D}}(M)} \xrightarrow{-\mathrm{d}_{\mathrm{dR}}} \overset{(0)}{\Omega^1_{\mathscr{D}}(M)} \xrightarrow{-\mathrm{d}_{\mathrm{dR}}} \overset{(1)}{\Omega^2_{\mathscr{D}}(M)} \xrightarrow{-\mathrm{d}_{\mathrm{dR}}} \overset{(2)}{\Omega^3_{\mathscr{D}}(M)} \Big) . \tag{6.14}$$

The signs in the differential come from our convention on shifts, see (2.13). Homogeneous elements in $\mathfrak{F}_{\mathrm{CS}}(M)$ have the following field theoretical interpretation: The 0-cochains $A \in \mathfrak{F}^0_{\mathrm{CS}}(M) = \Omega^1(M)$ are the gauge fields, the (-1)-cochains $c \in \mathfrak{F}^{-1}_{\mathrm{CS}}(M) = \Omega^0(M)$ are the ghost fields, while the 1-cochains $A^\ddagger \in \mathfrak{F}^1_{\mathrm{CS}}(M) = \Omega^2(M)$ and the 2-cochains $c^\ddagger \in \mathfrak{F}^2_{\mathrm{CS}}(M) = \Omega^3(M)$ are the antifields of the gauge and of the ghost fields, respectively.

The complex of differential operators $(F_{\mathrm{CS}}, Q_{\mathrm{CS}})$ may be endowed with a differential pairing $(-,-)_{\mathrm{CS}} : \mathfrak{F}_{\mathrm{CS}}(M)^{\otimes 2} \to \Omega^\bullet(M)[2]$ in $\mathbf{Ch}_{\mathbb{R}}$ arising as a suitably shifted version of the graded commutative wedge multiplication $\wedge : \Omega^\bullet(M)^{\otimes 2} \to \Omega^\bullet(M)$. More explicitly, the non-vanishing contributions, which are concentrated in degrees $n = -2, \dots, 1$ for degree reasons, are

$$\begin{aligned}
(c_1, c_2)^{-2}_{\mathrm{CS}} &:= c_1 \wedge c_2\,, \\
(A, c)^{-1}_{\mathrm{CS}} &:= -A \wedge c =: -(c, A)^{-1}_{\mathrm{CS}}\,, \\
(A^\ddagger, c)^{0}_{\mathrm{CS}} &:= A^\ddagger \wedge c =: (c, A^\ddagger)^{0}_{\mathrm{CS}}\,, \\
(A_1, A_2)^{0}_{\mathrm{CS}} &:= -A_1 \wedge A_2\,, \\
(c^\ddagger, c)^{1}_{\mathrm{CS}} &:= -c^\ddagger \wedge c =: -(c, c^\ddagger)^{1}_{\mathrm{CS}}\,, \\
(A^\ddagger, A)^{1}_{\mathrm{CS}} &:= A^\ddagger \wedge A =: -(A, A^\ddagger)^{1}_{\mathrm{CS}}\,,
\end{aligned} \tag{6.15}$$

for all $c, c_1, c_2 \in \mathfrak{F}^{-1}_{\mathrm{CS}}(M) = \Omega^0(M)$, $A, A_1, A_2 \in \mathfrak{F}^{0}_{\mathrm{CS}}(M) = \Omega^1(M)$, $A^\ddagger \in \mathfrak{F}^{1}_{\mathrm{CS}}(M) = \Omega^2(M)$ and $c^\ddagger \in \mathfrak{F}^{2}_{\mathrm{CS}}(M) = \Omega^3(M)$. The signs appearing in the formulae (6.15) are due to our convention on the shift. The pairing $(-,-)_{\mathrm{CS}}$ is clearly a bi-differential operator (of order 0 in both entries) and it is manifestly graded antisymmetric as a consequence of the graded commutativity of the multiplication $\wedge$ (with respect to the differential form degree) and of the definition. If we denote by $|\omega|_{\mathrm{dR}} := p$ the degree of $\omega \in \Omega^p(M)$ in the de Rham complex, the pairing $(-,-)_{\mathrm{CS}}$ may be expressed more compactly as $(\omega_1, \omega_2)_{\mathrm{CS}} = (-1)^{|\omega_1|_{\mathrm{dR}}} \omega_1 \wedge \omega_2$, for all homogeneous $\omega_1, \omega_2 \in \mathfrak{F}_{\mathrm{CS}}(M)$. This makes easy to check the compatibility with differentials. Indeed, one computes

$$\begin{aligned}
(-,-)_{\mathrm{CS}} \circ Q_{\mathrm{CS}\otimes}(\omega_1 \otimes \omega_2) &= (\mathrm{d}_{\mathrm{dR}[1]}\omega_1, \omega_2)_{\mathrm{CS}} + (-1)^{|\omega_1|}(\omega_1, \mathrm{d}_{\mathrm{dR}[1]}\omega_2)_{\mathrm{CS}} \\
&= (-1)^{|\omega_1|_{\mathrm{dR}}}(\mathrm{d}_{\mathrm{dR}}\omega_1) \wedge \omega_2 + \omega_1 \wedge (\mathrm{d}_{\mathrm{dR}}\omega_2) \\
&= (-1)^{|\omega_1|_{\mathrm{dR}}}\mathrm{d}_{\mathrm{dR}}(\omega_1 \wedge \omega_2) \\
&= \mathrm{d}_{\mathrm{dR}[2]}(\omega_1, \omega_2)_{\mathrm{CS}}\,.
\end{aligned} \tag{6.16}$$

The first step used the definition of the differential of a tensor product, see (2.3), the second and last steps used the definition of $(-,-)_{\mathrm{CS}}$. In the third step we used that the de Rham differential d_{dR} is a graded derivation with respect to the $\wedge$ multiplication and the de Rham gradings. This shows that $(-,-)_{\mathrm{CS}}$ is compatible with differentials, hence it is a differential pairing.

Since $M \in \mathbf{Loc}_3$ is an oriented Lorentzian manifold, we can build out of its orientation and metric the codifferential $\delta_{\mathrm{dR}} : \Omega^p(M) \to \Omega^{p-1}(M)$. It is the key ingredient of the Green's witness W_{CS}. We define $W_{\mathrm{CS}} \in [\mathfrak{F}_{\mathrm{CS}}(M), \mathfrak{F}_{\mathrm{CS}}(M)]^{-1}$ as the degree decreasing graded linear map whose components which are not necessarily zero are given by the 1-shift of the codifferential, $W^n_{\mathrm{CS}} := -\delta_{\mathrm{dR}}$ for $n = 0, 1, 2$. Since the components of W_{CS} are manifestly linear differential operators, in order to show that it is a Green's witness for $(F_{\mathrm{CS}}, Q_{\mathrm{CS}})$ it is enough to check that its 'adjoint' differential ∂W_{CS} :

$\mathfrak{F}_{\mathrm{CS}}(M) \to \mathfrak{F}_{\mathrm{CS}}(M)$ in $\mathbf{Ch}_{\mathbb{R}}$ is a degree-wise Green hyperbolic operator. This is immediate since one finds that $(\partial W_{\mathrm{CS}})^n = \mathrm{d}_{\mathrm{dR}}\delta_{\mathrm{dR}} + \delta_{\mathrm{dR}}\mathrm{d}_{\mathrm{dR}} = \Box$ is the Laplace-de Rham operator acting on the $(n+1)$-forms, for $n = -1, \dots, 2$. Since the Laplace-de Rham operator $\Box$ is normally hyperbolic, hence in particular Green hyperbolic, see Example 1.3.10, we conclude. Moreover, the Green witness W_{CS} is also formally self-adjoint with respect to the differential pairing $(-,-)_{\mathrm{CS}}$ in the sense of Definition 4.2.1: Item 1 in the definition is trivially satisfied since $(W_{\mathrm{CS}})^2 = 0$ as a consequence of the fact that the codifferential $\delta_{\mathrm{dR}}^2 = 0$ squares to zero. The formal self-adjointness with respect to the integrated differential pairing from item 2 is proved as follows. Let $\omega_1, \omega_2 \in \mathfrak{F}_{\mathrm{CS}}(M)$ be homogeneous sections with compact overlapping support, then

$$\begin{aligned}
\int_M (W_{\mathrm{CS}}\omega_1, \omega_2)_{\mathrm{CS}} &= -(-1)^{|\omega_1|} \int_M (\delta_{\mathrm{dR}}\omega_1) \wedge \omega_2 \\
&= -(-1)^{|\omega_1|} \int_M \omega_1 \wedge *\mathrm{d}_{\mathrm{dR}} *^{-1} \omega_2 \\
&= -(-1)^{|\omega_1|} (-1)^{|\omega_2|_{\mathrm{dR}}} \int_M \omega_1 \wedge \delta_{\mathrm{dR}}\omega_2 \\
&= (-1)^{|\omega_1|} \int_M (\omega_1, W_{\mathrm{CS}}\omega_2)_{\mathrm{CS}} \,,
\end{aligned} \tag{6.17}$$

where in the first step we used the definitions of the Green's witness W_{CS} and of the differential pairing $(-,-)_{\mathrm{CS}}$, the second step used that the codifferential δ_{dR} is the formal adjoint of the de Rham differential d_{dR} with respect to integral pairing $\int_M - \wedge *-$. Third step used the definition of the codifferential $\delta_{\mathrm{dR}} = (-1)^p *^{-1} \mathrm{d}_{\mathrm{dR}}*$ acting on differential p-forms. Recall that, since M is of odd dimension and the metric is of Lorentzian signature, $*^{-1} = -*$. Last step used the definition of the Green's witness W_{CS} and of the differential pairing $(-,-)_{\mathrm{CS}}$. Note that for degree reasons $|\omega_1| + |\omega_2| = 2$.

Since $(F_{\mathrm{CS}}, Q_{\mathrm{CS}})$ is endowed with a (formally self-adjoint) Green's witness W_{CS}, Theorem 4.1.5 yields that the complex of differential operators associated with the Abelian Chern-Simons theory is Green hyperbolic. Moreover, a special choice of retarded/advanced Green's homotopies for it is provided by the (-1)-cochain

$$\Lambda_{\mathrm{CS}}^{\pm} \in \underline{\mathrm{hom}}(\mathfrak{F}_{\mathrm{CS}\,J_M^{\pm}(-)}(M), \mathfrak{F}_{\mathrm{CS}\,J_M^{\pm}(-)}(M))^{-1} \tag{6.18a}$$

in the enriched hom, whose K-component, for each compact subset $K \subseteq M$, is given by the diagonal arrows in the diagram below

$$
\begin{tikzcd}
\Omega^0_{J^\pm_M(K)}(M) \arrow[r, "-\mathrm{d}_{\mathrm{dR}}", "(-1)"'] \arrow[r, phantom] & \Omega^1_{J^\pm_M(K)}(M) \arrow[r, "-\mathrm{d}_{\mathrm{dR}}"] \arrow[dl, "-\delta_{\mathrm{dR}}G^\pm_\Box"'] & \Omega^2_{J^\pm_M(K)}(M) \arrow[r, "-\mathrm{d}_{\mathrm{dR}}"] \arrow[dl, "-\delta_{\mathrm{dR}}G^\pm_\Box"'] & \Omega^3_{J^\pm_M(K)}(M) \arrow[dl, "-\delta_{\mathrm{dR}}G^\pm_\Box"'] \\
\Omega^0_{J^\pm_M(K)}(M) \arrow[r, "-\mathrm{d}_{\mathrm{dR}}"'] & \Omega^1_{J^\pm_M(K)}(M) \arrow[r, "-\mathrm{d}_{\mathrm{dR}}"'] & \Omega^2_{J^\pm_M(K)}(M) \arrow[r, "-\mathrm{d}_{\mathrm{dR}}"'] & \Omega^3_{J^\pm_M(K)}(M)
\end{tikzcd}
\tag{6.18b}
$$

where $G^\pm_\Box$ denotes appropriate retarded/advanced Green's operator of the Laplace-de Rham operator $\Box$ acting on differential p-forms. Note that $-\delta_{\mathrm{dR}}G^\pm_\Box = -G^\pm_\Box\delta_{\mathrm{dR}}$ as consequence of general properties of the Laplace-de Rham operator. Hence, the retarded/advanced Green's homotopies above may be expressed both as $\Lambda^\pm_{\mathrm{CS}} = W_{\mathrm{CS}}G^\pm_\Box$ and as $\Lambda^\pm_{\mathrm{CS}} = G^\pm_\Box W_{\mathrm{CS}}$, which is always true when a formally self-adjoint Green's witness is provided, see item i of Remark 4.2.2. The associated retarded-minus-advanced quasi-isomorphism

$$\Lambda_{\mathrm{CS}} : \mathfrak{F}_{\mathrm{CSc}}(M)[1] \longrightarrow \mathfrak{F}_{\mathrm{CSsc}}(M) \tag{6.19a}$$

in $\mathbf{Ch}_{\mathbb{R}}$, descended to colimits according to (4.14), has as its only non-vanishing components the linear maps

$$\begin{aligned} \Lambda^n_{\mathrm{CS}} : \mathfrak{F}_{\mathrm{CSc}}(M)[1]^n = \Omega^{n+2}_{\mathrm{c}}(M) &\longrightarrow \mathfrak{F}_{\mathrm{CSsc}}(M)^n = \Omega^{n+1}_{\mathrm{sc}}(M) \\ \omega &\longmapsto \Lambda^n_{\mathrm{CS}}\omega := -\delta_{\mathrm{dR}}G_\Box\omega\,, \end{aligned} \tag{6.19b}$$

for $n = \pm 1, 0$, where $G_\Box := G^+_\Box - G^-_\Box$ denotes the retarded-minus-advanced propagator of the appropriate Laplace-de Rham operator $\Box$. The associated unshifted Poisson structure $\tau_{\mathrm{CS}M}$ on $\mathfrak{F}_{\mathrm{CSc}}(M)$ is defined by

$$\begin{aligned} \tau_{\mathrm{CS}M}(\alpha^\ddagger \otimes \gamma) &:= \int_M (\alpha^\ddagger, \Lambda^1_{\mathrm{CS}}\gamma)_{\mathrm{CS}} = \int_M \alpha^\ddagger \wedge \delta_{\mathrm{dR}}G_\Box\gamma =: \tau_{\mathrm{CS}M}(\gamma \otimes \alpha^\ddagger)\,, \\ \tau_{\mathrm{CS}M}(\alpha_1 \otimes \alpha_2) &:= \int_M (\alpha_1, \Lambda^0_{\mathrm{CS}}\alpha_2)_{\mathrm{CS}} = -\int_M \alpha_1 \wedge \delta_{\mathrm{dR}}G_\Box\alpha_2\,, \end{aligned} \tag{6.20}$$

for all $\alpha^\ddagger \in \mathfrak{F}_{\mathrm{CSc}}(M)[1]^{-1} = \Omega^1_{\mathrm{c}}(M)$, $\gamma \in \mathfrak{F}_{\mathrm{CSc}}(M)[1]^1 = \Omega^3_{\mathrm{c}}(M)$ and $\alpha_1, \alpha_2 \in \mathfrak{F}_{\mathrm{CSc}}(M)[1]^0 = \Omega^2_{\mathrm{c}}(M)$. Let $\Sigma \stackrel{\iota}{\hookrightarrow} M$ be a spacelike Cauchy surface, the fixed-time Poisson structure $\sigma_{\mathrm{CS}\Sigma}$ on $\mathfrak{F}_{\mathrm{CSsc}}(M)$ is given by the only non-vanishing evaluations

$$\begin{aligned} \sigma_{\mathrm{CS}\Sigma}(c \otimes A^\ddagger) &:= \int_\Sigma \iota^*(c, A^\ddagger)_{\mathrm{CS}} = \int_\Sigma \iota^*(c \wedge A^\ddagger) =: \sigma_{\mathrm{CS}\Sigma}(A^\ddagger \otimes c)\,, \\ \sigma_{\mathrm{CS}\Sigma}(A_1 \otimes A_2) &:= \int_\Sigma \iota^*(A_1, A_2)_{\mathrm{CS}} = -\int_\Sigma \iota^*(A_1 \wedge A_2)\,, \end{aligned} \tag{6.21}$$

for all $c \in \mathfrak{F}^{-1}_{\mathrm{CSsc}}(M) = \Omega^0_{\mathrm{sc}}(M)$, $A^\ddagger \in \mathfrak{F}^1_{\mathrm{CSsc}}(M) = \Omega^2_{\mathrm{sc}}(M)$ and $A_1, A_2 \in \mathfrak{F}^0_{\mathrm{CSsc}}(M) = \Omega^1_{\mathrm{sc}}(M)$.

We would like to observe that we used only the degrees 1 and 0 of the differential pairing $(-,-)_{\mathrm{CS}}$ from (6.15), which appeared in the definition of the covariant Poisson structure $\tau_{\mathrm{CS}M}$ from (6.20) (and of the Dirac pairing (6.23) and the (-1)-shifted Poisson structure (6.24) below) and of the fixed-time Poisson structure $\sigma_{\mathrm{CS}\Sigma}$ from (6.2), respectively. The lower degrees, that are needed to guarantee the sought compatibility with differentials, may capture phenomena that happen on submanifolds of higher codimension, as it is suggested by the fact that they produce differential forms of lower order.

The previous constructions admit an easy upgrade to the covariant free BV theory $(\mathsf{F}_{\mathrm{CS}}, Q_{\mathrm{CS}}, (-,-)_{\mathrm{CS}}, W_{\mathrm{CS}})$, which describes the covariant Abelian Chern-Simons theory on $\mathbf{Loc}_3$. Here, $\mathsf{F}_{\mathrm{CS}} := \Lambda^\bullet[1]$ is the natural graded vector bundle of differential forms, given component-wise by $\mathsf{F}_{\mathrm{CS}}(M) := \Lambda^\bullet M[1] \to M$, for all $M \in \mathbf{Loc}_3$. It is natural since the morphisms in $\mathbf{Loc}_3$ are, in particular, open embeddings. The natural differential Q_{CS}, the natural differential pairing $(-,-)_{\mathrm{CS}}$ and the natural Green's witness W_{CS} are given for each $M \in \mathbf{Loc}_3$ by the differential, differential pairing and Green's witness defined above for the case of a fixed manifold M. Naturality of these data follows from the naturality of the de Rham differential d_{dR}, of the wedge product $\wedge$ of differential forms and of the Hodge operator $*$ with respect to morphisms in $\mathbf{Loc}_3$. These are due to the fact that the morphisms are, in particular, orientation preserving isometries.

Note in particular that the unshifted Poisson structure (6.20) upgrades to the natural unshifted Poisson structure

$$\tau_{\mathrm{CS}} : \mathfrak{F}_{\mathrm{CSc}}[1]^{\otimes 2} \longrightarrow \Delta\mathbb{R}\,, \tag{6.22a}$$

whose M-component, for $M \in \mathbf{Loc}_3$, is the cochain map

$$\tau_{\mathrm{CS}M} : \mathfrak{F}_{\mathrm{CSc}}(M)[1]^{\otimes 2} \longrightarrow \mathbb{R}\,, \tag{6.22b}$$

in $\mathbf{Ch}_{\mathbb{R}}$ defined in (6.20). In the same way, one has the natural Dirac pairing (4.38),

$$\tau^D_{\mathrm{CS}} \in \underline{\mathrm{hom}}(\mathfrak{F}_{\mathrm{CSc}}[1]^{\otimes 2}, \Delta\mathbb{R})^0\,, \tag{6.23a}$$

whose M-component, for each $M \in \mathbf{Loc}_3$, is the 0-cochain

$$\tau^D_{\mathrm{CS}M} \in [\mathfrak{F}_{\mathrm{CSc}}(M)[1]^{\otimes 2}, \mathbb{R}]^0 \tag{6.23b}$$

in the internal hom, which is defined by the evaluations

$$\begin{aligned} \tau^D_{\mathrm{CS}M}(\alpha^\ddagger \otimes \gamma) &:= \int_M \alpha^\ddagger \wedge \delta_{\mathrm{dR}} G^D_\Box \gamma =: -\tau^D_{\mathrm{CS}M}(\gamma \otimes \alpha^\ddagger)\,, \\ \tau^D_{\mathrm{CS}M}(\alpha_1 \otimes \alpha_2) &:= -\int_M \alpha_1 \wedge \delta_{\mathrm{dR}} G^D_\Box \alpha_2\,, \end{aligned} \tag{6.23c}$$

for all $\alpha^\ddagger \in \mathfrak{F}_{\mathrm{CSc}}(M)[1]^{-1} = \Omega^1_{\mathrm{c}}(M)$, $\gamma \in \mathfrak{F}_{\mathrm{CSc}}(M)[1]^1 = \Omega^3_{\mathrm{c}}(M)$ and $\alpha_1, \alpha_2 \in \mathfrak{F}_{\mathrm{CSc}}(M)[1]^0 = \Omega^2_{\mathrm{c}}(M)$. Here, $G^D_\square := \frac{1}{2}(G^+_\square + G^-_\square)$ denotes the Dirac propagator associated with the Laplace-de Rham operator $\square$. Finally, the natural (-1)-shifted Poisson structure

$$\tau^{(-1)}_{\mathrm{CS}} : \mathfrak{F}_{\mathrm{CSc}}[1]^{\otimes 2} \longrightarrow \Delta\mathbb{R}[1] \tag{6.24a}$$

from (4.41) is given, for each $M \in \mathbf{Loc}_3$, by the cochain map

$$\tau^{(-1)}_{\mathrm{CS}\,M} : \mathfrak{F}_{\mathrm{CSc}}(M)[1]^{\otimes 2} \longrightarrow \mathbb{R}[1] \tag{6.24b}$$

in $\mathbf{Ch}_{\mathbb{R}}$, whose only non-vanishing components are defined as follows:

$$\begin{aligned} \tau^{(-1)}_{\mathrm{CS}\,M}(\gamma^\ddagger \otimes \gamma) &:= \int_M \gamma^\ddagger \wedge \gamma =: \tau^{(-1)}_{\mathrm{CS}\,M}(\gamma \otimes \gamma^\ddagger)\,, \\ \tau^{(-1)}_{\mathrm{CS}\,M}(\alpha^\ddagger \otimes \alpha) &:= \int_M \alpha^\ddagger \wedge \alpha =: \tau^{(-1)}_{\mathrm{CS}\,M}(\alpha \otimes \alpha^\ddagger)\,, \end{aligned} \tag{6.24c}$$

for all sections $\gamma^\ddagger \in \mathfrak{F}_{\mathrm{CSc}}(M)[1]^{-2} = \Omega^0_{\mathrm{c}}(M)$, $\gamma \in \mathfrak{F}_{\mathrm{CSc}}(M)[1]^1 = \Omega^3_{\mathrm{c}}(M)$ and $\alpha^\ddagger \in \mathfrak{F}_{\mathrm{CSc}}(M)[1]^{-1} = \Omega^1_{\mathrm{c}}(M)$, $\alpha \in \mathfrak{F}_{\mathrm{CSc}}(M)[1]^0 = \Omega^2_{\mathrm{c}}(M)$ with compact support.

6.3 Maxwell p-forms

A last example is provided by the Maxwell p-forms on a manifold $M \in \mathbf{Loc}_m$, for $p \leq m-1$. The constructions below recover the earlier approach in [AB23] and for $p = 1$ one finds the linear Yang-Mills theory considered in [BBS20]. Note that this is a richer example than the previous one since for $p > 1$ it describes a higher gauge field theory, admitting also higher gauge transformations between (higher) gauge transformations. The relevant complex of differential operators $(F_{\mathrm{MW}^p}, Q_{\mathrm{MW}^p})$ is given by the graded vector bundle $F_{\mathrm{MW}^p} \to M$ concentrated between degrees $-p$ and $p+1$ and degree-wise defined by

$$F^n_{\mathrm{MW}^p} := \begin{cases} \Lambda^{n+p} M\,, & \text{for } n = -p, \dots, 0\,, \\ \Lambda^{p+1-n} M\,, & \text{for } n = 1, \dots, p+1\,, \end{cases} \tag{6.25}$$

where $\Lambda^k M \to M$ denotes the vector bundle of the k-th exterior power of the cotangent bundle of M, and by the family Q_{MW^p} of degree increasing linear differential operators, whose only non-vanishing components are

$$Q^n_{\mathrm{MW}^p} := \begin{cases} \mathrm{d}_{\mathrm{dR}} & \text{for } n = -p, \dots, -1\,, \\ \delta_{\mathrm{dR}}\mathrm{d}_{\mathrm{dR}}\,, & \text{for } n = 0\,, \\ \delta_{\mathrm{dR}}\,, & \text{for } n = 1, \dots, p\,, \end{cases} \tag{6.26}$$

where d_{dR} denotes the de Rham differential and δ_{dR} the codifferential acting on differential k-forms. Equation (6.26) defines a differential, i.e. $(Q_{\mathrm{MW}^p})^2 = 0$, since both the de Rham differential, $\mathrm{d}_{\mathrm{dR}}^2 = 0$, and the codifferential, $\delta_{\mathrm{dR}}^2 = 0$, square to zero. Let us write explicitly the associated cochain complex

$$\mathfrak{F}_{\mathrm{MW}^p\mathscr{D}}(M) \tag{6.27}$$
$$= \Big(\overset{(-p)}{\Omega^0_{\mathscr{D}}(M)} \xrightarrow{\mathrm{d}_{\mathrm{dR}}} \cdots \xrightarrow{\mathrm{d}_{\mathrm{dR}}} \overset{(0)}{\Omega^p_{\mathscr{D}}(M)} \xrightarrow{\delta_{\mathrm{dR}}\mathrm{d}_{\mathrm{dR}}} \overset{(1)}{\Omega^p_{\mathscr{D}}(M)} \xrightarrow{\delta_{\mathrm{dR}}} \cdots \xrightarrow{\delta_{\mathrm{dR}}} \overset{(p+1)}{\Omega^0_{\mathscr{D}}(M)}\Big)$$

of the fields with $\mathscr{D}$-support, for a (non-empty) directed subset $\mathscr{D} \subseteq \mathrm{cl}$ of the directed set cl of closed subsets of M. Let us give an interpretation of the n-cochains in $\mathfrak{F}_{\mathrm{MW}^p}(M) \in \mathbf{Ch}_{\mathbb{R}}$, as suggested by the BV formalism. The 0-cochains $A \in \mathfrak{F}^0_{\mathrm{MW}^p}(M) = \Omega^p(M)$ are the gauge fields, the (-1)-cochains $c_{(1)} \in \mathfrak{F}^{-1}_{\mathrm{MW}^p}(M) = \Omega^{p-1}(M)$ are the ghost fields and the $(-n)$-cochains $c_{(n)} \in \mathfrak{F}^{-n}_{\mathrm{MW}^p}(M) = \Omega^{p-n}(M)$, for $n = 2, \dots, p$, are the ghost of ghosts. The 1-cochains $A^\ddagger \in \mathfrak{F}^1_{\mathrm{MW}^p}(M) = \Omega^p(M)$ are interpreted as the antifields of the gauge fields, the 2-cochains $c^\ddagger_{(1)} \in \mathfrak{F}^2_{\mathrm{MW}^p}(M) = \Omega^{p-1}(M)$ are the antifields of the ghosts and, finally, the $(n+1)$-cochains $c^\ddagger_{(n)} \in \mathfrak{F}^{n+1}_{\mathrm{MW}^p}(M) = \Omega^{p-n}(M)$, for $n = 2, \dots, p$, are interpreted as the antifields of the higher ghosts.

Let us consider the differential pairing

$$(-,-)_{\mathrm{MW}^p} : \mathfrak{F}_{\mathrm{MW}^p}(M)^{\otimes 2} \longrightarrow \Omega^\bullet(M)[m-1]\,, \tag{6.28a}$$

in $\mathbf{Ch}_{\mathbb{R}}$, whose only non-vanishing components are concentrated in degree $j = -p, \dots, 1$ and they are defined as follows: In degree $j = 1$, we have, for all $k = 0, \dots, p$,

$$(a^\ddagger, a)^1_{\mathrm{MW}^p} := s_{k+1} a^\ddagger \wedge *a =: -(a, a^\ddagger)^1_{\mathrm{MW}^p}\,, \tag{6.28b}$$

for all $a^\ddagger \in \mathfrak{F}^{k+1}_{\mathrm{MW}^p}(M) = \Omega^{p-k}(M)$ and $a \in \mathfrak{F}^{-k}_{\mathrm{MW}^p}(M) = \Omega^{p-k}(M)$, where $s_1 := 1$ and $s_k := (-1)^k s_{k-1}$, for $k = 2, \dots, p+1$. In degree $j = 0$, we define

$$\begin{aligned}(A_1, A_2)^0_{\mathrm{MW}^p} &:= (-1)^{m-1}(A_1 \wedge *\mathrm{d}_{\mathrm{dR}} A_2 - A_2 \wedge *\mathrm{d}_{\mathrm{dR}} A_1)\,,\\ (c^\ddagger, c)^0_{\mathrm{MW}^p} &:= (-1)^m s_{k+2}\, c \wedge *c^\ddagger =: (-1)^k (c, c^\ddagger)^0_{\mathrm{MW}^p}\end{aligned} \tag{6.28c}$$

for all $k = 0, \dots, p-1$, $A_1, A_2 \in \mathfrak{F}^0_{\mathrm{MW}^p}(M) = \Omega^p(M)$, $c^\ddagger \in \mathfrak{F}^{k+1}_{\mathrm{MW}^p}(M) = \Omega^{p-k}(M)$ and $c \in \mathfrak{F}^{-k-1}_{\mathrm{MW}^p}(M) = \Omega^{p-k-1}(M)$. In degrees $j \leq -1$, we have

$$\begin{aligned}(\widetilde{c}, A)^j_{\mathrm{MW}^p} &:= (-1)^{(j-1)(m-1)} \widetilde{c} \wedge *\mathrm{d}_{\mathrm{dR}} A =: -(A, \widetilde{c})^j_{\mathrm{MW}^p}\,,\\ (c^\ddagger, c)^j_{\mathrm{MW}^p} &:= -(-1)^{(j-1)(m-1)} s_{1-j} s_{k+2-j} c \wedge *c^\ddagger =: (-1)^{kj+k+j} (c, c^\ddagger)^j_{\mathrm{MW}^p}\,,\end{aligned} \tag{6.28d}$$

for all $k = 0, \dots, p+j-1$, $A \in \mathfrak{F}^0_{\mathrm{MW}^p}(M) = \Omega^p(M)$, $\widetilde{c} \in \mathfrak{F}^j_{\mathrm{MW}^p}(M) = \Omega^{p+j}(M)$, $c^\ddagger \in \mathfrak{F}^{k+1}_{\mathrm{MW}^p}(M) = \Omega^{p-k}(M)$ and $c \in \mathfrak{F}^{-k-1+j}_{\mathrm{MW}^p}(M) = \Omega^{p-k-1+j}(M)$.

It is not difficult to realize that the equations displayed above define a differential pairing for the complex of differential operators $(F_{\mathrm{MW}^p}, Q_{\mathrm{MW}^p})$. Clearly, it is degree-wise a bi-differential operator and it is graded antisymmetric by definition. Moreover, it is not hard to show that it is compatible with differentials, although it requires quite long computations. Since they are not particularly difficult, we will avoid to write them out explicitly. Let us just mention that a key ingredient for the proof is the identity $\mathrm{d}_{\mathrm{dR}}(a_1 \wedge *a_2) = (\mathrm{d}_{\mathrm{dR}}a_1) \wedge *a_2 + (-1)^{k+q} a_1 \wedge *\delta_{\mathrm{dR}} a_2$, which holds for all $a_1 \in \Omega^k(M)$ and $a_2 \in \Omega^q(M)$.

The complex $(F_{\mathrm{MW}^p}, Q_{\mathrm{MW}^p})$ of Maxwell p-forms on M may be also endowed with a Green's witness W_{MW^p}. This is the degree decreasing graded linear map on $\mathfrak{F}_{\mathrm{MW}^p}(M)$ whose only non-vanishing components are defined by

$$W^n_{\mathrm{MW}^p} := \begin{cases} \delta_{\mathrm{dR}}\,, & \text{for } n = -p+1, \dots, 0\,, \\ \mathrm{id}\,, & \text{for } n = 1\,, \\ \mathrm{d}_{\mathrm{dR}}\,, & \text{for } n = 2, \dots, p+1\,. \end{cases} \tag{6.29}$$

Direct inspection shows that each non-vanishing component of the 'adjoint' differential $(\partial W_{\mathrm{MW}^p})^n = \delta_{\mathrm{dR}}\mathrm{d}_{\mathrm{dR}} + \mathrm{d}_{\mathrm{dR}}\delta_{\mathrm{dR}} = \Box$ coincides with the Laplace-de Rham operator acting on differential forms, for $n = -p, \dots, p+1$. Since the Laplace-de Rham operators $\Box$ are normally hyperbolic, hence Green hyperbolic, see Example 1.3.10, we conclude that the W_{MW^p} defined above is a Green's witness for $(F_{\mathrm{MW}^p}, Q_{\mathrm{MW}^p})$. Let us check that it is also formally self-adjoint with respect to the differential pairing $(-,-)_{\mathrm{MW}^p}$ from (6.28). Item 1 of Definition 4.2.1 follows from direct inspection, recalling that the square of the de Rham differential $(\mathrm{d}_{\mathrm{dR}})^2 = 0$ and of the codifferential $(\delta_{\mathrm{dR}})^2 = 0$ are identically zero. Note that in this case we have a Green's witness W_{MW^p} which does not square to zero as in the previously discussed examples. Item 2 requires the following straightforward computations. First, consider homogeneous sections $A_1^\ddagger, A_2^\ddagger \in \mathfrak{F}^1_{\mathrm{MW}^p}(M) = \Omega^p(M)$ of compact overlapping support, one finds

$$\int_M (W^1_{\mathrm{MW}^p} A_1^\ddagger, A_2^\ddagger)_{\mathrm{MW}^p} = -\int_M A_2^\ddagger \wedge *A_1^\ddagger = -\int_M (A_1^\ddagger, W^1_{\mathrm{MW}^p} A_2^\ddagger)_{\mathrm{MW}^p}\,, \tag{6.30a}$$

exploiting (6.28b) and that $A_2^\ddagger \wedge *A_1^\ddagger = A_1^\ddagger \wedge *A_2^\ddagger$. Then, for $n = 2, \dots, p+1$ let $c^\ddagger \in \mathfrak{F}^n_{\mathrm{MW}^p}(M) = \Omega^{p+1-n}(M)$ and $c \in \mathfrak{F}^{2-n}_{\mathrm{MW}^p}(M) = \Omega^{2-n+p}(M)$ be homogeneous sections with compact overlapping support. One computes

$$\begin{aligned} \int_M (W^n_{\mathrm{MW}^p} c^\ddagger, c)_{\mathrm{MW}^p} &= s_{n-1} \int_M \mathrm{d}_{\mathrm{dR}} c^\ddagger \wedge *c \\ &= s_{n-1} \int_M c^\ddagger \wedge *\delta_{\mathrm{dR}} c \\ &= (-1)^n \int_M (c^\ddagger, W^{2-n}_{\mathrm{MW}^p} c)_{\mathrm{MW}^p}\,, \end{aligned} \tag{6.30b}$$

where in the first and last steps we used the definitions (6.28b) of the differential pairing and (6.29) of the Green's witness. The second step used that the codifferential δ_{dR} is the formal adjoint of the de Rham differential d_{dR} with respect to the integration pairing $\int_M - \wedge *-$. Finally, the cases for $n = -p+1, \dots, 0$ may be obtained from (6.30b) exploiting the graded antisymmetry of the differential pairing $(-,-)_{\mathrm{MW}^p}$.

The (formally self-adjoint) Green's witness W_{MW^p} from (6.29) allows us to conclude that the complex of differential operators $(F_{\mathrm{MW}^p}, Q_{\mathrm{MW}^p})$ associated to Maxwell p-forms on M is Green hyperbolic. Theorem 4.1.5 provides us also with the strictly natural Green's homotopies

$$\Lambda^{\pm}_{\mathrm{MW}^p} \in \underline{\mathrm{hom}}(\mathfrak{F}_{\mathrm{MW}^p J^{\pm}_M(-)}(M), \mathfrak{F}_{\mathrm{MW}^p J^{\pm}_M(-)}(M))^{-1}\,, \tag{6.31a}$$

whose K-component, for each compact subset $K \subseteq M$, is the degree decreasing graded linear map represented by the diagonal arrows displayed in the diagram below

$$\begin{tikzcd}
\overset{(-p)}{\Omega^0_{J^{\pm}_M(K)}} \arrow[r, "\mathrm{d}_{\mathrm{dR}}"] & \cdots \arrow[r, "\mathrm{d}_{\mathrm{dR}}"] \arrow[dl, "\delta_{\mathrm{dR}} G^{\pm}_{\Box}"'] & \overset{(0)}{\Omega^p_{J^{\pm}_M(K)}} \arrow[r, "\delta_{\mathrm{dR}}\mathrm{d}_{\mathrm{dR}}"] \arrow[dl, "\delta_{\mathrm{dR}} G^{\pm}_{\Box}"'] & \overset{(1)}{\Omega^p_{J^{\pm}_M(K)}} \arrow[r, "\delta_{\mathrm{dR}}"] \arrow[dl, "G^{\pm}_{\Box}"'] & \cdots \arrow[r, "\delta_{\mathrm{dR}}"] \arrow[dl, "\mathrm{d}_{\mathrm{dR}} G^{\pm}_{\Box}"'] & \overset{(p+1)}{\Omega^0_{J^{\pm}_M(K)}} \arrow[dl, "\mathrm{d}_{\mathrm{dR}} G^{\pm}_{\Box}"'] \\
\Omega^0_{J^{\pm}_M(K)} \arrow[r, "\mathrm{d}_{\mathrm{dR}}"'] & \cdots \arrow[r, "\mathrm{d}_{\mathrm{dR}}"'] & \Omega^p_{J^{\pm}_M(K)} \arrow[r, "\delta_{\mathrm{dR}}\mathrm{d}_{\mathrm{dR}}"'] & \Omega^p_{J^{\pm}_M(K)} \arrow[r, "\delta_{\mathrm{dR}}"'] & \cdots \arrow[r, "\delta_{\mathrm{dR}}"'] & \Omega^0_{J^{\pm}_M(K)}
\end{tikzcd} \tag{6.31b}$$

where $G^{\pm}_{\Box}$ is the retarded/advanced Green's operator of the appropriate Laplace-de Rham operator. General properties of the $\Box$ operator yield that the de Rham differential as well as the codifferential commute with the retarded/advanced Green's operator, $\mathrm{d}_{\mathrm{dR}} G^{\pm}_{\Box} = G^{\pm}_{\Box} \mathrm{d}_{\mathrm{dR}}$ and $\delta_{\mathrm{dR}} G^{\pm}_{\Box} = G^{\pm}_{\Box} \delta_{\mathrm{dR}}$. This entails that the retarded/advanced Green's homotopy $\Lambda^{\pm}_{\mathrm{MW}^p}$, that in (6.31b) is expressed as $\Lambda^{\pm}_{\mathrm{MW}^p} = W_{\mathrm{MW}^p} G^{\pm}_{\Box}$, can be equivalently expressed as $\Lambda^{\pm}_{\mathrm{MW}^p} = G^{\pm}_{\Box} W_{\mathrm{MW}^p}$, as it is usual in presence of a formally self-adjoint Green's witness, see Remark 4.2.2. The retarded-minus-advanced quasi-isomorphism associated with the said choice of retarded/advanced Green's homotopies is the cochain map

$$\Lambda_{\mathrm{MW}^p} : \mathfrak{F}_{\mathrm{MW}^p\mathrm{c}}(M)[1] \longrightarrow \mathfrak{F}_{\mathrm{MW}^p\mathrm{sc}}(M) \tag{6.32a}$$

in $\mathbf{Ch}_{\mathbb{R}}$, defined by the non-vanishing components

$$\Lambda^n_{\mathrm{MW}^p} := \begin{cases} \delta_{\mathrm{dR}} G_{\Box}\,, & \text{for } n = -p, \dots, -1\,, \\ G_{\Box}\,, & \text{for } n = 0\,, \\ \mathrm{d}_{\mathrm{dR}} G_{\Box}\,, & \text{for } n = 1, \dots, p\,, \end{cases} \tag{6.32b}$$

for $G_{\Box} := G^+_{\Box} - G^-_{\Box}$ the retarded-minus-advanced propagator of the Laplace-de Rham operator $\Box$. The unshifted Poisson structure

$$\tau_{\mathrm{MW}^p M} : \mathfrak{F}_{\mathrm{MW}^p\mathrm{c}}(M)[1]^{\otimes 2} \longrightarrow \mathbb{R}\,, \tag{6.33a}$$

built out of the retarded-minus-advanced quasi-isomorphism Λ_{MW^p} according to Proposition 4.2.4, has as its only non-vanishing components the ones defined below, for all $k = 1, \dots, p$,

$$
\begin{aligned}
\tau_{\mathrm{MW}^p M}(\alpha_1 \otimes \alpha_2) &:= \int_M (\alpha_1, \Lambda^0_{\mathrm{MW}^p}\alpha_2)_{\mathrm{MW}^p} = \int_M \alpha_1 \wedge *G_\square \alpha_2\,,\\
\tau_{\mathrm{MW}^p M}(\gamma^\ddagger \otimes \gamma) &:= \int_M (\gamma^\ddagger, \Lambda^k_{\mathrm{MW}^p}\gamma)_{\mathrm{MW}^p}\\
&= -s_k \int_M \gamma^\ddagger \wedge *\mathrm{d}_{\mathrm{dR}} G_\square \gamma =: -(-1)^k \tau_{\mathrm{MW}^p M}(\gamma \otimes \gamma^\ddagger)\,,
\end{aligned}
\tag{6.33b}
$$

for $\alpha_1, \alpha_2 \in \mathfrak{F}_{\mathrm{MW}^p\mathrm{c}}(M)[1]^0 = \Omega^p_{\mathrm{c}}(M)$, $\gamma^\ddagger \in \mathfrak{F}_{\mathrm{MW}^p\mathrm{c}}(M)[1]^{-k} = \Omega^{p-k+1}_{\mathrm{c}}(M)$ and $\gamma \in \mathfrak{F}_{\mathrm{MW}^p\mathrm{c}}(M)[1]^k = \Omega^{p-k}_{\mathrm{c}}(M)$. Then, let $\Sigma \xhookrightarrow{\iota} M$ be a spacelike Cauchy surface for M, the fixed-time Poisson structure

$$
\sigma_{\mathrm{MW}^p\Sigma} : \mathfrak{F}_{\mathrm{MW}^p\mathrm{sc}}(M)^{\otimes} \longrightarrow \mathbb{R}\,,
\tag{6.34a}
$$

is defined according to Proposition 4.2.6 by the non-vanishing evaluations

$$
\begin{aligned}
\sigma_{\mathrm{MW}^p\Sigma}(A_1 \otimes A_2) &:= (-1)^{m-1} \int_\Sigma \iota^*(A_1, A_2)_{\mathrm{MW}^p}\\
&= \int_\Sigma \iota^*(A_1 \wedge *\mathrm{d}_{\mathrm{dR}} A_2 - A_2 \wedge *\mathrm{d}_{\mathrm{dR}} A_1)\,,\\
\sigma_{\mathrm{MW}^p\Sigma}(c \otimes c^\ddagger) &:= (-1)^{m-1} \int_\Sigma \iota^*(c, c^\ddagger)_{\mathrm{MW}^p}\\
&= s_{k+2} \int_\Sigma \iota^*(c \wedge *c^\ddagger) =: -(-1)^k \sigma_{\mathrm{MW}^p\Sigma}(c^\ddagger \otimes c)\,,
\end{aligned}
\tag{6.34b}
$$

upon all $A_1, A_2 \in \mathfrak{F}^0_{\mathrm{MW}^p\mathrm{sc}}(M) = \Omega^p_{\mathrm{sc}}(M)$, $c \in \mathfrak{F}^{-k}_{\mathrm{MW}^p\mathrm{sc}}(M) = \Omega^{p-k}_{\mathrm{sc}}(M)$ and $c^\ddagger \in \mathfrak{F}^k_{\mathrm{MW}^p\mathrm{sc}}(M) = \Omega^{p+1-k}_{\mathrm{sc}}$, for all $k = 1, \dots, p$.

As already noted in Section 6.2, only the components of degrees 1 and 0 of the differential pairing $(-,-)_{\mathrm{MW}^p}$ from (6.28) enter our constructions. Respectively, in the definition of the covariant Poisson structure $\tau_{\mathrm{MW}^p M}$ from (6.33) (and of the Dirac pairing (4.38) and of the (-1)-shifted Poisson structure (6.37)) and of the fixed-time Poisson structure $\sigma_{\mathrm{MW}^p\Sigma}$ from (6.34). The components of degrees $j \leq 1$, which are already non-vanishing for $p = 1$, may capture phenomena happening on submanifolds of codimension $\mathrm{codim} = 1 - j$.

The previous constructions admit a straightforward upgrade to the covariant free BV theory $(\mathsf{F}_{\mathrm{MW}^p}, Q_{\mathrm{MW}^p}, (-,-)_{\mathrm{MW}^p}, W_{\mathrm{MW}^p})$ which captures the covariant theory of Maxwell p-forms on $\mathbf{Loc}_m$. The natural graded vector bundle $\mathsf{F}_{\mathrm{MW}^p}$ is given for each $M \in \mathbf{Loc}_m$ by the graded vector bundle $\mathsf{F}_{\mathrm{MW}^p}(M) := F_{\mathrm{MW}^p} \to M$ defined in (6.25). Its naturality follows from the fact that the morphisms in $\mathbf{Loc}_m$ are open embeddings. The family of natural differential operators Q_{MW^p}, the natural differential pairing $(-,-)_{\mathrm{MW}^p}$ and the natural Green's witness W_{MW^p} are defined, component-wise for each $M \in \mathbf{Loc}_m$, by (6.26), (6.28) and (6.29), respectively. Since the morphisms in $\mathbf{Loc}_m$ are orientation preserving isometries, the de Rham differential d_{dR}, the wedge product $\wedge$ of differential forms and the Hodge operator $*$, hence also the codifferential δ_{dR}, are natural. This entails that the data above are natural.

It follows that the unshifted Poisson structure $\tau_{\mathrm{MW}^p M}$ from (6.33) upgrades to the natural Poisson structure

$$\tau_{\mathrm{MW}^p} : \mathfrak{F}_{\mathrm{MW}^p\mathrm{c}}[1]^{\otimes 2} \longrightarrow \Delta\mathbb{R} \tag{6.35a}$$

in $\mathbf{Ch}_{\mathbb{R}}^{\mathbf{Loc}_m}$, whose M-component, for each $M \in \mathbf{Loc}_m$, is given by the cochain map

$$\tau_{\mathrm{MW}^p M} : \mathfrak{F}_{\mathrm{MW}^p\mathrm{c}}(M)[1]^{\otimes 2} \longrightarrow \mathbb{R} \tag{6.35b}$$

in $\mathbf{Ch}_{\mathbb{R}}$, defined by (6.33). The associated natural Dirac pairing from (4.38) is the 0-cochain

$$\tau^D_{\mathrm{MW}^p} \in \underline{\mathrm{hom}}(\mathfrak{F}_{\mathrm{MW}^p\mathrm{c}}[1]^{\otimes 2}, \Delta\mathbb{R})^0 \tag{6.36a}$$

in the enriched hom, defined component-wise for each $M \in \mathbf{Loc}_m$ by the 0-cochains

$$\tau^D_{\mathrm{MW}^p M} \in [\mathfrak{F}_{\mathrm{MW}^p\mathrm{c}}(M)[1]^{\otimes 2}, \mathbb{R}]^0 \tag{6.36b}$$

in the internal hom, whose only non-vanishing components are defined, for all $k = 1, \dots, p$, by

$$\tau^D_{\mathrm{MW}^p M}(\alpha_1 \otimes \alpha_2) := \int_M \alpha_1 \wedge * G^D_{\Box} \alpha_2 \,, \tag{6.36c}$$

$$\tau^D_{\mathrm{MW}^p M}(\gamma^\ddagger \otimes \gamma) := -s_k \int_M \gamma^\ddagger \wedge * \mathrm{d}_{\mathrm{dR}} G^D_{\Box} \gamma =: (-1)^k \tau^D_{\mathrm{MW}^p M}(\gamma \otimes \gamma^\ddagger) \,,$$

for all compactly supported sections $\alpha_1, \alpha_2 \in \mathfrak{F}_{\mathrm{MW}^p\mathrm{c}}(M)[1]^0 = \Omega^p_{\mathrm{c}}(M)$, $\gamma^\ddagger \in \mathfrak{F}_{\mathrm{MW}^p\mathrm{c}}(M)[1]^{-k} = \Omega^{p-k+1}_{\mathrm{c}}(M)$ and $\gamma \in \mathfrak{F}_{\mathrm{MW}^p\mathrm{c}}(M)[1]^k = \Omega^{p-k}_{\mathrm{c}}(M)$. Here, $G^D_{\Box} := \frac{1}{2}(G^+_{\Box} + G^-_{\Box})$ denotes the Dirac propagator associated with the Laplace-de Rham operator $\Box$. Finally, the natural (-1)-shifted Poisson structure

$$\tau^{(-1)}_{\mathrm{MW}^p} : \mathfrak{F}_{\mathrm{MW}^p\mathrm{c}}[1]^{\otimes 2} \longrightarrow \Delta\mathbb{R}[1] \tag{6.37a}$$

in $\mathbf{Ch}_{\mathbb{R}}^{\mathbf{Loc}_m}$ is given, for all $M \in \mathbf{Loc}_m$, by the cochain map

$$\tau^{(-1)}_{\mathrm{MW}^p M} : \mathfrak{F}_{\mathrm{MW}^p\mathrm{c}}(M)[1]^{\otimes 2} \longrightarrow \mathbb{R}[1] \tag{6.37b}$$

in $\mathbf{Ch}_{\mathbb{R}}$, which is defined, according to (4.41), by

$$\tau^{(-1)}_{\mathrm{MW}^p M}(\gamma^\ddagger \otimes \gamma) := s_{k+2} \int_M \gamma \wedge * \gamma^\ddagger =: \tau^{(-1)}_{\mathrm{MW}^p M}(\gamma \otimes \gamma^\ddagger) \,, \tag{6.37c}$$

for all $k = 0, \dots, p$, and compactly supported sections $\gamma \in \mathfrak{F}_{\mathrm{MW}^p\mathrm{c}}(M)[1]^k = \Omega^{p-k}_{\mathrm{c}}(M)$ and $\gamma^\ddagger \in \mathfrak{F}_{\mathrm{MW}^p\mathrm{c}}(M)[1]^{-k-1} = \Omega^{p-k}_{\mathrm{c}}(M)$.

Appendix A

Proofs from Section 2.3

This Appendix collects the proofs of some technical results stated in Section 2.3.

Proofs from Subsection 2.3.1

Proof of Lemma 2.3.7. We have to show that, for all $n, i \in \mathbb{Z}$,

$$\delta(\zeta \circ \eta) = \delta\zeta \circ \eta + (-1)^i \zeta \circ \delta\eta \,, \tag{A.1}$$

for all homogeneous $\zeta \in \underline{\mathrm{map}}(\mathcal{W}, \mathcal{Z})^i$ and $\eta \in \underline{\mathrm{map}}(\mathcal{V}, \mathcal{W})^{n-i}$. We will prove this by showing that the vertical δ_{v} and horizontal δ_{h} differentials separately satisfy the Leibniz rule with respect to the enriched composition $\circ$. Checking that this is indeed the case for δ_{v} is quite straightforward. Let us compute, for any integer $q \geq 0$ and functor $\underline{c} : [q] \to \mathbf{C}$, the projection

$$\begin{aligned}
\mathrm{pr}_{q,\underline{c}}\, \delta_{\mathrm{v}}(\zeta \circ \eta) &= (-1)^q \partial(\mathrm{pr}_{q,\underline{c}}(\zeta \circ \eta)) \\
&= \sum_{k=0}^{q} (-1)^{q+k(q-k+i)} \partial(\mathrm{pr}_{q-k,\underline{c}^{\geq k}}\, \zeta \circ \mathrm{pr}_{k,\underline{c}^{\leq k}}\, \eta) \\
&= \sum_{k=0}^{q} (-1)^{q+k(q-k+i)} \{\partial(\mathrm{pr}_{q-k,\underline{c}^{\geq k}}\, \zeta) \circ \mathrm{pr}_{k,\underline{c}^{\leq k}}\, \eta \\
&\qquad + (-1)^{i-q+k}\, \mathrm{pr}_{q-k,\underline{c}^{\geq k}}\, \zeta \circ \partial(\mathrm{pr}_{k,\underline{c}^{\leq k}}\, \eta)\} \\
&= \sum_{k=0}^{q} (-1)^{k(q-k+i+1)}\, \mathrm{pr}_{q-k,\underline{c}^{\geq k}}\, \delta_{\mathrm{v}}\zeta \circ \eta \\
&\qquad + (-1)^i \sum_{k=0}^{q} (-1)^{k(q-k+i)}\, \mathrm{pr}_{q-k,\underline{c}^{\geq k}}\, \zeta \circ \mathrm{pr}_{k,\underline{c}^{\leq k}}\, \delta_{\mathrm{v}}\eta \\
&= \delta_{\mathrm{v}}\zeta \circ \eta + (-1)^i \zeta \circ \delta_{\mathrm{v}}\eta \,,
\end{aligned} \tag{A.2}$$

where the first and fourth steps used just the definition (2.81c) of δ_{v}, the second and last steps used the definition of the composition map (2.92) and, finally, the third step used that the ‘adjoint’ differential is a derivation with respect to the standard composition in the internal hom. Recall, in particular, that the degree $|\mathrm{pr}_{q-k,\underline{c}^{\geq k}}\,\zeta| = |\zeta| - q + k$.

The computation for the horizontal differential is not morally different, yet it is more annoying since it involves the interplay of the coface maps in δ_{h}, which shorten the tuple $\underline{c}$, and the composition map $\circ$, which splits it. Note that for $q = 0$, the graded Leibniz rule $\mathrm{pr}_0\,\delta_{\mathrm{h}}(\zeta\circ\eta) = 0 = \mathrm{pr}_0\,(\delta_h\zeta\circ\eta + (-1)^i\zeta\circ\delta_h\eta)$ is trivially fulfilled by δ_{h}. Let us consider the case of $q \geq 1$. By exploiting the definitions of the horizontal differential δ_{h}, of the cofaces maps (2.79) and of the composition map, one computes, for all integer $q \geq 0$ and functor $\underline{c} : [q] \to \mathbf{C}$,

$$\begin{aligned}
&\mathrm{pr}_{q,\underline{c}}\,\delta_{\mathrm{h}}(\zeta\circ\eta)\\
&= \mathcal{Z}(\gamma_{q-1})\circ\mathrm{pr}_{q-1,\underline{c}\circ\widehat{q}}\,(\zeta\circ\eta) + \sum_{k=1}^{q-1}(-1)^k\,\mathrm{pr}_{q-1,\underline{c}\circ\widehat{q-k}}\,(\zeta\circ\eta)\\
&\qquad + (-1)^q\,\mathrm{pr}_{q-1,\underline{c}\circ\widehat{0}}\,(\zeta\circ\eta)\circ\mathcal{V}(\gamma_0)\\
&= \mathcal{Z}(\gamma_{q-1})\circ\sum_{j=0}^{q-1}(-1)^{j(q-1-j+i)}\,\mathrm{pr}_{q-1-j,(\underline{c}\circ\widehat{q})^{\geq j}}\,\zeta\circ\mathrm{pr}_{j,(\underline{c}\circ\widehat{q})^{\leq j}}\,\eta\\
&\qquad + \sum_{k=1}^{q-1}\sum_{j=0}^{q-k-1}(-1)^{j(q-1-j+i)+k}\,\mathrm{pr}_{q-1-j,(\underline{c}\circ\widehat{q-k})^{\geq j}}\,\zeta\circ\mathrm{pr}_{j,(\underline{c}\circ\widehat{q-k})^{\leq j}}\,\eta\\
&\qquad + \sum_{k=1}^{q-1}\sum_{j=q-k}^{q-1}(-1)^{j(q-1-j+i)+k}\,\mathrm{pr}_{q-1-j,(\underline{c}\circ\widehat{q-k})^{\geq j}}\,\zeta\circ\mathrm{pr}_{j,(\underline{c}\circ\widehat{q-k})^{\leq j}}\,\eta\\
&\qquad + (-1)^q\sum_{j=0}^{q-1}(-1)^{j(q-1-j+i)}\,\mathrm{pr}_{q-1-j,(\underline{c}\circ\widehat{0})^{\geq j}}\,\zeta\circ\mathrm{pr}_{j,(\underline{c}\circ\widehat{0})^{\leq j}}\,\eta\circ\mathcal{V}(\gamma_0)\\
&=: I_1 + I_2 + I_3 + I_4\,,
\end{aligned}\tag{A.3}$$

where I_l denotes the l-th summand in the equation displayed above, for $l = 1,\dots,4$. Note that the third summand in the second step gives rise to two contributions depending on whether the composition map splits the chain of composable morphisms to the left (I_2) or to the right (I_3) of the object c_{q-k} removed by δ_{h}. Consider the tuples of composable morphisms in $\mathbf{C}$ in the projection above. They are obtained by first removing an object from $\underline{c}$ and afterwards splitting it at a certain height. To reverse the order of such operations, one has to carefully taking account of the relative positions of the dropped object and of the split of the tuple. If the split happens at the position $j < p$, to the left of the removed object c_p, then one has $(\underline{c}\circ\widehat{p})^{\leq j} = \underline{c}^{\leq j}$ and $(\underline{c}\circ\widehat{p})^{\geq j} = \underline{c}^{\geq j}\circ\widehat{p-j}$. This is the case for the tuples in I_1 and I_2. On the other hand, when $j \geq p$ one has that $(\underline{c}\circ\widehat{p})^{\leq j} = \underline{c}^{\leq j+1}\circ\widehat{p}$ and $(\underline{c}\circ\widehat{p})^{\geq j} = \underline{c}^{\geq j+1}$. This includes the contributions in I_3 and I_4.

Expanding in a similar way $\delta_{\mathrm{h}}\zeta \circ \eta$ and $\zeta \circ \delta_{\mathrm{h}}\eta$, one finds

$$
\begin{aligned}
&\mathrm{pr}_{q,\underline{c}}(\delta_{\mathrm{h}}\zeta \circ \eta) \\
&= \sum_{j=0}^{q-1} (-1)^{j(q-j+i+1)} \Big(\mathcal{Z}(\gamma_{q-1}) \circ \mathrm{pr}_{q-j-1,\underline{c}^{\geq j}\circ \widehat{q-j}}\, \zeta \circ \mathrm{pr}_{j,\underline{c}^{\leq j}}\, \eta \\
&\qquad + \sum_{k=1}^{q-j-1} (-1)^k \, \mathrm{pr}_{q-j-1,\underline{c}^{\geq j}\circ \widehat{q-j-k}}\, \zeta \circ \mathrm{pr}_{j,\underline{c}^{\leq j}}\, \eta \\
&\qquad + (-1)^{q-j}\, \mathrm{pr}_{q-j-1,\underline{c}^{\geq j}\circ \widehat{0}}\, \zeta \circ \mathcal{W}(\gamma_j) \circ \mathrm{pr}_{j,\underline{c}^{\leq j}}\, \eta \Big) \\
&= I_1 + I_2 + \sum_{j=0}^{q-1} (-1)^{(j+1)(q-j)+j(i+1)}\, \mathrm{pr}_{q-j-1,\underline{c}^{\geq j}\circ \widehat{0}}\, \zeta \circ \mathcal{W}(\gamma_j) \circ \mathrm{pr}_{j,\underline{c}^{\leq j}}\, \eta
\end{aligned}
\tag{A.4}
$$

and

$$
\begin{aligned}
&\mathrm{pr}_{q,\underline{c}}\,(\zeta \circ \delta_{\mathrm{h}}\eta) \\
&= \sum_{j=1}^{q} (-1)^{j(q-j+i)} \Big(\mathrm{pr}_{q-j,\underline{c}^{\geq j}}\, \zeta \circ \mathcal{W}(\gamma_{j-1}) \circ \mathrm{pr}_{j-1,\underline{c}^{\leq j}\circ \widehat{j}}\, \eta \\
&\qquad + \sum_{k=1}^{j-1} (-1)^k\, \mathrm{pr}_{q-j,\underline{c}^{\geq j}}\, \zeta \circ \mathrm{pr}_{j-1,\underline{c}^{\leq j}\circ \widehat{j-k}}\, \eta \\
&\qquad + (-1)^j\, \mathrm{pr}_{q-j,\underline{c}^{\geq j}}\, \zeta \circ \mathrm{pr}_{j-1,\underline{c}^{\leq j}\circ \widehat{0}}\, \eta \circ \mathcal{V}(\gamma_0) \Big) \\
&= \sum_{j=1}^{q} (-1)^{j(q-j+i)}\, \mathrm{pr}_{q-j,\underline{c}^{\geq j}}\, \zeta \circ \mathcal{W}(\gamma_{j-1}) \circ \mathrm{pr}_{j-1,\underline{c}^{\leq j}\circ \widehat{j}}\, \eta + (-1)^i (I_3 + I_4)\,,
\end{aligned}
\tag{A.5}
$$

where for the identification in the last steps one needs also to change the order of the summations. Putting together Equations (A.3), (A.4) and (A.5), we conclude that the graded Leibniz rule $\delta_{\mathrm{h}}(\zeta \circ \eta) = \delta_h \zeta \circ \eta + (-1)^i \zeta \circ \delta_{\mathrm{h}}\eta$ is fulfilled by the horizontal differential δ_{h} as well. $\square$

Proof of Proposition 2.3.8. Let us start by proving the unitality. More explicitly, we have to show that the triangles

$$
\begin{tikzcd}
\underline{\mathrm{map}}(\mathcal{W},\mathcal{W}) \otimes \underline{\mathrm{map}}(\mathcal{V},\mathcal{W}) \arrow[dr, "\circ_{\mathcal{V},\mathcal{W},\mathcal{W}}"] & & \underline{\mathrm{map}}(\mathcal{V},\mathcal{W}) \otimes \underline{\mathrm{map}}(\mathcal{V},\mathcal{V}) \arrow[dl, "\circ_{\mathcal{V},\mathcal{V},\mathcal{W}}"'] \\
\mathbb{K} \otimes \underline{\mathrm{map}}(\mathcal{V},\mathcal{W}) \arrow[u, "j_{\mathcal{W}}\otimes \mathrm{id}"] \arrow[r, "\cong"'] & \underline{\mathrm{map}}(\mathcal{V},\mathcal{W}) & \underline{\mathrm{map}}(\mathcal{V},\mathcal{W}) \otimes \mathbb{K} \arrow[u, "\mathrm{id}\otimes j_{\mathcal{V}}"'] \arrow[l, "\cong"]
\end{tikzcd}
\tag{A.6}
$$

in $\mathbf{Ch}_{\mathbb{K}}$ commute, for all $\mathcal{V},\mathcal{W} \in \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$. Let us just consider the left triangle. The commutativity of the right one is proved in a completely analogous way. Let $\eta \in \underline{\mathrm{map}}(\mathcal{V},\mathcal{W})^i$ be an arbitrary i-cochain in the mapping complex. Then, for all $q \geq 0$ and $\underline{c} : [q] \to \mathbf{C}$, one computes

$$\begin{aligned}
\mathrm{pr}_{q,\underline{c}}\big(\circ_{\mathcal{V},\mathcal{W},\mathcal{W}}(j_{\mathcal{W}} \otimes \mathrm{id})(1 \otimes \eta)\big) &= \mathrm{pr}_{q,\underline{c}}(\mathrm{id} \circ \eta) \\
&= \sum_{k=0}^{q} (-1)^{k(q-k)}\, \mathrm{pr}_{q-k,\underline{c}^{\geq k}}\, \mathrm{id} \circ \mathrm{pr}_{k,\underline{c}^{\leq k}}\, \eta \\
&= \mathrm{pr}_{q,\underline{c}}\, \eta \,,
\end{aligned} \tag{A.7}$$

where in the last step we used that the only non-vanishing component of the 0-cochain $\mathrm{id} \in \underline{\mathrm{map}}(\mathcal{W},\mathcal{W})$ is $\mathrm{pr}_{0,c}\, \mathrm{id} = \mathrm{id} : \mathcal{W}(c) \to \mathcal{W}(c)$, for all $c \in \mathbf{C}$. Let us move on to the associativity. Consider functors $\mathcal{U},\mathcal{V},\mathcal{W},\mathcal{Z} \in \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$, one has to show that the pentagon

$$\begin{tikzcd}
 & \underline{\mathrm{map}}(\mathcal{V},\mathcal{Z}) \otimes \underline{\mathrm{map}}(\mathcal{U},\mathcal{V}) \arrow[dd, "\circ_{\mathcal{U},\mathcal{V},\mathcal{Z}}"] \\
\big(\underline{\mathrm{map}}(\mathcal{W},\mathcal{Z}) \otimes \underline{\mathrm{map}}(\mathcal{V},\mathcal{W})\big) \otimes \underline{\mathrm{map}}(\mathcal{U},\mathcal{V}) \arrow[ur, "\circ_{\mathcal{V},\mathcal{W},\mathcal{Z}} \otimes \mathrm{id}"] \arrow[dd, "\cong"'] & \\
 & \underline{\mathrm{map}}(\mathcal{U},\mathcal{Z}) \\
\underline{\mathrm{map}}(\mathcal{W},\mathcal{Z}) \otimes \big(\underline{\mathrm{map}}(\mathcal{V},\mathcal{W}) \otimes \underline{\mathrm{map}}(\mathcal{U},\mathcal{V})\big) \arrow[dr, "\mathrm{id} \otimes \circ_{\mathcal{U},\mathcal{V},\mathcal{W}}"'] & \\
 & \underline{\mathrm{map}}(\mathcal{W},\mathcal{Z}) \otimes \underline{\mathrm{map}}(\mathcal{U},\mathcal{W}) \arrow[uu, "\circ_{\mathcal{U},\mathcal{W},\mathcal{Z}}"']
\end{tikzcd} \tag{A.8}$$

in $\mathbf{Ch}_{\mathbb{K}}$ commutes. Consider homogeneous elements $\eta \in \underline{\mathrm{map}}(\mathcal{U},\mathcal{V})^i$, $\zeta \in \underline{\mathrm{map}}(\mathcal{V},\mathcal{W})^r$ and $\xi \in \underline{\mathrm{map}}(\mathcal{W},\mathcal{Z})^s$ in the mapping complexes. For any $q \geq 0$ and $\underline{c} : [q] \to \mathbf{C}$, one computes the $(q,\underline{c})$-projection

$$\begin{aligned}
\mathrm{pr}_{q,\underline{c}}\big((\xi \circ \zeta) \circ \eta\big) &= \sum_{k=0}^{q} \varsigma_{k,q,r+s}\, \mathrm{pr}_{q-k,\underline{c}^{\geq k}}(\xi \circ \zeta) \circ \mathrm{pr}_{k,\underline{c}^{\leq k}}\, \eta \\
&= \sum_{k=0}^{q} \sum_{j=0}^{q-k} \varsigma_{k,q,r+s} \varsigma_{j,q-k,s}\, \mathrm{pr}_{q-k-j,(\underline{c}^{\geq k})^{\geq j}}\, \xi \circ \mathrm{pr}_{j,(\underline{c}^{\geq k})^{\leq j}}\, \zeta \circ \mathrm{pr}_{k,\underline{c}^{\leq k}}\, \eta
\end{aligned} \tag{A.9}$$

by exploiting twice the definition (2.92b) of the composition cochain map, where, $\varsigma_{i,j,k} := (-1)^{i(j-i+k)}$. Notice that the first tuple of composable $\mathbf{C}$-morphisms in (A.9) is obtained by first keeping the last $q-k$ arrows and then by keeping only the last $q-k-j$, therefore $(\underline{c}^{\geq k})^{\geq j} = \underline{c}^{\geq k+j}$. The second tuple is obtained by first keeping the last $q-k$ arrows and then by keeping only the first remaining j arrows, which is the same as first selecting the first $j+k$ arrows and then keeping only the last remaining j arrows, namely $(\underline{c}^{\geq k})^{\leq j} = (\underline{c}^{\leq j+k})^{\geq k}$. Hence, Equation (A.9) may be rewritten, by also sending $j \to j-k$ and reversing the summation order, as

$$= \sum_{j=0}^{q} \sum_{k=0}^{j} \varsigma_{j,q,s} \varsigma_{k,j,r} \, \mathrm{pr}_{q-j,\underline{c}^{\geq j}} \, \xi \circ \mathrm{pr}_{j-k,(\underline{c}^{\leq j})^{\geq k}} \, \zeta \circ \mathrm{pr}_{k,\underline{c}^{\leq k}} \, \eta$$

$$= \sum_{j=0}^{q} \varsigma_{j,q,s} \, \mathrm{pr}_{q-j,\underline{c}^{\geq j}} \, \xi \circ \mathrm{pr}_{j,\underline{c}^{\leq j}} \, (\zeta \circ \eta) = \mathrm{pr}_{q,\underline{c}} \, (\xi \circ (\zeta \circ \eta)) \,, \tag{A.10}$$

where we exploited the definition of the composition $\circ$, and in the first step the fact that $(\underline{c}^{\leq j})^{\leq k} = \underline{c}^{\leq k}$ for $k \leq j$. This proves the associativity of the composition and concludes our proof. □

A proof from Subsection 2.3.2

Proof of Proposition 2.3.11. In order to show that the homotopy colimit functor $\mathrm{hocolim} : \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}} \to \mathbf{Ch}_{\mathbb{K}}$ gets upgraded to a dg-functor by the definition (2.105) of its action

$$\mathrm{hocolim} : \underline{\mathrm{map}}(\mathcal{V}, \mathcal{W}) \longrightarrow [\mathrm{hocolim}(\mathcal{V}), \mathrm{hocolim}(\mathcal{W})] \tag{A.11}$$

on the hom-objects, for all $\mathcal{V}, \mathcal{W} \in \mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$, we have to check that (1) it is a cochain map and (2) it is compatible with the enriched identities and (3) compositions.

Compatibility with differentials. Let us start by proving that (A.11) is a cochain map by showing that its adjunct (2.105) is compatible with differentials, namely that for all $n \in \mathbb{Z}$,

$$\mathrm{hocolim}(\delta\eta \otimes \iota_{p,\underline{c}} v) + (-1)^i \, \mathrm{hocolim}(\eta \otimes \mathrm{d}\iota_{p,\underline{c}} v) = \mathrm{d} \; \mathrm{hocolim}(\eta \otimes \iota_{p,\underline{c}} v) \tag{A.12}$$

for all $\eta \in \underline{\mathrm{map}}(\mathcal{V}, \mathcal{W})^i$, $p \geq 0$, $\underline{c} : [p] \to \mathbf{C}$ and $v \in \mathcal{V}(c_0)^{n-i+p}$. We consider the contributions coming from the horizontal and vertical differentials separately. For the vertical differentials one has,

$$\begin{aligned}
&\mathrm{hocolim}(\delta_{\mathrm{v}}\eta \otimes \iota_{p,\underline{c}} v + (-1)^i \eta \otimes \mathrm{d}_{\mathrm{v}}(\iota_{p,\underline{c}} v)) \\
&\quad = \sum_{k=0}^{p} (-1)^{-p(i+1)+k(p-k+1)} \iota_{p-k,\underline{c}^{\geq k}} (\partial (\mathrm{pr}_{k,\underline{c}^{\leq k}} \, \eta) \, v) \\
&\qquad\quad + (-1)^{i-k} \sum_{k=0}^{p} (-1)^{-p(i+1)+k(p-k+1)} \iota_{p-k,\underline{c}^{\geq k}} ((\mathrm{pr}_{k,\underline{c}^{\leq k}} \, \eta) \, Qv) \\
&\quad = \sum_{k=0}^{p} (-1)^{-p(i+1)+k(p-k+1)} \iota_{p-k,\underline{c}^{\geq k}} \circ Q((\mathrm{pr}_{k,\underline{c}^{\leq k}} \, \eta) \, v) \\
&= \mathrm{d}_{\mathrm{v}} \, \mathrm{hocolim}(\eta \otimes \iota_{p,\underline{c}} v) \,,
\end{aligned} \tag{A.13}$$

where the first step uses definitions (2.81c) and (2.99) of the vertical differentials δ_{v} and d_{v}, respectively, and the definition (2.105) of the graded linear map hocolim. Second step follows from the definition (2.5b) of the 'adjoint' differential ∂, recalling that $\mathrm{pr}_{k,\underline{c}^{\leq k}}\,\eta$ has degree $i-k$. Last step exploits again the definition of hocolim and of the vertical differential d_{v}.

Computing the contributions coming from the horizontal differentials requires a little more care since it involves the interplay of splitting the tuple $\underline{c}$, due to the homotopy colimit, and dropping one of its nodes, due to the action of both the horizontal differentials. This is similar to the proof of Lemma 2.3.7. For $p=0$ there is nothing to show since all horizontal differentials vanish. For $p \geq 1$, we calculate

$$
\begin{aligned}
&\mathrm{hocolim}\big(\delta_{\mathrm{h}}\eta\otimes\iota_{p,\underline{c}}v+(-1)^{i}\eta\otimes \mathrm{d}_{\mathrm{h}}(\iota_{p,\underline{c}}v)\big)\\
&\quad=\sum_{k=1}^{p}(-1)^{-p(i+1)+k(p-k)}\,\iota_{p-k,\underline{c}^{\geq k}}\big(\mathcal{W}(\gamma_{k-1})(\mathrm{pr}_{k-1,(\underline{c}^{\leq k})\circ\widehat{k}}\,\eta)\,v\big)\\
&\qquad+\sum_{k=1}^{p}\sum_{j=1}^{k-1}(-1)^{-p(i+1)+k(p-k)+j}\,\iota_{p-k,\underline{c}^{\geq k}}\big(\mathrm{pr}_{k-1,(\underline{c}^{\leq k})\circ\widehat{k-j}}\,\eta\big)\,v\big)\\
&\qquad+\sum_{k=1}^{p}(-1)^{-p(i+1)+k(p-k+1)}\,\iota_{p-k,\underline{c}^{\geq k}}\big((\mathrm{pr}_{k-1,(\underline{c}^{\leq k})\circ\widehat{0}}\,\eta)\mathcal{V}(\gamma_0)v\big)\\
&\qquad-\sum_{k=0}^{p-1}(-1)^{-pi+k(p-1-k)}\,\iota_{p-1-k,(\underline{c}\circ\widehat{0})^{\geq k}}\big((\mathrm{pr}_{k,(\underline{c}\circ\widehat{0})^{\leq k}}\,\eta)\mathcal{V}(\gamma_0)v\big)\\
&\qquad-\sum_{k=0}^{p-1}\sum_{j=1}^{p}(-1)^{-pi+k(p-1-k)+j}\,\iota_{p-1-k,(\underline{c}\circ\widehat{j})^{\geq k}}\big((\mathrm{pr}_{k,(\underline{c}\circ\widehat{j})^{\leq k}}\,\eta)\,v\big)
\end{aligned}
$$

by exploiting the definitions (2.81b) and (2.98) of the horizontal differential δ_{h} of the mapping complex and d_{h} of the homotopy colimit, respectively, and the definition (2.105) of the action of the homotopy colimit functor on the hom complexes. In particular, the first three rows come from the explicit expressions for the coface maps (2.79) in δ_{h}, while the last two from the definition of the face maps (2.95) in d_{h}. Recalling from the proof of Lemma 2.3.7 that $(\underline{c}^{\leq k})\circ\widehat{k}=\underline{c}^{\leq k-1}$, $\underline{c}^{\geq k+1}=\underline{c}^{\geq k}\circ\widehat{0}$ and that

$$
(\underline{c}\circ\widehat{j})^{\leq k}=\begin{cases}\underline{c}^{\leq k+1}\circ\widehat{j}\\ \underline{c}^{\leq k}\end{cases},\quad (\underline{c}\circ\widehat{j})^{\geq k}=\begin{cases}\underline{c}^{\geq k+1} & \text{if}\quad j\leq k\\ \underline{c}^{\geq k}\circ\widehat{j-k} & \text{if}\quad j>k\end{cases} \tag{A.14}
$$

one realizes that the third and fourth rows cancel each other and that the equation displayed above can be rewritten, by splitting at $j=k$ the inner sum in the last row, as

$$
\begin{aligned}
&= -\sum_{k=0}^{p-1} (-1)^{-pi+k(p-k)} \iota_{p-k-1,\underline{c}^{\geq k}\circ\hat{0}}(\mathcal{W}(\gamma_k)(\mathrm{pr}_{k,\underline{c}^{\leq k}}\,\eta)\,v) \\
&\quad + \sum_{k=1}^{p}\sum_{j=1}^{k-1} (-1)^{-p(i+1)+k(p-k)+j} \iota_{p-k,\underline{c}^{\geq k}}(\mathrm{pr}_{k-1,\underline{c}^{\leq k}\circ\widehat{k-j}}\,\eta)\,v) \\
&\quad - \sum_{k=0}^{p-1}\sum_{j=1}^{k} (-1)^{-pi+k(p-1-k)+j} \iota_{p-1-k,\underline{c}^{\geq k+1}}((\mathrm{pr}_{k,\underline{c}^{\leq k+1}\circ\hat{j}}\,\eta)\,v) \\
&\quad - \sum_{k=0}^{p-1}\sum_{j=k+1}^{p} (-1)^{-pi+k(p-1-k)+j} \iota_{p-1-k,\underline{c}^{\geq k}\circ\widehat{j-k}}((\mathrm{pr}_{k,\underline{c}^{\leq k}}\,\eta)\,v)
\end{aligned}
$$

By a change of indices in the sums, one sees that the second and the third contributions are one the opposite of the other, and that we can finally write

$$
\begin{aligned}
&= -\sum_{k=0}^{p-1} (-1)^{-pi+k(p-k)} \iota_{p-k-1,\underline{c}^{\geq k}\circ\hat{0}}(\mathcal{W}(\gamma_k)(\mathrm{pr}_{k,\underline{c}^{\leq k}}\,\eta)\,v) \\
&\quad - \sum_{k=0}^{p-1}\sum_{j=1}^{p-k} (-1)^{-pi+k(p-k)+j} \iota_{p-1-k,\underline{c}^{\geq k}\circ\hat{j}}((\mathrm{pr}_{k,\underline{c}^{\leq k}}\,\eta)\,v) \\
&= \mathrm{d_h} \sum_{k=0}^{p-1} (-1)^{-pi+k(p-k)} \iota_{p-k,\underline{c}^{\geq k}}((\mathrm{pr}_{k,\underline{c}^{\leq k}}\,\eta)\,v) \\
&= \mathrm{d_h}\,\mathrm{hocolim}(\eta\otimes\iota_{p,\underline{c}}v)\,,
\end{aligned}
\tag{A.15}
$$

where the first step exploits the explicit definition of the horizontal differential $\mathrm{d_h}$ and the last one that of hocolim. By putting together (A.13) and (A.15), one shows that the graded linear map hocolim defined in (2.105) is actually a cochain map as claimed.

Compatibility with enriched identities. Compatibility with the enriched identities boils down to check that the identity of cochain maps

$$
\mathrm{hocolim}\circ j_{\mathcal{V}} = j_{\mathrm{hocolim}(\mathcal{V})} : \mathbb{K} \longrightarrow [\mathrm{hocolim}(\mathcal{V}), \mathrm{hocolim}(\mathcal{V})] \tag{A.16}
$$

holds for all objects $\mathcal{V}\in\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$. This readily follows from the definitions of the dg-functor hocolim and of the enriched identity of $\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$. Indeed, for all $p\geq 0$, $\underline{c}:[p]\to\mathbf{C}$ and $v\in\mathcal{V}(c_0)^{n+p}$, one has

$$
\mathrm{hocolim}(j_{\mathcal{V}}(1))(\iota_{p,\underline{c}}v) = \sum_{k=0}^{p}(-1)^{k(p-k)}\iota_{p-k,\underline{c}^{\geq k}}((\mathrm{pr}_{k,\underline{c}^{\leq k}}\,j_{\mathcal{V}})\,v) = \iota_{p,\underline{c}}v\,, \tag{A.17}
$$

since only the 0-th components $\mathrm{pr}_{0,c_0}\,j_{\mathcal{V}} = \mathrm{id}:\mathcal{V}(c_0)\to\mathcal{V}(c_0)$, for all $c_0\in\mathbf{C}$, are non-vanishing.

Compatibility with enriched compositions. We just have to prove that for all $\mathcal{V},\mathcal{W},\mathcal{Z}\in\mathbf{Ch}_{\mathbb{K}}^{\mathbf{C}}$ the identity

$$\mathrm{hocolim}(\zeta)\circ\mathrm{hocolim}(\eta)=\mathrm{hocolim}(\zeta\circ\eta) \tag{A.18}$$

holds true for all homogeneous $\eta\in\underline{\mathrm{map}}(\mathcal{V},\mathcal{W})^i$ and $\zeta\in\underline{\mathrm{map}}(\mathcal{W},\mathcal{Z})^q$. Let then $v\in\mathcal{V}(c_0)^{n+p}$. For any $p\geq 0$ and $\underline{c}:[p]\to\mathbf{C}$, one computes

$$\begin{aligned}
&\mathrm{hocolim}(\zeta)(\mathrm{hocolim}(\eta)(\iota_{p,\underline{c}}v))\\
&\quad=\mathrm{hocolim}(\zeta)\Big(\sum_{j=0}^{p}(-1)^{-pi}\varsigma_{j,p,0}\,\iota_{p-j,\underline{c}^{\geq j}}\big((\mathrm{pr}_{j,\underline{c}^{\leq j}}\,\eta)\,v\big)\Big)\\
&\quad=\sum_{j=0}^{p}\sum_{k=0}^{p-j}(-1)^{-p(i+q)}\varsigma_{j,p,q}\,\varsigma_{k,p,j}\,\iota_{p-j-k,\underline{c}^{\geq j+k}}\big((\mathrm{pr}_{k,(\underline{c}^{\geq j})^{\leq k}}\,\zeta)((\mathrm{pr}_{j,\underline{c}^{\leq j}}\,\eta)\,v)\big)\\
&\quad=\sum_{j=0}^{p}\sum_{k=j}^{p}(-1)^{-p(i+q)}\varsigma_{k,p,0}\,\varsigma_{j,k,q}\,\iota_{p-k,\underline{c}^{\geq k}}\big((\mathrm{pr}_{k-j,(\underline{c}^{\leq k})^{\geq j}}\,\zeta)((\mathrm{pr}_{j,\underline{c}^{\leq j}}\,\eta)\,v)\big)\\
&\quad=\sum_{k=0}^{p}(-1)^{-p(i+q)}\varsigma_{k,p,0}\,\iota_{p-k,\underline{c}^{\geq k}}\Big(\sum_{j=0}^{k}\varsigma_{j,k,q}\big((\mathrm{pr}_{k-j,(\underline{c}^{\leq k})^{\geq j}}\,\zeta)(\mathrm{pr}_{j,\underline{c}^{\leq j}}\,\eta)\big)\,v\Big)\\
&\quad=\sum_{k=0}^{p}(-1)^{-p(i+q)}\varsigma_{k,p,0}\,\iota_{p-k,\underline{c}^{\geq k}}\big(\mathrm{pr}_{j,\underline{c}^{\leq k}}(\zeta\circ\eta)\,v\big)\\
&\quad=\mathrm{hocolim}(\zeta\circ\eta)(\iota_{p,\underline{c}}v)\,,
\end{aligned} \tag{A.19}$$

where $\varsigma_{i,j,k}:=(-1)^{i(j-i+k)}$. The second step used that $(\underline{c}^{\geq j})^{\geq k}=\underline{c}^{\geq j+k}$. In the first and second steps we used (the adjunct of) Equation (2.105). The third step follows from noticing that $(\underline{c}^{\geq j})^{\leq k}=(\underline{c}^{\leq j+k})^{\geq j}$ and sending the index $k\to k-j$. Fourth step includes reversing the sum order and suitably grouping the contributions. The fifth step uses the definition (2.92b) of the composition after noticing that $(\underline{c}^{\leq k})^{\leq j}=\underline{c}^{\leq j}$. Last step uses just the definition of hocolim. □

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