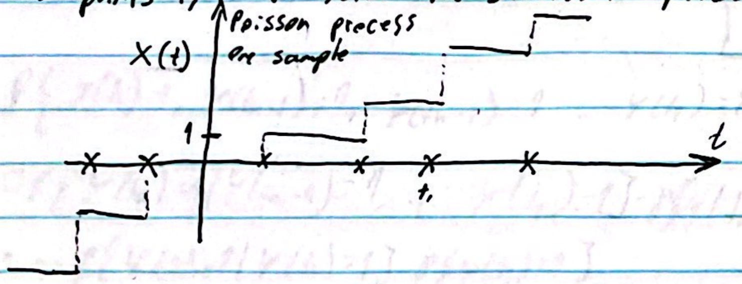
1

1. The number $n(t_1, t_2)$ of the points t_i in the interval $(t_1, t_2]$ of length $t = t_2 - t_1$ is a Poissen random variable with parameter λt:

$$P(n(t_1,t_2) = k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}$$

2. If the intervals (t_1, t_2) and (t_3, t_4) are non-overlapping, then the random variables $n(t_1, t_2)$ and $n(t_3, t_4)$ are independent

3. At a time moment t_i there is no more than one Poisson point. Using the points t_i we form the stochastic process $X(t) = n(0, t)$



First-order probability masses: $P(x(t) = k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}$

Mean: $E\{X(t)\} = \lambda t$

Autocorrelation: $R(t_1, t_2) = \lambda^2 t_1 t_2 + \lambda \min(t_1, t_2)$

Auto-covariance: $C(t_1, t_2) = \lambda^2 t_1 t_2 + \lambda \min(t_1, t_2) - (\lambda t_1)(\lambda t_2) = \lambda \min(t_1, t_2)$

2 Poisson increment process is defined as $Y(t) = \frac{X(t+\varepsilon) - X(t)}{\varepsilon}$ where $X(t)$ is Poissen process

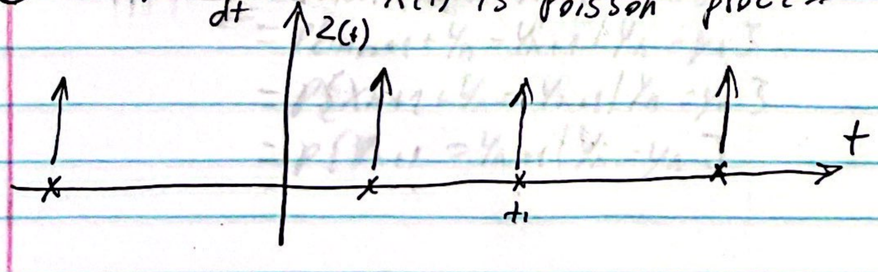
First-order probability masses: $P(Y(t) = \frac{n}{\varepsilon}) = e^{-\lambda\varepsilon}\frac{(\lambda\varepsilon)^n}{n!}$

Mean: $E\{Y(t)\} = \lambda$

Auto correlation: $R(t_1, t_2) = \begin{cases} \lambda^2 & |t_1 - t_2| \geq \varepsilon \\ \lambda^2 + \left(\frac{\lambda}{\varepsilon} - \frac{\lambda|t_1 - t_2|}{\varepsilon^2}\right) & |t_1 - t_2| < \varepsilon \end{cases}$

Auto-covariance $C(t_1, t_2) = \begin{cases} 0, & |t_1 - t_2| \geq \varepsilon \\ \frac{\lambda}{\varepsilon} - \frac{\lambda|t_1 - t_2|}{\varepsilon^2} & |t_1 - t_2| < \varepsilon \end{cases}$

3 $Z(t) = \frac{dX(t)}{dt}$ $X(t)$ is Poisson process



Probability masses: No definition.

Mean $E\{z(t)\} = \lambda$

Autocorrelation: $R(t_1, t_2) = \lambda^2 + \lambda\delta(t_1 - t_2)$

Auto-covariance: $C(t_1, t_2) = \lambda\delta(t_1 - t_2)$

$z(t) - \lambda$ has zero mean and ~~zero~~ Autocorrelation $R(t_1\ t_2) = \lambda\delta(t_1, t_2)$

$z(t) - \lambda$ is a white noise

(4) $n(t_1, t_2)$ of the points t_i in the interval (t_1, t_2) of length $t = t_2 - t_1$ is a Poisson random variable with parameter λt:

$$P(n(t_1, t_2) = k) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}$$

$$X(t) = \begin{cases} 1, & k \text{ is even in } (0, t) \\ -1, & k \text{ is odd in } (0, t) \end{cases}$$

first-order probability masses:

$$P(X(t) = 1) = e^{-\lambda t} \frac{e^{\lambda t} + e^{-\lambda t}}{2}, \quad P(X(t) = -1) = e^{-\lambda t} \frac{e^{\lambda t} - e^{-\lambda t}}{2}$$

mean: $E\{X(t)\} = e^{-2\lambda t}$

Autocorrelation: $R(t_1 t_2) = e^{-2\lambda|t_1 - t_2|}$

Auto-covariance: $C(t_1, t_2) = e^{-2\lambda|t_1 - t_2|} - e^{-2\lambda(t_1 + t_2)}$

(5)

$Y(t) = aX(t)$, $P(a = 1) = P(a = -1) = \frac{1}{2}$, $X(t)$ is semi-random telegraph signal.

First-order probability masses: $P(Y(t) = 1) = P(Y(t) = -1) = \frac{1}{2}$

$$P(Y(t) = 1) = P(aX(t) = 1) = P(X(t) = 1, a = 1) + P(X(t) = -1, a = -1)$$

$$= P(X(t) = 1)P(a = 1) + P(X(t) = -1)\ P(a = -1) = \frac{1}{2})$$

mean: $E\{Y(t)\} = 0$

Autocorrelation: $R(t_1\ t_2) = e^{-2\lambda|t_1 - t_2|}$

Auto-covariance: $C(t_1, t_2) = e^{-2\lambda|t_1 - t_2|}$

(6) Toss a fair coin n times. k is the number of heads in n tosses $x(nT) = ks - (n-k)s = (2k-n)s$

First-order probability masses

$$P(x(t) = X(nT) = rs) = \binom{n}{\frac{n+r}{2}} \frac{1}{2^n}, \text{ for } nT \le t < (n+1)T$$

mean: $E\{X(t)\} = 0$

Autocorrelation: $R(t_1, t_2) = E\{X(t_1) X(t_2)\} = E\{X(nT) X(mT)\} = \min(n, m) s^2$

Auto-covariance: $C(t_1, t_2) = E\{X(t_1) X(t_2)\} = E\{X(nT) X(mT)\} = \min(n, m) s^2$

(7) $W(t) = \lim\limits_{T \to 0,\ \frac{s^2}{T} \to a,\ n \to \infty} x(t)$ $\quad x(t)$ is random walk

First-order density function: $f(w, t) = \frac{1}{\sqrt{2\pi a t}} e^{-\frac{w^2}{2at}}$

mean: $E\{W(t)\} = 0$

Autocorrelation: $R(t_1, t_2) = \min(t_1, t_2) a$

Auto-covariance $C(t_1, t_2) = \min(t_1, t_2) a$

(2)

1. A stochastic process is called wide-sense-stationary process if its mean is a constant and autocorrelation is a function of time difference

2. Independent increment process means $X(t_1)-X(t_2)$ and $X(t_3)-X(t_4)$ on non-overlapping intervals are independent.

3. We say that a process $X(t)$ is a Markov process if the past has no influence on the future when the present is known

4. A process $X(t)$ is said to be normal if the random variables $X(t_1), X(t_2), \dots X(t_n)$ are jointly normal for any n, any $t_1 t_2 \dots t_n$ that is

$$f_X(x_1, x_2 \dots x_n; t_1, t_2, \dots t_n) = \frac{1}{\sqrt{2\pi}^n \sqrt{\det C}} e^{-\frac{1}{2}(x-\eta)^T C^{-1}(x-\eta)}$$

where $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $\underline{X} = \begin{bmatrix} X(t_1) \\ \vdots \\ X(t_n) \end{bmatrix}$, $\underline{\eta} = E\{\underline{X}\}$, $C = E\{(\underline{X}-\underline{\eta})(\underline{X}-\underline{\eta})^T\}$

5. A process is said to be a strict-sense-stationary process if its statistics is not affected by a shift on the time axis

6. White noise: for Wiener process $W(t)$, if we wish to discuss its derivative $V(t) = \frac{dW(t)}{dt}$, we have $E\{V(t)\} = \frac{dE\{W(t)\}}{dt} = 0$ and $R_V(t_1, t_2) = \alpha\delta(t_1 - t_2)$. Wiener process $W(t)$ does have derivative. We use $V(t) = \frac{dW(t)}{dt}$ just as a notation. When we need to do further discussions instead of $V(t)dt$, we use $dW(t)$. $W(t)$ is continuous with probability one. It has differential $dW(t)$. $V(t) = \frac{dW(t)}{dt}$ is Gaussian white noise with zero mean and impulse autocorrelation function. We wish to define more general white noise with zero mean and impulse autocorrelation function

Stochastic process $V(t)$ is said to be a White Noise if $E\{V(t)\}=0$ and $R_V(t_1, t_2) = q(t_1)\delta(t_1 - t_2)$

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7. A stochastic process $X(t)$ is said to be Mean-ergodic process if $\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X(t)dt = E\{X(t)\}$ in mean-square-sense

8. A stochastic process $X(t)$ is said to be a Martingale if $E\{X(t) \mid X(s)=x_1\} = x_1$ where $s \le t$

(3)

$X(t)$ is mean-ergodic if $\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X(t)dt = E\{X(t)\}$ in mss
$E\{X(t)\}$ must be a constant $E\{X(t)\} = \eta$

$X(t)$ is mean-ergodic iff $\lim_{T\to\infty} \frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} C(t_1,t_2)dt_1 dt_2 = 0$
A wss $X(t)$ is mean-ergodic iff $\lim_{T\to\infty} \frac{1}{2T}\int_{-2T}^{2T} C(\tau)\left(1-\frac{|\tau|}{2T}\right)d\tau = 0$

- If $X(t)$ is wss and $\int_{-\infty}^{\infty} |C(\tau)|d\tau < \infty$, then $X(t)$ is mean-ergodic
If $X(t)$ is wss and $\lim_{\tau\to\infty} C(\tau) = 0$ then $X(t)$ is mean-ergodic

1. Poison process has its mean $E\{X(t)\} = \lambda t$ not a constant. It cannot be mean-ergodic.

2. Poisson increment process has mean $E\{Y(t)\} = \lambda$
autocorrelation: $R(t_1,t_2) = \begin{cases} \lambda^2 & |t_1-t_2| \ge \varepsilon \\ \lambda^2 + \left(\frac{\lambda}{\varepsilon} - \frac{\lambda|t_1-t_2|}{\varepsilon^2}\right) & |t_1-t_2| < \varepsilon \end{cases}$

auto-covariance: $C(t_1,t_2) = \begin{cases} 0 & |t_1-t_2| \ge \varepsilon \\ \frac{\lambda}{\varepsilon} - \frac{\lambda|t_1-t_2|}{\varepsilon^2} & |t_1-t_2| < \varepsilon \end{cases}$

We see that $Y(t)$ is wss with $\int_{-\infty}^{\infty} |C(\tau)|d\tau = \lambda < \infty$
therefore, by sufficient condition 1, $Y(t)$ is mean-ergodic.

3. Poisson impulse process has mean $E\{Z(t)\} = \lambda$, autocorrelation $R(t_1,t_2) = \lambda^2 + \lambda\delta(t_1-t_2)$ and auto-covariance: $C(t_1,t_2) = \lambda\delta(t_1-t_2)$
$Z(t)$ is wss
$\lim_{T\to\infty} \frac{1}{2T}\int_{-2T}^{2T} C(\tau)\left(1-\frac{|\tau|}{2T}\right)d\tau = \lim_{T\to\infty} \frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|\tau|}{2T}\right)d\tau = \lim_{T\to\infty}$

$\frac{1}{2T}\int_{-2T}^{2T} \lambda\delta(\tau)d\tau = \lim_{T\to\infty} \frac{\lambda}{2T} = 0$ $\qquad Z(t)$ is mean-ergodic

(4) Semi-random telegraph signal $X(t)$ has mean $E\{X(t)\}=e^{-2\lambda t}$. It is not a constant. Therefore, it is not mean-ergodic

(5) Random telegraph signal $Y(t)$ has mean $E\{Y(t)\}=0$, autocorrelation $R(t_1,t_2)=e^{-2\lambda|t_1-t_2|}$ and auto-covariance $C(t_1,t_2)=e^{-2\lambda|t_1-t_2|}$

we see that $Y(t)$ is wss

$\int_{-\infty}^{\infty}|C(\tau)|d\tau=\int_{-\infty}^{\infty}e^{-2\lambda|\tau|}d\tau=2\int_0^{\infty}e^{-2\lambda\tau}d\tau=\frac{1}{\lambda}<\infty$ therefore by sufficient condition 1, $Y(t)$ is mean-ergodic

(4) $E\{X(t)Y(t)\}=E\{X(t)(X(t)\cdot h(t))\}=E\{X(t)\int_{-\infty}^{\infty}X(t-\tau)h(\tau)d\tau\}=\int_{-\infty}^{\infty}E\{X(t)X(t-\tau)\}h(\tau)d\tau=\int_{-\infty}^{\infty}R_{xx}(t,t-\tau)h(\tau)d\tau$
$=\int_{-\infty}^{\infty}q(t)\delta(t-t+\tau)h(\tau)d\tau=q(t)\int_{-\infty}^{\infty}\delta(\tau)h(\tau)d\tau=q(t)\int_{-\infty}^{\infty}\delta(\tau)h(0)d\tau=$
$=q(t)h(0)$

(5) a. if $R_{xx}(\tau)=5\delta(\tau)$ and

$Y'(t)+2Y(t)=X(t)$ for all t

B. if (1) holds for $t\geq 0$ only and $Y(0)=0$

(a) $E\{Y^2(t)\}=5\int_{-\infty}^{\infty}|h(\alpha)|^2d\alpha=5\int_{-\infty}^{\infty}(e^{-2\alpha})^2u(\alpha)d\alpha=5\int_0^{\infty}e^{-4\alpha}d\alpha=\frac{5}{4}$

(B) suppose $V(t)$ is white noise with $R_{VV}(\tau)=5\delta(\tau)$. Then for this case, the input is $X(t)=V(t)u(t)$. $R_{xx}(t_1,t_2)u(t_1)u(t_2)=5u(t_1)\delta(t_1-t_2)$ therefore $X(t)$ is also a white noise with average intensity $5u(\tau)$. Then, $E\{Y^2(t)\}=5\int_{-\infty}^{\infty}u(t-\alpha)h^2(\alpha)d\alpha=5\int_{-\infty}^{\infty}u(t-\alpha)e^{-4\alpha}u(\alpha)d\alpha=5\int_0^t e^{-4\alpha}d\alpha u(t)=\frac{5}{4}(1-e^{-4t})u(t)$

(6) $E\{Y_{n+1}|Y_n=y_n\}=E\{X_{n+1}+Y_n|Y_n=y_n\}=E\{X_{n+1}+Y_n|Y_n=y_n\}=E\{X_{n+1}|Y_n=y_n\}+E\{Y_n|Y_n=y_n\}=E\{X_{n+1}\}+y_n=y_n$

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7

1) $R^*(\tau) = E\{X^*(t+\tau)X(t)\} = E\{X(t)X^*(t+\tau)\} = R(-\tau)$

2) If $X(t)$ is real, then $R(-\tau) = R^*(\tau) = R(\tau)$

3) $R_{YX}(\tau) = E\{Y(t+\tau)X^*(t)\}$

$R^*_{YX}(\tau) = E\{Y^*(t+\tau)X(t)\} = E\{X(t)Y^*(t+\tau)\} = R_{XY}(-\tau)$

4) if $x(t)$ and $Y(t)$ are real then $R_{YX}(\tau) = R_{YX}(\tau) = R_{XY}(-\tau)$

8.

1) $X(t)$ is ms periodic

2) $R(T) = R(0)$

3) $R(\tau + mT) = R(\tau)$ for all τ, all integer m

1) <=> 2) $X(t)$ is ms periodic $\Longleftrightarrow E\{(X(t+T) - X(t))^2\} = 2(R(0) - R(T)) = 0 \Longleftrightarrow R(T) = R(0)$

3) => 2) just let $\tau = 0$, $m = 1$ in 3)

2) => 3)

$E^2\{XY\} \leq E\{X^2\}\ E\{Y^2\}$

$E^2\{(X(t+\tau+T) - X(t+\tau))X(t)\}$

$= (R(\tau+T) - R(\tau))^2 \leq E\{(X(t+\tau+T) - X(t+\tau))^2\}\ E\{X^2(t)\}$

$= 2(R(0) R(T)) \cdot R(0)$