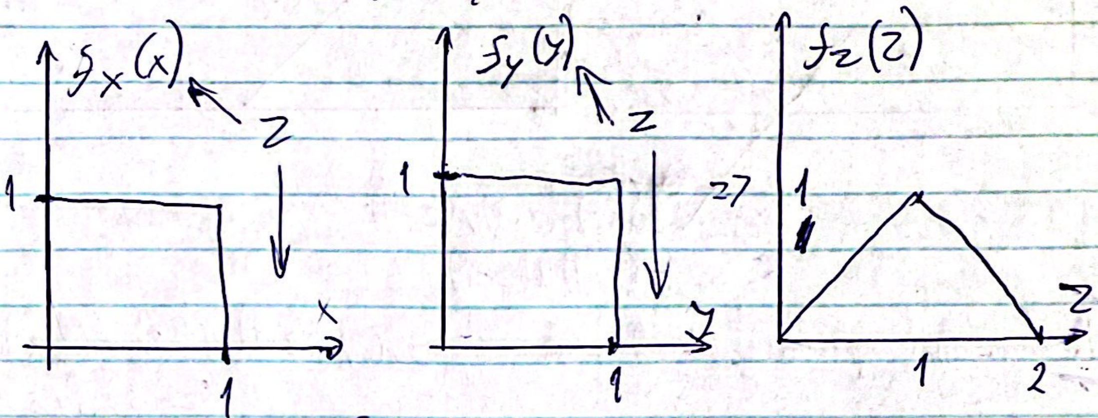
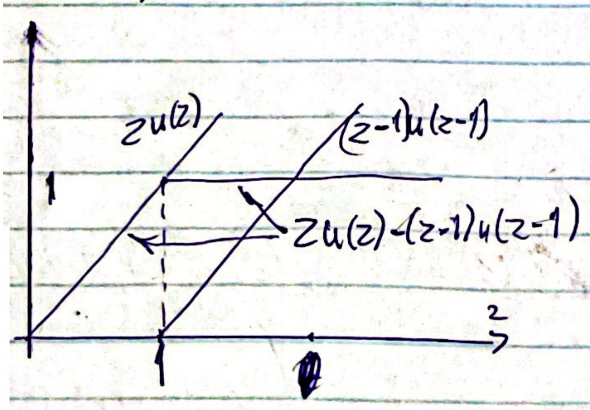
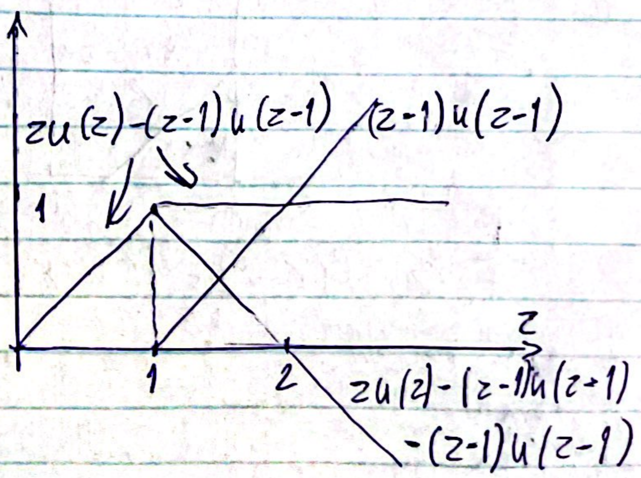
(1)

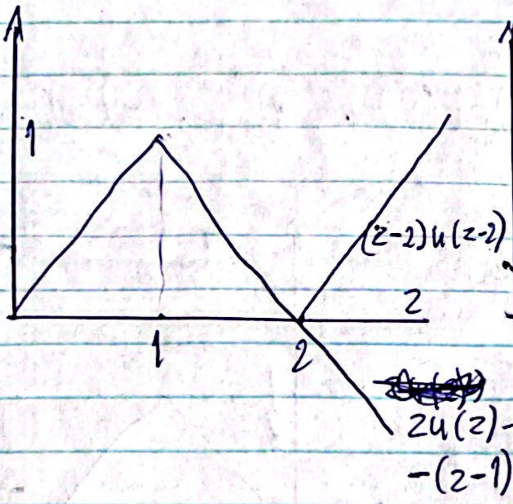
calculate $f_z(z)$
for $z=x+y$ with $f_x(x)=u(x)-u(x-1)$, $f_y(y)=u(y)-u(y-1)$
shift result to right by 1.

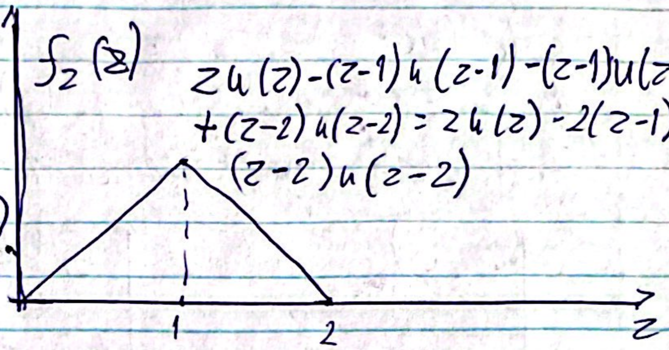


$f_z(z)=f_x(z)*f_y(z)=\int_{-\infty}^{\infty} f_x(a)f_y(z-a)da=\int_{-\infty}^{\infty}(u(a)-u(a-1))$
$(u(z-a)-u(z-a-1))da=\int_{-\infty}^{\infty}u(a)u(z-a)da-\int_{-\infty}^{\infty}u(a)\,u(z-1-a)da$
$-\int_{-\infty}^{\infty}u(a-1)u(z-a)da+\int_{-\infty}^{\infty}u(a-1)u(z-1-a)da=\int_0^{z}u(a)u(z-a)$
$da\cdot u(z)-\int_0^{z-1}u(a)u(z-1-a)\,da\,u(z)-\int_1^{z}u(a-1)u(z-a)da\cdot u(z-1)+$
$\int_1^{z-1}u(a-1)u(z-1-a)da\cdot u(z-2)=\int_0^{z}1da\cdot u(z)-\int_0^{z-1}1da\,u(z)-$
$\int_1^{z}1da\cdot u(z-1)+\int_1^{z-1}1da\cdot u(z-2)=zu(z)-(z-1)u(z-1)-(z-1)$
$u(z-1)+(z-2)u(z-2)$



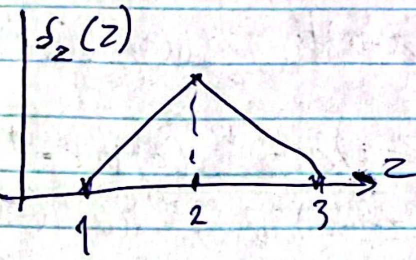






$zu(z)-(z-1)u(z-1)-(z-1)u(z-1)$
$+(z-2)u(z-2) = zu(z)-2(z-1)u(z-1)+(z-2)u(z-2)$

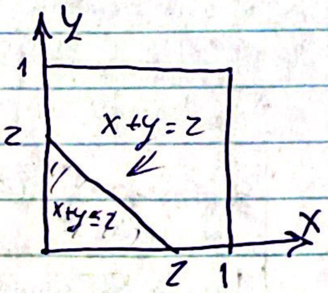
te origild problem answer is $f_Z(z)=(z-1)u(z-1)-2(z-2)u(z-2)+(z-3)u(z-3)$



2. Geometric

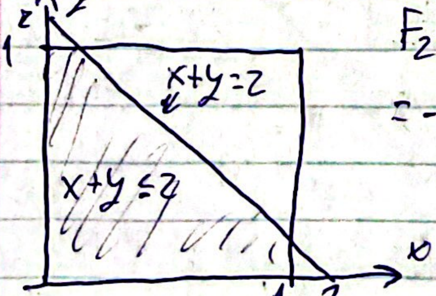
$F_Z(z)=P(Z\le z)=P(X+Y\le z)$

(i) $z\le 1$



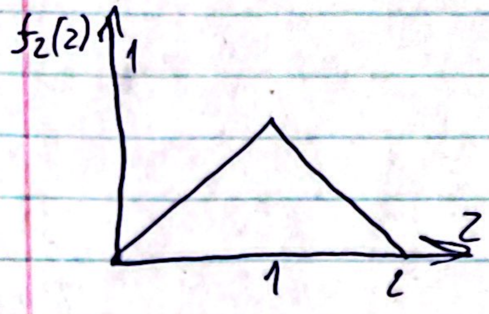
$F_Z(z)=P(Z\le z)=P(X+Y\le z)=\frac{z^2}{2} \Rightarrow f_Z(z)=\frac{d}{dz}F_Z(z)=z$

(ii) $1<z\le 2$



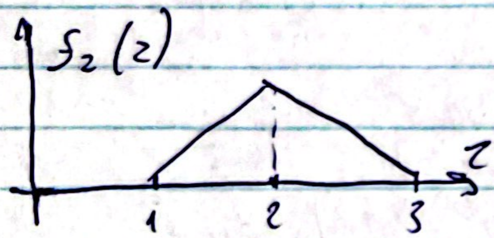
$F_Z(z)=P(Z\le z)=P(X+Y\le z)=\frac{z^2}{2}-2\frac{(z-1)^2}{2}=$
$=-\frac{z^2}{2}+2z-1 \Rightarrow f_Z(z)=\frac{d}{dz}F_Z(z)=-z+2$

(vi) $2 < z$ $\quad F_Z(z) = P(Z \leq z) = 1 \Rightarrow f_Z(z) = \frac{d}{dz} F_Z(z) = 0$



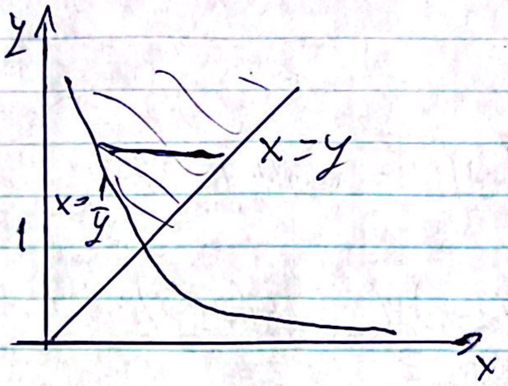
shift to right by one

$f_Z(z) = (z-1)u(z-1) - 2(z-2)u(z-2) + (z-3)u(z-3)$



②

$f(x,y)=\frac{1}{Ay^2x}$, $y\geq 1$ $\frac{1}{y}<x<y$



(2) $f(x,y)=\frac{1}{Ay^2x}$ $y\geq 1$ $\frac{1}{y}<x<y$

$\int_1^\infty \{\int_{\frac{1}{y}}^{y} \frac{1}{Ay^2x}dx\}dy=\int_1^\infty \frac{1}{Ay^2}(\int_{\frac{1}{y}}^{y}\frac{1}{x}dx)dy$

$=\int_1^\infty \frac{1}{Ay^2}(\ln x|_{\frac{1}{y}}^{y})dy=\int_1^\infty \frac{1}{Ay^2}(\ln y-\ln(\frac{1}{y}))dy$

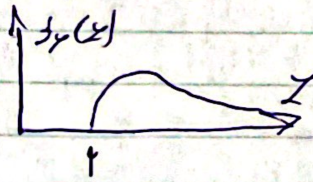
$=\frac{2}{A}\int_1^\infty \frac{\ln y}{y^2}dx \underset{y=e^w}{\overset{w=\ln y}{=}} \frac{2}{A}\int_0^\infty \frac{w}{e^{2w}}de^w=\frac{2}{A}\int_0^\infty \frac{w}{e^{2w}}e^w dw$

$=\frac{2}{A}\int_0^\infty we^{-w}dw=-\frac{2}{A}\int_0^\infty w\,de^{-w}$

$=-\frac{2}{A}\{we^{-w}|_0^\infty-\int_0^\infty e^{-w}dw\}=1 \Rightarrow A=2$

3) and 4

$f(y)=\int_{-\infty}^{\infty}f(x,y)dx=\int_{\frac{1}{y}}^{y}\frac{1}{y^2x}dx=\frac{\ln y}{y^2}$ $1\leq y<\infty$; 0 otherwise

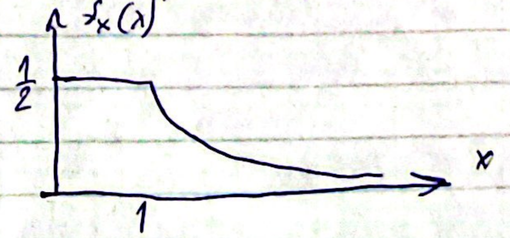


$\int_1^\infty \frac{\ln y}{y^2}dy=1$ Yes $f(y)$ is a valid probability density function

$f(x)=\int_{-\infty}^{\infty}f(x,y)dy$

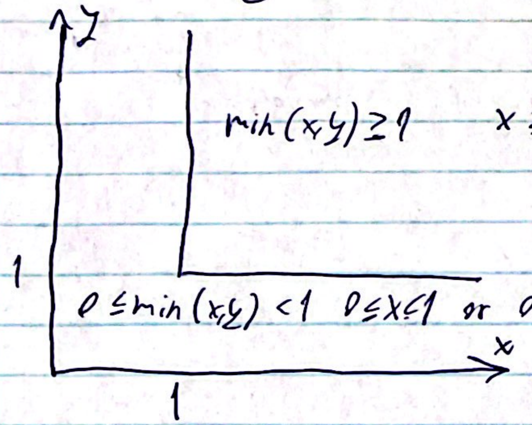
$0<x<1$, $f(x)=\int[(\frac{1}{x}),\infty]\frac{1}{2(y^2)x}dy=\frac{1}{2}$

$1<x<\infty$, $f(x)=\int[x,\infty]\frac{1}{2y^2x}dy=\frac{1}{2x^2}$

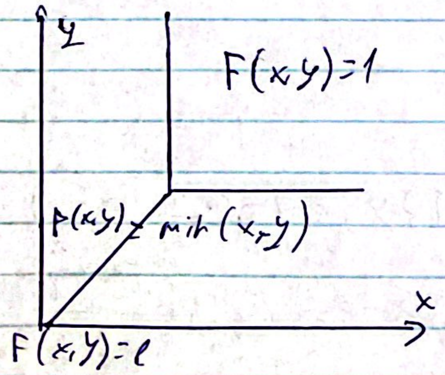


(3)

$$F_{X,Y}(x,y) = \begin{cases} 0, & \min(x,y) < 0 & x<0 \text{ or } y<0 \\ \min(x,y) & 0 \le \min(x,y) < 1 & 0\le x<1 \text{ or } 0 \le y < 1 \\ 1 & \min(x,y) \ge 1 & x \ge 1 \text{ and } y \ge 1 \end{cases}$$



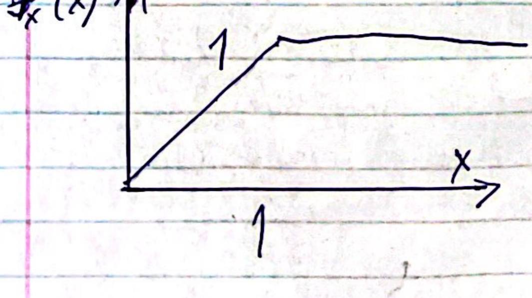
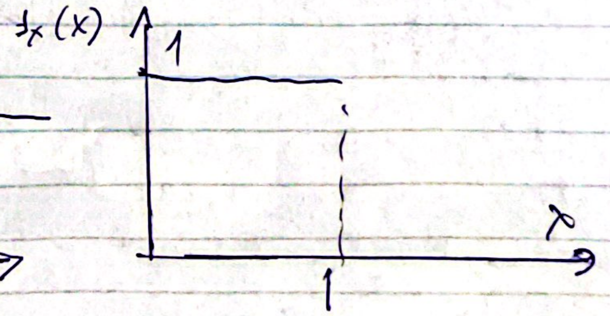
$\min(x,y) < 0 \quad x<0$ or $y<0$



$h(0,1)$ $\quad f_X(x) = F_{XY}(x,y)\big|_{y=\infty} = F_{XY}(x,y)\big|_{y=1} = \min(x,1) = x$

therefore $F_X(x) = \begin{cases} 0 & x<0 \\ x & 0\le x<1 \\ 1 & x \ge 1 \end{cases} \overset{\text{or}}{=} x u(x) - (x-1)u(x-1).$

$f_X(x) = u(x) - u(x-1) = \begin{cases} 1 & 0 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$

(4)

$f(x,y) = \frac{1}{\pi} e^{-(x^2+y^2)}$ $1 + \sin x \sin y$ is not in the form of Gaussian

therefore X, Y are not jointly normal

$f(x) = \int[y \text{ from } -\infty \text{ to } \infty] \frac{1}{\pi} e^{-(x^2+y^2)} (1 + \sin x \sin y) dx dy$

$e^{-y^2} \sin y$ is an odd function

$\int[y \text{ from } -\infty \text{ to } \infty] e^{-y^2} \sin y \, dy = 0$

$f(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}$ Gaussian

$f(y) = \frac{1}{\sqrt{\pi}} e^{-y^2}$ Gaussian

(5)

1. $A_1 = 1$ and $A_2 = 2$

2. $E\{Z\} = E\{X + 3Y\} = E\{X\} + 3E\{Y\} = 1 + 3 \cdot \frac{1}{2} = 2.5$

3. $f_Y(y) = 2e^{-2y} u(y)$ let $w = 3y \Rightarrow f_w(w) = \frac{2e^{-\frac{2}{3}w} u(w)}{3}$

$Z = X + 3Y$. $f_Z(z) = f_X(z) * f_w(z)$

$\mathcal{L}\{f_X(z)\} = \mathcal{L}\{e^{-z} u(z)\} = \frac{1}{s+1}$, $\mathcal{L}\{f_w(z)\} = \mathcal{L}\{\frac{2}{3} e^{-\frac{2}{3}z} u(z)\} =$

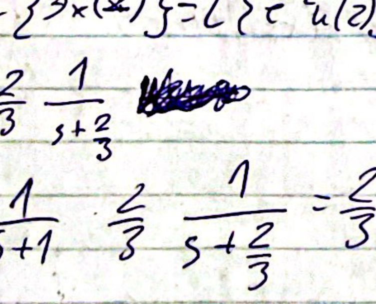
$\frac{2}{3} \frac{1}{s + \frac{2}{3}}$

$\frac{1}{s+1} \frac{2}{3} \frac{1}{s + \frac{2}{3}} = \frac{2}{3} \frac{1}{(s + \frac{2}{3})(s+1)} = 2\left(\frac{1}{s + \frac{2}{3}} - \frac{1}{s+1}\right)$

$2(e^{-\frac{2}{3}z} - e^{-z}) u(z)$

$f_X(x)$ has component e^{-x}, $f_w(w)$ has $e^{-\frac{2}{3}w}$

due to linear transformation $f_{X+w}(z)$ and $f_{X+w}(z)$ are linear combinations of e^{-z} and $e^{-\frac{2}{3}z}$



(6)

1. $A=1$

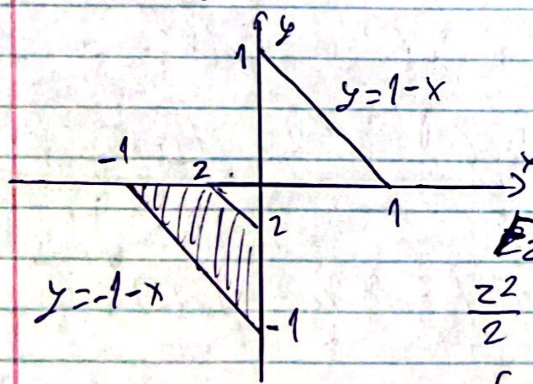
2. $f(y)=\int_y^{2-y} dx = 2-2y,\ 0<y<1;\ =0$, otherwise

$$f(x)=\begin{cases}\int_0^x 1dy & 0\le x<1\\ \int_0^{2-x} 1dy & 1\le x<2\\ 0 & \text{otherwise}\end{cases} = \begin{cases}x, & 0\le x<1\\ 2-x & 1\le x<2\\ 0, & \text{otherwise}\end{cases}$$

(7)

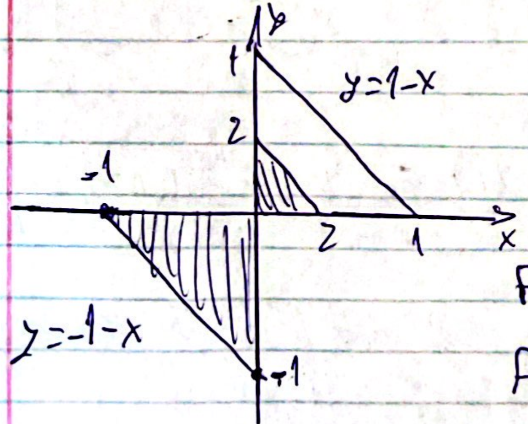
1. $A=1$

2. $-1\le z<0$



$F_Z(z)=P(Z\le z)=P(X+Y\le z)=\frac{1}{2}-\frac{z^2}{2}$. $f_Z(z)=-z$

for $-1\le z\le 0$



$F_Z(z)=P(Z\le z)=P(X+Y\le z)=\frac{1}{2}+\frac{z^2}{2}$

$f_Z(z)=z$

$$f_Z(z)=\begin{cases}|z|, & |z|\le 1\\ 0, & \text{otherwise}\end{cases} \text{ and } F_Z(z)=\begin{cases}0, & z<-1\\ \frac{1}{2}(1-z^2), & -1\le z<0\\ \frac{1}{2}(1+z^2) & 0\le z<1\\ 1, & z\ge 1\end{cases}$$

(8)

a. $Z = X + Y$

$f_Z(z) = f_X(z) * f_Y(z) = \int_{-\infty}^{\infty} e^{-x} u(x) e^{-(z-x)} u(z-x)\,dx$

$= \int_0^z e^{-x} e^{-(z-x)}\,dx \cdot u(z) = e^{-z} x|_0^z u(z) = ze^{-z} u(z)$

b. $Z = X - Y$

density of $-Y$ equal $f(-y)$

$f_Z(z) = f_X(z) * f_Y(-z) = \int_{-\infty}^{\infty} e^{x} u(-x) e^{-(z-x)} u(z-x)\,dx$

$= \int_{-\infty}^{\infty} e^{x}(1-u(x)) e^{-(z-x)} u(z-x)\,dx$

$= \int_{-\infty}^{\infty} e^{x} e^{-(z-x)} u(z-x)\,dx - \int_{-\infty}^{\infty} e^{x} u(x) e^{-(z-x)} u(z-x)\,dx$

$= e^{-z} \int_{-\infty}^{\infty} e^{2x} u(z-x)\,dx - e^{-z} \int_{-\infty}^{\infty} e^{2x} u(x) u(z-x)\,dx$

$= e^{-z} \int_{-\infty}^{z} e^{2x}\,dx - e^{-z} \int_0^z e^{2x}\,dx\, u(z) = e^{-z} \frac{1}{2} e^{2x}\Big|_{-\infty}^{z} - e^{-z} \frac{1}{2} e^{2x}\Big|_0^z u(z)$

$= e^{-z} \frac{1}{2} e^{2z} - (e^{-z} \frac{1}{2} e^{2z} - e^{-z} \frac{1}{2}) u(z) = e^{-z} \frac{1}{2} \{ e^{2z} - (e^{2z} - 1) u(z) \}$

$= \begin{cases} e^{z} \frac{1}{2} & z < 0 \\ e^{-z} \frac{1}{2} & z > 0 \end{cases} = \frac{1}{2} e^{-|z|}$

c. $\min(X, Y)$

$f_Z(z) = f_X(z)(1 - F_Y(z)) + f_Y(z)(1 - F_X(z)) = 2e^{-2z} u(z)$

d. $\max(X, Y)$

$F_Z(z) = f_X(z) F_Y(z) + f_X(z) F_Y(z) = 2e^{-z}(1 - e^{-z}) u(z)$