ELECTRICAL & COMPUTER ENGINEERING

Telecommunication Systems II

Assignment 1

Due by: December 12th 2025
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Contents

Task 1

(1a) (`task_1a.py`)

The spectrum of the LTI impulse response is

$$H(f) = \Pi\left(\frac{f}{2W}\right) = \begin{cases} 1, & |f| \le W, \\ 0, & |f| > W, \end{cases}$$

we take the inverse continuous–time Fourier transform:

$$h(t) = \int_{-\infty}^{\infty} H(f)\, e^{j2\pi ft}\, df = \int_{-W}^{W} e^{j2\pi ft}\, df.$$

Evaluating the integral,

$$h(t) = \left. \frac{e^{j2\pi ft}}{j2\pi t} \right|_{-W}^{W} = \frac{\sin(2\pi Wt)}{\pi t}.$$

Using the normalized sinc function $\mathrm{sinc}(u) = \frac{\sin(\pi u)}{\pi u}$, this becomes

$$h(t) = 2W\, \mathrm{sinc}(2Wt).$$

As for the noise, from **Lecture 6a**, the continuous–time white Gaussian noise process $n(t)$ has the properties:

- It is a zero–mean random process: $E\{n(t)\} = 0$.
- It is WSS, with autocorrelation

$$R_{nn}(t) = \mathbb{E}\{n(a)n(a-t)\} = R_{nn}(t) = \sigma^2\delta(t) = N_0\delta(t),$$

$$S_{nn}(f) = \mathcal{F}\{R_{nn}(t)\} = N_0$$

The simulation was implemented in Python using the provided `util.py` helpers (`iarange`, `convxaxis`, `fftxaxis`, etc.). The provided parameters N_0, W, Th, Fs.

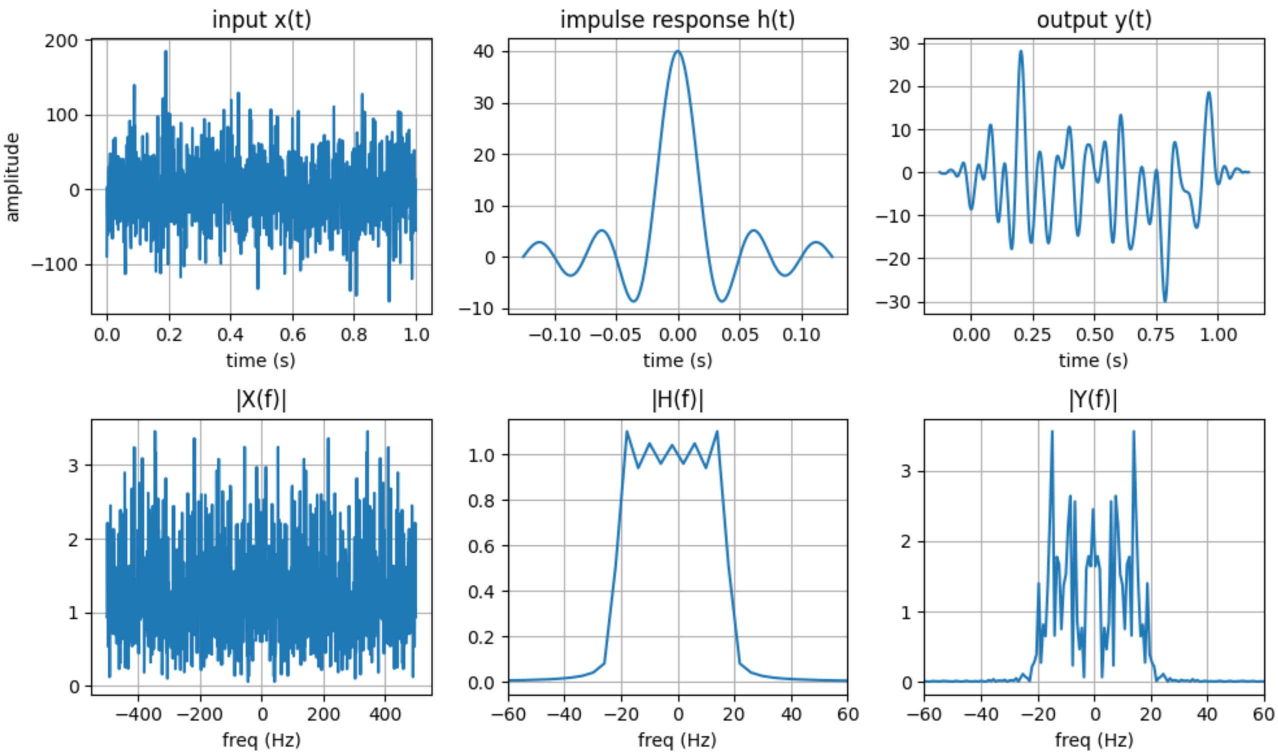
The input signal was implemented in [0,1]s using `iarange(0,1,Ts)`. In discrete time, the variance N_0 becomes N_0/Ts, so the input is:
`x=sqrt(N0/Ts)*randn(len(t_x))`.
That is because randn() generates a random process $\sim N(0,1)$, so the multiplication by N_0/Ts gives us the input WGN process $x[n] \sim N(0, N_0/Ts)$.

The impulse response was implemented in the finite interval $[-T_h/2, T_h/2]$ using
`t_h = iarange(-Th/2, Th/2, Ts)` and `h = 2*W*sinc(2*W*t_h)`.

The output signal was obtained by discrete-time convolution:
$y(t) = (x * h)(t) \quad \Rightarrow$ `y = conv(x, h) * Ts`,
and the corresponding time axis `t_y` was computed using `convxaxis(t_x, t_h)`.
For the spectral analysis, the FFT of $x(t)$, $h(t)$, and $y(t)$ were computed using `fft`, applied `fftshift` to center the spectrums, and scaled by T_s to approximate the continuous-time Fourier transform.

Both the convolution and the FFT were done using our own implementation of `fftxaxis(N, Ts)` and `convxaxis(x_axis, h_axis)` in utils.py.

For $W = 20Hz$, $Th = 0.25s$, $Fs = 1000Hz$, $N_0 = 2$, the result was:



Since the input signal is [0,1]sec and the impulse response is [-0.125, 0.125], we see that the output signal, which is the convolution of the input with the impulse response, is in $[0-0.125,\ 1+0.125] = [-0.125, 1.125]$. $|H(f)|$ is correctly ≈ 1 in $[-W, W]$, and ≈ 0 elsewhere, and so is the $|Y(f)|$, since $Y(f) = X(f)H(f)$. **(Lecture 1)**

(1b) (`task_1b.py`)

Having solved exercise 4.7.2 in class, we had found:

$$Rss(t) = \frac{1}{2T}\Lambda(\frac{t}{T}) - \frac{1}{4T}\Lambda(\frac{t+2T}{T}) - \frac{1}{4T}\Lambda(\frac{t-2T}{T})$$

The PSD is simply the fourier transform of $Rss(t)$: $S_{ss}(f) = \mathcal{F}\{R_{ss}(t)\}$ **(Chapter 3, (3.14))** We know that

$$\Lambda\left(\frac{t}{T}\right) \overset{\mathcal{F}}{\longleftrightarrow} T\text{sinc}^2(fT),$$

therefore we can calculate the theoretical PSD.

$$S_{ss}(f) = sinc^2(fT) \ \cdot sin^2(2\pi fT)$$

Now, to estimate the experimental PSD, we have to average the periodograms of multiple implementations of the stochastic process $s(t)$.

We have a symbol period $T = 0.01s$, oversampling = 100, therefore $Ts = \frac{T}{100} = 0.0001s$. The simulation generates 100 symbols, however since $S_i = b_i - b_{i-2}$ we need 102 bits.

We create the rectangular pulse and scale the amplitude by $1/\sqrt{T}$ to ensure unit energy. Our notes describe the periodogram as

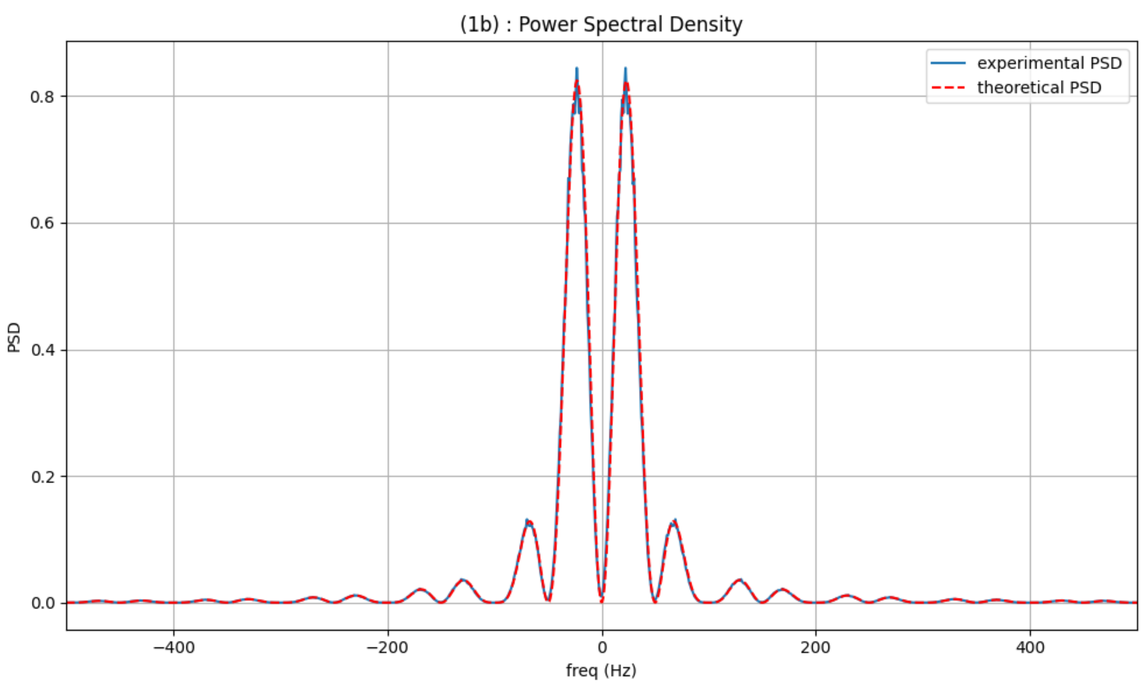
$$\frac{1}{T}\{|X_T(f)|^2\}, \qquad (3.12 \ - \ Chapter\ 3)$$

and since we use the FFT, we must scale the transform by the sampling period T_s. The squared magnitude becomes:

$$\frac{1}{T_{sim}}|S_f \ \cdot T_s|^2 = \frac{|T_s^2 \ \cdot S_f^2|}{T_{sim}} = \frac{T_s^2 \ \cdot |S_f^2|}{T_{sim}}$$

since T_{sim} is the total length of time of our signal, which is roughly 100 $symbols \ \cdot 0.01s/symbol = 1s$, the periodogram is approximately $T_s^2 \ \cdot |S_f^2|$.

We average the periodograms across the 1000 iterations and reveal the underlying shape, which we compare against the theoretical computation of the PSD.



Task 2 (`task_2.py`)

According to our theory notes (the entire Chapter 4), each complex symbol $s_i = s_i^I + js_i^Q \in \mathbb{C}$ is multiplied by the constant $A > 0$, and then multiplied by the pulse $g(t)$, shifted by iT seconds.
In our case, 4-QAM, each symbol contains 2 bits.
The **baseband signal** is

$$s(t) = A \sum_i s_i \; g(t - iT) \quad (4.6)$$

which according to the properties of the convolution, is

$$s(t) = \left[A \sum_i s_i \; \delta(t - iT)\right] \; * \; g(t)$$

In the **transmitter**, we upconvert by multiplying by the phasor

$$2e^{j2\pi f_c t} = 2cos(2\pi f_c t) + 2jsin(2\pi f_c t)$$

to retain the amplitude magnitude, then only keep the real counterparts of the signal, since the physical signal transmitted over the channel must be a real quantity.
This means that

$$\begin{aligned}
Re\{s(t)2e^{j2\pi f_c t}\} &= 2Re\{s(t)\left[\cos(2\pi f_c t) + j\sin(2\pi f_c t)\right]\} \\
&= 2Re\{[s^I(t) + js^Q(t)]\left[\cos(2\pi f_c t) + j\sin(2\pi f_c t)\right]\} \\
&= 2Re\{s^I(t)\cos(2\pi f_c t) + js^I(t)\sin(2\pi f_c t) + js^Q(t)\cos(2\pi f_c t) + j^2 s^Q(t)\sin(2\pi f_c t)\} \\
&= 2Re\{s^I(t)\cos(2\pi f_c t) - s^Q(t)\sin(2\pi f_c t) + j(s^I(t)\sin(2\pi f_c t) + s^Q(t)\cos(2\pi f_c t)) \\
&= 2s^I(t)\cos(2\pi f_c t) - 2s^Q(t)\sin(2\pi f_c t).\}
\end{aligned}$$

So, this gives us the **band pass signal** $s_{BP}(t)$, which is then transmitted.

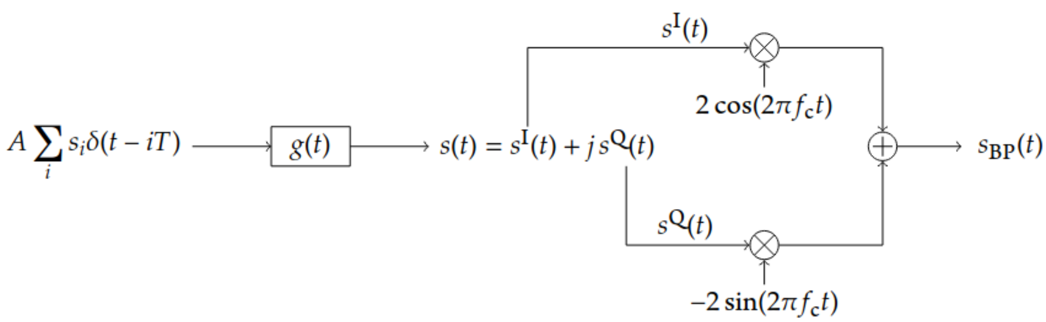


figure (4.6) - chapter 4

At **the receiver**, we get the signal $r_{BP}(t) = s_{BP}(t)$, since the channel in this case is ideal.
We downconvert by multiplying with the phasor $e^{-j2\pi f_c t} = cos(2\pi f_c t) - jsin(2\pi f_c t)$, then filter by convolving with the $g^*(-t)$, which can be simplified to $g^*(-t) = g(-t) = g(t)$, since the SRRC is real and symmetric.
Finally, we sample the signal at each symbol period kT, which gives us the samples As_k

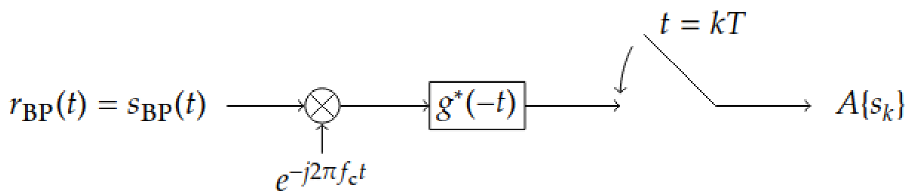
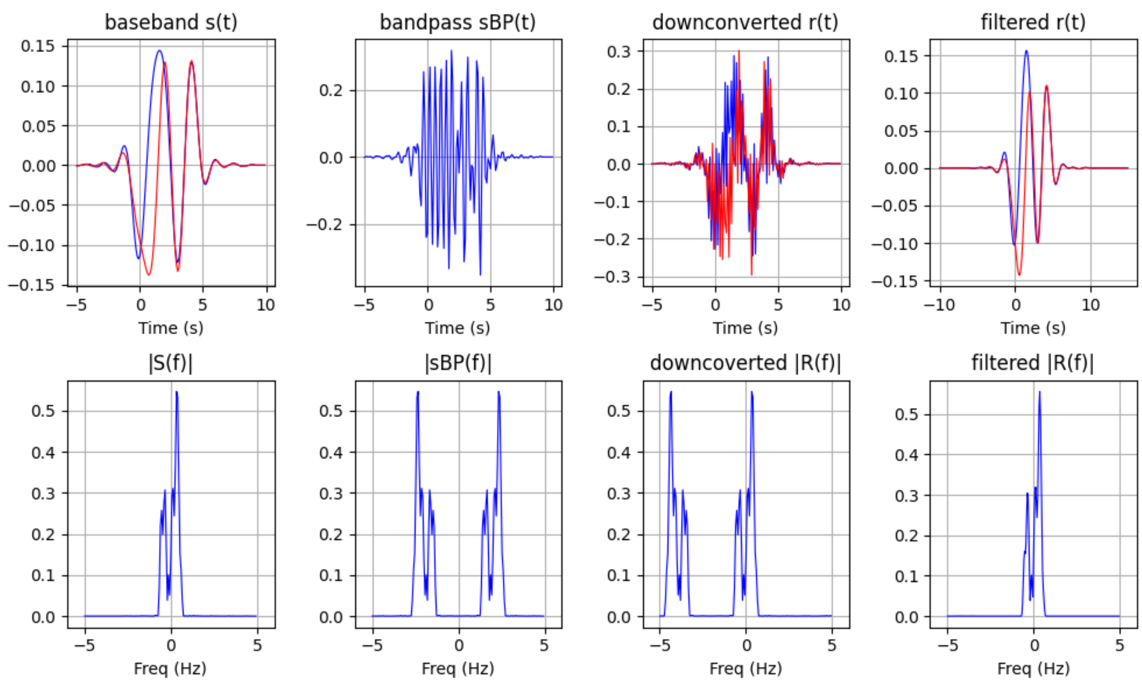


figure 4.10 - chapter 4

This was the plot printed by the code:



We will now proceed to see whether the diagrams are correct.

- We can see that the baseband $S(f)$ is centered at 0Hz, while the band pass $S_{BP}(f)$ is centered at $\pm$2Hz.

- At the receiver, the downcoverted unfiltered $R(f)$ has two identical lobes at -4Hz and 0Hz. The filtering removes the duplicate component at -4Hz and keeps the one at 0Hz.

Task 3

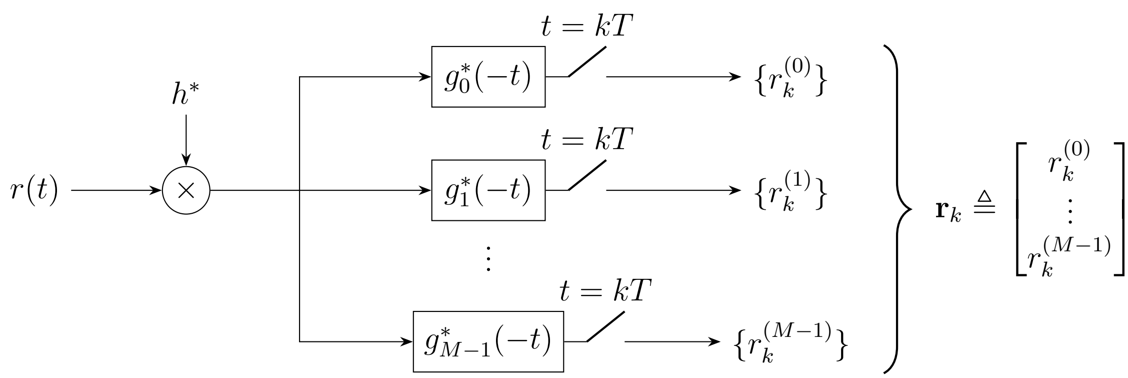
Our conventions for the following part are to use :

- $< f, g > \triangleq \int_{-\infty}^{\infty} f(\tau)g^*(\tau)d\tau = f(t) * \overleftarrow{g}^*(t)\Big|_{t=0}$
- $\overleftarrow{f}(t) \triangleq f(-t)$

For M-FSK:
In **the transmitter**, $log_2 M$ bits are packed into symbols. Each symbol goes through its own pulse, is multiplied by the constant A, and this gives us the baseband

$$s(t) = \sum_{k=0}^{K-1} A g_{m_k} g(t - kT) \quad (4.29,\ ch4)$$

which is then upconverted to $s_{BP}(t)$.
We consider **the channel** to be flat-fading, therefore the signal is scaled by $h \in \mathbb{C}$ and the noise $n(t)$ is added at the receiver end.
The receiver downconverts and sees $r(t) = hs(t) + n(t)$, then multiplies h^* to reverse the phase distortion by the channel, which also scales it by $|h|$. Each symbol in the signal $h^*r(t)$ is then passed through the bank of M parallel matching filters, where the m-th filter has the impulse response $g_m^*(-t)$, one of them will match the symbol we sent. Then, we sample the output of each matching filter at each symbol period kT.



The output of this is a vector r_k, defined as:

$$\mathbf{r}_k \triangleq \begin{bmatrix} r_k^{(0)} \\ \vdots \\ r_k^{(M-1)} \end{bmatrix}$$

(3a)

We want to prove that the sampled output of the m-th matched filter at time $t = kT$ is:

$$r_k^{(m)} = A|h|^2 \sum_{k'=0}^{K-1} \langle g_{m_{k'},k'}, g_{m,k} \rangle + n_k^{(m)}$$

with the noise component defined as:

$$n_k^{(m)} \triangleq h^* \left(n(t) * \overleftarrow{g^*}_m(t) \right) \Big|_{t=kT}$$

We begin by defining the signal AT EACH matching filter, right before sampling, after it has been multiplied by h^* and passed through $g^*_{(m)}(t)$:

$$y_m(t) = h^*(r(t) * \overleftarrow{g^*}_m(t))$$

We replace $r(t) = hs(t) + n(t)$:

$$\begin{aligned} y_m(t) &= h^*([hs(t) + n(t)] * \overleftarrow{g^*}_m(t)) \\ &= h^* hs(t) * \overleftarrow{g^*}_m(t) + h^*[n(t) * \overleftarrow{g^*}_m(t)] \\ &= |h|^2 [s(t) * \overleftarrow{g^*}_m(t)] + h^*[n(t) * \overleftarrow{g^*}_m(t)] \end{aligned}$$

This was right before sampling. Now, when we sample at each symbol period (at time t=kT), this becomes:

$$\begin{aligned} r_k^{(m)} &= y_m(t = kT) = y_m(kT) \\ &= |h|^2 [s(t) * \overleftarrow{g^*}_m(t)] + h^*[n(t) * \overleftarrow{g^*}_m(t)] \Big|_{t=kT} \\ &= |h|^2 [s(t) * \overleftarrow{g^*}_m(t)] \Big|_{t=kT} + n_k^{(m)} \end{aligned}$$

Therefore we proved the noise component. Now we can isolate the signal to prove the $r_k^{(m)}$. We've got

$$s(t) = \sum_{k'=0}^{K-1} A g_{m_{k'}} g(t - k'T)$$

Replacing this into the relation we found above:

$$\begin{aligned} & |h|^2 \left[s(t) * \overleftarrow{g^*}_m(t) \right] \Big|_{t=kT} \\ &= |h|^2 \Big[\sum_{k'=0}^{K-1} A g_{m_{k'}} g(t - k'T) \Big] * {g^*}_m(-t) \Big|_{t=kT} \\ &= A|h|^2 \sum_{k'=0}^{K-1} \left[g_{m_{k'}}(t - k'T) * g^*_m(-t) \right]_{t=kT} \end{aligned}$$

$$= A|h|^2 \sum_{k'=0}^{K-1} \int_{-\infty}^{\infty} g_{m_{k'}}(\tau - k'T) g_m^*(-(t-\tau)) d\tau \Big|_{t=kT}$$

$$= A|h|^2 \sum_{k'=0}^{K-1} \int_{-\infty}^{\infty} g_{m_{k'}}(\tau - k'T) g_m^*(\tau - t) d\tau \Big|_{t=kT}$$

$$= A|h|^2 \sum_{k'=0}^{K-1} \int_{-\infty}^{\infty} g_{m_{k'}}(\tau - k'T) g_m^*(\tau - kT) d\tau$$

Using the definition $g_{m,k}(t) \triangleq g_m(t - kT)$ inside the integral:

$$\int_{-\infty}^{\infty} g_{m_{k'},k'}(\tau) g_{m,k}^*(\tau) d\tau = \langle g_{m_{k'},k'}, g_{m,k} \rangle$$

We put this back into the total expression for $r_k^{(m)}$:

$$r_k^{(m)} = A|h|^2 \sum_{k'=0}^{K-1} \langle g_{m_{k'},k'}, g_{m,k} \rangle + n_k^{(m)}$$

,where $n_k^{(m)} = h^*[n(t) * \overleftarrow{g}^*{}_m(t)] \Big|_{t=kT}$

(3b)

We want to prove that:

$$\langle g_{m',k'},\, g_{m,k} \rangle \;=\; \langle g_{m'},\, g_m \rangle\, \delta_{k,k'}$$

for

- $m, m' \in \mathbb{Z}_M$ and $k, k' \in \mathbb{Z}_K$,
  having defoned $\mathbb{Z}_n \triangleq \{m \in \mathbb{Z}_{\geq 0} \mid m < n\} = \{0, 1, \ldots, n-1\}$.
- ϕ being zero outside $[0, T)$,
  having defined $\phi : \mathbb{R} \to \mathbb{R}$ such that $\langle \phi, \phi \rangle = 1$ for symbol period $T > 0$.

We have also defined the shaping pulse for each symbol s_m as

$$g_m(t) \triangleq \phi(t)\, e^{j2\pi s_m \Delta F t},$$

so each $g_m(t)$ is zero outside the interval $[0, T)$.

The time–shifted pulse used for symbol index k is

$$g_{m,k}(t) = g_m(t - kT).$$

Solving for the inner product:

$$\langle g_{m',k'}, g_{m,k} \rangle = \int_{-\infty}^{\infty} g_{m'}(\tau - k'T) g_m^*(\tau - kT) d\tau$$

We now distinguish two cases:

- $k = k'$:

$$\langle g_{m',k'}, g_{m,k} \rangle = \int_{-\infty}^{\infty} g_{m'}(\tau - k'T) g_m^*(\tau - kT) d\tau = \int_{-\infty}^{\infty} g_{m'}(\tau - kT) g_m^*(\tau - kT) d\tau = \langle g_{m'}, g_m \rangle$$

- $k' \neq k$: The pulse $g_{m,k}(t) = g_m(t - kT)$ is active in the time duration $[kT, (k+1)T]$.
  The pulse $g_{m',k'}(t) = g_{m'}(t - k'T)$ is active in the time duration $[k'T, (k'+1)T]$.
  Since $k \neq k'$, these two pulses do not overlap, meaning that their product is always 0, making their inner product also 0.
  $\langle g_{m',k'}, g_{m,k} \rangle = 0$

Since the inner product is

$$\langle g_{m',k'}, g_{m,k} \rangle = \begin{cases} 0, & k \neq k' \\ \langle g_{m'}, g_m \rangle, & k = k' \end{cases}$$

we can symbolize this as $\langle g_{m',k'}, g_{m,k} \rangle = \langle g_{m'}, g_m \rangle \delta_{k,k'}$

(3c)

We want to prove that $\langle g_{m'}, g_m \rangle = e^{j\pi \Delta F \Delta s T}\ sinc(\Delta F \Delta s T)$, where $\Delta s \triangleq s_{m'} - s_m$, for :

- $m, m' \in \mathbb{Z}_M = \{0, 1, .., M-1\}$
- ϕ being constant in $[0, T)$ and 0 elsewhere.

To begin with, let's remember again that $g_m(t) = \phi(t) e^{j2\pi s_m \Delta F t}$.
The pulse is constant in $[0, T)$, 0 elsewhere, and has unit energy. $\Rightarrow \int_{-\infty}^{\infty} |\phi(t)|^2 dt = 1 \Rightarrow \int_0^T A^2 dt = 1 \Rightarrow A^2 T = 1 \Rightarrow A = 1/\sqrt{T}$.
This makes

$$g_m(t) = \frac{1}{\sqrt{T}} e^{j2\pi s_m \Delta F t}, t \in [0, T)$$

and the inner product becomes

$$\begin{aligned}
\langle g_{m'}, g_m \rangle &= \int_0^T g_{m'}(t) g_m^*(t) dt = \int_0^T (\frac{1}{\sqrt{T}} e^{j2\pi s_{m'} \Delta F t})(\frac{1}{\sqrt{T}} e^{-j2\pi s_m \Delta F t}) \\
&= \int_0^T \frac{1}{T} e^{j2\pi s_{m'} \Delta F t - j2\pi s_m \Delta F t} = \frac{1}{T} \int_0^T e^{j2\pi (s_{m'} - s_m) \Delta F t} = \frac{1}{T} \int_0^T e^{j2\pi \Delta s \Delta F t} \\
&= \frac{1}{T} \Big[\frac{1}{j2\pi \Delta s \Delta F t} \cdot e^{j2\pi \Delta s \Delta F t} \Big]_0^T = \frac{1}{T} \cdot \Big(\frac{e^{j2\pi \Delta s \Delta F T} - 1}{j2\pi \Delta s \Delta F T} \Big) \\
&= \frac{1}{T(j2\pi \Delta s \Delta F)} \cdot e^{j\pi \Delta s \Delta F T} \big(e^{j\pi \Delta s \Delta F T} - e^{-j\pi \Delta s \Delta F T} \big) \\
&= \frac{e^{j\pi \Delta s \Delta F T}}{T(\pi \Delta s \Delta F)} \cdot \frac{e^{j\pi \Delta s \Delta F T} - e^{-j\pi \Delta s \Delta F T}}{2j} = \frac{e^{j\pi \Delta s \Delta F T}}{T(\pi \Delta s \Delta F)} \cdot \sin(\pi \Delta s \Delta F T) \\
&= e^{j\pi \Delta s \Delta F T} \cdot \frac{\sin(\pi \Delta s \Delta F T)}{\pi \Delta s \Delta F T} \\
&= e^{j\pi \Delta F \Delta s T} \mathrm{sinc}(\Delta F \Delta s T)
\end{aligned}$$

As for the minimum value of ΔF that guarantees the orthogonality of $\langle g_{m'}, g_m \rangle$, in order to guarantee orthogonality between different symbols m, m', the inner product has to be 0.

$$\langle g_{m'}, g_m \rangle = e^{j\pi\Delta F\Delta sT}\text{sinc}(\Delta F\Delta sT) = 0$$

$sinc(x) = \frac{\sin(\pi x)}{\pi x} = 0$ only for $x \in \mathbb{Z} - \{0\}$. So, we require $\Delta F\Delta sT \in \mathbb{Z} - \{0\}$
As mentioned in our assignment prompt, $m \mapsto s_m$ ensures integer differences, so the minimum $\Delta s = 1$. This gives us

$$\Delta F\Delta sT = 1 \Rightarrow \Delta FT = 1 \Rightarrow \Delta F = \frac{1}{T}$$

Now for $r_k^{(m)}$: For this value of ΔF, the inner product simplifies to

$$\langle g_{m',k'}, g_{m,k} \rangle = \langle g_{m'}, g_m \rangle \delta_{k,k'}$$

We substitute this into the expression derived in 3a:

$$r_k^{(m)} = A|h|^2 \sum_{k'=0}^{K-1} \langle g_{m'}, g_m \rangle \delta_{k,k'} + n_k^{(m)}$$

The sum over k' eliminates the time-shift terms, leaving only the term for $k' = k$:

$$r_k^{(m)} = A|h|^2 \langle g_{m_k}, g_m \rangle + n_k^{(m)}$$

Since we showed that $\Delta F = 1/T$ ensures orthogonality between different frequencies ($\langle g_{m'}, g_m \rangle = \delta_{m',m}$), this simplifies to:

$$r_k^{(m)} = A|h|^2 \delta_{m_k,m} + n_k^{(m)}$$

This effectively means that the m-th filter output contains signal energy $A|h|^2$ only if the transmitted symbol m_k is equal to the filter m. Otherwise it's just noise in the output.

(3d)

For the same provided requirements as in 3c, with the added requirement $\Delta F = 1/2T$, we need to prove that $\langle g_{m'}, g_m \rangle = \delta^{FSK}_{m,m'}$, where $\delta^{FSK}_{m,m'} = \delta_{m,m'} + j\frac{1/2\pi}{\Delta s + \delta_{m,m'}}[\Delta s]_2$.
$[\cdot]_2$ has been defined in our assignment prompt as the remainder of input, modulo 2.
This means that $[\Delta s]_2 = 1$ if Δs is odd, 0 if even.
We already proved in 3c that

$$\langle g_{m'}, g_m \rangle = e^{j\pi\Delta F\Delta sT}\text{sinc}(\Delta F\Delta sT)$$

So now we can just plug $\Delta F = 1/2T$ into this formula:

$$\langle g_{m'}, g_m \rangle = e^{j\frac{\pi}{2}\Delta s}\text{sinc}(\frac{\Delta s}{2})$$

So now we can take cases based on this formula:

- **Same symbol** ($m = m' \Rightarrow \Delta s = 0$)

  $$\langle g_{m'}, g_m \rangle = e^{j\frac{\pi}{2}0}\text{sinc}(\frac{0}{2}) = 1 \cdot 1 = 1$$

  And $\Delta s = 0$, therefore $[\Delta s]_2 = 0$

- **Different symbol** ($\Delta s \neq 0$)

  - Δs **even**
    If Δs is even, then the argument $\Delta s/2$ of the *sinc* is a non zero integer. We mentioned before in 3c, that $sinc(x) = 0$ for $x \in Z - \{0\}$.
    Therefore this makes $\langle g_{m'}, g_m \rangle = 0$
    Also Δs is even, therefore $[\Delta s]_2 = 0$
  - Δs **odd**
    If Δs is odd, then the the argument $\Delta s/2$ of the *sinc* is no longer an integer, due to the 1/2. This means that *sinc* will be non zero, and we can use Euler's formula to proceed with the formula.

    $$\langle g_{m'}, g_m \rangle = e^{j\frac{\pi}{2}\Delta s}\text{sinc}(\frac{\Delta s}{2}) = [cos(\frac{\pi}{2}\Delta s) + jsin(\frac{\pi}{2}\Delta s)]\ \frac{sin(\frac{\pi}{2}\Delta s)}{\frac{\pi}{2}\Delta s}$$
    $$= [0 + jsin(\frac{\pi}{2}\Delta s)]\frac{sin(\frac{\pi}{2}\Delta s)}{\frac{\pi}{2}\Delta s} = \frac{jsin^2(\frac{\pi}{2}\Delta s)}{\frac{\pi}{2}\Delta s}$$

    Since Δs is odd, $\frac{\pi\Delta s}{2}$ is an odd multiple of $\frac{\pi}{2}$. So $\sin(\frac{\pi\Delta s}{2}) = \pm 1, \Rightarrow \sin^2(\frac{\pi\Delta s}{2}) = 1$.

    $$\langle g_{m'}, g_m \rangle = j\frac{1}{\frac{\pi\Delta s}{2}} = j\frac{2}{\pi\Delta s}$$

    And Δs odd, therefore $[\Delta s]_2 = 1$

Taking all of the cases above into consideration, we can unify them all into the formula:

$$\langle g_{m'}, g_m \rangle = \delta_{m,m'} + j\frac{2/\pi}{\Delta s + \delta_{m,m'}}[\Delta s]_2$$

Now for the formula of $r_k^{(m)}$ from task 3a, we subsittute the formula we derived for the inner product, inside $r_k^{(m)}$
Using the time-orthogonality result ($\delta_{k,k'}$), the sum over k' leaves us with $k' = k$:

$$r_k^{(m)} = A|h|^2 \langle g_{m'}, g_m \rangle + n_k^{(m)}$$

Substituting the specific inner product for $\Delta F = 1/2T$:

$$r_k^{(m)} = A|h|^2 \left(\delta_{m,m'} + j \frac{2/\pi}{\Delta s + \delta_{m,m'}} [\Delta s]_2 \right) + n_k^{(m)}$$

where $\Delta s = s_{m'} - s_m$.
This shows that when we use tighter frequency spacing $\Delta F = 1/2T$, we introduce inter-carrier interference (the imaginary term) whenever Δs is odd. However, since the interference is only imaginary, we can remove it by discarding the imaginary part.

(3e)

The shaping pulse $g_m(t) = \varphi(t)e^{j2\pi s_m \Delta F t}$, now has $\varphi(t)$ as the real-valued SRRC pulse with roll-off β, instead of the rectangular pulse.
We assume a single symbol per shaped signal ($K = 1$), so we no longer care about the index k.
We remember that the inner product of two pulses with different frequency indexes m and m' is:

$$\langle g_{m'}, g_m \rangle = \int_{-\infty}^{\infty} g_{m'}(t) g_m^*(t)\, dt$$

Substituting $g_m(t)$ inside it:

$$\langle g_{m'}, g_m \rangle = \int_{-\infty}^{\infty} \left(\varphi(t)e^{j2\pi s_{m'} \Delta F t}\right) \left(\varphi(t)e^{-j2\pi s_m \Delta F t}\right)^* dt$$

$\varphi^*(t) = \varphi(t)$ since $\phi(t)$ is real:

$$\langle g_{m'}, g_m \rangle = \int_{-\infty}^{\infty} \varphi^2(t) e^{j2\pi \Delta s \Delta F t} dt$$

This is exactly the fourier of $\varphi^2(t)$ at $f = -\Delta s \Delta F$:

$$\langle g_{m'}, g_m \rangle = \int_{-\infty}^{\infty} \varphi^2(t) e^{j2\pi f t} dt \Big|_{f=-\Delta s \Delta F} = \mathcal{F}\{\varphi^2(t)\} \Big|_{f=-\Delta s \Delta F}$$

We know that multiplication in time is convolution in frequency, and vice versa.

$$\mathcal{F}\{\varphi(t)\varphi(t)\} = \Phi(f) * \Phi(f)$$

Since the spectrum of SRRC $\Phi(f)$ is symmetric around 0, the convolution $(\Phi * \Phi)(f)$ is also symmetric around 0. This means that $(\Phi * \Phi)(-f) = (\Phi * \Phi)(f)$, and it doesn;t matter whether $f = \Delta s \Delta F$ or $f = -\Delta s \Delta F$:

$$\langle g_{m'}, g_m \rangle = (\Phi * \Phi)(\Delta s \Delta F) \Big|_{f=\Delta s \Delta F}$$

As for the minimum ΔF to guarantee orthogonality of $\langle g_{m',k'}, g_{m,k}\rangle = \langle g_{m'}, g_m\rangle$:
We require the inner product to be zero for any distinct symbols ($|\Delta s| \geq 1$). This means we need:

$$(\Phi * \Phi)(\Delta s \Delta F) = 0 \quad \text{for } |\Delta s| \geq 1$$

The BW of $\Phi(f)$ is

$$f \in [-\frac{1+\beta}{2T}, \frac{1+\beta}{2T}]$$

The BW of the $(\Phi * \Phi)(f)$ is the sum of each $\Phi(f)$'s BW.

$$(\Phi * \Phi) = \left[-\frac{1+\beta}{T}, \frac{1+\beta}{T}\right]$$

The minimum value for distinct symbols is $\Delta s = 1$.
For the value of the inner product to be zero, so as to preserve orthogonality, the frequency shift ΔF for $\Delta s = 1$ must be outside the BW of the convolution we just mentioned above.

$$\Delta F \geq |\frac{1+\beta}{T}| \Rightarrow \Delta F \geq \frac{1+\beta}{T}$$

since $(1+a)$ always positive between 1 and 2, and $T > 0$.
The minimum value of ΔF that guarantees orthogonality is:

$$\Delta F = \frac{1+\beta}{T}$$

(3f)

We just derived the minimum spacing required to preserve orthogonality : $\Delta F = \frac{1+\beta}{T}$

Using the original mapping $s_m = m$, where $m \in \mathbb{Z}_M = \{0, 1, \ldots, M-1\}$:
The spectrum of a single SRRC pulse $\Phi(f)$ is centered at 0 Hz and its BW is:

$$\left[-\frac{1+\beta}{2T}, \frac{1+\beta}{2T}\right],$$

which is also essentially the spectrum of the very first symbol's pulse.
The spectrum of the adjacent symbol's pulse is shifted by ΔF. To generalize this a bit, the spectrum of the m-th symbol's pulse $g_m(t)$ is shifted by $s_m \Delta F$. Therefore, its BW is:

$$\left[s_m \Delta F - \frac{1+\beta}{2T}, \quad s_m \Delta F + \frac{1+\beta}{2T}\right]$$

To find the total BW range for this mapping in the baseband, we take the union of the spectrums, for the lowest symbol ($m = 0$) and the highest symbol ($m = M-1$).

- Lowest frequency ($m = 0$):

$$f_{\min} = 0 \cdot \Delta F - \frac{1+\beta}{2T} = -\frac{1+\beta}{2T}$$

- Highest frequency ($m = M-1$):

$$f_{\max} = (M-1)\Delta F + \frac{1+\beta}{2T} = (M-1)\frac{1+\beta}{T} + \frac{1+\beta}{2T}$$

This gives us the total range for the baseband:

$$f \in [f_{min}, f_{max}] = \left[-\frac{1+\beta}{2T}, \quad (M-1)\frac{1+\beta}{T} + \frac{1+\beta}{2T}\right]$$

We can clearly see that this baseband range is completely asymmetric and extends heavily towards the positive frequencies.
For example, if we use the current mapping for 5-FSK, then we have 5 symbols s_0, s_1, s_2, s_3, s_4, therefore the multipliers for the different frequencies that the symbols use are $0, 1, 2, 3, 4$. This gives us $0, \Delta F, 2\Delta F, 3\Delta F, 4\Delta F$. This is clearly skewed to the positive frequencies.

We need to rethink the mapping, so as to center the BW around 0Hz, while maintaining the BW.
To do this, we will need to shift the indexes of the symbols, so their average is 0.
The middle index in the set $\{0, 1, \ldots, M-1\}$ is $\frac{M-1}{2}$.

If we use this mapping instead :

$$s_m = m - \frac{M-1}{2}$$

then we satisfy the required condition of integer differences between the symbols, since

$$s_{m+1} - s_m = \left((m+1) - \frac{M-1}{2}\right) - \left(m - \frac{M-1}{2}\right) = 1$$

The new frequency range is now symmetric

$$f \in \left[-\left(\frac{M-1}{2}\Delta F + \frac{1+\beta}{2T}\right), \quad +\left(\frac{M-1}{2}\Delta F + \frac{1+\beta}{2T}\right)\right]$$

I'd like to provide another example for the proposed mapping, using 5-FSK, since it's also what has to implemented next. The symbol indexes are $\{0, 1, 2, 3, 4\}$ and this makes the midddle index 2.
We now use the mapping $s_m = m - 2$
For the symbols s_0, s_1, s_2, s_3, s_4, the frequency multipliers will be $\{0-2, 1-2, 2-2, 3-2, 4-2\} = \{-2, -1, 0, 1, 2\}$, making the frequencies for the symbols $-2\Delta F, -\Delta F, 0, \Delta F, 2\Delta F$.
This range is symmetrical, since the average frequency is 0Hz.

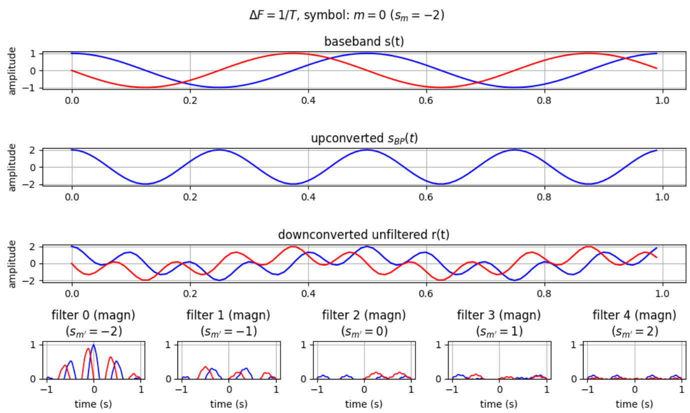
(3g) - Simulation (`task_3.py`)

For the coding part, we declare all the needed paremeters.
$f_c = 6Hz, T = 1sec, T_s = T/100 = 0.01sec$, we're back to rectangular pulse $\phi(t) = 1/\sqrt{T}$, the channel is ideal ($r(t) = hs(t) \Rightarrow r(t) = s(t)$) and there's no noise at the receiver side.

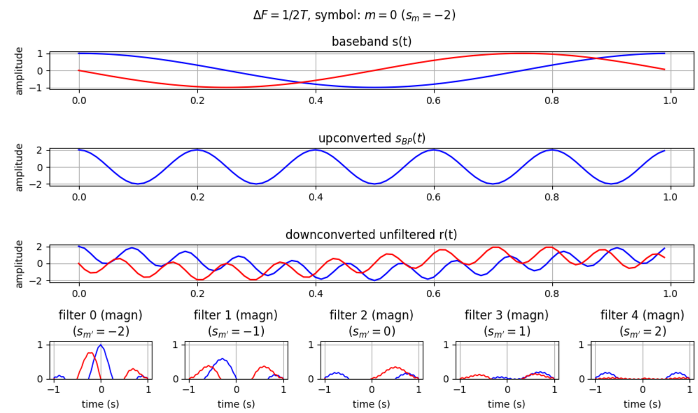
We declare the symbols, using the proposed mapping as detailed right above in the example in (3g) : $s_m = m - 2$, therefore the frequency multipliers are $\{-2, -1, 0, 1, 2\}$, making the frequency spacing for the 5 symbols : $-2\Delta F, -\Delta F,\ 0\ , \Delta F,\ 2\Delta F$.

We now proceed with the plotting.
For each value of $\Delta F = \{\frac{1}{T}, \frac{1}{2T}\}$, we have a figure for each of the 5 symbols, and that brings us to a total of 10 figures. Each of the 10 figures contains: the baseband, the upconverted, the downconverted unfiltered signal, as well as the downconverted signal filtered by $g_0^*(-t), g_1^*(-t), g_2^*(-t), g_3^*(-t), g_4^*(-t)$ separately.

These were the plots that appeared, and we will show both cases of ΔF per symbol.

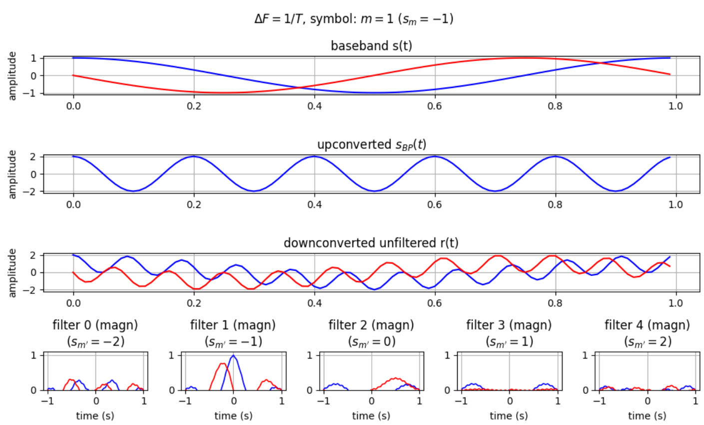


(a) $\Delta F = 1/T$

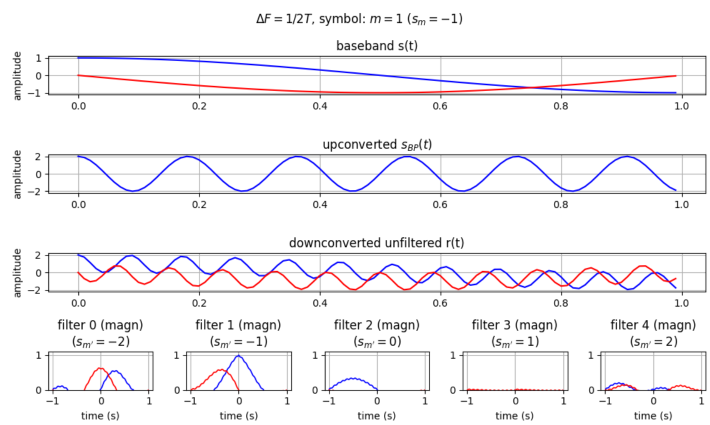


(b) $\Delta F = 1/2T$

symbol ($s_0 = -2\ , -2\Delta F$).

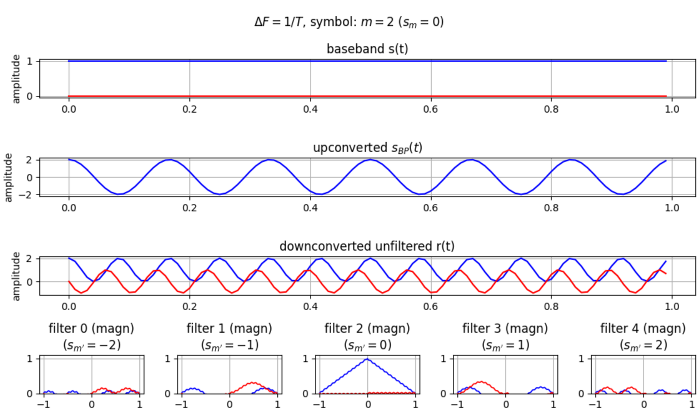


(a) $\Delta F = 1/T$

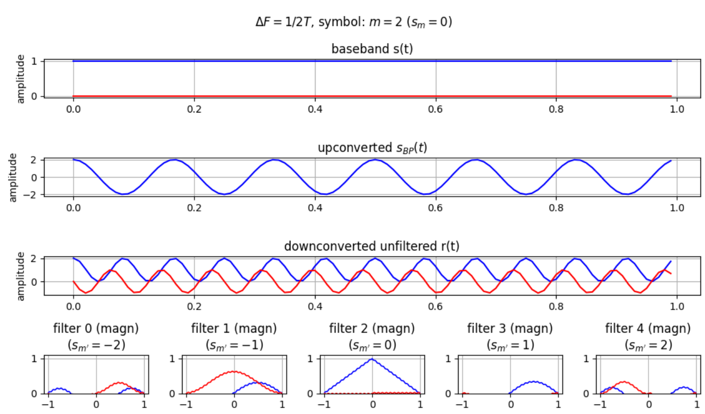


(b) $\Delta F = 1/2T$

symbol ($s_1 = -1\ , -\Delta F$).

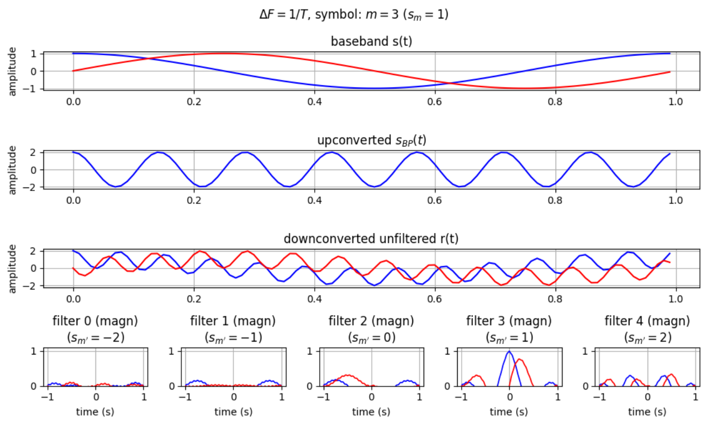


(a) $\Delta F = 1/T$

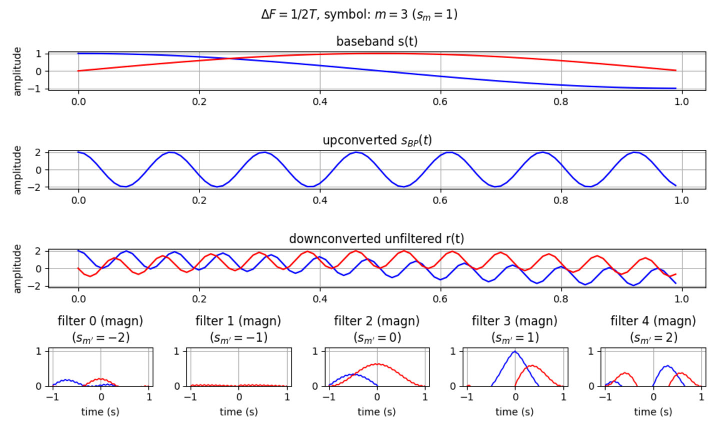


(b) $\Delta F = 1/2T$

symbol ($s_2 = 0,\, 0\Delta F$).

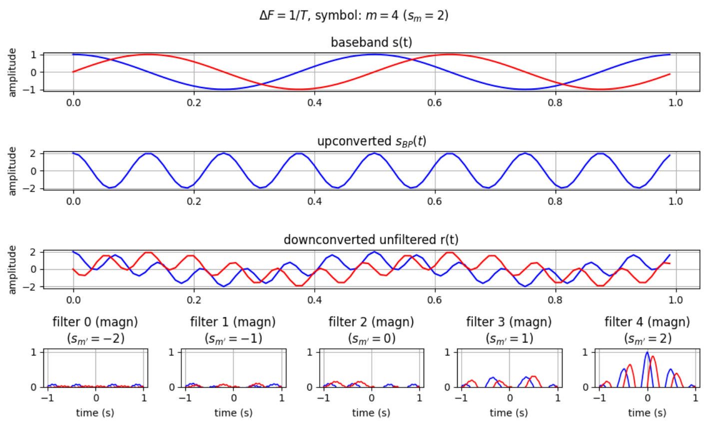


(a) $\Delta F = 1/T$

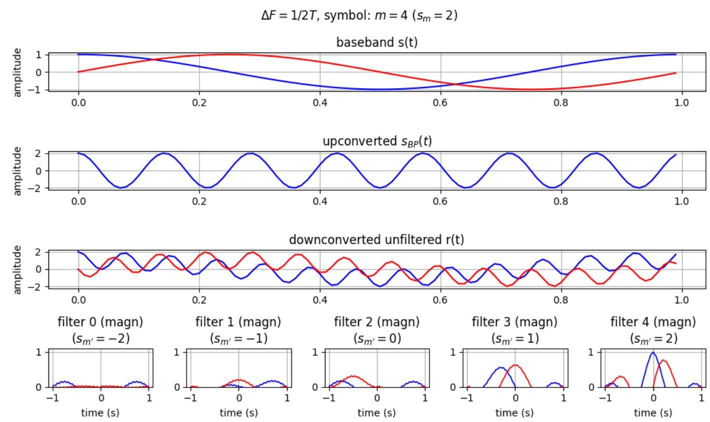


(b) $\Delta F = 1/2T$

symbol ($s_3 = -1\ , \Delta F$).



(a) $\Delta F = 1/T$



(b) $\Delta F = 1/2T$

symbol ($s_4 = 2\ , 2\Delta F$).

Now, to answer the questions in the assignment:

- **Observations on the transmitted signals between different symbols**
  The transmitted signal is $s_{BP}(t)$.

  For $\boldsymbol{\Delta F = 1/T}$,
  we can see that the signal completes integer numbers of cycles within the period $T = 1s$ (4 cycles for s_0, 5 cycles for s_1, 6 cycles for s_2, 7 cycles for s_3, 8 cycles for s_4).
  That's because our carrier freq is $f_c = 6$Hz, and the spacing is $\Delta F = 1/T = 1$Hz. So, for each transmitted symbol the frequency is $f = f_c + s_m\Delta F = 6 + s_m\Delta F$.
  That makes it $f = 6 - 2\Delta F = 4$Hz $= 4$ cycles for s_0, $f = 6 - \Delta F = 5$Hz $= 4$ cycles for s_1, and so on.
  We can see that the phase starts and ends at the same point. So, for each symbol, the phase is reset to start with the next symbol.

For $\boldsymbol{\Delta F = 1/2T}$,
In the same manner as above, the frequency is $f = f_c + s_m \Delta F$.
The carrier frequency is $f_c = 6\text{Hz}$, while the spacing $\Delta F = 0.5\text{Hz}$. We proved before that orthogonality is not guaranteed for this value.
The signal will complete integer numbers of cycles for even s_m values and non-integers for odd s_m values.
Let's examine this a bit further:
$s_0 : f = f_c + s_0 \Delta F = 6 - 2\Delta F = 5$ cycles (Hz)
$s_1 : f = f_c + s_1 \Delta F = 6 - \Delta F = 5.5$ cycles (Hz)
$s_2 : f = f_c + s_2 \Delta F = 6 + 0\Delta F = 6$ cycles (Hz)
$s_3 : f = f_c + s_3 \Delta F = 6 + \Delta F = 6.5$ cycles (Hz)
$s_4 : f = f_c + s_4 \Delta F = 6 + 2\Delta F = 7$ cycles (Hz)
This means that for the symbols s_1, s_3, the signal ends with an inverted phase relative to the one it began with, resulting in a phase discontinuity with the next transmission.

- **Observations on the filtered signal for each sent symbol**

For $\boldsymbol{\Delta F = 1/T}$,
This spacing ensures orthogonality as proven before.
In the output of the $g_m^*(-t)$ filter matching the transmitted symbol $s_{m'}$, therefore $m = m'$, we observe an amplitude of 1 at the moment of sampling.
For all other filters where $m \neq m'$, we observe that there are ripples in the symbol during the symbol period, but the output at the exact moment of sampling is 0. That happens for every symbol when the spacing is $\Delta F = 1/T$.
Now as for the ripples that we see, there's two observations to be made:
1) What they are, and that's simply the result of the mathematical similarity between the signal we sent adn the signal we're checking for. In this case, pulse orthogonality is guaranteed, thus there's zero intercarrier interference. The negative and positive interference cancels out, and we can see that at the moment of sampling, the ripples have 0 amplitude.
2) We also observe that for filters further away from the the correct symbol index, the ripples appear to have lower amplitude and higher frequency. This happens because the frequency difference Δf between the signal and the filter increases. For the amplitude, as the frequencies diverge, the signal and filter become less correlated.

For $\boldsymbol{\Delta F = 1/2T}$,
This spacing no longer guarantees orthogonality.
We observe the same behavior as above when $\boldsymbol{m = m'}$, a peak amplitude of 1 at the sampling moment, and the ripples in the output of the other filters have a amplitude of 0 at the sampling moment.
When $\boldsymbol{m \neq m'}$, the behavior now depends on whether s_m has even or odd m. (reminder that our mapping is $s_m = m - 2$ and the symbols are $s_0 = -2, s_1 = -1, s_0 = 0, s_1 = 1, s_2 = 2$)
We mentioned that for the symbols s_1, s_3, the signal ends with an inverted phase relative to the one it began with. For those same symbols, the orthogonal case does not hold anymore and we observe interference, with the ripples having a non-zero amplitude (≈ 0.6) at sampling time.
This ties back to what we proved in (3d), that an imaginary interference term arises.

- **For a consecutive transmission of symbols (continuous signal), which value for ΔF would we choose? Why?** We examined right before that $\Delta F = 1/T$ always allows for seamless phase transition from one symbol to the other, due to the signal completing integer numbers of cycles for each symbol. That's why we would choose $\Delta F = 1/T$

- **Which value for ΔF would we choose if the continuous signal requirement was not a concern? why?** If the continuous signal is no longer a concern, then we no longer care about the phase discontinuity caused by $\Delta F = 1/2T$ in odd symbols.
  If we reduce the spacing from $1/T$ to $1/2T$, then each transmission uses half the bandwidth, which doubles our spectral efficiency (bits/sec/Hz).
  Also, the interference we had noticed for odd symbols for $\Delta F = 1/2T$ can be dropped by discarding the imaginary part, as the assignment prompt assures. Therefore, a receiver that only extracts the real part of the matched filter output can eliminate the interference, which means that with $\Delta F = 1/2T$ we can also achieve error-free communication, using half the bandwidth.

Task 4

Our assumption for the channel is flat-fading for this task, therefore when sampling at the receiver for a single symbol X, we see:

$$Y = hX + N,$$

where

- Y, X, N are RVs, and h is complex deterministic $\in \mathbb{C}$
- X, N are independent
- N is white Gaussian noise, $N \sim \mathbb{C}_N(0, N_0)$

The SNR is defined as follows:

- Transmitter side

$$\text{SNR}_{T_X} \triangleq \frac{E[\ |X|^2\ ]}{E[\ |N|^2\ ]}$$

- Receiver side :

$$\text{SNR}_{R_X} \triangleq \frac{E[\ |hX|^2\ ]}{E[\ |N|^2\ ]} = |h|^2 \frac{E[\ |X|^2\ ]}{E[\ |N|^2\ ]} = |h|^2 \cdot \text{SNR}_{T_X}$$

For tasks 4a, 4b, 4c : As denoted in the chapter 4 pdf, in M-PSK the symbols contain $log_2 M$ bits each, and the constellation is

$$\{1, e^{j\frac{2\pi}{M}}, e^{j2\frac{2\pi}{M}}, .., e^{j(M-1)\frac{2\pi}{M}}\} \qquad (4.18)$$

In our case of BPSK specifically, the constellation will be

$$\{a, ae^{j(2-1)\frac{2\pi}{2}}\} = \{a, ae^{j\pi}\} = \{a, -a\},$$

where the bit to symbol mapping is $0 \to a, 1 \to -a$.

(4a)

For BPSK, X is uniformly distributed over $\{a, -a\}$. This means that $P(X = a) = P(X = -a) = 1/2$. The squared magnitude for both values of the symbol is the same, $|a|^2 = |-a|^2 = a^2$.

- $E[\,|X|^2\,] = \frac{1}{2}a^2 + \frac{1}{2}a^2 = a^2$
- $E[\,|N|^2\,] = N_0$

which makes

$$\text{SNR}_{T_X,BPSK} = \frac{a^2}{N_0}$$

(4b)

Since we have BPSK, the points on the constellation align with BPAM. They're both on the real axis, with the points being at $-a$ and a.
This means that mathematically, our decision rule is the same for BPAM and BPSK.
The optimal receiver for BPAM can be seen in lecture 6a from our notes on eclass, which also contains the decision rule.
The ML rule simplifies to the minimum Euclidean distance rule in this case.
If the received sample Y is closer to $P_0 = ha$ than to $P_1 = -ha$, the receiver decides $\hat{X} = a$ ($\hat{b} = 0$). Otherwise, it decides $\hat{X} = -a$ ($\hat{b} = 1$).
We have:

$$R_{|b=0} \sim \mathbb{C}_N(ha, \sigma^2), \quad R_{|b=1} \sim \mathbb{C}_N(-ha, \sigma^2)$$

Therefore,

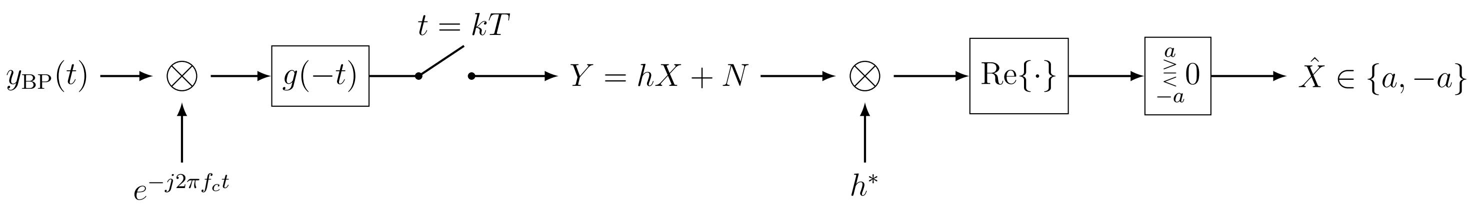
$$\begin{aligned}
&\text{If } f_{R|b=0} > f_{R|b=1}, \text{ decide } \hat{X} = a \ (\hat{b} = 0).\\
&\Leftrightarrow \frac{1}{\pi\sigma^2}e^{-\frac{|Y-ha|^2}{\sigma^2}} > \frac{1}{\pi\sigma^2}e^{-\frac{|Y-(-ha)|^2}{\sigma^2}} \Leftrightarrow e^{-\frac{|Y-ha|^2}{\sigma^2}} > e^{-\frac{|Y+ha|^2}{\sigma^2}}\\
&\Leftrightarrow -\frac{|Y-ha|^2}{\sigma^2} > -\frac{|Y+ha|^2}{\sigma^2} \Leftrightarrow -|Y-ha|^2 > -|Y+ha|^2\\
&\Leftrightarrow |Y-ha|^2 < |Y+ha|^2
\end{aligned}$$

To proceed, we will use $|z + \omega|^2 = |z|^2 + |\omega|^2 + 2Re(z\omega^*)$, also seen in lecture 6a.

$$\begin{aligned}
&\Leftrightarrow |Y|^2 + |-ha|^2 + 2Re(Y(-ha)^*) < |Y|^2 + |ha|^2 + 2Re(Y(ha)^*)\\
&\Leftrightarrow 2Re(Y(-ha)^*) < 2Re(Y(ha)^*)\\
&\Leftrightarrow -Re(Yh^*a) < Re(Yh^*a) \Leftrightarrow -aRe(Yh^*) < aRe(Yh^*)\\
&\Leftrightarrow 2Re(Yh^*) > 0 \Leftrightarrow Re(Yh^*) > 0
\end{aligned}$$

So, the decision rule is $Z = Re(Yh^*) \underset{-a}{\overset{a}{\gtrless}} 0$

This points us to the implementation of the optimal receiver.
As seen in the optimal receiver for BPAM (Lecture 6a), the same applies now.
The band pass signal $y_{BP}(t)$ is received, downconverted, passed through $g^*(-t) = g(-t)$ (since it's a linear modulation), then sampled at each symbol period.
The sampled signal is then multiplied by h^*, the real part is kept and the decision rule is applied.



(4c)

The probability of error Pe is the probability that the symbol $\hat{X}$ (or bit $\hat{b}$) that the receiver decided, is not the symbol X (or bit b) that was actually sent by the transmitter.
We want to derive Pe as a function of $\mathrm{SNR}_{T_X,BPSK}$ and h.
Once again, this can be found in lecture 6a.
We found the decision rule as $Z = Re(Yh^*) = Re((hX+N)h^*) = Re(hh^*X + h^*N) = Re(|h|^2X + h^*N) = |h|^2X + Re(h^*N) = |h|^2X + W$,
where $W \sim \mathcal{N}(0, \frac{|h|^2N_0}{2})$, since it is the noise N, scaled by $|h^2|$, and the real part is kept only (no complex distribution).

$$
\begin{aligned}
&P_e = P(\hat{b} \neq b)\\
&= p(\hat{b} \neq b \mid b=0)p(b=0) + p(\hat{b} \neq b \mid b=1)p(b=1)\\
&= p(\hat{b} = 1 \mid b=0)p(b=0) + p(\hat{b} = 0 \mid b=1)p(b=1)\\
&= p(Z<0 \mid b=0)p(b=0) + p(Z>0 \mid b=1)p(b=1)\\
&= p(|h^2|X + W < 0 \mid b=0)p(b=0) + p(|h^2|X + W > 0 \mid b=1)p(b=1)\\
&\text{h deterministic, X is \{-a,a\}. Since the only RV here is W, and it's independent of the bit b,}\\
&\text{we can remove the conditional.}\\
&= p(W < -|h^2|a)p(b=0) + p(W > -|h^2|(-a))p(b=1)\\
&= p(W < -|h^2|a)p(b=0) + p(W > |h^2|a)p(b=1) \qquad (*)
\end{aligned}
$$

We'll return to the equation above shortly.
$Q(a)$ is defined as $P(X > a) = Q(a)$. $Q(-a) = 1 - Q(a)$
For an RV $Y \sim \mathcal{N}(\mu, N_0) \Rightarrow P(Y > a) = Q(\frac{a-\mu}{\sigma})$

We also have

$$W \sim \mathcal{N}(0, \frac{|h|^2N_0}{2}) \Rightarrow \; p(W > |h|^2a) = Q\Big(\frac{|h|^2a - 0}{\sqrt{\frac{|h|^2N_0}{2}}}\Big) = Q(\sqrt{\frac{2|h|^2a^2}{N_0}})$$

$$p(W < -|h|^2a) = 1 - p(W > -|h|^2a) = 1 - Q(-\sqrt{\frac{2|h|^2a^2}{N_0}}) = Q(\sqrt{\frac{2|h|^2a^2}{N_0}})$$

$$(*) \Rightarrow p(W < -|h^2|a)p(b=0) + p(W > |h^2|a)p(b=1) = Q(\sqrt{\frac{2|h|^2a^2}{N_0}})/2 + Q(\sqrt{\frac{2|h|^2a^2}{N_0}})/2$$
$$= Q(\sqrt{\frac{2|h|^2a^2}{N_0}}) = Q(\sqrt{2|h|^2 \cdot \mathrm{SNR}_{T_X,BPSK}})$$

(4d)

For 4-QAM, the symbol set is $X \in \{\pm a \pm ja\}$.

- $\mathbb{E}[|X|^2]$:
  Again, the probability of all 4 symbols is uniform. The square of the magnitude of any of the 4 symbols is the same, $|X|^2 = |\pm a \pm ja|^2 = a^2 + a^2 = 2a^2$. So,
  $$E[\ |X|^2\ ] = 2a^2$$
- $E[\ |N|^2\ ] = N_0$, already provided.

Substituting these, the SNR is:
$$SNR_{Tx,4QAM} = \frac{2a^2}{N_0}$$

We can see that the SNR of 4-QAM is double the SNR of BPSK, as we use both the in-phase and quadrature components in contrast to BPSK where the symbols were exclusively on the real axis.
We can think of the 4-QAM stream as two BPSK streams, one in the quadrature and one in the in-phase, each with SNR $= a^2/N_0$

(4e)

As hinted in the assignment, we can make the BPSK detector pop up, instead of brute-forcing the detection for the 4 points in the QAM constellation.
Each 4-QAM symbol can be decomposed to the in-phase and quadrature component : $X = X_I + jX_Q$, where $X_I \in \{-a, a\}$, $X_Q \in \{-a, a\}$, just like in BPSK.

The received signal is $Y = hX + N$. Next, we make up for channel's phase distortion by multiplying by h^*, just like we did in BPSK:

$$Z = Yh^* = (hX + N)h^* = |h|^2 X + Nh^*$$

Substituting $X = X_I + jX_Q$:

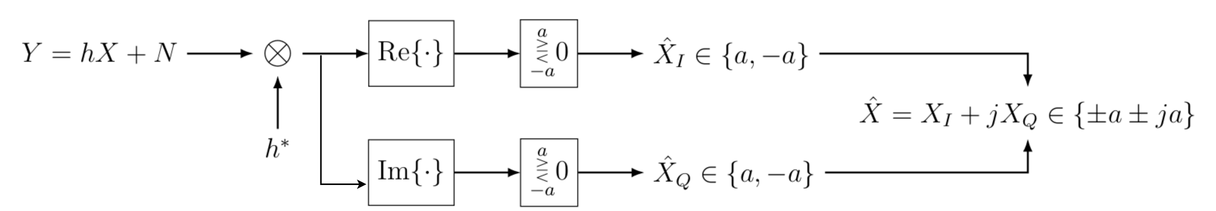
$$Z = |h|^2(X_I + jX_Q) + W = |h|^2 X_I + j|h|^2 X_Q + W$$

where $W = Nh^*$ is the rotated noise.
And finally, this is the part where we can break this down to the BPSK detector. Since the real and imaginary parts of complex Gaussian noise are independent, we can detect X_I and X_Q separately.

- **For X_I:** This is identical to the BPSK case derived in Task 4.b. We check the sign of the real part of the statistic:
$$\text{Re}\{Z\} = \text{Re}\{Yh^*\} \underset{-a}{\overset{a}{\gtrless}} 0 \Rightarrow \hat{X}_I = \pm a$$

- **For X_Q:** Similarly, we check the sign of the imaginary part:
$$\text{Im}\{Z\} = \text{Im}\{Yh^*\} \underset{-a}{\overset{a}{\gtrless}} 0 \Rightarrow \hat{X}_Q = \pm a$$

So this confirms that the optimal 4-QAM detector is just two parallel BPSK detectors.



(4f)

The error probability, whether that's from the real or the imaginary branch, is simply the error probability we found for BPSK in (4c):

$$p = Q\left(\sqrt{\frac{2a^2|h|^2}{N_0}}\right)$$

In (4d), we found $SNR_{T_X,4QAM} = \frac{2a^2}{N_0}$. Substituting into p:

$$p = Q\left(\sqrt{|h|^2 \cdot SNR_{T_X,4QAM}}\right)$$

And that is the bit error probability for 4-QAM.

(4g) - Simulation (task_4.py)

To implement the simulation of the 4QAM system, we defined the parameters:

- symbol parameter $a = 1$, so the symbols are $s_k \in \{\pm 1 \pm j\}$
- SRRC pulse with roll-off $\alpha = 0.35$ and half-duration $A = 4$ symbol periods.
- carrier frequency $f_c = 800$ kHz, pass band bandwidth $W_{pass} = 80$ kHz.
- the symbol period T was derived from the SRRC bandwidth requirement

$$W_{pass} = \frac{1+\alpha}{T} \implies T = \frac{1.35}{80\text{ kHz}} = 16.875\,\mu\text{s}$$

- **o**versampling factor 200, so $T_s = T/200$.

THE TRANSMITTER

The transmission begins by generating a sequence of random bits, which are mapped to independent complex 4-QAM symbols s_k. The baseband signal $s(t)$ is constructed by passing the symbol impulse train through the pulse filter:

$$s(t) = \sum_k s_k g(t - kT)$$

In the discrete-time simulation, this is implemented by upsampling the symbol vector by the oversampling factor and convolving it with the sampled pulse $g[n]$. The discrete pulse $g[n]$ is normalized to have unit energy ($\sum |g[n]|^2 = 1$).

THE 2-RAY CHANNEL

The channel has two rays, a primary and a delayed. The received bandpass signal is given by:

$$r_{BP}(t) = a_0 s_{BP}(t) + a_1 s_{BP}(t - t_1) + n(t) \qquad \text{(Lecture 5)}$$

where a_0, a_1 are the ray amplitudes and t_1 is the delay of the second ray (the delay of the first ray is 0). The equivalent baseband channel impulse response is:

$$h(t) = a_0 \delta(t) + a_1 e^{-j2\pi f_c t_1} \delta(t - t_1) \qquad \text{(Lecture 5)}$$

Note the term $e^{-j2\pi f_c t_1}$, which represents the phase rotation of the baseband envelope caused by the delay in the pass band. The frequency response of this channel is:

$$H(f) = a_0 + a_1 e^{-j2\pi f_c t_1} e^{-j2\pi f t_1} \qquad \text{(Lecture 5)}$$

THE RECEIVER

Since our assumption is the flat-fading model, we approximate the channel as a a complex scalar h. We estimate h by setting $f = 0$ in $H(f)$:

$$h \approx H(0) = a_0 + a_1 e^{-j2\pi f_c t_1} \qquad \text{(Lecture 5)}$$

This is the h whose conjugate we use to apply the optimal decision rule for 4QAM, that we found in (4e).

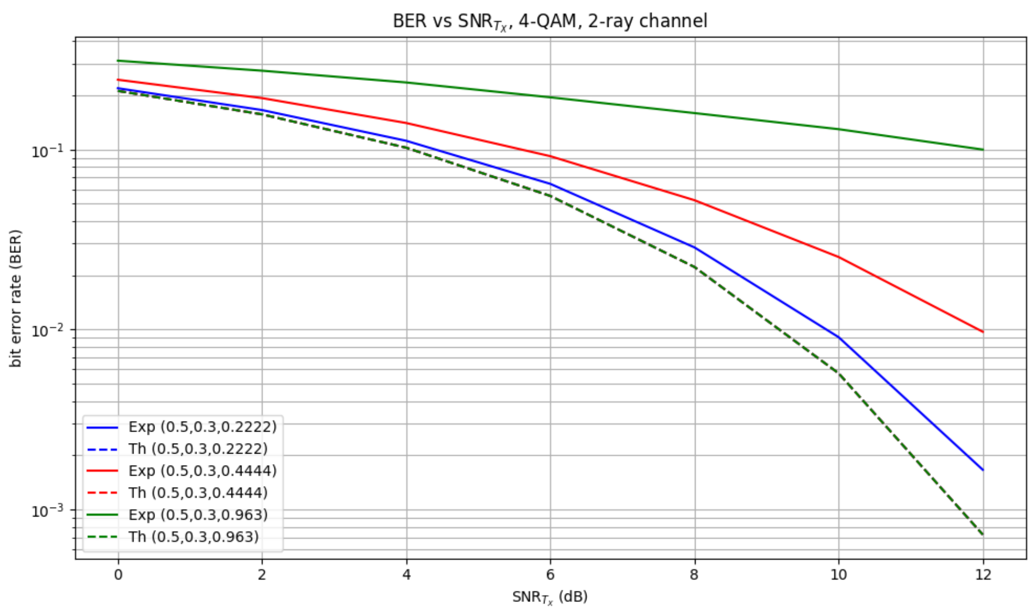
We will now evaluate the bit error rate (BER) performance of this receiver across different channel scenarios (a_0, a_1, t_1) and SNR levels.

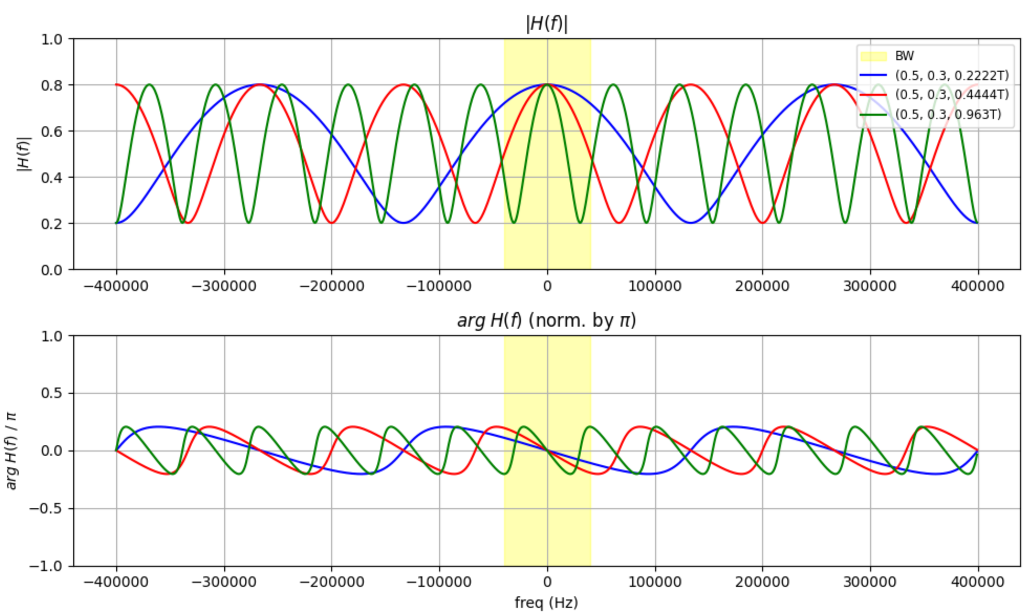
FIRST GROUP

We define the channel parameters (a_0, a_1, t_1) for three scenarios where the delay t_1 varies significantly relative to the symbol period T.

- **Blue:** $(0.5, 0.3, 0.2222T)$
- **Red:** $(0.5, 0.3, 0.4444T)$
- **Green:** $(0.5, 0.3, 0.9630T)$

Experimental BER: We observe that the experimental BER curves (solid lines) are as expected. The blue case, where the delay is the closest to 0 than the other two cases (0.2222, compared to 0.4444 and 0.963), begins with a smaller BER, and the BER drops faster than the other two as the SNR rises.





Looking at the spectrum of $H(f)$, we understand why.
We observe that all 3 magnitudes oscillate within our desired bandwidth. However, the oscillations are considerably worse as the delays of the rays worsen. The case with the least delay (blue, $0.2222T$) is the one that is closest to being flat in our bandwidth. And we can see that the blue case is the one with the best SNR curve.
What we understand from this is that the **smaller the delay, the less oscillations in the magnitude of $H(f)$, therefore less signal distortion by the channel, which equals less probability of error during detection at the receiver**.

Comparison between theoretical and experimental BER

The **theoretical BER** (dashed lines) is identical for all 3 cases. The green, red and blue lines overlap entirely.
The identical performance can be explained by examining the channel frequency response $H(f)$. The receiver approximates the channel as $h \approx H(0) = a_0 + a_1 e^{-j2\pi f_c t_1}$. Although the delays t_1 are different, all 3 values chosen for t_1 result in the phasor $e^{-j2\pi f_c t_1} = 1$
We can verify this: Given $f_c = 800$ kHz and $T \approx 16.875\mu$s, the product $f_c T = 13.5$. The phase rotation is $\theta = 2\pi f_c t_1 = 2\pi(13.5)\frac{t_1}{T} = 27\pi\tau$.

- **Blue** ($\tau = 0.222 \approx 2/9$)**:** $\theta = 27\pi(2/9) = 6\pi \implies e^{-j6\pi} = 1$.
- **Red** ($\tau = 0.4444 \approx 4/9$)**:** $\theta = 27\pi(4/9) = 12\pi \implies e^{-j12\pi} = 1$.
- **Green** ($\tau = 0.963 \approx 26/27$)**:** $\theta = 27\pi(26/27) = 26\pi \implies e^{-j26\pi} = 1$.

Since the exponential term is 1 in all cases, the h is identical in all cases:

$$h = H(0) = 0.5 + 0.3 \cdot 1 = 0.8$$

Therefore the error probability is the same for all 3 cases,

$$P_e \approx 2Q\left(\sqrt{|h|^2 \cdot SNR_{Tx}}\right),$$

which makes the theoretical BER

$$BER = \frac{P_e}{log_2 M} = P_e/2 = Q\left(\sqrt{|h|^2 \cdot SNR_{Tx}}\right)$$

, where

$$\text{SNR}_{T_X} = \frac{2a^2}{N_0} = \frac{2}{N_0}$$

As the SNR rises in the x-axis, this means that the N_0 gets smaller. Expectedly, the theoretical BER is a curve that decreases as the SNR gets better. The better the signal-to-noise ratio, the less frequent the detection errors for the bits.

The **experimental BER** in contrast to the theoretical one, is affected by the phenomenon we observed before : a larger delay causes greater oscillations in the magnitude of $H(f)$, which in turn distort our signal more, causing more bit errors (larger BER).
The theoretical BER doesn't see this, it only sees a flat value for the channel, $h = 0.8$. This means that in any case, the theoretical BER will:

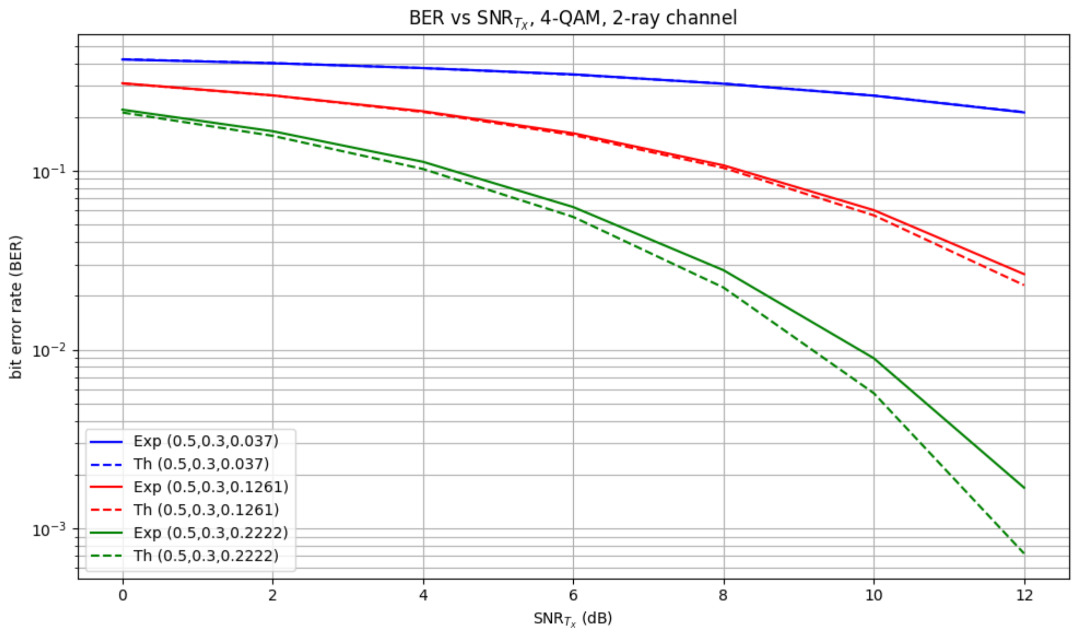
- START with a lower BER value than all 3 experimental cases,at $\text{SNR}_{T_X}(dB) = 0$
- END with a lower BER value than all 3 experimental cases, at $\text{SNR}_{T_X}(dB) = 12$

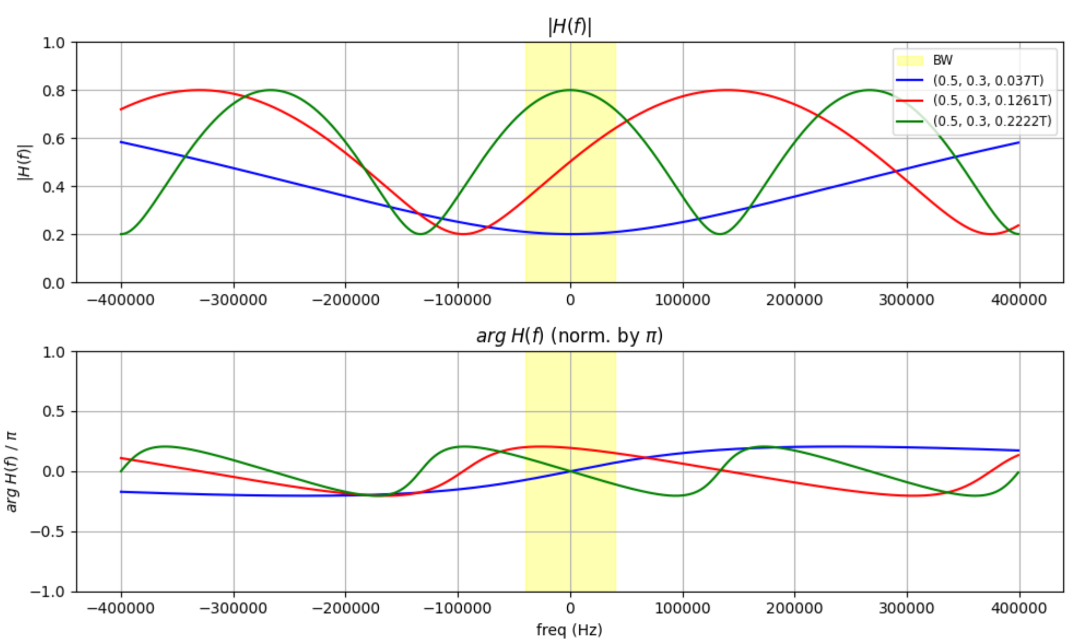
, simply because in theory, the channel is approximated as flat, while in practice it isn't purely flat.

SECOND GROUP

This time, we have :

Blue: $(0.5, 0.3, 0.037T)$, **Red:** $(0.5, 0.3, 0.1261T)$, **Green:** $(0.5, 0.3, 0.2222T)$





- **The BERs are getting better, as the delay increases. Why?**
  We observe a clear separation in the BER curves. The case with the smallest delay ($0.037T$) has the worst performance, while the case with the largest delay ($0.222T$) exhibits the best performance. This is a paradox, given that in class we generally consider that the smaller the delay, the better. But, it can be explained by the carrier phase interference.

  So how do we explain the paradox?
  The receiver's effective signal power depends on $H(0) = a_0 + a_1 e^{-j\theta}$, where $\theta = 2\pi f_c t_1$.

  - Blue ($0.037T$): The phase rotation is $\theta = 27\pi(0.037) \approx \pi$. This creates destructive interference ($e^{-j\pi} = -1$). The channel magnitude drops to $|h| = |0.5 + 0.3(-1)| = 0.2$. This massive loss in signal power causes the high BER.
  - Green ($0.222T$): The phase rotation is $\theta = 27\pi(0.222) \approx 6\pi$. This creates constructive interference ($e^{-j6\pi} = 1$). The channel magnitude is maximized, at $|h| = |0.5 + 0.3| = 0.8$. This high SNR results in excellent BER compared to the blue case.
  - The red case ($0.1261T$) lies in between the two.

  And that is why the BER improves, as the delays become bigger, due to the carrier phase interference, which either plummets or maximizes the channel magnitude.

- **Intuitive relation between the rays of the blue case (worst case):**
  In the blue case ($t_1 = 0.037T$), the delay corresponds to a phase shift of π rad at the carrier frequency. This causes the second ray to arrive with completely opposite phase to the primary ray. As a consequence, the two rays end up cancelling each other out: $|H(0)| = |a_0 - a_1|$.
  This implies that if the second ray had the same magnitude as the first ($a_1 = a_0$), the channel would completely annihilate any signal, since it would be $|h| = 0$

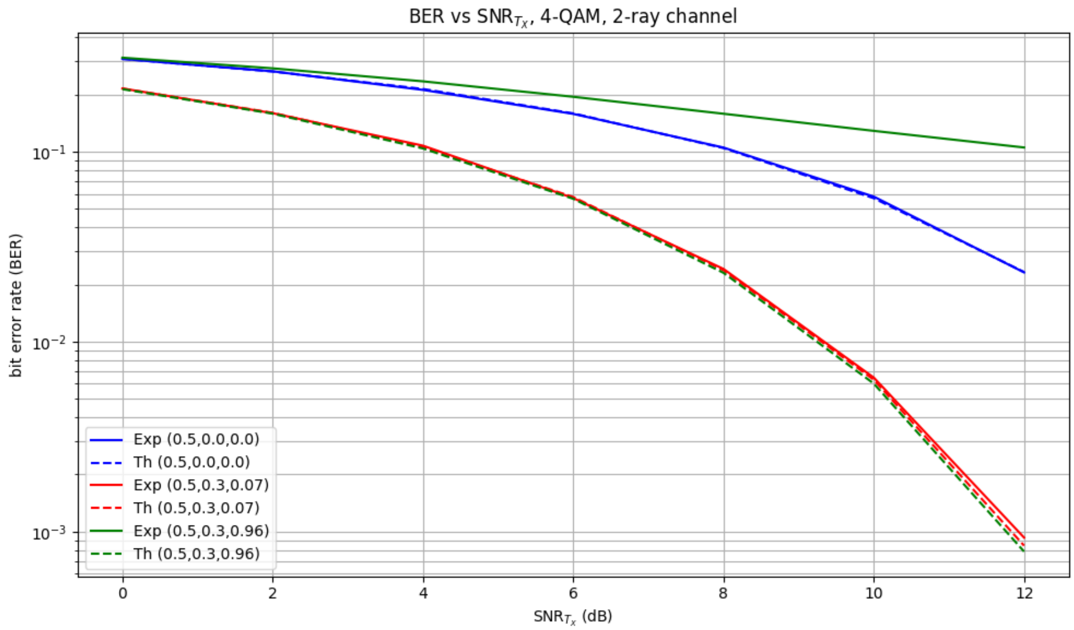
- **Theoretical vs. Experimental Comparison:** Our main concern when comparing again, is the factor we mentioned when comparing theoretical vs. experimenntal for the previous group : the flat channel in theory, and the oscillations in the channel in the experiment.
  As we can see in the spectrums plot :

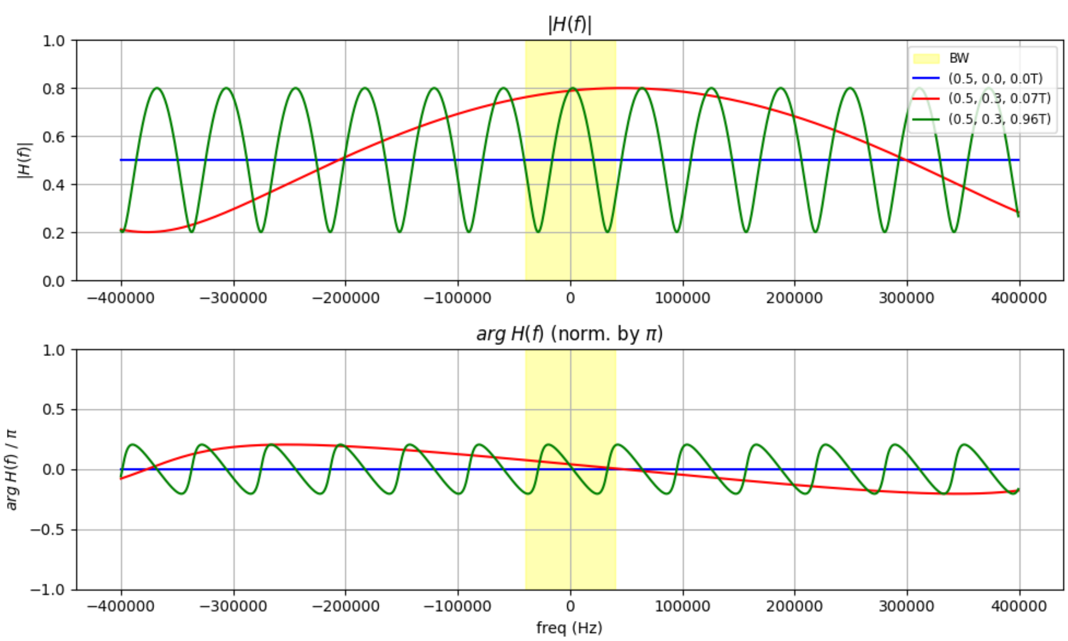
  - For the **blue case ($0.037T$)**, despite the fact that the $|h|$ is very low due to the carrier phase interference, the very small value of the delay results the magnitude of the channel oscillating at the lowest frequency compared to the other two cases. Therefore, the experimental channel is relatively flat as well, which means the experimental BER approximates the theoretical BER very closely.
  - For the **green case ($0.2222T$)**, we observe that the magnitude oscillates at a higher frequency than the blue and red case, which is why the theoretical BER deviates the most from the experimental, compared to the other two cases : because the flat channel assumption in theory doesn't translate entirely in practice.

THIRD GROUP

Now for:

Blue: $(0.5, 0, 0T)$, **Red:** $(0.5, 0.3, 0.07T)$, **Green:** $(0.5, 0.3, 0.96T)$





Comparing theoretical vs. experimental BERs
We've got $f_c T = 13.5$, the phase rotation is $\theta = 27\pi\tau$.

- **Blue case** $(0T)$: One ray, no second ray, therefore no delay.
  $|h| = |H(0)| = |a_0| = 0.5$ constantly. Both the magnitude and the phase are constant, no oscillatios whatsoever.
  This effectively means that the theoretical and experimental BER are in complete agreement, as we can see in the BER plot.

- **Red case** $(0.07T)$
  $\theta = 27\pi(0.07) = 1.89\pi \Rightarrow e^{-j1.89\pi} \approx 1$
  $|h| \approx |H(0)| = |a_0 + a_1| = 0.8$
  In the spectrum plot, we can see that due to the existent (yet very small) delay, the magnitude $|H(f)|$ oscillates, but tends to flatten within the bandwidth.
  That is exactly why the theoretical BER is slightly lower than the experimental, but they're still really close.

- **Green case** $(0.96T)$
  $\theta = 27\pi(0.96) = 26\pi \Rightarrow e^{-j26\pi} = 1$
  $|h| \approx |H(0)| = |a_0 + a_1| = 0.8$
  This is where the experimental and theoretical BER are completely off. This case has a channel approximation $|h| = 0.8$.
  The channel magnitude estimation is the same for both cases, but the huge delay (almost an entire period) in the experimental side creates destructive oscillations within the bandwidth.
  This is what creates the giant contrast between the theoretical and the experimental BER.

Can the presence of a second ray be advantageous?
Yes, it can and it is advantageous. As we see in the plots of the spectrum, in **the blue case**, the lack of a second delayed ray leads to a constant magnitude of 0.5 for the channel.
On the other hand, the **red case** has 2 rays, where the phase rotation of the phasor doesn't cause the two rays to cancel out each other, instead it adds them. As a result, the magnitude of $H(f)$ does oscillate, but is relatively flat in our desired bandwidth, at a total magnitude of ≈ 0.8.
This means that the presence of the second ray has helped, because the interference was constructive and the delay was very small, less than 1/10th of the period T.

Can the presence of a second ray be damaging?
Also yes.
This shows when comparing the green with the red case.
The second ray in the **red case** has a very small delay, which gives us a relatively flat channel and a magnitude of 0.8 (summed ray magnitudes).
The **green case** has a delay of almost one period, which still ends up adding the ray magnitudes (due to phase rotation) and giving us $|h| = 0.8$ at H(0), but the oscillations heavily distort the signal and cause extremely poor BER compared to the red case.