1

1. $\int_{-1}^{1}\int_{-1}^{1}\left(\frac{1}{A}\right)(1+xy)\,dx\,dy = \frac{4}{A} = 1 \Rightarrow A = 4$

2. and 3 $E\{XY\} = \int_{-1}^{1}\int_{-1}^{1} xy\frac{1+xy}{4}\,dx\,dy = \frac{1}{4}\left(\int_{-1}^{1}\int_{-1}^{1} xy\,dx\,dy + \int_{-1}^{1}\int_{-1}^{1} x^2y^2\,dx\,dy\right) = \frac{1}{9}$

$E\{X\} = \int_{-1}^{1}\int_{-1}^{1} x\frac{1+xy}{4}\,dx\,dy = \frac{1}{4}\left(\int_{-1}^{1}\int_{-1}^{1} x\,dx\,dy + \int_{-1}^{1}\int_{-1}^{1} x^2y\,dx\,dy\right) = 0$

$E\{XY\} \neq E\{X\}E\{Y\} \Rightarrow X, Y$ are not uncorrelated $\Rightarrow$
X, Y are not independand.

4. $\begin{cases} z = x^2 \\ w = y^2 \end{cases}$, $J = \begin{vmatrix} 2x & 0 \\ 0 & 2y \end{vmatrix} = 4xy$, $\begin{cases} z = x^2 \\ w = y^2 \end{cases}$ has four roots

when $0 \le z \le 1$, $0 \le w \le 1$, $|J(i)| = 4\sqrt{z}\sqrt{w}$, $i = 1,2,3,4$ and

$$f_{zw}(z,w) = \frac{f_{xy}(\sqrt{z},\sqrt{w})}{|J(1)|} + \frac{f_{xy}(-\sqrt{z},\sqrt{w})}{|J(2)|} + \frac{f_{xy}(\sqrt{z},-\sqrt{w})}{|J(3)|} + \frac{f_{xy}(-\sqrt{z},-\sqrt{w})}{|J(4)|}$$

$$= \frac{1}{4\sqrt{z}\sqrt{w}}\frac{1}{4}\left(1+\sqrt{z}\sqrt{w}+1-\sqrt{z}\sqrt{w}+1-\sqrt{z}\sqrt{w}+1+\sqrt{z}\sqrt{w}\right) = \frac{1}{2\sqrt{z}}\frac{1}{2\sqrt{w}}$$

When $z<0$ or $z>1$ and $w<0$ or $w>1$, $f_{zw}(z,w) = 0$

$$f_{zw}(z,w) = \begin{cases} \frac{1}{2\sqrt{z}}\frac{1}{2\sqrt{w}}, & \text{for } 0 \le z \le 1,\ 0 \le w \le 1 \\ 0, & \text{otherwise} \end{cases} = f_z(z)\cdot f_w(w)$$

with $f_z(z) = \begin{cases} \frac{1}{2\sqrt{z}} & \text{for } 0 \le z \le 1 \\ 0 & \text{otherwise} \end{cases}$ and $f_w(w) = \begin{cases} \frac{1}{2\sqrt{w}}, & \text{for } 0 \le w \le 1 \\ 0, & \text{otherwise} \end{cases}$

therefore, X^2, Y^2 are independent

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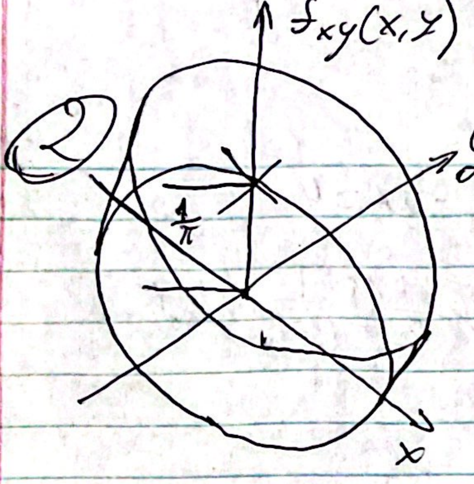
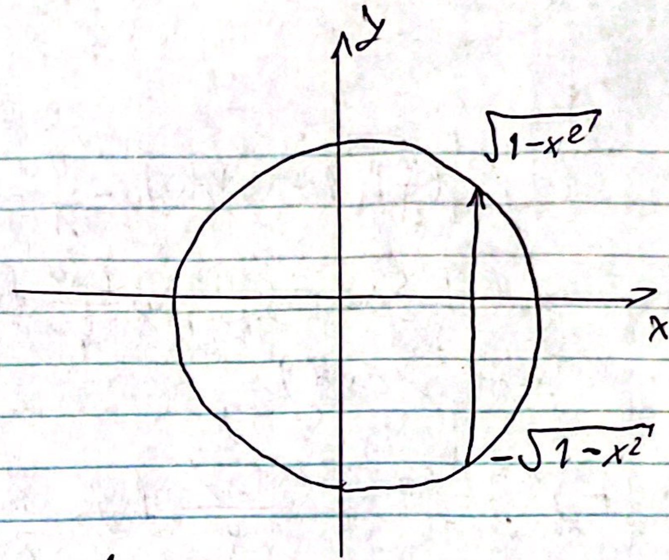
(2)





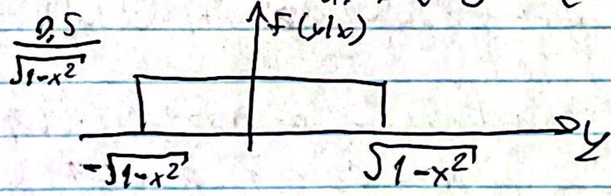
a) it is easy to see $B=\frac{1}{\pi}$

b) $E\{X\}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{xy}(x,y)\,dx\,dy=\left(\frac{1}{\pi}\right)\int_{-1}^{1} x\left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1\,dy\right)dx=$

$=\left(\frac{2}{\pi}\right)\int_{-1}^{1} \underbrace{x\sqrt{1-x^2}}_{\text{odd function}}\,dx=0$

It is easy to see $E\{X|y\}=0$, for every y. $E\{X|Y\}=0$

Or, $E\{X\}$ can be found: $E\{X\}=E\{E\{X|Y\}\}=E\{0\}=0$



due to symmetry, $E\{Y\}=0$. $E\{Y|X\}=0$. $E\{Y|x\}=0$

$E\{XY\}=\left(\frac{1}{\pi}\right)\int_{-1}^{1}\left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} xy\,dy\right)dx=\left(\frac{1}{\pi}\right)\int_{-1}^{1} x\left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \underbrace{y}_{\text{odd}}\,dy\right)dx=0$

Or $E\{XY\}=E\{E\{XY|X\}\}=E\{X\underbrace{E\{Y|X\}}_{0}\}=0$

That is $E\{XY\}=E\{X\}E\{Y\}$. We conclude that X,Y are uncorrelated.

$f_X(x)=\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f_{XY}(x,y)\,dy=\left(\frac{2}{\pi}\right)\sqrt{1-x^2}$ for $|x|\le 1$, and zero, otherwise

similarly $f_Y(y)=\left(\frac{2}{\pi}\right)\sqrt{1-y^2}$ for $|y|\le 1$ and zero, otherwise

$f_X(x)\cdot f_Y(y)=\left(\frac{2}{\pi}\right)\sqrt{1-x^2}\left(\frac{2}{\pi}\right)\sqrt{1-y^2}\ne f_{XY}(x,y)$

X, Y are not independent. Or we can explain it is: when $x=-1$, we must have $Y=0$. That is X, Y are not independent

③

X, Y with $f_X(x)=e^{-x}u(x)$, $f_Y(y)=e^{-y}u(y)$

a) $Z=X-Y$

Let $W=-Y$, $f_W(w)=\frac{f_Y(y)}{|g'(y)|}=\frac{f_Y(-w)}{|-1|}=f_Y(-w)$

Therefore, the density of $-Y$ equals $f_Y(-y)$

Characteristic function method

$f_Z(z)=f_X(z) * f_Y(-z)$

$f_X(z)=e^{-z}u(z) \Leftrightarrow \Phi_X(jw)=\frac{1}{1-jw}$, $f_W(z)=e^{z}u(-z) \Leftrightarrow \Phi_W(jw)=\Phi_X(-jw)=\frac{1}{1+jw}$

$f_Z(z)=f_X(z) * f_W(z) \Leftrightarrow \Phi_X(jw)\Phi_W(jw)=\frac{1}{1-jw}\frac{1}{1+jw}=\frac{1}{1+w^2}$

Fourier transform pair: $\frac{1}{2}e^{-|z|} \Leftrightarrow \frac{1}{1+w^2}$

therefore, $f_Z(z)=\frac{1}{2}e^{-|z|}$

b) $Z=XY$.

introduce an auxiliary variable $W=Y$

then $\begin{cases} z=xy \\ w=y \end{cases}$, $J(x,y)=\begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{vmatrix}=\begin{vmatrix} y & x \\ 0 & 1 \end{vmatrix}=y$

$f_{ZW}(z,w)=\frac{f_{ZW}(x_i,y_i)}{|J(x_i,y_i)|}=\frac{1}{|w|}f_{XY}(\frac{z}{w},w)=\frac{1}{|w|}f_X(\frac{z}{w})f_Y(w)$

$f_Z(z)=\int_{-\infty}^{\infty}f_{ZW}(z,w)dw=\int_{-\infty}^{\infty}\frac{1}{|w|}e^{-\frac{z}{w}}u(\frac{z}{w})e^{-w}u(w)dw$

$=\int_0^{\infty}\frac{1}{w}e^{-(\frac{z}{w}+w)}dw \cdot u(z)$

$\int_{-\infty}^{\infty}f_Z(z)dz=\int_0^{\infty}\int_0^{\infty}\frac{1}{w}e^{-(\frac{z}{w}+w)}dwdz=\int_0^{\infty}\frac{1}{w}e^{-w}(\int_0^{\infty}e^{-\frac{z}{w}}dz)dw$

$=\int_0^{\infty}\frac{1}{w}e^{-w}w\,dw=1$

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(4)

$$\begin{cases} z = x+y \\ w = \frac{x}{y} \end{cases} \quad x \geq 0, y \geq 0 \Rightarrow z \geq 0, w \geq 0, \quad \begin{cases} x = \frac{wz}{w+1} \\ y = \frac{z}{w+1} \end{cases}$$

$$|J(x,y)| = \left| \begin{vmatrix} 1 & 1 \\ \frac{1}{y} & \frac{-x}{y^2} \end{vmatrix} \right| = \left| -\left(\frac{x}{y^2} + \frac{1}{y}\right) \right| = \frac{(w+1)^2}{z}$$

$$f_{zw}(z,w) = \frac{f_{xy}(x,y)}{|J|} \overset{x,y \text{ indep}}{=} \frac{z}{(w+1)^2} e^{-(x+y)} = \frac{z}{(w+1)^2} e^{-z}$$

$$f_{zw}(z,w) = ze^{-z}U(z) \cdot \frac{1}{(w+1)^2} u(w)$$

We see that z, w, are independent

$ze^{-z}u(z)$ and $\frac{1}{(w+1)^2}u(w)$ are density functions.

(5)

$$E\{|x|x\} = \int_{-\infty}^{\infty} |x| x f(x) dx = 0 \text{ and } E\{x\} = \int_{-\infty}^{\infty} x f(x) dx = 0$$

$\uparrow$ odd function

$$C_{|x|,x} = E\{|x|x\} - E\{|x|\}E\{x\} = 0$$

$|x|$ and x are ~~~~ uncorrelated

When x is given, $|x|$ is fixed value so they are not independent

(6)

X, Y are normal and independent

X, Y are jointly normal $\quad \begin{cases} Z = aX + bY \\ W = cX + dY \end{cases} \Rightarrow Z, W$ are jointly normal

$a = c = \alpha$, $b = \beta$, $d = -\beta$ therefore Z, W are jointly normal

let's show Z, W are independent

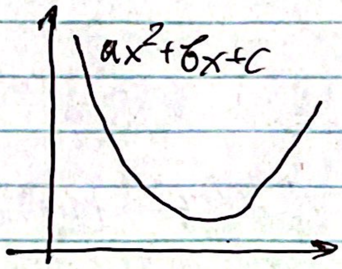
the covariance of Z, W is

$C_{zw} = E\{(Z - \eta_z)(W - \eta_w)\} = E\{(aX + \beta Y - a\eta_x - \beta\eta_y)(aX - \beta Y - a\eta_x + \beta\eta_y)\} = E\{[a(X - \eta_x) + \beta(Y - \eta_y)][a(X - \eta_x) + \beta(Y - \eta_y)]\}$

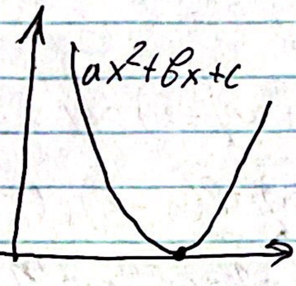
$= a^2\sigma_x^2 - aBE\{(X-\eta_x)(Y-\eta_y)\} + aBE\{(X-\eta_x)(Y-\eta_y)\} - B^2\sigma_Y^2 =$
$a^2\sigma_x^2 - B^2\sigma_Y^2$ $a^2\sigma_x^2 = B^2\sigma_Y^2$, $C_{zw}=0$ z, w are uncorrelated
and z, w are jointly normal, so z, w are independent

7.

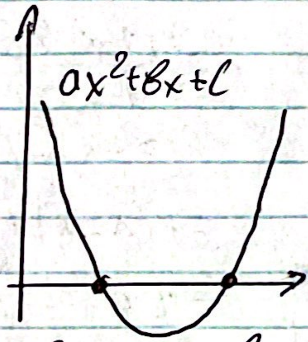
a) Since $|E\{XY\}| \le E\{|X||Y|\}$ let's prove that
$E\{|X||Y|\}^2 \le E\{|X|^2\}E\{|Y|^2\}$



$ax^2+bx+c=0$
has no real roots



$ax^2+bx+c=0$
has repeated real roots



$ax^2+bx+c=0$
has two real roots

$ax^2+bx+c \ge 0$ for any $x \Rightarrow ax^2+bx+c=0$ has no real root or has repeated real roots $\Rightarrow$ the discriminant
$b^2-4ac \le 0$

Now let us apply the above analysis to our problem. It is easy to see that

$$E\{(z|X|+|Y|)^2\} \ge 0 \quad \text{for any } z$$

$$= z^2 \underbrace{E\{|X|^2\}}_{a} + \underbrace{2E\{|X||Y|\}}_{b} z + \underbrace{E\{|Y|^2\}}_{c} \ge 0$$

$$\Rightarrow \underbrace{(2E\{|X||Y|\})^2}_{b} - 4E\{\underbrace{|X|^2}_{a}\}\underbrace{E\{|Y|^2\}}_{c} \le 0$$

$$(2E\{|X||Y|\})^2 \le 4E\{|X|^2\}E\{|Y|^2\}$$

b) $\left(\sqrt{E\{|X|^2\}} + \sqrt{E\{|Y|^2\}}\right)^2$

$= E\{|X|^2\} + 2\sqrt{E\{|X|^2\}E\{|Y|^2\}} + E\{|Y|^2\}$

$\ge E\{|X|^2\} + 2\sqrt{E\{|X||Y|\}^2} + E\{|Y|^2\} = E\{|X|^2\} + 2E\{|X||Y|\} +$
$E\{|Y|^2\} = E\{(|X|+|Y|)^2\} \ge E\{|X+Y|^2\}$

(8)

$1 = E\left\{\frac{x+y+z}{x+y+z}\right\} = E\left\{\frac{x}{x+y+z}\right\} + E\left\{\frac{y}{x+y+z}\right\} + E\left\{\frac{z}{x+y+z}\right\} =$

$= 3E\left\{\frac{x}{x+y+z}\right\} \Rightarrow E\left\{\frac{x}{x+y+z}\right\} = \frac{1}{3}$

(9) $P(X \le n) = p + pq + pq^2 + pq^3 + \dots + pq^n = p(1 + q + q^2 + q^3 + \dots + q^n)$

$= p\frac{1-q^{n+1}}{1-q} = 1 - q^{n+1}$

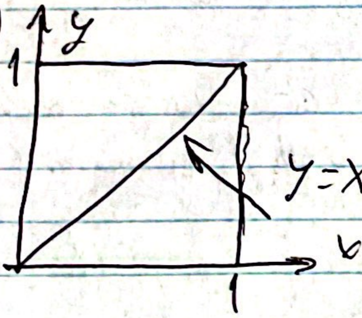
$P(Z \le n) = P(\min(X, Y) \le n) = P(X \le n \text{ or } Y \le n)$

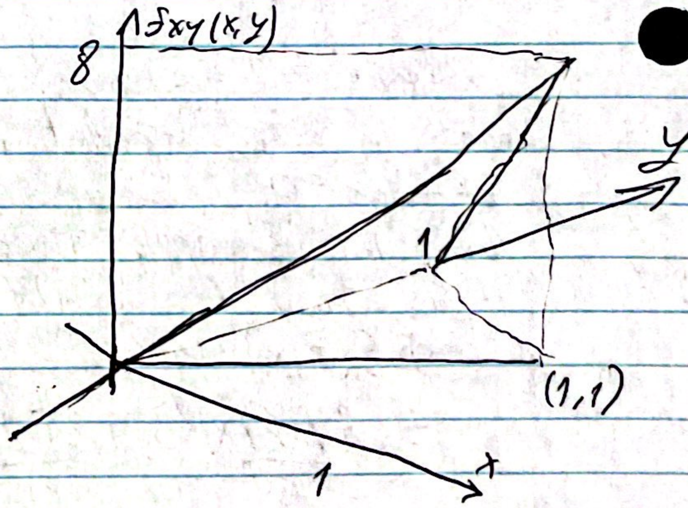
$= P(X \le n) + P(Y \le n) - P(X \le n)P(Y \le n) = 1 - q^{n+1} + 1 - q^{n+1} - (1 - q^{n+1})^2$

$= 1 - q^{n+1} + 1 - q^{n+1} - 1 + 2q^{n+1} - q^{2(n+1)} = 1 - q^{2(n+1)}$

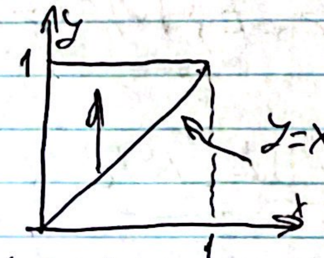
$P(Z = n) = P(Z \le n) - P(Z \le n-1) = 1 - q^{2(n+1)} - (1 - q^{2n}) = q^{2n}(1 - q^2)$

(10)





a)



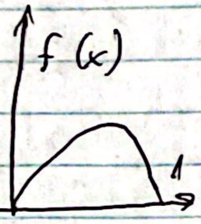
$\int_0^1 \left(\int_x^1 Axy\,dy\right)dx = A\int_0^1 \left(x\int_x^1 y\,dy\right)dx = A\int_0^1 \left(x\frac{1}{2}y^2\Big|_x^1\right)dx$

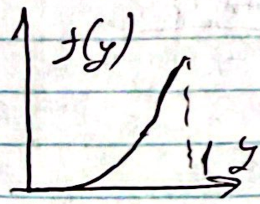
$\frac{A}{2}\int_0^1 x(1-x^2)\,dx = \frac{A}{2}\left(\int_0^1 x\,dx - \int_0^1 x^3\,dx\right) = \frac{A}{8} = 1 \quad A = 8$

$f_{XY}(x,y) = \begin{cases} 8xy, & 0 \le x \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$

b) $f_X(x) = \int_x^1 8xy\,dy = 4(1-x^2)$, for $0 \le x \le 1$; zero otherwise

$f_Y(y) = \int_0^y 8xy\,dx = 4y^3$ for $0 \le y \le 1$ zero otherwise





c) We see that $f_{XY}(x,y) \neq f_X(x) f_Y(y)$
therefore X and Y are not independent

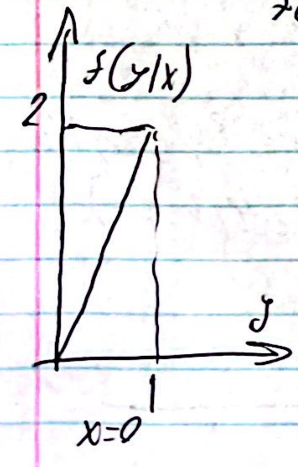
d) $E\{X\} = \int_0^1 4x^2(1-x^2)\,dx = \frac{8}{15}$. $E\{Y\} = \int_0^1 4y^4\,dy = \frac{4}{5}$

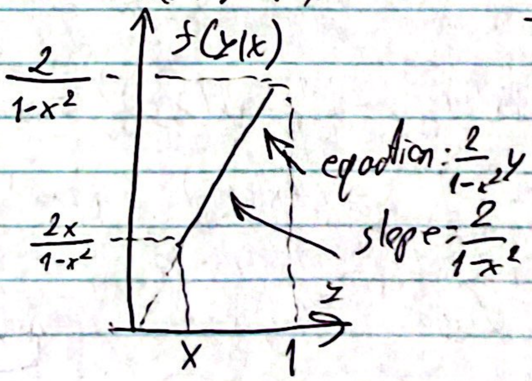
$E\{XY\} = \int_0^1 \left(\int_x^1 xy\,8xy\,dy\right)dx = 8\int_0^1 \left(x^2 \int_x^1 y^2\,dy\right)dx =$

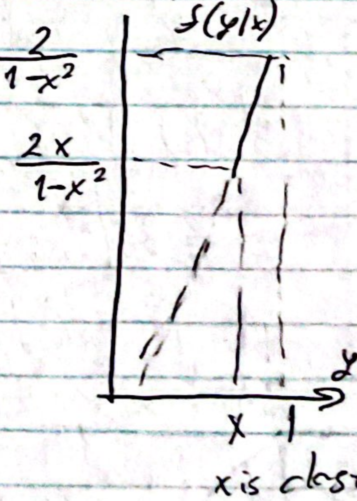
$= \frac{8}{3}\int_0^1 x^2(1-x^3)\,dx = \frac{8}{3}\left(\frac{1}{3} - \frac{1}{6}\right) = \frac{4}{9}$

e) $E\{XY\} \neq E\{X\}E\{Y\}$ therefore X and Y are not uncorrelated

f) $f(y|x) = \frac{f(x,y)}{f(x)} = \frac{8xy}{4x(1-x^2)} = \frac{2y}{1-x^2}$, for $x \le y \le 1$ zero otherwise

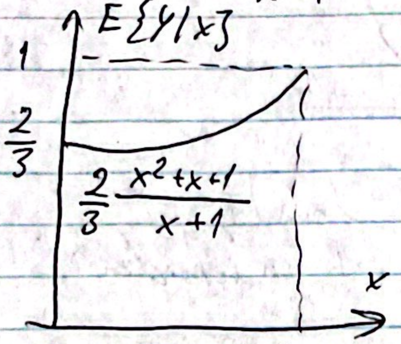






9) $E\{y|x\} = \int_x^1 y f(y|x)\,dy = \int_x^1 \frac{2y^2}{1-x^2}\,dx = \frac{2}{1-x^2}\int_x^1 y^2\,dy = \frac{2}{3}\frac{1}{1-x^2}$

$(1-x^3) = \frac{2}{3}\frac{x^2+x+1}{x+1}$

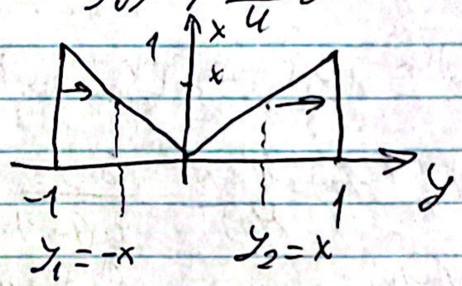


$E\{y|x\} = \frac{2}{3}\frac{x^2+x+1}{x+1}$

11

$A\int_{-1}^1 \left(\int_0^{|y|} y^2\,dx\right)dy = A\int_{-1}^1 y^2|y|\,dy = 2A\int_0^1 y^3\,dy = \frac{2A}{4} = 1$ thus $A=2$

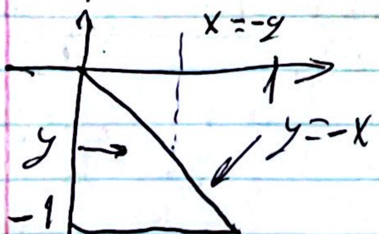
$f_{xy}(x,y) = \begin{cases} 2y^2 & 0<x<|y|<1 \\ 0, & \text{otherwise} \end{cases}$

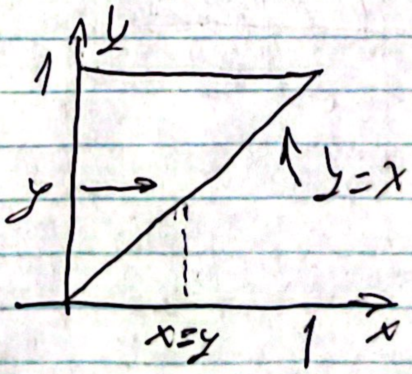


$f_x(x) = \int_{-\infty}^{\infty} f_{xy}(x,y)\,dy = \int_{-1}^{1} f_{xy}(x,y)\,dy = \int_{-1}^{-x} 2y^2\,dy + \int_x^1 2y^2\,dy$

$= \frac{2}{3}y^3\Big|_{-1}^{-x} + \frac{2}{3}y^3\Big|_x^1 = \frac{2}{3}(-x^3+1) + \frac{2}{3}(1-x^3) = \frac{4}{3}(1-x^3)$

that is $f_x(x) = \begin{cases} \frac{4}{3}(1-x^3), & \text{for } 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$





$f_y(y) = \int_{-\infty}^{\infty} f_{xy}(x,y)\,dx = \int_0^{-y} 2y^2\,dx = -2y^3$

$f_y(y) = \int_{-\infty}^{\infty} f_{xy}(x,y)\,dx = \int_0^{y} 2y^2\,dx = 2y^3$

$|y| \le 1$, $f_y(y) = 2|y|^3$; zero, otherwise

a) $E\{X\} = \int_0^1 x \frac{4}{3}(1-x^3)\,dx = \frac{2}{5}$

$E\{Y\} = \int_{-1}^1 y\, 2|y|^3\, dy = 0$ because $y|y|^3$ is an odd function

$E\{XY\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{XY}(x,y)\,dx\,dy = \int_{-1}^1 \left(\int_{-\infty}^{\infty} xy\, f_{XY}(x,y)\,dx\right)dy$

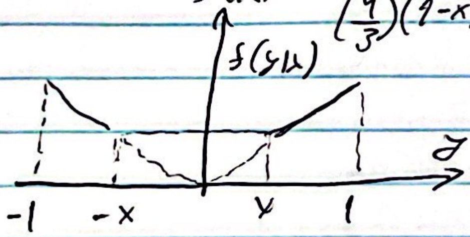
$= \int_{-1}^0 \left(\int_0^{-y} xy\, 2y^2\, dx\right)dy + \int_0^1 \left(\int_0^{y} xy\, 2y^2\, dx\right)dy = 2\int_{-1}^0 y^3 \left(\int_0^{-y} x\,dx\right)dy + 2\int_0^1 y^3 \left(\int_0^{y} x\,dx\right)dy$

$= \int_{-1}^0 y^3 (x^2)\big|_0^{-y}\, dy + \int_0^1 y^3 (x^2)\big|_0^{-y}\, dy = \int_{-1}^0 y^5\, dy + \int_0^1 y^5\, dy = \frac{1}{6} y^6\big|_{-1}^0 + \frac{1}{6} y^6\big|_0^1 = -\frac{1}{6} + \frac{1}{6} = 0$

Or, $E\{XY\} = E\{E\{XY|X\}\} = E\{X \cdot E\{Y|X\}\} = E\{0\} = 0$ because $E\{Y|X\} = 0$

b) $E\{XY\} = E\{X\}E\{Y\}$. So X and Y are uncorrelated. however they are not independent because when $X=1$ $|Y|=1$

c) $f(y|x) = \frac{f(x,y)}{f(x)} = \frac{2y^2}{\left(\frac{4}{3}\right)(1-x^3)} = \frac{3}{2}\frac{y^2}{1-x^3}$ for $0 \le x \le |y| \le 1$, zero otherwise



d) $E\{Y|X\} = \int_{-1}^1 y\, f(y|x)\,dy = \int_{-1}^1 y \frac{3}{2}\frac{y^2}{1-x^3}\,dy = \frac{3}{2}\frac{1}{1-x^3}\int_{-1}^1 y^3\,dy = 0$

$E\{Y|X\} = 0$

$E\{X|y\} = \int_0^{|y|} x\, f(x|y)\,dx = \int_0^{|y|} x \frac{1}{|y|}\,dx = \frac{|y|}{2}$, for $|y| \le 1$ zero, otherwise

$E\{X|y\} = \frac{|y|}{2}$