1. $E[X^2] = E[Y^2] = E[XY]$

$E[(X-Y)^2] = E[X^2 - 2XY + Y^2]$

$E[XY] - 2E[XY] + E[XY]$

$E[XY] - E[XY] - E[XY] + E[XY]$

thus, $E[(X-Y)^2] = 0$

$X = Y$ in the MS sense

2. $P(|X-Y| \geq \varepsilon) \leq \frac{1}{\varepsilon^2} E[(X-Y)^2]$

$E[(X-Y)^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-y)^2 f_{X,Y}(x,y)\,dx\,dy$

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-y)^2 f_{X,Y}(x,y)\,dx\,dy \geq \int_{|x-y|\geq\varepsilon} \varepsilon^2 f_{X,Y}(x,y)\,dx\,dy$

$\varepsilon^2 \int_{|x-y|\geq\varepsilon} f_{X,Y}(x,y)\,dx\,dy = \varepsilon^2 P(|X-Y| \geq \varepsilon)$

$P(|X-Y| \geq \varepsilon) \leq \frac{E[(X-Y)^2]}{\varepsilon^2}$

3. $\frac{\partial}{\partial A} E[(Y-AX)^2] = 0$

$E[Y^2 - 2AXY + A^2X^2]$

$-2E[XY] + 2AE[X^2]$

$-2E[XY] + 2AE[X^2] = 0$

$A = \frac{E[XY]}{E[X^2]}$

4. $a_n \to a$

$E\left[\left(\frac{X_n - a_n}{n}\right)^2\right] \to 0$ as $n \to \infty$

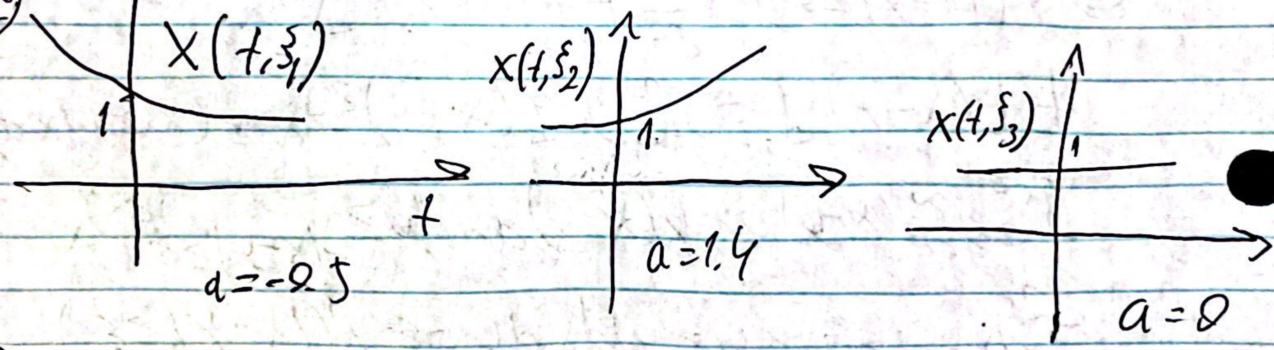
$E[(X_n - a)^2] = E\left[\left(\frac{X_n - a_n}{n} + \frac{a_n - a}{n}\right)^2\right]$

$= E\left[\left(\frac{X_n - a_n}{n}\right)^2\right] + 2E\left[\frac{X_n - a_n}{n}\,\frac{a_n - a}{n}\right] + E\left[\left(\frac{a_n - a}{n}\right)^2\right]$

$E[(X_n - a)^2] \to 0$, $X_n \to a$ in the MS sense

5.

a)



b. $\eta(t) = \int_{-\infty}^{\infty} e^{at} f_A(a)\,da$

$R(t_1, t_2) = \int_{-\infty}^{\infty} e^{at_1} e^{at_2} f_A(a)\,da$

$f_{x(t)}(x) = f_A\left(\frac{1}{t}\ln x\right)\left|\frac{d}{dx}\left(\frac{1}{t}\ln x\right)\right|$

$f_{x(t)}(x) = \frac{1}{|x|t} f_A\left(\frac{\ln x}{t}\right)$

$x > 0 \quad t \neq 0$

$\int_0^{\infty} f_{x(t)}(x)\,dx = \int_0^{\infty} \frac{1}{xt} f_A\left(\frac{\ln x}{t}\right) dx$

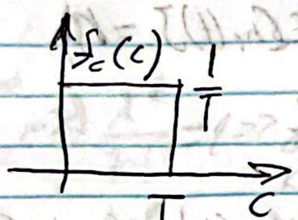
$a = \frac{1}{t}\ln x$

$\int_{-\infty}^{\infty} f_A(a) e^{at}\,da = 1$

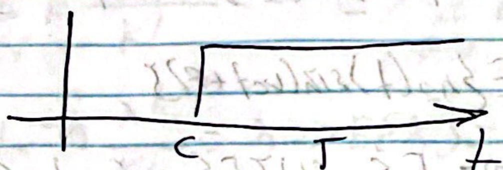
Scanned with CamScanner

6

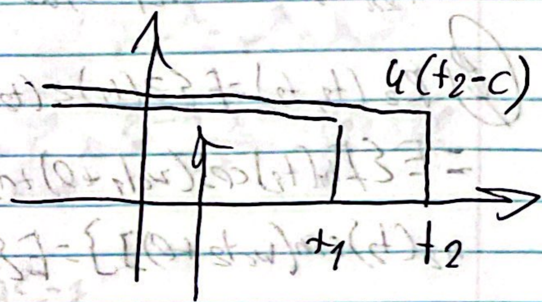
a)



$x(t)=u(t-c)$



for t_1 or $t_2<0$, $R(t_1,t_2)=0$, for t_1 and $t_2>T$, $R(t_1,t_2)=1$,



$R(t_1,t_2)=E\{u(t_1-c)u(t_2-c)\}=$

$=\int_{-\infty}^{\infty} u(t_1-c)\,u(t_2-c)f_c(c)\,dc=\frac{1}{T}\int_0^T u(t_1-c)u(t_2-c)\,dc=$

$=\frac{1}{T}\int_0^{\min(t_1,t_2)} 1\,dc=\frac{1}{T}\min(t_1,t_2)$

b) $R(t_1,t_2)=E\{\delta(t_1-c)\delta(t_2-c)\}=E\{\frac{d}{dt_1}u(t_1-c)\frac{d}{dt_2}u(t_2-c)\}$

$=\frac{d}{dt_1}\frac{d}{dt_2}E\{u(t_1-c)u(t_2-c)\}=\frac{d}{dt_1}\frac{d}{dt_2}\frac{1}{T}\min(t_1,t_2)=\frac{1}{T}\delta(t_1-t_2)$

$\frac{d}{dt_2}\min(t_1,t_2)=\frac{d}{dt_2}\{t_1u(t_2-t_1)+t_2u(t_1-t_2)\}$

$=t_1\delta(t_2-t_1)+u(t_1-t_2)-t_2\delta(t_1-t_2)$

$=t_1\delta(t_2-t_1)+u(t_1-t_2)-t_2\delta(t_2-t_1)$

$=t_1\delta(t_2-t_1)+u(t_1-t_2)-t_1\delta(t_2-t_1)$

$\frac{d}{dt_1}\frac{d}{dt_2}\min(t_1,t_2)=\frac{d}{dt_1}u(t_1-t_2)=\delta(t_1-t_2)$

7)

a) $n_1(t)$ and $n_2(t)$ are WSS $\Rightarrow E\{n_1(t)\} = \eta_1$

and $E\{n_2(t)\} = \eta_2$ are constants

$E\{Z(t)\} = E\{n_1(t)\cos(w_c t+\theta) + n_2(t)\sin(w_c t+\theta)\}$

$= E\{n_1(t)\cos(w_c t+\theta)\} = E\{n_2(t)\sin(w_c t+\theta)\}$

$= E\{n_1(t)\} E\{\cos(w_c t+\theta)\} + E\{n_2(t)\} E\{\sin(w_c t+\theta)\}$

$= \eta_1 \frac{1}{2\pi}\int_0^{2\pi} \cos(w_c t+\theta)d\theta + \eta_2 \frac{1}{2\pi}\int_0^{2\pi} \sin(w_c t+\theta)d\theta = \eta_1\cdot 0 + \eta_2 \cdot 0 = 0$

b) $R_Z(t_1, t_2) = E\{Z(t_1)Z(t_2)\}$

$= E\{[n_1(t_1)\cos(w_c t_1+\theta) + n_2(t_1)\sin(w_c t_1+\theta)][n_1(t_2)\cos(w_c t_2+\theta) + n_2(t_2)\sin(w_c t_2+\theta)]\} = E\{n_1(t_1)\cos(w_c t_1+\theta) n_1(t_2)\cos(w_c t_2+\theta)\} + E\{n_1(t_1)\cos(w_c t_1+\theta)\cdot n_2(t_2)\sin(w_c t_2+\theta)\} + E\{n_2(t_1)\sin(w_c t_1+\theta)\cdot n_1(t_2)\cos(w_c t_2+\theta)\} + E\{n_2(t_1)\sin(w_c t_1+\theta)\cdot n_2(t_2)\sin(w_c t_2+\theta)\}$

$= R_{n_1}(T) E\{\cos(w_c t_1+\theta)\cdot\cos(w_c t_2+\theta)\} + \eta_1\eta_2 E\{\cos(w_c t_1+\theta)\cdot\sin(w_c t_2+\theta)\} + \eta_1\eta_2 E\{\sin(w_c t_1+\theta)\cdot\cos(w_c t_2+\theta)\} + R_{n_2}(T) E\{\sin(w_c t_1+\theta)\cdot\sin(w_c t_2+\theta)\} = \frac{1}{2} R_{n_1}(T)\left[\frac{1}{2\pi}\int_0^{2\pi}\cos(w_c(t_1+t_2)+2\theta)d\theta + \cos w_c T\right] + \frac{1}{2}\eta_1\eta_2\left[\frac{1}{2\pi}\int_0^{2\pi}\sin(w_c(t_1+t_2)+2\theta)d\theta - \sin w_c T\right] + \frac{1}{2}\eta_1\eta_2\left[\frac{1}{2\pi}\int_0^{2\pi}\sin(w_c(t_1+t_2)+2\theta)d\theta + \sin w_c T\right] - \frac{1}{2}R_{n_2}(T)\left[\frac{1}{2\pi}\int_0^{2\pi}\cos(w_c(t_1+t_2)+2\theta)d\theta - \cos w_c T\right] = \frac{1}{2}(R_{n_1}(T) + R_{n_2}(T))\cdot \cos w_c T = R_Z(T)$

$Z(t)$ is WSS

c) $R_Z(0) = \frac{1}{2}(R_{n_1}(0) + R_{n_2}(0)) = \frac{1}{2}(E\{n_1^2(t)\} + E\{n_2^2(t)\})$

9 a)

$$\sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} = 1 \Rightarrow \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{\lambda} \Rightarrow \frac{d}{d\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = \frac{d}{d\lambda} e^{\lambda}$$

$$\Rightarrow \sum_{k=1}^{\infty} k \frac{\lambda^{k-1}}{k!} = e^{\lambda} \Rightarrow \sum_{k=1}^{\infty} k \frac{\lambda^k}{k!} = \lambda e^{\lambda} \Rightarrow$$

$$\sum_{k=0}^{\infty} k e^{-\lambda} \frac{\lambda^k}{k!} = \lambda \Rightarrow E\{X\} = \sum_{k=0}^{\infty} k e^{-\lambda} \frac{\lambda^k}{k!} = \lambda$$

$$\sigma^2 = E\{(x-\eta)^2\} = E\{x^2\} - \eta^2 = E\{x^2\} - \lambda^2$$

$$\sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} = \lambda e^{\lambda} \Rightarrow \frac{d}{d\lambda} \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} = \frac{d}{d\lambda} \lambda e^{\lambda} = e^{\lambda} + \lambda e^{\lambda}$$

$$\Rightarrow \sum_{k=0}^{\infty} k^2 \frac{\lambda^{k-1}}{k!} = (1+\lambda) e^{\lambda} \Rightarrow \sum_{k=0}^{\infty} k^2 e^{-\lambda} \frac{\lambda^k}{k!} = (1+\lambda)\lambda$$

$$\sigma^2 = E\{x^2\} - \lambda^2 = (1+\lambda)\lambda - \lambda^2 = \lambda$$

$$E\{x(t)\} = \lambda t, \quad E\{x^2(t)\} = (1+\lambda t)\lambda t \quad \sigma_x^2(t) = \lambda t$$

b)

$$R(t_1, t_2) = E\{x(t_1) x(t_2)\} = E\{x(t_1)(x(t_2) - x(t_1) + x(t_1))\}$$

$$= E\{x(t_1)\} E\{x(t_2) - x(t_1)\} + E\{x^2(t_1)\}$$

$$= \lambda t_1 (\lambda t_2 - \lambda t_1) + (1 + \lambda t_1)\lambda t_1 = \lambda (t_1 + \lambda t_1 t_2)$$

For $t_1 \geq t_2$ $\quad R(t_1, t_2) = \lambda (t_2 + \lambda t_1 t_2)$

$$R(t_1, t_2) = \lambda(\min(t_1, t_2) + \lambda t_1 t_2)$$

(9)

$E\{x(t)\} = \lambda t \Rightarrow E\{x(9)\} = 9\lambda = 6 \Rightarrow \lambda = \frac{2}{3}$

(a) $E\{x(t)\} = \lambda t \Rightarrow E\{x(8)\} = 8\lambda = 8\left(\frac{2}{3}\right) = \frac{16}{3} \approx 5.33$

$\sigma_x^2(t) = \lambda t \Rightarrow \sigma_x^2(8) = 8\lambda = \frac{16}{3} \approx 5.33$

(b) $P(x(t) \le k) = \sum_{i=0}^{k} P(n(0,t) = i) = \sum_{i=0}^{k} e^{-\lambda t} \frac{(\lambda t)^i}{i!}$

$P(x(2) \le 3) = \sum_{k=0}^{3} e^{-\frac{4}{3}} \frac{\left(\frac{4}{3}\right)^k}{k!} = e^{-\frac{4}{3}}\left(1 + \frac{4}{3} + \frac{1}{2}\left(\frac{4}{3}\right)^2 + \frac{1}{6}\left(\frac{4}{3}\right)^3\right) = 0.264 \cdot 3.62 = 0.955$