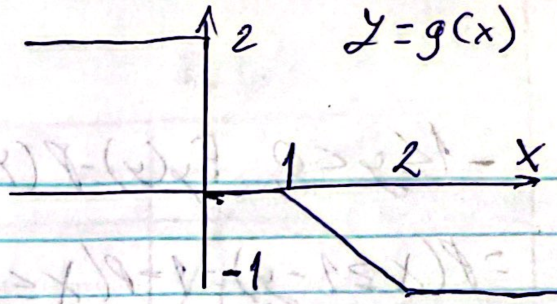
1



$$y = g(x) = \begin{cases} 2, & x<0 \\ 0, & 0 \le x < 1 \\ -x+1, & 1 \le x \le 2 \\ -1, & x > 2 \end{cases}$$

$-1 \le y \le 1$, so, $F_Y(y) = \begin{cases} 0, & y<-1 \\ 1, & y>1 \end{cases}$

for $-1 \le y \le 1$

$y=-1$, $\quad P(Y=-1) = P(X>2) = 1 - F_X(2)$

$y=0$ $\quad P(Y=0) = P(0 < X < 1) = F_X(1) - F_X(0)$

$y=2$ $\quad P(Y=2) = P(X<0) = F_X(0)$

In $(-1, 0)$, $y = -x+1$, $f_Y(y) = \dfrac{f_X(x)}{|g'(x)|} = \dfrac{f_X(1-y)}{|-1|} = f_X(1-y)$

$$f_Y(y) = \begin{cases} (1 - F_X(2))\delta(y+1), & y=-1 \\ f_X(1-y), & -1 < y < 0 \\ (F_X(1) - F_X(0))\delta(y), & y=0 \\ F_X(0)\delta(y-2), & y=2 \\ 0, & \text{otherwise} \end{cases}$$

$$F_Y(y) = \begin{cases} 0 & y<-1 \\ 1 - F_X(1-y) & -1 \le y < 0 \\ 1 - F_X(0), & 0 \le y < 2 \\ 1, & 2 \le y \end{cases}$$

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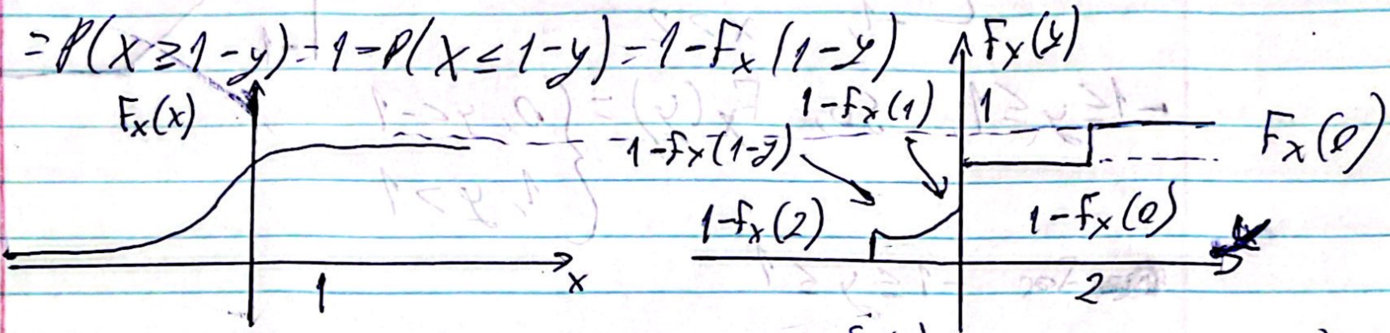
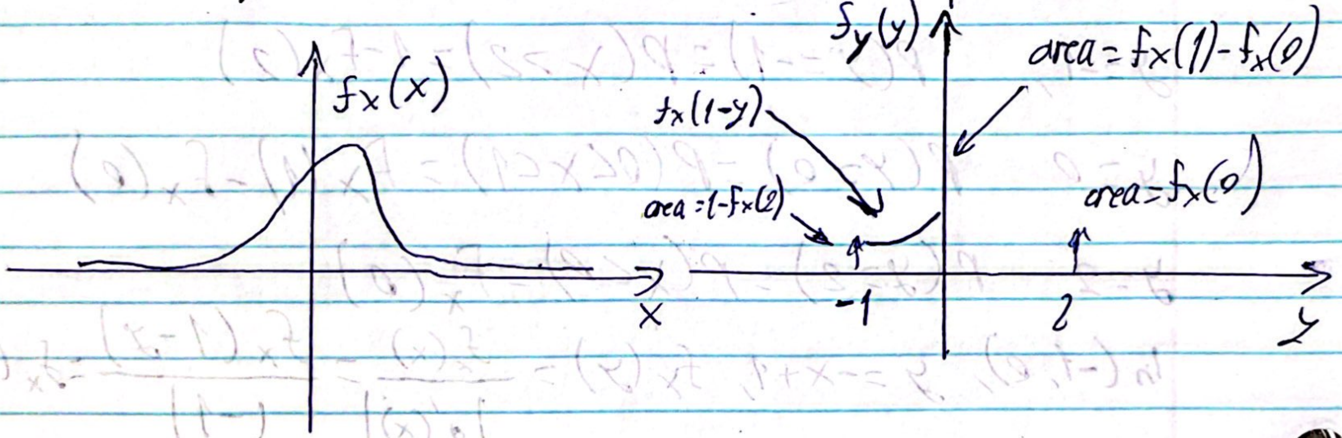
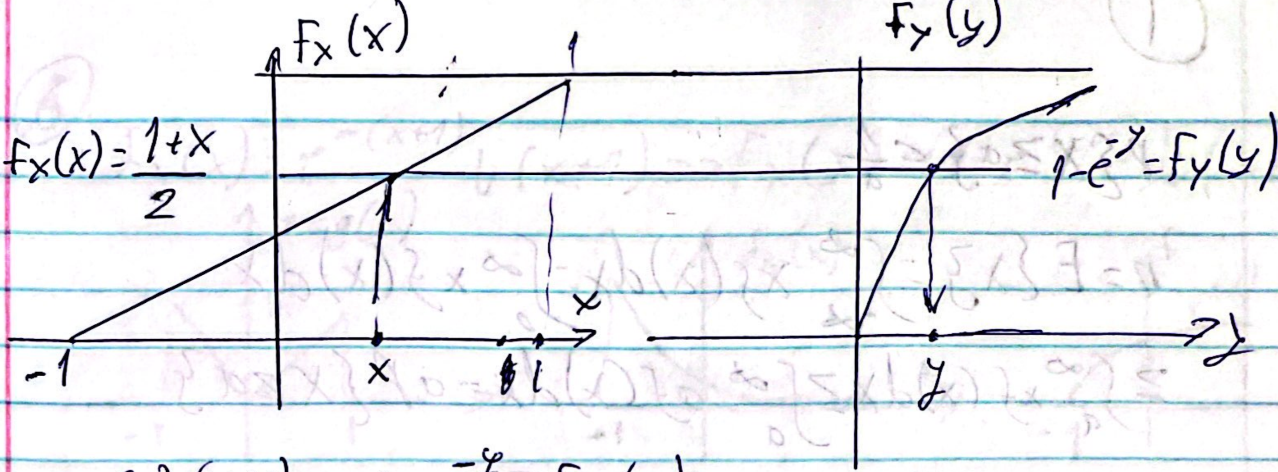
$-1 \le y < 0, \quad F_Y(y) = P(Y \le y) = P(1 - X \le y) = P(-X \le y - 1)$

$= P(X \ge 1 - y) = 1 - P(X \le 1 - y) = 1 - F_X(1 - y)$





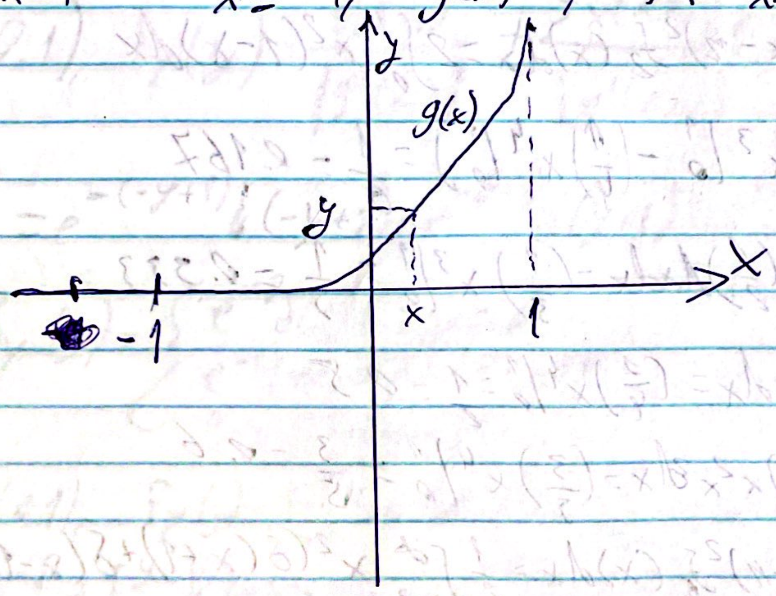
2

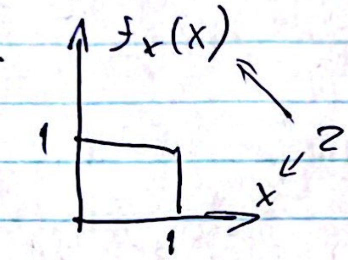


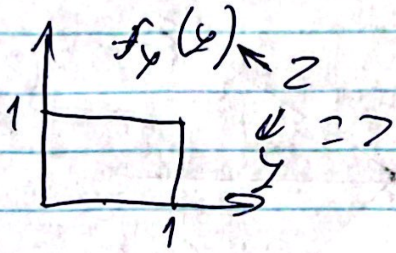
$F_X(x) = \frac{(1+x)}{2} = 1-e^{-y} = F_Y(y)$

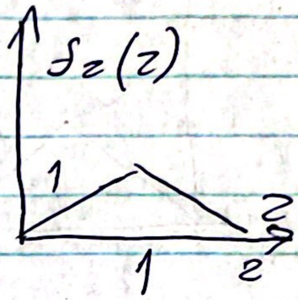
$y = -\ln \cancel{x} \frac{1-x}{2} = g(x)$

$-1<x<1$. $x \le -1$, $g(x)=0$, for $x \ge 1$ $g(x)=\infty$







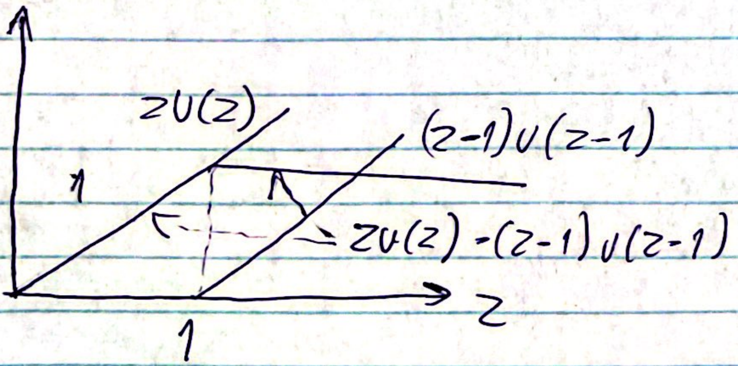


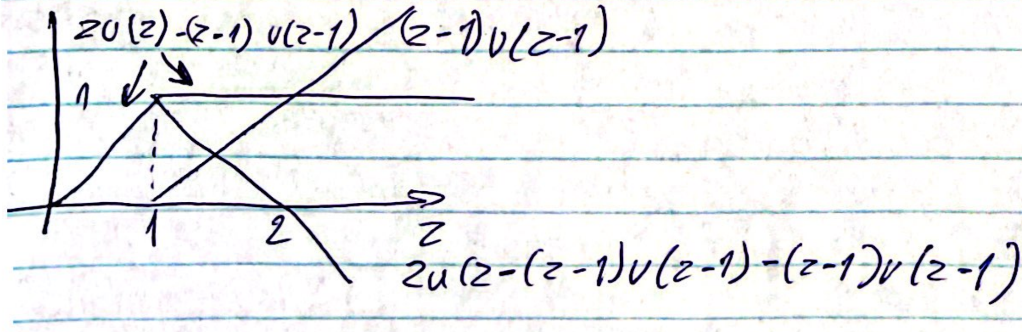
$$f_z(z) = f_x(z) * f_y(z) = \int_{-\infty}^{\infty} f_x(a) f_y(z-a)\,da = \int_{-\infty}^{\infty} (u(a) - u(a-1))(u(z-a) - u(z-a-1))\,da = \int_{-\infty}^{\infty} u(a)u(z-a)\,da - \int_{-\infty}^{\infty} u(a)u(z-1-a)\,da - \int_{-\infty}^{\infty} u(a-1)u(z-a)\,da + \int_{-\infty}^{\infty} u(a-1)u(z-1-a)\,da$$

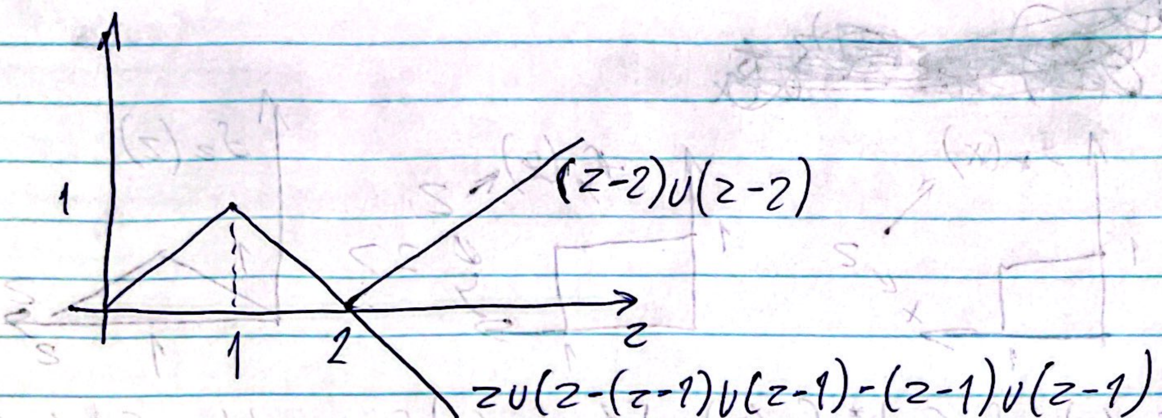
$$= \int_0^z u(a)u(z-a)\,da \cdot u(z) - \int_0^{z-1} u(a)u(z-1-a)\,da\,u(z) - \int_1^z u(a-1)u(z-a)\,da\,u(z-1) + \int_1^{z-1} u(a-1)u(z-1-a)\,da \cdot u(z-2)$$

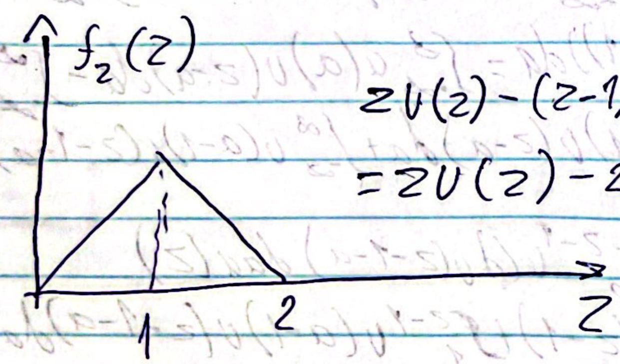
$$= \int_0^z 1\,da \cdot u(z) - \int_0^{z-1} 1\,da\,u(z) - \int_1^z 1\,da \cdot u(z-1) + \int_1^{z-1} 1\,da\,u(z-2)$$

$$= zu(z) - (z-1)u(z-1) - (z-1)u(z-1) + (z-2)u(z-2)$$









$$zu(z)-(z-1)u(z-1)-(z-1)u(z-1)+(z-2)u(z-2)$$
$$=zu(z)-2(z-1)u(z-1)+z-2)u(z-2)$$

4

$P\{X \geq a\} \leq \frac{\eta}{a}$

$\eta = E\{X\} = \int_{-\infty}^{\infty} x f(x)dx = \int_{0}^{\infty} x f(x) dx$

$\geq \int_a^{\infty} x f(x)dx \geq \int_a^{\infty} a f(x) dx = aP\{X \geq a\}$

let $a = \sqrt{\eta}$, $P\{X \geq \sqrt{\eta}\} \leq \frac{\eta}{\sqrt{\eta}} = \sqrt{\eta}$

5

1. $\sigma_1^2 = \int_{-\infty}^{\infty} (x-\eta)^2 f_1(x)dx = \int_{-\infty}^{\infty} x^2 \delta(x)dx = \int_{-\infty}^{\infty} 0^2 \delta(x)dx = 0$

2. $\sigma_2^2 = \int_{-\infty}^{\infty} (x-\eta)^2 f_2(x)dx = 2\int_0^1 x^2(1-x)dx$

$= 2((\frac{1}{3})x^3|_0^1 - (\frac{1}{4})x^4|_0^1) = \frac{1}{6} = 0.167$

3. $\sigma_3^2 = 2\int_0^1 x^2(\frac{1}{2})dxdx = (\frac{1}{3})x^3|_0^1 = \frac{1}{3} = 0.333$

4. $\sigma_4^2 = 2\int_0^1 x^2 x dx = (\frac{1}{4})x^4|_0^1 = \frac{1}{2} = 0.5$

5. $\sigma_5^2 = 2\int_0^1 (\frac{3}{2})x^2 x^2 dx = (\frac{3}{5})x^4|_0^1 = \frac{3}{5} = 0.6$

6. $\sigma_6^2 = \int_{-\infty}^{\infty} (x-\eta)^2 f_6(x)dx = \frac{1}{2}\int_{-\infty}^{\infty} x^2(\delta(x+1)+\delta(x-1))dx$

$= (\frac{1}{2})(\int_{-\infty}^{\infty} x^2\delta(x+1)dx + \int_{-\infty}^{\infty} x^2\delta(x-1)dx)$

$= (\frac{1}{2})(\int_{-\infty}^{\infty} (-1)^2\delta(x+1)dx + \int_{-\infty}^{\infty} 1^2\delta(x-1)dx)$

$= (\frac{1}{2})(\int_{-\infty}^{\infty} \delta(x+1)dx + \int_{-\infty}^{\infty} \delta(x-1)dx) = (\frac{1}{2})(1+1) = 1$

7. $\sigma_7^2 = 2\int_0^{0.5} 0.75x^2dx + 2\int_{0.5}^1 0.25x^2dx = 2(0.75\frac{1}{3}\frac{1}{8} + 0.75\frac{1}{3}\frac{7}{8}) = \frac{5}{24} \approx 0.208$

8. $\sigma_8^2 = 2\int_{0.5}^1 x^2dx = \frac{7}{12} \approx 0.583$

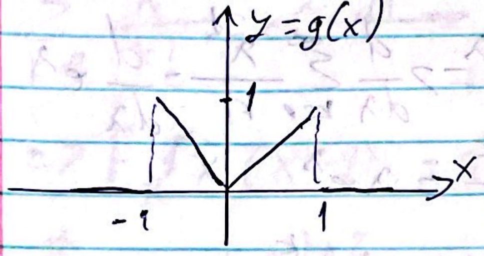
9. $\int_0^1 a(1-x^2)dx = 1 \Rightarrow a = \frac{3}{4}$ $f_9(x) = (\frac{3}{4})(1-x^2)$

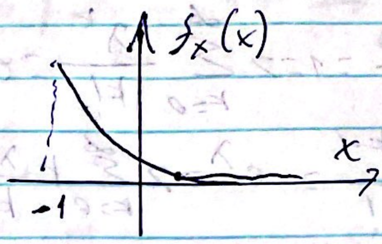
$\sigma_9^2 = 2(\frac{3}{4})\int_0^1 x^2(1-x^2)dx = (\frac{3}{2})(\frac{1}{3} - \frac{1}{5}) = \frac{1}{5} = 0.2$

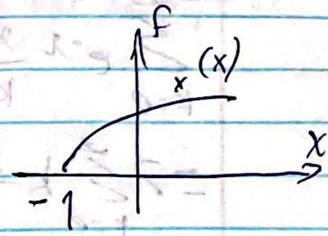
$\sigma_1^2, \sigma_2^2, \sigma_9^2, \sigma_7^2, \sigma_3^2, \sigma_4^2$ σ_8^2 σ_5^2 σ_6

8

$$f_X(x)=e^{-(x+1)}u(x+1) \Rightarrow F_X(x)=\left(1-e^{-(x+1)}\right)u(x+1)$$







$$P(Y=0)=P(X<-1)+P(X>1)=P(X<-1)+P(X>1)$$

$$=F_X(-1)+(1-F_X(1))=0+1-\left(1-e^{-(1+1)}\right)u(1+1)=e^{-2}$$

In $(0,1)$, $f_Y(y)=\dfrac{f_X(x_1)}{|g'(x_1)|}+\dfrac{f_X(x_2)}{|g'(x_2)|}=f_X(-y)+f_X(y)$

$$=e^{-(-y+1)}u(-y+1)+e^{-(y+1)}u(y+1)=e^{-1}\left(e^{y}+e^{-y}\right)$$

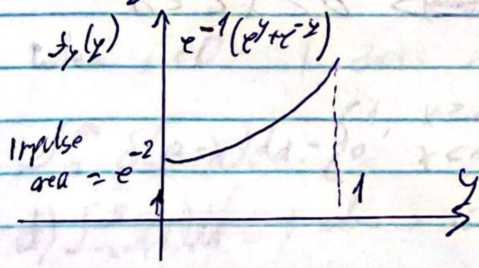
$$f_Y(y)=\begin{cases} e^{-2}\delta(y), & y=0 \\ e^{-1}(e^{y}+e^{-y}) & 0<y<1 \\ 0, & \text{otherwise} \end{cases}$$

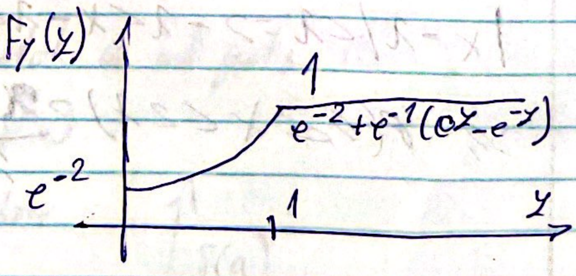
In $(0,1)$, $F_Y(y)=\int_{-\infty}^{y} f_Y(a)\,da=\int_{-\infty}^{0^+} f_Y(a)\,da+\int_{0^+}^{y} f_Y(a)\,da$

$$=e^{-2}+\int_{0^+}^{y} f_Y(a)\,da=e^{-2}+e^{-1}\int_{0^+}^{y}\left(e^{a}+e^{-a}\right)da$$

$$=e^{-2}+e^{-1}\left(e^{a}\Big|_{0^+}^{y}-e^{-a}\Big|_{0^+}^{y}\right)=e^{-2}+e^{-1}\left(e^{y}-1-e^{-y}+1\right)=e^{-2}+e^{-1}\left(e^{y}-e^{-y}\right)$$

for $y\geq 1$ $F_Y(y)=1$





7

$$P(X=k) = e^{-\lambda}\frac{\lambda^k}{k!}$$

$$\sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} = 1 \Rightarrow \sum_{k=0}^{\infty}\frac{\lambda^k}{k!} = e^{\lambda} \Rightarrow \frac{d}{d\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} = \frac{d}{d\lambda}e^{\lambda}$$

$$\Rightarrow \sum_{k=0}^{\infty} k\frac{\lambda^{k-1}}{k!} = e^{\lambda} \Rightarrow \sum_{k=0}^{\infty} k\frac{\lambda^k}{k!} = \lambda e^{\lambda}$$

$$\Rightarrow \sum_{k=0}^{\infty} k e^{-\lambda}\frac{\lambda^k}{k!} = \lambda \Rightarrow E\{X\} = \sum_{k=0}^{\infty} k e^{-\lambda}\frac{\lambda^k}{k!} = \lambda$$

$$\sigma^2 = E\{(x-\eta)^2\} = E\{x^2\} - \eta^2 = E\{x^2\} - \lambda^2$$

$$E\{x^2\} = \sum_{k=0}^{\infty} k^2 e^{-\lambda}\frac{\lambda^k}{k!} = ?$$

$$\sum_{k=0}^{\infty} k\frac{\lambda^k}{k!} = \lambda e^{\lambda} \Rightarrow \frac{d}{d\lambda}\sum_{k=0}^{\infty} k\frac{\lambda^k}{k!} = \frac{d}{d\lambda}\lambda e^{\lambda} = e^{\lambda} + \lambda e^{\lambda}$$

$$\Rightarrow \sum_{k=0}^{\infty} k^2\frac{\lambda^{k-1}}{k!} = (1+\lambda)e^{\lambda} \Rightarrow \sum_{k=0}^{\infty} k^2 e^{-\lambda}\frac{\lambda^k}{k!} = (1+\lambda)\lambda$$

$$\sigma^2 = E\{x^2\} - \lambda^2 = (1+\lambda)\lambda - \lambda^2 = \lambda$$

$$P\{|x-\eta| \geq \varepsilon\} \leq \frac{\sigma^2}{\varepsilon^2}$$

$$\Rightarrow P\{|x-\eta| < \varepsilon\} \geq 1 - \frac{\sigma^2}{\varepsilon^2}$$

let $\varepsilon = \lambda$, $P\{|x-\lambda| < \lambda\} \geq 1 - \frac{\lambda}{\lambda^2} = \frac{\lambda - 1}{\lambda}$

$$|x-\lambda| < \lambda \Rightarrow -\lambda < x - \lambda < \lambda \Leftrightarrow 0 < x < 2\lambda$$

so, $P(0 < x < 2\lambda) \geq \frac{\lambda - 1}{\lambda}$

(7b)

$$\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{\lambda} \xrightarrow{\frac{d}{d\lambda}} \sum_{k=0}^{\infty} k\frac{\lambda^{k-1}}{k!} = e^{\lambda} \xrightarrow{\frac{d}{d\lambda}} \sum_{k=0}^{\infty} k(k-1)\frac{\lambda^{k-2}}{k!} = e^{\lambda}$$

$$\Rightarrow E\{X(X-1)\} = \lambda^2$$

$$\sum_{k=0}^{\infty} k(k-1)\frac{\lambda^{k-2}}{k!} = e^{\lambda} \xrightarrow{\frac{d}{d\lambda}} \sum_{k=0}^{\infty} k(k-1)(k-2)\frac{\lambda^{k-3}}{k!} = e^{\lambda} \Rightarrow$$

$$E[X(X-1)(X-2)] = \lambda^3$$

8.

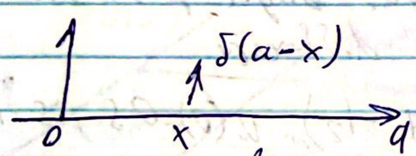
$$P(|X-\eta| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2} \Rightarrow P(|X-\eta| \geq k\sigma) \leq \frac{\sigma^2}{k^2\sigma^2} = \frac{1}{k^2}$$

$$\Rightarrow P(|X-\eta| \leq k\sigma) \geq 1 - \frac{1}{k^2} = \frac{k^2-1}{k^2} \quad \text{In (d), } k=4 \quad (d)$$

9. a) $\int_0^{\infty} \delta(x-a)\,da = -\int_x^{\infty} \delta(\beta)\,d\beta = \int_{-\infty}^{x} \delta(\beta)\,d\beta = u(x)$

b) α and β are same. impulse function is even

c)



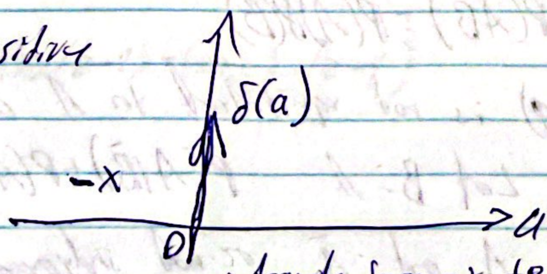
integrate from 0 to ∞

when $x \geq 0$ integral catches the impulse and gets 1

when $x < 0$ it does not catch up and gets 0

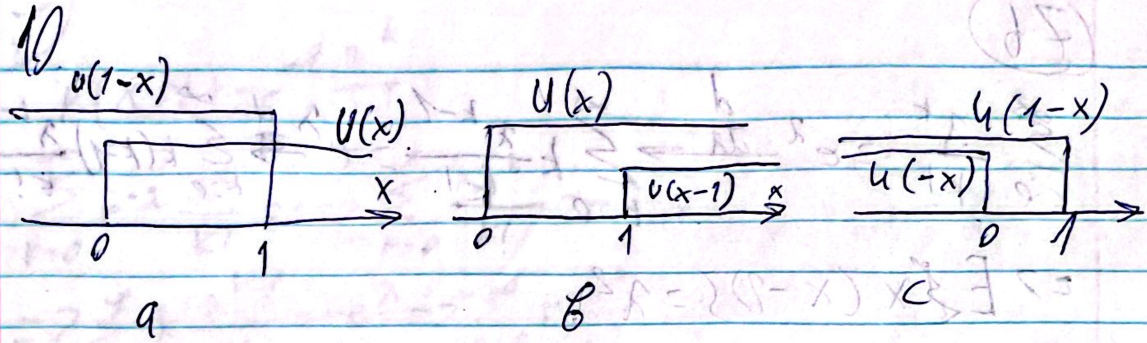
$$\int_0^{\infty} \delta(a-x)\,da = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases} = u(x)$$

d) $\int_{-x}^{\infty} \delta(a)\,da = 1$ when x is positive

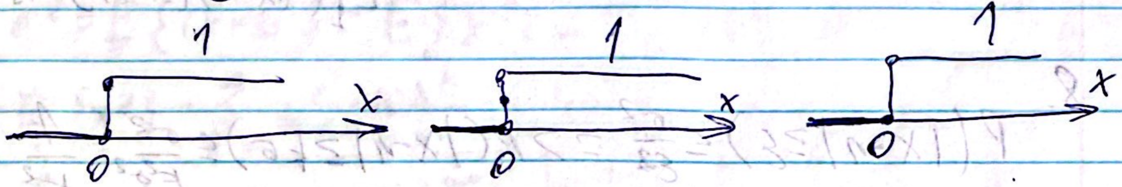


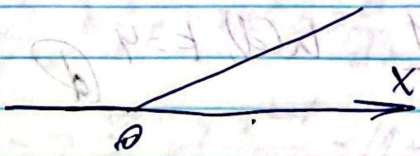
integrate from $-x$ to ∞

10



$$u(x) = \begin{cases} 1 & x > 0 \\ A, & x = 0 \\ 0, & x < 0 \end{cases}$$





$$\int_{-\infty}^{x} u(a)\, da = x u(x)$$

A can be any finite value

When x is continuous variable, single point with finite value does not make any difference

$\min(t_1, t_2) = t_1 u(t_2 - t_1) + t_2 u(t_1 - t_2)$, $u(0) = 0.5$ is more convenient

11. $P(A|B) = P(A|\bar{B}) \Rightarrow \frac{P(AB)}{P(B)} = \frac{P(A\bar{B})}{P(\bar{B})}$

$P(AB(1 - P(B))) = P(B)P(A\bar{B}) = P(B)(P(A) - P(AB))$

$P(AB) - P(AB)P(B) = P(A)P(B) - P(B)P(AB)$

$P(AB) = P(A)P(B)$

g) is not equivalent to A and B are independent

Let $B = A$ $\quad P(A|\bar{B}) + P(\bar{A}|\bar{B}) = P(A|\bar{A}) + P(\bar{A}|\bar{A}) = 0 + 1 = 1$

A and $\bar{A}$ are not independent