Applied Stochastic Processes
Assignment I

PART I - Theory

A high-tech data processing unit at a research lab is responsible for executing a computational task multiple times each day. However, the number of times X that this task must be processed is random and varies from day to day. The expected number of executions μ is itself uncertain, as it depends on factors like incoming data volume and system workload. This expectation follows a Gamma distribution with parameters α (shape) and β (rate). The random variable X is hence modelled according to the following hierarchical structure:

$$\mu \sim \Gamma(\alpha, \beta) \quad \text{and} \quad X \mid \mu \sim \text{Poisson}(\mu)$$

0.1 ex1

Describe the statistical properties of the random variable μ. Specifically, provide its probability density function, expectation, and variance.
The random variable μ follows a Gamma distribution with shape parameter α and rate parameter β. The probability density function of μ is given by:

$$f_\mu(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \quad \text{for } x > 0$$

The expectation and variance of μ are given by:

$$E[\mu] = \frac{\alpha}{\beta} \quad \text{and} \quad \text{Var}(\mu) = \frac{\alpha}{\beta^2}$$

0.1.1 Derivation of the EV

Since I dont know how differenciate between mu uppercase (RV) and mu lowercase (realization) I ll call the RV Y for this proof, referring to the gamma distribution: $Y \sim \Gamma(\alpha, \beta)$. The expected value is defined as:

$$\mathbb{E}[Y] = \int_0^\infty y f(y) dy.$$

Substituting the Gamma PDF:

$$\mathbb{E}[Y] = \int_0^\infty y \cdot \frac{\beta^\alpha y^{\alpha-1} e^{-\beta y}}{\Gamma(\alpha)} dy.$$

Rearrange:

$$\mathbb{E}[Y] = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty y^\alpha e^{-\beta y} dy.$$

Using the identity:

$$\int_0^\infty y^n e^{-\lambda y} dy = \frac{\Gamma(n+1)}{\lambda^{n+1}}, \quad \text{for } n > -1.$$

Set $n = \alpha$ and $\lambda = \beta$:

$$\int_0^\infty y^\alpha e^{-\beta y} dy = \frac{\Gamma(\alpha+1)}{\beta^{\alpha+1}}.$$

Since $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, we get:

$$\mathbb{E}[Y] = \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \frac{\alpha\Gamma(\alpha)}{\beta^{\alpha+1}}.$$

Cancel $\Gamma(\alpha)$ and simplify:

$$\mathbb{E}[Y] = \frac{\alpha}{\beta}.$$

0.1.2 Derivation of $\mathbf{E}[\mathbf{Y}^2]$

$$\mathbb{E}[Y^2] = \int_0^\infty y^2 f(y) dy.$$

Substituting the PDF:

$$\mathbb{E}[Y^2] = \int_0^\infty y^2 \cdot \frac{\beta^\alpha y^{\alpha-1} e^{-\beta y}}{\Gamma(\alpha)} dy.$$

Rearrange:

$$\mathbb{E}[Y^2] = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty y^{\alpha+1} e^{-\beta y} dy.$$

Using the integral identity:

$$\int_0^\infty y^n e^{-\lambda y} dy = \frac{\Gamma(n+1)}{\lambda^{n+1}},$$

with $n = \alpha + 1$ and $\lambda = \beta$:

$$\int_0^\infty y^{\alpha+1} e^{-\beta y} dy = \frac{\Gamma(\alpha+2)}{\beta^{\alpha+2}}.$$

Since $\Gamma(\alpha+2) = (\alpha+1)\alpha\Gamma(\alpha)$, we get:

$$\mathbb{E}[Y^2] = \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \frac{(\alpha+1)\alpha\Gamma(\alpha)}{\beta^{\alpha+2}}.$$

Cancel $\Gamma(\alpha)$:

$$\mathbb{E}[Y^2] = \frac{(\alpha+1)\alpha}{\beta^2}.$$

0.1.3 Derivation of the Variance

By definition:

$$\mathrm{Var}(Y) = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2.$$

Substituting the computed values:

$$\mathrm{Var}(Y) = \frac{(\alpha+1)\alpha}{\beta^2} - \left(\frac{\alpha}{\beta}\right)^2.$$

Simplify:

$$\mathrm{Var}(Y) = \frac{(\alpha+1)\alpha}{\beta^2} - \frac{\alpha^2}{\beta^2}.$$

Factor out $\frac{1}{\beta^2}$:

$$\mathrm{Var}(Y) = \frac{1}{\beta^2}\left[(\alpha+1)\alpha - \alpha^2\right].$$

Since $(\alpha+1)\alpha - \alpha^2 = \alpha$, we get:

$$\mathrm{Var}(Y) = \frac{\alpha}{\beta^2}.$$

0.2 ex2

Derive the expectation and variance of the times a task has to be performed on a random day, that are $E[X]$ and $\mathrm{Var}(X)$ respectively. Justify rigorously all the steps of your derivation. (Hint: use the law of total expectation and the law of total variance)
The expectation and variance of the random variable X can be derived using the law of total expectation and the law of total variance. The expectation of X is given by:

$$E[X] = E[E[X \mid \mu]]$$

Now, since $X \mid \mu \sim Poisson(\mu)$ and $E[X \mid \mu] = \mu$ and $E[\mu] = \frac{\alpha}{\beta}$

$$\begin{aligned} E[X] &= E[E[X \mid \mu]] && (1)\\ &= E[\mu] && (2)\\ &= \frac{\alpha}{\beta} && (3) \end{aligned}$$

Accorsing to the law of total variance, the variance of X is given by:

$$\begin{aligned} \mathrm{Var}(X) &= E[\mathrm{Var}(X \mid \mu)] + \mathrm{Var}(E[X \mid \mu])\\ &= E[\mu] + \mathrm{Var}(\mu)\\ &= \frac{\alpha}{\beta} + \frac{\alpha}{\beta^2} \end{aligned}$$

0.3 ex3

The duration required to execute once the assigned task, denoted as T, depends on the server assigned for the day. The system can be allocated one of two servers, each with distinct average processing times (τ_1 and τ_2 respectively).
Server A: High-speed processing, average time τ_1, assigned with probability w
Server B: Standard processing, average time τ_2, assigned with probability $1 - w$
Hence the expectation τ of the random time T is random, drawn independently each day, and follows a categorical distribution fully defined by:

$$P(\tau = \tau_1) = w$$
$$P(\tau = \tau_2) = 1 - w$$

The random variable T is modelled conditioning on τ accordingly:

$$T \mid \tau \sim \mathrm{Exp}(1/\tau)$$

Derive the expectation and variance of the time needed to process the task once on a random day, that are $E[T]$ and $\mathrm{Var}(T)$ respectively. Justify rigorously all the steps of your derivation. (Hint: use the law of total expectation and the law of total variance)

Expectation of T

Using the **law of total expectation**:

$$E[T] = E[E[T \mid \tau]].$$

Since $T \mid \tau \sim \mathrm{Exp}(1/\tau)$, we recall that the EV of an exponential distr. with parameter a is $\frac{1}{a}$, therefore:

$$E[T \mid \tau] = \tau.$$

Thus,

$$E[T] = E[\tau].$$

Computing $E[\tau]$ from the problem formulation:

$$E[\tau] = E[T] = \tau_1 w + \tau_2(1 - w).$$

Variance of T

Using the **law of total variance**:

$$\mathrm{Var}(T) = E[\mathrm{Var}(T \mid \tau)] + \mathrm{Var}(E[T \mid \tau]).$$

First term: Since $\mathrm{Var}(T \mid \tau)$ for an exponential distribution is:

$$\mathrm{Var}(T \mid \tau) = \tau^2,$$

taking expectation:

$$E[\mathrm{Var}(T \mid \tau)] = E[\tau^2].$$

From the problem formulation:

$$E[\tau^2] = \tau_1^2 w + \tau_2^2 (1-w).$$

Second term: Since $E[T \mid \tau] = \tau$,

$$\mathrm{Var}(E[T \mid \tau]) = \mathrm{Var}(\tau).$$

We compute:

$$\mathrm{Var}(\tau) = E[\tau^2] - (E[\tau])^2$$

Substituting:

$$\mathrm{Var}(T) = E[\tau^2] + E[\tau^2] - (E[\tau])^2 = 2E[\tau^2] - (E[\tau])^2$$

$$\mathrm{Var}(T) = 2(\tau_1^2 w + \tau_2^2 (1-w)) - (\tau_1 w + \tau_2 (1-w))^2.$$

Assume that X and T are independent.

0.4 ex4

Derive the expectation of the time taken by the unit to process all tasks of a single day, i.e. $E[TX]$. Justify rigorously all the steps of your derivation.

Since T and X are independent, we apply the property:

$$E[TX] = E[T]E[X].$$

We have already computed $E[T]$ and $E[X]$ in previous questions. Substituting their values:

$$E[TX] = (\tau_1 w + \tau_2 (1-w)) \left(\frac{\alpha}{\beta}\right).$$

0.5 ex 5

Use the law of total variance to show in full generality that, given two independent random variables T and X, the following equality holds:

$$\mathrm{Var}(TX) = E^2[X]\,\mathrm{Var}(T) + E^2[T]\,\mathrm{Var}(X) + \mathrm{Var}(X)\,\mathrm{Var}(T).$$

Proof:

By the law of total variance:

$$\mathrm{Var}(TX) = E[\mathrm{Var}(TX \mid X)] + \mathrm{Var}(E[TX \mid X]).$$

Let s consider the second term:

- Since T and X are independent:

$$E[TX \mid X] = E[T \mid X]E[X \mid X]$$

- Still because of independence:

$$E[T \mid X] = E[T]$$

- and trivially:

$$E[X \mid X] = X$$

- Therefore:

$$E[TX \mid X] = XE[T].$$

Taking variance:

$$\begin{aligned} \operatorname{Var}(E[TX \mid X]) &= \operatorname{Var}(XE[T]) && \text{Since } E[T] \text{ is a constant, we can factor it out.} \\ &= E^2[T]\operatorname{Var}(X). \end{aligned}$$

For the first term, we use:

$$\begin{aligned} \operatorname{Var}(TX \mid X) &= X^2 \operatorname{Var}(T). \\ E[\operatorname{Var}(TX \mid X)] &= E[X^2]\operatorname{Var}(T) && \text{Taking expectatinos} \\ &= (\operatorname{Var}(X) + E^2[X])\operatorname{Var}(T). && \text{Using the identity } E[X^2] = \operatorname{Var}(X) + E^2[X] \end{aligned}$$

Substituting both terms into the total variance equation:

$$\begin{aligned} \operatorname{Var}(TX) &= (\operatorname{Var}(X) + E^2[X])\operatorname{Var}(T) + E^2[T]\operatorname{Var}(X). \\ \operatorname{Var}(TX) &= E^2[X]\operatorname{Var}(T) + E^2[T]\operatorname{Var}(X) + \operatorname{Var}(X)\operatorname{Var}(T) && \text{sokving the product} \end{aligned}$$

PART II - Computation

Considering the parameters values $\alpha = 10, \beta = 1, \tau_1 = 10$ min, $\tau_2 = 25$ min and $w = 0.6$. You are interested in studying in more depth the distributions of the random variables X, T and XT.

1

Calculate expectation and variance of the three random variables.
Given:

$$\mu \sim \text{Gamma}(\alpha, \beta)$$

$$X \mid \mu \sim \text{Poisson}(\mu)$$

where $\alpha = 10$, $\beta = 1$.

$$E[X] = \frac{\alpha}{\beta} = \frac{10}{1} = 10.$$

$$\operatorname{Var}(X) = \frac{\alpha}{\beta^2} + \frac{\alpha}{\beta} = 10 + 10 = 20.$$

$$E[T] = w\tau_1 + (1 - w)\tau_2.$$

$$E[T] = 0.6 \times 10 + 0.4 \times 25 = 6 + 10 = 16.$$

Using the given formula for variance:

$$\operatorname{Var}(T) = 2(\tau_1^2 w + \tau_2^2(1 - w)) - (\tau_1 w + \tau_2(1 - w))^2.$$

$$\begin{aligned} \operatorname{Var}(T) &= 2(10^2 \times 0.6 + 25^2 \times 0.4) - (10 \times 0.6 + 25 \times 0.4)^2 \\ &= 2(60 + 250) - (6 + 10)^2 \\ &= 2(310) - 16^2 \\ &= 620 - 256 \end{aligned}$$

$$= 364.$$

$$E[XT] = E[X]E[T] = 10 \times 16 = 160.$$

Using the variance formula:

$$\text{Var}(XT) = E[X]^2\text{Var}(T) + E[T]^2\text{Var}(X) + \text{Var}(X)\text{Var}(T).$$

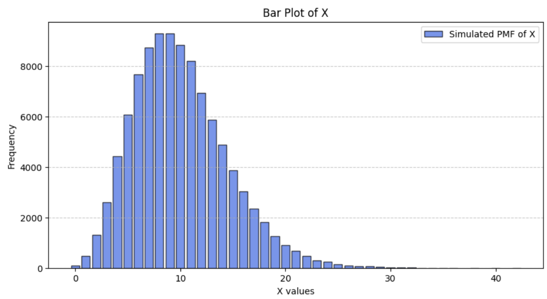
$$= 10^2 \times 364 + 16^2 \times 20 + 20 \times 364.$$

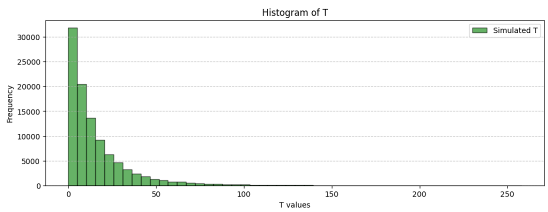
$$= 100 \times 364 + 256 \times 20 + 7280.$$

$$= 36400 + 5120 + 7280 = 48800.$$

2

Produce three plots to visualise the (approximated) distribution of the three random variables.
(Hint: computing explicitly the distributions in terms of their pdf and cdf might be difficult or not feasible. Use instead stochastic simulation.
Produce a large sample drawn from each random variable. Exploit the hierarchical structure of the model. A histogram of such large sample constitute a reasonable approximation of the respective variable distribution i.e. pdf)





3

Estimate the probability that the time taken by the unit to process all tasks of a single day is above 5 hours, i.e. $P(TX > 5$ hours).
(Hint: approximate this probability using stochastic simulation. First simulate a large sample from TX. Then evaluate the proportions of sample above 5 hours)

The Probability that XT is bigger than 5 is 0.14782 (See notebook)

4

Estimate three intervals such that the probability that the time taken by the unit to process all tasks of a single day is in the interval is (approximately) $0.90, 0.95$ and 0.99 respectively.
(Hint: keep using your simulated sample from TX. Find such intervals by looking at the suitable quantiles of the sample)

- 90% confidence interval: [5.11721826, 560.35799788]
- 95% confidence interval: [2.49693427, 766.01247145]
- 99% confidence interval: [4.10587499e-01, 1.30415111e+03]

Assume the cost per hour to run the unit is 100€ for the first hour, and increases by 30€ each hour: the second hour it costs 130€, the third 160€, ...
For example, the cost to run the unit for 3 hours is 390€. Also, the unit cannot be deployed for fraction of an hour, so the cost to run it for e.g. 3 hours is the same as the cost to run it for 2 hours and 30 minutes.

5

Estimate the expectation and variance of the cost to run the unit on a random day.
Expected Cost: 621.2217 Variance of Cost: 2559087.5533

6

Produce a suitable visualisation of the distribution of the cost to run the unit on a random day.