ALCUNE CONSIDERAZIONI:

- É possibile che una piccola parte degli esercizi sia scorretta
- Alcuni esercizi possono essere risolti in modi diversi, ma ugualmente corretti
- Gli esercizi sono svolti in ordine cronologico, quindi più si va in fondo nel file più sarà probabile che con più esperienza alle spalle saranno corretti.

BUON LAVORO!

$$f(z) = \frac{2z}{(z^2+1)(2z^2-5z+2)}$$

$$\frac{2z}{(z^2+1)\cdot(4z-5)}$$

$$\frac{5}{4}\cdot(-3)$$

γ: Circonferenza di raggio $\sqrt{2}$ e centro 0.

f è olomorfa in $\mathbb{C} \setminus \{$zeri del denominatore$\}$

$z_1 = i \quad z_2 = -i$

$2z^2 - 5z + 2 = 0$

$$z_{1,2} = \frac{5 \pm \sqrt{25-16}}{4} = \frac{5 \pm 3}{4} = \begin{cases} 2 \\ \frac{1}{2} \end{cases}$$

E sono tutti poli di ordine 1.

$$\lim_{z \to i} (z-i) f(z) = \lim_{z \to i} \frac{2z}{\cancel{(z-i)}(z+i)(z-2)(z-\frac{1}{2})} \cdot \cancel{(z-i)} =$$

$$= \frac{2i}{2i(i-2)(i-\frac{1}{2})} = \frac{1}{i^2 - 2i - \frac{1}{2}i + 1} = \frac{1}{-\frac{1}{2} - \frac{5}{2}i} = -\frac{1}{13} + \frac{5}{13}i$$

$$\lim_{z \to -i} (z+i) f(z) = \lim_{z \to -i} \frac{2z}{(z-i)(z-2)(z-\frac{1}{2})} = \frac{-2i}{-2i(z-2)(z-\frac{1}{2})} = -\frac{1}{13} + \frac{5}{13}i$$

$$\lim_{z \to \frac{1}{2}} (z-\tfrac{1}{2}) f(z) = \lim_{z \to \frac{1}{2}} \frac{2z}{(z-i)(z+i)(z-2)} = \frac{1}{\frac{5}{4} \quad (-\frac{3}{2})} = -\frac{16}{15}$$

$$\lim_{z \to \frac{1}{2}} \frac{\cancel{(z-\frac{1}{2})} 2z}{(z^2+1)\cancel{(z-\frac{1}{2})}(z-2)} = \left(\frac{5}{4}\right)^{-1} \left(-\frac{3}{2}\right)^{-1} = \left(-\frac{15}{8}\right)^{-1} = -\frac{8}{15}$$

2)

$$f(z) = \frac{e^{i\pi z}}{z(z-2)^2}$$

$$C = \{ z \in \mathbb{C} \,/\, x^2+y^2-2x-2y-2=0 \}$$

$P(1,1)$ $\quad r=?$

Se $x=0 \Rightarrow$ $y^2-2y-2=0$

$y_{1,2} = 1 \pm \sqrt{1+2} = 1+\sqrt{3}$

$(0, 1+\sqrt{3})$ appartiene a γ.

$r = \sqrt{1+3} = 2$

$\Rightarrow$ $f(z)$ ha singolarità in 0 e 2, appartenenti a γ.

$$\text{Res}(f(z),0) = \left[\frac{e^{i\pi z}}{(z-2)^2} \right]_{z=0} = \frac{1}{4}$$

$$\text{Res}(f(z),2) = \frac{d}{dz}\left[\cancel{(z-2)^2} \cdot \frac{e^{i\pi z}}{z\cancel{(z-2)^2}} \right]_{z=2} = \left[\frac{i\pi e^{i\pi z} z - e^{i\pi z}}{z^2} \right]_{z=2} =$$

$$= \left[\frac{2i\pi - 1}{4} \right]$$

Integrale: $2\pi i \left[\frac{1}{4} + \frac{2\pi i - 1}{4} \right] = -4\pi^2$

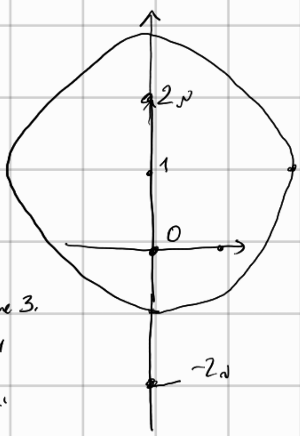
3)

$$f(z) = \frac{z^2+2z-3}{(z^4-z^3)(z^2+4)}$$

$$C = \{ z \in \mathbb{C} \mid x^2+y^2-2y-3=0 \} \qquad R = \sqrt{\tfrac{a^2}{4}+\tfrac{b^2}{4}-c}$$

$P(0,1)$

$r = \sqrt{1+3} = 2$



$$f(z) = \frac{z^2+2z-3}{z^3(z-1)(z^2+4)}$$

$z_0 = 0$ polo di ordine 3.

$z_1 = 1$ polo di ordine 1

$z_2 = -2i$ " "

$z_3 = +2i$ " "

$z_0 = 0$, $z_1 = 1$, $z_3 = 2i$ interni

$$\mathrm{Res}(f(z), 1) = \left[\frac{z^2+2z-3}{z^3(z^2+4)}\right]_{z=1} = 0$$

$$\mathrm{Res}(f(z), 2i) = \left[\frac{z^2+2z-3}{z^3(z-1)\cdot 2z}\right]_{z=2i} = \frac{-5+4i}{-2i\cdot 2i\,(2i-1)} =$$

$$= \frac{-5+4i}{8i-4} = \frac{13}{20} + \frac{3}{10}i$$

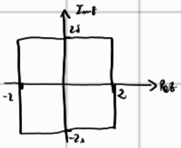
$$\mathrm{Res}(f(z), 0) =$$

$$= \frac{d^2}{dz^2}\left(\frac{z^2+2z-3}{\underbrace{(z-1)(z^2+4)}_{z^3-z^2+4z-4}}\right)_{z=0} = \frac{d^2}{dz^2}\left[\frac{(2z+2)(z^3-z^2+4z-4)-(z^2+2z-3)(3z^2-2z+4)}{\quad}\right.$$

3)

$$\int_{S^+D} \frac{\cos(z)}{z(z^2+8)}\,dz$$

$D = \{ z \in \mathbb{C} \mid |\mathrm{Re}(z)| \le 2,\ |\mathrm{Im}(z)| \le 2 \}$



$f(z)$ ha singolarità in $0, 2\sqrt{2}i, -2\sqrt{2}i$

$\Downarrow$ Non rientrano nell'insieme di integraz.

$$\mathrm{Res}(f(z), 0) = \lim_{z\to 0} \frac{\cos(z)}{z^2+8} = \frac{1}{8}$$

$$\Rightarrow \int_{S^+D} \frac{\cos(z)}{z(z^2+8)}\,dz = \frac{\pi i}{4}$$

4)

$$\int_{+SD} \frac{e^{-z}}{z-\frac{\pi i}{2}}\,dz \qquad D = \{ z \in \mathbb{C} \mid |\mathrm{Im}(z)| \le 3;\ |\mathrm{Re}(z)| \le 3 \}$$

Singolare in $\frac{\pi i}{2}$

$$\mathrm{Res}\left(f(z), \frac{\pi i}{2}\right) = e^{-\frac{\pi i}{2}} = -i$$

$$\Rightarrow \int_{S^+D} \frac{e^{-z}}{z-\frac{i\pi}{2}}\,dz = 2\pi$$

5)

$$\int_{S^+D} \frac{\cosh(z)}{z^4}\,dz$$

$D = \{ z \in \mathbb{C} \,/\, |z| \leq 2 \}$

$f(z)$ ha singolarità in $z=0$. Polo di ordine 4.

$$\mathrm{Res}(f(z),0) = \frac{1}{3!}\frac{d^3}{dz^3}\left(\cosh(z)\right)\Big|_{z=0} =$$

$$\frac{1}{6}\sinh(z)\Big|_{z=0} = 0$$

$$\Rightarrow \int_{S^+D} \frac{\cosh(z)}{z^4}\,dz = 0$$

1.8)

$$\int_0^{+\infty} \frac{\cos(\pi x)}{(x^2+1)^2}\,dx = I$$

$$f(z) = \frac{\cos(\pi z)}{(z^2+1)^2}$$

$I = II/2$ con $II = \int_{-\infty}^{+\infty} f(x)\,dx$

Singolarità: $\pm i$, poli di ordine 2.

$+i$ è unico polo nel semipiano positivo.

$$\mathrm{Res}(f(z), i) = \frac{d}{dz}\left(\frac{\cos(\pi z)}{(z^2+1)^2}\cdot(z-i)^2\right)\Big|_{z=i} =$$

$$= \frac{d}{dz}\left(\frac{\cos(\pi z)}{(z+i)^2}\right)\Big|_{z=i} = \left(\frac{-\pi\sin(\pi z)(z+i)^2 - \cos(\pi z)\cdot 2(z+i)}{(z+i)^2}\right)\Big|_{z=i} =$$

$$= \left(\frac{-\cos(i\pi)\, 2\cdot 2i}{-4}\right) =$$

$$= \frac{-\frac{1}{2}\left(e^{i\pi}+e^{-i\pi}\right)\cdot 4i}{-4} =$$

$$= \frac{4i}{-4} = -i$$

$$II = 2\pi i(-i) = 2\pi \Rightarrow I = \pi$$

3)

$$f(z)=\frac{\sin \pi z}{z^2\left(z-\frac{3}{2}\right)^2}$$

$z=0$ e $z=\frac{3}{2}$ sono singolarità.

$$\lim_{z\to 0} z\cdot\frac{\sin \pi z}{z^2\left(z-\frac{3}{2}\right)^2}=\lim_{z\to 0}\frac{\pi}{\left(z-\frac{3}{2}\right)^2}=\frac{\pi}{\frac{9}{4}}$$

Polo singolo.

$$\mathrm{Res}(f(z),0)=\lim_{z\to 0}\frac{\sin\pi z}{z^2\left(z-\frac{3}{2}\right)^2}=\frac{4\pi}{9}$$

$$\mathrm{Res}\left(f(z),\tfrac{3}{2}\right)=\frac{d}{dz}\left(\frac{\sin\pi z}{z^2}\right)\Bigg|_{z=\frac{3}{2}}=\left(\frac{\pi\cos\pi z\cdot z^2-2z\cdot\sin\pi z}{z^4}\right)=$$

$$=\left(\frac{2\cdot\frac{3}{2}}{\left(\frac{3}{2}\right)^4}\right)=2\cdot\left(\frac{2}{3}\right)^3=\frac{2\cdot 8}{27}=\frac{16}{27}$$

2)

$$f(z)=\frac{z-4}{(z+i)(z-i)^3}$$

$z=i$ polo 3 $\quad z=-i$ polo 1

$$\mathrm{Res}(f(z),-i)=\left[\frac{z-4}{(z-i)^3}\right]_{z=-i}=\left[\frac{-i-4}{8i}\right]=-\frac{1}{8}+\frac{1}{2}i$$

$$\mathrm{Res}(f(z),i)=\frac{d^2}{dz^2}\left[\frac{z-4}{z+i}\right]_{z=i}=\frac{d}{dz}\left[\frac{(z+i)-(z-4)}{(z+i)^2}\right]_{z=i}=$$

$$\frac{d}{dz}\left[\frac{i+4}{(z+i)^2}\right]_{z=i}=\left[\frac{-2(i+4)}{(z+i)^3}\right]_{z=i}=\left[\frac{-2i-8}{-8i}\right]=\frac{1}{4}-i$$

1.7.3)

$$f(z)=\frac{1}{z(e^z-1)}$$

$$\lim_{z\to 0} z^2\cdot\frac{1}{z(e^z-1)}=\lim_{z\to 0}\frac{\cancel{z^2}}{\cancel{z}(z)}=1$$

Polo di ordine 2.

$$\mathrm{Res}(f(z),0)=\frac{d}{dz}\left(z^2\frac{}{z(e^z-1)}\right)_{z=0}=\left[\frac{(e^z-1)-ze^z}{(e^z-1)^2}\right]_{z=0}=\left[\frac{e^z-1-ze^z}{(e^z-1)^2}\right]_{z=0}$$

$$=\frac{d}{dz}\left(\frac{z}{(e^z-1)}\right)_{z=0}=\left(\frac{e^z-1-ze^z}{(e^z-1)^2}\right)$$

3)

$$f(z)=\frac{(\log(z))^4}{1+z^2}$$

$\log i=-1$

$z=\pm i$ poli di ordine 1.

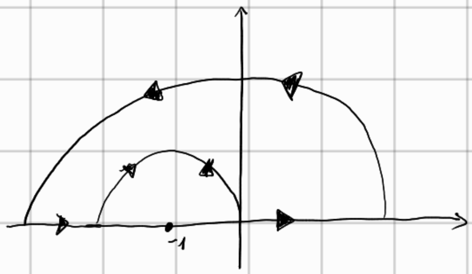
$$\mathrm{Res}(f(z),i)=\left[\frac{(\log(z))^4}{2z}\right]_{z=i}=\frac{1}{2i}=-\frac{1}{2}i$$

Se $z=-i$, $\mathrm{Res}(f(z),-i)=\frac{1}{2}i$

4)

$$\int_{-\infty}^{+\infty} \frac{e^{i\pi x}}{1+x^3}\,dx$$

In $x=-1$ ho problemi:



$(x+1)(1+x+x^2)$

$\Rightarrow x^2+x+1=0 \Rightarrow x_{1,2}=\frac{-1\pm\sqrt{1-4}}{2} = -\frac{1}{2}\pm\frac{\sqrt{3}}{2}i \longrightarrow$ Nella parte $\mathrm{Im}>0$ c'è solo $-\frac{1}{2}+\frac{\sqrt{3}}{2}i$

$$\Gamma_{r,R} = [-R,-r] \cup \left(-\gamma_r^+(-1)\right) \cup [r,R] \cup \left(\gamma_R^+(0)\right)$$

Polo del 1° ordine

$$\int_{\Gamma_{r,R}} \frac{e^{i\pi z}}{1+z^3}\,dz = 2\pi i\,\mathrm{Res}\left(f(z), -\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) =$$

$$= 2\pi i \cdot \left[\frac{e^{i\pi z}}{3z^2}\right]_{z=-\frac{1}{2}+\frac{\sqrt{3}}{2}i} = \alpha + i\beta$$

Va a 0 per il lemma del grande cerchio

$$\int_{\Gamma_{r,R}} \frac{e^{i\pi z}}{1+z^3}\,dz = \int_{-R}^{-r} f(z)\,dz + \int_r^R f(z)\,dz - \int_{\gamma_r^+(-1)} f(z)\,dz + \int_{\gamma_R^+(0)} f(z)\,dz$$

Tende a $\pi i\,\mathrm{Res}(f(z),-1)$

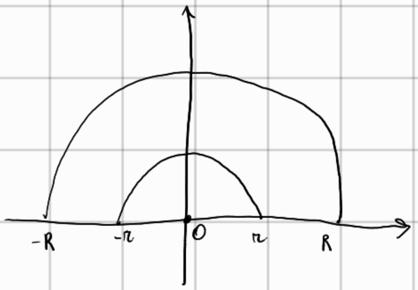
Con $r\to 0^+$ e $R\to+\infty$

$$\int_{\mathbb{R}} f(z)\,dz = 2\pi i\,\mathrm{Res}\left(f(z), -\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) + \pi i\,\mathrm{Res}(f(z),-1) =$$

$$= \frac{1}{6}e^{-\frac{\pi\sqrt{3}}{2}}\left(\sqrt{3}-i\right)\cdot 2\pi i - \frac{\pi i}{3}$$

5)

$$\int_{\mathbb{R}} \frac{e^{ix}}{x}\, dx$$



$$\Gamma_{r,R} = [-R,-r] \cup \left(-\gamma_r^+(0)\right) \cup [r,R] \cup \left(\gamma_R^+(0)\right)$$

$$\int_{\Gamma_{r,R}} \frac{e^{iz}}{z}\, dz = 0 \overset{\substack{R\to+\infty \\ r\to 0^+}}{=} \int_{\mathbb{R}} \frac{e^{ix}}{x}\, dx - \pi i \overset{1}{\mathrm{Res}(f(z),0)}$$

4)

$$\int_{-\infty}^{+\infty} \frac{1}{1+x+x^2}\, dx =$$

$$f(z) = \frac{1}{z^2+z+1}$$

Singolarità?

$$z^2+z+1=0$$

$$z_{1,2} = \frac{-1 \pm \sqrt{1-4}}{2} = \begin{cases} -\frac{1}{2} - \frac{\sqrt{3}}{2} i \\ -\frac{1}{2} + \frac{\sqrt{3}}{2} i \end{cases}$$

$$\Gamma_R = [-R,R] \cup \gamma_R^+(0)$$

$$\int_{\Gamma_R} f(z)\, dz = \int_{-R}^{R} f(x)\, dx + \int_{\gamma_R^+(0)} f(z)\, dz$$

Con $R \to +\infty$

$$\int_{\Gamma_R} f(z)\, dz = \int_{\mathbb{R}} f(x)\, dx$$

$$= 2\pi i\, \mathrm{Res}\left(f(z), -\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)$$

$$\Rightarrow \mathrm{Res}\left(f(z), -\frac{1}{2} + \frac{\sqrt{3}}{2} i\right) = \left.\frac{1}{2z+1}\right|_{-\frac{1}{2}+\frac{\sqrt{3}}{2} i} = \frac{1}{-1+\sqrt{3} i+1} = -\frac{1}{\sqrt{3}} i$$

$$\Rightarrow \int_{\mathbb{R}} \frac{1}{x^2+x+1}\, dx = \frac{2\pi}{\sqrt{3}} \quad \square$$

3)

$$\int_{\mathbb{R}} \frac{1}{x^4+1}\,dx =$$

$$f(z) = \frac{1}{z^4+1}$$

$$z_1 = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \qquad z_2 = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$$

$$z_3 = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} \qquad z_4 = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$$

$$\int_{\mathbb{R}} \frac{1}{x^4+1}\,dx = 2\pi i \sum_{\operatorname{Im} z_k > 0} \operatorname{Res}(f(z), z_k)$$

$$\Rightarrow \operatorname{Res}\left(f(z), \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \lim_{z \to \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}} \frac{1}{\left(z+\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)\cdot\left(z+\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)\left(z-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)} =$$

$$= \frac{1}{\sqrt{2}\left(\sqrt{2}+i\sqrt{2}\right)\cdot\sqrt{2}\,i} = \frac{1}{2i\left(\sqrt{2}+i\sqrt{2}\right)} = \frac{1}{2\sqrt{2}\,i - 2\sqrt{2}} = \frac{2\sqrt{2} + 2\sqrt{2}\,i}{-8-8} = -\frac{\sqrt{2}}{8} - \frac{\sqrt{2}}{8}i$$

$$\operatorname{Res}\left(f(z), -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = \lim_{z \to \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}} \frac{1}{\left(z-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)\cdot\left(z+\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)\left(z-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)} =$$

$$= \frac{1}{-\sqrt{2}\cdot i\sqrt{2}\left(-\sqrt{2}+i\sqrt{2}\right)} = \frac{1}{-2i\left(-\sqrt{2}+i\sqrt{2}\right)} = \frac{1}{2\sqrt{2}\,i + 2\sqrt{2}} = \frac{2\sqrt{2} - 2\sqrt{2}\,i}{8+8} = \frac{\sqrt{2}}{8} - \frac{\sqrt{2}}{8}i$$

$$\Rightarrow \int_{\mathbb{R}} \frac{1}{x^4+1}\,dx = 2\pi i\left(\cancel{-\frac{\sqrt{2}}{8}} - \frac{\sqrt{2}}{8}i + \cancel{\frac{\sqrt{2}}{8}} - \frac{\sqrt{2}}{8}i\right) = 2\pi i\left(-\frac{\sqrt{2}}{4}i\right) = \pi\cdot\frac{\sqrt{2}}{2} = \frac{\pi}{\sqrt{2}}$$

1)

$$u(x,y) = x^2 + 2xy - y^2$$

Verifico se u é armonica:

$$\Delta u = 0\,? \qquad \frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0 \Rightarrow 1 - 1 = 0 \checkmark$$

$u \in C^\infty(\mathbb{R}^2)$

devo trovare la sua armonica coniugata.

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned} \Rightarrow \quad \nabla v = (-u_y, u_x)$$

Verifico se campo é irrotazionale: lo é per armonicità di u.

Trovo potenziale:

$\nabla V = (2y-2x,\ 2x+2y)$

$V_x = 2y-2x$

$\Rightarrow V = \int (2y-2x)\,dx = 2xy - x^2 + c(y)$

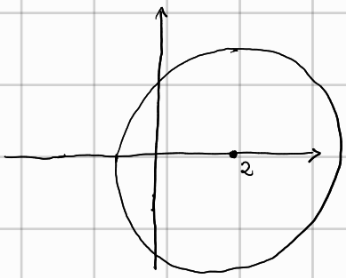
$V_y = 2x + c'(y) \Rightarrow c'(y) = 2y \Rightarrow c(y) = y^2 + c$

$V = 2xy - x^2 + y^2$

$f(x+iy) = x^2+2xy-y^2 + i(2xy - x^2 + y^2)$

2)

$$\int_\gamma \frac{5z^2-63z}{(z-1)^3}\,dz \qquad \gamma = \{ z \in \mathbb{C} : |z-2| = 3 \}$$



Funzione ha singolarità in $z=1$

Polo di ordine 3.

$$\mathrm{Res}(f(z), 1) = \frac{1}{2} \cdot \frac{d^2}{dz^2}\left(5z^2-63z\right)_{z=1} = \frac{5}{2}$$

$$\int_\gamma \frac{5z^2-63z}{(z-1)^3}\,dz = 2\pi i \cdot \frac{5}{2} = 5\pi i$$

3)

$$\int_\gamma \tan(z)\,dz$$

1) $\gamma_1(0)$:

$$\tan(z) = \frac{\sin z}{\cos z} = \frac{\frac{e^{iz}-e^{-iz}}{i}}{e^{iz}+e^{-iz}} = \frac{e^{iz}-e^{-iz}}{i\,(e^{iz}+e^{-iz})}$$

$e^{2x}+e^{-2x}$

$$|\cos z| = \sqrt{\cos^2 x + \sinh^2 y}$$

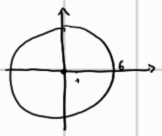
$\Rightarrow \quad x = \frac{\pi}{2}+k\pi \qquad y=0$ Zeri della funzione.

$$\mathrm{Res}\left(f(z), \frac{\pi}{2}\right) = ?$$

$$\lim_{z\to\frac{\pi}{2}} \frac{\sin z}{\cos z}\cdot\left(z-\frac{\pi}{2}\right) = \frac{\sin z}{\quad}$$

4)

$$\int_{|z|=6} \frac{dz}{1-\cos z}$$



$\cos z = 1 \Rightarrow$

$\cos x \cosh y - i(\sin x \sinh y) = 1$

$$\begin{cases} \sin x \sinh y = 0 \\ \cos x \cosh y = 1 \end{cases}$$

- $y=0 \quad \cos x = 1$
- $x=0 \quad \cosh y = 1$

$\rightarrow z_k = 2k\pi$

Th dei residui: solo 0 è contenuto.

$$\mathrm{Res}\left(\frac{1}{1-\cos z}, 0\right)$$

$$\cos z = \sum_{k=0}^{+\infty} (-1)^k \frac{z^{2k}}{(2k)!}$$

$$1-\cos z = 1-\sum_{k=0}^{+\infty} (-1)^k \frac{z^{2k}}{(2k)!}$$

$$\frac{1}{1-\cos z} = \frac{1}{1-\sum_{k=0}^{\infty}(-1)^k \frac{z^{2k}}{(2k)!}} = \frac{1}{-\sum_{k=1}^{\infty}(-1)^k \frac{z^{2k}}{2k!}} = -\left(\sum_{k=1}^{\infty}(-1)^k \frac{z^{2k}}{2k!}\right)^{-1} = ?$$

$\int_\gamma \tan z \, dz =$

$\frac{\sin z}{\cos z}$?

$\cos z = 0 \rightarrow \begin{cases} \cos x \operatorname{Ch} y = 0 \\ \sin x \operatorname{Sh} y = 0 \end{cases}$ $\quad x = \frac{\pi}{2} + k\pi$ $\quad y = 0$

$z_k = \frac{\pi}{2} + k\pi$

$\operatorname{Res}\left(\frac{\sin z}{\cos z}, \frac{\pi}{2}\right)$?

considero $\frac{\sin(z-\frac{\pi}{2})}{\cos(z-\frac{\pi}{2})}$

Σ

1)

$$\int_\gamma (6z^5+7z^6)dz$$

$z_0 = 1+i \qquad z_1 = 2-i$

Nota: $6z^5+7z^6$ é olomorfa su $\mathbb{C}$ e

$F(z) = z^6+z^7$ é una sua primitiva.

$$\rightarrow \int_\gamma (6z^5+7z^6)dz = (2-i)^6+(2-i)^7-(1+i)^6-(1+i)^7$$

$$\Rightarrow (2-i)^6(2-i+1)-(1+i)^6(1+i+1)$$

2)

$u(x,y) = \cos(e^{\alpha y}+e^{-y})$

1) Se f olomorfa, u armonica.

$$\Rightarrow \frac{\delta^2 u}{\delta x^2}+\frac{\delta^2 u}{\delta y^2}=0$$

$$\rightarrow 0+\frac{\delta}{\delta y}\left(-\sin(e^{\alpha y}+e^{-y})\cdot(\alpha e^{\alpha y}-e^{-y})\right)=0$$

$$\cos(e^{\alpha y}+e^{-y})(e^{-y}-\alpha e^{\alpha y})-\sin(e^{\alpha y}+e^{-y})\cdot(\alpha^2 e^{\alpha y}+e^{-y})=0$$

$$e^{-y}\cos(e^{\alpha y}+e^{-y})-\alpha e^{\alpha y}\cos(e^{\alpha y}+e^{-y})-\alpha^2 e^{\alpha y}\sin(e^{\alpha y}+e^{-y})+e^{-y}\sin(e^{\alpha y}+e^{-y})=0$$

$$e^{-y}\left(\cos(e^{\alpha y}+e^{-y})+\sin(e^{\alpha y}+e^{-y})\right)$$

$$-\alpha e^{\alpha y}\left(\cos(e^{\alpha y}+e^{-y})+\alpha\sin(e^{\alpha y}+e^{-y})\right)=0$$

$\alpha=1$ é l'unico valore che garantisce uguaglianza.

$u(x,y) = \cos(e^{y}+e^{-y})$

$u_x = 0 \qquad u_y = -\sin(e^{y}+e^{-y})\cdot(e^{y}-e^{-y})$

$$\begin{cases} u_x = V_y \\ u_y = -V_x \end{cases} \Rightarrow \quad V_y = 0 \Rightarrow V = c(x)$$

$\Rightarrow c'(x) = \sin(e^{y}+e^{-y})(e^{y}-e^{-y})$

$\Rightarrow c(x) = x\cdot\sin(e^{y}+e^{-y})(e^{y}-e^{-y})+c$

$$f(z) = \cos(e^{y}+e^{-y})+i\left(x\sin(e^{y}+e^{-y})(e^{y}-e^{-y})+c\right)$$

3)

$$f(z) = \frac{1-e^{i\pi z}}{z(z+i)(z-1)^2}$$

$e^{i\pi z} = 1$?

$\uparrow$

$$e^{i\pi z} = e^{i\pi(x+iy)} = e^{i\pi x - \pi y} =$$

$$= e^{-\pi y} e^{i\pi x} =$$

$$= e^{-\pi y}\cos(\pi x) + i e^{-\pi y}\sin(\pi x) = 1$$

$x = 2k \quad k \in \mathbb{Z}$

$e^{-\pi y} = 1 \Rightarrow y = 0$

$z(z+i)(z-1)^2 \Rightarrow$

$z=0$ Eliminabile?

$z=-i$ Polo 1

$z=1$ Polo 2

$$\lim_{z\to 0} \frac{1-e^{i\pi z}}{z(z+i)(z-1)^2} = \lim_{z\to 0} \frac{-i\pi \cancel{z}}{\cancel{z}(z+i)(z-1)^2} = \frac{-i\pi}{i} = -\pi$$

$z=0$ eliminabile

$z=-i$? Polo 1:

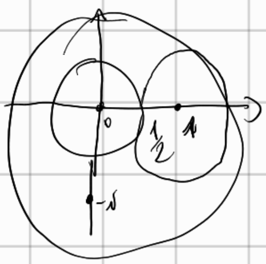
$$\lim_{z\to -i} f(z) = \frac{1-e^{\pi}}{-i\cdot(-i-1)^2} \neq 0$$

$$Res(f(z),-i) = \left(\frac{1-e^{i\pi z}}{z(z-1)^2}\right)_{z=-i} = \frac{1-e^{\pi}}{-i(1+i)^2}$$

$$Res(f(z),1) = \frac{d}{dz}\left(\frac{1-e^{i\pi z}}{z(z+i)}\right)_{z=1} =$$

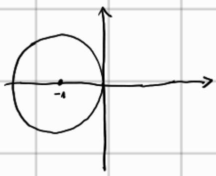
$$= \left.\frac{-i\pi e^{i\pi z}(z(z+i)) - (1-e^{i\pi z})[(z+i)+z]}{z^2(z+i)^2}\right|_{z=1} =$$

$$= \frac{i\pi(1+i) - 2(2+i)}{2i} = \frac{i\pi - \pi - 4 - 2i}{2i}$$



1)

$$\int_{|z+1|=1} \frac{1}{(z^3+1)(z^3-8)}\,dz$$



Singolarità:

- $z^3+1=0$

$z^3=-1 \Rightarrow z^3 = e^{i\pi}$

$z_k = e^{i\left(\frac{\pi+2k\pi}{3}\right)}$

$z_0 = e^{i\frac{\pi}{3}} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$

$z_1 = e^{i\pi} = -1$ $\longrightarrow$ unica singolarità nel cerchio.

$z_2 = e^{i\frac{5}{3}\pi} = \frac{1}{2} - i\frac{\sqrt{3}}{2}$

$z^3=8$

$\Rightarrow z^3 = 8e^{i\cdot 0}$

$z_k = 2e^{i\frac{2k\pi}{3}}$

$z_0 = 2 \quad z_1 = -1+\sqrt{3}i$

$z_2 = -1-\sqrt{3}i$

$|z_1+1|<1? \Rightarrow |\sqrt{3}i|<1?$ No.

$|-\sqrt{3}i|<1?$ No.

$$\int_{|z+1|=1} \frac{1}{(z^3+1)(z^3-8)}\,dz = 2\pi i \operatorname{Res}(f(z),-1)$$

$$\operatorname{Res}(f(z),-1) = \left[\frac{1}{(z^3-8)\cdot 3z^2}\right]_{z=-1} = \frac{1}{-9\cdot 3} = -\frac{1}{27}$$

$$\int_{|z+1|=1} \frac{1}{(z^3+1)(z^3-8)}\,dz = -\frac{2\pi i}{27}$$

2)

$$\int_{|z|=1} \frac{e^z-e^{-z}}{z^4}\,dz$$

$z=0$ polo di ordine 4? No.

$$f(z) = \frac{e^{-z}(e^{2z}-1)}{z^4}$$

$$\lim_{z\to 0} z^3\cdot\frac{e^{-z}(e^{2z}-1)}{z^4} = \lim_{z\to 0}\frac{e^{-z}(e^{2z}-1)}{z} = \lim_{z\to 0}\frac{e^{-z}\cdot 2\cancel{z}}{\cancel{z}} = 2$$

Polo di ordine 3.

$$\operatorname{Res}(f(z),0) = \frac{1}{2}\frac{d^2}{dz^2}\left[\frac{e^z-e^{-z}}{z}\right]_{z=0} =$$

$$\frac{1}{2}\left[\frac{(e^z+e^{-z})z - e^z+e^{-z}}{z^2}\right]' =$$

$$=\frac{1}{2}\left[\frac{(e^z-e^{-z})z^3+(e^z+e^{-z})z^2-e^z z^2-e^{-z}z^2-2z\left[(e^z+e^{-z})z-e^z+e^{-z}\right]}{z^4}\right]_{z=0} =$$

$$e^z z^3 - e^{-z}z^3 - 2z\left[ze^z+e^{-z}z-e^z+e^{-z}\right] =$$

$$e^z\cdot z^3 - e^{-z}z^3 - 2z^2e^z - 2z^2e^{-z} + 2ze^z - 2ze^{-z}$$

$$= \frac{1}{2} \frac{(e^{z}-e^{-z})z^2 - 2z(e^{z}+e^{-z}) + 2(e^{z}-e^{-z})}{z^3}$$

$$\lim_{z\to 0} \frac{1}{2} \frac{(e^{z}-e^{-z})z^2 - 2z(e^{z}+e^{-z}) + 2(e^{z}-e^{-z})}{z^3} =$$

$$= \lim_{z\to 0} \frac{1}{2} \frac{e^{-z}(e^{2z}-1)z^2 - 2z e^{-z}(e^{2z}+1) + 2e^{-z}(e^{2z}-1)}{z^3} =$$

$$= \lim_{z\to 0} \frac{1}{2} \frac{e^{-z} \cdot 2z^3 - 2z e^{-z}(e^{2z}+1) + 2e^{-z} 2z}{z^3} =$$

$$= \lim_{z\to 0} \frac{1}{2} \left(\frac{2z^2 e^{-z} - 2(e^{z}+e^{-z}) + 4e^{-z}}{z^2} \right) =$$

Hopital:

$$= \lim_{z\to 0} \frac{1}{2} \left[\frac{4z e^{-z} - 2z^2 e^{-z} - 2(e^{z}-e^{-z}) - 4e^{-z}}{2z} \right] =$$

Hopital:

$$= \lim_{z\to 0} \frac{1}{4} \left[4e^{-z} - 4ze^{-z} - 4ze^{-z} + 2z^2 e^{-z} - 2(e^{z}+e^{-z}) + 4e^{-z} \right] =$$

$$= \frac{1}{4} \left[4 - 4 + 4 \right] = 1$$

Comunque calcolando si trova.

3) $$\int_{|z|=1} \frac{1}{\sin^3 z} \, dz$$

Il seno si annulla in $z=0$ nella circonferenza.

Polo di ordine 3:

$$\lim_{z\to 0} z^3 \cdot \frac{1}{\sin^3 z} = 1$$

$$\mathrm{Res}\left(\frac{1}{\sin^3 z}, 0\right) = \frac{1}{2} \frac{d^2}{dz^2} \left(\frac{z^3}{\sin^3 z} \right)_{z\to 0} =$$

$$\frac{1}{2} \left[\frac{3z^2(\sin^3 z) - z^3 3\sin^2 z \cos z}{\sin^6 z} \right]' =$$

$$\frac{1}{2} \left[\frac{\sin^6 z \left(6z(\sin^3 z) + \cancel{3z^2 3\sin^2 z \cos z} - \cancel{9z^2 \sin^2 z \cos z} - 3z^3(2\sin z \cos^2 z - \sin^3 z)\right) - 6\sin^5 z \left(3z^2(\sin^3 z) - z^3 3\sin^2 z \cos z\right)}{\sin^{12} z} \right]_{z\to 0} =$$

$$= \frac{1}{2} \left[\frac{z^6 \left(6z^4 - 3z^3(2z\cos^2 z - z^3)\right) - 6z^5 \left(3z^5 - 3z^5 \cos z\right)}{z^{12}} \right]_{z\to 0} =$$

$$= \frac{1}{2} \left[\frac{6z^{10} - 3z^9(2z\cos^2 z - z^3) - 18z^{10} + 18z^{10}\cos z}{z^{12}} \right]_{z\to 0} =$$

$$= \frac{1}{2} \left[\frac{6 - 6\cos^2 z + 3z^2 - 18 + 18\cos z}{z^2} \right]_{z\to 0} =$$

Hopital:

$$\frac{1}{2} \left[\frac{12\cos z \sin z + 6z - 18\sin z}{2z} \right]_{z\to 0} = \frac{1}{4} \left[12\cos z + 6 - 18 \right]$$

4.4)

$$f(z)=\frac{z}{4z^2+1} \longrightarrow$$ FRATTI SEMPLICI:

$4z^2+1=0$

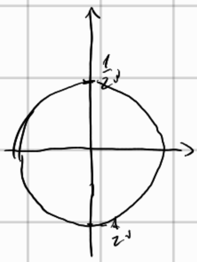
$z^2=-\frac{1}{4} \Rightarrow z=\pm\frac{1}{2}i$

$$f(z)=\frac{z}{(z+\frac{1}{2}i)(z-\frac{1}{2}i)}=\frac{A}{z+\frac{1}{2}i}+\frac{B}{z-\frac{1}{2}i} \Rightarrow \left(z-\frac{1}{2}i\right)A+\left(z+\frac{1}{2}i\right)B=z$$

$\Rightarrow Az+Bz=z$ $\qquad A=\frac{1}{2}\quad B=\frac{1}{2}$

$-\frac{1}{2}iA+\frac{1}{2}iB=0 \Rightarrow iB=iA \Rightarrow A=B$

$$f(z)=\frac{1}{2}\cdot\frac{1}{z+\frac{1}{2}i}+\frac{1}{2}\cdot\frac{1}{z-\frac{1}{2}i}$$



$$\frac{\frac{1}{2}}{\frac{1}{2}i}\cdot\frac{1}{1-\left(-\frac{2}{i}z\right)}+\frac{\frac{1}{2}}{-\frac{1}{2}i}\cdot\frac{1}{1-\left(\frac{2}{i}z\right)}=$$

$$=\frac{-i}{1-(2iz)}+\frac{i}{1-(-2iz)}=$$

$|2z|<1 \Rightarrow$

$|z|<\frac{1}{2}$

$$=-i\sum_{k=0}^{\infty}(2iz)^k+i\sum_{k=0}^{\infty}(-2ik)^k=$$

$$i\sum_{k=0}^{\infty}\left[-(2iz)^k+(-1)^k(2iz)^k\right]=$$

$$i\sum_{k=0}^{\infty}\left[(2iz)^k\left((-1)^k-1\right)\right]=i\sum_{k=0}^{\infty}(2iz)^{2k+1}$$

3)

$$f(z) = \frac{e^{z^2}}{z^3}$$

Res $(f(z),0)=1$

$$= \frac{1}{z^3}\sum_{k=0}^{\infty}\frac{(z^2)^k}{k!} = \sum_{k=0}^{\infty}\frac{z^{2k-3}}{k!} = \frac{1}{z^3}+\frac{1}{z}+\frac{z}{2}+\dots$$

Polo di ordine 3: $\text{Res}(f(z),0) = \frac{1}{2}\frac{d^2}{dz^2}\left[e^{z^2}\right]_{z=0} =$

$$= \frac{1}{2}\left[2ze^{z^2}\right]'_{z=0} =$$

$$= \frac{1}{2}\left[2e^{z^2}+4z^2e^{z^2}\right]_{z=0} = 1$$

2)

$$f(z) = \frac{1-\cos z}{z^2}$$ singolarità eliminabile.

$$1-\cos z = \sum_{m=1}^{\infty}\frac{(-1)^{m+1}(z)^{2m}}{(2m)!}$$

$$\Rightarrow f(z) = \sum_{k=1}^{\infty}\frac{(-1)^{m+1}z^{2m-2}}{(2m)!}$$

3)

$$f(z) = \frac{e^z}{z^2+1} =$$

$z = \pm i$ singolarità.

Polo di ordine 1.

$$\text{Res}(f(z), i) = \left[\frac{e^z}{2z}\right]_{z=i} = \frac{e^i}{2i}$$

$$\text{Res}(f(z), -i) = \frac{e^{-i}}{-2i}$$

4) $$f(z) = \frac{z^2+3}{z^6+1}$$ $\text{Res}(f(z), i)$?

i polo del primo ordine, perché z^6+1 ha 6 radici semplici e numeratore non si annulla in i.

$$\text{Res}(f(z), i) = \left[\frac{z^2+3}{6z^5}\right]_{z=i} = \frac{2}{6i} = -\frac{1}{3}i$$

5)

$$f(z)=\frac{e^z}{(z^4-1)(z^2+2)^2}$$

Singularności:

$z_0=+\sqrt{2}i$ pole do orderu 2
$z_1=-\sqrt{2}i$ pole do orderu 2

$\left.\begin{array}{l} z_2=1 \\ z_3=-1 \\ z_4=i \\ z_5=-i \end{array}\right\}$ Pole do orderu 1

$$Res(f(z),1)=\left[\frac{e^z}{4z^3(z^2+2)^2}\right]_{z=1}=\frac{e}{36}$$

$$Res(f(z),-1)=\frac{e^{-1}}{-36}$$

$$Res(f(z),i)=\frac{e^i}{-4i}$$

$$Res(f(z),-i)=\frac{e^{-i}}{4i}$$

$$Res(f(z),\sqrt{2}i)=\frac{d}{dz}\left[\frac{e^z}{4z^3(z+\sqrt{2}i)^2}\right]_{z=\sqrt{2}i}=$$

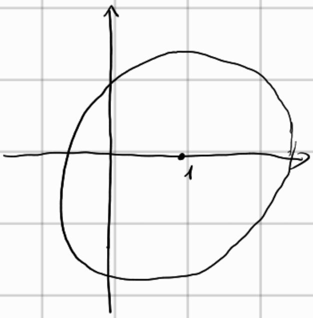
$$=\left[\frac{e^z\cdot 4z^3(z+\sqrt{2}i)^2-e^z\left[12z^2(z+\sqrt{2}i)^2+4z^3\cdot 2(z+\sqrt{2}i)\right]}{16z^6(z+\sqrt{2}i)^4}\right]=$$

$=\ldots$

$$z^{-3}\sum_{k=0}^{\infty}(-1)\frac{z^{2k+1}}{(2k+1)!}=\sum_{k=0}^{\infty}(-1)\frac{z^{2k-2}}{(2k+1)!}$$

4)

$$\int_{\gamma^+_{\frac{3}{2}}(1)} \frac{z^2}{z^3-8}\,dz = \frac{2\pi i}{3}$$



$z^3=8$ składniki:

$z_0=2 \quad z_1=-1+\sqrt{3}i$

$\quad\quad\quad z_2=-1-\sqrt{3}i$

$z_0=2$ calaki:

$$Res(f(z),2)= \left[\frac{z^2}{3z^2}\right]_{z=2}=\frac{1}{3}$$

1)

$$\int_\gamma \frac{e^z+e^{-z}}{z^4}\,dz = 0$$

$\gamma: |z|=1$

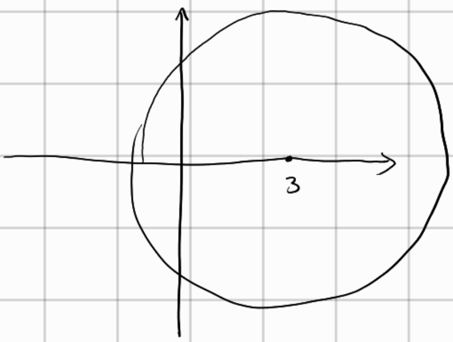
$f(z)=\dfrac{e^z+e^{-z}}{z^4}$ $\qquad \lim_{z\to 0} \cancel{z^4}\cdot \dfrac{e^z+e^{-z}}{\cancel{z^4}} = 2$ polo 4

$$\text{Res}(f(z),0) = \frac{1}{6}\cdot\frac{d^3}{dz^3}\left(e^z+e^{-z}\right)\Big|_{z=0} =$$

$$= \frac{1}{6}\left(e^z-e^{-z}\right)'' = \frac{1}{6}\left(e^z+e^{-z}\right)' = \frac{1}{6}\left(e^z-e^{-z}\right)\Big|_{z=0} = 0$$

2)

$$\int_\gamma \frac{z^2+1}{z(z-8)}\,dz$$



$\gamma: |z-3|=4$

Poli: $z=0$ $z=8$

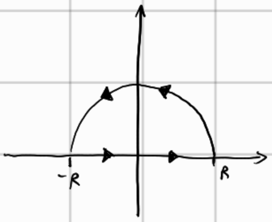
$$\lim_{z\to 0} \frac{z^2+1}{z-8} = -\frac{1}{8} = \text{Res}(f(z),0)$$

3)

$$\int_0^{+\infty} \frac{t^3 \sin t}{(1+t^2)(4+t^2)}\, dt = I$$

$$\frac{t^3 \sin t}{(1+t^2)(4+t^2)} \tilde{\approx} \frac{\sin t}{t}$$ INTEGRABILE SU $\mathbb{R}$

$$\int_{-\infty}^{+\infty} f(x)\,dx = 2I \qquad f(z) = \frac{z^3 e^{iz}}{(1+z^2)(4+z^2)}$$



$\Gamma_R = [-R, R] \cup \gamma_R^+(0)$

$$\int_{\Gamma_R} f(z)\,dz = \int_{-R}^{R} f(z)\,dz + \int_{\gamma_R^+(0)} f(z)\,dz$$

Se $R \to +\infty$

Il lemma del grande cerchio dice che $\int_{\gamma_R^+(0)} f(z)dz = 0$ perche $\sup_{|z|=R} |f(z)| \to 0$ se $R \to +\infty$

Se $R \to +\infty$

$$\int_{\Gamma_R} f(z)dz = \sum_{\operatorname{Im} z_k > 0} 2\pi i \operatorname{Res}(f(z), z_k) = \int_{\mathbb{R}} f(x)dx$$

$$f(z) = \frac{z^3 e^{iz}}{(1+z^2)(4+z^2)}$$

Poli: $z_1 = -i \qquad z_2 = -2i$

$\qquad z_2 = +i \qquad z_3 = 2i$

$$\lim_{z \to i} \frac{z^3 e^{iz}}{(z+i)(4+z^2)} = \frac{-i e^{-1}}{2\cancel{i} \cdot 3} \; z_0 \text{ Polo } 1 = -\frac{1}{6e}$$

$$\lim_{z \to 2i} \frac{z^3 e^{iz}}{(z^2+1)(z+2i)} = \frac{-8\cancel{i} e^{-2}}{-3 \cdot 4\cancel{i}} \; z_0 \text{ Polo } 1 = +\frac{2}{3e^2}$$

$$\operatorname{Im}\left(\int_{\mathbb{R}} f(x)\,dx\right) = \operatorname{Im}\left[2\pi i\left(-\frac{1}{6e} + \frac{2}{3e^2}\right)\right] = 2\pi\left(\frac{2}{3e^2} - \frac{1}{6e}\right)$$

$$I = \pi\left(\frac{2}{3e^2} - \frac{1}{6e}\right) = \frac{4\pi - e\pi}{6e^2}$$

6)

$$\int_{-\infty}^{+\infty} \frac{\sin^2(\pi x)}{x^2+1}\,dx$$

$f(x) \approx \frac{1}{x^2+1} \Rightarrow$ É integrabile su $\mathbb{R}$ perché va a 0 come $\frac{1}{x^2}$.

$$\sin^2 x = \frac{1-\cos(2\pi x)}{2}$$

$$\int_{-\infty}^{+\infty} \frac{1}{2(x^2+1)}\,dx - \int_{-\infty}^{+\infty} \frac{\cos(2\pi x)}{2(x^2+1)}\,dx$$

$$\int_{-\infty}^{+\infty} \frac{\cos(2\pi x)}{2(x^2+1)}\,dx = Re\left(\int_{-\infty}^{+\infty} \frac{e^{2i\pi x}}{2(x^2+1)}\,dx\right)$$

$\Gamma_R = [-R, R] \cup \gamma_R^+(0)$

Se $R \to +\infty$

$$\int_{\Gamma_R} f(z)dz = \int_{\mathbb{R}} \frac{e^{2i\pi x}}{2(x^2+1)}\,dx + \underbrace{\int_{\gamma_R^+(0)} f(z)dz}_{0 \text{ per lemma del grande cerchio}}$$

$$\| $$

$$2\pi i \sum_{Im\, z_k > 0} Res\left(f(z), z_k\right)$$ con i unico residuo.

$$\lim_{z \to i} \frac{e^{2i\pi z}}{2(x+i)} = \frac{e^{-2\pi}}{4i} \neq 0.$$

$$\int_{\mathbb{R}} \frac{e^{2ix}}{2(x^2+1)}\,dx = \frac{2\pi e^{-2}}{4} = \frac{\pi}{2e^{2\pi}} = Re\left(\int_{\mathbb{R}} \frac{e^{2i\pi x}}{2(x^2+1)}\,dx\right) = \int_{\mathbb{R}} \frac{\cos(2\pi x)}{2(x^2+1)}\,dx$$

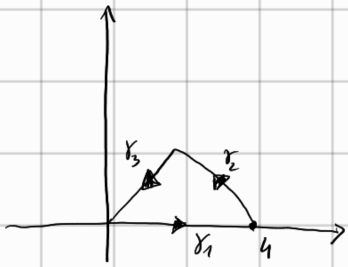
$$\lim_{z \to i} \frac{1}{2(x+i)} = \frac{1}{4i}.$$

$$\int_{\mathbb{R}} \frac{1}{2(x^2+1)}\,dx = \frac{\pi}{2}$$

$$\int_{-\infty}^{+\infty} \frac{\sin^2(\pi x)}{x^2+1}\,dx = \frac{\pi}{2} - \frac{\pi}{2e^{2\pi}} = \frac{\pi(e^{2\pi}-1)}{2e^{2\pi}}$$

1)

$$\int_\gamma |z|\,dz$$



$\gamma_1: \gamma_1(t) = t \quad t \in [0,4]$

$\gamma_2: \gamma_2(t) = 4e^{it} \quad t \in [0, \frac{\pi}{4}]$

$-\gamma_3: \gamma_3(t) = t + it \quad t \in [0, 4\frac{\sqrt{2}}{2}]$ $\qquad t^2 + t^2 \Rightarrow \sqrt{2}t$

$$\gamma_1): \int_0^4 t\,dt = \frac{t^2}{2}\Big]_0^4 = 8$$

$$\gamma_2: \int_0^{\frac{\pi}{4}} 4 \cdot 4ie^{it}\,dt = 16\cancel{i} \frac{e^{it}}{\cancel{i}}\Big|_0^{\frac{\pi}{4}} = 16\left[e^{i\frac{\pi}{4}} - e^0\right] = 16\left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i - 1\right] = 8\sqrt{2} + 8\sqrt{2}i - 16$$

$$\gamma_3: -\int_0^{\frac{4\sqrt{2}}{2}} \sqrt{2}t \cdot (1+i)\,dt = -(1+i) \cdot \sqrt{2} \cdot \frac{t^2}{2}\Big]_0^{\frac{4\sqrt{2}}{2}} = -(1+i)\sqrt{2} \cdot 4$$

3)

$$f(z) = \frac{\sin(2z)}{z(z-\pi)}$$

$z=0$ e $z=\pi$ singolarità.

- $z=0$ singolarità eliminabile:

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{\sin(2z)}{z(z-\pi)} = \lim_{z\to 0} \frac{2z}{z(z-\pi)} = -\frac{2}{\pi}$$

- π è eliminabile:

$$\lim_{z\to\pi} f(z) = \lim_{z\to\pi} \frac{\sin(2z)}{z(z-\pi)} = \lim_{z\to\pi} \frac{2\cos(2z)}{(z-\pi)+z} = \frac{2}{\pi}$$

$$g(z) = (z+1)\sinh\left(\frac{1}{z}\right)$$

$$g(z) = (z+1)\cdot\frac{e^{\frac{1}{z}} - e^{-\frac{1}{z}}}{2}$$ dove 0 é singolarità essenziale perché $e^{\frac{1}{z}}$ ha sing. essenziale in 0.

2)

$$\int_{-\infty}^{+\infty} \frac{\cos(2x) + \sin(-x)}{x^2+4}\,dx =$$

$$= \int_{-\infty}^{+\infty} \frac{\cos(2x)}{x^2+4}\,dx - \int_{-\infty}^{+\infty} \frac{\sin x}{x^2+4}\,dx$$

$|f(x)|$ e $|g(x)|$ con $x \to \pm\infty$ vanno a 0 come $\frac{1}{x^2}$. Sono integrabili.

$$\int_{-\infty}^{+\infty} \frac{\cos(2x)}{x^2+4}\,dx = Re\left(\int_{-\infty}^{+\infty} \frac{e^{i2x}}{x^2+4}\,dx\right)$$

$$\Gamma_R = [-R,R] \cup \gamma_R^+(0)$$

Va a 0 se $R \to +\infty$ per lemma di Jordan

$$\int_{\Gamma_R} f(z)\,dz = \underbrace{\int_{-R}^{R} \frac{e^{i2x}}{x^2+4}\,dx}_{\text{Tende a } \int_{\mathbb{R}} \frac{e^{i2x}}{x^2+4}dx} + \int_{\gamma_R^+(0)} \frac{e^{i2z}}{z^2+4}\,dz$$

$$\int_{\Gamma_R} f(z)\,dz = 2\pi i \sum_{Im z_k > 0} Res(f(z), z_k)$$

Poli: $z_1 = 2i \to$ Calcolo Residuo
$z_2 = -2i$

$$\lim_{z\to 2i} (z-2i) f(z) = \lim_{z \to 2i} \frac{e^{2iz}}{(z+2i)} = \frac{e^{-4}}{4i}$$

$$\Rightarrow \int_{\Gamma_R} f(z)\,dz = \frac{\pi}{2e^{+4}} = Re\left(\int_{\mathbb{R}} f(z)\,dz\right)$$

Nel secondo caso mi serve $Im\left(\int_{\mathbb{R}} \frac{e^{ix}}{x^2+4}\,dx\right)$

$$Res(f(z), 2i) = \lim_{z\to 2i} \frac{e^{iz}}{(z+2i)} = \frac{e^{-2}}{4i}$$

$$\int_{\mathbb{R}} \frac{e^{ix}}{x^2+4}dx = \frac{\pi}{2e^2} \Rightarrow \mathrm{Im}\left(\int_{\mathbb{R}} \frac{e^{ix}}{x^2+4}dx\right) = 0$$

1)

$$\mathrm{Re}\left(\frac{z+i}{z-1}\right) = 0 \qquad z \neq 1$$

$$\frac{x+iy+i}{x+iy-1} = \frac{x+iy+i}{(x-1)+iy} = \frac{[x+i(y+1)][(x-1)-iy]}{x^2+1-2x-y^2} =$$

$$\frac{x(x-1) - ixy + y(y+1) + i(x-1)(y+1)}{x^2+1-2x-y^2} = 0$$

$$\frac{x^2-x-ixy+y^2+y+i(xy+x-y-1)}{x^2+1-2x-y^2} = 0$$

$$\frac{x^2-x+y^2+y+i(x-y-1)}{x^2+1-2x-y^2} = 0$$

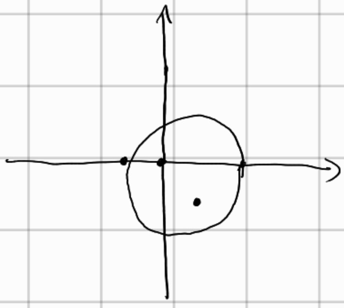
$$\mathrm{Re}\left(\frac{z+i}{z-1}\right) = \frac{x^2-x+y^2+y}{x^2+1-2x-y^2} = 0$$

$x \neq 1 \quad y \neq 0$

$\Rightarrow x^2-x+y^2+y=0$

$x^2+y^2-x+y=0$

$C\left(+\frac{1}{2}; -\frac{1}{2}\right) \quad r = \frac{\sqrt{2}}{2}$



6)

$u(x,y) = x^2 - y^2 + e^{-x}\cos y$ $\quad f: \mathbb{C} \to \mathbb{C}$ di cui $u(x,y)$ sia parte reale.

Verifico che u sia armonica

$\Delta u = 2 + e^{-x}\cos y - 2 - e^{-x}\cos y = 0$ ✓ ed é $C^\infty(\mathbb{R}^2)$, $\hookrightarrow$ Semplic. connesso. Esiste armonic coniugata.

Devo trovare $V(x,y)$ /:

$$\begin{cases} u_x = V_y \\ u_y = -V_x \end{cases}$$

$\Rightarrow \nabla V = (-u_y, u_x) = (2y + e^{-x}\sin y, 2x - e^{-x}\cos y)$

$V(x,y) = \int V_x dx = \int (2y + e^{-x}\sin y) dx = 2xy - e^{-x}\sin y + C(y)$

$V_y = 2x - e^{-x}\cos y + C'(y) \Rightarrow C'(y) = 0 \Rightarrow C(y) = c$

$V(x,y) = 2xy - e^{-x}\sin y + c$

$f(x+iy) = x^2 - y^2 + e^{-x}\cos y + i(2xy - e^{-x}\sin y)$

1)

$$f(z)=\frac{z}{1+z^2}$$

$$f(z)=\frac{z}{(z+i)(z-i)}=\frac{A}{z+i}+\frac{B}{z-i}=$$

$$\frac{Az-iA+Bz+Bi}{(z+i)(z-i)}=\frac{1}{2}\cdot\frac{1}{z+i}+\frac{1}{2}\cdot\frac{1}{z-i}=$$

$$=\frac{1}{2i}\cdot\frac{1}{1+\frac{z}{i}}-\frac{1}{2i}\cdot\frac{1}{1-\frac{z}{i}}$$

$$=\frac{1}{2i}\sum_{k=0}^{\infty}z^k-\frac{1}{2i}\sum_{k=0}^{\infty}(-1)^k z^k$$

Vale per $|z|<1$

$$(z+1)(z^2-z+1)$$

$$f(z)=\frac{z}{1+z^3}$$

$$(z+1)\left(z+\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)=$$

$$z^2+\frac{1}{2}z-\frac{\sqrt{3}}{2}iz+z+\frac{1}{2}-\frac{\sqrt{3}}{2}i$$

$$1+z^3=0 \Rightarrow z_0=-1$$

$$z_1=e^{\frac{2}{3}i\pi}=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$$

$$z_2=e^{\frac{4}{3}i\pi}=-\frac{1}{2}-\frac{\sqrt{3}}{2}i$$

$$f(z)=\frac{A}{z+1}+\frac{B}{z+\frac{1}{2}-\frac{\sqrt{3}}{2}i}+\frac{C}{z+\frac{1}{2}+\frac{\sqrt{3}}{2}i}=$$

$$\Rightarrow \cancel{A}z^2-Az+A+\cancel{B}z^2+\frac{B}{2}z+\frac{\sqrt{3}}{2}izB+zB+\frac{1}{2}B+\frac{\sqrt{3}}{2}iB+\cancel{C}z^2+\frac{C}{2}z-\frac{\sqrt{3}}{2}izC+Cz+\frac{C}{2}-\frac{\sqrt{3}}{2}iC$$

$$\begin{cases} A+B+C=0 \\ -A+\frac{B}{2}+\frac{\sqrt{3}}{2}iB+B+\frac{C}{2}-\frac{\sqrt{3}}{2}iC+C=0 \\ A+\frac{B}{2}+\frac{\sqrt{3}}{2}iB+\frac{C}{2}-\frac{\sqrt{3}}{2}iC=0 \end{cases}$$

$$\begin{cases} A+B+C=0 \\ -2A+B+\sqrt{3}iB+2B+C-\sqrt{3}iC+2C=1 \\ 2A+B+\sqrt{3}iB+C-\sqrt{3}iC=0 \end{cases}$$

$$\begin{cases} A=-B-C \\ 2B+2C+B+\sqrt{3}iB+2B+3C-\sqrt{3}iC=1 \\ B+C-2B-2C+\sqrt{3}iB-\sqrt{3}iC=0 \end{cases}$$

$$\begin{cases} A=-B-C \\ 5B+5C+\sqrt{3}iB-\sqrt{3}iC=1 \\ -B-C+\sqrt{3}iB-\sqrt{3}iC=0 \end{cases}$$

$\Downarrow$

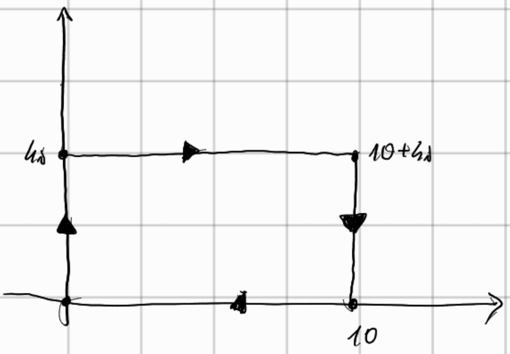
$$C(1+\sqrt{3}i)=B(-1+\sqrt{3}i)$$

$$C=\frac{B(-1+\sqrt{3}i)}{1+\sqrt{3}i}=\frac{B(1+3+2\sqrt{3}i)}{4}=B+\frac{\sqrt{3}}{2}iB$$

$$5B+5B+\frac{5\sqrt{3}}{2}iB+\cancel{\sqrt{3}i}B-\cancel{\sqrt{3}i}B+\frac{3}{2}B=1$$

$$20B+5\sqrt{3}iB+3B=2$$

$$(23+5\sqrt{3}i)B=2$$



$$f(z) = \frac{1}{z^2 - 3z + 5}$$

$$z_{1,2} = \frac{3 \pm \sqrt{9-20}}{2} = \frac{3 \pm i\sqrt{11}}{2}$$

$z_1 = \frac{3}{2} + \frac{\sqrt{11}}{2} i$ é nell'insieme di integrazione

Polo di ordine 1.

$$\operatorname{Res}(f(z), z_1) = \left[\frac{1}{2z-3} \right]_{z = \frac{3}{2} + \frac{\sqrt{11}}{2} i} = \frac{1}{3 + \sqrt{11} i - 3} = \frac{1}{\sqrt{11} i} = -\frac{\sqrt{11}}{11} i$$

PRIMO INTEGRALE: $-2\pi i \cdot \left(-\frac{\sqrt{11}}{11} i\right) = -\frac{2\pi}{11}\sqrt{11}$

2) $f(z) = \frac{1}{z^2 + z + 1}$

$z_{1,2} = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$ Nessun polo nell'insieme. Int. $= 0$

3) $f(z) = \frac{1}{z^2 - z + 1}$

$z_{1,2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i \qquad z_1 = \frac{1}{2} + \frac{\sqrt{3}}{2} i.$

$$\operatorname{Res}\left(f(z), \frac{1}{2} + \frac{\sqrt{3}}{2} i\right) = \left[\frac{1}{2z-1} \right]_{z = \frac{1}{2} + \frac{\sqrt{3}}{2} i} = \frac{1}{1 + \sqrt{3} i - 1} = -\frac{\sqrt{3}}{3} i$$

INTEGRALE: $-\frac{2\pi}{3}\sqrt{3}$

3)

$$\int_0^{+\infty} \frac{1-\cos(2x)}{x^2}\,dx =$$

$$= \int_0^{+\infty} \frac{1-\cos^2 x+\sin^2 x}{x^2}\,dx = \int_{-\infty}^{+\infty} \frac{\sin^2 x}{x^2}\,dx$$

$$\sin^2 x = -\frac{e^{2ix}}{4} - \frac{e^{-2ix}}{4} + \frac{1}{2}$$

1)

$$f(z) = \frac{1-e^{i\pi z}}{z(z+i)(z-1)^2}$$

Singolarità: $z=0$ eliminabile: $\lim_{z\to 0} f(z) = \pi$

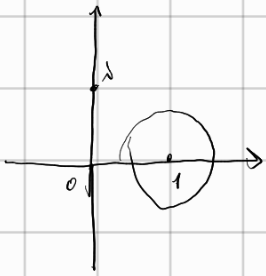
$z=-i$ Polo di ordine 1:

$$\lim_{z\to i} \frac{1-e^{i\pi z}}{z(z-1)^2} = \frac{1-e^{+\pi}}{-i(1-1+2i)} = \frac{1-e^{+\pi}}{2} \neq 0$$

$z=1$: Polo 2

$$\lim_{z\to 1} \frac{1-e^{i\pi z}}{z(z+i)} = \frac{1+1}{1+i} = \frac{2(1-i)}{2} = 1-i$$

Integrale con centro origine e raggio $\frac{1}{2} = 0$ perché la funz. è olomorfa



Centro 1 e raggio $\frac{1}{2}$: $2\pi i \operatorname{Res}(f(z), 1)$

$$\int_\gamma \frac{1-e^{i\pi z}}{z(z-1)^2(z+i)}\,dz = \frac{2\pi i}{1!} g'(z_0) = 2\pi i \cdot \frac{(1-e^{i\pi z})(z+i+z) + i\pi e^{i\pi z}(z(z+i))}{z^2(z+i)^2} =$$

$n=1 \quad z_0=1 \qquad$ Con $g(z) = \frac{1-e^{i\pi z}}{z(z+i)}$

$$\Downarrow$$

$$= \cancel{2}\pi\cancel{i} \cdot \frac{2(2+i) - \cancel{i}\pi(1+i)}{\cancel{2}\cancel{i}} =$$

3)

$$\int_\gamma \frac{z^2+4}{z(z^2+1)}\,dz \qquad \gamma(t)=4e^{it} \quad t\in[0,2\pi)$$

$$f(z)=\frac{z^2+4}{z(z^2+1)} \qquad \begin{array}{l} z_0=0 \\ z_1=+i \\ z_2=-i \end{array}$$

Sono poli 1:

$$Res(f(z),i)=\lim_{z\to i}\frac{z^2+4}{z(z+i)}=\frac{3}{-2}=-\frac{3}{2}$$

$$Res(f(z),-i)=\lim_{z\to -i}\frac{z^2+4}{z(z-i)}=\frac{3}{-2}=-\frac{3}{2}$$

$$Res(f(z),0)=\lim_{z\to 0}\frac{z^2+4}{z^2+1}=4$$

1) $$\int_{-\infty}^{+\infty} \frac{x-2}{x^2-4x+3} \sin(2x)\, dx$$

$f(x) \approx \frac{\sin(2x)}{x}$ all'infinito. Integrale converge.

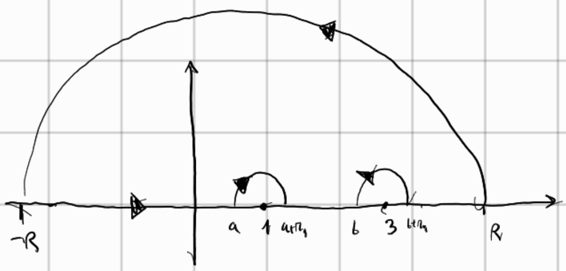
$$f(z) = \frac{z-2}{z^2-4z+3} e^{i2z}$$

$$\int_{-\infty}^{+\infty} \frac{x-2}{x^2-4x+3} \sin(2x)\, dx = \operatorname{Im}\left(\int_{-\infty}^{+\infty} \frac{x-2}{x^2-4x+3} e^{i2x} dx\right)$$

Verifico sing:

$z^2-4z+3=0$

$z_{1,2} = 2 \pm \sqrt{4-3} = 2 \pm 1 = \begin{cases} 3 \\ 1 \end{cases}$



$$\Gamma_{R,r_1,r_2} = [-R, 1-r_1] \cup \gamma_{r_1}^-(1) \cup [1+r_1, 3-r_2] \cup \gamma_{r_2}^-(3) \cup [3+r_2, R] \cup \gamma_R^+(0)$$

$$\int_{\Gamma_{R,r_1,r_2}} f(z)\,dz = \int_{[-R,1-r_1]} f(z)\,dz - \int_{\gamma_{r_1}(1)} f(z)\,dz + \int_{[1+r_1,3-r_2]} f(z)\,dz - \int_{\gamma_{r_2}(3)} f(z)\,dz + \int_{[3+r_2,R]} f(z)\,dz + \int_{\gamma_R^+(0)} f(z)\,dz = 0$$

$z=1$ e $z=3$ sono poli di ordine 1.

$$\operatorname{Res}(f(z), 1) = \lim_{z\to 1} \frac{z-2}{(z-3)} e^{i2z} = \frac{2}{3} e^{2i}$$

$$\operatorname{Res}(f(z), 3) = \lim_{z\to 3} f(z)(z-3) = \frac{1}{2} e^{6i}$$

Se $r_1 \to 0^+$ e $r_2 \to 0^+$, integrali tendono a $\pi i \operatorname{Res}(f(z), z_0)$.

$$-\int_{\gamma_{z_1}(1)} f(z)\,dz = -\frac{2}{3}\pi i e^{2i}$$

$$-\int_{\gamma_{z_2}(3)} f(z)\,dz = -\frac{1}{2}\pi i e^{6i}$$

$$\int_{\mathbb{R}} f(z)\,dz = \frac{1}{2}\pi i e^{6i} + \frac{2}{3}\pi i e^{2i}$$

$$\Rightarrow \int_{-\infty}^{+\infty} \frac{x-2}{x^2-4x+3}\sin(2x)\,dx = \operatorname{Im}\left(\pi i e^{2i}\left(\frac{1}{2}e^{4i}+\frac{2}{3}\right)\right) \approx 0.63665$$

1)

$$\int_\gamma \frac{e^z+e^{-z}}{z^4}\,dz \qquad \gamma: |z|=1$$

La funzione $g(z)=e^z+e^{-z}$ é olomorfa su $\mathbb{C}$

$n=3 \quad z_0=0.$

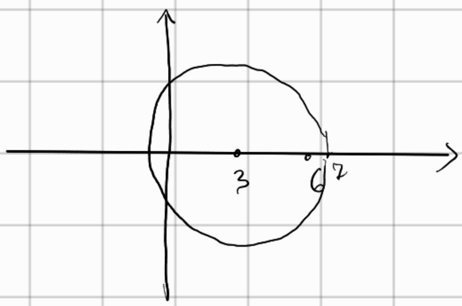
$$f^{(3)}(0)=\frac{3!}{2\pi i}\int_\gamma \frac{e^z+e^{-z}}{z^4}\,dz$$

$$\Rightarrow f^{(3)}(z)=e^z-e^{-z} \qquad f^{(3)}(0)=0$$

Integrale é 0.

2)

$$\int_\gamma \frac{z^2+1}{z(z-8)}\,dz \qquad \gamma: |z-3|=4$$



La funzione $\frac{z^2+1}{z-8}$ é olomorfa nell'interno di γ.

$z_0=0 \quad n=0$

$$f(0)=\frac{1}{2\pi i}\int_\gamma \frac{z^2+1}{z(z-8)}\,dz \Rightarrow \int_\gamma \frac{z^2+1}{z(z-8)}\,dz=2\pi i\left(-\frac{1}{8}\right)=-\frac{\pi i}{4}$$

2)

$u(x,y) = ax^2 + y^2$

$\Delta u = 2a + 2 = 0 \Rightarrow u$ é armonica $\Leftrightarrow a = -1$

Considero $u(x,y) = y^2 - x^2$

Devo trovare $v(x,y)$ /

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

$\Rightarrow \nabla v = (-u_y, u_x)$ $\qquad -u_y = -2y$

$V = \int -2y \, dx = -2xy + C(y)$

$V_y = -2x + C'(y) = -2x \Rightarrow C'(y) = 0$

$V(x,y) = -2xy + C$

$f_k(x+iy) = (y^2 - x^2) + i(-2xy + C)$

2) $$\int_\gamma \tan z \, dz$$

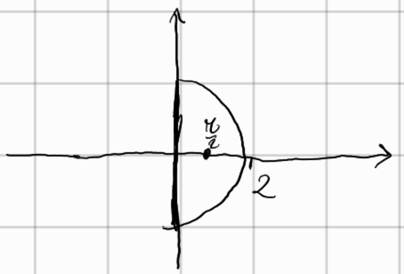
$\gamma \subset \mathbb{C}$ é circonf. di centro origine e $r=1$

$$\tan z = \frac{\sin z}{\cos z}$$

$\cos z = 0$

$$z = \frac{\pi}{2} + 2k\pi$$

Primo integrale é 0 perche $f(z)$ olomorfa all'interno.

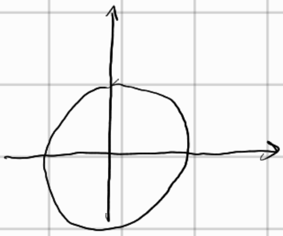


$$\tan\left(z-\frac{\pi}{2}\right) = \frac{\sin\left(z-\frac{\pi}{2}\right)}{\cos\left(z-\frac{\pi}{2}\right)} = \frac{-\cos z}{\sin z} = -\frac{\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}}{\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}} = -\frac{1-\frac{z^2}{2}+\frac{z^4}{24}-\dots}{z-\frac{z^3}{6}+\frac{z^5}{120}-\dots}$$

$$\lim_{z\to\frac{\pi}{2}} \left(z-\frac{\pi}{2}\right)\frac{\sin z}{\cos z} = \lim_{z\to\frac{\pi}{2}} \frac{\sin z + \cos z\left(z-\frac{\pi}{2}\right)}{-\sin z} = -1$$

3)

$$\int_\gamma \frac{z(z+1)}{\sin(z+1)}\,dz$$



$\gamma: |z|=3$

$f(z)$ ammette singolarità se

$z+1 = 0+k\pi$

$z = -1+k\pi.$

$z_0=-1 \quad z_1=\pi-1$

Calcolo Residui:

Hp: -1 é eliminabile

$$\lim_{z\to -1} f(z) = -1 = \frac{(z+1)+z}{\cos(z+1)}$$

$\Rightarrow \mathrm{Res}(f(z),-1)=0$

$\pi-1$ é polo 1:

$$\lim_{z\to\pi-1} f(z)(z+1-\pi) = \frac{(z^2+z)(z+1-\pi)}{\sin(z+1)} = \frac{(2z+1)(z+1-\pi)+(z^2+z)}{\cos(z+1)} = -\left[(\pi-1)^2+\cancel{z_1}\right] = \left[\pi^2+\cancel{1}-2\pi+\pi\cancel{-1}\right] = \pi-\pi^2$$

3)

$$\int_0^\infty \frac{x^2}{x^4+1}\,dx$$

$f(x)$ é pari: calcolo integrale su $\mathbb{R}$.

$f(x)$ non si annulla mai su $\mathbb{R}$ e va a

0 con ordine $\approx \frac{1}{x^2}$. Converge.

Considero $f(z) = \dfrac{z^2}{z^4+1}$

e $\Gamma_R = [-R,R] \cup \gamma_R^+(0)$

$$\int_{\Gamma_R} f(z)dz = \int_{[-R,R]} f(z)dz + \int_{\gamma_R^+(0)} f(z)dz$$

Va a 0 se $R\to+\infty$ per teorema di integraz. del rapporto di polinom.

$$\Rightarrow \int_{\mathbb{R}} f(x)dx = 2\pi i \sum_{z_k \in \mathrm{Im}\, z>0} \mathrm{Res}(f(z), z_k)$$

$z_0 = \frac{\sqrt2}{2} + \frac{\sqrt2}{2} i$ $\quad\Rightarrow z_0$ e z_1 hanno $\mathrm{Im}\, z > 0$

$z_1 = -\frac{\sqrt2}{2} + \frac{\sqrt2}{2} i$

$z_2 = -\frac{\sqrt2}{2} - \frac{\sqrt2}{2} i$

$z_3 = \frac{\sqrt2}{2} - \frac{\sqrt2}{2} i$

$$\mathrm{Res}(f(z), z_0) = \left[\frac{z^2}{4z^3}\right]_{z=z_0} = \frac{1}{4 z_0} = \frac14 \cdot \frac{1}{\frac{\sqrt2}{2}+\frac{\sqrt2}{2}i} = \frac14 \frac{\frac{\sqrt2}{2} - \frac{\sqrt2}{2}i}{1} = \frac{\sqrt2}{8} - \frac{\sqrt2}{8}i$$

$$\mathrm{Res}(f(z), z_0) = \frac{1}{4 z_0} = \frac14 \cdot \frac{1}{-\frac{\sqrt2}{2}+\frac{\sqrt2}{2}i} = \frac14 \cdot \frac{\frac{\sqrt2}{2}+\frac{\sqrt2}{2}i}{-1} = -\frac{\sqrt2}{8} - \frac{\sqrt2}{8}i$$

$$\Rightarrow \int_{\mathbb{R}} f(x)dx = 2\pi i\left(-\frac{\sqrt2}{4} i\right) = \frac{\sqrt2}{2}\pi$$

3)

$$\int_\gamma \frac{\sin(2z^2+3z+1)}{z-\pi}\,dz$$

$\gamma: |z-\pi|=1$

$g(z)=\sin(2z^2+3z+1)$ è olomorfa in int. di γ.

$z_0=\pi \quad n=0$

$$g(\pi)=\frac{1}{2\pi i}\int_\gamma \frac{\sin(2z^2+3z+1)}{z-\pi}\,dz$$

$$\int_\gamma \frac{\sin(2z^2+3z+1)}{z-\pi}\,dz = 2\pi i \sin(2\pi^2+3\pi+1)$$

$$f(z)=\frac{1-e^{i\pi z}}{z(z+i)(z-1)^2}$$

Singolarità: $z=0$
$z=-i$
$z=1$

$$\lim_{z\to 0} f(z)=\lim_{z\to 0}\frac{-i\pi}{i(0-1)^2}=-\frac{\pi}{1}=-\pi$$

$z=-i$ Hp: Polo 1:

$$\lim_{z\to -i}\frac{1-e^{i\pi z}}{z(z-1)^2}=\frac{1-e^{\pi}}{-i\cdot 2i}=\frac{1-e^{\pi}}{2}$$

$z=1$ polo di ordine 2:

$$\mathrm{Res}(f(z),1)=\frac{d}{dz}\left[\frac{1-e^{i\pi z}}{z(z+i)}\right]_{z=1}=\left[\frac{-i\pi e^{i\pi z}(z^2+iz)-(2z+i)(1-e^{i\pi z})}{z(z+i)}\right]_{z=1}=$$

$$=\left[\frac{-i\pi e^{i\pi}(1+i)-(2+i)(1-e^{i\pi})}{1+i}\right]=$$

$$=\frac{-i\pi e^{i\pi}+\pi e^{i\pi}-(2-2e^{i\pi}+i-ie^{i\pi})}{1+i}=$$

1)

$$f(z)=\frac{e^{\frac{1}{z}}}{z+2}=\left(\sum_{k=0}^{\infty}\left(\frac{1}{z}\right)^{k}\cdot\frac{1}{k!}\right)\cdot\frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{z}{2}\right)^{n}=\frac{1}{2}\sum_{k=0}^{-\infty}\frac{z^{k}}{(-k)!}\cdot\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^{n}z^{n}$$

$$\frac{1}{z+2}=\frac{1}{2}\cdot\frac{1}{\frac{z}{2}+1}$$

2)

$$f(z)=\frac{z}{1+z^{3}}$$

$$f(z)=z\cdot\frac{1}{1+z^{3}}=z\sum_{k=0}^{\infty}(-z^{3})^{k}=z\sum_{k=0}^{\infty}(-1)^{k}z^{3k}=\sum_{k=0}^{\infty}(-1)^{k}\cdot z^{3k+1}$$

Vale per $|z^{3}|<1 \Rightarrow |z|<1$

$$f(z)=\frac{z}{z^{3}}\cdot\frac{1}{\frac{1}{z^{3}}+1}=\frac{1}{z^{2}}\cdot\sum_{k=0}^{\infty}\left(\frac{1}{z^{3}}\right)^{n}=\frac{1}{z^{2}}\sum_{k=0}^{\infty}(-1)^{k}\cdot z^{-3k}=\sum_{k=0}^{\infty}(-1)^{k}\cdot z^{-3k-2}=$$

$$=\sum_{k=0}^{-\infty}(-1)^{-k}z^{3k-2}$$

$\left|\frac{1}{z^{3}}\right|<1$

$|z|>1$

3)

$$f(z)=\frac{e^{\frac{1}{z}}}{z+2}$$

Centrata in 0:

$$=\frac{1}{2}\sum_{k=0}^{-\infty}\frac{z^{k}}{(-k)!}\cdot\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^{n}z^{n}=\frac{1}{2}\left(1+\frac{1}{z}+\frac{1}{2z^{2}}+\frac{1}{6z^{3}}+\ldots\right)\left(z-\frac{1}{2}z+\frac{1}{4}z^{2}-\ldots\right)$$

$\frac{1}{24z^{4}}$ $\quad$ $\frac{1}{8}$

$$\frac{1}{n!\,2^{n}}$$

1)

$$f(z) = \frac{e^{\frac{1}{z}}}{z^2} = \frac{1}{z^2}\sum_{k=0}^{\infty}\left(\frac{1}{z}\right)^k \cdot \frac{1}{k!} = \frac{1}{z^2}\sum_{k=0}^{-\infty}\frac{z^k}{(-k)!} = \sum_{k=0}^{-\infty}\frac{z^{k-2}}{(-k)!} = \frac{1}{z^2} + \frac{1}{z^3} + \dots$$

2)

$$f(z) = \frac{z+1}{z-1} = \frac{z}{z-1} + \frac{1}{z-1} = \frac{-z}{1-z} - \frac{1}{1-z} =$$

$$= -z\sum_{k=0}^{\infty} z^k - 1\sum_{k=0}^{\infty} z^k = -\sum_{k=0}^{\infty} z^{k+1} - \sum_{k=0}^{\infty} z^k = -1 - 2\sum_{k=1}^{\infty} z^k$$

$|z|<1$

Se $|z|>1$

$$f(z) = \frac{z}{z-1} + \frac{1}{z-1} = \frac{z}{z}\cdot\frac{1}{1-\frac{1}{z}} + \frac{1}{z}\,\frac{1}{1-\frac{1}{z}} = \sum_{k=0}^{\infty}\left(\frac{1}{z}\right)^k + \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{1}{z}\right)^k =$$

$$= \sum_{k=0}^{-\infty} z^{k-1} + \sum_{k=0}^{-\infty} z^k = 1 + 2\sum_{k=1}^{-\infty} z^k$$

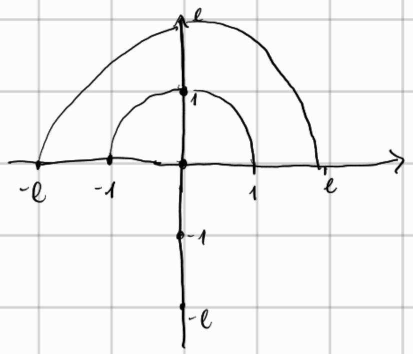
$$\left(\frac{1}{z} + \frac{1}{z^2} + \dots + 1 + \frac{1}{z}\dots\right.$$

1)

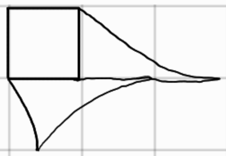
$f(z) = e^z$

$z = x + iy \;/\; x \leq 1 \text{ e } 0 \leq y \leq \pi$

$e^{x+iy} = e^x\cos y + i e^x \sin y$



2)



$$\int_{\mathbb{R}} \frac{1}{x^4+x^2+1}\,dx$$

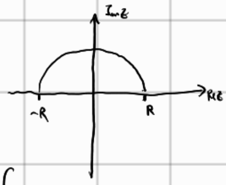
- $x^4+x^2+1=0$

$y=x^2$

$y^2+y+1=0$

Ha soluzioni complesse, posso usare residui.

$\Gamma_R = [-R,R] \cup \gamma_R^+(0)$



$$\int_{\Gamma_R} f(z)\,dz = \int_{[-R,R]} f(z)\,dz + \int_{\gamma_R^+(0)} f(z)\,dz$$

Se $R\to+\infty$ $\downarrow$ va a 0 per Th. di conv. di rapp. polin.

$$\int_{\Gamma_R} f(z)\,dz = \int_{\mathbb{R}} f(x)\,dx = \sum_{\operatorname{Im} z_k>0} 2\pi i \operatorname{Res}(f(z),z_k)$$

Trovo le singolarità:

$x^4+x^2+1=0$

$\Rightarrow x^2=y$

$y^2+y+1=0$

$y_{1,2} = \frac{-1\pm\sqrt{1-4}}{2} = -\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$

$y_1 = -\frac{1}{2}+\frac{\sqrt{3}}{2}i \qquad y_2 = -\frac{1}{2}-\frac{\sqrt{3}}{2}i$

$y_1 = e^{\frac{2}{3}\pi} \qquad y_2 = e^{\frac{4}{3}\pi}$

$\Rightarrow x_1 = e^{\frac{\pi}{3}} \qquad x_3 = e^{\frac{2}{3}\pi}$

$x_2 = e^{\frac{4}{3}\pi} \qquad x_4 = e^{\frac{5}{3}\pi}$

$z_1 = \frac{1}{2}+\frac{\sqrt{3}}{2}i \qquad z_3 = -\frac{1}{2}+\frac{\sqrt{3}}{2}i$

$z_2 = -\frac{1}{2}-\frac{\sqrt{3}}{2}i \qquad z_4 = \frac{1}{2}-\frac{\sqrt{3}}{2}i$

$z_k \,/\, \operatorname{Im} z_k>0 : z_1, z_3$

Sono poli di ordine 1:

$$\operatorname{Res}(f(z),z_1) = \lim_{z\to\frac{1}{2}+\frac{\sqrt{3}}{2}i} \frac{1}{\left(z+\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\left(z-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\left(z+\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)} =$$

$$= \frac{1}{(1)(\sqrt{3}i)(1+\sqrt{3}i)} = \frac{1}{\sqrt{3}i-3} = \frac{\sqrt{3}i+3}{-3-9} = -\frac{1}{4}-\frac{\sqrt{3}}{12}i$$

$$\operatorname{Res}(f(z),z_3) = \lim_{z\to-\frac{1}{2}+\frac{\sqrt{3}}{2}i} \frac{1}{\left(z-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\left(z+\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\left(z-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)} =$$

$$= \frac{1}{(+1)(\sqrt{3}i)(+1-\sqrt{3}i)} = \frac{1}{\sqrt{3}i+3} = \frac{3-\sqrt{3}i}{9+3} = \frac{1}{4}-\frac{\sqrt{3}}{12}i$$

$$\int_{\mathbb{R}} f(x)\,dx = 2\pi i\left(\frac{1}{4}-\frac{\sqrt{3}}{12}i-\frac{1}{1}-\frac{\sqrt{3}}{12}i\right) = 2\pi\cdot\frac{2\sqrt{3}}{12} = \frac{\sqrt{3}}{3}\pi$$

2)

$$\int_{\gamma} \frac{\sin(2z^2+3z+1)}{z-\pi}\,dz \qquad \gamma=\{z\in\mathbb{C} : |z-\pi|=1\}$$

funzione $g(z)=\sin(2z^2+3z+1)$ olomorfa su $\mathbb{C}$.

Applico Cauchy:

$z_0=\pi \quad n=0$

$$g(z_0)=\frac{1}{2\pi i}\int_{\gamma}\frac{\sin(2z^2+3z+1)}{z-\pi}\,dz$$

$$\int_{\gamma}\frac{\sin(2z^2+3z+1)}{z-\pi}\,dz = 2\pi i\sin(2\pi^2+3\pi+1)$$

$$\operatorname{Res}(f(z),\pi) = \lim_{z\to\pi}\sin(2z^2+3z+1)$$

b)

$$\int_{\gamma}\frac{3z^2+2z+\sin(z+1)}{(z-2)^2}\,dz \qquad \gamma=\{z\in\mathbb{C} : |z-2|=1\}$$

$g(z)=3z^2+2z+\sin(z+1)$ olomorfa in $\mathbb{C}$

$z_0=2 \quad n=1$

$$g'(2)=\frac{2}{2\pi i}\int_{\gamma}\frac{3z^2+2z+\sin(z+1)}{(z-2)^2}\,dz$$

$$\int_{\gamma}\frac{3z^2+2z+\sin(z+1)}{(z-2)^2}\,dz = \pi i\left(6z+2+\cos(z+1)\right)\Big|_{z=2} = \pi i(14+\cos(3))$$

3) $$\int_\gamma \frac{z(z+1)}{\sin(z+1)}\,dz \qquad \gamma: \{ z\in\mathbb{C} \mid |z|=3 \}$$

Singolarità: $z_0=-1$, $z_1=\pi-1$ contenute in γ.

$z_0=-1$ eliminabile: $$\lim_{z\to -1} f(z) = \frac{z(z+1)}{\sin(z+1)} = \frac{(z+1)+z}{\cos(z+1)} = -1$$

$z_1=\pi-1$ polo 1:

$$\lim_{z\to\pi-1} \frac{(z-\pi+1)(z^2+z)}{\sin(z+1)} = \lim_{z\to\pi-1} \frac{(z^2+z)+(2z+1)(z-\pi+1)}{\cos(z+1)} = -(\pi^2+1-2\pi+\pi-1) = \pi-\pi^2$$

1)

$$\int_0^{+\infty} \frac{x^2}{x^6+1}\,dx$$

L'integrale converge

$$f(z)=\frac{z^2}{z^6+1}$$

Poli: $\sqrt[6]{-1} = e^{\frac{\pi+2k\pi}{6} i}$

$z_0 = e^{\frac{\pi}{6} i}$ $z_1 = e^{\frac{\pi}{2} i}$ $z_2 = e^{\frac{5}{6}\pi i}$ $z_3 = e^{\frac{7}{6}\pi i}$

$z_4 = e^{\frac{3}{2}\pi i}$ $z_5 = e^{\frac{11}{6}\pi i}$

$z_K \mid I_m(z_K)>0 = z_0, z_1, z_2$

$\Gamma_R = [-R, R] \cup \gamma_R^+(0)$

$$\int_{-\infty}^{+\infty} f(x)\,dx = \lim_{R\to+\infty}\left(\int_{\Gamma_R} f(z)\,dz\right) = \sum_{I_m(z_k)>0} 2\pi i \operatorname{Res}(f(z), z_K)$$

$$\operatorname{Res}(f(z), z_0) = \left[\frac{z^2}{6z^5}\right]_{z=z_0} = \frac{1}{6(z_0)^3} = \frac{1}{6(e^{\frac{\pi}{2}})} = \frac{-i}{6}$$

$$\operatorname{Res}(f(z), z_1) = \frac{1}{6(e^{\frac{3}{2}\pi i})} = \frac{i}{6}$$

$$\operatorname{Res}(f(z), z_2) = \frac{1}{6(e^{\frac{5}{2}\pi i})} = -\frac{i}{6}$$

$$\int_{-\infty}^{+\infty} f(x)\,dx = 2\pi i\left(-\frac{i}{6}\right) = \frac{2\pi}{6} = \frac{\pi}{3}$$

3)

$$f(z) = \frac{z+1}{z-1}$$

$|z|<1$

$z+z^2+\ldots$

$1+z+z^2+\ldots$

$$f(z) = \frac{z}{z-1} + \frac{1}{z-1} = -\frac{z}{1-z} - \frac{1}{1-z} = -z\sum_{k=0}^{\infty} z^k - \sum_{k=0}^{\infty} z^k =$$

$$= 1 + 2\sum_{k=1}^{\infty} z^k$$

$|z|>1$:

$$f(z) = \frac{z}{z-1} + \frac{1}{z-1} = \frac{z}{z}\cdot\frac{1}{1-\frac{1}{z}} + \frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}} = \sum_{k=0}^{\infty}\left(\frac{1}{z}\right)^k + \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{1}{z}\right)^k =$$

$$\sum_{k=0}^{-\infty} z^k + \frac{1}{z}\sum_{k=0}^{-\infty} z^k = 1 + 2\sum_{k=-1}^{-\infty} z^k$$

3)

$$\int_0^{2\pi} \frac{2}{2+\cos\theta}\, d\theta$$

$z = e^{i\theta}$ $\qquad$ $z = e^{i\theta}$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \Rightarrow z = e^{i\theta} \Rightarrow \frac{z+\frac{1}{z}}{2}$$

Se θ varia da 0 a 2π, z varia su circonf di raggio 1

$$\int_{\gamma_1(0)} \frac{2}{2+\frac{z+\frac{1}{z}}{2}}\, dz$$

$$f(z) = \frac{4}{4+z+\frac{1}{z}} = \frac{4}{\frac{4z+z^2+1}{z}}$$

$z^2+4z+1=0$

$$z_{1_2} = \frac{-2\pm\sqrt{4-1}}{1} = -2\pm\sqrt{3}$$

$z_0=0$ é eliminabile:

$$\lim_{z\to 0} \frac{4z}{4z+z^2+1} = 0$$

$z_1 = -2+\sqrt{3}$ é polo di ordine 1

$$\lim_{z\to -2+\sqrt{3}} \frac{4z}{(z+2+\sqrt{3})} = \frac{-8+4\sqrt{3}}{2\sqrt{3}} = -\frac{4}{3}\sqrt{3}+2$$

$z_2 = -2-\sqrt{3}$ polo 1:

$$\lim_{z\to -2-\sqrt{3}} \frac{4z}{(z+2-\sqrt{3})} = \frac{-8-4\sqrt{3}}{-2\sqrt{3}} = +\frac{4\sqrt{3}}{3}+2$$

$$\int_0^{2\pi} \frac{2}{2+\cos\theta}\, d\theta$$

$$= \frac{2}{2+\frac{e^{i\theta}+\frac{1}{e^{i\theta}}}{2}} = \frac{4}{4+e^{i\theta}+\frac{1}{e^{i\theta}}} = \frac{4e^{i\theta}}{4e^{i\theta}+e^{2i\theta}+1}$$

$z = e^{i\theta} \qquad dz = i e^{i\theta}\, d\theta$

$$\int_0^{2\pi} \frac{2}{2+\cos\theta}\, d\theta = \int_{\gamma_1(0)} \frac{-4i}{4z+z^2+1}\, dz$$

$$f(z) = \frac{-4i}{z^2+4z+1} = \frac{-4i}{(z+2+\sqrt{3})(z+2-\sqrt{3})}$$

$z_0 = -2+\sqrt{3} \qquad z_1 = -2-\sqrt{3}$

$$\lim_{z\to 2+\sqrt{3}} \frac{-4i}{(z+2+\sqrt{3})} = \frac{-4i}{2\sqrt{3}} = -\frac{2}{\sqrt{3}}i$$

$$\int_{\gamma_1(0)} f(z)\, dz = 2\pi i\left(-\frac{2}{\sqrt{3}}i\right) = \frac{4\pi}{\sqrt{3}}$$

3)

$$\int_0^{2\pi} \frac{\sin^2\left(\frac{5\theta}{2}\right)}{\sin^2\left(\frac{\theta}{2}\right)}\, d\theta$$

$$\frac{e}{2i} \quad \frac{e}{2i} \to \frac{2e}{-4}$$

$$\sin^2\left(\frac{5}{2}\theta\right) = \left(\frac{e^{\frac{5}{2}i\theta} - e^{-\frac{5}{2}i\theta}}{2i}\right)^2 = \frac{e^{5i\theta}}{-4} + \frac{e^{-5i\theta}}{-4} + \frac{1}{2} = -\frac{e^{5i\theta}}{4} - \frac{e^{-5i\theta}}{4} + \frac{1}{2} = -\frac{1}{4}\left(e^{5i\theta} + e^{-5i\theta} - 2\right)$$

$$\sin^2\left(\frac{\theta}{2}\right) = \left(\frac{e^{\frac{\theta}{2}i} - e^{-i\frac{\theta}{2}}}{2i}\right) = \frac{e^{\theta i}}{-4} + \frac{e^{-\theta i}}{-4} + \frac{1}{2} = -\frac{1}{4}\left(e^{i\theta} + e^{-i\theta} - 2\right)$$

$$f(\theta) = \frac{e^{5i\theta} + e^{-5i\theta} - 2}{e^{i\theta} + e^{-i\theta} - 2} = \frac{e^{i\theta}\left(e^{5i\theta} + e^{-5i\theta} - 2\right)}{e^{2i\theta} - 2e^{i\theta} + 1} = \frac{\left(\frac{e^{10i\theta} - 2e^{5i\theta} + 1}{e^{5i\theta}}\right) e^{i\theta}}{e^{2i\theta} - 2e^{i\theta} + 1}$$

$$z = e^{i\theta} \qquad dz = i e^{i\theta} d\theta$$

$$-i$$

$$\int_0^{2\pi} \frac{\sin^2\left(\frac{5\theta}{2}\right)}{\sin^2\left(\frac{\theta}{2}\right)}\, d\theta = \int_{\gamma_1(0)} \frac{-i\left(z^{10} - 2z^5 + 1\right)}{\left(z^2 - 2z + 1\right) z^5}\, dz$$

$z = 1$ singolarità eliminabile: $z^{10} - 2z^5 + 1 = (z^5 - 1)(z^5 - 1) = (z-1)(z-1)P_8(z)$

$$\lim_{z \to 1} \frac{-i(z-1)^2 P_8(z)}{(z-1)^2 z^5} \neq 0$$

	1	0	0	0	0	-1
+1		1	1	1	1	1
	1	1	1	1	1	0

$z = 0$ Polo di ordine 5:

$$\lim_{z \to 0} \frac{-i\left(z^{10} - 2z^5 + 1\right)}{\left(z^2 - 2z + 1\right)} = -i$$

NOTA: $(z^5-1)(z^5-1) = (z-1)^2(z^4+z^3+z^2+z+1)^2$

$$\mathrm{Res}(f(z), 0) = \frac{1}{4!}\frac{d^4}{dz^4}\left(-i\left(z^4+z^3+z^2+z+1\right)^2\right)$$

$$= \frac{-i}{24}\left[(2z^4+2z^3+2z^2+2z+1)(4z^3+3z^2+2z+1)\right]'''$$

$$= \frac{-i}{24}\left[2z^3 + 4z^3 + 6z^3 + 4z^3\right]'''$$

$$= \frac{-i}{24}\left[2+4+6+4\right] = -\frac{16}{24}i = -\frac{2}{3}i$$

$$\int_0^{2\pi} \frac{\sin^2\left(\frac{5\theta}{2}\right)}{\sin^2\left(\frac{\theta}{2}\right)}\, d\theta = 2\pi i\left(-\frac{2}{3}i\right) = \frac{4}{3}\pi$$

1)

$$\int_\gamma \frac{s e^{\frac{1}{s}}}{(s-1)(s-3)}\, ds$$

$\gamma(t) = 6e^{it} \quad t \in [0, 2\pi]$

$s=0$ singolarità essenziale

$$\frac{1}{(z-1)(z-3)} = \frac{A}{z-1} + \frac{B}{z-3} = A(z-3) + B(z-1)$$

$Az - 3A + Bz - B$

$3A + B = 1 \qquad -2B = 1 \qquad B = -\frac{1}{2}$

$A + B = 0 \Rightarrow A = -B$

$$f(z) = \frac{z e^{\frac{1}{z}}}{2} \cdot \frac{1}{z-1} - \frac{z e^{\frac{1}{z}}}{2} \cdot \frac{1}{z-3}$$

$$= -\frac{1}{2} z e^{\frac{1}{z}} \cdot \sum_{k=0}^{\infty} z^k + \frac{z e^{\frac{1}{z}}}{6} \sum_{p=0}^{\infty} \left(\frac{z}{3}\right)^p$$

$$= -\frac{1}{2} e^{\frac{1}{z}} \sum_{k=0}^{\infty} z^{k+1} + \frac{e^{\frac{1}{z}}}{6} \sum_{p=0}^{\infty} \left(\frac{1}{3}\right)^p z^{p+1}$$

$$= -\frac{1}{2} \sum_{m=0}^{\infty} \frac{1}{m!} \cdot z^{-m}$$

Bro what 💀

1)

$$g(z) = \frac{\sin(z^2)}{z^3 - \sqrt{\frac{\pi}{2}}\, z^2} = \frac{\sin(z^2)}{z^2\left(z - \sqrt{\frac{\pi}{2}}\right)}$$

$z = \sqrt{\frac{\pi}{2}}$ singolarità di tipo polo 1:

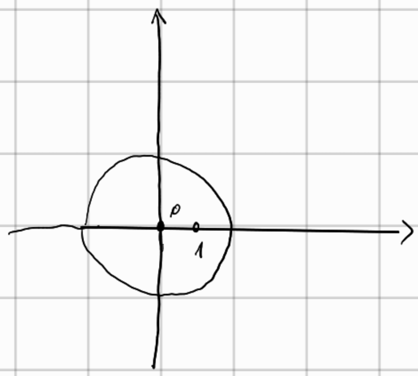
$$\lim_{z \to \sqrt{\frac{\pi}{2}}} \frac{\sin(z^2)}{z^2} = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}$$

$z = 0$ eliminabile:

$$\lim_{z \to 0} \frac{\sin(z^2)}{z^2\left(z - \sqrt{\frac{\pi}{2}}\right)} = -\frac{1}{\sqrt{\frac{\pi}{2}}}$$

2)

$$\int_{\gamma^+} \frac{3z+1}{z(z-1)^3}\, dz \qquad \gamma: \{z : |z| = 2\}$$



$z=0$ Polo 1:

$$\lim_{z \to 0} \frac{3z+1}{(z-1)^3} = -1 = \mathrm{Res}(f(z), 0)$$

$$\lim_{z \to 1} \frac{3z+1}{z} = 4 \quad \text{Polo } 3$$

$$\mathrm{Res}(f(z), 1) = \frac{1}{2!} \frac{d^2}{dz^2}\left(\frac{3z+1}{z}\right)_{z=1} = \frac{1}{2}\left[\frac{\cancel{3z} - \cancel{3z} - 1}{z^2}\right]' = \frac{1}{2}\left[+2z^{-3}\right]_{z=1} = +1$$

3)

$$\int_{-\infty}^{+\infty} \frac{\sin^2(\pi x)}{x^2+1}\,dx = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{1}{x^2+1}\,dx - \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\cos(2\pi x)}{x^2+1}\,dx$$

$$\int_{-\infty}^{+\infty} \frac{1}{x^2+1}\,dx =$$

$$\Gamma_R = [-R,R] \cup \gamma_R^+(0)$$

$$\underbrace{\int_{\Gamma_R} f(z)\,dz}_{R\to+\infty} = \underbrace{\int_{[-R,R]} f(z)\,dz}_{\text{Va a integrale}} + \underbrace{\int_{\gamma_R^+(0)} f(z)\,dz}_{\text{Va a } 0}$$

$$= 2\pi i \operatorname{Res}\left(f(z), i\right) = 2\pi i \cdot \left(\frac{1}{2i}\right) = \pi$$

$$\int_{-\infty}^{+\infty} \frac{\cos(2\pi x)}{x^2+1}\,dx = \operatorname{Re}\left(\int_{-\infty}^{+\infty} \frac{e^{2\pi i x}}{x^2+1}\,dx\right)$$

$$\Gamma_R = [-R,R] \cup \gamma_R^+(0) \qquad f(z) = \frac{e^{2\pi i z}}{z^2+1}$$

$$\int_{\Gamma_R} f(z)\,dz = \int_{[-R,R]} f(z)\,dz + \int_{\gamma_R^+(0)} f(z)\,dz$$

Se $R \to \infty$: $\int_{[-R,R]} f(z)\,dz = \int_{\mathbb{R}} \frac{e^{2\pi i x}}{x^2+1}\,dx$; $\int_{\gamma_R^+(0)} f(z)\,dz \to 0$ per lemma grande cerchio.

$$\int_{-\infty}^{+\infty} \frac{e^{2\pi i x}}{x^2+1}\,dx = 2\pi i \operatorname{Res}(f(z), i) = \frac{\pi}{e^{2\pi}} = \operatorname{Re}\left(\int_{-\infty}^{+\infty} \frac{e^{2\pi i x}}{x^2+1}\,dx\right)$$

$$\operatorname{Res}(f(z), i) = \lim_{z\to i} \frac{e^{2\pi i z}}{(z+i)} = \frac{e^{-2\pi}}{2i}$$

$$\int_{-\infty}^{+\infty} \frac{\sin^2(\pi x)}{x^2+1}\,dx = \pi - \frac{\pi}{e^{2\pi}} = \frac{e^{2\pi}\pi - \pi}{2e^{2\pi}} = \frac{\pi(e^{2\pi}-1)}{2e^{2\pi}}$$

4)

$$f(z) = \frac{\sin(2z)}{z(z-\pi)}$$

$z=0$ singolarità eliminabile

$$\lim_{z\to 0} \frac{\sin(2z)}{z(z-\pi)} = \lim_{z\to 0} \frac{2\cos(2z)}{2z-\pi} = -\frac{2}{\pi}$$

$$f(z) = \frac{\sin(2z-2\pi)}{z(z-\pi)}$$ $z=\pi$ sing. eliminabile

$$\lim_{z\to\pi} \frac{\sin(2z)}{z(z-\pi)} = \frac{2\cos(2z)}{2z-\pi} = \frac{2}{\pi}$$

4.1)

$$g(z) = (z+1)\sinh\left(\frac{1}{z}\right)$$

$$= z\sum_{k=0}^{\infty} \frac{1}{(2n+1)!}\left(\frac{1}{z}\right)^{2k+1} + \sum_{k=0}^{\infty} \frac{1}{(2n+1)!}\left(\frac{1}{z}\right)^{2k+1}$$

che ha infiniti termini negativi. Singolarità essenziale.

2)

$$\int_{-\infty}^{+\infty} \frac{\cos(2x) + \sin(-x)}{x^2+4}\,dx =$$

$$\int_{-\infty}^{+\infty} \frac{\cos(2x)}{x^2+4}\,dx + \int_{-\infty}^{+\infty} \frac{\sin(-x)}{x^2+4}\,dx$$

dispari: integrale è 0.

$$\parallel$$

$$= \mathrm{Re}\left(\int_{-\infty}^{+\infty} \frac{e^{2ix}}{x^2+4}\,dx\right)$$

$$\Gamma_R = [-R,R] \cup \gamma_R^+(0) \qquad f(z) = \frac{e^{2iz}}{z^2+4}$$

$$\int_{\Gamma_R} f(z)\,dz = \int_{[-R,R]} f(z)\,dz + \int_{\gamma_R^+(0)} f(z)\,dz$$

$$\downarrow \qquad\qquad \downarrow$$

$$\to \int_{\mathbb{R}} f(z)\,dz$$

lemma del grande cerchio $\to 0$

$$\int_{\Gamma_R} f(z)\,dz = 2\pi i\,\mathrm{Res}(f(z), 2i) = \frac{\pi}{2e^4} = \mathrm{Re}\left(\int_{-\infty}^{+\infty} \frac{e^{2ix}}{x^2+4}\,dx\right)$$

Polo 1: $\mathrm{Res}(f(z), 2i) = \lim_{z\to 2i} \frac{e^{2iz}}{(z+2i)} = \frac{e^{-4}}{4i}$

2)

$$\int_\gamma \frac{z^2+4}{z(z^2+1)}\,dz \qquad \gamma(t)=4e^{it} \quad t\in[0,2\pi)$$

Singolarità: $z_0=0$, $z_1=i$, $z_2=-i$

$$f(z)=\frac{z^2+4}{z(z^2+1)} \qquad z=0 \text{ eliminabile}$$

$$\lim_{z\to 0} \frac{z^2+4}{z^2+1}=4$$

$$\lim_{z\to i} \frac{z^2+4}{z(z+i)}=\frac{3}{i\cdot 2i}=-\frac{3}{2}$$

$$\lim_{z\to -i} \frac{z^2+4}{z(z-i)}=\frac{3}{2}$$

1)

$$u(x,y)=e^x \sin(y)$$

$\Delta u=0$?

$e^x\sin y-e^x\sin y=0$ ✓

Devo trovare V

$$\begin{cases} u_x=V_y \\ u_y=-V_x \end{cases} \qquad \nabla V(-u_y,\ u_x)=(-e^x\cos y,\ e^x\sin y)$$

$$V=\int -e^x\cos y\,dx=-e^x\cos y+c(y)$$

$$V_y=e^x\sin y+c'(y)$$

$V(x,y)=-e^x\cos y+c$ $\qquad f(x+iy)=e^x\sin y-i(e^x\cos y+c$

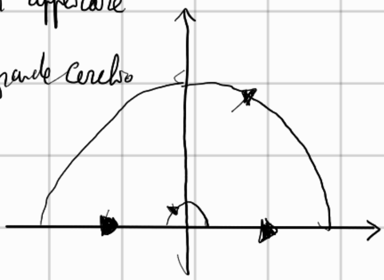
10)

$$\int_{-\infty}^{+\infty} \frac{1-\cos(2x)}{x^2}\,dx$$

$$\int_{-\infty}^{+\infty} \frac{1-\cos(2x)}{x^2}\,dx = \mathrm{Re}\left(\int_{-\infty}^{+\infty} \frac{1-e^{2ix}}{x^2}\,dx\right)$$

$$f(z) = \frac{1-e^{2iz}}{z^2} = \frac{e^{iz}(1-e^{2iz})}{e^{iz}z^2}$$

Per poter applicare piccolo e grande cerchio



$$\Gamma_{r,R} = [-R,-r] \cup \gamma_r^-(0) \cup [r,R] \cup \gamma_R^+(0)$$

$$\int_{\Gamma_{r,R}} f(z)\,dz = \int_{[-R,-r]} f(z)\,dz - \int_{\gamma_r^+(0)} f(z)\,dz + \int_{[r,R]} f(z)\,dz + \int_{\gamma_R^+(0)} f(z)\,dz$$

Se $R \to +\infty$:

$$g(z) = \frac{(1-e^{2iz})}{e^{iz}z^2}$$ Se $R \to +\infty$ $|g(z)| \to 0$? Esponenziale come va a ∞?

Vale lemma grande cerchio?

Valuto $z=0$ per verificare che sia un polo 1:

$$\lim_{z\to 0} \frac{1-e^{2iz}}{z} = \frac{-2iz}{z} = -2i \quad \checkmark \text{ Vale piccolo cerchio}$$

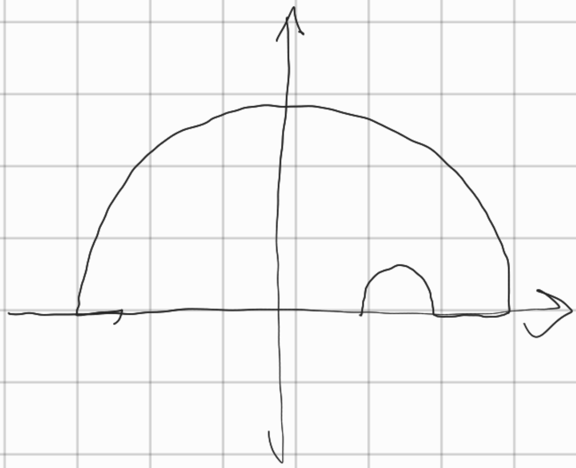
$$\int_{\gamma_r^+(0)} f(z)\,dz = \pi i(-2i) = 2\pi$$

Valuto:

$$\int_{\Gamma_{r,R}} f(z)\,dz = 0$$ perché non ho residui nel circuito

Se $r \to 0^+$ e $R \to +\infty$

$$2\pi = \mathrm{Re}\left(\int_{-\infty}^{+\infty} \frac{1-e^{2ix}}{x^2}\,dx\right)$$



1)

$$z=\sqrt[3]{i}$$

$$i = e^{\frac{\pi}{2} i}$$

$$\sqrt[3]{i} = e^{i\cdot\frac{\frac{\pi}{2}+2k\pi}{3}} \quad \begin{cases} e^{\frac{\pi}{6} i} \\ e^{\frac{5}{6}\pi i} \\ e^{\frac{9}{6}\pi i} = e^{\frac{3}{2}\pi i} \end{cases}$$

2)

$$z \neq i$$

$$\mathrm{Re}\left(\frac{z+i}{z-i}\right)=0$$

$$y^2-1$$

$$\frac{x+iy+i}{x+iy-i} = \frac{x+(y+1)i}{x+(y-1)i} = \frac{[x+(y+1)i][x-(y-1)i]}{x^2+(y-1)^2} = \frac{x^2-x(y-1)i+x(y+1)i+(y-1)(y+1)}{x^2+(y-1)^2}=0$$

$$\frac{x^2+y^2-1+i[\cancel{xy}+x-\cancel{xy}+x]}{x^2+(y-1)^2}=0$$

$$x^2+y^2=1 \qquad y\neq 1$$

3)

$$u(x,y)=e^{(x^2-y^2)}\cos(2xy)=$$

$$=e^{x^2}\cos(2xy)+e^{-y^2}\cos(2xy)$$

$$u_x = 2xe^{x^2}\cos(2xy)-e^{x^2}\sin(2xy)\cdot 2y - e^{-y^2}\sin(2xy)2y$$

$$u_y = -e^{x^2}\sin(2xy)\cdot 2x - 2ye^{-y^2}\cos(2xy) - e^{-y^2}\sin(2xy)2x$$

$$\begin{cases} u_x = V_y \\ u_y = -V_x \end{cases} \qquad \nabla V(-u_y,\ u_x)$$

$$V=\int e^{x^2}\sin(2xy)\cdot 2x+2ye^{-y}\cos(2xy)+e^{-y^2}\sin(2xy)2x\,dx$$

$$\cos(2xy) = \frac{e^{2ixy} + e^{-2ixy}}{2}$$

$$\frac{e^{x^2-y^2} \cdot e^{2ixy}}{2} + \frac{e^{x^2-y^2} \cdot e^{-2ixy}}{2} =$$

$$= \frac{1}{2} e^{(x^2-y^2+2ixy)} + \frac{1}{2} e^{(x^2-y^2-2ixy)} = \frac{1}{2} e^{(x+iy)^2} + \frac{1}{2} e^{(x-iy)^2}$$

$$u_x = (x+iy) e^{(x+iy)^2} + (x-iy) e^{(x-iy)^2}$$
$$u_y = i(x+iy) e^{(x+iy)^2} - i(x-iy) e^{(x-iy)^2}$$

$$V = \int i(x-iy)e^{(x-iy)^2} - i(x+iy)e^{(x+iy)^2} dx =$$

$$= i \cdot \frac{1}{2} e^{(x-iy)^2} - i \frac{1}{2} e^{(x+iy)^2} + c(y)$$

$$V_y = i(x-iy) \cdot (-i) e^{(x-iy)^2} - i(x+iy) i e^{(x+iy)^2} + c'(y)$$

$$c'(y) = 0 \qquad c(y) = c$$

$$V(x,y) = \frac{1}{2} i e^{(x-iy)^2} - \frac{1}{2} i e^{(x+iy)^2} + c$$

$$= \frac{1}{2} i \left(e^{x^2-y^2-2ixy} - e^{x^2-y^2+2ixy} \right) + c =$$

$$\frac{1}{2i} \left(e^{x^2-y^2} \left(e^{2ixy} - e^{-2ixy} \right) \right) + c \Rightarrow$$

$$V(x,y) = e^{x^2-y^2} \sin(2xy) + c$$

$$u_x = (x+iy)\,e^{(x+iy)^2} + (x-iy)\,e^{(x-iy)^2}$$

$$u_y = i(x+iy)\,e^{(x+iy)^2} - i(x-iy)\,e^{(x-iy)^2}$$

$$u_{xx} = e^{(x+iy)^2} + 2(x+iy)^2 e^{(x+iy)^2} + e^{(x-iy)^2} + 2(x-iy)^2 e^{(x-iy)^2}$$

$$u_{yy} = -e^{(x+iy)^2} - 2(x+iy)^2 e^{(x+iy)^2} - e^{(x-iy)^2} - 2(x-iy)^2 e^{(x-iy)^2}$$

4)

$$I_\gamma := \int_\gamma \frac{\cos(2z)}{z}\,dz$$

$f(z) = \dfrac{\cos(2z)}{z}$ $z_0 = 0$ polo di ordine 1:

$$\lim_{z\to 0} \cos(2z) = 1$$

Se γ é una curva il cui interno contiene 0; $n=0$ $z_0=0$ $g(z) = \cos(2z)$

$$\frac{1}{2\pi i}\int_\gamma \frac{\cos(2z)}{z}\,dz = g(0)$$

$$\int_\gamma \frac{\cos(2z)}{z}\,dz = 2\pi i$$

5)

$$f(z) = z\cos\left(\frac{1}{z}\right) \qquad z_0 = 0$$

$$f(z) = z\sum_{n=0}^{+\infty}(-1)^n\cdot\frac{(z)^{-2n}}{(2n)!} = \sum_{n=0}^{+\infty}(-1)^n\frac{(z)^{-2n+1}}{(2n)!} = \left(z - \frac{1}{2z} + \dots\right)$$

$$\operatorname{Res}(f(z), 0) = -\frac{1}{2}$$

6)

$$I = \int_\gamma \frac{\sin(z+1)}{z(z+1)}\,dz \qquad \gamma: \{|z| = 3\}$$

Singolarità: $z=0$, $z=-1$. Entrambe in interno di γ.

$z=-1$ sing. eliminabile:

$$\lim_{z\to-1}\frac{\sin(z+1)}{z(z+1)} = \lim_{z\to-1}\frac{\cos(z+1)}{2z+1} = -1$$

$z=0$ Polo 1:

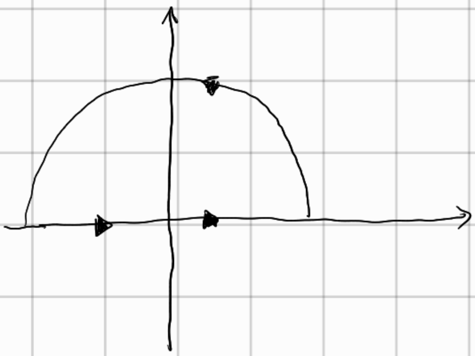
$$\lim_{z\to0} f(z)z = \sin(1)$$

7)

$$I = \int_0^{+\infty} \frac{x^2}{(x^2+4)^2(x^2+9)}\,dx$$

La funzione è dispari:

$$2I = \int_{\mathbb{R}} \frac{x^2}{(x^2+4)^2(x^2+9)}\,dx$$



Considero $f(z) = \frac{z^2}{(z^2+4)^2(z^2+9)}$

$\Gamma_R = [-R, R] \cup \gamma_R^+(0)$

$$\int_{\Gamma_R} f(z)\,dz = \int_{[-R,R]} f(z)\,dz + \int_{\gamma_R^+(0)} f(z)\,dz$$

Se $R \to +\infty$ $\qquad\qquad$ $\downarrow$ va a 0 per convergenza di f. razionali

$$\int_{\Gamma_R} f(z)\,dz = \int_{\mathbb{R}} f(x)\,dx$$

$$\int_{\Gamma_R} f(z)\,dz = 2\pi i \sum_{\operatorname{Im} z_k > 0} \operatorname{Res}(f(z), z_k)$$

$$(z^2+4)^2 = (z-2i)^2(z+2i)^2$$

TROVO SINGOLARITÀ:

$$f(z) = \frac{z^2}{(z^2+4)^2(z^2+9)}$$

$z_0 = 2i \quad z_1 = -2i \quad$ Poli di ordine 2.

$z_2 = 3i \quad z_3 = -3i \quad$ Poli di ordine 1.

$$\operatorname{Res}(f(z), z_2) = \lim_{z \to 3i} (z-3i)\, f(z) = \lim_{z \to 3i} \frac{z^2}{(z^2+4)^2(z+3i)} = \frac{-\cancel{9}\,3}{25 \cdot \cancel{6}_2 i} = \frac{3}{50} i$$

$$\operatorname{Res}(f(z), 2i) = \frac{d}{dz}\left[\frac{z^2}{(z+2i)^2(z^2+9)}\right]_{z=2i} =$$

$$= \left[\frac{2z\,(z+2i)^2(z^2+9) - z^2\left[2(z+2i)(z^2+9) + (z+2i)^2 \cdot 2z\right]}{(z+2i)^4(z^2+9)^2}\right]_{z=2i} =$$

$$= \left[\frac{4i(4i)^2(5) - (-4)\left[2(4i)(5) + (4i)^2 \cdot 4i\right]}{(4i)^4(5)^2}\right] =$$

$$-\frac{13}{200} + \frac{3}{50} = -\frac{1}{200}$$

$$= \frac{-320i + 4\left[40i - 64i\right]}{6400} = -\frac{13}{200} i$$

$$\int_{\Gamma_R} f(z)\,dz = 2\pi i \left[-\frac{13}{128} i + \frac{3}{50} i\right] = \frac{\pi}{100}$$

1)

$$g(z) = \cosh\left(3z - \frac{1}{2}\right) + z^2 + 1$$

$$g'(z) = \sinh\left(3z - \frac{1}{2}\right) \cdot 3 + 2z$$

2)

$$f(z) = (1+z)\sin(z)$$

$$f(\rho e^{i\theta}) = (1+\rho e^{i\theta})\sin(\rho e^{i\theta}) = \sin(\rho e^{i\theta}) + \rho e^{i\theta}\sin(\rho e^{i\theta})$$

$$\frac{\delta f}{\delta \rho} = e^{i\theta}\cos(\rho e^{i\theta}) + e^{i\theta}\sin(\rho e^{i\theta}) + \rho e^{i\theta}\cos(\rho e^{i\theta})e^{i\theta}$$

$$\frac{\delta f}{\delta \theta} = \cos(\rho e^{i\theta})i\rho e^{i\theta} + \rho i e^{i\theta}\sin(\rho e^{i\theta}) + \rho e^{i\theta}\cos(\rho e^{i\theta})\rho i e^{i\theta}$$

$$\frac{\delta f}{\delta \rho} = i\rho\frac{\delta f}{\delta \theta} \quad \checkmark$$

3)

$$\gamma: \begin{cases} x(t) = \cos(t) + 1 \\ y(t) = 2\sin(t) \end{cases} \qquad t \in \left[\frac{\pi}{2}, \pi\right]$$

$$\int_\gamma \frac{1+\sinh(2z-1)}{2}\,dz =$$

$$\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi}\left(1+\sinh(2\cos t + 2 + 4i\sin t - 1)\right)\cdot\left[-\sin t + 2i\cos t\right]dt =$$

$$= \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi}(-\sin t + 2i\cos t)\,dt + \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi}\left(\sinh(2\cos t + 4i\sin t + 1)\right)(-\sin t + 2i\cos t)\,dt =$$

$$= \frac{1}{2}\left[\cos t + 2i\sin t\right]_{\frac{\pi}{2}}^{\pi} + \frac{1}{4}\left[\cosh(2\cos t + 4i\sin t + 1)\right]_{\frac{\pi}{2}}^{\pi} =$$

$$= \frac{1}{2}\left[-1-2i\right] + \frac{1}{4}\left[\cosh(-z+1) - \cosh(4i-1)\right]$$

4)

$$\text{Res}(f(z), z_0) = 2\pi i\, C_{-1}$$

$$f(z) = 1 + (z-1)^2 \cosh\left(\frac{1}{z+1}\right)$$

$$f(z) = 1 + (z^2 - 2z + 1) \sum_{n=0}^{\infty} \left(\frac{1}{z+1}\right)^{2n} \cdot \frac{1}{(2n)!} =$$

$$1 + (z^2 - 2z + 1) \cdot \left[1 + \frac{1}{2} \cdot \frac{1}{z^2+2z+1} + \frac{1}{(z+1)^4} \cdot \frac{1}{4!} + \dots\right]$$

5)

$$\int_{\Gamma} \frac{z^2}{(z-\sqrt{10}\,i)(z-1)^2}\,dz =$$

$z_0=1$ è unica singolarità che sta nella

$g(z)=\dfrac{z^2}{z-\sqrt{10}\,i}$ olomorfa nell'interno di Γ.

$n=1 \qquad z_0=1$

$$g'(z_0)=\frac{1}{2\pi i}\int_{\Gamma}\frac{z^2}{(z-\sqrt{10}\,i)(z-1)^2}\,dz$$

$$g'(z)=\frac{2z(z-\sqrt{10}\,i)-z^2}{(z-\sqrt{10}\,i)^2}$$

$$g'(1)=\frac{2(1-\sqrt{10}\,i)-1}{(1-\sqrt{10}\,i)^2}=\frac{1-2\sqrt{10}\,i}{1-10-2\sqrt{10}\,i}=\frac{1-2\sqrt{10}\,i}{-9-2\sqrt{10}\,i}=\frac{2\sqrt{10}\,i-1}{9+2\sqrt{10}\,i}=\frac{(2\sqrt{10}\,i-1)(9-2\sqrt{10}\,i)}{81+40}=$$

$$\frac{(2\sqrt{10}\,i-1)(9-2\sqrt{10}\,i)}{81+40}=\frac{18\sqrt{10}\,i-9+40+2\sqrt{10}\,i}{121}=\frac{20\sqrt{10}\,i+31}{121}$$

$$\int_{\Gamma} f(z)\,dz = 2\pi i\cdot\frac{20\sqrt{10}\,i+31}{121}$$

6)

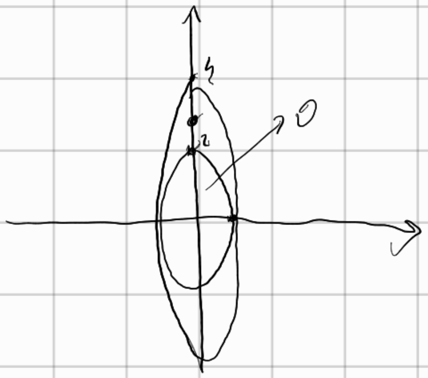
$$f(z) = \frac{2}{(z-\sqrt{\pi})^2 (z+2\pi)}$$

$z_0 = -2\pi$ $z_1 = \sqrt{\pi}$ polo 2.

polo 1

$$\operatorname{Res}(f(z), z_0) = \lim_{z \to -2\pi} \frac{2}{(z-\sqrt{\pi})^2} = \frac{2}{(2\pi+\sqrt{\pi})^2} = \frac{2}{4\pi^2-\pi^2+8\sqrt{\pi}\pi^2} = \frac{2}{3\pi^2+8\sqrt{\pi}\pi^2}$$

$$\operatorname{Res}(f(z), z_1) = \frac{d}{dz}\left[2(z+2\pi)^{-1}\right]_{z=\sqrt{\pi}} = \left[-2(z+2\pi)^{-2}\right] = \left[\frac{-2}{(\sqrt{\pi}+2\pi)^2}\right] = \frac{-2}{3\pi^2+8\sqrt{\pi}\pi^2}$$



7)

$$f(z) = \frac{\sin(2z)}{z(z-\pi)}$$

$z=0$ eliminabile $z=\pi$ eliminabile:

$$\lim_{z\to\pi} \frac{\sin(2z)}{z(z-\pi)} = \frac{2\cos(2z)}{(z-\pi)+z} = \frac{2}{\pi}$$

$$g(z) = (z+1)\sinh\left(\frac{1}{z}\right)$$

$z=0$ é essenziale:

$$g(z) = (z+1)\sum_{k=0}^{\infty} \frac{z^{-2n-1}}{(2n+1)!}$$ ho infiniti termini negativi

1)

$$g(z) = \cosh\left(3z - \frac{1}{2}\right) + z^2 + 1$$

$$\lim_{h \to 0} \frac{\cosh\left(3z + 3h - \frac{1}{2}\right) + \cancel{z^2} + h^2 + 2zh + \cancel{1} - \cosh\left(3z - \frac{1}{2}\right) - \cancel{z^2} - \cancel{1}}{h}$$

$$\lim_{h \to 0} \frac{\cosh\left(3z + 3h - \frac{1}{2}\right) - \cosh\left(3z - \frac{1}{2}\right)}{h} =$$

$$= \lim_{h \to 0} \frac{e^{3z+3h-\frac{1}{2}} + e^{-3z-3h+\frac{1}{2}} - e^{3z-\frac{1}{2}} - e^{-3z+\frac{1}{2}}}{2h} =$$

$$= \lim_{h \to 0} \frac{e^{3z-\frac{1}{2}}\left(e^{3h} - 1\right) + e^{-3z+\frac{1}{2}}\left(e^{-3h} - 1\right)}{2h} =$$

$$= \frac{3e^{3z-\frac{1}{2}}}{2} - \frac{3e^{-3z+\frac{1}{2}}}{2} = 3\sinh\left(3z - \frac{1}{2}\right) + 2z$$

3)

$$f(z) = \frac{1 - e^{i\pi z}}{z(z+i)(z-1)^2}$$

Sing: $z_0 = 0$ elimin.
$z_1 = -i$ polo 1
$z_2 = 1$ polo 2

1) $\lim_{z\to 0} \frac{1-e^{i\pi z}}{z(z+i)(z-1)^2} = \lim_{z\to 0} \frac{-i\pi z}{z(z+i)(z-1)^2} = \frac{-i\pi}{i} = -\pi$

2) $\lim_{z\to -i} \frac{1-e^{i\pi z}}{z(z-1)^2} = \frac{1-e^{\pi}}{-i(2i)} = \frac{1-e^{\pi}}{2}$

3) $\text{Res}(f(z), z_2) = \frac{d}{dz}\left(\frac{1-e^{i\pi z}}{z^2+iz}\right)_{z=1} = \frac{d}{dz}\left[\frac{-i\pi e^{i\pi z}(z^2+iz) - (2z+i)(1-e^{i\pi z})}{(z^2+iz)^2}\right]_{z=1} =$

$$= \frac{-i\pi e^{i\pi}(1+i) - (2+i)(1-e^{i\pi})}{2i} =$$

$$= \frac{-i\pi e^{i\pi} + \pi e^{i\pi} - (2 - 2e^{i\pi} + i - ie^{i\pi})}{2i} =$$

$$= \frac{-i\pi e^{i\pi} + \pi e^{i\pi} - 2 + 2e^{i\pi} - i + ie^{i\pi}}{2i} =$$

$$= \frac{ie^{i\pi} - i\pi e^{i\pi} + \pi e^{i\pi} + 2e^{i\pi} - 2 - i}{2i} =$$

$$= \frac{-i + i\pi - \pi - 2 - 2 - i}{2i} = \frac{-2i + i\pi - \pi - 4}{2i} = -1 + \frac{\pi}{2} + \frac{\pi}{2}i + 2i$$

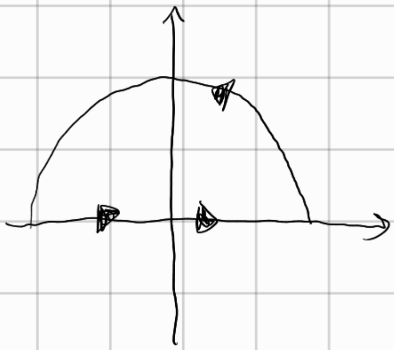
3)

$$I = \int_0^{+\infty} \frac{t^3 \sin t}{(1+t^2)(4+t^2)}\, dt$$

Se $t \to \pm\infty$, $f(t) \approx \frac{\sin t}{t} \Rightarrow$ integrabile.

$$2I = \int_{-\infty}^{+\infty} f(t)\, dt = \operatorname{Im}\left(\int_{-\infty}^{+\infty} \frac{t^3 e^{it}}{(1+t^2)(4+t^2)}\, dt \right)$$

$$f(z) = \frac{z^3 e^{iz}}{(1+z^2)(4+z^2)}$$



$$\Gamma_R = [-R, R] \cup \gamma_R^+(0)$$

$$\int_{\Gamma_R} f(z)\, dz = \int_{[-R,R]} f(x)\, dx + \underbrace{\int_{\gamma_R^+(0)} f(z)\, dz}_{\downarrow 0}$$

per lemma del grande cerchio

$$\int_{\Gamma_R} f(z)\, dz = 2\pi i \sum_{\operatorname{Im} z_k > 0} \operatorname{Res}(f(z), z_k)$$

$z_0 = i \quad z_1 = 2i$

$$\operatorname{Res}(f(z), i) = \lim_{z \to i} \frac{z^3 e^{iz}}{(z+i)(4+z^2)} = \frac{-i e^{-1}}{2i \cdot 3} = \frac{-e^{-1}}{6}$$

$$\operatorname{Res}(f(z), 2i) = \lim_{z \to 2i} \frac{z^3 e^{iz}}{(1+z^2)(z+2i)} = \frac{-8 i e^{-2}}{-3 \cdot 4i} = \frac{2}{3} e^{-2}$$

Ciao Come stai

io ! sto bene e tu?

$$\int_{\Gamma_R} f(z)\,dz = 2\pi i\left(-\frac{1}{6e} + \frac{2}{3e^2}\right) = 2\pi i\left(\frac{-e+4}{6e^2}\right) = 2\pi i\left(\frac{4-e}{6e^2}\right)$$

$$\int_{\mathbb{R}} f(x)\,dx = 2\pi\left(\frac{4-e}{6e^2}\right)$$

1)

$$\int_\gamma |z|\,dz$$

$\gamma_1(t) = t \quad t \in [0,4]$

$2x^2 = 16$

$x^2 = 8$

$x = 2\sqrt{2}$

$\gamma_2(t) = 4e^{it} \quad t \in [0, \frac{\pi}{4}]$

$-\gamma_3(t) = t + it \quad t \in [0, \sqrt{2}]$

$$\int_{\gamma_1} |z|\,dz = \int_0^4 t\,dt = \left[\frac{t^2}{3}\right]_0^4 = \frac{16}{3}$$

$$\int_{\gamma_2} |z|\,dz = \int_0^{\frac{\pi}{4}} 4 \cdot 4ie^{it}\,dt = \left[16e^{it}\right]_0^{\frac{\pi}{4}} = 16\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) - 16$$

$$-\int_{\gamma_3} |z|\,dz = -\int_0^{\sqrt{2}} \sqrt{2}\,t \cdot (1+i)\,dt = -\left[\frac{t^2}{2}\right]_0^{\sqrt{2}} (\sqrt{2} + i\sqrt{2}) = -(\sqrt{2} + i\sqrt{2}) = -\sqrt{2} - i\sqrt{2}$$

1)

$$f(z) = \frac{z}{1+z^3}$$

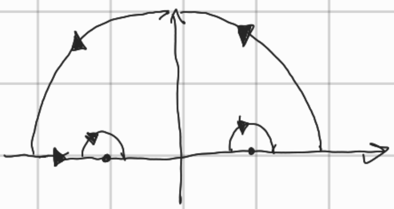
2)

$$\int_{-2}^{+\infty} \frac{1}{x^3+8}\, dx$$

$y = x - 2 \qquad dy = dx$

$(x+1)(x-1)$

$$\frac{1}{(z-1)}$$



$$\mathrm{Res}(f(z), -1) = \lim_{z \to -1} \frac{1}{z-1} = \frac{-1}{2}$$

$$\mathrm{Res}(f(z), 1) = \lim_{z \to 1} \frac{1}{z+1} = \frac{1}{2}$$

$$\frac{1}{z}$$

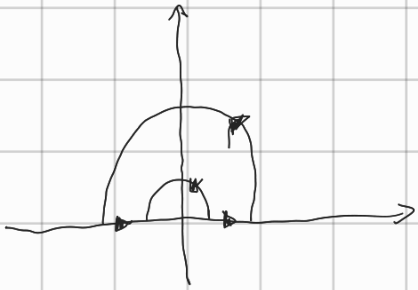
4)

$$\int_{-\infty}^{+\infty} \frac{\sin x}{x(1+x^2)}\,dx$$

$f(x)$ va a 0 come $\frac{1}{x^3}$ $\qquad \frac{|\sin x|}{|x|(1+x^2)} \le \frac{1}{|x|(1+x^2)}$ Converge

$\Gamma_R = [-R,-r] \cup \gamma_r^-(0) \cup [r,R] \cup \gamma_R^+(0)$

$$\int_{\Gamma_R} f(z)\,dz = \int_{[-R,-r]} f(z)\,dz + \int_{\gamma_R^+(0)} f(z)\,dz + \int_{[r,R]} f(z)\,dz - \int_{\gamma_r^+(0)} f(z)\,dz$$



$$f(z) = \frac{e^{iz}}{z(1+z^2)}$$

Posso applicare grande cerchio e va a 0 integrale

Integrale su $\gamma_r^+(0)$ risolto con piccolo cerchio.

$$\lim_{r\to 0^+} \int_{\gamma_r^+(0)} f(z)\,dz = \pi i \operatorname{Res}(f(z), 0) = \pi i$$

$$\operatorname{Res}(f(z), 0) = 1$$

$$\operatorname{Res}(f(z), i) = \frac{e^{-1}}{-2} = -\frac{1}{2e}$$

$$\int_{\Gamma_R} f(z)\,dz = 2\pi i\left(-\frac{1}{2e}\right) = -\frac{\pi i}{e}$$

$$-\frac{\pi i}{e} = \int_{\mathbb{R}} \frac{e^{ix}}{x(1+x^2)}\,dx - \pi i$$

$$\int_{\mathbb{R}} \frac{e^{ix}}{x(1+x^2)}\,dx = \pi i\left(1-\frac{1}{e}\right) = \pi i\left(\frac{e-1}{e}\right)$$

$$\int_{\mathbb{R}} \frac{\sin x}{x(1+x^2)}\,dx = \frac{\pi(e-1)}{e}$$