1 $\sum_{k=0}^{\infty} p^k = 1+p+p^2+\ldots = \frac{1}{1-p}$.

$\sum_{k=0}^{\infty} kp^{k-1} = \frac{1}{(1-p)^2} \Rightarrow \sum_{k=0}^{\infty} kp^k = \frac{p}{(1-p)^2}$

$\sum_{k=1}^{\infty} k^2p^{k-1} = \frac{1+p}{(1-p)^3} \Rightarrow \sum_{k=1}^{\infty} k^2p^k = \frac{p+p^2}{(1-p)^3}$

$\sum_{k=1}^{\infty} k^3p^{k-1} = \frac{1+4p+p^2}{(1-p)^4}$

$\sum_{k=1}^{\infty} kp^{k-1} = \frac{1}{(1-p)^2} \Rightarrow \sum_{k=1}^{\infty} Bkp^{k-1} = 1 \Rightarrow B = (1-p)^2$

$\sum_{k=1}^{\infty} k^2p^{k-1} = \frac{1+p}{(1-p)^3} \Rightarrow E(x) = \sum_{k=1}^{\infty} Bk^2p^{k-1} = \frac{1+p}{1-p}$

$\sum_{k=1}^{\infty} k^3p^{k-1} = \frac{1+4p+p^2}{(1-p)^4} \Rightarrow E(x^2) = \sum_{k=1}^{\infty} Bk^3p^{k-1} = \frac{1+4p+p^2}{(1-p)^2}$

a) $B = (1-p)^2$

b) $E(x) = \sum_{k=0}^{\infty} kP(X=k) = \sum_{k=1}^{\infty} Bk^2p^{k-1} = \frac{1+p}{1-p}$

c) $E(x^2) = \sum_{k=0}^{\infty} k^2P(X=k) = \sum_{k=1}^{\infty} Bk^3p^{k-1} = \frac{1+4p+p^2}{(1-p)^2}$

d) $B = (1-p)^2 = \frac{4}{9}$ $\quad P(X=k) = Bkp^{k-1}, k=1,2\ldots\infty;\ p=\frac{1}{3}$

$$p(x=k \mid 2\le X\le 5) = \frac{P(x=k, 2\le X\le 5)}{\sum_{k=2}^{5} P(X=k)} = \frac{P(x=k, 2\le X\le 5)}{\sum_{k=2}^{5}\left(\frac{4}{9}\right)k\left(\frac{1}{3}\right)^{k-1}}$$

$$\approx \frac{1}{0.537}\left(\frac{4}{9}\right)k\left(\frac{1}{3}\right)^{k-1}, \quad k=2,3,\ldots 5 \qquad 0, \text{ otherwise}$$

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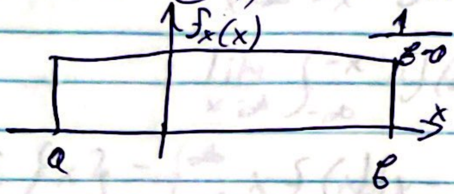
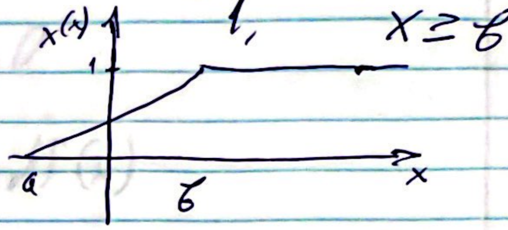
2.

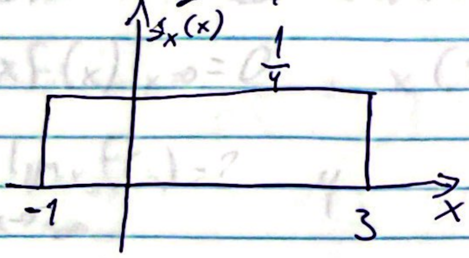
a) $f_X(x) = \begin{cases} \frac{1}{(b-a)} & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$ $\quad F_X(x) = \begin{cases} 0, & x<a \\ \frac{1}{(b-a)}(x-a) & a \le x < b \\ 1, & x \ge b \end{cases}$

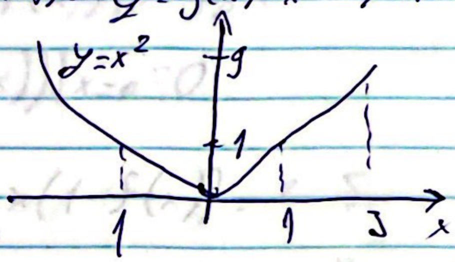




b) (c) and (d) $f_Y(y) = 0$, for $y<0$ and for $y>9$

for $0 \le y < 1$, there are two roots: $y = g(x) = x^2 \Rightarrow x = \pm\sqrt{y}$





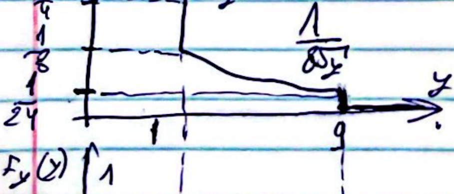
$$f_Y(y) = \frac{f_X(x_1)}{|g'(x_1)|} + \frac{f_X(x_2)}{|g'(x_2)|} = \frac{\frac{1}{4}}{|-2\sqrt{y}|} + \frac{\frac{1}{4}}{|+2\sqrt{y}|} = \frac{1}{4\sqrt{y}}$$

For $1 \le y < 9$, there is one root: $f_Y(y) = \frac{f_X(x)}{g'(x)} = \frac{\frac{1}{4}}{|2\sqrt{y}|} = \frac{1}{8\sqrt{y}}$

$$f_Y(y) = \begin{cases} \frac{1}{4\sqrt{y}} & \text{for } 0 \le y < 1 \\ \frac{1}{8\sqrt{y}} & \text{for } 1 \le y < 9 \\ 0, & \text{otherwise} \end{cases} \qquad F_Y(y) = \begin{cases} 0 & \text{for } y<0 \\ \frac{1}{2}\sqrt{y} & \text{for } 0 \le y < 1 \\ \frac{1}{4}(\sqrt{y}+1) & \text{for } 1 \le y < 9 \\ 1 & \text{for } 9 \le y \end{cases}$$

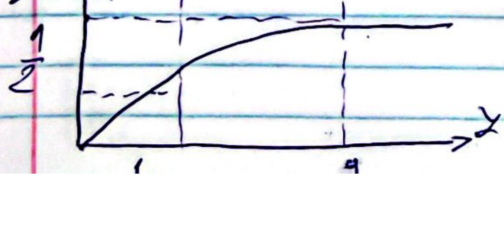
$$f_Y(y) = \frac{1}{4\sqrt{y}}(u(y) - u(y-1)) + \frac{1}{8\sqrt{y}}(u(y-1) - u(y-9))$$

$$F_Y(y) = \frac{1}{2}\sqrt{y}(u(y) - u(y-1)) + \frac{1}{4}(\sqrt{y}+1)(u(y-1) - u(y-9))$$



$$E\{Y\} = \frac{1}{4}\int_0^1 y\left(\frac{1}{\sqrt{y}}\right)dy + \frac{1}{8}\int_1^9 y\left(\frac{1}{\sqrt{y}}\right)dy = \frac{1}{4}\int_0^1 \sqrt{y}\,dy + \frac{1}{8}\int_1^9 \sqrt{y}\,dy$$

$$= \frac{1}{4}\cdot\frac{2}{3}y^{\frac{3}{2}}\Big|_0^1 + \frac{1}{8}\cdot\frac{2}{3}y^{\frac{3}{2}}\Big|_1^9 = \frac{1}{6} + \frac{1}{12}(27-1) = \frac{7}{3}$$



3.
$E\{x\}$ s.t. $\lim_{x\to\infty}\int_x^\infty a f(a)\,da = 0$

$\lim_{x\to\infty}\int_{-\infty}^{-x} a f(a)\,da = 0$

$E\{x\} = \int_{-\infty}^{\infty} x f(x)dx = \int_{-\infty}^{\infty} x\,dF(x)$

$= \int_{-\infty}^{0} x\,dF(x) + \int_0^\infty x\,dF(x) = \int_{-\infty}^{0} x\,dF(x) - \int_0^\infty x\,d(1-F(x))$

$= xF(x)\Big|_{-\infty}^{0} - \int_{-\infty}^{0} F(x)dx - x(1-F(x))\Big|_0^\infty + \int_0^\infty (1-F(x))dx$

$xF(x)\big|_{x=0} = 0 \qquad x(1-F(x))\big|_{x=0} = 0$

$\lim_{x\to-\infty} xF(x) = ?$ 4 $\qquad \lim_{x\to\infty} x(1-F(x)) = ?$ 5

$\lim_{x\to-\infty} xF(x) = -\lim xF(-x)$ $\qquad \lim_{x\to\infty} x(1-F(x)) = \lim_{x\to\infty} x\int_x^\infty f(a)\,da$

$0 \le \lim_{x\to\infty} xF(-x) = \lim_{x\to\infty} x\int_{-\infty}^{-x} f(a)\,da$ $\qquad \le \lim_{x\to\infty}\int_x^\infty a f(a)\,da = 0$ (from (1))

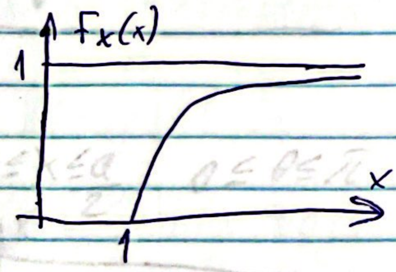
$\le \lim_{x\to\infty}\left(-\int_{-\infty}^{-x} a f(a)\,da\right) = 0$ (from (2)) $\qquad$ Now (3) becomes $E\{x\} = \int_0^\infty (1-F(x))dx$

$-\int_{-\infty}^{0} F(x)dx$

4.

1) $\lambda = 1$

2) $f_X(x) = e^{-(x-1)} U(x-1) \Rightarrow F_X(x) = 1 - e^{-(x-1)}) U(x-1)$

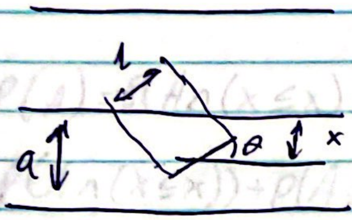


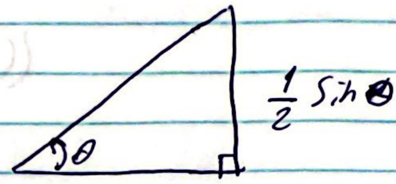
3) $f_Y(y) = \dfrac{f_X(x)}{|g'(x)|} = \dfrac{1}{3y^{\frac{2}{3}}} e^{-(y^{\frac{1}{3}}-1)} U(y^{\frac{1}{3}}-1) = \dfrac{1}{3y^{\frac{2}{3}}} e^{-(y^{\frac{1}{3}}-1)} U(y-1)$

because for $y \geq 1$ $U(y^{\frac{1}{3}}-1) = U(y-1)$

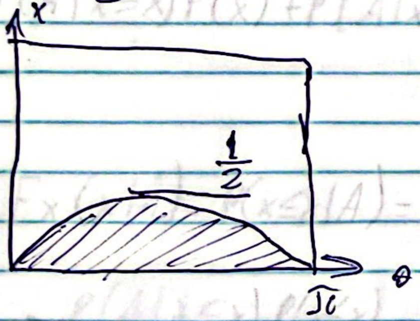
4) $E\{\sqrt[3]{y}\} = E\{x\} = 2$

5.





$0 \le x \le \frac{a}{2}$ $\quad 0 \le \theta \le \pi$ $\quad$ to intersect $x \le \left(\frac{l}{2}\right)\sin\theta$



$$P(\text{intersection}) = \frac{\int_0^{\pi}\left(\frac{l}{2}\right)\sin\theta\,d\theta}{\left(\frac{a}{2}\right)\pi} = \frac{2l}{\pi a}$$

2) P(needle vertically intersects any of the lines) = 0

3) P(needle intersects a line with the angle between needle and line is within 0°~30° or 150°~180°)

$$= P(\text{intersection}) = \frac{\int_0^{\frac{\pi}{6}}\left(\frac{l}{2}\right)\sin\theta\,d\theta + \int_{\frac{2\pi}{6}}^{\pi}\left(\frac{l}{2}\right)\sin\theta\,d\theta}{\left(\frac{a}{2}\right)\pi} = \frac{l}{\pi a}\left(2-\sqrt{3}\right)$$

6

1) $f_y(y) = \frac{f_x(x)}{|g'(x)|} = \frac{1}{y\sqrt{2\pi n\sigma^2}} e^{\frac{-(\ln y - n\mu)^2}{2n\sigma^2}}$

2) $E\{y\} = E\{e^x\} = \int_{-\infty}^{\infty} e^x f(x)dx$

$= \int_{-\infty}^{\infty} e^x \frac{1}{\sqrt{2\pi n\sigma^2}} e^{-\frac{(x-n\mu)^2}{2n\sigma^2}} dx$

let $\frac{x-n\mu}{\sqrt{2n\sigma^2}} = z \Rightarrow x = \sqrt{2n\sigma^2}z + n\mu$

$\int_{-\infty}^{\infty} e^{(\sqrt{2n\sigma^2})z} e^{n\mu} \frac{1}{\sqrt{2\pi n\sigma^2}} e^{-z^2} d(\sqrt{2n\sigma^2}z + n\mu) =$

$= e^{n\mu} \frac{1}{\sqrt{2\pi n\sigma^2}} \sqrt{2n\sigma^2} \int_{-\infty}^{\infty} e^{(\sqrt{2n\sigma^2})z} e^{-z^2} dz$

$= e^{n\mu} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{(\sqrt{2n\sigma^2}z - z^2)} dz = e^{n\mu} \frac{1}{\sqrt{\pi}} \sqrt{\pi} e^{\frac{n\sigma^2}{2}} = e^{n(\mu + \frac{\sigma^2}{2})}$

3) $E\{y^2\} = E\{e^{2x}\} = \int_{-\infty}^{\infty} e^{2x} f(x)dx = e^{2n(\mu+\sigma^2)}$

7.

$P(A) = P(A \cap (x \leq x) + A \cap (x \leq x))$

$= P(A \cap (x \leq x)) + P(A \cap (x \leq x))$

$= P(A | x \leq x) P(x \leq x) + P(A | x > x) P(x > x)$

$= P(A | x \leq x) F(x) + P(A | x > x) [1 - F(x)]$

8

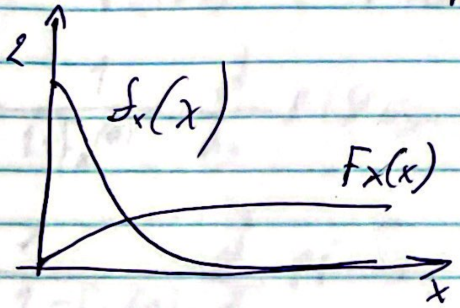
$$F_x(x | A) = P(x \leq x | A) = \frac{P((x \leq x) \cap A)}{P(A)}$$

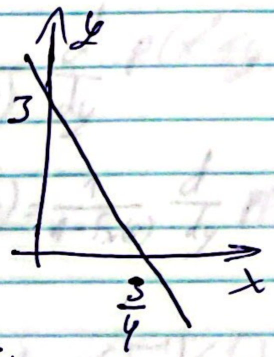
$$= \frac{P(A | x \leq x) P(x \leq x)}{P(A)} = \frac{P(A | x \leq x) f_x(x)}{P(A)}$$

9

$$F_X(x) = \int_{-\infty}^{x} f_X(\beta)\,d\beta = \int_{-\infty}^{x} 2e^{-2\beta}u(\beta)\,d\beta = \int_0^x 2e^{-2\beta}u(\beta)\,d\beta\, u(x)$$

$$\int_0^x 2e^{-2\beta}\,d\beta\, u(x) = -e^{-2\beta}\Big|_0^x u(x) = (1-e^{-2x})\,u(x)$$



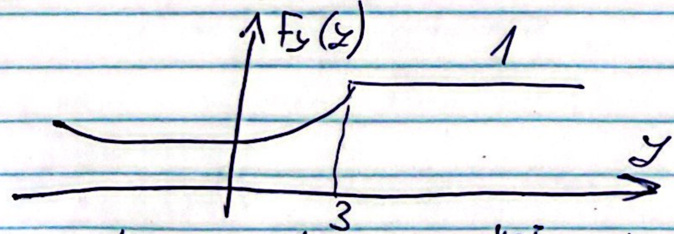


$$F_Y(y) = P(Y \le y) = P(-4x+3 \le y) = P\left(-x \le \frac{y-3}{4}\right) = P\left(x \ge -\frac{y-3}{4}\right) = 1 - P\left(x < -\frac{y-3}{4}\right)$$

$$= 1 - F_X\left(-\frac{y-3}{4}\right) = 1-(1-e^{-2x})u(x)\Big|_{x=\frac{y-3}{4}} = 1-\left(1-e^{-2\left(-\frac{y-3}{4}\right)}\right)u\left(-\frac{y-3}{4}\right)$$

$$= 1-\left(1-e^{-2\left(-\frac{y-3}{4}\right)}\right)u(3-y) = 1-\left(1-e^{\frac{y-3}{2}}\right)u(3-y) = 1-u(3-y)+e^{\frac{y-3}{2}}u(3-y) = u(y-3)+e^{\frac{y-3}{2}}u(3-y)$$

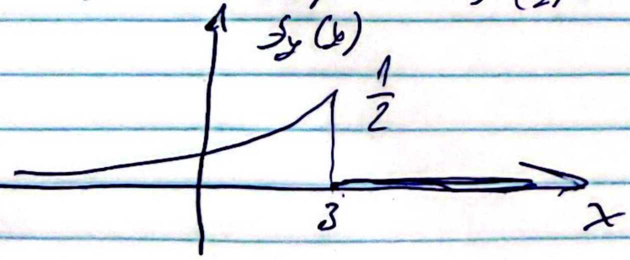
$$= \begin{cases} e^{\frac{y-3}{2}} & \text{for } y \le 3 \\ 1 & \text{for } y > 3 \end{cases}$$



$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}\left(u(y-3) - e^{\frac{y-3}{2}}u(3-y)\right) = \delta(y-3) + \left(\frac{d}{dy}e^{\frac{y-3}{2}}\right)u(3-y) + e^{\frac{y-3}{2}}\frac{d}{dy}u(3-y)$$

$$= \delta(y-3) + \left(\frac{1}{2}\right)e^{\frac{y-3}{2}}u(3-y) - e^{\frac{y-3}{2}}\delta(3-y) = \delta(y-3) + \left(\frac{1}{2}\right)e^{\frac{y-3}{2}}u(3-y) - \delta(3-y)$$

$$= \delta(y-3) + \left(\frac{1}{2}\right)e^{\frac{y-3}{2}}u(3-y) - \delta(y-3) = \left(\frac{1}{2}\right)e^{\frac{y-3}{2}}u(3-y)$$



10.

$$f_Y(y \mid X \geq 0) = \frac{d}{dy} F_Y(y \mid X \geq 0) = \frac{d}{dy} P(Y \leq y \mid X \geq 0)$$

$$= \frac{d}{dy} \frac{P(Y \leq y, X \geq 0)}{P(X \geq 0)} = \frac{1}{1 - P(X < 0)} \frac{d}{dy} P(Y \leq y, X \geq 0)$$

$$= \frac{1}{1 - F_X(0)} \frac{d}{dy} P(Y \leq y, X \geq 0) = \frac{1}{1 - F_0(0)} \frac{d}{dy} P(X^2 \leq y, X \geq 0)$$

$$= \frac{1}{1 - F_X(0)} \frac{d}{dy} P(-\sqrt{y} \leq X \leq \sqrt{y}, X \geq 0) = \frac{1}{1 - F_X(0)} \frac{d}{dy} P(0 \leq X \leq \sqrt{y})$$

$$= \frac{1}{1 - F_Y(0)} \frac{d}{dy} (F_X(\sqrt{y}) - F_X(0)) = \frac{1}{1 - F_X(0)} f_X(\sqrt{y})\left(\frac{1}{2}\right) \frac{1}{\sqrt{y}}$$

$$= \frac{U(y)}{1 - F_X(0)} \frac{f_X(\sqrt{y})}{2\sqrt{y}}$$ y is non-negative