On some quantitative aspects of categorical Lagrangian topology

Giovanni Ambrosioni

CHAPTER 1

Algebraic preliminaries

1.1. A_∞-categories

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CHAPTER 2

Filtered Fukaya categories

2.1. Preliminaries

2.1.1. Monotone Lagrangians. Let (M, ω) be a closed symplectic manifold, fixed for the rest of the paper. Let L be a Lagrangian submanifold of M. The Maslov class of L induces a map

$$\mu : \pi_2(M, L) \to \mathbb{Z}.$$

We say that L is monotone if there is a positive constant $\tau > 0$ such that

$$\omega = \tau\mu,$$

where we see ω as a map on $\pi_2(M, L)$, and if $N_L \geq 2$ where N_L generates the image of μ in $\mathbb{Z}$. We refer to τ as the monotonicity constant of L.
We will denote the standard Novikov field over $\mathbb{Z}_2$ as

$$\Lambda := \left\{ \sum_{k=0}^{\infty} a_k T^{\lambda_k} : \ a_k \in \mathbb{Z}_2, \ \lambda_k \in \mathbb{R} \ \lim_{k\to\infty} \lambda_k = \infty \right\}$$

and by Λ_0 the positive Novikov ring over $\mathbb{Z}_2$, that is

$$\Lambda_0 := \left\{ \sum_{k=0}^{\infty} a_k T^{\lambda_k} : \ a_k \in \mathbb{Z}_2, \ \lambda_k \geq 0, \ \lim_{k\to\infty} \lambda_k = \infty \right\}$$

Let L be a monotone Lagrangian and assume in addition that it is closed. Then for a generic choice of almost complex structure J on M the count of J-holomorphic disks with boundary on L, Maslov index equal to 2 and passing through a generic point $p \in L$ weighted by symplectic area is well-defined, and independent from the choice of J and of the point $p \in L$. We denote this count by $\mathbf{d}_L \in \Lambda_0$ (see [Laz11]). Let $\mathbf{d} \in \Lambda_0$, then we define $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M, \omega)$, where the m stands for *monotone*, as the set of closed, connected and monotone Lagrangians L og M with $\mathbf{d}_L = \mathbf{d}$. Note that if $\mathbf{d} \neq 0$ all Lagrangians in $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M, \omega)$ share the same monotonicity constant.

2.1.2. Tuples of Lagrangians. In this subsection, we introduce many definitions and notations that will turn out to be very useful when defining our geometric structures in Section 2.2 and in Section 2.4.

Let $d \geq 1$ and pick a tuple $(L_0, \dots, L_d)$ of Lagranians in M. In the following, we will often denote such a tuple by $\vec{L}$.

Definition 2.1.2.1. We say that $\vec{L}$ is made of cyclically different Lagrangians (or, simply, is a cyclically different tuple) if $L_i \neq L_{i+1}$ for each $i \in \{0, \dots, d\}$[1], while we say that it is made of almost cyclically different Lagrangians (or, simply, is an almost cyclically different tuple) if $L_i \neq L_{i+1}$ for any $i \in \{0, \dots, d-1\}$ but $L_0 = L_d$.

Definition 2.1.2.2. Assume now that the tuple $\vec{L}$ is not cyclically different: each time there are consecutive indices $i, i+1, \dots, i+k$ indexing the same Lagrangian, we subtract k from all the indices bigger than $i+k+1$ and write the new tuple as $L_0, \dots, L_{i-1}, (\overline{L}_i, k), \overline{L}_{i+1}, \dots, \overline{L}_{d-k}$, with $L_{i-1} \neq \overline{L}_i \neq \overline{L}_{i+1}$ (we always work modulo $d+1$); we repeat this process until we get a tuple

$$\vec{L}_{\text{red}} := \big((\overline{L}_0, m_0), \dots, (\overline{L}_{d^R}, m_{d^R})\big)$$

of (d^R+1)-many cyclically different Lagrangians with multiplicities $m_i \geq 1$, which we will call the *reduced tuple* of $\vec{L}$.

Of course the number d^R satisfies $0 \leq d^R \leq d$. Notice that the multiplicity $m_0 \geq 1$ can be split as a sum $m_0 = m_0^b + m_0^e$ where $m_0^b \geq 1$ is the number of subsequent Lagrangians equal to L_0 at the beginning of the tuple $\vec{L}$, while $m_0^e \geq 0$ is the number of subsequent Lagrangians equal to L_0 at the end of the tuple. For notational convenience we will write $\overline{m_0} := m_0^b, \overline{m_i} := m_i$ for $i = 1, \dots, d^R$ and $\overline{m_{d^R+1}} := m_0^e$. In the following we will often omit multiplicities from the notation of the reduced tuple $\vec{L}_{\text{red}}$.

Definition 2.1.2.3. Given $\vec{L} = (L_0, \dots, L_d)$ we define a tuple $\vec{L^F} = (L_0^F, \dots, L_{d^F}^F)$ of length d^F+1, where $d^F \geq 0$, as follows: $L_0^F := L_0$ while $L_k^F := \overline{L_m}$, where

$$m := \min\big\{l \in \{k, \dots, d^R-1\} \ : \ \overline{L_l} \neq L_j^F \ \forall j \in \{0, \dots, k-1\}\big\}$$

until there is no Lagrangian left to index. We call $\vec{L^F}$ the *fundamental tuple* of $\vec{L}$.

Of course the number d^F satisfies $0 \leq d^F \leq d^R \leq d$. In words, given a tuple $\vec{L}$, the reduced tuple $\vec{L}_{\text{red}}$ is the tuple obtained by merging pairs of equal subsequent Lagrangians in $\vec{L}$, while the fundamental tuple $\vec{L^F}$ is obtained by keeping geometrically different Lagrangians only (for an example see the next Example and Figure 1).

Example 2.1.2.4. Let L_0, L_1, L_2, L_3, L_4 denote five different Lagrangians in (M, ω) and consider the tuple

$$\vec{L} = \big(L_0, L_0, L_1, L_1, L_3, L_3, L_3, L_2, L_4, L_1, L_3, L_2, L_0, L_0\big).$$

In this case $d = 13$. The corresponding reduced tuple is

$$\vec{L}_{\text{red}} = \big((L_0, 4), (L_1, 2), (L_3, 3), (L_2, 1), (L_4, 1), (L_1, 1), (L_3, 1), (L_2, 1)\big)$$

that is, $m_0^b = 2$ and $m_0^e = 2$. In particular

$$\overline{L}_0 = L_0,\ \overline{L}_1 = L_1,\ \overline{L}_2 = L_3,\ \overline{L}_3 = L_2,\ \overline{L}_4 = L_4,\ \overline{L}_5 = L_1,\ \overline{L}_6 = L_3, \overline{L}_7 = L_2.$$

[1]Here and in the following definitions like these have to be taken modulo d. In this case this means $L_d \neq L_{d+1} = L_0$.

The fundamental tuple of $\vec{L}$ is then

$$\vec{L}^F = (L_0, L_1, L_3, L_2, L_4)$$

that is

$$L_0^F = L_0,\ L_1^F = L_1,\ L_2^F = L_3,\ L_3^F = L_2,\ L_4^F = L_4.$$

In conclusion, in this case we have $d^F = 4 < d^R = 7 < d = 13$.

Consider a tuple $\vec{L} = (L_0, \dots, L_d)$ of Lagrangians in M. We will write

$$\pi_2(M, \vec{L}) := \pi_2\left(M, \bigcup_{i=0}^{d} L_i\right)$$

where π_2 denotes the second fundamental group. Notice that for any subtuple $(\tilde{L_0}, \dots, \tilde{L_k})$ of $\vec{L}$ of any length we have a map

$$\pi_2\left(M, \bigcup_{i=0}^{k} \tilde{L}_i\right) \to \pi_2(M, \vec{L})$$

induced by the inclusion $\cup\tilde{L}_i \to \cup L_i$. In the rest of the paper, we will see each possible $\pi_2(M, \cup\tilde{L}_i)$ as a subset of $\pi_2(M, \vec{L})$ and omit the inclusions above.

2.1.3. Trees. Let $d \geq 2$. We define, as in [Sei08], a d-leafed tree to be a properly embedded planar tree $T \subset \mathbb{R}^2$ with $d+1$ semi-infinite edges, which we call exterior edges, one of which is distinguished and called the root of T, denoted e_T, while the others are numbered clockwise starting from the root and denoted

$$e_0(T) = e_T, e_1(T), \dots, e_d(T)$$

The unique vertex attached to the root will be called the root vertex and denoted by v_T. For us, all leafed trees are oriented from the non-root leaves to the root. For $d = 1$, a leafed tree is just an infinite edge, oriented from leaf to root.

REMARK 2.1.3.1. Note that, as in [Sei08] but contrary to most of the literature, we do not consider leaves to be vertices.

Notice that a d-leafed tree cuts $\mathbb{R}^2$ in $d+1$ connected components, which we number clockwise, starting from the one nearest to the root when moving clockwise. We abuse notation and denote those connected components also by $e_0(T), \dots, e_d(T)$. Next we introduce a bunch of notations and definition for leafed trees.

DEFINITION 2.1.3.2. Let T be a d-leafed tree.

(1) We denote by $V(T)$ the set of its vertices, by $E(T)$ the set of its edges and by $E^{\text{int}}(T) \subset E(T)$ the subset of edges which are not exterior, and which we call interior.

(2) We write $|v|$ for the number of edges wich are attached to a vertex $v \in V(T)$, and call it the *valency* of v, $|v|_{\text{int}}$ for the number of interior edges attached to v and $|v|_e := |v| - |v|_{\text{int}}$ for the number of exterior edges attached to v. We denote by $V^i(T) \subset V(T)$ the subset of vertices of T having valency i.

(3) Assume that T has no vertices of valency equal to 1, then a vertex $v \in V(T)$ touches $|v|$ connected components of $\mathbb{R}^2 - T$: we number them clockwise starting from the one associated to the edge attached to v which is nearest to the root and denote them by

$$e_0(v), \ldots, e_{|v|-1}(v)$$

for any vertex $v \in V(T)$.

(4) T is called *stable* if the minimal valency of a vertex of T is 3, and it is called *binary* if each vertex has valency equal to 3. We denote by $\mathcal{T}^{d+1}$ the space of stable d-leafed trees, where two trees are identified if there exists an isomorphism of planar trees between them.

(5) A *flag* of T is a couple $(v, e) \in V(T) \times E(T)$ such that the edge e is attached to the vertex v. Given a vertex $v \in V(T)$ we denote by $f_0(v) \in E(T)$ the unique edge that exits from v (with respect to the orientation introduced above), and by $f_1(v), \ldots, f_{|v|-1}(v) \in E(T)$ the remaining edges attached to v, ordered in clockwise order starting from $f_0(v)$. Conversely, given an edge $e \in E(T)$ we define $t(e) \in V(T)$ as the start-vertex of e and by $h(e) \in V(T)$ the end-vertex of e (of course, one between h or t is not defined for exterior edges). We denote by $F(T)$ the set of flags of T and by $F^{\text{int}}(T) \subset F(T)$ the subset of flags made of interior edges.

(6) A metric on T is a map $\lambda : E(T) \to [0, \infty]$ such that

$$\lambda(e_i(T)) = \infty \text{ for any } i = 0, \ldots, d \text{ and } \lambda(e) < \infty \text{ for any } e \in E^{\text{int}}(T)$$

We call a couple (T, λ) a metric tree and we denote by $\lambda(T)$ the space of metrics of T. We also define the space $\overline{\lambda(T)}$ of maps $\lambda : E(T) \to [0, \infty]$ such that

$$\lambda(e_i(T)) = \infty \text{ for any } i = 0, \ldots, d$$

Let $k \in \mathbb{Z}$, then we define $\lambda^k(T) \subset \overline{\lambda(T)}$ to be the space of metrics with exactly k interior edges of infinite length. Note that if $T \in \mathcal{T}^{d+1}$ then $\lambda^k(T) = \emptyset$ for any $k > d - 1$.

(7) If $T' \subset T$ is a subtree of T and $\lambda \in \lambda(T)$ is a metric on T, then λ induces a metric on T', which we still denote by λ, and call (T', λ) a metric subtree of (T, λ), even if T' itself is not leafed or unstable. We denote by $\lambda(T')$ the space of metrics on T' in the sense above, and by $\overline{\lambda(T')}$ the obvious analogous of $\overline{\lambda(T)}$.

2.1.4. Trees with Lagrangian labels. Let $\vec{L} = (L_0, \ldots, L_d)$ be a tuple of Lagrangians in $\mathcal{L}\text{ag}^{(\text{mon},\mathbf{d})}(M, \omega)$ and consider a d-leafed tree T.

Definition 2.1.4.1. The assignement $L_i \mapsto e_i(T)$, where $e_i(T)$ is seen as the ith connected component of $\mathbb{R}^2 \setminus T$ in the sense of Section 2.1.3, is called a *labelling* by L for T. We denote by $\mathcal{T}^{d+1}(\vec{L})$ the space of *stable* d-leafed trees labelled by $\vec{L}$.

If $\vec{L^F} = L$, i.e $\vec{L}$ contains the same Lagrangian $d+1$ times, we simply write $\mathcal{T}^{d+1}(L)$ for $\mathcal{T}^{d+1}(\vec{L})$.
Let now $T \in \mathcal{T}^{d+1}(\vec{L})$ be a labelled tree. Any vertex $v \in V(T)$ naturally inherits a labelling $\vec{L}_v$ by the conventions above[2]. Any edge $e \in E(T)$ also inherits a labelling by a couple of Lagrangians in an obvious way.

Definition 2.1.4.2. An edge $e \in E(T)$ is said to be *unilabelled* if it is labelled by a couple of equal Lagrangians.

We introduce the following notations for various subsets of the set of edges $E(T)$ of T:

- we denote by $E_U(T) \subset E(T)$ the set of unilabelled edges of T,
- we denote by $E_F(T) := E(T) \setminus E_U(T)$ the set of non-unilabelled edges of T,
- for any $i \in \{0, \dots, d^F\}$, we denote by $E_i(T) \subset E(T)$ the set of edges of T that are (uni)labelled by the Lagrangian L_i^F (see Definition 2.1.2.3 for the definition of $\vec{L}_F$ and d^F),
- we set $E_U^{\text{int}}(T) := E_U(T) \cap E^{\text{int}}(T)$, $E_F^{\text{int}} := E_F(T) \cap E^{\text{int}}(T)$ and $E_i^{\text{int}}(T) := E_i(T) \cap E^{\text{int}}(T)$ for any $i \in \{0, \dots, d^F\}$.

Remark 2.1.4.3. The concept of unilabelled edge will be relevant for the following reason: as in [Sei08] in order to efficiently deal with Floer-curves in the definition of the Fukaya category in Section 2.2 we will use the language of trees; however, in our case those curves will be a combination of Floer-polygons (to use the terminology from [Sei08] (and trees of Morse trajectories, where the latter will be controlled precisely by trees of unilabelled edges.

Definition 2.1.4.4. We define $\mathcal{T}_U^{d+1}(\vec{L}) \subset \mathcal{T}^{d+1}(\vec{L})$ to be the set of stable trees labelled by $\vec{L}$ with only unilabelled interior edges, that is

$$\mathcal{T}_U^{d+1}(\vec{L}) := \left\{ T \in \mathcal{T}^{d+1}(\vec{L}) : \; E_U^{\text{int}}(T) = E^{\text{int}}(T) \right\}.$$

Given a labelled tree $T \in \mathcal{T}^{d+1}(\vec{L})$ we define T_{red} to be T with unilabelled edges removed.

Definition 2.1.4.5. A metric $\lambda \in \lambda(T)$ on a labelled tree $T \in \mathcal{T}^{d+1}(\vec{L})$ is said to be unilabelled if $\lambda(e) = 0$ for any non-unilabelled interior edge $e \in E^{\text{int}}(T)$. We denote by $\lambda_U(T)$ the space of unilabelled metrics on T, by $\overline{\lambda_U(T)}$ its closure in $\overline{\lambda(T)}$ and set $\lambda_U^k(T) := \lambda_U(T) \cap \lambda^k(T)$ for any k, where $\lambda^k(T) \subset \overline{\lambda(T)}$ is the subspace of metrics with exactly k interior edges with infinite length (see Definition 2.1.3.2).

The stability requirement in the definition of trees in the subset $\mathcal{T}_U^{d+1}(\vec{L})$ will be crucial in order to deal with breaking of Floer-clusters correctly in Section 2.2. Note that if $T \in \mathcal{T}_U^{d+1}(\vec{L})$

[2] That is, $L_i \in \vec{L}_v$ if and only if v touches the connected component $e_i(T)$.

is unilabelled, then $\lambda_U(T) = \lambda(T)$. In general, the subtree T_{red} may not be connected and its connected components may be unstable leafed trees. Notice that if T lies in $\mathcal{T}_U^{d+1}(\vec{L})$, then T_{red} obviously does not have any interior edges. Anyway, the fact that T is a planar connected tree implies that the numbering of leaves of T induces a unique numbering

$$e_0(T_{\text{red}}), \ldots, e_{d^R}(T_{\text{red}})$$

of the leaves of (the subtrees of) T_{red}, even if the latter is not connected (recall that the number d^R comes from the definition of the reduced tuple $\vec{L}_{\text{red}}$ of $\vec{L}$, see Definition 2.1.2.2). Assume now that $T \in \mathcal{T}_U^{d+1}(\vec{L})$, then the connected components (i.e. the vertices) of T_{red} induce a splitting

$$\{0, \ldots, d^R\} = \bigcup_{v \in V(T_{\text{red}})} \Lambda_v$$

where $i \in \Lambda_v$ if and only if $\overline{L_i} \in \vec{L_v}$. Note that the above union is not disjoint in general.
We have the following decomposition of $T \setminus T_{\text{red}}$ indexed by the fundamental tuple $\vec{L}_F$ of $\vec{L}$ (see Definition 2.1.2.3 for the definition of the fundamental tuple $\vec{L}_F$ of $\vec{L}$ and of the number d^F):

$$T \setminus T_{\text{red}} = \bigcup_{i=0}^{d^F} T_j^F$$

where for any $j \in \{0, \ldots, d^F\}$, T_j^F is the union of all the subtrees of $T \setminus T_{\text{red}}$ with edges unilabelled by L_j^F.

Definition 2.1.4.6. Given a labelled tree $T \in \mathcal{T}^{d+1}(\vec{L})$ we call the decomposition

$$T = T_{\text{red}} \cup \bigcup_{i=0}^{d^F} T_j^F$$

described above the *fundamental decomposition* of the labelled tree T. Moreover, we define T_{uni} as the union of the trees T_j^F from the fundamental decomposition of T.

Although T_{uni} is not a leafed tree in general, the planar structure of T induces the following ordering of edges of T_{uni} which are exterior edges of T: we denote by $e_j^i(T_{\text{uni}})$ as the jth edge (in clockwise order) of the subtree with label $\overline{L_i}$, that is

$$e_j^i(T_{\text{uni}}) := e_{\sum_{k=0}^{i-1} \overline{m_k} + j}(T)$$

(see page 6 for the definition of $\overline{m_k}$). In Figure 1 we sketched an example of a labelled tree and of its reduced tree.

Definition 2.1.4.7. We define $\lambda^{d+1}(\vec{L})$ as the space of metric trees (T, λ) (see Section 2.1.3), where $T \in \mathcal{T}^{d+1}(\vec{L})$ is a labelled tree and $\lambda \in \lambda(T)$, up to the relation that identifies identical metric trees, i.e. $(T_1, \lambda_1) \sim (T_2, \lambda_2)$ if and only if there is a planar isomorphism

$$\varphi : T_1 \setminus \{e \in E(T_1) : \ \lambda_1(e) = 0\} \to T_2 \setminus \{e \in E(T_2) : \ \lambda_2(e) = 0\}$$

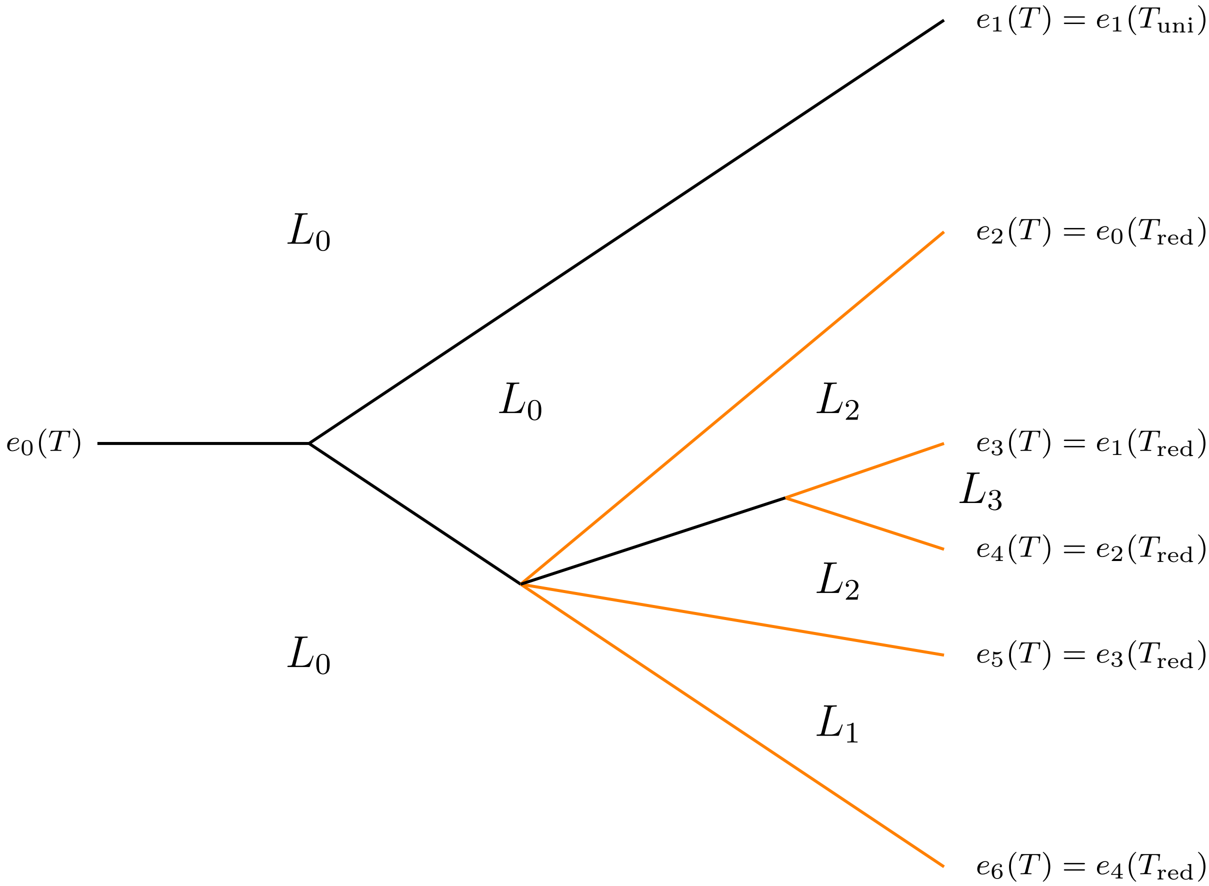


FIGURE 1. An element $T \in \mathcal{T}^7(\vec{L})$ for $\vec{L} = (L_0, L_0, L_2, L_3, L_2, L_1, L_0)$. We colored in orange the components of T_{red}. Note that in this case $\vec{L_{\mathrm{red}}} = ((L_0, 2+1), L_2, L_3, L_2, L_1)$ and $\vec{L^F} = (L_0, L_2, L_3, L_1)$.

such that $\lambda_2(\varphi(e)) = \lambda_1(e)$ for any remaining edge.

REMARK 2.1.4.8. Note that here we use the letter λ to denote a set of metric trees, and not only of metrics on a tree.

We will often write elements of $\lambda^{d+1}(\vec{L})$ simply as (T, λ). Note that $\lambda^{d+1}(\vec{T})$ has an obvious conformal structure. We define $\lambda_U^{d+1}(\vec{L}) \subset \lambda^{d+1}(\vec{L})$ as the subspace containing metric trees which can be represented by a tuple (T, λ) with $\lambda \in \lambda_U(T)$, or equivalently $T \in \mathcal{T}_U^{d+1}(\vec{L})$. Notice that the space $\lambda_U^{d+1}(\vec{L})$ may of course have boundary, depending on the nature of the tuple $\vec{L}$.

2.1.5. Systems of ends for trees with Lagrangian labels. We introduce the notion of system of ends (originally introduced in a different form in [Cha12]).

DEFINITION 2.1.5.1. Let $\vec{L} = (L_0, \dots, L_d)$ be a tuple of Lagrangians, $T \in \mathcal{T}^{d+1}(\vec{L})$ a labelled tree and $\lambda \in \lambda_U(T)$ a unilabelled metric on T.

- A system of ends for the metric tree (T, λ) is a map
$$s_{T,\lambda} : E(T) \to [0, \infty)$$
such that for any edge $e \in E(T)$ of T it satisfies $s_{T,\lambda}(e) < \frac{\lambda(e)}{2}$ if $\lambda(e) > 0$ and $s_{T,\lambda}(e) = 0$ otherwise.
- A system of ends for T is a map
$$s_T : E(T) \times \lambda_U(T) \to \mathbb{R}$$
such that $s_{T,\lambda} := s_T(\lambda)$ is a system of ends for (T, λ) and $s_T(e) : \lambda(T) \to \mathbb{R}$ is smooth for any $e \in E_U(T)$.
- A system of ends for $\vec{L}$ is a smooth map[3]
$$s : \lambda_U^{d+1}(\vec{L}) \to \mathbb{R}^{2d-1}$$
such that $s_{T,\lambda} := s(T, \lambda)$ is a system of ends for any $(T, \lambda) \in \lambda_U^{d+1}(\vec{L})$.
- A universal choice of system of ends is a choice of system of ends for any tuple $\vec{L}$ of Lagrangians of any length $d + 1$.

We recall that, given a tree T, in Section 2.1.3 we defined the space $\overline{\lambda(T)}$ by allowing metrics to take infinite value away from exterior edges. In the following, we will see an interior edge of T with infinite length as a 'breaking' of T into two leafed trees. We will now describe gluing of genuine labelled trees (that is, not endowed with metrics). Let $\vec{L_1}$ and $\vec{L_2}$ be tuples of Lagrangians of length $d_1 + 1$ and $d_2 + 1$ respectively.

DEFINITION 2.1.5.2. We say that the tuple $\vec{L_1}$ is compatible with the tuple $\vec{L_2}$ if one of the following (mutually exclusive) conditions hold:

(1) we have $m_0^e(\vec{L_1}) = 0$ (see page 6) and there is $i \in \{1, \dots, d_2\}$ such that
$$L_0^1 = L_i^2 \text{ and } L_{i+1}^2 = L_{d_1}^1,$$
(2) we have $m_0^e(\vec{L_1}) > 0$, that is in particular $L_0^1 = L_{d_1}^1$, and there is $i \in \{1, \dots, d_2\}$ such that
$$L_0^1 = L_i^2 = L_{i+1}^2 = L_{d_1}^1.$$

In both cases, we call such an $i \in \{1, \dots, d_2\}$ where the compatibility is verified, an admissible index.

REMARK 2.1.5.3. Note that formally the notion of compatibility is not symmetric.

[3]Here the map s is well-defined if we think of any element of λ_U^{d+1} as represented by a tree with $d - 2$ internal edges, which is possible by definition (indeed, recall that we are dealing with stable trees here). The fact that s lands in $\mathbb{R}^{2d-1}$ is due to the presence of the $d + 1$ exterior edges.

Assume that $\vec{L_1}$ is compatible with $\vec{L_2}$ and pick labelled trees $T_1 \in \mathcal{T}^{d_1+1}(\vec{L_1})$ and $T_2 \in \mathcal{T}^{d_2+1}(\vec{L_2})$. Let $i \in \{1,\dots,d_2\}$ be an admissible index, then we define the tree $T_1\#_i T_2$ as the tree obtained by gluing the root $e_0(T_1)$ of T_1 to the ith exterior edge $e_i(T_2)$ of T_2 g. We call $T_1\#_i T_2$ the tree obtained by gluing T_1 to T_2 at the admissible index i. Note that we have $T_1\#_i T_2 \in \mathcal{T}^{d_1+d_2+1}(\vec{L_1}\#_i\vec{L_2})$, where

$$\vec{L}^1\#_i\vec{L}^2 := \left(L_0^2,\dots,L_i^2,L_1^1,\dots,L_{d_1-1}^1,L_{i+1}^2,\dots,L_{d_2}^2\right).$$

Moreover, we can see $T_1 \setminus e_0(T_1)$ and $T_2 \setminus e_i(T_2)$ as subsets of $T_1\#_i T_2$ in an obvious way; the new edge resulting from the gluing we just described will be denoted by $e_g \in E(T_1\#_i T_2)$.

We next describe gluing of metrics on labelled trees. Let $\lambda_1 \in \lambda(T_1)$ and $\lambda_2 \in \lambda(T_2)$ and consider $\rho \in [-1,0)$. Suppose again that $\vec{L_1}$ is compatible with $\vec{L_2}$ and let $i \in \{1,\dots,d_2\}$ be an admissible index. Then we can define a metric

$$\gamma^{T_1,T_2;i}(\rho,\lambda_1,\lambda_2) \in \lambda(T_1\#_i T_2)$$

on the labelled tree $T_1\#_i T_2$ via

$$\gamma^{T_1,T_2;i}(\rho,\lambda_1,\lambda_2)|_{T_1\setminus e_0(T_1)} :\equiv \lambda_1,\ \ \gamma^{T_1,T_2;i}(\rho,\lambda_1,\lambda_2)|_{T_2\setminus e_i(T_2)} :\equiv \lambda_2 \text{ and } \gamma^{T_1,T_2;i}(\rho,\lambda_1,\lambda_2)(e_g) := -\ln(-\rho)$$

This way we defined for any admissible $i \in \{0,\dots,d_2\}$ a map

$$\gamma^{T_1,T_2;i} : [-1,0) \times \lambda(T_1) \times \lambda(T_2) \to \lambda(T_1\#_i T_2)$$

Note that such maps extend to maps

$$\overline{\gamma^{T_1,T_2;i}} : [-1,0] \times \lambda(T_1) \times \lambda(T_2) \to \overline{\lambda(T_1\#_i T_2)}$$

by declaring $\overline{\gamma^{T_1,T_2;i}}|_{0\times\lambda(T_1)\times\lambda(T_2)}$ to be the trivial gluing. This explains what we meant above by seeing interior edges of infinite length as 'broken edges'.
Let now $\vec{L}$ be a tuple of Lagrangians of length $d+1$. By considering all the decompositions of all trees $T \in \mathcal{T}^{d+1}(\vec{L})$ into two and more leafed trees and by packing all the maps of the form $\overline{\gamma^{T_1,T_2;i}}$ as above we get boundary charts for $\overline{\lambda^{d+1}(\vec{L})}$ by looking at 'small enough neighbourhoods of the trivial gluing', similarly to what is done for moduli spaces of punctured disks in [**Sei08**]. We skip the details of this construction. As it will be apparent from the constructions in Section 2.2 we are particularly interested in $\overline{\lambda_U^{d+1}(\vec{L})}$, that is the closure of $\lambda_U^{d+1}(\vec{L})$ in $\overline{\lambda^{d+1}(\vec{L})}$, which inherits conformal structure and boundary charts from $\overline{\lambda^{d+1}(\vec{L})}$.

We can now introduce the notion of consistency for system of ends.

Definition 2.1.5.4. A system of ends s on $\vec{L}$ is said to be consistent if it extends smoothly to a map $\overline{s}$ on $\overline{\lambda_U^{d+1}(\vec{L})}$.

Note that the fact that our trees are labelled is not of central importance for the notion of consistency. In particular, the following result can be proved by constructing an explicit system of ends as in [**Cha12**].

LEMMA 2.1.5.5. *Universal choices of consistent system of ends exist.*

PROOF. An example is constructed in [**Cha12**, Section 1.3]. □

Lastly, given a metric labelled tree $T \in \mathcal{T}^{d+1}(\vec{L})$ and a metric $\lambda \in \overline{\lambda(T)}$ on it, we will identify edges with intervals according to the metric and the orientation described above, that is:

- The non-root leaves will be identified with $(-\infty, 0]$, while the root with $[0, \infty)$;
- interior edges $e \in E^{\text{int}}(T)$ with $\lambda(e) < \infty$ will be identified with $[0, \lambda(e)]$;
- interior edges $e \in E^{\text{int}}(T)$ with $\lambda(e) = \infty$ will be identified with the disjoint union $[0, \infty) \sqcup (-\infty, 0]$.

Given $e \in E(T)$ we will often write the point t in the interval representation of e as $e(t)$ for notational convenience. Given a system of ends s on $\vec{L}$, we will abuse notation and often identiy $s_T(e) \in \mathbb{R}$ as an interval in the following way:

$$s_T(e) = \left[\frac{\lambda(e)}{2} - s_T(e), \frac{\lambda(e)}{2} + s_T(e)\right] \subset e$$

if e has finite length, $s_T(e) = [s_T(e), \infty) \subset e$ if e is the root $e_0(T)$ and $s_T(e) = (-\infty, -s_T(e)] \subset e$ if e is a non-root exterior edge.

2.2. A Morse-Bott model for $Fuk(X)$ and its weakly filtered structure

In this section, we construct a Morse-Bott model for the Fukaya category. Similar construction have appeared in [**CL, She11, Cha12**]. The idea of using such a model to construct filtered Fukaya categories already appeared in [**BCZ24b**]. We fix once and for all a closed and connected symplectic manifold (M, ω). Recall from Section 2.1.1 that given a positive Novikov serie $d \in \Lambda_0$ the set $\mathcal{L}\text{ag}^{(\text{mon},\mathbf{d})}(M, \omega)$ consists of closed, connected and monotone Lagrangians $L \subset M$ with $d_L := d$.

2.2.1. Source spaces: moduli spaces of clusters. Let $d \geq 2$. We recall the definition of moduli spaces of (configurations of) disks with d boundary marked points $\mathcal{R}^{d+1}$ following [**Sei08**, Chapter 9].

We denote by $D := \{z \in \mathbb{C} \ : \ |z| \leq 1\}$ and $\partial D := \{z \in \mathbb{C} \ : \ |z| = 1\}$ the unit disk and the unit circle in the complex plane. We define

$$\text{conf}_{d+1}(\partial D) \subset (\partial D)^{d+1}$$

as the space of ordered tuples of $d+1$ distinct points on ∂D. An element of $\text{conf}_{d+1}(\partial D)$ will be usually denoted as $(z_0, \ldots, z_d)$. We then define

$$\mathcal{R}^{d+1} := \frac{\text{conf}_{d+1}(\partial D)}{\text{Aut}(D)}$$

where $\text{Aut}(D)$ denotes the group of automorphisms of the unit disk D, which acts on $\text{conf}_{d+1}(\partial D)$ in the standard way. Recall that $\mathcal{R}^{d+1}$ is a smooth manifold of dimension $d-2$ and admits a compactification into a manifold with corners $\overline{\mathcal{R}^{d+1}}$ which realizes Stasheff's associahedron.

Let $\vec{L} = (L_0, \ldots, L_d)$ be a tuple of Lagrangians in $\mathcal{L}^{m,\mathbf{d}}(M, \omega)$.

DEFINITION 2.2.1.1. We denote by $\mathcal{R}^{d+1}(\vec{L})$ the space of disks $r \in \mathcal{R}^{d+1}$ equipped with the Lagrangian label $\vec{L}$, that is, for any $i = 0, \ldots, d$ we view the boundary arc of r between z_i and z_{i+1} as labelled by L_i. An element $r \in \mathcal{R}^{d+1}$ will be called a (labelled) disk configuration.

The difference between $\mathcal{R}^{d+1}(\vec{L})$ and $\mathcal{R}^{d+1}$ is hence purely formal, but the a-priori specification of Lagrangian labels will be of help when defining moduli spaces of clusters.

DEFINITION 2.2.1.2. Let $r \in \mathcal{R}^{d+1}(\vec{L})$ be a disk configuration: the marked point z_i will be called of type I if $L_{i-1} \neq L_i$, and of type II otherwise.

Over $\mathcal{R}^{d+1}(\vec{L})$ we have a bundle

$$\pi^{d+1}(\vec{L}) : \mathcal{S}^{d+1}(\vec{L}) \to \mathcal{R}^{d+1}(\vec{L})$$

where the fiber $S_r := \left(\pi^{d+1}(\vec{L})\right)^{-1}(r)$ over $r \in \mathcal{R}^{d+1}(\vec{L})$ is the equivalence class of punctured disk representing the configuration r, with punctures at points of type I and smooth marked points at points of type II (see [**Sei08**] for the formal definition of disk with punctures). We call the family $\left(\pi^{d+1}(\vec{L})\right)_{d,\vec{L}}$ over all tuples of Lagrangians in $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M, \omega)$ of any length a *universal family of disks.*

In the remaining of this subsection we will describe a partial compactification for $\pi^{d+1}(\vec{L})$, which requires the notion of strip-like ends. We briefly recall from [**Sei08**] what strip-like ends are. Let S be a punctured disk and let z be either a point on ∂S or a puncture (viewed in the compactification of S). A positive strip-like end for S at z is a proper holomorphic embedding $\epsilon : [0, \infty) \times [0, 1] \to S$ such that

$$\epsilon^{-1}(\partial S) = [0, \infty) \times \{0, 1\} \text{ and } \lim_{s \to \infty} \epsilon(s, \cdot) = z$$

A negative strip-like end is a strip-like end modeled on $(-\infty, 0]$. A choice of strip-like ends for $\pi^{d+1}(\vec{L})$ consists of proper embeddings $\epsilon_0 : \mathcal{R}^{d+1}(\vec{L}) \times [0, \infty) \times [0, 1] \to \mathcal{S}^{d+1}(\vec{L})$ and $\epsilon_i : \mathcal{R}^{d+1}(\vec{L}) \times (-\infty, 0] \times [0, 1] \to \mathcal{S}^{d+1}(\vec{L})$ for $i \in \{1, \ldots, d\}$ that restrict to strip-like ends on fibers at the associated $z_i(r)$ such that the images are pairwise disjoint. Given a choice $(\epsilon_0, \ldots, \epsilon_d)$ of strip-like ends on $\pi^{d+1}(\vec{L})$ we denote by $(\overline{\epsilon_1}, \ldots, \overline{\epsilon_{d^R}})$ the set of negative strip-like ends at the points z_i of type I (i.e. at punctures). We will often omit strip-like ends from the notation and identify half-strips as subsets of disks. Moreover, for any $c \geq 0$ we will write $|s| \geq c$ for the subset $[c, \infty] \times [0, 1]$ (resp. $(-\infty, -c] \times [0, 1]$) of the standard positive half-strip (resp. negative half-strip). There is a notion of consistency for a universal choice of strip-like ends on the universal family $\left(\pi^{d+1}(\vec{L})\right)$ which requires strip-like ends to be compatible with breaking and gluing of disks (see [**Sei08**, Section 9g]). We do not recall here the definition but require any of our choices of strip-like ends to be consistent.

Let $T \in \mathcal{T}^{d+1}(\vec{L})$ be a d-leafed tree labeled by $\vec{L}$. Recall (see Section 2.1.3) that elements of $\mathcal{T}^{d+1}(\vec{L})$ are assumed to be stable (that is, with no vertices of valency less than 3). A T-disk configuration is a tuple

$$\big((r_v)_{v\in V(T)}, (z_{h(e)}, z_{t(e)})_{e\in E^{\mathrm{int}}(T)}\big)$$

where $r_v \in \mathcal{R}^{|v|}(\vec{L_v})$ and $z_{h(e)}$ (resp. $z_{t(e)}$) is a distinguished point in the tuple $r_{h(e)}$ (resp. $r_{t(e)}$). We denote by $\mathcal{R}^T$ the space of T-disk configurations and we set

$$\overline{\mathcal{R}^{d+1}(\vec{L})} := \bigcup_{T\in\mathcal{T}^{d+1}(\vec{L})} \mathcal{R}^T.$$

The following result is Lemma 9.2 in [Sei08].

LEMMA 2.2.1.3. *The space $\overline{\mathcal{R}^{d+1}(\vec{L})}$ admits the structure of a smooth manifold with corners. Moreover, it realizes Stasheff's $(d-2)$-associahedron.*

The partial compatictification of π^{d+1} is

$$\overline{\pi^{d+1}(\vec{L})} : \overline{\mathcal{S}^{d+1}(\vec{L})} \to \overline{\mathcal{R}^{d+1}(\vec{L})}$$

where fibers are disjoint unions of nodal disks representing elements of the base.

We mimic an idea contained in [Cha12] in order to construct moduli spaces of cluster of punctured disks with marked points. Basically, what we do to define source spaces for the Floer maps defining the A_∞-maps of our Fukaya category is adding a collar neighbourhood to certain boundary components of $\overline{\mathcal{R}^{d+1}(\vec{L})}$. Recall from Section 2.1.4 that given a labelled tree $T \in \mathcal{T}^{d+1}(\vec{L})$ we defined the space

$$\lambda_U(T) = \{\lambda \in \lambda(T) \ : \ \lambda(e) = 0 \text{ for any non unilabelled } e \in E^{\mathrm{int}}(T)\}.$$

Moreover, we defined $\mathcal{T}_U^{d+1}(\vec{L}) \subset \mathcal{T}^{d+1}(\vec{L})$ as the subset of labelled trees with no non unilabelled interior edges. We define

$$\mathcal{R}_C^{d+1}(\vec{L}) := \bigcup_{T\in\mathcal{T}_U^{d+1}(\vec{L})} \mathcal{R}^T \times \lambda(T).$$

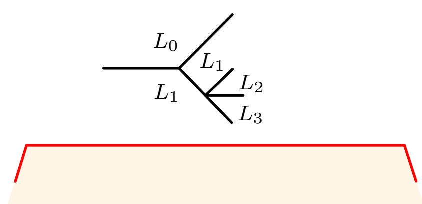
An element of $\mathcal{R}_C^{d+1}(\vec{L})$ will be usually denoted as (r, T, λ), where $r \in \mathcal{R}^T$ is a T-disk configuration and $\lambda \in \lambda(T)$ is a metric on the labelled tree T. Notice that $\mathcal{R}_C^{d+1}(\vec{L})$ is the interior of

$$\bigsqcup_{T\in\mathcal{T}^{d+1}(\vec{L})} \mathcal{R}^T \times \lambda_U(T)$$

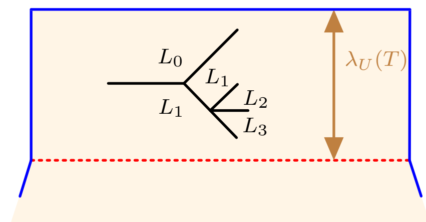
We also define

$$\overline{\mathcal{R}_C^{d+1}(\vec{L})} := \bigsqcup_{T\in\mathcal{T}^{d+1}(\vec{L})} \mathcal{R}^T \times \overline{\lambda_U(T)}.$$

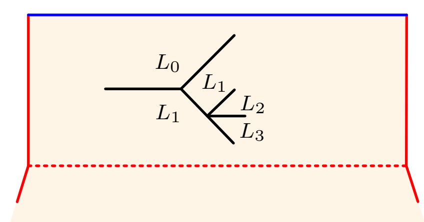
To help the reader understand the difference between the various moduli spaces defined above, we add a graphical description of the construction of part of the boundary of $\mathcal{R}_C^d(\vec{L})$ in the case $d = 4$ for a particular choice of tuple of Lagrangians in Figure 2.



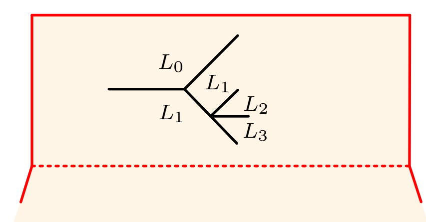
(A) A detail of a graphical description of the moduli space $\overline{\mathcal{R}^5(\vec{L})}$ (cfr. [**Sei08**]).



(B) A detail of a graphical description of the moduli space of clusters $\mathcal{R}^5_C(\vec{L})$. Note that this space is open. The dotted red line is there only to highlight where the boundary component of $\overline{\mathcal{R}^5(\vec{L})}$ was, before the addition of the $\lambda_U(T)$ factor (in brown).



(C) A detail of a graphical description of the moduli space $\sqcup_{T\in\mathcal{T}^5(\vec{L})}\mathcal{R}^T \times \lambda_U(T)$.



(D) A detail of a graphical description of the compactified moduli space od clusters $\overline{\mathcal{R}^5_C(\vec{L})}$.

FIGURE 2. A graphical description of the construction of the compactification $\overline{\mathcal{R}^5_C(\vec{L})}$ of the moduli space of cluster $\mathcal{R}^5_C(\vec{L})$ for $\vec{L} = (L_0, L_1, L_2, L_3, L_1)$ starting from $\overline{\mathcal{R}^5(\vec{L})}$ (cfr. the drawing on page 121 of [**Sei08**]). For simplicity, we zoomed only on the boundary component corresponding to the depicted tree, which has one interior edge unilabelled by the Lagrangian L_1. We adopted the convention that red lines correspond to closed and blue correspond to open.

We define the bundle of clusters disks labelled by $\vec{L}$

$$\pi_C^{d+1}(\vec{L}) : \mathcal{S}_C^{d+1}(\vec{L}) \to \mathcal{R}_C^{d+1}(\vec{L})$$

where the fiber $S_{r,T,\lambda} := (\pi_C^{d+1}(\vec{L}))^{-1}(r,T,\lambda)$ over an element $(r,T,\lambda) \in \mathcal{R}_C^{d+1}(\vec{L})$ is obtained by modifying $S_r \in \mathcal{S}^{d+1}(\vec{L})$ in the following way: any nodal point of S_r is replaced by a line segment of length $\lambda(e)$ oriented as e, while at any marked point (of type II) we attach a semi-infinite line segment $(-\infty, 0]$ or $[0,\infty)$ depending on weather the marked point is an entry or an exit.

To describe partial compactifications of the universal families $\pi_C^{d+1}(\vec{L})$, we introduce strip-like ends for clusters and then define gluing. Let $d \geq 2$ and $\vec{L} = (L_0, \ldots, L_d)$ be a tuple of Lagrangians in $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M,\omega)$. A choice of strip-like ends on $\pi_C^{d+1}(\vec{L})$ is a choice of strip-like ends on each $\pi^{|v|+1}(\vec{L_v})$ for any $T \in \mathcal{T}^{d+1}(\vec{L})$ and any $v \in V(T)$, which is smooth in the following sense: if $(r,T,\lambda) \in \mathcal{R}_C^{d+1}(\vec{L})$ is a configurations such that there is an interior edge $e \in E^{\mathrm{int}}(T)$ of T such that $\lambda(e) = 0$, then both marked points $z_{h(e)} \in S_{r,T,\lambda}(h(e))$ and $z_{t(e)} \in S_{r,T,\lambda}(t(e))$ do not lie in the image of a strip-like end. A universal choice of strip-like ends for clusters is a choice of strip-like ends on $\pi^{d+1}(\vec{L})$ for any tuple of Lagrangians in $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M,\omega)$ of any finite length $d+1$. We will always assume that our universal choices of strip-like ends for clusters are consistent, that is, vertex-wise consistent.

We fix once and for all a consistent universal choice of system of ends as well as a consistent universal choice of strip-like ends for clusters on $\mathcal{L}^{m,\mathbf{d}}(M,\omega)$. We can now adapt the gluing procedure for punctured disks described in [Sei08, Section (9e)] to the case of clusters of disks. The aim of the following is not really to precisely describe gluing, but more to set up the necessary notation for later stages in this paper.
Let $d_1, d_2 \geq 2$ and consider two tuples

$$\vec{L^1} = (L_0^1, \ldots, L_{d_1}^1) \text{ and } \vec{L^2} = (L_0^2, \ldots, L_{d_2}^2)$$

of Lagrangians in $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M,\omega)$, whose reduced tuples we denote by $\vec{L^1}_{\mathrm{red}}$ and $\vec{L^2}_{\mathrm{red}}$. Consider

$$(r_1, T_1, \lambda_1) \in \mathcal{R}^{d_1+1}(\vec{L^1}) \text{ and } (r_2, T_2, \lambda_2) \in \mathcal{R}^{d_2+1}(\vec{L^2})$$

and let $\rho \in [-1, 0)$. As in the case of gluing of labelled trees in Section 2.1.5, in order to define gluing of clusters we consider the cases $m_0^e(\vec{L^1}) = 0$ and $m_0^e(\vec{L^1}) > 0$ (see page 6 for the definition of m_0^e) separately:

(1) First, we assume $m_0^e(\vec{L^1}) = 0$, that is $L_0^1 \neq L_0^d$. Consider an admissible (see Definition 2.1.5.2) index $i \in \{1, \ldots, d_2\}$ and denote by $j \in \{1, \ldots, d_2^R\}$ the index such that $L_i^2 = \overline{L_j^2}$ (that is, the position of L_i^2 in the reduced tuple $\vec{L^2}_{\mathrm{red}}$, see page 6). The gluing of (r_1, T_1, λ_1) with (r_2, T_2, λ_2) at i with length ρ is defined as the tuple

$$\big(r, T_1 \#_i T_2, \gamma^{T_1, T_1; i}[-1, \lambda_1, \lambda_2]\big) \in \mathcal{R}^{d_1+d_2+1}(\vec{L_1} \#_i \vec{L^2})$$

where:

- $T_1\#_i T_2$ is the tree obtained by gluing the root of T_1 to the ith exterior edge of T_2 (see page 12),
- $\gamma^{T_1,T_1,i}$ is the gluing map defined in Section 2.1.4 and
- $r \in \mathcal{R}^{T_1\#_i T_2}$ is defined as follows: the map $\gamma^{T_1,t_2,i}$ applied with length -1 identifies (as it produces an edge with vanishing length by definition) the unique vertex v_1 of T_1 attached to the root $e_0(T_1)$ with the unique vertex v_2 of T_2 attached to the exterior edge $e_j((T_2)_{\mathrm{red}})$[4] to produce a vertex v_g of $T_1\#_i T_2$; the configuration r is the cluster configuration that:
  - agrees with r_1 on $T_1 \setminus v_1 \subset T_1\#_i T_2$,
  - agrees with r_2 on $T_2 \setminus v_2 \subset T_1\#_i T_2$,
  - and on v_g represents the puntured disks obtained by gluing the exit of $S_{r_1,T_1,\lambda_1}(v_1)$ to the ith entry of $S_{r_2,T_2,\lambda_2}(v_2)$ with gluing length ρ (see [Sei08, Section (9e)]).

(2) Assume now $m_0^e(\vec{L^1}) > 0$, that is in particular $L_0^1 = L_{d_1}^1$, and consider indices i and j as above. The gluing of (r_1, T_1, λ_1) with (r_2, T_2, λ_2) at i with length ρ is defined as the tuple
$$\left(r, T_1\#_i T_2, \gamma^{T_1,T_1,i}(-\ln(-\rho), \lambda_1, \lambda_2)\right)$$
where $T_1\#_i T_2$ and $\gamma^{T_1,T_1,i}$ are as above and r agrees with r_1 on $T_1 \setminus e_0(T_1) \subset T_1\#_i T_2$ and agrees with r_2 on $T_2 \setminus e_0(T_2) \subset T_1\#_i T_2$.

In summary, we defined maps
$$\gamma^{\vec{L^1},\vec{L^2},i} : [-1.0) \times \mathcal{R}^{d_1+1}(\vec{L^1}) \times \mathcal{R}^{d_2+1}(\vec{L^2}) \to \mathcal{R}_C^{d_1+d_2}(\vec{L^1}\#_i\vec{L^2})$$
for any tuples of Lagrangians as above and any admissible index $i \in \{1, \ldots, d_2\}$. It is easy to see that those maps extend to maps
$$\overline{\gamma^{\vec{L^1},\vec{L^2},i}} : [-1,0] \times \mathcal{R}^{d_1+1}(\vec{L^1}) \times \mathcal{R}^{d_2+1}(\vec{L^2}) \to \overline{\mathcal{R}_C^{d_1+d_2}(\vec{L^1}\#_i\vec{L^2})}$$
by trivial gluing.

Let now $\vec{L} := (L_0, \ldots, L_d)$ be a tuple of Lagrangians in $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M,\omega)$, and pick a labelled tree $T \in \mathcal{T}^{d+1}(\vec{L})$ and a number $k \in \{0, \ldots, d-2\}$. We define a gluing map
$$\gamma^{T,k} : [-1,0)^{|E_F^{\mathrm{int}}(T)|+k} \times \mathcal{R}^T \times \lambda_U^k(T) \to \mathcal{R}_C^{d+1}(\vec{L})$$
where $E_F^{\mathrm{int}}(T) = E(T) \setminus E_U(T)$ is the set of non-unilabelled edges of T and $\lambda_U^k(T)$ is the set of unilabelled metrics on T such that exactly k interior edges have infinite length (both concepts were introduced on page 8), by composing various maps of the form $\gamma^{\vec{L^1},\vec{L^2},i}$ for $k-1$ admissible tuples of Lagrangian decomposing $\vec{L}$. For instance, if $T \in \mathcal{T}_U^{d+1}(\vec{L})$ is unilabelled, then there

[4]Another characterization of v_1 and v_2 using the partition $(\Lambda_v)_v$ introduced on page 10 is the following: $v_1 \in V((T_1)_{\mathrm{red}})$ is the unique vertex such that $0, d_1^R \in \Lambda_{v_1}$ and $v_2 \in V((T_2)_{\mathrm{red}})$ is the unique vertex such that $j, j+1 \in \Lambda_{v_2}$.

are tuples $\vec{L_1}$, of length d_1+1, and $\vec{L^2}$, of length d_2+1, of Lagrangians in $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M,\omega)$ such that $d_1+d_2=d+2$ and an index $i\in\{0,\dots,d_2\}$ such that $\gamma^{T,1}=\gamma^{\vec{L_0},\vec{L_1},i}$.
Note that the maps $\gamma^{T,k}$ are well-defined since we are working with consistent choices of strip-like end and of system of ends, and the two are independent concepts. Each of these maps extends to maps of the form

$$\overline{\gamma^{T,k}}:[-1,0]^{|E_F^{\mathrm{int}}(T)|+k}\times\mathcal{R}^T\times\lambda_U^k(T)\to\overline{\mathcal{R}_C^{d+1}(\vec{L})}$$

again by trivial gluing.

By obvious considerations and repeatedly applying [**Sei08**, Lemma 9.2] we see that for any $(d+1)$-tuple $\vec{L}$ of Lagrangians in $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M,\omega)$ there are maps $\overline{\gamma^{\vec{L^1},\vec{L^2},i}}$ whose restriction to small enough neighbourhood of the trivial gluing in the domain define boundary charts for $\mathcal{R}_C^{d+1}(\vec{L})$.

LEMMA 2.2.1.4. *The space $\mathcal{R}_C^{d+1}(\vec{L})$ admits the structure of a smooth manifold of dimension $d-2$. Moreover, $\overline{\mathcal{R}_C^{d+1}(\vec{L})}$ admits the structure of a smooth manifold with corners which realizes Stasheff's associahedron.*

Notice that the fact that our moduli spaces of clusters $\mathcal{R}_C^{d+1}(\vec{L})$ realize the associahedra comes for free from the fact that to define them we just added some collar neighbourhoods to the compactified moduli spaces of punctured disks with Lagrangian labels $\overline{\mathcal{R}^{d+1}(\vec{L})}$. We remark that for $T\in\mathcal{T}^{d+1}(\vec{L})$ and $k\in\{0,\dots,d-2\}$, the subspace $\mathcal{R}^T\times\lambda^k(T)\subset\overline{\mathcal{R}_C^{d+1}(\vec{L})}$ is a boundary component of codimension $|E_F(T)|+k$ (where $E_F(T)$ denotes the subset of non unilabelled edges of T).

We now introduce the following piece of notation which will be useful later on. Given $(r,T,\lambda)\in\mathcal{R}_C^{d+1}(\vec{L})$ we define

$$\partial S_{r,T,\lambda}=\partial_F S_{r,T,\lambda}\cup\partial_M S_{r,T,\lambda}$$

where $\partial_F S_{r,T,\lambda}$ is the union of the boundaries of the disk components of $S_{r,T,\lambda}$, while $\partial_M S_{r,T,\lambda}$ is the union of all the line components in $S_{r,T,\lambda}$. Moreover, we denote by $S_{r,T,\lambda}(v)$ the disk with punctures and marked points corresponding to the vertex $v\in V(T)$, define

$$S^p_{r,T,\lambda}:=\bigsqcup_{v\in V(T)}S_{r,T,\lambda}(v)$$

and for any $i\in\{0,\dots,d^R\}$ we denote by $\partial_i S_{r,T,\lambda}(v)$ to be the ith boundary component of $S_{r,T,\lambda}(v)$, that is the one corresponding to the boundary component $e_i(v)$ (see Section 2.1.3), and hence labelled by $\overline{L_i^v}$ (here we are again differentiating between $\overline{L_i}$ and $\overline{L_j}$ for $i\neq j$ even if they agree as Lagrangians).
A universal choice of strip-like ends for $\pi_C^{d+1}(\vec{L})$ is the pullback from $\overline{\pi^{d+1}(\vec{L})}$ of the compactification of a universal and consistent choice of strip-like ends on $\pi^{d+1}(\vec{L})$. Hence, consistency of strip-like ends for clusters comes for free by definition. Given a choice of strip-like ends on

$S_{r,T,\lambda}$ we number the strip-like ends following the numbering of the exterior edges of T_{red} and denote them by $\epsilon_0, \ldots, \epsilon_{d^R}$ or $\epsilon_1, \ldots, \epsilon_{d^R}$ depending on weather or not T_{red} contains the root or not (i.e. weather $L_0 \neq L_d$ or $L_0 = L_d$).
$\pi_C^{d+1}(\vec{L})$ admits a partial compactification

$$\overline{\pi_C^{d+1}(\vec{L})} : \overline{\mathcal{S}_C^{d+1}(\vec{L})} \to \overline{\mathcal{R}_C^{d+1}(\vec{L})}$$

defined by allowing line segments between disks to have infinite length. Those edges will be identified with broken edges following the conventions defined in Section 2.1.5.

2.2.2. Perturbation data for clusters. We define the concept of perturbation data in our cluster-setup. Fix coherent choices of strip-like ends and of system of ends $(s_T)_T$ (see page 11). Let (L_0, L_1) be a couple of Lagrangians in $\mathcal{L}^{m,\mathbf{d}}(M,\omega)$.

Definition 2.2.2.1. A Floer datum for (L_0, L_1) consists of:

- if $L_0 = L_1 = L$: a triple (f^L, g^L, J^L) where (f^L, g^L) is a Morse-Smale pair on L such that f^L has a unique maximum and J^L is an ω-compatible almost complex structure on M;
- if $L_0 \neq L_1$: a couple $(H^{L_0,L_1}, J^{L_0,L_1})$ where $H : M \times [0,1] \to \mathbb{R}$ is a time dependent Hamiltonian on M such that

$$\varphi_1^{L_0,L_1}(L_0) \pitchfork L_1$$

  where $\varphi_t^{L_0,L_1}$ is the flow of the Hamiltonian vector field generated by H^{L_0,L_1}, and J is a $[0,1]$-family of ω-compatible almost complex structures on M which equals J^{L_0} near 0 and J^{L_1} near 1.

Given $L \in \mathcal{L}\text{ag}^{(\text{mon},\mathbf{d})}(M,\omega)$ we denote by $\text{Crit}(f^L)$ the (finite) set of critical points of a Morse function $f^L : L \to \mathbb{R}$ on L, and given $x \in \text{Crit}(f^L)$ we denote by $|x|$ its Morse index, by $W^u(x)$ its unstable manifold and by $W^s(x)$ its stable manifold. Given a couple (L_0, L_1) of different Lagrangians in $\mathcal{L}\text{ag}^{(\text{mon},\mathbf{d})}(M,\omega)$ with a choice H^{L_0,L_1} of Hamiltonian Floer datum we denote by $\mathcal{O}(H^{L_0,L_1})$ the (finite) set of orbits $\gamma : [0,1] \to M$ of the Hamiltonian vector field of H^{L_0,L_1} such that

$$\gamma(0) \in L_0 \text{ and } \gamma(1) \in L_1.$$

Fix a choice of Floer data for any couple of Lagrangians in $\mathcal{L}^{m,\mathbf{d}}(M,\omega)$. In order to state the following definition in an easier way, we arbitrarily set $H^{L,L} := 0$ and $J^{L,L} := J^L$ for any $L \in \mathcal{L}\text{ag}^{(\text{mon},\mathbf{d})}(M,\omega)$ and keep this convention for the whole paper. Let $\vec{L} = (L_0, \ldots, L_d)$ be a tuple of Lagrangians in $\mathcal{L}^{m,\mathbf{d}}(M,\omega)$ and let $(r,T,\lambda) \in \mathcal{R}_C^{d+1}(\vec{L})$ be a cluster configuration.

Definition 2.2.2.2. A perturbation datum for $\vec{L}$ on the cluster $S_{r,T,\lambda}$ consists of the following data:

- For any $i = 0, \dots, d^F$ (see page 6 for the definition of d^F) a choice of tuple
$$\left(f^{r,T,\lambda}_{(T^F_i,\lambda_i)}, g^{r,T,\lambda}_{(T^F_i,\lambda_i)}\right)$$
where the (T^F_i, λ_i)'s are the metric subtrees of T coming from the fundamental decomposition of T (see Definition 2.1.4) and:
  - $f^{r,T,\lambda}_{(T^F_i,\lambda_i)} : T^F_i \times \overline{L_i} \to \mathbb{R}$ is a map smooth on edges and continuous on vertices such that
$$f^{r,T,\lambda}_{(T^F_i,\lambda_i)}(\tau) : \overline{L_i} \to \mathbb{R}$$
is Morse for any[5] $\tau \in T^F_i$ and
$$f^{r,T,\lambda}_{(T^F_i,\lambda_i)}(\tau) = f^{\overline{L_i}}$$
for any $\tau \in s_{T^F_i}$;
  - $g^{r,T,\lambda}_{(T^F_i,\lambda_i)}$ assigns to any $\tau \in T^F_i$ a Riemannian metric $g^{r,T,\lambda}_{(T^F_i,\lambda_i)}(\tau)$ on L such that the pair
$$(f^{r,T,\lambda}_{(T^F_i,\lambda_i)}(\tau), g^{r,T,\lambda}_{(T^F_i,\lambda_i)})$$
is Morse-Smale for any $\tau \in T^F_i$ and
$$g^{r,T,\lambda}_{(T^F_i,\lambda_i)}(\tau) = g^{\overline{L_i}}$$
for anu $\tau \in s_{T^F_i}$.
- For any $v \in V(T)$ a choice of couples
$$(K^{r,T,\lambda}_v, J^{r,T,\lambda}_v)$$
such that:
  - $K^{r,T,\lambda}_v \in \Omega^1(S_{r,T,\lambda}(v), C^\infty(M))$ is an Hamiltonian-valued one-form which vanishes identically if $v \notin T_{\text{red}}$ (i.e. if $S_{r,T,\lambda}(v)$ does not have any punctures), while for $v \in T_{\text{red}}$ is such that for any $i = 0, \dots, d^R$ it satisfies
$$K^{r,T,\lambda}_v(\xi)|_{\overline{L_i}} = 0 \text{ for any } \xi \in T(\partial_i S_{r,T,\lambda}(v))$$
and for any $i = 0, \dots, d$ on the strip-like ϵ_i we have[6]
$$K^{r,T,\lambda}_v = H^{L_i,L_{i+1}}dt \text{ for any } |s| \geq 1.$$
  - $J^{r,T,\lambda}_v$ is a domain-dependent ω compatible almost complex structure which, if $v \in T^F_i \subset T \setminus T_{\text{red}}$ for some $i \in \{0, \dots, d^F\}$ (see Section 2.1.4) it is identical to $J^{L^F_i}$, while if $v \in T_{\text{red}}$, it is such that on the ith strip-like end ϵ_i we have
$$J^{r,T,\lambda}_v = J^{L_i,L_{i+1}} \text{ for any } |s| \geq 1$$
for any $i = 0, \dots, d$.

[5]Here we identify system of ends as subintervals of edges via the convention introduced at the end of Section 2.1.3.

[6]See page 15 for the definition of $|s| \geq 1$.

We define a perturbation datum for $\vec{L}$ as a smooth choice of perturbation data for $\vec{L}$ on $\mathcal{R}_C^{d+1}(\vec{L})$ and denote it by

$$\left((f^{\vec{L}}, g^{\vec{L}}), (K^{\vec{L}}, J^{\vec{L}})\right).$$

The couple $(f^{\vec{L}}, g^{\vec{L}})$ will be called the Morse part of the perturbation datum, while $(K^{\vec{L}}, J^{\vec{L}})$ will be called its Floer part. A universal choice of perturbation data is a choice of perturbation datum for any tuple $\vec{L}$.

We have the following consistency conditions for perturbation data (see [**Mes18, She11**]).

Definition 2.2.2.3. We say that a choice of perturbation data is consistent if for any tuple $\vec{L}$ of Lagrangians in $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M,\omega)$, say of length $d+1$, we have:

(1) for any $T \in \mathcal{T}(\vec{L})$ and $k \in \{0, \ldots, d-2\}$, there is a subset

$$U \subset \mathcal{R}^T \times \lambda_U^k(T) \times [-1,0)^{|E_F^{\mathrm{int}}(T)|+k}$$

whose closure is a neighbourhood of the trivial gluing where the gluing parameters are small such that the perturbation data for clusters over U agree with perturbation data induced by gluing on thin parts;

(2) all perturbation data extend smoothly to $\partial\overline{\mathcal{R}_C^{d+1}(\vec{L})}$ and agree there with perturbation data coming from (trivial) gluing.

We define the space E of consistent universal choices of perturbation data for clusters (with respect to a fixed choice of strip-like ends and of system of ends).

Remark 2.2.2.4. It is not hard to see that E is non-empty. Indeed, there are plenty of consistent universal choices of perturbation data for puntured disks (á la [**Sei08**]) and of consistent universal choices of perturbation data for Morse trees (see [**Mes18**]). As those two types of perturbation data are independent enough (indeed, equality for glued perturbation data has to hold only on small neighbourhoods of the trivial gluing) in our definition of perturbation data for clusters, it follows that the space E is non-empty.

Given $p \in E$ and a tuple $\vec{L}$ of Lagrangians in $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M,\omega)$ we will often write the perturbation data on $\vec{L}$ induced by p as

$$\mathcal{D}_p^{\vec{L}} = \left(\left(f^{\vec{L}}, g^{\vec{L}}\right), \left(K^{\vec{L}}, J^{\vec{L}}\right)\right).$$

Moreover, given $(r, T, \lambda) \in \mathcal{R}_C^{d+1}(\vec{L})$ we will write the associated perturbation datum for $\vec{L}$ on $S_{r,T,\lambda}$ as $\mathcal{D}_p^{\vec{L}}(r, T, \lambda)$.

2.2.3. Moduli spaces of Floer clusters with Lagrangian boundary. Let $p \in E$ be a consistent universal choice of perturbation data for clusters as defined in the previous section. In the following, given a disk $u : D \to M$ in M we write $[u] \in \pi_2(M, L)$ for the image of the fundamental class of D under the pushforward of u.

First, we define moduli spaces of pearly-edges (see [**BC09a**]). Let L be a monotone Lagrangian in $\mathcal{L}^{m,\mathbf{d}}(M,\omega)$ and $d \geq 1$. Pick a tree $T \in \mathcal{T}^{d+1}(L)$ labelled by the single Lagrangian L, an interior edge $e \in E^{\text{int}}(T)$ of T, points $a, b \in L \setminus \text{Crit}(f^L)$ and a class $A \in \pi_2(M,L)$.

Definition 2.2.3.1. We define the moduli space

$$\mathcal{P}(a,b;p;e;A)$$

of so-called pearly trajectories in the class A modeled on the edge e and joining a to b as the space of tuples

$$(\lambda(e),(u_1,\ldots,u_k))$$

where

(1) $\lambda \in \lambda(T)$ is a metric on T,
(2) for any $i = 1,\ldots,k$, $u_i : D \to M$ is a non-constant J-holomorphic disk such that $u_i(\partial D) \subset L$,
(3) we have $\sum_{i=1}^k [u_i] = A$,
(4) there are $t_0,\ldots,t_k \in [0,\infty)$ such that $\sum_{i=0}^k t_i = \lambda(e)$ and we have the relations

$$\varphi^{t_0}_{f^L_{(T,\lambda)|_e}, g^L_{(T,\lambda)|_e}}(a) = u_1(-1) \text{ and } \varphi^{-t_k}_{f^L_{(T,\lambda)}|_e, g^L_{(T,\lambda)}|_e}(b) = u_k(1)$$

as well as

$$\varphi^{t_i}_{f^L_{(T,\lambda)}|_e, g^L_{(T,\lambda)}|_e}(u_i(1)) = u_{i+1}(-1)$$

for any $i = 1,\ldots,k-1$, where $\varphi^t_{f^L_{(T,\lambda)}|_e, g^L_{(T,\lambda)}|_e}$ is the time t flow map of the negative gradient of the time dependent map $f^L_{(T,\lambda)}|_e$ with respect to the time dependent Riemannian metric $g^L_{(T,\lambda)}|_e$, where $f^L_{(T,\lambda)}$ and $g^L_{(T,\lambda)}$ are Morse data for L associated to $p \in E$, as defined in Section 2.2.2.

up to reparametrization, i.e. $(\lambda(e),u_1,\ldots,u_k)$ is identified with $(\lambda'(e),u'_1,\ldots,u'_l)$ if and only if $k = l$, $\lambda(e) = \lambda'(e)$ and there are automorphisms $\sigma_1,\ldots,\sigma_k \in \text{Aut}(D)$ of the unit disk $D \subset \mathbb{C}$ fixing -1 and 1 such that $u'_i = \sigma_i \circ u_i$ for any $i = 1,\ldots,k$.
Moreover, we define $\mathcal{P}(a,a;p;e;A) = \emptyset$ for any choice of parameters. The definition of $\mathcal{P}(a,b;p;e;A)$ extends also to the case where a, b are critical points of the function f_L in a standard way: if for instance $a \in \text{Crit}(f_L)$, then we ask that $t_0 = \infty$ (i.e. $u_1(-1) \in W^u(a)$) and $\sum_{i=1}^k t_i \in [0,\infty)$.

The virtual dimension of $\mathcal{P}(a,b;p;e;A)$ is $n+\mu(A)-1$, where μ denotes the Maslov index. If both a and b are critical points, then the virtual dimension of $\mathcal{P}(a,b;p;e;A)$ is $|a|-|b|+\mu(A)-1$, where $|\cdot|$ denotes the Morse index.

Consider now a tuple $\vec{L} = (L_0,\ldots,L_d)$ of Lagrangians in $\mathcal{L}^{m,\mathbf{d}}(M,\omega)$. We define moduli spaces of Floer clusters with boundary on $\vec{L}$.
Assume first $L_0 \neq L_d$. Pick $\vec{x}_i := (x^i_1,\ldots,x^i_{m_i-1})$ for any $i = 0,\ldots,d^R$, where $x^i_j \in \text{Crit}(f^{\overline{L^i}})$ are critical points, orbits $\gamma_j \in \mathcal{O}(H^{\overline{L_{j-1}},\overline{L_j}})$ for any $i = 1,\ldots,d^R$ and $\gamma_+ \in \mathcal{O}(H^{\overline{L_0},\overline{L_{d^R}}})$ and

a class $A \in \pi_2(M, \vec{L})$. Recall that we defined $\pi_2(M, \vec{L}) := \pi_2(M, \cup_i L_i)$ in Section 2.1.2. We define the moduli space

$$\mathcal{M}^{d+1}\left(\vec{x}_0, \gamma_1, \vec{x}_1, \ldots, \gamma_{d^R}, \vec{x}_{d^R}; \gamma^+; A; p\right)$$

of Floer clusters joining $\vec{x}_0, \gamma_1, \vec{x}_1, \ldots, \gamma_{d^R}, \vec{x}_{d^R}$ to γ^+ in the class A as the space of tuples $((r, T, \lambda), u)$ where

$$u = \left((u_v)_{v \in V(T)}, (u_e)_{e \in E(T)}\right) : S_{r,T,\lambda} \to M$$

satisfies

(1) for any vertex $v \in V(T)$, $u_v : S_{r,T,\lambda}(v) \to M$ satisfies the (K^v, J^v)-Floer equation and the boundary conditions $u(\partial_i S_{r,T,\lambda}) \subset \overline{L_i}$,

(2) for any $i = 1, \ldots, d^R$ we have

$$\lim_{s \to \infty} u_{h(e_i(T_{\mathrm{red}}))}(\overline{\epsilon_i}(s,t)) = \gamma_i(t)$$

and

$$\lim_{s \to \infty} u_{h(e_0(T_{\mathrm{red}}))}(\epsilon_0(s,t)) = \gamma^+(t),$$

(3) for any $i = 0, \ldots, d^R$ and any interior edge $e \in E_i^{\mathrm{int}}(T)$ (uni)labelled by $\overline{L_i}$ there is a class $B_e \in \pi_2(M, L)$ such that

$$u_e \in \mathcal{P}(u_{t(e)}(z_{t(e)}), u_{h(e)}(z_{h(e)}); p; e; B_e),$$

(4) for any $i = 0, \ldots, d^R$ and any $j = 1, \ldots, m_i$ (see page 6) there is a class $B_j^i \in \pi_2(M, \overline{L_i})$ such that

$$u_{e_j^i(T_{\mathrm{uni}})} \in \mathcal{P}(x_j^i, u_{h(e_j(T_i^F))}(z_{h(e_j(T_i^F))}); p; e_j^i(T_{\mathrm{uni}}); B_j^i),$$

(5) We have the relation

$$A = \sum_{v \in V(T)} [u_v] + \sum_{e \in E^{\mathrm{int}}(T)} B_e + \sum_{i=0}^{d^R} \sum_{j=1}^{m_i} B_j^i$$

on[7] $\pi_2(M, \vec{L})$.

Assume now $L_0 = L_d$. Pick $x^+ \in \mathrm{Crit}(f^{\overline{L_0}})$, $\vec{x}_i := (x_1^i, \ldots, x_{\overline{m_i}}^i)$ for $i = 0, \ldots, d^R + 1$, where $x_j^i \in \mathrm{Crit}(f^{\overline{L_i}})$ are critical points[8], orbits $\gamma_j \in \mathcal{O}(H^{\overline{L_{j-1}}, \overline{L_j}})$ for any $i = 1, \ldots, d^R + 1$ and a class $A \in \pi_2(M, \vec{L})$. We define the moduli space

$$\mathcal{M}^{d+1}\left(\vec{x}_0, \gamma_1, \vec{x}_2, \ldots, \gamma_{d^R+1}, \vec{x}_{d^R+1}; x^+; A; p\right)$$

of Floer clusters joining $\vec{x}_0, \gamma_1, \vec{x}_1, \ldots, \gamma_{d^R+1}, \vec{x}_{d^R+1}$ to x^+ in the class A as the space of tuples $((r, T, \lambda), u)$ where

$$u = ((u_v)_{v \in V(T)}, (u_e)_{e \in E(T)}) : S_{r,T,\lambda} \to M$$

satisfies

[7]Recall that we ignore inclusions on π_2, see Section 2.1.2.

[8]Recall that we work modulo d^R, so that $\overline{L_{d^R+1}} = \overline{L_0}$.

(1) for any vertex $v \in V(T)$, $u_v : S_{r,T,\lambda}(v) \to M$ satisfies the (K^v, J^v)-Floer equation and the boundary conditions $u(\partial_i S_{r,T,\lambda}) \subset \overline{L_i}$,
(2) for any $i = 1, \ldots, d^R$ we have
$$\lim_{s\to\infty} u_{h(e_i(T_{\mathrm{red}}))}(\overline{\epsilon_i}(s,t)) = \gamma_i(t),$$
(3) for any $i = 0, \ldots, d^R$ and any interior edge $e \in E_i^{\mathrm{int}}(T)$ (uni)labelled by $\overline{L_i}$ there is a class $B_e \in \pi_2(M, L)$ such that
$$u_e \in \mathcal{P}(u_{t(e)}(z_{t(e)}), u_{h(e)}(z_{h(e)}); p; e; B_e),$$
(4) for any $i = 0, \ldots, d^R$ and any $j = 1, \ldots, m_i$ there is a class $B^i_j \in \pi_2(M, \overline{L_i})$ such that
$$u_{e^i_j(T_{\mathrm{uni}})} \in \mathcal{P}(x^i_j, u_{h(e^i_j(T_{\mathrm{uni}}))}(z_{h(e^i_j(T_{\mathrm{uni}}))}); p; e^i_j(T_{\mathrm{uni}}); B^i_j)$$
and there is a class $B^0_0 \in \pi_2(M, \overline{L_0})$ such that
$$u_{e_0(T)} \in \mathcal{P}(u_{t(e_0(T))}, x^+; p; e_0(T); B^0_0),$$
(5) We have the relation
$$A = \sum_{v\in V(T)} [u_v] + \sum_{e\in E^{\mathrm{int}}(T)} B_e + \sum_{i=0}^{d^R}\sum_{j=1}^{m_i} B^i_j + B^0_0$$
on $\pi_2(M, \vec{L})$.

A schematic representation of a Floer cluster in the case $L_0 = L_d$ is depicted in Figure 3.

Given a Floer cluster (r, T, λ, u) we write $u_F := (u_v)_{v\in V(T_{\mathrm{red}})}$ for the collection of curves contributing to u which are not purely pseudoholomorphic, and
$$\omega(u) := \sum_{v\in V(T)} \omega(u_v) + \sum_{e\in E(T)} \omega(u_e) = \omega(A) + \omega(u_F)$$
fot its total symplectic area.

2.2.4. Transversality for Floer clusters. Let $d \geq 1$ and $\vec{L}$ be a tuple of pairwise different Lagrangians in $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M,\omega)$, i.e. with $\vec{L^F} = \vec{L}$ in the notation introduced in Section 2.1.2, of length $d+1$. Consider Hamiltonian orbits $\gamma_i \in \mathcal{O}(H^{L_{i-1},L_i})$ for $i \in \{1, \ldots, d\}$ and $\gamma^+ \in \mathcal{O}(H^{L_0,L_d})$ as well as a class $A \in \pi_2(M, \vec{L})$ (recall that we defined $\pi_2(M, \vec{L}) := \pi_2(M, \cup_i L_i)$, see page 7). Then, associated to any Floer polygon u connecting $\gamma_1, \ldots, \gamma_d$ to γ^+ in the class A there is a polygonal Maslov index $\mu(u) \in \mathbb{Z}$ (see [**FOOO09**] or [**Oh15**, Chapter 13]). It is known that this index only depends on the elements $\gamma_1, \ldots, \gamma_d, \gamma^+$ and on the class A, and hence will be denoted by $\mu(\gamma_1, \ldots, \gamma_d, \gamma^+; A)$. Moreover, it is additive under breaking and bubbling of Floer polygons and bubbling of pseudoholomorphic disks (and in this case reduces to the standard Maslov index). In particular, it follows directly that the Maslov index is defined for clusters too.

In this section we sketch the proof of the following result.

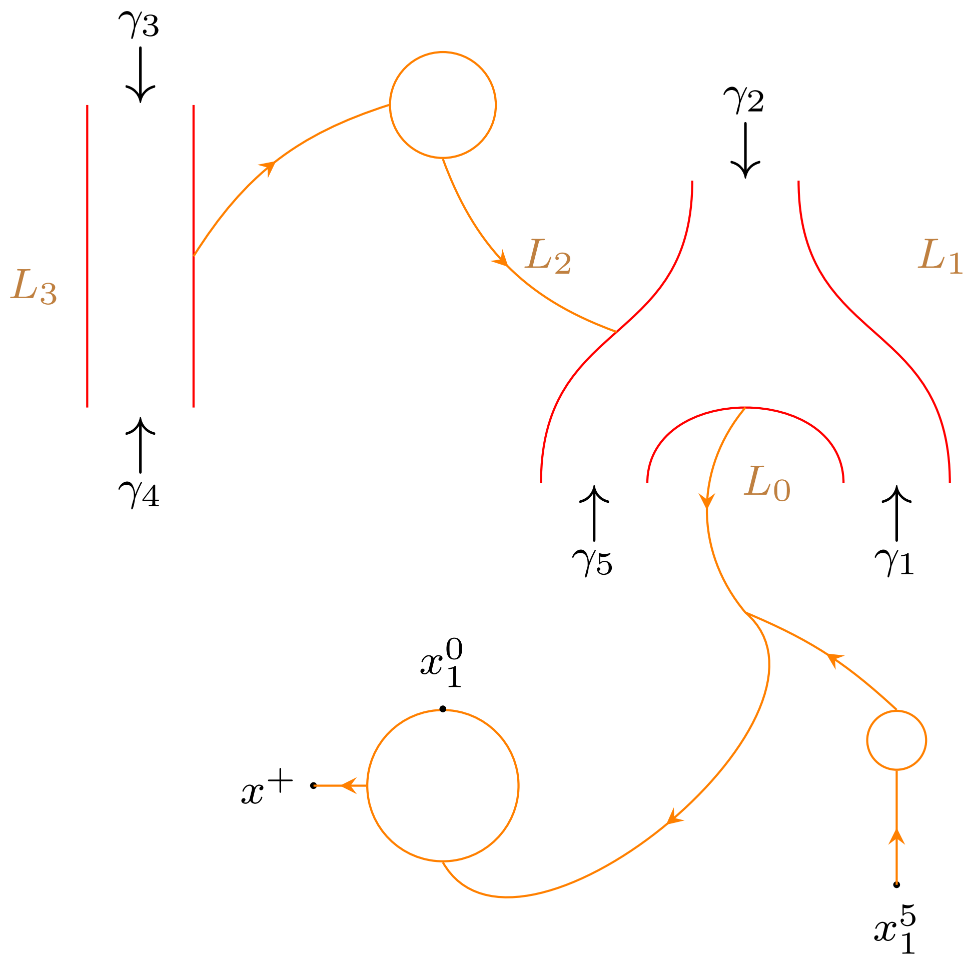


FIGURE 3. A schematic representation of a Floer cluster in the moduli space $\mathcal{M}^8(x_1^0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, x_1^{4+1}; x^+; A)$ for some class $A \in \pi_2(M, \vec{L})$ for the tuple $\vec{L} = (L_0, L_0, L_1, L_2, L_3, L_2, L_0, L_0)$. Morse trajectories and smooth disks are colored in orange, critical points in black and Floer polygons in red. Note that in this case $\vec{L_{\text{red}}} = ((L_0, 2+2), L_1, L_2, L_3, L_2)$ and $\vec{L^F} = (L_0, L_1, L_2, L_3)$.

PROPOSITION 2.2.4.1. *There exists a residual subset $E_{\text{reg}} \subset E$ such that for any $p \in E_{\text{reg}}$ and any tuple $\vec{L} = (L_0, \ldots, L_d)$ of Lagrangians in $\mathcal{L}ag^{(mon,\mathbf{d})}(M, \omega)$ of any length $d \geq 1$ the following holds:*

(1) *if $L_0 \neq L_d$ then for any $\vec{x}_i := (x_1^i, \ldots, x_{m_i-1}^i)$, $i = 0, \ldots, d^R$, where $x_j^i \in \mathrm{Crit}(f^{\overline{L_i}})$ are critical points, any orbits $\gamma_j \in \mathcal{O}\left(H^{\overline{L_{j-1}}, \overline{L_j}}\right)$, $i = 1, \ldots, d^R$ and $\gamma^+ \in \mathcal{O}\left(H^{\overline{L_0}, \overline{L_{d^R}}}\right)$*

and any class $A \in \pi_2(M, \vec{L})$ *satisfying*

$$d_A^{(\vec{x}_i)_i,(\gamma_j)_j,\gamma^+} := \mu(\gamma_1, \dots, \gamma_d, \gamma^+, A) + \sum_{i=0}^{d^R} \sum_{j=1}^{m_i - 1} |x_j^i| - n(d - d^R - 1) + d - 2 \leq 1 \tag{1}$$

then the moduli space

$$\mathcal{M}^{d+1}\left(\vec{x}_0, \gamma_1, \vec{x}_1, \dots, \gamma_{d^R}, \vec{x}_{d^R}; \gamma^+; A; p\right)$$

is a smooth manifold of dimension $d_A^{(\vec{x}_i)_i,(\gamma_j)_j,\gamma^+}$.

(2) *if* $L_0 = L_d$ *then for any* $x^+ \in \mathrm{Crit}(f^{\overline{L_0}})$, $\vec{x}_i := (x_1^i, \dots, x_{\overline{m_i}}^i)$ *for* $i = 0, \dots, d^R + 1$, *where* $x_j^i \in \mathrm{Crit}(f^{\overline{L_i}})$, *any orbits* $\gamma_j \in \mathcal{O}\left(H^{\overline{L_{j-1}}, \overline{L_j}}\right)$, $i = 1, \dots, d^R$ *and any class* $A \in \pi_2(M, \vec{L})$ *satisfying*

$$d_A^{(\vec{x}_i)_i,(\gamma_j)_j,x^+} := \mu(\gamma_1, \dots, \gamma_d, \gamma^+, A) + \sum_{i=0}^{d^R} \sum_{j=1}^{m_i - 1} |x_j^i| - |x^+| - n(d - d^R - 1) + d - 2 \leq 1 \tag{2}$$

then the moduli space

$$\mathcal{M}^{d+1}\left(\vec{x}_0, \gamma_1, \vec{x}_1, \dots, \gamma_{d^R+1}, \vec{x}_{d^R+1}; x^+; A; p\right)$$

is a smooth manifold of dimension $d_A^{(\vec{x}_i)_i,(\gamma_j)_j,x^+}$.

Definition 2.2.4.2. The elements of $E_{\mathrm{reg}} \subset E$ will be called regular perturbation data.

The proof of Proposition 2.2.4.1 is analogous to the proof of Proposition 4.7 in [**She11**], with the difference that we are not using Hamiltonian perturbations for pearly trees. As it will be apparent from the discussion below, this is the reason why we are dealing with moduli spaces of virtual dimension ≤ 1 only (cfr. [**BC09a**]). We now explain how to account for this difference.

Fix $p \in E$ (which we will omit from the notation of Floer and perturbation data). Assume first that $\vec{L^F} = L$, that is $L = L_i$ for any i, and pick critical points $x_1, \dots, x_d, x^+$ of f^L and a class $A \in \pi_2(M, \vec{L}) = \pi_2(M, L)$. Then the virtual dimension of

$$\mathcal{M}^{d+1}(x_1, \dots, x_d; x^+; A; ((f_T^L, g_T^L)_T, J^L))$$

where $(f_T^L, g_T^L)_T$ is some choice of (Morse) perturbation datum for L (and the notation makes sense as $d^F = 1$) and J^L is part of the Floer data for L, is

$$\mu(A) + \sum_{i=1}^{d}(|x_i| - n) - |x^+| + d - 2.$$

Moreover, assuming that the virtual dimension is ≤ 1, then for a generic choice of the almost complex structure J^L and (Morse) perturbation data, the Morse functions are in generic position, the evaluation maps from the holomorphic discs to the Lagrangian is transverse to stable and unstable manifolds of our Morse functions, and our moduli spaces only contain configurations made of simple and absolutely distinct disks joined by absolutely distinct Morse flowlines

(see [**Mes18**] for the Morse part, and [**Cha12**] for simpleness of holomorphic disks, proved by applying the results from [**Laz11**] following ideas from [**BC09a**]), which are the conditions needed in order to ensure regularity of moduli spaces (see [**MS12**]). In this case, it is crucial that the minimal Maslov number N_L of L is at least 2, as bubbling of pseudoholomorphic disks is a codimension 1 phenomenon.
We now take care of the general situation. Consider $\vec{L}$ such that $m_0^e(\vec{L}) = 0$, i.e. $L_0 \neq L_d$. Pick $\vec{x}_i := (x_1^i, \ldots, x_{m_i-1}^i)$, $i = 0, \ldots, d^R$, $\gamma_j \in \mathcal{O}\left(H^{\overline{L_{j-1}}, \overline{L_j}}\right)$, $i = 1, \ldots, d^R$, $\gamma_+ \in \mathcal{O}\left(H^{\overline{L_0}, \overline{L_{d^R}}}\right)$ and $A \in \pi_2(M, \vec{L})$ such that $d_A^{(\vec{x}_i)_i, (\gamma_j)_j, \gamma^+} \leq 1$. Then, due to the fact that the Maslov index for polygonal maps respects breaking and disk bubbling (see [**Oh15**]), it follows from Estimate 1 that for any $i \in \{0, \ldots, d^F\}$ we have

$$\mu\left(\sum_{v \in V(T_i^F)} [u_v] + \sum_{e \in E_i^{\mathrm{int}}(T)} B_e + \sum_{\substack{k \in \{0, \ldots, d^R\} \\ k^F = i}} \sum_{j=1}^{\overline{m_k}} B_j^k\right) + \sum_{\substack{k \in \{0, \ldots, d^R\} \\ k^F = i}} \left(\sum_{j=1}^{\overline{m_k}} (|x_j^k| - 1) + m_k\right) - 2 \leq 1$$

(see page 10 for the definition of the subtrees T_i^F). In particular, as above, we can apply the results contained in [**Mes18, Cha12**] to any configuration associated to any connected component of T_i^F. This arguments fills in the gap between the situation in [**She11**] and ours.

2.2.5. Definition of $Fuk(M)$. Let $p \in E_{\mathrm{reg}}$ be a regular perturbation datum. We recall that Λ denotes the standard Novikov field over $\mathbb{Z}_2$ (see Section 2.1.1). Given a couple (L_0, L_1) of Lagrangians in $\mathcal{L}^{m,\mathbf{d}}(M, \omega)$ we define its p-Floer vector spaces $CF(L_0, L_1; p)$ as follows:

(1) if $L_0 \neq L_1$ we define

$$CF(L_0, L_1; p) := \bigoplus_{\gamma \in \mathcal{O}(H_p^{L_0, L_1})} \Lambda \cdot \gamma$$

where $H_p^{L_0, L_1}$ is part of the Floer datum for (L_0, L_1) prescribed by p;

(2) if $L_0 = L_1 =: L$ we define

$$CF(L_0, L_1; p) := \bigoplus_{x \in \mathrm{Crit}(f_p^L)} \Lambda \cdot x$$

where f_p^L is part of the Floer datum for (L_0, L_1) prescribed by p.

We will often suppress p from the notation when there is no risk of confusion.
Let $d \geq 1$ and consider a tuple $\vec{L} = (L_0, \ldots, L_d)$ of Lagrangians in $\mathcal{L}^{m,\mathbf{d}}(M, \omega)$. In the following we define a map

$$\mu_d : CF(L_0, L_1) \otimes \cdots \otimes CF(L_{d-1}, L_d) \to CF(L_0, L_d)$$

which depends on the choice of the perturbation datum $p \in E_{\mathrm{reg}}$. First, we assume $L_0 \neq L_d$. Let $\vec{x}_i := (x_1^i, \ldots, x_{m_i-1}^i)$, $i = 0, \ldots, d^R$, where $x_j^i \in \mathrm{Crit}(f^{\overline{L_i}})$ are critical points, $\gamma_j \in$

$\mathcal{O}(H^{\overline{L_{j-1}},\overline{L_j}})$, $i = 1, \ldots, d^R$ are orbits (see page 6 for the definition of d^R and m_i). Then we define

$$\mu_d(\vec{x}_0, \gamma_1, \vec{x}_1, \ldots, \gamma_{d^R}, \vec{x}_{d^R}) := \sum_{\gamma^+, A} \sum_u T^{\omega(u)} \cdot \gamma^+$$

where the first sum runs over orbits $\gamma^+ \in \mathcal{O}(H^{L_0,L_d})$ and classes $A \in \pi_2(M, \vec{L})$ such that

$$d_A^{(\vec{x}_i)_i,(\gamma_j)_j,\gamma^+} = 0,$$

and second sum runs over Floer clusters

$$u \in \mathcal{M}^{d+1}\left(\vec{x}_0, \gamma_1, \vec{x}_1, \ldots, \gamma_{d^R}, \vec{x}_{d^R}; \gamma^+; A; p\right).$$

Assume now $L_0 = L_d =: L$. Let $\vec{x}_i := (x_1^i, \ldots, x_{\overline{m_i}}^i)$ for $i = 0, \ldots, d^R + 1$, where $x_j^i \in \mathrm{Crit}(f^{\overline{L_i}})$ are critical points, orbits $\gamma_j \in \mathcal{O}(H^{\overline{L_{j-1}},\overline{L_j}})$ for any $i = 1, \ldots, d^R$. Then we define

$$\mu_d(\vec{x}_0, \gamma_1, \vec{x}_1, \ldots, \gamma_{d^R}, \vec{x}_{d^R}) := \sum_{x^+, A} \sum_u T^{\omega(u)} \cdot x^+$$

where the first sum runs over critical points $x^+ \in \mathrm{Crit}(f^{L_0})$ and classes $A \in \pi_2(M, \vec{L})$ such that

$$d_A^{(\vec{x}_i)_i,(\gamma_j)_j,x^+} = 0,$$

and the second sum runs over Floer clusters

$$u \in \mathcal{M}^{d+1}\left(\vec{x}_0, \gamma_1, \vec{x}_1, \ldots, \gamma_{d^R}, \vec{x}_{d^R}; x^+; A; p\right).$$

Before defining the p-Fukaya category of M, we show that the maps μ_d above are well-defined and satisfy the expected properties, that is the A_∞-equations (see Equation **??**). We have the following standard compactness result (see [**Sei08**, **Cha12**, **?**]).

Proposition 2.2.5.1. *Let $p \in E_{\mathrm{reg}}$. Then:*

(1) assume $L_0 \neq L_d$ and consider generators $\vec{x}_i := (x_1^i, \ldots, x_{m_i-1}^i)$, $i = 0, \ldots, d^R$, $\gamma_j \in \mathcal{O}\left(H^{\overline{L_{j-1}},\overline{L_j}}\right)$, $i = 1, \ldots, d^R$ and $\gamma_+ \in \mathcal{O}\left(H^{\overline{L_0},\overline{L_{d^R}}}\right)$ and a class $A \in \pi_2(M, \vec{L})$ as in Proposition 2.2.4.1, then:

(a) if $d_A^{(\vec{x}_i)_i,(\gamma_j)_j,\gamma^+} = 0$, the moduli space

$$\mathcal{M}^{d+1}\left(\vec{x}_0, \gamma_1, \vec{x}_1, \ldots, \gamma_{d^R}, \vec{x}_{d^R}; \gamma^+; A; p\right)$$

is compact,

(b) if $d_A^{(\vec{x}_i)_i,(\gamma_j)_j,\gamma^+} = 1$ it admits a compactification into a manifold with boundary

$$\overline{\mathcal{M}}^{d+1}\left(\vec{x}_0, \gamma_1, \vec{x}_1, \ldots, \gamma_{d^R}, \vec{x}_{d^R}; \gamma^+; A; p\right)$$

whose boundary points are in one-to-one correspondence with the terms in the A_∞-equation **??** *for $(\vec{x}_0, \gamma_1, \vec{x}_1, \ldots, \gamma_{d^R}, \vec{x}_{d^R}; \gamma^+)$*

(2) assume $L_0 = L_d$ and consider generators $x^+ \in \mathrm{Crit}(f^{L_0})$, $\vec{x}_i := (x_1^i, \ldots, x_{\overline{m_i}}^i)$ for $i = 0, \ldots, d^R + 1$, $\gamma_j \in \mathcal{O}\left(H^{\overline{L_{j-1}},\overline{L_j}}\right)$, $i = 1, \ldots, d^R$ and a class $A \in \pi_2(M, \vec{L})$ as in Proposition 2.2.4.1, then:

(a) if $d_A^{(\vec{x}_i)_i,(\gamma_j)_j,x^+} = 0$*, the moduli space*

$$\mathcal{M}^{d+1}\left(\vec{x}_0, \gamma_1, \vec{x}_1, \ldots, \gamma_{d^R}, \vec{x}_{d^R}; x^+; A; p\right)$$

is compact,

(b) if $d_A^{(\vec{x}_i)_i,(\gamma_j)_j,x^+} = 1$ *it admits a compactification into a manifold with boundary*

$$\overline{\mathcal{M}}^{d+1}\left(\vec{x}_0, \gamma_1, \vec{x}_1, \ldots, \gamma_{d^R}, \vec{x}_{d^R}; x^+; A; p\right)$$

whose boundary points are in one-to-one correspondence with the terms in the A_∞*-equation* **??** *for* $(\vec{x}_0, \gamma_1, \vec{x}_1, \ldots, \gamma_{d^R}, \vec{x}_{d^R}; x^+)$.

Remark 2.2.5.2. The statement about the boundary of 1-dimensional components of moduli spaces of Floer clusters is not presented in the most accurate form to avoid notational complexity. However, the meaning should be clear as results of this kind are very standard in various construction of Fukaya categories (see for instance [**Sei08**, Section 9l]).

The proof of Proposition 2.2.5.1 is a mix of standard arguments (for Morse trees [**?, Mes18**] and for Floer polygons [**Sei08**]) and the structure of the boundary of 1-dimensional components comes from the fact that compactification of sources spaces for Floer clusters realize associahedra (see Section 2.2.1) and from breaking of Morse flowlines along leaves as well as concentration of energy along strip-like ends. Notice that we do not have to care about shrinking of Morse trajectories between pseudoholomorphic disks as in [**BC**], as by construction the limit lies in the interior of our moduli spaces. There is just one case of bubbling that may a priori happen in 1-dimensional case but we want to avoid in order for the boundary to look 'as it should be' (that is, so that it realizes A_∞-operations): what in [**BC**] is referred to as 'side bubbling', that is bubbling of pseudoholomorphic disks away from marked points. This cannot happen in our situation because of our assumption on the minimal Maslov number of Lagrangians in $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M,\omega)$, and because the polygonal Maslov index respects bubbling and breaking, as it would give rise to a Floer cluster lying in a smooth manifold of negative dimension (exactly as in [**BC**]).

We define the p-Fukaya category $Fuk^{\mathbf{d}}(M,\omega;p)$ of (M,ω) as follows: the objects are monotone Lagrangians in $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M,\omega)$, the morphism space between any two objects $L_0, L_1 \in \mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M,\omega)$ is the associated p-Floer complex $CF(L_0,L_1;p)$ and the μ_d-maps are those defined above. When there is no risk of confusion we will drop $\mathbf{d}$, ω and p from the notation. The following is the main result of this section.

Proposition 2.2.5.3. *For any* $p \in E_{\mathrm{reg}}$*,* $Fuk^{\mathbf{d}}(M,\omega;p)$ *is a strictly unital* A_∞*-category.*

Proof. The fact that $Fuk^{\mathbf{d}}(M,\omega;p)$ is an A_∞-category follows directly from Proposition 2.2.5.1. We show that it is a strictly unital one. Let $L \in \mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M,\omega)$ and denote by $e_L \in \mathrm{Crit}(f_p^L)$ the (unique) maximum of the Morse function $f_p^L\colon L \to \mathbb{R}$ associated to the perturbation datum p. Note that for any critical point $x^+ \in \mathrm{Crit}(f_p^L)$ and any non-trivial class $0 \neq A \in \pi_2(M,L)$ the moduli space $\mathcal{M}^2(e_L, x^+; A; p)$ is at least one dimensional. From this and the well known fact that the top Morse homology of L is isomorphic to the coefficient field Λ it follows that e_L is a cycle in $CF(L, L; p)$.

We first prove that $\mu_2(e_L, -)$ acts as the identity on chain complexes. The proof for $\mu_2(-, e_L)$ is analogous. Let $x^- \in \mathrm{Crit}(f_p^L)$. Assume there is a non-trivial class $0 \neq A \in \pi_2(M, L)$ and another critical point $x^+ \in \mathrm{Crit}(f_p^L)$ such that

$$|e_L| + |x^-| - |x^+| + \mu(A) - n = 0 \text{ and } \mathcal{M}^3(e_L, x^-; x^+; A; p) \neq \emptyset$$

As e_L is a maximum it has index $|e_L| = n$ and the first equation above may be rewritten as $\mu(A) + |x^-| - |x^+| = 0$. At this point however, the existence of a trajectory in $\mathcal{M}^3(e_L, x^-; x^+; A; p)$ implies the existence of a trajectory in $\mathcal{M}^2(x^-; x^+; A'; p)$ for some class $A' \in \pi_2(M, L)$ such that $\mu(A') \leq \mu(A)$. In particular, the dimension of the latter moduli space is non-negative, as p is regular by assumption, that is

$$\mu(A') + |x^-| - |x^+| - 1 \geq 0.$$

Combining this estimate with the above identity we get

$$0 \leq \mu(A') + |x^-| - |x^+| - 1 \leq \mu(A) + |x^-| - |x^+| - 1 = -1,$$

a contradiction. We showed that no trajectory with non-constant pseudoholomorphic discs can contribute to the term $\mu_2(e_L, x^-)$. Assume now that there is a trajectory in $\mathcal{M}^3(e_L, x^-; x^+; 0; p)$. This implies $|x^-| = |x^+|$ (as $\mu(0) = 0$) and that there is a Morse trajectory from x^- to x^+, so that $x^- = x^+$. We hence proved $\mu_2(e_L, x^-) = x^-$ for any $x^- \in \mathrm{Crit}(f_p^L)$.
Let now $L_1 \neq L$ and consider an orbit $\gamma^- \in \mathcal{O}(H_p^{L,L_1})$. Assume there is some orbit $\gamma^+ \in \mathcal{O}(H_p^{L,L_1})$ and some class $A \in \pi_2(M, \vec{L})$ such that

$$\mu(\gamma^-, \gamma^+; A) + |e_L| - n = 0 \text{ and } \mathcal{M}_0^3(e_L, \gamma^-; \gamma^+; A; p) \neq 0$$

This implies in particular the existence of an unparametrized Floer $(H_p^{L,L_1}, J_p^{L,L_1})$-strip of zero index from γ^- to γ^+, which is possible if and only if $\gamma^- = \gamma^+$. We hence proved $\mu_2(e_L, \gamma^-) = \gamma^-$ for any $\gamma^- \in \mathcal{O}(H_p^{L,L_1})$.

Let now $d \geq 3$. To finish the proof of strict unitality, one needs to show that the maps $\mu_d(-, e_L, -)$ (with the maximum e_L as an entry in ith position for $i = 1, \ldots, d$) vanishes identically. We omit the details of this part, but the proof follows the same idea as for μ_2: we assume the existence of a rigid configuration contributing to some term of the form $\mu_d(-, e_L, -)$; this implies non-negativity of the dimension of a moduli space of clusters with $d-1$ inputs and one output; then, a simple estimate leads to a contradiction. This completes the proof of the claim that e_L is a strict unit, and hence of the proposition. □

The following result, whose proof we omit, can be proved in the same way as the analogous result for standard Fukaya categories in [**Sei08**, Chapter 10]. Notice that it may be proved also by slightly extending the construction of continuation functors developed in Section 2.4.

LEMMA 2.2.5.4. *Let $p, q \in E_{\mathrm{reg}}$. Then $Fuk(M; p)$ is quasi equivalent to $Fuk(M; q)$.*

It can moreover be proved that our construction of the Fukaya category is quasi-equivalent to the standard one contained in [Sei08]. This may be proved as in [She11] or via the machinery of PSS functors, which will appear in the author's PhD thesis.

2.2.6. Weakly-filtered structure on $Fuk(M)$. The content of this subsection is an adaptation of [BCS21, Section 3.3] to the cluster setting. Fix a regular perturbation datum $p \in E_{\mathrm{reg}}$. Let (L_0, L_1) be a tuple of Lagrangians in $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M,\omega)$. We define the p-action functional

$$\mathbb{A}_p : CF(L_0, L_1; p) \to \mathbb{R}$$

for (L_0, L_1) as follows: consider a generator $g \in CF(L_0, L_1; p)$ and a Novikov series $P := \sum a_i T^{\lambda_i} \in \Lambda$ with $a_i \neq 0$ for all i, ordered in such a way that $\lambda_0 < \lambda_i$ for any $i \in \mathbb{Z}_{\geq 1}$, then we set

$$\mathbb{A}_p(Pg) := -\lambda_0 + \int_0^1 H^{L_0,L_1} \circ g \; dt, \quad \mathbb{A}_p(0) := -\infty$$

and extend it for $\sum P_i g_i \in CF(L_0, L_1 : p)$ as

$$\mathbb{A}_p\Big(\sum P_i g_i\Big) := \max_i \mathbb{A}_p(P_i g_i)$$

We use $\mathbb{A}_p$ to define an increasing $\mathbb{R}$-filtration on $CF(L_0, L_1; p)$ via

$$CF^{\leq \alpha}(L_0, L_1; p) := \mathbb{A}_p^{-1}(-\infty, \alpha]$$

for any $\alpha \in \mathbb{R}$. It is well known that this filtration endows $(CF(L_0, L_1; p), \mu_1)$ with the structure of a filtered chain complex (see for instance [BCS21]) in the case $L_0 \neq L_1$, and is trivial to see that it does also in the case $L_0 = L_1$ (indeed in this case all generators lie at filtration level 0 since we arbitrarily set $H^{L,L} = 0$ at the beginning of Section 2.2.2). We have the following central result.

PROPOSITION 2.2.6.1. *There is a non-empty subset $E^{\mathrm{wf}}_{\mathrm{reg}} \subset E_{\mathrm{reg}}$ such that for any $p \in E^{\mathrm{wf}}_{\mathrm{reg}}$ the filtrations described above on p-Floer complexes induce on $Fuk(M; p)$ the structure of a weakly-filtered A_∞-category with units at filtration level ≤ 0.*

REMARK 2.2.6.2. The above result is the analogous of [BCS21, Proposition 3.1] in the cluster setting. The main difference is that we get units at vanishing filtration level as a result of the use of the cluster setting, as opposed to the standard setting used in the cited paper.

In the remaining of this section, we sketch a proof of why in general the action functional only induces a weakly-filtered structure instead of a genuine filtered one (see Section **??** for the definition of filtered and weakly-filtered A_∞-categories). Central to this kind of results are energy computations for Floer-like curves with asymptotic conditions. Recall that the energy of a Floer like curve $u : S \to M$ (where S is some Riemann surface with punctures) is defined via

$$E(u) := \int_S |Du - X^K|^2 \sigma$$

where K is some Hamiltonian perturbation datum on S, X^K is the indued Hamiltonian vector field and σ is some area form on S (from the choice of which the value of E is independent, see [MS12]) and the norm involved is defined on the space of linear maps $TS \to u^*TM$ as

$$|L| := \sqrt{\frac{\omega(L(v), J(L(v))) + \omega(L(jv), J(L(jv)))}{\sigma(v, jv)}}$$

where j is the almost complex structure on S (from which E definitely depends on). Given a Floer cluster (r, T, λ, u) (in some moduli space of clusters) we write

$$E(u) := \sum_{v \in V(T)} E(u_v) + \sum_{e \in E(T)} E(u_e) = A + E(u_F)$$

where u_F is defined on page 26.

We start by showing that μ_1 preserves filtration. Let (L_0, L_1) be a couple of different Lagrangians in $\mathcal{L}^{m,\mathbf{d}}(M, \omega)$. Notice that for the standard strip $\mathbb{R} \times [0,1]$, the norm in the definition of the energy is defined with respect to the Riemannian metric $g := g_{\omega, J_p^{L_0,L_1}}$ induced by ω and $J_p^{L_0,L_1}$. Consider two orbits $\gamma_-, \gamma_+ \in \mathcal{O}(H_p^{L_0,L_1})$, a class $A \in \pi_2(M, L_0 \cup L_1)$ such that $\mu(\gamma_-, \gamma_+; A) - 1 = 0$ and a Floer strip $u \in \mathcal{M}^2(\gamma_-, \gamma_+; A; p)$, then:

$$\begin{aligned}
E(u) &= \int_{\mathbb{R}\times[0,1]} |\partial_s u|^2 \, dsdt = \int_{\mathbb{R}\times[0,1]} g(\partial_s u, J_p^{L_0,L_1}(-J_p^{L_0,L_1}\partial u - \nabla H_p^{L_0,L_1}(u)) dsdt \\
&= \int_{\mathbb{R}\times[0,1]} \omega(\partial_s u, \partial_t u) \, dsdt + \int_0^1 H_p^{L_0,L_1}(t, u(s,t)) \, dt|_{s=-\infty}^{s=+\infty} \\
&= \omega(u) - \int_0^1 H_p^{L_0,L_1} \circ \gamma_+ \, dt + \int_0^1 H_p^{L_0,L_1} \circ \gamma_- \, dt \\
&= \mathbb{A}_p(\gamma_-) - \mathbb{A}_p(T^{\omega(u)}\gamma_+).
\end{aligned}$$

From this and the definition on the filtration on Floer complexes it follows that

$$\mathbb{A}_p(\mu_1(\gamma_-)) \leq \mathbb{A}_p(\gamma_-)$$

as claimed.

Consider now a couple (L, L) of identical Lagrangians in $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M, \omega)$. Consider critical points $x_-, x_+ \in \mathrm{Crit}(f_p^L)$ and a class $A \in \pi_2(M, L)$ such that $\mu(A) + |x_-| - |x_+| - 1 = 0$ and consider a pearly trajectory $u \in \mathcal{M}^2(x_-; x_+; A; p)$. Then

$$0 \leq E(u) = \omega(u) = \mathbb{A}_p(x_-) - \mathbb{A}_p(T^{\omega(u)}x_+)$$

and hence $\mathbb{A}_p(\mu_1(x_-)) \leq \mathbb{A}_p(x_-)$.

We perform similar calculations in the case of higher operations (see [Sei08, BC14]). Let $d \geq 2$ and consider first the $(d+1)$-tuple $\vec{L} := (L, \ldots, L)$ where $L \in \mathcal{L}^{m,\mathbf{d}}(M, \omega)$. Consider critical points $x_1, \ldots, x_d, x^+ \in \mathrm{Crit}(f_p^L)$ and a class $A \in \pi_2(M, L)$ such that $d_A^{(x_i)_i, x^+} = 0$ and pick a Floer cluster

$$(r, T, \lambda, u) \in \mathcal{M}^{d+1}(x_-^1, \ldots, x_-^d; x^+; A; p).$$

This case is very similar to the case of a couple of equal Lagrangians above: we have

$$0 \leq E(u) = \omega(u) = \omega(A) = \sum_{i=1}^{d} \mathbb{A}_p(x_-^i) - \mathbb{A}_p(T^{\omega(u)}x^+)$$

by definition of the action functional, and hence

$$\mathbb{A}_p(\mu^d(x_-^1, \dots, x_-^d)) \leq \sum_{i=1}^{d} \mathbb{A}_p(x_-^i)$$

that is, μ^d restricted to tuples made of equal Lagrangians preserves filtration.

Let $d \geq 2$ and $\vec{L} = (L_0, \dots, L_d)$ be an arbitrary tuple of Lagrangians in $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M,\omega)$. Assume $L_0 \neq L_d$ first and consider $\vec{x}_i := (x_1^i, \dots, x_{m_i-1}^i)$, $i = 0, \dots, d^R$, where $x_j^i \in \mathrm{Crit}(f^{\overline{L_i}})$ are critical points, orbits $\gamma_j \in \mathcal{O}(H^{\overline{L_{j-1}},\overline{L_j}})$, $i = 1, \dots, d^R$ and $\gamma_+ \in \mathcal{O}(H_p^{L_0,L_d})$ and a class $A \in \pi_2(M, \vec{L})$ such that $d_A^{(\vec{x}_i)_i,(\gamma_j)_j,\gamma^+} = 0$ (see page 27). Pick a Floer cluster

$$(r, T, \lambda, u) \in \mathcal{M}_0^{d+1}\left(\vec{x}_0, \gamma_1, \vec{x}_1, \dots, \gamma_{d^R}, \vec{x}_{d^R}; \gamma^+; A; p\right).$$

For any $v \in V(T)$ choose conformal coordinates (s,t) on $S_{r,T,\lambda}(v)$ and write the area form as $\sigma_{r,T,\lambda}(v) = \rho_{r,T,\lambda}(v) ds \wedge dt$ in those coordinates. Locally, we can write the Hamiltonian term of the perturbation datum on v induced by p (see page 21) as

$$K_{r,T,\lambda}^p(v) = F_{r,T,\lambda}(v)ds + G_{r,T,\lambda}(v)dt$$

for domain dependent functions of the form

$$F_{r,T,\lambda}(v)_{s,t}, G_{r,T,\lambda}(v)_{s,t} : M \to \mathbb{R}$$

for local coordinates (s,t). Recall that $F_{r,T,\lambda}(v) = G_{r,T,\lambda}(v) = 0$ for $v \notin V(T_{\mathrm{red}})$ by definition of perturbation data for Floer clusters (see page 21). For any $v \in V(T_{\mathrm{red}})$ we have that on conformal patches

$$\begin{aligned}\int |du_v - X^{K_{r,T,\lambda}^p(v)}|^2 \rho_{r,T,\lambda}\, dsdt = \int \omega(\partial_s u_v, \partial_t u_v) + \omega(X^{F_{r,T,\lambda}(v)}, X^{G_{r,T,\lambda}(v)}) + \\ + dF_{r,T,\lambda}(v)(\partial_t u_v) - dG_{r,T,\lambda}(v)(\partial_s u_v)\, dsdt\end{aligned}$$

holds. With a couple more calculations and summing over different patches it can be shown that

$$\begin{aligned}0 \leq E(u_v) &= \int_{S_{r,T,\lambda}(v)} |du_v - X^{K_{r,T,\lambda}^p(v)}|^2 \rho_{r,T,\lambda}\, dsdt = \\ &= \omega(u_v) + \int_{S_{r,T,\lambda}(v)} d(u_v^* K_{r,T,\lambda}^p(v)) + \int_{S_{r,T,\lambda}(v)} R_{r,T,\lambda}^p(v) \circ u_v\end{aligned}$$

where $R_{r,T,\lambda}^p(v) \in \Omega^2(S_{r,T,\lambda}(v), C^\infty(M))$ is the so called *curvature form* of $K_{r,T,\lambda}^p(v)$ which in conformal coordinates can be written as

$$R_{r,T,\lambda}^p(v) = \left(\partial_s G_{r,T,\lambda}(v) - \partial_t F_{r,T,\lambda}(v) + \omega\left(X^{F_{r,T,\lambda}(v)}, X^{G_{r,T,\lambda}(v)}\right)\right)\, ds \wedge dt$$

From this it follows

$$0 \leq E(u_F) = \omega(u_F) - \int_0^1 H_p^{L_0,L_d} \circ \gamma_+ \, dt + \sum \int_0^1 H_p^{L_{i-1},L_i} \circ \gamma_-^i \, dt + \sum_{v \in V(T)} \int_{S_{r,T,\lambda}(v)} R^p_{r,T,\lambda}(v) \circ u_v$$

and hence

$$0 \leq E(u) = \omega(u) - \int_0^1 H_p^{L_0,L_d} \circ \gamma_+ \, dt + \sum \int_0^1 H_p^{L_{i-1},L_i} \circ \gamma_-^i \, dt + \sum_{v \in V(T)} \int_{S_{r,T,\lambda}(v)} R^p_{r,T,\lambda}(v) \circ u_v$$

by the definition of u_F on page 26.

We call the integral $\int_{S_{r,T,\lambda}(v)} R^p_{r,T,\lambda}(v) \circ u_v$ the *curvature term of the Floer disk* u_v, while the sum

$$\sum_{v \in V(T)} \int_{S_{r,T,\lambda}(v)} R^p_{r,T,\lambda}(v) \circ u_v$$

will be called the *curvature term of the Floer cluster* u.

Applying the arguments contained in the proof of Proposition 3.1 in [**BCS21**] vertexwise, we get that there is a non-empty subset $E^{\mathrm{wf}}_{\mathrm{reg}} \subset E_{\mathrm{reg}}$ of regular perturbation data such that for any $p \in E^{\mathrm{wf}}_{\mathrm{reg}}$, any $d \geq 2$, any tuple of Lagrangians $\vec{L} = (L_0, \dots, L_d)$ and any possible Floer cluster u on $\vec{L}$ of index 0, the curvature term of u can be absolutely bounded by a term $\epsilon_d = \epsilon_d(p) \in \mathbb{R}$ which does only depend on the number d and on the perturbation data $p \in E^{\mathrm{wf}}_{\mathrm{reg}}$. Hence we conclude

$$\mathbb{A}_p(\mu_d(\vec{x}_0, \gamma_1, \vec{x}_1, \dots, \gamma_{d^R}, \vec{x}_{d^R})) \leq \sum_{i=1}^{d^R} \mathbb{A}_p(\gamma_-^i) + \sum_{i=0}^{d^R} \sum_{j=1}^{m_i} \mathbb{A}_p(x_j^i) + \epsilon_d$$

as desired. For tuples with $L_0 = L_d$ the result follows by analogous means. As the (strict) units lie at filtration level ≤ 0 by definition, this shows that the above defined action functional endows $Fuk(M;p)$ with the structure of weakly-filtered A_∞-category.

Remark 2.2.6.3.

(1) As we will show in Section 2.3, by choosing the perturbation data carefully we can arrange that the curvature term of a Floer clusters are non-positive.

(2) Proposition 2.2.6.1 applies to the standard (i.e. without Morse trees, see [**Sei08**]) construction of Fukaya categories too. What is not apparent from the statement of the proposition, but is the reason for which we developed the Morse-Floer model for $Fuk(M)$ is the following trivial but crucial observation: in our case, units lie at filtration level ≤ 0, while this is not true in the standard case, due to the presence of Hamiltonian perturbation data. More about this will be explained in Section 2.3.3.

2.3. ϵ-Perturbation data and filtered structure on $Fuk(M)$

Recall from Section 2.2.2 that, given a symplectic manifold (M, ω) and a choice of family of monotone Lagrangians $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M, \omega)$, we denote by E the space of perturbation data for the cluster model of $Fuk(M)$ and by $E_{\mathrm{reg}} \subset E$ the subset of regular perturbation data, that is those perturbation data leading to a well-defined strictly-unital Fukaya category (see Definition 2.2.4.2). In this section we construct families of perturbation data $E^{\varepsilon}_{\mathrm{reg}} \subset E_{\mathrm{reg}}$, one for any positive real number $\varepsilon > 0$, such that for any $\varepsilon > 0$ and any $p \in E^{\varepsilon}_{\mathrm{reg}}$ the A_∞-category $Fuk(M, p)$ is filtered, i.e. the discrepancies $\varepsilon_d(p)$ (defined in Section 2.2.6) of the maps μ_d can be taken to be zero (see Section **??**). In order to achieve this, we will restrict ourselves to perturbation data encapsulating Floer data whose Hamiltonian part is uniformly bounded: this will be the role of the parameter $\varepsilon > 0$. The following theorem, which is a more accurate version of Thorem **??**, summarizes the results and constructions contained in this section.

THEOREM 2.3.0.1. *For any $\varepsilon > 0$ there is a non-empty family of perturbation data $E^{\varepsilon}_{\mathrm{reg}} \subset E_{\mathrm{reg}}$ such that for any $p \in E^{\varepsilon}_{\mathrm{reg}}$ the associated Fukaya category $Fuk(M; p)$ developed in Section 2.2 is a filtered A_∞-category.*

The proof of Theorem 2.3.0.1 occupies Section 2.3.1 and 2.3.2. The idea for the proof is to construct ε-perturbation data for strips and triangles first, and then extend the construction to more complicated clusters via gluing. After the proof of Theorem 2.3.0.1, we will explain in Section 2.3.3 why it seems necessary to work with a hybrid Floer-Morse model for the Fukaya category in order to obtain an A_∞-category in which the μ_d-operations are filtered and also its units lie in filtration zero.

REMARK 2.3.0.2. As hinted above and as it will be clear from the construction, the role of ε in our construction will be to control the oscillation of Hamiltonians in the Floer data of our Lagrangians. Note that the filtered structure of $Fuk(M, p)$ (and hence the structure of triangulated persistence category [**BCZ24b**] of the derived Fukaya category) will depend on the choice of $\varepsilon > 0$ and $p \in E^{\varepsilon}_{\mathrm{reg}}$ (although in a quantifiable way, see Section 2.4). Still, for some fixed choice of ε-perturbation data, the filtered structure on $Fuk(M, p)$ does contain interesting informations about M and $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M, \omega)$ and the limit for $\varepsilon \to 0$ can be computed for some invariants arising from filtrations, as we will show in forthcoming work. Anyway we plan to use the technology of continuation functors to define a limit Fukaya category for $\varepsilon \to 0$ in future work.

2.3.1. ε-Perturbation data for $d^R \leq 2$. Fix $\varepsilon > 0$ and $\delta \in \left(\frac{1}{2}, 1\right)$. The basic idea for the definition of our class E^{ε} of perturbation data is to construct homotopies on strip-like ends between Floer data and the zero form, similarly to the case of continuation maps. Note that, as mentioned in Section 2.2, since Morse flowlines and homolomorphic disks do not carry Hamiltonian perturbations, we just have to choose special Floer data for couples made of different Lagrangians and special perturbation data for Floer polygons, which, via gluing, amounts to choose perturbation data for what we will call ‘fundamental’ polygons: strips and 3-punctured disks (with marked points).

REMARK 2.3.1.1. From their definition in Section 2.2.2, it is clear that perturbation data on some universal family depend on the a priori choice of (consistent) strip-like ends and system of ends on such family. However, the (explicit or inductive) definition of such perturbation data does not really depend on the choice of ends on a formal level, but only on the fact that such a choice has been done. In other words, if we perturb strip-like ends, then perturbation data change, altough their formal definition does not.

Let $\mathcal{F}$ be the set of smooth and increasing functions $\beta : \mathbb{R} \to [0,1]$ such that $\beta|_{(-\infty,0]} = 0$ and $\beta|_{[1,\infty)} = 1$.

DEFINITION 2.3.1.2. Let $L_0, L_1 \in \mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M,\omega)$ such that $L_0 \neq L_1$. An (ε,δ)-Floer datum for (L_0, L_1) is a choice of Floer datum $(H^{L_0,L_1}, J^{L_0,L_1})$ as in Section 2.2.2 such that

$$\mathrm{image}(H^{L_0,L_1}) \subset (\delta\varepsilon, \varepsilon)$$

Meanwhile, an (ε,δ)-Floer datum for a couple (L, L) of identical Lagrangians in $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M,\omega)$ is just a choice of Floer datum as in Section 2.2.2.

Fix a choice of (ε,δ)-Floer datum for any couple of Lagrangians in $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M,\omega)$; all the perturbation data we will define in the following are to be taken with respect to this choice of Floer data.

Let $d \geq 2$ and consider a tuple $\vec{L} = (L_0, \dots, L_d)$ of Lagrangians in $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M,\omega)$ such that $d^R \leq 2$. We split the definition of (ε,δ)-perturbation data for this case in five subcases. Recall that given a universal choice of strip-like ends and an element $(r,T,\lambda) \in \mathcal{R}^{d+1}_C(\vec{L})$ we denote by $\epsilon_i = \epsilon_i^{r,T,\lambda}$ for $i = 0, \dots, d$ the induced strip-like ends on $S_{r,T,\lambda}$ and by $\overline{\epsilon}_i$ for $i = 0, \dots, d^R$ the strip-like ends at positive marked points of type I (i.e. near punctures, see Definition 2.2.1.2).

Case 1 Assume first that $\vec{L_{\mathrm{red}}} = (L_0)$, i.e. $\vec{L} = (L_0, \dots, L_0)$. Then a choice of (ε,δ)-perturbation datum for $\vec{L}$ is just a choice of perturbation datum for $\vec{L}$ in the sense of Definition 2.2.2.2.

Case 2 Assume now that $\vec{L_{\mathrm{red}}} = (\overline{L_0}, \overline{L_1})$ and $m_0^e(\vec{L}) = 0$, that is $L_0 \neq L_d$ (see page 6 for the definition of the numbers m_i, m_0^b and m_0^e). Then a choice of (ε,δ)-perturbation datum for $\vec{L}$ is just a choice of perturbation datum for $\vec{L}$.

Case 3 Assume now that $\vec{L_{\mathrm{red}}} = (\overline{L_0}, \overline{L_1})$ and $m_0^e(\vec{L}) > 0$, that is, in particular, $L_0 = L_d$. Then a choice of (ε,δ)-perturbation datum for $\vec{L}$ is a choice $\Big((f^{\vec{L}}, g^{\vec{L}}), (K^{\vec{L}}, J^{\vec{L}})\Big)$ of perturbation datum for $\vec{L}$ such that for any $(r,T,\lambda) \in \mathcal{R}^{d+1}_C(\vec{L})$ and for the unique vertex $v \in V(T_{\mathrm{red}})$ of T_{red} we have that:

(1) $K_v^{r,T,\lambda}$ vanishes away from the strip-like end $\overline{\epsilon_i}$ for any $i \in \{0,1\}$;

(2) On the ith ($i = 0, 1$) negative strip-like end $\overline{\epsilon_i}$ we have

$$K_v^{r,T,\lambda} = \overline{H_{r,T,\lambda}^{L_{i-1},L_i}}(s,t)dt$$

where

$$\overline{H_{r,T,\lambda}^{L_{i-1},L_i}} : (-\infty, 0] \times [0,1] \times M \longrightarrow \mathbb{R}$$

is of the form

$$\overline{H_{r,T,\lambda}^{L_{i-1},L_i}}(s,t) = (1 - \beta_{r,T,\lambda}^{L_{i-1},L_i}(s+1))H_t^{L_{i-1},L_i}$$

for some $\beta_{r,T,\lambda}^{L_{i-1},L_i} \in \mathcal{F}$.

Case 4 Assume now that $\vec{L}_{\text{red}} = (\overline{L_0}, \overline{L_1}, \overline{L_2})$ and $m_0^e(\vec{L}) = 0$. A choice of (ε, δ)-perturbation for $\vec{L}$ is a choice $\Big((f^{\vec{L}}, g^{\vec{L}}), (K^{\vec{L}}, J^{\vec{L}})\Big)$ of perturbation datum for $\vec{L}$ such that for any $(r, T, \lambda) \in \mathcal{R}_C^{d+1}(\vec{L})$ and for the unique vertex $v \in V(T_{\text{red}})$ of T_{red} we have that:

(1) $K_v^{r,T,\lambda}$ vanishes away from the strip-like end $\overline{\epsilon_i}$ for any $i \in \{0, 1, 2\}$;

(2) On the ith ($i = 1, 2$) negative strip-like end $\overline{\epsilon_i}$ we have

$$K_v^{r,T,\lambda} = \overline{H_{r,T,\lambda}^{L_{i-1},L_i}}(s,t)dt$$

where

$$\overline{H_{r,T,\lambda}^{L_{i-1},L_i}} : (-\infty, 0] \times [0,1] \times M \longrightarrow \mathbb{R}$$

is of the form

$$\overline{H_{r,T,\lambda}^{L_{i-1},L_i}}(s,t) = (1 - \beta_{r,T,\lambda}^{L_{i-1},L_i}(s+1))H_t^{L_{i-1},L_i}$$

for some $\beta_{r,T,\lambda}^{L_{i-1},L_i} \in \mathcal{F}$;

(3) On the unique positive strip-like end we have

$$K_v^r = \overline{H_{r,T,\lambda}^{L_0,L_d}}dt$$

where

$$\overline{H_{r,T,\lambda}^{L_0,L_d}} : [0, +\infty) \times [0,1] \times M \longrightarrow \mathbb{R}$$

is of the form

$$\overline{H_{r,T,\lambda}^{L_0,L_d}}(s,t) = \beta_{r,T,\lambda}^{L_0,L_2}(s)H_t^{L_0,L_2}$$

for some $\beta_{r,T,\lambda}^{L_0,L_2} \in \mathcal{F}$.

Case 5 Assume now that $\vec{L}_{\text{red}} = (\overline{L_0}, \overline{L_1}, \overline{L_2})$ and $m_0^e(\vec{L}) > 0$. A choice of (ε, δ)-perturbation for $\vec{L}$ is a choice $\Big((f^{\vec{L}}, g^{\vec{L}}), (K^{\vec{L}}, J^{\vec{L}})\Big)$ of perturbation datum for $\vec{L}$ such that for any $(r, T, \lambda) \in \mathcal{R}_C^{d+1}(\vec{L})$ and for the unique vertex $v \in V(T_{\text{red}})$ of T_{red} we have that:

(1) $K_v^{r,T,\lambda}$ vanishes away from the strip-like ends $\overline{\epsilon_i}$ for $i \in \{0, 1, 2\}$;

(2) On the ith ($i = 0, 1, 2$) negative strip-like $\overline{\epsilon_i}$ end we have

$$K_v^{r,T,\lambda} = \overline{H_{r,T,\lambda}^{L_{i-1},L_i}}dt$$

where

$$\overline{H_{r,T,\lambda}^{L_{i-1},L_i}} : (-\infty, 0] \times [0,1] \times M \longrightarrow \mathbb{R}$$

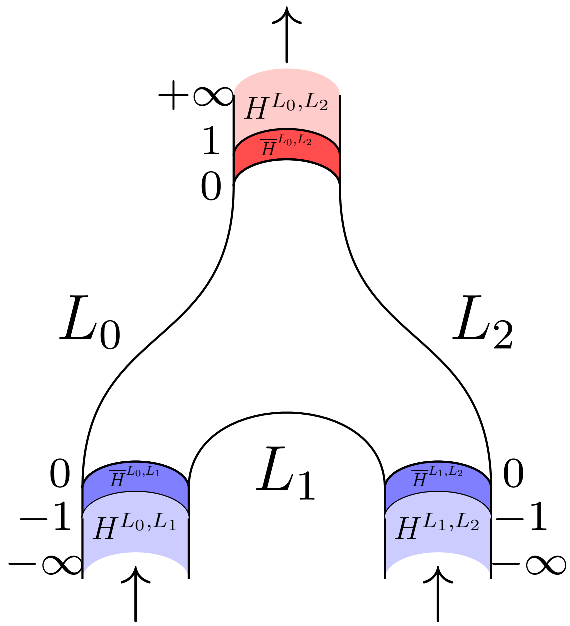


FIGURE 4. **Case 4**: A schematic representation of the Hamiltonian part a choice of (ε, δ)-perturbation datum on a punctured disk in $\mathcal{R}^3_C(\vec{L})$ for a triple $\vec{L} = (L_0, L_1, L_2)$ made of different Lagrangians. White corresponds to vanishing Hamiltonian perturbation.

is of the form

$$\overline{H^{L_{i-1},L_i}_{r,T,\lambda}}(s,t) = (1 - \beta^{L_{i-1},L_i}_{r,T,\lambda}(s+1))H^{L_{i-1},L_i}_t$$

for some $\beta^{L_{i-1},L_i}_{r,T,\lambda} \in \mathcal{F}$.

Before defining (ε, δ)-perturbation data for tuples of Lagrangians in $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M, \omega)$ with reduced tuple of length greater than 4, we stop for a moment to explain why the above construction is useful for the filtration point of view. Until the end of this subsection we assume transversality of all Floer and perturbation data. The map μ_d associated with tuples of Lagrangians as in **Case 1** and **Case 2** above preserves action filtrations, as discussed in Section 2.2.6. Let $\vec{L}$ be a tuple of Lagrangians in $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M, \omega)$ as in **Case 4** above. As it will be apparent from the following computations, this is the most delicate case of the three remaining ones, as it is the only one with a positive contribution to the curvature term coming from the (unique) positive strip-like end. For simplicity, we will assume that $d^R = d = 3$; the treatment of the general case is similar to this one, the only complication is a pure formalism: the presence of quantum-Morse trees, which do not interfere with curvature terms of Floer polygons.

Let $(K^{\vec{L}}, J^{\vec{L}})$ be a choice of (ε,δ)-perturbation data for $\vec{L}$. We show that the associated map

$$\mu_2 : CF(L_0, L_1) \otimes CF(L_1, L_2) \to CF(L_0, L_2)$$

preserves filtrations. Consider orbits $\gamma_1 \in \mathcal{O}(H^{L_0,L_1})$, $\gamma_2 \in \mathcal{O}(H^{L_1,L_2})$ and $\gamma^+ \in \mathcal{O}(H^{L_0,L_2})$ and a class $A \in \pi_2(M, \vec{L})$, and assume there is a Floer polygon with respect to the above chosen (ε,δ)-perturbation data connecting γ_1^- and γ_2^- to γ^+ in the class A. Note that $\mathcal{R}^3_C(\vec{L}) = \mathcal{R}^3(\vec{L})$ is a singleton (made of a 3-punctured disk with no marked points), say $\{pt\}$ we write $K^{\vec{L}} = K^{pt}_v$ and omit $pt \in \mathcal{R}^3(\vec{L})$ from the notation from now on. We estimate the cuvature term $\int_S R^{K^{\vec{L}}} \circ u$ of $K^{\vec{L}}$ on u. For any $i \in \{0,1,2\}$ we denote by $\int_i R^{K^{\vec{L}}} \circ u$ the integral over the strip-like end $\overline{\epsilon_i}$ of $S := S_{pt}$. By definition of (ε,δ)-perturbation data, we have the equality

$$\int_S R^{K^{\vec{L}}} \circ u = \sum_{i=0}^{3} \int_i R^{K^{\vec{L}}} \circ u$$

Consider $i \in \{1,2\}$ first, then with respect to the coordinates on the respective strip-like ends (which are the negative one) we have

$$\begin{aligned}\int_i R^{K^{\vec{L}}} \circ u &= \int_0^1 \int_{-\infty}^0 R^{K^{\vec{L}}} \circ u = \int_0^1 \int_{-1}^0 R^{K^{\vec{L}}} \circ u = \int_0^1 \int_{-1}^0 \partial_s \overline{H^{L_0,L_1}_{s,t}} \circ u \; dsdt \\ &= -\int_0^1 \int_0^1 \partial_s \beta^{L_0,L_1} H^{L_0,L_1}_t \circ u \; dsdt \leq -\int_0^1 \int_0^1 \partial_s \beta^{L_0,L_1} \; ds \; \min_M H^{L_0,L_1}_t \; dt \\ &< -\delta\varepsilon\end{aligned}$$

On the other hand, for $i = 0$, i.e. the unique positive end, we have

$$\begin{aligned}\int_0 R^{K^{\vec{L}}} \circ u &= \int_0^1 \int_0^{+\infty} R^{K^{\vec{L}}} \circ u = \int_0^1 \int_0^1 R^{K^{\vec{L}}} \circ u = \int_0^1 \int_0^1 \partial_s \overline{H^{L_0,L_2}_{s,t}} \circ u \; dsdt \\ &= \int_0^1 \int_0^1 \partial_s \beta^{L_0,L_2} H^{L_0,L_2}_t \circ u \; dsdt \\ &\leq \left(\int_0^1 \int_0^1 \partial_s \beta^{L_0,L_2} \; ds \right) \max_M H^{L_0,L_2}_t \; dt \\ &< \varepsilon\end{aligned}$$

Hence, we conclude

$$\int_S R^{K^{\vec{L}}} \circ u < \varepsilon(1 - 2\delta) < 0$$

as $\delta > \frac{1}{2}$ by assumption. By the computations carried out in Section 2.2.6 and the definition of the filtrations on Floer complexes via the action functional we conclude that μ_2 is a filtration preserving map. As explained above, this implies that the μ_d-maps defined for tuples of Lagrangians as in **Case 4** via (ε,δ)-perturbation data are filtered (if we assume transversality). Moreover, similar (but easier) computations to the ones presented above show that the μ_d-maps for tuples of Lagrangians as in **Case 3** and **Case 5** defined via (ε,δ)-perturbation data are filtered too, because Floer polygons for such tuples only have entries and no exit (the exit will be a Morse flowline).

2.3.2. ε-Perturbation data for $d^R > 2$. Let $\varepsilon > 0$, $\delta \in \left(\frac{1}{2}, 1\right)$ and $d^R > 2$. We assume that (ε, δ)-perturbation data have been defined for any tuple of Lagrangians in $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M, \omega)$ such that its reduced tuples has length $< d^R+1$. Let $\vec{L} = (L_0, \ldots, L_d)$ be a tuple of Lagrangians in $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M, \omega)$ such that its reduced tuple $\vec{L_{\mathrm{red}}}$ has length $d^R + 1$. Near the boundary $\partial\mathcal{R}_C^{d+1}(\vec{L})$ of $\overline{\mathcal{R}_C^{d+1}(\vec{L})}$, more precisely on an union of neighbourhoods of boundary strata where the associated gluing maps are diffeomorphisms onto their image, we define (ε, δ)-perturbation data to be images of lower order (ε, δ)-perturbation data for different tuples of Lagrangians under the gluing maps. In particular, near vertices of $\overline{\mathcal{R}_C^{d+1}(\vec{L})}$ (ε, δ)-perturbation data are obtained by gluing of 'fundamental' (ε, δ)-perturbation data, i.e. those we explicitly defined in Section 2.3.1 for strips and 3-punctured disks with marked points. Note that consistency of strip-like ends and of system of ends implies that this construction is well-defined. Before extending the definition to the whole $\mathcal{R}_C^{d+1}(\vec{L})$ we make the following (obvious) remark, which is however fundamental from the point of view of our aim to achieve a filtered Fukaya category.

Remark 2.3.2.1. A first observation is that Floer clusters defined on clusters lying near vertices of $\partial\mathcal{R}_C^{d+1}(\vec{L})$ are the most problematic from the filtration point of view, as they are the ones with most inherited positive strip-like ends on thin parts (which contribute positively to the total curvature term). The further away we go from vertices, the less positive contributions to the total curvature term a Floer cluster will inherit. In this remark, we want to show that, although being the most problematic case, cluster near vertices have negative curvature term.

(1) Assume first that $\vec{L}$ is cyclically different (i.e. $d = d^R$ in the notation of Section 2.1.2) so that in particular $L_0 \neq L_d$. Let $T \in \mathcal{T}_U^{d+1}(\vec{L})$ such that $|V(T_{\mathrm{red}})| = 1$ (note that in this case T_{red} has $d^R+1 = d+1$ exterior edges, among which exactly one is outgoing), and assume that $(r, T, \lambda) \in \mathcal{R}_C^{d+1}(\vec{L})$ lies near $\partial\mathcal{R}_C^{d+1}(\vec{L})$ in the sense above. Then $S_{r,T,\lambda}$ has at most $2d - 1$ thin parts ($d + 1$ 'exterior' ones plus $d - 2$ 'interior' ones, one for each codimension of $\mathcal{R}_C^{d+1}(\vec{L})$). Indeed, $S_{r,T,\lambda}$ has $2d - 1$ thin parts exactly when (r, T, λ) lies near a vertex of $\overline{\mathcal{R}_C^{d+1}(\vec{L})}$. In this case (assuming transversality), a Floer polygon defined on $S_{r,T,\lambda}$ endowed with the above defined (ε, δ)-perturbation data carries a curvature term strictly bounded above by
$$\varepsilon - d\delta\varepsilon + (d-2)\left(\varepsilon - \delta\varepsilon\right) = (d-1)\varepsilon(1-2\delta) < 0$$
as $\delta > \frac{1}{2}$ (the $\varepsilon - \delta\varepsilon$ term coming from thin parts in the interior). Note that the last term goes to 0 as $\delta \to \frac{1}{2}$.

(2) Assume now that $L_0 \neq L_d$ but $\vec{L}$ is not necessarily cyclically different, then a similar reasoning to the above (with d^R instead of d) shows that Floer polygons associated to $\vec{L}$ carry a curvature term bounded above by
$$(d^R - 1)\varepsilon(1 - 2\delta) < 0.$$

(3) Assume now that $\vec{L}$ is almost cyclically different, in particular $L_0 = L_d$ and $d^R = d-1$. Let $T \in \mathcal{T}_U^{d+1}(\vec{L})$ such that $|V(T_{\text{red}})| = 1$ (note that in this case T_{red} has $d^R = d-1$ exterior edges, all of which are oriented towards the only vertex of T_{red}), and assume that $(r, T, \lambda) \in \mathcal{R}_C^{d+1}(\vec{L})$ lies near $\partial\mathcal{R}_C^{d+1}(\vec{L})$. Then $S_{r,T,\lambda}$ has at most $d + (d-2) = 2d - 2 = 2d^R$ thin parts (d 'exterior' ones plus $d-2$ 'interior' ones). Indeed, $S_{r,T,\lambda}$ has $2d^R$ thin parts exactly when (r, T, λ) lies near a vertex of $\overline{\mathcal{R}_C^{d+1}(\vec{L})}$. In this case (assuming transversality), a Floer polygon defined on $S_{r,T,\lambda}$ endowed with the above defined (ε, δ)-perturbation data carries a curvature term bounded above by

$$-(d_R + 1)\delta\varepsilon + (d_R - 1)\varepsilon(1 - 2\delta) = d^R\varepsilon(1 - 2\delta) - \varepsilon < 0.$$

(4) Assume now that $L_0 = L_d$ but $\vec{L}$ is not necessarily cyclically different. Then it is easy to see that the same estimate as in point (3) holds.

In view of the above remark, after having defined (ε, δ)-perturbation data near the boundary of $\mathcal{R}_C^{d+1}(\vec{L})$, we interpolate on the whole $\mathcal{R}_C^{d+1}(\vec{L})$ while keeping the requirement that the total curvature term is non-positive.

Summing things up, we just defined a family $E^{\varepsilon,\delta} \subset E$ of consistent perturbation data for any positive real $\varepsilon > 0$ and any $\delta \in \left(\frac{1}{2}, 1\right)$. Indeed, notice that the inductive definition of (ε, δ)-perturbation data implies that we get consistency for free. Moreover, our discussions above imply that, assuming transversality, for any $\varepsilon > 0$, any $\delta \in \left(\frac{1}{2}, 1\right)$ and any $p \in E^{\varepsilon,\delta}$ the associated Fukaya category $Fuk(M; p)$ is a strictly unital and filtered A_∞-category. We set

$$E^\varepsilon := \bigcup_{\frac{1}{2} < \delta < 1} E^{\varepsilon,\delta}$$

and refer to elements of E^ε as ε-perturbation data. In the remaining of the section we discuss transversality of our perturbation data, which is the last ingredient missing in the proof of Theorem 2.3.0.1. Regularity of quantum trees is generic for (ε, δ)-perturbation data, exactly as explained in the general case in Section 2.2.4. It only remains to deal with regularity of Floer polygons defined via (ε, δ)-perturbation data.
Let $d \geq 2$, $\vec{L} = (L_0, \ldots, L_d)$ be a tuple of Lagrangians in $\mathcal{L}\text{ag}^{(\text{mon},\mathbf{d})}(M, \omega)$ and (K, J) the Floer part of p restricted to $\mathcal{R}_C^{d+1}(\vec{L})$. We generalize [**Sei08**, Chapter 9k] and define an admissible deformation of (K, J) as a couple

$$(\Delta K, \Delta J) \in \left(\Omega^1(S_{r,T,\lambda}(v), C^\infty(M)) \times C^\infty(S_{r,T,\lambda}(v), T_J\mathcal{J})\right)_{v \in V(T_{\text{red}}), (r,T,\lambda) \in \mathcal{R}_C^{d+1}(\vec{L})}$$

(where $\mathcal{J}$ is the space of ω-compatible almost complex structures on M) smooth in the $\mathcal{R}_C^{d+1}(\vec{L})$ direction and such that $(\Delta K_v^{r,T,\lambda}, J_v^{r,T,\lambda})$ (that is, the couple $(\Delta K, \Delta J)$ restricted to the polygon corresponing to the vertex $v \in V(T_{\text{red}})$ on the cluster $S_{r,T,\lambda}$) is supported on the thick parts of the polygons $S_{r,T,\lambda}(v)$ and $\Delta K_v^{r,T,\lambda}$ vanishes along vectors tangent to the boundary of $S_{r,T,\lambda}(v)$. The deformation of (K, J) via $(\Delta K, \Delta J)$ is defined on $S_{r,T,\lambda}(v)$, for

$(r,T,\lambda) \in \mathcal{R}_C^{d+1}(\vec{L})$ and $v \in V(T_{\text{red}})$ as

$$\left(K_v^{r,T,\lambda} + \Delta K_v^{r,T,\lambda}, J_v^{r,T,\lambda} \exp(-J_v^{r,T,\lambda} \Delta J_v^{r,T,\lambda})\right).$$

This way we defined the concept of admissible Hamiltonian deformation for the elements of E. Note that the requirement that $(\Delta K, \Delta J)$ is supported on thick parts of polygons is fundamental in order to keep consistency for the deformed Hamiltonian perturbation datum.

Definition 2.3.2.2. We define $E_{\text{reg}}^{\varepsilon,\delta} \subset E_{\text{reg}}$ to be the space of regular perturbation data $p \in E_{\text{reg}}$ such that the Floer parts of p are obtained via an admissible deformation of the Floer parts of some (ε,δ)-perturbation datum $q \in E^{\varepsilon,\delta}$ and such that the associated (well-defined) Fukaya category $Fuk(M;p)$ is filtered. Moreover, we define

$$E_{\text{reg}}^{\varepsilon} := \bigcup_{\frac{1}{2}<\delta<1} E_{\text{reg}}^{\varepsilon,\delta}$$

and refer to its elements as regular ε-perturbation data.

To conclude the proof of Theorem 2.3.0.1 we show that for any $\varepsilon > 0$ the space $E_{\text{reg}}^{\varepsilon}$ is non-empty. Let $q \in E^{\varepsilon,\delta}$ for some $\varepsilon > 0$ and $\delta \in \left(\frac{1}{2}, 1\right)$. Let $d \geq 2$, $\vec{L} = (L_0, \dots, L_d)$ be a tuple of Lagrangians in $\mathcal{L}\text{ag}^{(\text{mon},\mathbf{d})}(M,\omega)$ and (K,J) the Floer part of p restricted to $\mathcal{R}_C^{d+1}(\vec{L})$. As proved[9] in [**Sei08**, Chapter 9k], a generic admissible deformation $(\Delta K, \Delta J)$ of (K,J) turns (K,J) into a regular Hamiltonian perturbation datum on $\mathcal{R}_C^{d+1}(\vec{L})$. Thus we can choose a generic admissible Hamiltonian 1-form ΔK supported on the thick parts of punctured disks such that the associated function

$$(r,T,\lambda) \in \mathcal{R}_C^{d+1}(\vec{L}) \longmapsto \sum_{v \in T_{\text{red}}} \int_{S_{r,T,\lambda}(v)} \max_{x \in M} R_{r,T,\lambda}^{\Delta K}(v)$$

is bounded above by

$$\begin{cases} (d^R - 1)\varepsilon(2\delta - 1), & \text{if } L_0 \neq L_d \\ d^R \varepsilon(2\delta - 1) + \varepsilon, & \text{if } L_0 = L_d \end{cases}$$

implying that the deformed Hamiltonian $K + \Delta K$ has a negative curvature term overall, as the supports of K and ΔK are disjoint. In particular, the map μ_d defined via $K + \Delta K$ is filtered.

This concludes the proof of Theorem 2.3.0.1

Remark 2.3.2.3. (1) The above recipe for transversality allows us to define μ_d-maps shifting filtration by ≤ 0. By restricting the possible choice of deformations to the ΔK with arbitrary small curvature, say with curvature function bounded by a small number $\eta > 0$, we obtain for any $d \geq 2$ that the map μ_d (for instance applied to a cyclically different tuple of length $d+1$) shifts filtration by

$$\leq (d-1)\varepsilon(1 - 2\delta) + \eta.$$

[9]To see that Seidel's proof keep on working in the cluster setting it is enough to apply it to any vertex of T_{red}.

This control over the negative shift might be useful in some situations.

(2) Note that in general $E^{\varepsilon,\delta}_{\text{reg}} \not\subset E^{\varepsilon,\delta}$. However, we conjecture that it is possible to modify the above transversality argument to define a residual subset $\tilde{E}^{\varepsilon,\delta}_{\text{reg}} \subset E^{\varepsilon,\delta}$ of regular (ε,δ)-perturbation data. The idea is to fix the Floer part of the perturbation data and, via an extension of the arguments contained in [**MS12**], to show that there are adequate deformations of the almost complex structures turning (ε,δ)-perturbation data into regular ones. Moreover, in that case we could define (ε,δ)-perturbation data for $\delta = \frac{1}{2}$ too.

2.3.3. Observation: the case of the unit in the standard model of Fukaya categories. In this subsection we explain the main reason for working with a hybrid Floer-Morse (or 'cluster') model for the Fukaya category, in order to obtain a genuinely filtered A_∞-category and not only a weakly-filtered one, as in [**BCS21**].

We recall that in the standard model of the Fukaya category (that is, the one presented in [**Sei08**], see e.g. [**BC14**] for a monotone version) *all* Floer chain complexes are defined via Hamiltonian perturbations (also the ones associated to couple of identical Lagrangians) and the μ_d-maps are defined by counting Floer polygons joining Hamiltonian orbits (and no Morse trees). Note that the construction of the space E^ε of ε-perturbation data generalizes without much effort to this standard model and it is easy to see that the associated maps μ_d are filtered in this case too. However, one of the main differences between the hybrid Floer-Morse model defined in this paper and the standard one is that the definition of the (representatives of) the units in the latter involves counting homolomorphic disks with Hamiltonian perturbations, and this may a priori lead to curvature terms taking positive values and hence to representatives of units that may lie at positive filtration levels (see [**BCS21**, Proposition 3.1(ii)]). Indeed, in the following we sketch a proof of the fact that one *cannot* define representatives of the units lying at vanishing filtration level using ε-perturbation data on the standard model of $Fuk(M)$. The same proof can be expanded to show that it is not possible to have a filtered Fukaya category via the standard model: one has to give up either filtering the maps μ_d or having units at vanishing filtration levels.

Remark 2.3.3.1. Not having filtered units may seem a marginal fact compared to not having filtered μ_d-maps. However, upcoming work will show why filtration-zero units are desiderable, leading to a 'minimal energy' Fukaya category and interesting results. Moreover, for the derived Fukaya category of a monotone symplectic manifold to fit the definition of a triangulated persistence category (TPC) (see the recent work [**BCZ24b**]), vanishing filtration levels for units are crucial.

We briefly recall how representatives of the units are constructed in [**Sei08**]. Let $\varepsilon > 0$, $\delta \in \left(\frac{1}{2}, 1\right)$ and $L \in \mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M,\omega)$ be a monotone Lagrangian. Choose a Floer datum $(H^{L,L}, J^{L,L})$ for (L,L) in the sense of Seidel, i.e. a time-dependent Hamiltonian $H^{L,L}$ on M such that $\varphi^1_H(L) \pitchfork L$, where φ^t_H denotes the Hamiltonian flow of H, and an ω-compatible time dependent almost complex structure $J^{L,L}$ on M. We assume that $(H^{L,L}, J^{L,L})$ fits the definition of a (ε, δ)-Floer datum, i.e. image$(H^{L,L}) \subset (\delta\varepsilon, \varepsilon)$. We define

$$CF^S(L, L) := \bigoplus_{\gamma \in \mathcal{O}(H^{L,L})} \Lambda \cdot \gamma$$

where the superscript S stands for 'standard', $\mathcal{O}(H^{L,L})$ denotes the set of Hamiltonian orbits of $H^{L,L}$ and Λ the Novikov field over $\mathbb{Z}_2$. We further assume that $(H^{L,L}, J^{L,L})$ is regular, i.e. $CF^S(L, L)$ is a chain complex when endowed with the standard differential counting Floer strips. Consider the standard unit disk $D \subset \mathbb{C}$, define $D_1 := D \setminus \{1\}$ and pick a positive strip-like end (see page 15) on D_1 near the point $1 \in D$. We define ε-perturbation data for L on D_1 analogously to ε-perturbation data for $(d+1)$-punctured disks where $d \geq 2$ as introduced in Section 2.3.1 and 2.3.2. The notion of curvature term is also defined analogously. Let (K^L, J^L) be such a perturbation datum. Let $\gamma \in \mathcal{O}(H^{L,L})$ and $A \in \pi_2(M, L)$ such that $\mu(\gamma; A) = 0$ and consider the space $\mathcal{M}(\gamma; A)$ of (K^L, J^L)-perturbed Floer 1-gons $u : D_1 \to M$ such that on the unique strip-like end of D_1 we have

$$\lim_{s \to \infty} u(s, t) = \gamma(t).$$

We assume regularity of the perturbation datum (K^L, J^L), so that $\mathcal{M}(\gamma; A)$ is a smooth manifold of dimension 0 for any orbit $\gamma \in \mathcal{O}(H^{L,L})$ and any class $A \in \pi_2(M, L)$ as above. Standard Gromov-compactness arguments show that in such cases $\mathcal{M}(\gamma; A)$ is compact. We define

$$e_L := \sum_{\gamma, A} \sum_{u} T^{\omega(u)} \gamma \in CF^S(L, L)$$

where the first sum runs over classes $\gamma \in \mathcal{O}(H^{L,L})$ and classes $A \in \pi_2(M, L)$ such that $\mu(\gamma; A) = 0$ and the second sum runs over Floer 1-gons $u \in \mathcal{M}(\gamma; A)$. It is well-known that e_L is a representative of the homological unit of L in the standard model of $Fuk(M)$ (see [**Sei08**, Section 8d]).

In the remaining of this section we sketch a proof of the fact that e_L lies at a positive filtration level in $CF^S(L, L)$, that is $\mathbb{A}(e_L) > 0$. Assume by contradiction that that $\mathbb{A}(e_L) \leq 0$. Then by definition of the action functional $\mathbb{A}$ (see page 33) we have

$$C := \max_{u \in \mathcal{M}_0(\gamma)} \int_{D_1} R^{K^L} \circ u \leq 0$$

where R^{K^L} denotes the curvature two form (introduced on page 35) associated to the Hamiltonian perturbation datum K^L. Let now $\tilde{H}^{L,L}$ be another Hamiltonian on M such that $\varphi_{\tilde{H}^{L,L}}(L) \pitchfork L$ and assume that $0 < \tilde{H} < H$. It is well known that (assuming regularity) the Floer homology defined using $(H^{L,L}, J^{L,L})$ is quasi-isomorphic as a chain complex to the one defined using $(\tilde{H}^{L,L}, J^{L,L})$. However, those two homologies are *not* isomorphic as persistence modules, as the filtrations at the chain level may differ dramatically in general. However we sketch a proof of the fact that given our assumption on $\mathbb{A}(e_L)$ we can construct an isomorphism of persistence modules, leading to a contradiction.

It is easy to see that there is a filtration preserving chain map $CF(L, L, H) \to CF(L, L, \tilde{H})$

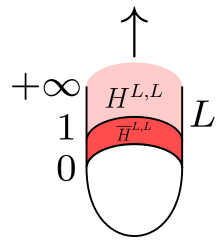


FIGURE 5. **Case 4**: A schematic representation of a representative of the unit in $CF^S(L, L)$ when endowed with some choice of ε-perturbation data.

(e.g. by choosing a monotone homotopy from H to $\tilde{H}$, see [**BPS03**]). We construct a filtration preserving map

$$\psi : CF(L, L, \tilde{H}) \to CF(L, L, H)$$

using our assumption on $\mathbb{A}(e_L)$. Pick D_1 and puncture it in -1 to get $D_{-1,1} \cong \mathbb{R} \times [0, 1]$. Then, we can see K^L as a one-form on $\mathbb{R} \times [0, 1]$ such that $K^L = 0$ (on a strip-like end) near $-\infty$. We define ψ as the continuation map associated with the Hamiltonian perturbation defined as the concatenation of a monotone homotopy from $\tilde{H}$ to 0 and K^L (seen as a one-form on the strip). Standard methods show that ψ is a chain map. Moreover, as $\tilde{H}$ is positive, the monotone homotopy has negative curvature term, and, as K has non-positive curvature term, energies idendities similar to the ones in Section 2.3.1 tell us that the map ψ is a filtered chain map. Hence it follows that $HF(L, L, H)$ and $HF(L, L, \tilde{H})$ are isomorphic as persistence modules. This contradicts the assumption $\tilde{H} < H$ and hence proves that the representative $e_L \in CF^S(L, L)$ of the unit satisfies $\mathbb{A}(e_L) > 0$, as claimed.

2.4. Continuation functors

It is well known that between two Floer complexes of the same objects (ambient symplectic manifold or couple of Lagrangians) defined using different Floer data one can construct chain maps, called continuation maps, defined by counting strips which homotope between the different data. Moreover these chain maps are quasi-isomorphisms. In particular, this shows that Floer homology is well-defined in the sense that it is independent of the auxiliary Floer data. In [**Syl19**], Sylvan extended continuation maps to A_∞-functors on (partially wrapped) Fukaya categories. In particular, he geometrically showed that Fukaya categories do not depend on choices up to quasi-equivalence of A_∞-categories. This approach differs from the well-known proof of this fact contained in [**Sei08**, Chapter 10], which heavily relies on algebraic machineries. In this section we will construct continuation functors for monotone Fukaya categories defined using the Morse-Bott model developed in Section 2.2 following the main ideas contained in Sylvain's work. In Section 2.4.7 we discuss how the filtered structure of filtered Fukaya categories defined using ε-perturbation data (as defined in Section 2.3) behaves under continuation functors. The main results of this section are summarized in the following theorem, which is an expanded version of Theorem **??**.

THEOREM 2.4.0.1. *Consider two regular perturbation data $p, q \in E_{\mathrm{reg}}$, and assume that they share the same Morse perturbation data. Then there is a non-empty space $E^{p,q}$ of so-called interpolation data and a residual subset $E^{p,q}_{\mathrm{reg}} \subset E^{p,q}$ such that for any $h \in E^{p,q}_{\mathrm{reg}}$ there is a weakly-filtered A_∞-functor*

$$\mathcal{F}^{p,q}_h \colon Fuk(M;p) \to Fuk(M;q)$$

called the continuation functor from $Fuk(M;p)$ to $Fuk(M;q)$ associated to h, which is a quasi-equivalence canonical up to quasi-isomorphism of A_∞-functors. Moreover, if $p \in E^{\varepsilon_1,\delta_1}_{\mathrm{reg}}$ and $q \in E^{\varepsilon_2}_{\mathrm{reg}}$ for some $\varepsilon_1, \varepsilon_2 > 0$ are ε-perturbation data sharing the same Morse part, then there is a non-empty subset $E^{p,q;f}_{\mathrm{reg}} \subset E^{p,q}_{\mathrm{reg}}$ such that for any $h \in E^{p,q,f}_{\mathrm{reg}}$ the associated functor $\mathcal{F}^{p,q}_h$ shifts filtration by $\leq \max(\varepsilon_2 - \delta_1\varepsilon_2, 0)$, in the sense of the definition appearing at the end of Section **??**. *In particular if $\varepsilon_2 \leq \delta_1\varepsilon_1$, then $\mathcal{F}^{p,q}_h$ is filtered for any choice of $h \in E^{p,q,f}_{\mathrm{reg}}$.*

Note that we define continuation functors only between Fukaya categories defined using the same Morse part of the perturbation data. This assumption may be easily dropped by defining Morse interpolation data, whose construction we omit to avoid notational complexity. In any case, Morse interpolation data play no role from the filtration point of view.

2.4.1. Colored trees. We introduce the notion of colored leafed tree following [**MW10**]. Let T be a d-leafed tree with no vertices of valency equal to 1 and finitely many interior edges. Given a metric $\lambda \in \lambda(T)$ and a vertex $v \in V(T)$ we define the distance $\lambda(v)$ of v from the root vertex v_T as the sum of the length $\lambda(e)$ of the edges $e \in E^{\mathrm{int}}(T)$ in the geodesic from v to v_T. A coloring on T is a choice of a subset $V^{\mathrm{col}}(T) \subset V(T)$ of vertices, which we call colored vertices, and of a metric $\lambda \in \lambda(T)$ such that:

(1) for any $i \in \{1, \ldots, d\}$, if we flow along the geodesic from $e_i(T)$ to the root $e_0(T)$ we meet exactly one colored vertex;
(2) each vertex of valency equal to 2 is colored;
(3) each colored vertex lies at the same distance from the root vertex v_T of T.

Two coloring are said to be equivalent if they carry the same set of colored vertices $V^{\mathrm{col}}(T)$ (and hence the metric is there only to make point (3) well-defined). A d-colored tree is a d-leafed tree T as above together with an equivalence class of colorings, which we denote by only writing the subset of colored vertices. We denote by $\mathcal{T}^{d+1}_{\mathrm{col}}$ the space of d-colored trees. Note that the notation might be a bit confusing, as $\mathcal{T}^{d+1}_{\mathrm{col}}$ contains non-stable trees, whereas $\mathcal{T}^{d+1}$ contains only stable trees by definition (see Definition 2.1.3.2(4)). Notice that there is a natural partition

$$V(T) = V^l(T) \sqcup V^{\mathrm{col}}(T) \sqcup V^r(T)$$

on colored trees by setting $V^l(T)$ to be the set of vertices of T lying at distance strictly smaller than vertices in $V^{\mathrm{col}}(T)$ from v_T (this notion does not depend on the choice of metric) and $V^r(T)$ to be the set of vertices of T lying at distance stritly larger than vertices in $V^{\mathrm{col}}(T)$ from v_T.

Let $V^{\mathrm{col}}(T) \subset V(T)$, then we denote by $\lambda(T, V^{\mathrm{col}}(T))$ the space of metrics $\lambda \in \lambda(T)$ such that $(V^{\mathrm{col}}(T), \lambda)$ is a coloring for T. The following lemma is proved in [**MW10**].

LEMMA 2.4.1.1. *For any choice of $V^{\mathrm{col}}(T) \subset V(T)$ the space $\lambda(T, V^{\mathrm{col}}(T))$ is a polyhedral set of dimension $|E^{\mathrm{int}}(T)| + 1 - |V^{\mathrm{col}}(T)|$.*

Let $d \geq 2$ and $\vec{L} = (L_0, \ldots, L_d)$ be a tuple of Lagrangians in $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M, \omega)$. We define by $\mathcal{T}^{d+1}_{\mathrm{col}}(\vec{L})$ the set of colored trees labelled by $\vec{L}$ and by $\mathcal{T}^{d+1}_{U,\mathrm{col}}(\vec{L})$ the set of colored trees labelled by $\vec{L}$ which can be represented by an unilabelled tree.

REMARK 2.4.1.2. As said above, our trees will be defined only between Fukaya categories constructed via perturbation data which share the same Morse part. As it will be apparent from the discussion below, this simplifies the definition of system of ends for colored labelled trees. The definition of universal system of ends for $\vec{L}$ (see Section 2.1.5 for the definition of system of ends), as well as that of consistency, readily translate to the case of colored labelled trees, as condition (3) in the definition implies that there are finitely many of those. We skip the details.

2.4.2. Stacked disks. To define the source spaces for continuation functors, we will adopt the same strategy as in Section 2.2.1: consider primitive source spaces consisting of some configuration of disks, compactify the moduli spaces of those and add a collar neighbourhood. The primitive source spaces in this case will be 'stacked' disks, whose moduli spaces will simply be a stack of the same moduli spaces of marked disks indexed by a positive real parameter.

We start from the case $d = 1$. Let (L_0, L_1) be a couple of Lagrangians in $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M, \omega)$. We define $\mathcal{R}^{2,2}(L_0, L_1) = \{pt\}$ to be a singleton and $\mathcal{S}^{2,2}(L_0, L_1) = \{S^{2,2}\}$, where $S^{2,2}$ is a strip if $L_0 \neq L_1$ and a disk with marked points in -1 and $+1$ if $L_0 = L_1$. In both cases we

assume that $S^{2,2}$ is endowed with two numbers $\varepsilon_- < 0$ and $\varepsilon_+ > 0$. We think of ε_- and ε_+ as two strip-like ends coming from conformal embeddings, and which may be positive or negative depending on the context.
Let now $d \geq 2$ and pick a tuple $\vec{L} = (L_0, \ldots, L_d)$ of Lagrangians in $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M,\omega)$. We define

$$\mathcal{R}^{d+1,2}(\vec{L}) := \mathcal{R}^{d+1}(\vec{L}) \times (0,\infty)$$

Over $\mathcal{R}^{d+1,2}(\vec{L})$ we have a fiber bundle

$$\pi^{d+1,2}(\vec{L})\colon \mathcal{S}^{d+1,2}(\vec{L}) \to \mathcal{R}^{d+1,2}(\vec{L})$$

where the fiber $S_{r,w} := \left(\pi^{d+1,2}(\vec{L})\right)^{-1}(r,w)$ over $(r,w) \in \mathcal{R}^{d+1,2}(\vec{L})$ equals (S_r, w), where $S_r \in \mathcal{S}^{d+1}(\vec{L})$ was defined in Section 2.2.1.
We fix a consistent choice of strip-like ends for the universal family $(\pi^{d+1}(\vec{L}))_{d,\vec{L}}$ for the rest of this section. A universal choice of strip-like ends on the universal family $(\pi^{d+1,2}(\vec{L}))_{d,\vec{L}}$ is said to be compatible with the above fixed universal choice of strip-like ends on $(\pi^{d+1}(\vec{L}))_{d,\vec{L}}$ if for any d, any $\vec{L}$ and any $(r,w) \in \mathcal{R}^{d+1,2}(\vec{L})$ the induced choice of strip-like ends on (S_r, w) agrees up to a shift with that on S_r.

Fix a tuple $\vec{L} = (L_0, \ldots, L_d)$ of Lagrangians in $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M,\omega)$. For any colored labelled tree $T \in \mathcal{T}^{d+1}_{\mathrm{col}}(\vec{L})$ we write

$$\mathcal{R}^{T,2} := \prod_{v \notin V^{\mathrm{col}}(T)} \mathcal{R}^{|v|}(\vec{L_v}) \times \prod_{v \in V^{\mathrm{col}}(T)} \mathcal{R}^{|v|,2}(\vec{L_v})$$

That is, to each uncolored vertex we associate a family of marked disks, while to each colored vertex we associate a family of stacked marked disks. We then define

$$\overline{\mathcal{R}^{d+1,2}(\vec{L})} := \bigcup_{T \in T^{d+1}_{\mathrm{col}}(\vec{L})} \mathcal{R}^{T,2}$$

We have the following standard-looking result.

Lemma 2.4.2.1. $\overline{\mathcal{R}^{d+1,2}(\vec{L})}$ *admits the structure of a generalized manifold with corners of dimension $d-1$ which realizes Stasheff's multiplihedron.*

As the definition of $\overline{\mathcal{R}^{d+1,2}(\vec{L})}$ involves two kind of moduli spaces the construction of boundary charts is more delicate than in the standard case of $\overline{\mathcal{R}^{d+1}(\vec{L})}$.
We split the construction of the boundary charts for $\overline{\mathcal{R}^{d+1,2}(\vec{L})}$ in three cases. A colored labelled tree is called:

(1) a lower tree if the root has valency equal to 2, and is hence in particular the only colored vertex;
(2) a middle tree if the root is colored and has valency greater than 2;
(3) an upper tree if the root is not colored.

We will see that constructing boundary charts near lower and middle trees is easy, while upper trees are more delicate to handle as there are multiple simultaneous splittings.

(1) Lower trees. First, we consider the colored labelled tree $T^2 \in \mathcal{T}^{d+1}(\vec{L})$ with only two vertices and two-valent root. Notice that $\mathcal{R}^{T^2,2} = \mathcal{R}^{2,2}(L_0, L_d) \times \mathcal{R}^{d+1}(\vec{L})$ and recall that we defined $\mathcal{R}^{2,2}(L_0, L_d)$ to be a singleton. We define a map

$$\gamma^{T^2,2}\colon (-1,0) \times \mathcal{R}^{T^2,2} \to \mathcal{R}^{d+1,2}(\vec{L})$$

as follows: given $r \in \mathcal{R}^{d+1}(\vec{L})$ and $\rho \in (-1,0)$ we set

$$\gamma^{T^2,2}(\rho, pt, r,) := (\tilde{r}, -\rho)$$

where $\tilde{r} \in \mathcal{R}^{d+1}(\vec{L})$ represents the surface obtained by gluing $pt \in \mathcal{R}^{2,2}(L_0, L_d)$ to the 0th marked point of r with length ρ. Notice that $r = \tilde{r}$ as elements of $\mathcal{R}^{d+1}(\vec{L})$, but they may come with different choices of strip-like ends.
Let now $T \in \mathcal{T}^{d+1}_{\text{col}}(\vec{L})$ be an arbitrary colored labelled tree with 2-valent root. Notice that $\mathcal{R}^{T,2} = \mathcal{R}^{2,2}(L_0, L_d) \times \mathcal{R}^{T'}$, where T' is obtained by collapsing the positive pointing edge attached to the root of T and can hence be viewed as an alement of $\mathcal{T}^{d+1}(\vec{L})$, as all the non root vertices of T are stable by definition of colored tree. We define a map

$$\gamma^{T,2}\colon (-1,0) \times (-1,0)^{|E^{\text{int}}(T)|-1} \times \mathcal{R}^{T,2} \to \mathcal{R}^{d+1,2}(\vec{L})$$

as follows: given $(r_v)_{v\in V(T')} \in \mathcal{R}^{T'}$, $\rho \in (-1,0)$ and $\rho_e \in (-1,0)$ for any $e \in E^{\text{int}}(T')$ we set

$$\gamma^{T,2}\left(\rho, (\rho_e)_e, (pt, (r_v)_v)\right) := \gamma^{T^2,2}\left(\rho, pt, \gamma^{T'}((\rho)_e, (r_v))\right)$$

where

$$\gamma^{T'}\colon (-1,0)^{|E^{\text{int}}(T')|} \times \mathcal{R}^{T'} \to \mathcal{R}^{d+1}(\vec{L})$$

is the gluing map defined in [**Sei08**]. Notice that by considering trivial gluing $\gamma^{T,2}$ extends to a map

$$\overline{\gamma^{T,2}}\colon (-1,0] \times (-1,0]^{|E^{\text{int}}(T)|-1} \times \mathcal{R}^{T,2} \to \overline{\mathcal{R}^{d+1,2}(\vec{L})}.$$

Remark 2.4.2.2. All these charts will correspond to genuine boundary charts in the sense of manifolds with corners. The only codimension one face arising in this subcase is parametrized by T_2, and in general, codimension k faces arising here are parametrized by bicolored trees with two-valent root and k uncolored vertices.

(2) Middle trees. Let $T \in \mathcal{T}^{d+1}_{\text{col}}(\vec{L})$ be a colored labelled tree with colored stable root. Notice that

$$\mathcal{R}^{T,2} = \prod_{v\neq v_T} \mathcal{R}^{|v|} \times \mathcal{R}^{|v_T|,2}.$$

Since the valency of v_T is at least 3 by assumption and v_T is colored (hence there are no two-valent vertices in T), we may see T as an element of $\mathcal{T}^{d+1}(\vec{L})$ by forgetting the coloring. In particular, $\mathcal{R}^T$ is well-defined. We define a map

$$\gamma^{T,2}\colon (-1,0)^{|E^{\text{int}}(T)|} \times \mathcal{R}^{T,2} \to \mathcal{R}^{d+1,2}(\vec{L})$$

as follows: given $\rho_e \in (-1,0)$ for any $e \in E^{\text{int}}(T)$, $r_v \in \mathcal{R}^{|v|}$ for any $v \neq v_T$ and $(r_{v_T}, w) \in \mathcal{R}^{|v_T|,2}$ we set

$$\gamma^{T,2}\left((\rho_e)_e, (r_{v_T}, w), (r_v)_v\right) := \left(\gamma^T\left((\rho_e)_e, (r_v)_{v\in V(T)}\right), w\right).$$

By considering trivial gluing $\gamma^{T,2}$ extends to a map

$$\overline{\gamma^{T,2}}\colon (-1,0]^{|E^{\text{int}}(T)|} \times \mathcal{R}^{T,2} \to \overline{\mathcal{R}^{d+1,2}(\vec{L})}.$$

Remark 2.4.2.3. All these charts correspond to genuine boundary charts in the sense of manifolds with corners. The codimension one faces arising in this subcase are parametrized by colored trees with colored root and one uncolored vertex, and in general, codimension k faces arising here are parametrized by bicolored trees with colored root and k uncolored vertices.

(3) Upper trees. As soon as the root is not colored, there may be complications in the choice of gluing lenghts, basically due to the fact that the coloring distance is unique, so that *colored vertices should remember what came before them* (where the concept of before is determined by the aforedefined orientation on trees).

Example 2.4.2.4. Consider the colored 3-tree labeled by the cyclically different (see page 6) tuple $\vec{L} = (L_0, L_1, L_2, L_3)$ of Lagrangians in $\mathcal{L}\text{ag}^{(\text{mon},\mathbf{d})}(M,\omega)$ and the associated stacked disk configuration in $\overline{\mathcal{R}^{4,2}(\vec{L})}$ depicted in Figure 6.

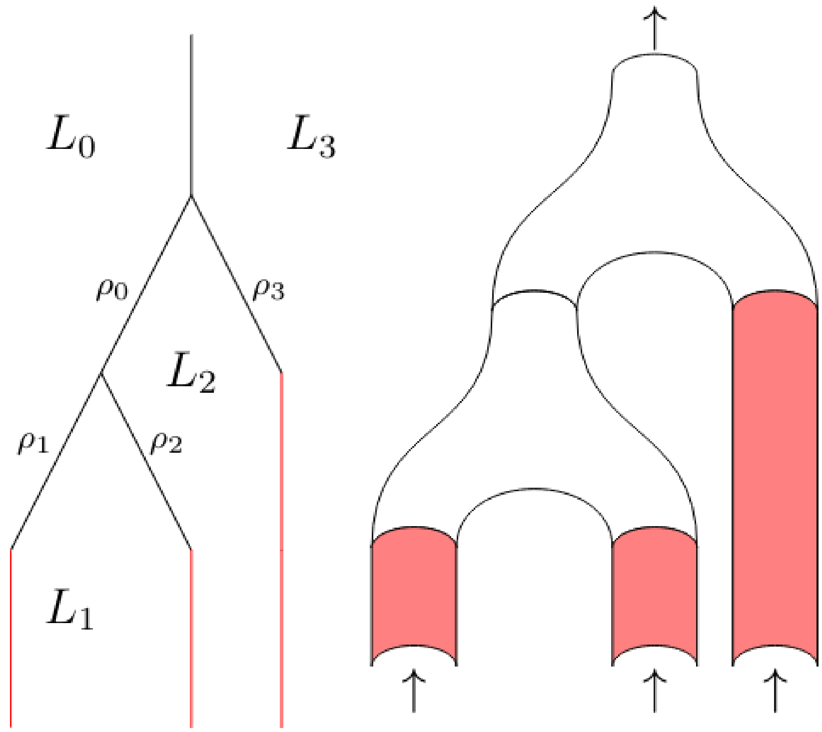


FIGURE 6. The colored tree we are discussing and the associated disks configuration in $\overline{\mathcal{R}^{4,2}(\vec{L})}$ for a cyclically different tuple $\vec{L} = (L_0, L_1, L_2, L_3)$. Black configurations correspond to elements of $\mathcal{R}^{k,1}$, while red configurations correspond to elements of $\mathcal{R}^{l,2}$ for some k, l.

The definition of colored tree forces $\rho_1 = \rho_2$ and $\rho_0 + \rho_1 = \rho_3$. Gluing the two non-colored disks following the only interior edge which does not touch colored vertices we get the tree/configuration depicted in Figure 7.

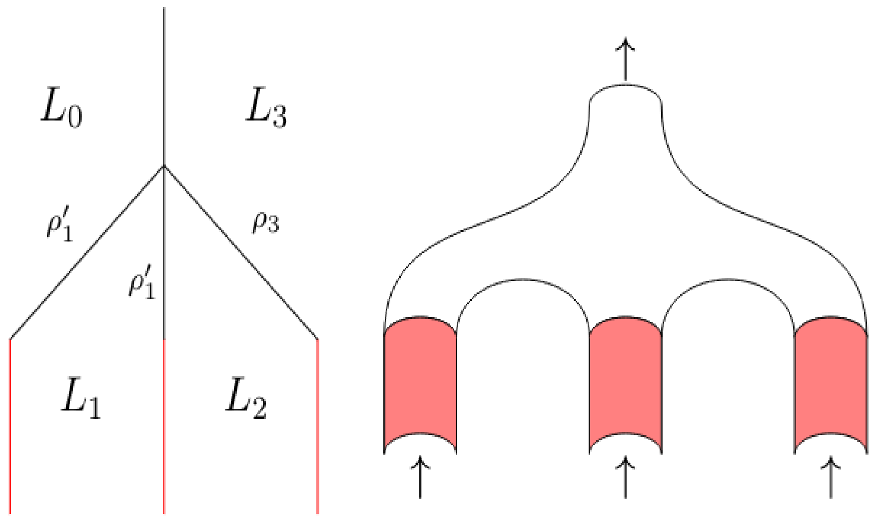


FIGURE 7

In this case, the definition of colored tree forces $\rho_1' = \rho_3 = \rho_0 + \rho_1$. On the other hand, if, starting again with the configuration from Figure 6, we glue along the interior edges touching a colored vertex, we get the tree/configuration depicted in Figure 8.

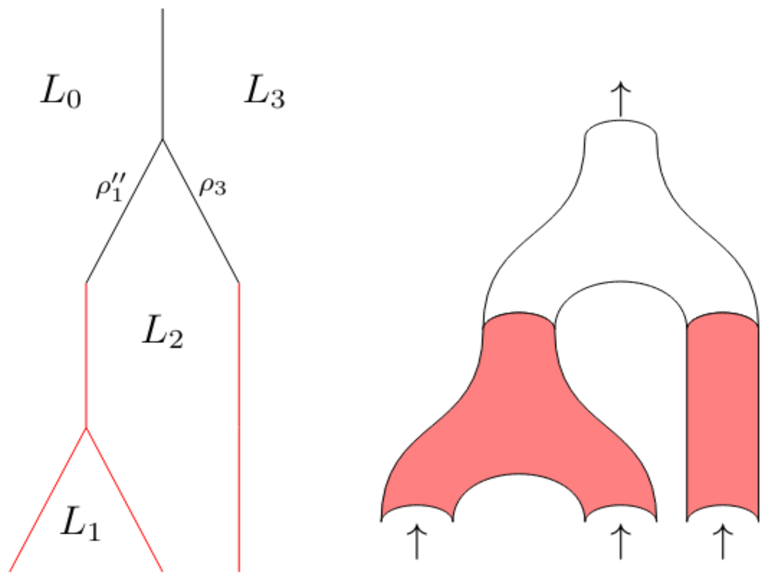


FIGURE 8. The colored tree we are discussing and the associated disks configuration in $\overline{\mathcal{R}^{3+1,2}}(\vec{L}$ for a cyclically different (see Page 6) tuple $\vec{L} = (L_0, \dots, L_3)$. Black configurations correspond to elements of $\mathcal{R}^{k,1}$, while red configurations correspond to elements of $\mathcal{R}^{l,2}$ for some k, l.

Again the definition of colored tree forces $\rho_1'' = \rho_3 = \rho_0 + \rho_1$. Performing one more gluing in both cases with lenght $\rho_1' = \rho_1''$ in order to get to an interior configuration in $\mathcal{R}^{3+1,2}$, we will in general get two different stacked disks, as in the first case it is ρ_0 that contributed to both gluings, while in the second one, it is ρ_1.

It follows from Example 2.4.2.4 that we have to refine the gluing process a bit, in order for non-root colored vertices to recall the gluing lenghts that were associated to preceedings (uncolored) vertices. The following definition comes from [Syl19].

DEFINITION 2.4.2.5 ([Syl19]). An intrinsic width function for the universal family $(\pi^{d+1}(\vec{L}))_{d,\vec{L}}$ consists of a family $\{w_i^{d,\vec{L}}\}$, $d \geq 2$ and $i \in \{1, ..., d\}$, of functions $w_i^{d,\vec{L}} \colon \mathcal{R}^{d+1}(\vec{L}) \to [0, \infty)$ such that:

(1) w_1^2 and w_2^2 are the zero function;
(2) let $S^{k+1} \in \mathcal{S}^{k+1}(\vec{L_1})$ and $S^{l+1} \in \mathcal{S}^{l+1}(\vec{L_2})$; if $S^{d+1} \in \mathcal{S}^{d+1}(\vec{L_1} \#_n \vec{L_2})$ is the disk[10] (diffeomorphic to the surface) obtained by gluing the root of S^{l+1} to the nth puncture

[10] See page 13 for the definition of the operation $\#$ on tuples of Lagrangians.

of S^{k+1} with lenght ρ (assuming this gluing is admissible), then

$$w_i^{d,\vec{L_1}\#_n\vec{L_2}}(S^{d+1}) = \begin{cases} w_i^{k,\vec{L_1}}(S^{k+1}), & \text{if } i<n \\ w_{i-n+1}^{l,\vec{L_2}}(S^{l+1})+\rho, & \text{if } n\leq i<n+l \\ w_{i-l+1}^{k,\vec{L_1}}(S^{k+1}), & \text{if } n+l\leq i \end{cases}$$

Lemma 2.4.2.6. *There is a unique choice of intrinsic width function.*

Proof. Build it by induction on d; the fact that the definition requires a fixed choice of function for $d=2$ implies uniqueness of the construction. See [**Syl19**]. $\square$

Back to the construction of boundary charts for $\overline{\mathcal{R}^{d+1,2}(\vec{L})}$ near upper trees. Let $k\geq 3$ and consider first a colored labelled tree $T^k\in\mathcal{T}^{d+1}_{\text{col}}(\vec{L})$ with only one non-colored vertex, the root, of valency k. In particular, T^k has k colored vertices. Notice that in this case we have

$$\mathcal{R}^{T^k,2}=\mathcal{R}^k\times\prod_{v\neq v_T}\mathcal{R}^{|v|,2}.$$

We define a map

$$\gamma^{T^k,2}\colon(-1,0)\times\mathcal{R}^{T^k,2}\to\mathcal{R}^{d+1,2}(\vec{L})$$

as follows: given $\rho\in(-1,0)$, $r\in\mathcal{R}^k$ and $(r_v,w_v)\in\mathcal{R}^{|v|,2}$ for any $v\neq v_T$ we set

$$\gamma^{T^k,2}\left(\rho,r,(r_v,w_v)_v\right):=(\tilde{r},e^{\frac{-1}{\rho}})$$

where $\tilde{r}$ corresponds to the configuration obtained by gluing the ith marked point of r to the negative marked point of $S_{r_{v_i}}$, where $v_i\in V(T^k)$ is the ith colored vertex of T^k in counterclockwise order starting from the root, with gluing length

$$l_i:=e^{-\frac{1}{\rho}}-w_{v_i}-w_i^k(S_r)$$

for any $i\in\{1,\dots,k\}$.
Let now $T\in\mathcal{T}^{d+1}_{\text{col}}(\vec{L})$ be a colored labelled tree with uncolored root. Consider the decomposition $V(T)=V^l(T)\sqcup V^{\text{col}}(T)\sqcup V^r(T)$ of the set of vertices of T introduced in Section 2.4.1: we define T^l to be the (uncolored) tree generated by $V^l(T)$ and the interior edges attached to those vertices (some of which become leaves), and by $T^r_1,\dots,T^r_n$ the connected components of the colored tree generated by $V^r(T)\sqcup V^{\text{col}}(T)$ (this involves a slight change in the induced metrics, but it is a nuance). Note that each T^r_i is either a lower or a middle tree. Since by construction the tree T^r_i has a colored root for any $i\in\{1,\dots,n\}$, it is either a lower or a middle tree. In particular any tree $T^r_i\setminus V^{\text{col}}(T)$ is a union of uncolored trees $T^r_{i,j}$, $j\in\{0,\dots,k_i\}$ for some k_i. Notice that we have

$$\mathcal{R}^{T,2}=\mathcal{R}^{T^l}\times\prod_{i=0}^n\mathcal{R}^{T^r_i,2}=\mathcal{R}^{T^l}\times\prod_{v\in V^{\text{col}}(T)}\mathcal{R}^{|v|,2}\times\prod_{i,j}\mathcal{R}^{T^r_{i,j}}$$

Remark 2.4.2.7. At this point, we would like to define a map

$$\gamma^{T,2}\colon (-1,0)^{|E^{\text{int}}(T^l)|} \times (-1,0) \times (-1,0)^{\sum_i |E^{\text{int}}(T_i^r)|} \times \mathcal{R}^{T,2} \longrightarrow \mathcal{R}^{d+1,2}(\vec{L})$$

however, in this case, there may be colored trees parametrizing generalized corners, e.g. the colored 4-tree associated to the configuration depicted in Figure 9 (cfr. [MW10]).

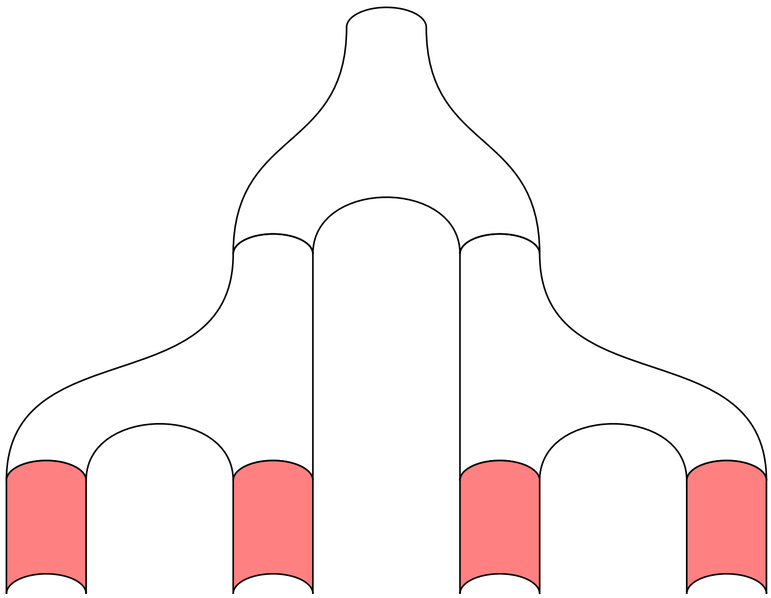


Figure 9. The singularity in the space $\overline{\mathcal{R}^{5,2}}$.

The codimension one faces arising in this subcase are parametrized by colored trees with the root as the only uncolored vertex, and, in general, codimension k faces are parametrized by colored trees with uncolored root and k total uncolored vertices.

Let T be a d-leafed tree with no vertices of valency 1 and finitely many interior edges, and let $V^{\text{col}}(T) \subset V(T)$ be a subset of vertices. Recall that by Lemma 2.4.1.1 the space $\lambda(T, V^{\text{col}}(T))$ is a polyhedral cone. For each $\lambda \in \lambda(T, V^{\text{col}}(T))$ define the map $\rho^\lambda\colon E^{\text{int}}(T) \to (-1,0)^{|E^{\text{int}}(T)|}$ by

$$\rho^\lambda := -e^{-\lambda}$$

and let $\rho(T, V^{\text{col}}(T))$ to be the space of all such maps; it is easy to see that $\rho(T, V^{\text{col}}(T))$ is a polyhedral cone in $(-1,0)^{|E^{\text{int}}(T^l)|} \times (-1,0) \times (-1,0)^{\sum_i |E^{\text{int}}(T_i^r)|}$, as by the decomposition above we get

$$|E^{\text{int}}(T)| - |V^{\text{col}}(T)| = |E^{\text{int}}(T^l)| + \sum_i |E^{\text{int}}(T_i^r)|.$$

We remark that the manifold $\mathcal{R}^{T,2}$ does not depend on the choice of the metric $\lambda \in \lambda(T, V^{\text{col}}(T))$ but only on the choice of the subset $V^{\text{col}}(T)$ by definition of colored tree. We hence define a map

$$\gamma^{T,2}\colon \rho(T, V^{\mathrm{col}}(T)) \times \mathcal{R}^{T,2} \to \mathcal{R}^{d+1,2}(\vec{L})$$

as follows: let

$$\begin{cases} (r_v^l)_{v\in V^l(T)} \in \mathcal{R}^{T^l}, \\ \left((r_{v_i}^r, w_i), (r_{i,v}^r)_{v\neq v_i \in V(T_i^r)}\right) \in \mathcal{R}^{T_i^r,2} = \mathcal{R}^{|v_i|,2}(\vec{L_{v_i}}) \times \prod_{v\neq v_i\in V(T_i^r)} \mathcal{R}^{|v|}(\vec{L_v}) \text{ for any } i \in \{1,\dots,n\}, \end{cases}$$

and write

$$\vec{r} := \left((r_v^l)_{v\in V^l(T)}, \left((r_{v_1}^r, w_1), (r_{1,v}^r)_{v\neq v_1}\right), \dots, \left((r_{v_n}^r, w_n), (r_{n,v}^r)_{v\neq v_n}\right)\right)$$

and consider $\rho \in \rho(T, V^{\mathrm{col}}(T))$, which we can write as

$$\rho = \left(\vec{\rho^l}, \tilde{\rho}, \vec{\rho_1^r}, \dots, \vec{\rho_n^r}\right) \in (-1,0)^{|E^{\mathrm{int}}(T^l)|} \times (-1,0) \times \prod_{i=1}^n (-1,0)^{|E^{\mathrm{int}}(T_i^r)|}$$

then we set

$$\gamma^{T,2}(\rho, \vec{r}) := \gamma^{T^n,2}\left(\rho, \gamma^{T^l}(\vec{\rho^r}, (r_v^l)_v), \gamma^{T_1^r,2}(\vec{\rho_1^r}, (r_{v_i}^r, w_i), (r_{1,v}^r)_{v\neq v_1}), \dots, \gamma^{T_n^r,2}(\vec{\rho_n^r}, (r_{v_n}^r, w_i), (r_{n,v}^r)_{v\neq v_n})\right)$$

In short, first we glue the all the uncolored disks, then we glue the resulting root with the colored disks following the receipt outlined just here above, and then we glue what's remaining, which is represented by a middle tree. Again, exactly as before, we extend $\gamma^{T,2}$ to a map

$$\overline{\gamma^{T,2}}\colon \mathcal{R}^{T,2} \times \overline{\rho(T)} \longrightarrow \overline{\mathcal{R}^{d+1,2}}$$

by considering trivial gluing.

We conclude this section with the following Lemma (cfr. [**MW10, Syl19**]).

Lemma 2.4.2.8. *The space $\overline{\mathcal{R}^{d+1,2}(\vec{L})}$ admits the structure of generalized manifold with corners by declaring the maps $\gamma^{T,2}$, $T \in \mathcal{T}_{\mathrm{col}}^{d+1}(\vec{L})$, to be boundary charts for small enough choice of parameters. Moreover, $\overline{\mathcal{R}^{d+1,2}(\vec{L})}$ realizes Stasheff's $(d-1)$-multiplihedron.*

2.4.3. Source spaces: moduli spaces of stacked clusters. In this section we define moduli spaces of stacked clusters of disks starting from (the compactification of) moduli spaces of stacked disks, similarly to how we constructed moduli spaces of clusters of disks from (the compactification of) moduli spaces of disks in Section 2.2.1. We will skip some details where the construction are the same.

Let $d \geq 1$ and pick a tuple $\vec{L} = (L_0, \dots, L_d)$ of Lagrangians in $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M,\omega)$. We define the moduli space of clusters of stacked disks (or stacked clusters of disks) with marked points as

$$\mathcal{R}_C^{d+1,2}(\vec{L}) := \bigcup_{T\in\mathcal{T}_{U,\mathrm{col}}^{d+1}(\vec{L})} \mathcal{R}^{T,2} \times \lambda(T)$$

and also set

$$\overline{\mathcal{R}_C^{d+1,2}(\vec{L})} := \bigcup_{T \in \mathcal{T}_{\mathrm{col}}^{d+1}(\vec{L})} \mathcal{R}^{T,2} \times \overline{\lambda_U(T)}$$

We will write elements of $\mathcal{R}_C^{d+1,2}(\vec{L})$ as (r, w, T, λ).
The following Lemma is the analogous of Lemma 2.2.1.4 for stacked clusters. Its proofs combines the construction from Section 2.4.2 with gluing of line segments as in Section 2.2.1.

LEMMA 2.4.3.1. *The space $\mathcal{R}_C^{d+1,2}(\vec{L})$ admits the structure of smooth manifold of dimension $d-1$. Moreover, the space $\overline{\mathcal{R}^{d+1,2}(\vec{L})}$ admits the structure of generalized manifold with corners and realizes Stasheff's $(d-1)$-multiplihedron.*

As in the case of Lemma 2.2.1.4, the fact that $\overline{\mathcal{R}^{d+1,2}(\vec{L})}$ realizes the multiplihedron comes for free from its construction.
We define bundles

$$\pi_C^{d+1,2}(\vec{L}) \colon \mathcal{S}_C^{d+1,2}(\vec{L}) \to \mathcal{R}_C^{d+1,2}(\vec{L})$$

where fibers are defined as in Section 2.2 by replacing nodal points by line segments of length controlled by the metric part of $\mathcal{R}_C^{d+1,2}(\vec{L})$.
A universal choice of strip-like ends for the universal family $(\pi_C^{d+1,2}(\vec{L}))_{d,\vec{L}}$ is the pullback of a universal choice of strip-like ends for the universal family $(\pi^{d+1,2}(\vec{L}))_{d,\vec{L}}$. A system of ends for (the trees involved in the definition of) $(\pi_C^{d+1,2}(\vec{L}))_{d,\vec{L}}$ is just a choice of system of ends for (the trees involved in the definition of) $(\pi_C^{d+1}(\vec{L}))_{d,\vec{L}}$ by forgetting about two valent vertices. This last definition relies on the fact that we will not construct continuation functors between perturbation data with different Morse parts (in particular, continuation maps between Floer complexes for a couple of identical Lagrangians will be the identity, so that we do not need system of ends for that case).

2.4.4. Interpolation data. We define the concept of interpolation datum, which is nothing else than the analogous of a perturbation datum but for stacked disks.
Fix a universal and consistent choice of strip-like ends for the universal family of cluster of disks $(\pi_C^{d+1}(\vec{L}))_{d,\vec{L}}$ and a universal, consistent and compatible choice of strip-like ends for the universal family of stacked clusters of disks $(\pi_C^{d+1,2}(\vec{L}))_{d,\vec{L}}$. We fix Floer data (f^L, g^L, J^L) for any couple (L, L) of identical Lagrangians in $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M,\omega)$ and associated Morse perturbation data for the universal family $(\pi_C^{d+1}(\vec{L}))_{d,\vec{L}}$ (in practice, for the family $(\mathcal{T}_U^{d+1}(\vec{L}))_{d,\vec{L}}$), which we denote by $(\boldsymbol{f}, \boldsymbol{g}) = (f^{\vec{L}}, g^{\vec{L}})_{d,\vec{L}}$. We denote by $E_{\mathrm{reg}}^{(\boldsymbol{f},\boldsymbol{g})} \subset E_{\mathrm{reg}}$ the subfamily of regular perturbation data whose Morse part agrees with $(\boldsymbol{f}, \boldsymbol{g})$. Note that for a generic choice of $(\boldsymbol{f}, \boldsymbol{g})$ this subfamily is non-empty, and we actually assume it is for our choice. We construct continuation functors between Fukaya categories defined via elements of $E_{\mathrm{reg}}^{(\boldsymbol{f},\boldsymbol{g})}$.

DEFINITION 2.4.4.1. Fix two perturbation data $p, q \in E_{\mathrm{reg}}^{(\boldsymbol{f},\boldsymbol{g})}$. Let $d \geq 1$ and consider a tuple $\vec{L} = (L_0, \ldots, L_d)$ of Lagrangians in $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M,\omega)$ and an element $(r, w, T, \lambda) \in$

$\mathcal{R}_C^{d+1,2}(\vec{L})$. An interpolation datum between the perturbation data $\mathcal{D}_p^{\vec{L}}(r,T,\lambda)$ and $\mathcal{D}_q^{\vec{L}}(r,T,\lambda)$ associated to p and q on $S_{r,T,\lambda}$ consists of a family of couples

$$\left(K_v^{r,w,T,\lambda}, J_v^{r,w,T,\lambda}\right)$$

indexed by vertices $v \in V(T)$ such that:

- if $v \in V^l(T)$ then:
  - $K_v^{r,w,T,\lambda} \in \Omega^1(S_{r,T,\lambda}(v), C^\infty(M))$ is an Hamiltonian-valued one-form which vanishes identically if $v \notin T_{\text{red}}$, while for $v \in T_{\text{red}}$ is such that for any $i \in \{0,\dots,|v|\}$ it satisfies
  $$K_v^{r,w,T,\lambda}|_{L_i^v} = 0 \text{ for any } \xi \in T(\partial_i S_{r,T,\lambda}(v)),$$
  and for any $i \in \{0,\dots,|v|\}$ on the strip-like end ϵ_i^v of $S_{r,T,\lambda}(v)$ we have[11]
  $$K_v^{r,w,T,\lambda} = H_p^{L_{i-1},L_i} dt \text{ for any } |s| \geq 1,$$
  - $J_v^{r,w,T,\lambda}$ is a domain-dependent ω-compatible almost complex structure such that if[12] $v \in T_i^F \subset T \setminus T_{\text{red}}$ for some $i \in \{0,\dots,d^F\}$ it is identical to $J_p^{L_i^F} = J_q^{L_i^F}$, while if $v \in T_{\text{red}}$ it is such that for any $i \in \{0,\dots,|v|\}$ on the ith strip-like end ϵ_i^v of $S_{r,T,\lambda}(v)$ we have
  $$J^{r,w,T,\lambda} = J_p^{L_{i-1},L_i};$$
- if $v \in V^{\text{col}}(T)$, then:
  - $K_v^{r,w,T,\lambda} \in \Omega^1(S_{r,T,\lambda}(v), C^\infty(M))$ is an Hamiltonian-valued one-form which vanishes identically if $v \notin T_{\text{red}}$, while for $v \in T_{\text{red}}$ is such that for any $i \in \{0,\dots,|v|\}$ it satisfies
  $$K_v^{r,w,T,\lambda}|_{L_i^v} = 0 \text{ for any } \xi \in T(\partial_i S_{r,T,\lambda}(v)),$$
  and for any $i \in \{1,\dots,|v|\}$ on the strip-like end ϵ_i^v of $S_{r,T,\lambda}(v)$ we have
  $$K_v^{r,w,T,\lambda} = H_p^{L_{i-1},L_i} dt \text{ for any } |s| \geq 1$$
  while on the 0th strip-like end we have
  $$K_v^{r,w,T,\lambda} = H_q^{L_0,L_d} dt,$$
  - $J_v^{r,w,T,\lambda}$ is a domain-dependent ω-compatible almost complex structure such that if $v \in T_i^F \subset T \setminus T_{\text{red}}$ for some $i \in \{0,\dots,d^F\}$ it is identical to $J_p^{L_i^F} = J_q^{L_i^F}$, while if $v \in T_{\text{red}}$ it is such that for any $i \in \{1,\dots,|v|\}$ on the ith strip-like end ϵ_i^v of $S_{r,T,\lambda}(v)$ we have
  $$J^{r,w,T,\lambda} = J_p^{L_i,L_{i+1}},$$
  while on the 0th strip-like end we have
  $$J_v^{r,w,T,\lambda} = J_q^{L_0,L_d};$$

[11] Recall that we set $H_p^{L,L} = H_q^{L,L} = 0$ and $J_p^{L,L} = J_q^{L,L} = J^L$ for any $L \in \mathcal{L}\text{ag}^{(\text{mon},\mathbf{d})}(M,\omega)$.

[12] See page 10 for the definition of the subtrees T_i^F.

- if $v \in V^r(T)$ then:
  - $K_v^{r,w,T,\lambda} \in \Omega^1(S_{r,T,\lambda}(v), C^\infty(M))$ is an Hamiltonian-valued one-form which vanishes identically if $v \notin T_{\text{red}}$, while for $v \in T_{\text{red}}$ is such that for any $i \in \{0,\dots,|v|\}$ it satisfies
$$K_v^{r,w,T,\lambda}|_{L_i^v} = 0 \text{ for any } \xi \in T(\partial_i S_{r,T,\lambda}(v)),$$
    and for any $i \in \{0,\dots,|v|\}$ on the strip-like end ϵ_i^v of $S_{r,T,\lambda}(v)$ we have
$$K_v^{r,w,T,\lambda} = H_q^{L_{i-1},L_i} dt \text{ for any } |s| \geq 1,$$
  - $J_v^{r,w,T,\lambda}$ is a domain-dependent ω-compatible almost complex structure such that if $v \in T_i^F \subset T \setminus T_{\text{red}}$ for some $i \in \{0,\dots,d^F\}$ it is identical to $J_p^{L_i^F} = J_q^{L_i^F}$, while if $v \in T_{\text{red}}$ it is such that for any $i \in \{0,\dots,|v|\}$ on the ith strip-like end ϵ_i^v of $S_{r,T,\lambda}(v)$ we have
$$J_v^{r,w,T,\lambda} = J_q^{L_{i-1},L_i}.$$

Remark 2.4.4.2. The above definition is long but intuitive. Assume $T = T_{\text{red}}$ for simplicity: interpolation data are just p- or q-perturbation data on uncolored vertices (depending on how far from the root the vertex lies), while on colored vertices interpolation data interpolate from Floer data with respect to p to a Floer datum with respect to q.

Remark 2.4.4.3. Notice that altough we want to interpolate between the *perturbation data* p and q, the definition of interpolation data between them makes no explicit mention of any perturbation datum, but only Floer ones. The connection to perturbation data will appear in the definition of the consistency condition for interpolation data which we introduce below.

We define an interpolation datum between p and q for $\vec{L}$ as a smooth choice of interpolation data on $\mathcal{R}_C^{d+1}(\vec{L})$. A universal choice of interpolation data from p to q is a choice of interpolation datum between p and q for any tuple $\vec{L}$ of Lagrangians in $\mathcal{L}\text{ag}^{(\text{mon},\mathbf{d})}(M,\omega)$ of any length.

Similarly to the case of perturbation data, given a universal choice of interpolation data from p to q, we might encounter some consistency problems (see Section 2.2.2). In this case, those problems are a bit more subtle compared to the case of perturbation data, as in the gluing process for stacked disks involves gluing of two kinds of clusters (in particular, disks and stacked disks, see Section 2.4.2).
Fix a universal choice of interpolation data from p to q on $(\pi_C^{d+1,2}(\vec{L}))_{d,\vec{L}}$. We say that this choice is consistent if for each $d \geq 1$ and for any tuple $\vec{L} = (L_0,\dots,L_d)$ of Lagrangians in $\mathcal{L}\text{ag}^{(\text{mon},\mathbf{d})}(M,\omega)$ we have:

(1) for any $T \in \mathcal{T}_{\text{col}}^{d+1}(\vec{L})$ and any $k \in \mathbb{Z}_{\geq 0}$ there is a subset
$$U \subset \mathcal{R}^{T,2} \times \lambda_U^k(T) \times [-1,0)^{|E_F^{\text{int}}(T)|+k}$$

whose closure is a neighbourhood of the trivial gluing, where the gluing parameter are small such that the interpolation data for stacked clusters over U agree with interpolation data induced by gluing on thin parts;

(2) all interpolation data extend smoothly to $\partial\overline{\mathcal{R}_C^{d+1,2}(\vec{L})}$ and agree there with perturbation data coming from (trivial) gluing.

Assume for a brief moment that $\vec{L}$ is made of different Lagrangians, i.e. $\vec{L} = \vec{L^F}$ in the notation of Section 2.1.2. In more detail, point (1) of the above definition means in this case the following: let $T \in \mathcal{T}_{\mathrm{col}}^{d+1}(\vec{L})$, (r,w) in the image of $\gamma^{T,2}$ restricted to a subset where the gluing parameters are small, and consider $e \in E^{\mathrm{int}}(T)$ (of course, e is not unilabelled) and the summand $\varepsilon_{f^+(e)}([0,-\ln(-\rho_e)] \times [0,1])$ of the thin part of $(S_r,w) \in \mathcal{S}^{d+1,2}(\vec{L})$, then

- if $f^+(e) = (v,e)$ for some $v \in V^l(T)$, then here we do want interpolation data restricted to (S_r,w) to match the perturbation data on $S_{r_v} \in \mathcal{S}^{|v|}(\vec{L_v})$ *as prescriped by the universal choice* p;
- if $f^+(e) = (v,e)$ for some $v \in V^{\mathrm{col}}(T)$, then here we do want interpolation data restricted to (S_r,w) to match the interpolation data on $(S_{r_v},w_v) \in \mathcal{S}^{|v|,2}(\vec{L_v})$;
- if $f^+(e) = (v,e)$ for some $v \in V^r(T)$, then here we do want interpolation data restricted to (S_r,w) to match the perturbation data on $S_{r_v} \in \mathcal{S}^{|v|}(\vec{L_v})$ *as prescriped by the universal choice* q.

The following result, the proof of which we omit, is an extension to stacked clusters of a result contained in [**Syl19**].

LEMMA 2.4.4.4. *Consistent choices of interpolation data exist.*

For any $p,q \in E_{\mathrm{reg}}^{(\boldsymbol{f},\boldsymbol{g})}$, we define the space $E^{p,q}$ of consistent universal choices of interpolation data for stacked clusters.

2.4.5. Moduli spaces of stacked Floer clusters with Lagrangian boundary. Fix regular perturbation data $p,q \in E_{\mathrm{reg}}^{(\boldsymbol{f},\boldsymbol{g})}$ sharing the same Morse part and an interpolation datum $h \in E^{p,q}$ between them. Let $d \geq 1$ and $\vec{L} = (L_0,\dots,L_d)$ be a tuple of Lagrangians in $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M,\omega)$. We define moduli spaces of stacked Floer clusters with boundary on $\vec{L}$ with respect to h.

Assume first $L_0 \neq L_d$. Pick $\vec{x}_i := (x_1^i,\dots,x_{m_i-1}^i)$ for any $i = 0,\dots,d^R$ (see page 2.1.2 for the definition of the numers m_i), where $x_j^i \in \mathrm{Crit}(f^{\overline{L^i}})$ are critical points, orbits $\gamma_j \in \mathcal{O}(H_p^{\overline{L_{j-1}},\overline{L_j}})$ for any $i = 1,\dots,d^R$ and $\gamma_+ \in \mathcal{O}(H_q^{\overline{L_0},\overline{L_{d^R}}})$ and a class $A \in \pi_2(M,\vec{L})$. We define the moduli space

$$\mathcal{M}^{d+1,2}\left(\vec{x}_0,\gamma_1,\vec{x}_1,\dots,\gamma_{d^R},\vec{x}_{d^R};\gamma^+;A;h\right)$$

of stacked Floer clusters joining $\vec{x}_0,\gamma_1,\vec{x}_1,\dots,\gamma_{d^R},\vec{x}_{d^R}$ to γ^+ in the class A as the space of tuples $((r,w,T,\lambda),u)$ where

$$u = \left((u_v)_{v\in V(T)},(u_e)_{e\in E(T)}\right) : S_{r,T,\lambda} \to M$$

satisfies

(1) for any vertex $v \in V(T)$, $u_v \colon S_{r,T,\lambda}(v) \to M$ satisfies the $(K_v^{r,w,T,\lambda}(h), J_v^{r,w,T,\lambda}(h))$-Floer equation and the boundary conditions $u(\partial_i S_{r,T,\lambda}) \subset \overline{L_i}$,

(2) for any $i = 1, \ldots, d^R$ we have

$$\lim_{s\to\infty} u_{h(e_i(T_{\mathrm{red}}))}(\overline{\epsilon_i}(s,t)) = \gamma_i(t)$$

and

$$\lim_{s\to\infty} u_{h(e_0(T_{\mathrm{red}}))}(\epsilon_0(s,t)) = \gamma^+(t),$$

(3) for any $i = 0, \ldots, d^R$ and any interior edge $e \in E_i^{\mathrm{int}}(T)$ (uni)labelled by $\overline{L_i}$ there is a class $B_e \in \pi_2(M, L)$ such that

$$u_e \in \mathcal{P}(u_{t(e)}(z_{t(e)}), u_{h(e)}(z_{h(e)}); (\boldsymbol{f}, \boldsymbol{g}); e; B_e),$$

(4) for any $i = 0, \ldots, d^R$ and any $j = 1, \ldots, m_i$ there is a class $B_j^i \in \pi_2(M, \overline{L_i})$ such that

$$u_{e_j^i(T_{\mathrm{uni}})} \in \mathcal{P}(x_j^i, u_{h(e_j(T_i^F))}(z_{h(e_j(T_i^F))}); (\boldsymbol{f}, \boldsymbol{g}); e_j^i(T_{\mathrm{uni}}); B_j^i),$$

(5) We have the relation

$$A = \sum_{v\in V(T)} [u_v] + \sum_{e\in E^{\mathrm{int}}(T)} B_e + \sum_{i=0}^{d^R}\sum_{j=1}^{m_i} B_j^i$$

on $\pi_2(M, \vec{L})$.

Assume now $L_0 = L_d$. Pick $x^+ \in \mathrm{Crit}(f^{\overline{L_0}})$, $\vec{x}_i := (x_1^i, \ldots, x_{\overline{m_i}}^i)$ for $i = 0, \ldots, d^R+1$, where $x_j^i \in \mathrm{Crit}(f^{\overline{L_i}})$ are critical points, orbits $\gamma_j \in \mathcal{O}(H_p^{\overline{L_{j-1}}, \overline{L_j}})$ for $i = 1, \ldots, d^R+1$ and a class $A \in \pi_2(M, \vec{L})$. We define the moduli space

$$\mathcal{M}^{d+1,2}\left(\vec{x}_0, \gamma_1, \vec{x}_2, \ldots, \gamma_{d^R+1}, \vec{x}_{d^R+1}; x^+; A; h\right)$$

of stacked Floer clusters joining $\vec{x}_0, \gamma_1, \vec{x}_1, \ldots, \gamma_{d^R+1}, \vec{x}_{d^R+1}$ to x^+ in the class A as the space of tuples $((r, w, T, \lambda), u)$ where

$$u = ((u_v)_{v\in V(T)}, (u_e)_{e\in E(T)}) \colon S_{r,T,\lambda} \to M$$

satisfies

(1) for any vertex $v \in V(T)$, $u_v \colon S_{r,T,\lambda}(v) \to M$ satisfies the $(K_v^{r,w,T,\lambda}(h), J_v^{r,w,T,\lambda}(h))$-Floer equation and the boundary conditions $u(\partial_i S_{r,T,\lambda}) \subset \overline{L_i}$,

(2) for any $i = 1, \ldots, d^R$ we have

$$\lim_{s\to\infty} u_{h(e_i(T_{\mathrm{red}}))}(\overline{\epsilon_i}(s,t)) = \gamma_i(t),$$

(3) for any $i = 0, \ldots, d^R$ and any interior edge $e \in E_i^{\mathrm{int}}(T)$ (uni)labelled by $\overline{L_i}$ there is a class $B_e \in \pi_2(M, L)$ such that

$$u_e \in \mathcal{P}(u_{t(e)}(z_{t(e)}), u_{h(e)}(z_{h(e)}); (\boldsymbol{f}, \boldsymbol{g}); e; B_e),$$

(4) for any $i = 0, \dots, d^R$ and any $j = 1, \dots, m_i$ there is a class $B^i_j \in \pi_2(M, \overline{L_i})$ such that

$$u_{e^i_j(T_{\mathrm{uni}})} \in \mathcal{P}(x^i_j, u_{h(e^i_j(T_{\mathrm{uni}}))}(z_{h(e^i_j(T_{\mathrm{uni}}))}); p; e^i_j(T_{\mathrm{uni}}); B^i_j)$$

and there is a class $B^0_0 \in \pi_2(M, \overline{L_0})$ such that

$$u_{e_0(T)} \in \mathcal{P}(u_{t(e_0(T))}, x^+; (\boldsymbol{f}, \boldsymbol{g}); e_0(T); B^0_0),$$

(5) We have the relation

$$A = \sum_{v \in V(T)} [u_v] + \sum_{e \in E^{\mathrm{int}}(T)} B_e + \sum_{i=0}^{d^R} \sum_{j=1}^{m_i} B^i_j + B^0_0$$

on $\pi_2(M, \vec{L})$.

We have the following transversality result for stacked Floer clusters.

PROPOSITION 2.4.5.1. *Let $(\boldsymbol{f}, \boldsymbol{g})$ be a choice of a Morse perturbation data in the sense of Section 2.4.4 and $p, q \in E^{(\boldsymbol{f},\boldsymbol{g})}_{\mathrm{reg}}$. Then there is a generic subset $E^{p,q}_{\mathrm{reg}} \subset E^{p,q}$ such that for any $h \in E^{p,q}_{\mathrm{reg}}$ and any tuple $\vec{L} = (L_0, \dots, L_d)$ of Lagrangians in $\mathcal{L}ag^{(mon,\mathbf{d})}(M, \omega)$ of any length $d \geq 1$ the following hold:*

(1) if $L_0 \neq L_d$ then for any $\vec{x}_i := (x^i_1, \dots, x^i_{m_i-1})$, $i = 0, \dots, d^R$, where $x^i_j \in \mathrm{Crit}(f^{\overline{L^i}})$ are critical points, any orbits $\gamma_j \in \mathcal{O}(H_p^{\overline{L_{j-1}}, \overline{L_j}})$, $i = 1, \dots, d^R$, and $\gamma_+ \in \mathcal{O}(H_q^{\overline{L_0}, \overline{L_{d^R}}})$ and any class $A \in \pi_2(M, \vec{L})$ satisfying

$$d_A^{(\vec{x}_i)_i, (\gamma_j)_j, \gamma^+} + 1 \leq 1$$

then the moduli space

$$\mathcal{M}^{d+1,2}\left(\vec{x}_0, \gamma_1, \vec{x}_1, \dots, \gamma_{d^R}, \vec{x}_{d^R}; \gamma^+; A; h\right)$$

is a smooth manifold of dimension $d_A^{(\vec{x}_i)_i, (\gamma_j)_j, \gamma^+} + 1$.

(2) if $L_0 = L_d$ then for any $x^+ \in \mathrm{Crit}(f^{\overline{L_0}})$, $\vec{x}_i := (x^i_1, \dots, x^i_{\overline{m_i}})$. $i = 0, \dots, d^R + 1$, where $x^i_j \in \mathrm{Crit}(f^{\overline{L_i}})$ are critical points, any orbits $\gamma_j \in \mathcal{O}(H_p^{\overline{L_{j-1}}, \overline{L_j}})$, $i = 1, \dots, d^R + 1$ and any class $A \in \pi_2(M, \vec{L})$ satisfying

$$d_A^{(\vec{x}_i)_i, (\gamma_j)_j, x^+} + 1 \leq 1$$

then the moduli space

$$\mathcal{M}^{d+1,2}\left(\vec{x}_0, \gamma_1, \vec{x}_2, \dots, \gamma_{d^R+1}, \vec{x}_{d^R+1}; x^+; A; h\right)$$

is a smooth manifold of dimension $d_A^{(\vec{x}_i)_i, (\gamma_j)_j, x^+} + 1$

The proof is a combinations of the arguments contained in [**Syl19**] with those sketched for the proof of Proposition 2.2.4.1 (cfr. Proposition 4.7 in [**She11**]).

REMARK 2.4.5.2. Let $h \in E^{p,q}_{\mathrm{reg}}$ and assume $\vec{L}$ is made of $d+1$ identical Lagrangians, i.e. $\vec{L^F} = (L)$, and pick $\vec{x}$, x^+ and $A \in \pi_2(M, L)$ as above such that $d_A^{\vec{x},x^+} + 1 = 0$. Then by a simple dimension argument all the moduli spaces $\mathcal{M}^{d+1,2}(\vec{x}, x^+; A; h)$ are empty except if $d = 1$, $x = x^+$ and $A = 0$, in which case $\mathcal{M}^{2,2}(x^+; x^+) = \mathcal{M}^2(x^+, x^+) = \{x^+\}$, as hidden in the definition of interpolation data there is the fact that we do not perturb Morse perturbation data.

2.4.6. Definition of continuation functors. Fix regular perturbation data $p, q \in E^{(\boldsymbol{f},\boldsymbol{g})}_{\mathrm{reg}}$ sharing the same Morse part and a regular interpolation datum $h \in E^{p,q}_{\mathrm{reg}}$ between them. By the results of Section 2.2 associated to p and q there are strictly unital A_∞-categories $Fuk(M;p)$ and $Fuk(M;q)$ which have the same set of objects. In this section we construct an A_∞-functor

$$\mathcal{F}^{p,q} \colon Fuk(M;p) \to Fuk(M;q)$$

which will depend on the choice of h.
First, we declare $\mathcal{F}^{p,q}$ to be the identity on objects.
Let $d \geq 1$ and $\vec{L} = (L_0, \ldots, L_d)$ be a tuple of Lagrangians in $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M, \omega)$. We define

$$\mathcal{F}^{p,q}_d \colon CF(L_0, L_1; p) \otimes \cdots \otimes CF(L_{d-1}, L_d; p) \to CF(L_0, L_d; q)$$

as follows. First, we assume $L_0 \neq L_d$. Let $\vec{x}_i := (x^i_1, \ldots, x^i_{m_i - 1})$, $i = 0, \ldots, d^R$, where $x^i_j \in \mathrm{Crit}(f^{\overline{L_i}})$ are critical points, $\gamma_j \in \mathcal{O}(H_p^{\overline{L_{j-1}}, \overline{L_j}})$, $i = 1, \ldots, d^R$ are orbits. Then we define

$$\mathcal{F}^{p,q}_d(\vec{x}_0, \gamma_1, \vec{x}_1, \ldots, \gamma_{d^R}, \vec{x}_{d^R}) := \sum_{\gamma^+, A} \sum_u T^{\omega(u)} \cdot \gamma^+$$

where the first sum runs over orbits $\gamma^+ \in \mathcal{O}(H^{L_0, L_d})$ and classes $A \in \pi_2(M, \vec{L})$ such that

$$d_A^{(\vec{x}_i)_i, (\gamma_j)_j, \gamma^+} + 1 = 0$$

and the second sum runs over stacked Floer clusters

$$(r, w, T, \lambda, u) \in \mathcal{M}^{d+1,2}\left(\vec{x}_0, \gamma_1, \vec{x}_1, \ldots, \gamma_{d^R}, \vec{x}_{d^R}; \gamma^+; A; h\right)$$

Assume now $L_0 = L_d$. Let $\vec{x}_i := (x^i_1, \ldots, x^i_{\overline{m_i}})$ for $i = 0, \ldots, d^R + 1$, where $x^i_j \in \mathrm{Crit}(f^{\overline{L_i}})$ are critical points, orbits $\gamma_j \in \mathcal{O}(H^{\overline{L_{j-1}}, \overline{L_j}})$ for any $i = 1, \ldots, d^R$. Then we define

$$\mathcal{F}^{p,q}_d(\vec{x}_0, \gamma_1, \vec{x}_1, \ldots, \gamma_{d^R}, \vec{x}_{d^R}) := \sum_{x^+, A} \sum_u T^{\omega(u)} \cdot x^+$$

where the first sum runs over critical points $x^+ \in \mathrm{Crit}(f^{L_0})$ and classes $A \in \pi_2(M, \vec{L})$ such that

$$d_A^{(\vec{x}_i)_i, (\gamma_j)_j, x^+} + 1 = 0$$

and the second sum runs over Floer clusters

$$(r, w, T, \lambda, u) \in \mathcal{M}^{d+1,2}\left(\vec{x}_0, \gamma_1, \vec{x}_2, \ldots, \gamma_{d^R+1}, \vec{x}_{d^R+1}; x^+; A; h\right)$$

Hidden in the next Proposition there is a compactness-type statement which follows from standard Gromov-compactness arguments up to the case where two or more configurations break simultaneosly: this case is a bit more delicate and is handled in [Lemma 3.28][**Syl19**].

PROPOSITION 2.4.6.1. *Let* $(\boldsymbol{f}, \boldsymbol{g})$ *as above and* $p, q \in E_{\mathrm{reg}}^{(\boldsymbol{f},\boldsymbol{g})}$. *Then, for any* $h \in E_{\mathrm{reg}}^{p,q}$, $\mathcal{F}^{p,q}$ *is a weakly-filtered unital* A_∞*-functor.*

Unitality of $\mathcal{F}^{p,q}$ is obvious since by dimension arguments $\mathcal{F}_1^{p,q}$ applied to a couple (L, L) of identical Lagrangians in $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M, \omega)$ is the identity. The fact that $\mathcal{F}^{p,q}$ is weakly-filtered comes from energy-action identities involving curvature terms identical to those carried out in Section 2.2.6 and uniform bounds can be found as in [**BCS21**]. The functor $\mathcal{F}^{p,q}$ will be called a continuation functor between the Fukaya categories $Fuk(M; p)$ and $Fuk(M; q)$.

Since, as it is well-known, continuation maps (that is, linear parts of continuation functors) induce isomorphisms in homology, we have the following result.

PROPOSITION 2.4.6.2. *Let* $(\boldsymbol{f}, \boldsymbol{g})$ *as above and* $p, q \in E_{\mathrm{reg}}^{(\boldsymbol{f},\boldsymbol{g})}$. *Then, for any* $h \in E_{\mathrm{reg}}^{p,q}$, $\mathcal{F}^{p,q}$ *is a quasi-equivalence between* $Fuk(M; p)$ *and* $Fuk(M; q)$. *Moreover,* $\mathcal{F}^{p,q}$ *does not depend on* h *up to quasi isomorphism of functors.*

The last part of the proposition has been proved in [**Syl19**] by constructing 'continuation homotopies'. As we mentioned above, Proposition 2.4.6.2 can be generalized to continuation functors between *any* two perturbation data, not only those sharing the same Morse part (see Proposition 2.2.5.4). Given the above machinery, this is not hard to achieve, but goes beyond our aim to use continuation functors to estimate 'distances' between filtered Fukaya categories.

2.4.7. $(\varepsilon_1, \varepsilon_2)$**-Continuation functors and filtrations.** Consider two positive real number $\varepsilon_1, \varepsilon_2 > 0$ and $\delta_1, \delta_2 \in \left(\frac{1}{2}, 1\right)$ and two perturbation data $p \in E^{\varepsilon_1,\delta_1}$ and $q \in E^{\varepsilon_2,\delta_2}$. Recall that by the main result of this paper, Theorem 2.3.0.1, the Fukaya categories $Fuk(M; p)$ and $Fuk(M; q)$ are strictly unital and filtered. Note that an arbitrary continuation functor defined via an element of $E^{p,q}$ might shift filtration in a way that is not controlled by the parameters ε_1, ε_2, δ_1 and δ_2. In this section, we will construct a subclass $E^{p,q;f} \subset E^{p,q}$ of so called $(\varepsilon_1, \varepsilon_2)$-interpolation data such that the associated continuation functors are still weakly-filtered, but with discrepancies controlled by the above parameters. Moreover, we describe choices of parameters such that the associated continuation functors are genuinely filtered. The construction of $(\varepsilon_1, \varepsilon_2)$-interpolation data is almost identical to the construction of ε-perturbation data, as all the Hamiltonian perturbation will still lie on strip-like ends as in Section 2.3, hence we will only explicitly define $(\varepsilon_1, \varepsilon_2)$-interpolation data for cyclically different tuples of Lagrangians of length 2 and 3, as the general case follows by obvious modifications and gluing. Let $\vec{L} = (L_0, L_1)$ be a couple of different Lagrangians on $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}(M, \omega)$ and consider the strip $S^{2,2}$ over $\mathcal{R}^{2,2}(\vec{L})$ coming with strip-like ends ϵ_1, ϵ_2. A choice of $(\varepsilon_1, \varepsilon_2)$-interpolation datum between p and q for $\vec{L}$ on the strip is a choice of interpolation datum (K, J) as defined in Section 2.4.4 but subject to the following restrictions:

(1) K vanishes away from the strip-like ends;
(2) On the negative strip-like end we have

$$K = \overline{H^{L_{i-1},L_i}}(s,t)dt$$

where

$$\overline{H^{L_{i-1},L_i}} \colon (-\infty, 0) \times [0,1] \times M \to \mathbb{R}$$

is of the form

$$\overline{H^{L_{i-1},L_i}}(s,t) = (1 - \beta^{L_{i-1},L_i}(s+1))H_p^{L_{i-1},L_i}(t)$$

for some $\beta^{L_{i-1},L_i} \in \mathcal{F}$ (see page 38 for the definition of the family $\mathcal{F}$);
(3) On the positive strip-like end we have

$$K = \overline{H^{L_0,L_2}}(s,t)dt$$

where

$$\overline{H^{L_0,L_2}} \colon [0,\infty) \times [0,1] \times M \to \mathbb{R}$$

is of the form

$$\overline{H^{L_0,L_2}}(s,t) = \beta^{L_0,L_2}(s)H_q^{L_0,L_2}(t)$$

for some $\beta^{L_0,L_2} \in \mathcal{F}$.

Let now $\vec{L} := (L_0, L_1, L_2)$ be a tuple of cyclically different monotone Lagrangians in $\mathcal{L}\text{ag}^{(\text{mon},\mathbf{d})}(M,\omega)$. Notice that $\mathcal{R}_C^{3,2}(\vec{L}) = \{pt\} \times (0,\infty)$. A choice of $(\varepsilon_1, \varepsilon_2)$-interpolation datum between p and q for $\vec{L}$ over $(pt, w) \in \mathcal{R}_C^{3,2}(\vec{L})$ is a choice of interpolation datum $(K^{pt,w}, J^{pt,w})$ as defined in Section 2.4.4 but subject to the following restrictions:

(1) $K^{pt,w}$ vanishes away from the strip-like ends;
(2) On the ith ($i \in \{1,2\}$) strip-like end we have

$$K^{pt,w} = \overline{H_{pt,w}^{L_{i-1},L_i}}(s,t)dt$$

where

$$\overline{H_{pt,w}^{L_{i-1},L_i}} \colon (-\infty,0) \times [0,1] \times M \to \mathbb{R}$$

is of the form

$$\overline{H_{pt,w}^{L_{i-1},L_i}}(s,t) = (1 - \beta_{pt,w}^{L_{i-1},L_i}(s+1))H_p^{L_{i-1},L_i}(t)$$

for some $\beta_{pt,w}^{L_{i-1},L_i} \in \mathcal{F}$;
(3) On the unique positive strip-like end we have

$$K^{pt,w} = \overline{H_{pt,w}^{L_0,L_2}}(s,t)dt$$

where

$$\overline{H_{pt,w}^{L_0,L_2}} \colon [0,\infty) \times [0,1] \times M \to \mathbb{R}$$

is of the form

$$\overline{H_{pt,w}^{L_0,L_2}}(s,t) = \beta_{pt,w}^{L_0,L_2}(s)H_q^{L_0,L_2}(t)$$

for some $\beta_{pt,w}^{L_0,L_2} \in \mathcal{F}$.

$(\varepsilon_1, \varepsilon_2)$-interpolation data for longer tuples of Lagrangians are constructed by gluing interpolation data for tuples $\vec{L}$ with reduced tuple of length 2 and 3. Assuming transversality, it follows by the construction of continuation functors that the dth term of a continuation functor defined via an interpolation datum with parameters $\varepsilon_1, \varepsilon_2, \delta_1$ and δ_2 applied to a tuple $\vec{L} = (L_0, \ldots, L_d)$ of Lagrangians of length $d+1$ with $d^R \geq 1$ (i.e. containing at least two geometrically different Lagrangians) shifts filtration by at most

$$\begin{cases} (d^R - 1)\varepsilon_2(1 - 2\delta_2) + d^R(\varepsilon_2 - \delta_1\varepsilon_1), & \text{if } L_0 \neq L_d, \\ d^R\varepsilon_2(1 - 2\delta_2) - \varepsilon_2 + d^R(\varepsilon_2 - \delta_1\varepsilon_1), & \text{if } L_0 \neq L_d \end{cases}$$

(cfr. Remark 2.3.2.1) since the least filtration preserving configurations are those endowed with interpolation data lying near the boundary part of $\partial\mathcal{R}^{d+1,2}(\vec{L})$ corresponding to

$$\mathcal{R}_C^{d+1}(\vec{L}) \times \prod_{i=1}^{d} \mathcal{R}^{2,2}(L_{i-1}, L_i).$$

If on the other hand $\vec{L}$ satisfies $d^R = 0$, i.e. it contains $d+1$ times the same Lagrangian, then it follows from Remark 2.4.5.2 that the dth term of a continuation functor vanishes unless $d = 1$, in which case it is just the identity, and, in particular, it does not shift filtration.
Similarly to the case of ε-perturbation data in Section 2.3.2, we define regular $(\varepsilon_1, \varepsilon_2)$-interpolation data as follows:

Definition 2.4.7.1. We define $E^{p,q;f}_{\text{reg}} \subset E^{p,q}_{\text{reg}}$ to be the space of regular interpolation data h such that the Floer parts of h is obtained via a deformation of the Floer parts of some $(\varepsilon_1, \varepsilon_2)$-interpolation datum $h' \in E^{p,q;f}$ in the sense of [**Sei08**, Chapter 9k] (that is, by deformation with support on the thich parts of the polygons, to keep consistency) and such that the associated (well-defined) continuation functor $\mathcal{F}_h^{p,q}$ shifts filtration by $\leq \max(\varepsilon_2 - \delta_1\varepsilon_1, 0)$ in the sense introduced at the end of Section **??**.

We show that $E^{p,q;f}_{\text{reg}}$ is non-empty. Pick an interpolation datum $h \in E^{p,q;f}$. It is well-known that there is no need to perturb the induced interpolation data on strips in order to get regularity for $d = 1$. Let $d \geq 2$, $\vec{L} = (L_0, \ldots, L_d)$ be a tuple of Lagrangians in $\mathcal{L}\text{ag}^{(\text{mon},\mathbf{d})}(M, \omega)$ with $d^R \geq 1$ and $\left(K^{\vec{L}}, J^{\vec{L}}\right)$ the Floer part of p restricted to $\mathcal{R}_C^{d+1,2}(\vec{L})$. As a generic deformation $(\Delta K^{\vec{L}}, \Delta J^{\vec{L}})$ (vanishing on thin parts) turns $\left(K^{\vec{L}}, J^{\vec{L}}\right)$ into a regular perturbation datum $(K^{\vec{L}} + \Delta K^{\vec{L}}, J^{\vec{L}} \exp(-J^{\vec{L}}\Delta J^{\vec{L}}))$ we can choose a generic Hamiltonian deformation $\Delta K^{\vec{L}}$ such that the associated function

$$(r, w, T, \lambda) \in \mathcal{R}_C^{d+1,2}(\vec{L}) \longmapsto \sum_{v \in T_{\text{red}}} \int_{S_{r,w,T,\lambda}(v)} \max_{x \in M} R^{\Delta K^{\vec{L}}}_{r,w,T,\lambda}(v)$$

is bounded above by the positive number

$$\begin{cases} (d^R - 1)\varepsilon_2(2\delta_2 - 1), & \text{if } L_0 \neq L_d, \\ d^R\varepsilon_2(2\delta_2 - 1) + \varepsilon_2, & \text{if } L_0 \neq L_d \end{cases}$$

It follows that for any $h \in E^{p,q;f}_{\mathrm{reg}}$ and any $d \geq 1$ the dth term $(\mathcal{F}^{p,q}_h)^d$ on the tuple of Lagrangians $(L_0, \dots, L_d)$ shifts filtration by $\leq d^R(\varepsilon_2 - \delta_1\varepsilon_1 \leq d(\varepsilon_2 - \delta_1\varepsilon_1)$, that is, $\mathcal{F}^{p,q}_h$ shifts filtration by $\leq \varepsilon_2 - \delta_1\varepsilon_1$ in the sense introduced at the end of Section **??**. As $\mathcal{F}^{p,q}$ applied to tuples of identical Lagrangians restricts to the identity functor (and hence obviously shifts filtration by ≤ 0), the claim follows.

CHAPTER 3

Approximability

3.1. Triangulated persistence refinements of Fukaya categories

The aim of this section is to fix the basic techniques and notation to deal with triangulated persistence categories and their applications to Fukaya categories. Persistence categories and some of their properties are discussed in §3.1.1. One important feature mentioned here is that the set of objects of such a category carries a class of natural pseudo-distances called *interleaving*, see (3). Triangulated persistence categories are briefly recalled in §3.1.2 where is also introduced TPC approximability in Definition 3.1.2.1. This is the version of approximability appearing in our main theorem, Theorem **??**. In §3.1.3 we start to discuss filtered A_∞ categories. The key section in this respect is §3.1.4 where are discussed derived categories associated to filtered A_∞ categories. Section 3.1.5 makes explicit the formal properties of a system of A_∞ categories (and the associated derived categories) that one obtains by making choices of perturbations that become smaller and smaller. In §3.1.5.10 a new, more precise, version of approximability is defined taking into account the system of categories with increasing precision discussed before. This is a stronger, but more technical, version of Definition 3.1.2.1 which is shown later in the paper to be satisfied in the cases of geometric interest. Finally, §3.1.6 contains the main steps in the construction of the filtered Fukaya categories to which can be applied the algebraic tools described earlier in the section. We refer to [**BCZ24b**] for further details on the filtered algebra background material in this section.

3.1.1. Persistence categories. In brief, a persistence category $\mathcal{C}$ is a category enriched over the category of persistence modules. We fix notations and give the main definitions below.

3.1.1.1. *Persistence modules.* A persistence module

$$M = \Big(\{M^\alpha\}_{\alpha\in\mathbb{R}},\ i_{\alpha,\beta} : M^\alpha \longrightarrow M^\beta,\ \forall\ \alpha \leq \beta\Big)$$

over a ring R (for us R will be often either a field or the positive Novikov ring) is a collection of R-modules M^α for each $\alpha \in \mathbb{R}$ and morphisms $i_{\alpha,\beta}$ such that $i_{\beta,\gamma} \circ i_{\alpha,\beta} = i_{\alpha,\gamma}$ whenever $\alpha \leq \beta \leq \gamma$ and $i_{\alpha,\alpha} = id$ (sometimes additional constraints are imposed on these modules, as needed). Let $u, v \in M^\alpha$ and $r \geq 0$. We write $u \underset{r}{=} v$ to indicate that $i_{\alpha,\alpha+r}(u) = i_{\alpha,\alpha+r}(v)$. We denote by

$$M_{\text{tot}} := \coprod_{\alpha\in\mathbb{R}} M^\alpha$$

the underlying set of M and we define

$$M^{\infty} := \varinjlim_{\alpha \to \infty} M^{\alpha},$$

where the colimit (or direct limit) is taken with respect to the structural maps $i_{\alpha,\beta}$, $\alpha \leq \beta$, and call it the ∞-limit of M.

REMARK 3.1.1.1. Let C be a filtered chain complex, with an increasing filtration $C^{\alpha} \subset C^{\beta} \subset C$, $\forall\, \alpha \leq \beta$. We will sometimes view C as a persistence module with $i_{\alpha,\beta} : C^{\alpha} \longrightarrow C^{\beta}$ being the inclusion maps. Let $u \in C^{\alpha}$ and consider the same element u but now viewed in C^{β} for some $\beta > \alpha$. For many practical purposes these two elements are viewed as one element that belongs to C. However, from the persistence module viewpoint $u \in C^{\alpha}$ and its image $i_{\alpha,\beta}(u) \in C^{\beta}$ should be viewed as distinct elements.

3.1.1.2. *Persistence categories.* (PC's in short) are (small) categories in which the morphisms between objects form persistence modules and the composition of morphisms is compatible with the persistence structures. The general theory of persistence categories is described in detail in [**BCZ24b**, Section 2] and here we will only recall several key concepts.

Unless otherwise stated, from now on we will always assume all the PC categories to be endowed with a shift functor (or, more precisely, a system of shift functors) $\Sigma = \{\Sigma^r\}_{r \in \mathbb{R}}$ (see [**BCZ24b**, Section 2.2.3] for the definition), and all the PC functors to commute with Σ^r for every $r \in \mathbb{R}$. Part of the shift functor structure are the natural transformations $\eta_{r,s} : \Sigma^r \longrightarrow \Sigma^s$, $s, r \in \mathbb{R}$. The special case $s = 0$ will be especially important and we denote it by $\eta_r := \eta_{r,0} : \Sigma^r \longrightarrow \mathrm{id}$. Moreover, all persistence functors $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ between PC's will be implicitly assumed to be compatible with the shift functors of these categories in the following sense: $\mathcal{F} \circ \Sigma^r = \Sigma^r \circ \mathcal{F}$ for all $r \in \mathbb{R}$, and $(\eta_{r,s})_{\mathcal{F}A} = \mathcal{F}((\eta_{r,s})_A)$ for all $A \in \mathrm{Obj}(\mathcal{C})$ and $r, s \in \mathbb{R}$.

Let $\mathcal{C}$ be a PC category. Denote by $\mathcal{C}^0$ the 0-level category associated with $\mathcal{C}$ and by $\mathcal{C}^{\infty}$ the limit (or ∞-level) category of $\mathcal{C}$. Isomorphisms in $\mathcal{C}^0$ will be called 0-isomorphisms and the property of being 0-isomorphic will be sometimes denoted by $\cong_0$. Similarly, the existence of an isomorphism in $\mathcal{C}^{\infty}$ will be denoted by $\cong_{\infty}$.

Most of the time we will assume our PC's $\mathcal{C}$ to be graded in the sense that the persistence modules $\hom_{\mathcal{C}}(A, B)$ are $\mathbb{Z}$-graded. We will mostly use cohomological grading conventions and denote by $\hom_{\mathcal{C}}(A, B)^k$, $k \in \mathbb{Z}$ the degree-k component of $\hom_{\mathcal{C}}(A, B)$. For $\alpha \in \mathbb{R}$ we denote by $\hom^{\alpha}_{\mathcal{C}}(A, B)^k$ the α-persistence level of $\hom_{\mathcal{C}}(A, B)^k$. We assume composition of morphisms to be degree-preserving and that identity morphisms are in degree 0.

Our PC's $\mathcal{C}$ will be often endowed with a *translation functor* $T : \mathcal{C} \longrightarrow \mathcal{C}$. This is a PC isomorphism functor with the property that for all $A, B \in \mathrm{Obj}(\mathcal{C})$, $k \in \mathbb{Z}$, we have PC-natural isomorphisms

$$\hom_{\mathcal{C}}(TA, B)^k \cong \hom_{\mathcal{C}}(A, B)^{k-1}, \quad \hom_{\mathcal{C}}(A, TB)^k \cong \hom_{\mathcal{C}}(A, B)^{k+1}.$$

As mentioned earlier, since T is a PC functor, we will implicitly assume that T commutes with the shift functors, i.e. $T \circ \Sigma^r = \Sigma^r \circ T$, $\forall\, r \in \mathbb{R}$.

3.1.1.3. *Acyclic objects.* Let $\mathcal{C}$ be a PC, $A \in \mathrm{Obj}(\mathcal{C})$ and $r \geq 0$. We say that A is r-acyclic if the morphism $\eta_r^A : \Sigma^r A \longrightarrow A$ equals 0, or, equivalently, if the identity id_A satifies $i_{0,r}(id_A) = 0$. Here, $\eta_r^A \in \hom_{\mathcal{C}^0}(\Sigma^r A, A)$ is given by $\eta_r^A = \eta_r(A)$. An object A is called acyclic if it is r-acyclic for some $r \geq 0$. Equivalently, A is acyclic if and only if A is isomorphic to 0 in the limit category $\mathcal{C}^\infty$. We denote by $A\mathcal{C}$ the full subcategory of $\mathcal{C}$ whose objects are the acyclic ones.

3.1.1.4. *Pseudo metrics associated with PC's.* Let $\mathcal{C}$ be a PC with a shift functor Σ. The collection of objects $\mathrm{Obj}(\mathcal{C})$ carries a natural pseudo-metric d_{int}, called the interleaving distance, which is defined as follows. For $X, Y \in \mathrm{Obj}(\mathcal{C})$ define:

$$\begin{aligned} d_{\mathrm{int}}(X,Y) = \inf\Big\{ r \geq 0 \ \Big| \ & \exists\, \varphi : \Sigma^r X \longrightarrow Y, \ \exists\, \psi : \Sigma^r Y \longrightarrow X \\ & \text{such that } \psi \circ \Sigma^r \varphi = \eta_{2r}^X, \ \varphi \circ \Sigma^r \psi = \eta_{2r}^Y \Big\}. \end{aligned} \tag{3}$$

Here, $\eta_{2r}^X \in \hom_{\mathcal{C}^0}(\Sigma^{2r} X, X)$ are the standard maps associated with the persistence structure of $\mathcal{C}$ and its shift functor and similarly for η_{2r}^Y.

Note that d_{int} is in general not a genuine metric but only a pseudo-metric and moreover it may have infinite values. Indeed, if X and Y are isomorphic in the subcategory $\mathcal{C}^0$, then $d_{\mathrm{int}}(X,Y) = 0$. Similarly, if X and Y are not isomorphic in $\mathcal{C}^\infty$ then $d_{\mathrm{int}}(X,Y) = \infty$. This is compatible with the convention that $\inf \emptyset = \infty$ which we use in (3).

There is another variant of d_{int} which measures how far an object R is from being a retract of another object X. More precisely, for $R, X \in \mathrm{Obj}(\mathcal{C})$ we define

$$d_{\text{r-int}}(R,X) = \inf\Big\{ r \geq 0 \ \Big| \ \exists\, \varphi : \Sigma^r R \longrightarrow X, \ \exists\, \psi : \Sigma^r X \longrightarrow R \ \text{ such that } \psi \circ \Sigma^r \varphi = \eta_{2r}^R \Big\}. \tag{4}$$

In contrast to d_{int}, the measurement $d_{\text{r-int}}(-,-)$ is in general not symmetric (such a structure is sometimes called a quasi-pseudometric).

The pseudo-metric d_{int}, admits a shift-invariant version $\bar{d}_{\mathrm{int}}$ that will play an important role further on:

$$\bar{d}_{\mathrm{int}}(X,Y) = \inf_{r,s} d_{\mathrm{int}}(\Sigma^r X, \Sigma^s Y) \tag{5}$$

It is easily seen that this is also a pseudo-metric. A similar construction applies also to $d_{\text{r-int}}$ and produces $\bar{d}_{\text{r-int}}$.

Here is an additional notation to be used later in the paper. Let Z be a set endowed with a (not necessarily symmetric) pseudo-metric d and $X, Y \subset Z$. We denote by

$$d(X,Y) := \sup_{x \in X} \inf_{y \in Y} d(x,y)$$

the longest distance between the points of X and those of Y. Note that this is not symmetric in (X,Y). We will sometimes use this measurement for $d = d_{\mathrm{int}}$ and $d = d_{\text{r-int}}$.

To end this subsection, we list a few basic metric properties of PC functors in the following lemma which is immediate to establish. In the statement the same notation d_{int} is used for the inteleaving distance in various PC categories.

LEMMA 3.1.1.2. *Let $\mathcal{C}', \mathcal{C}''$ be PC's and $\mathcal{F} : \mathcal{C}' \longrightarrow \mathcal{C}''$ a PC-functor.*

(1) For every $A, B \in Obj(\mathcal{C}')$ we have $d_{int}(\mathcal{F}A, \mathcal{F}B) \leq d_{int}(A, B)$.

(2) If there exists a PC functor $\mathcal{G} : \mathcal{C}'' \longrightarrow \mathcal{C}'$ such that $d_{int}(\mathcal{G} \circ \mathcal{F}, \mathrm{id}_{\mathcal{C}'}) \leq s$ then for every $A, B \in Obj(\mathcal{C}')$ we have $|d_{int}(\mathcal{F}A, \mathcal{F}B) - d_{int}(A, B)| \leq 2s$.

(3) If there is a PC-functor $\mathcal{H} : \mathcal{C}'' \longrightarrow \mathcal{C}'$ such that $d_{int}(\mathcal{F} \circ \mathcal{H}, \mathrm{id}_{\mathcal{C}''}) \leq r$ then

$$d_{int}(\mathcal{C}'', \operatorname{image} \mathcal{F}) \leq r.$$

3.1.2. Triangulated persistence categories and TPC approximability. We discuss here approximability in the setting of triangulated persistence categories.

3.1.2.1. *TPCs.* A *Triangulated Persistence Category* (TPC in short) $\mathcal{C}$ is a PC with an additional structure which makes its 0-level category $\mathcal{C}^0$ a triangulated category. The precise definition of a TPC involves several other axioms and we refer the reader to [**BCZ24b**] for a detailed exposition of the subject.

Let $\mathcal{C}$ be a TPC and $S \subset \mathrm{Obj}(\mathcal{C})$. We write $\langle S \rangle^{\Delta}$ for the minimal full sub-TPC of $\mathcal{C}$ which contains all the objects of S and is closed under 0-isomorphisms. (Note that in particular the shift and translation functors of $\mathcal{C}$ restrict to respective functors on $\langle S \rangle^{\Delta}$.)

3.1.2.2. *TPC approximability.* The key approximability definition for the paper is the following.

DEFINITION 3.1.2.1. A pseudo-metric space (X, d) is approximable through triangulated persistence categories (in short, TPC-approximable) if there exists a constant $A \geq 1$ such that for each $\epsilon > 0$ the following structure exists. For each $0 < \eta < \epsilon$ there is a TPC, $\mathcal{Y}_{\epsilon,\eta}$, with two properties:

i. There exists an (A, η)-quasi-isometric embedding (see Remark 3.1.2.2):

$$\Phi_{\epsilon,\eta} : (X, d) \to (\mathrm{Obj}(\mathcal{Y}_{\epsilon,\eta}), \bar{d}_{\mathrm{int}}^{\,\epsilon,\eta})$$

ii. There exists a *finite* family $\mathcal{F}_{\epsilon,\eta} \subset \mathrm{Obj}(\mathcal{Y}_{\epsilon,\eta})$ with the property:

$$\bar{d}_{\mathrm{int}}^{\,\epsilon,\eta}(\Phi_\eta(x), \mathrm{Obj}(\langle \mathcal{F}_{\epsilon,\eta} \rangle^{\Delta})) < \epsilon, \ \forall x \in X$$

where $\bar{d}_{\mathrm{int}}^{\,\epsilon,\eta}$ is the shift invariant interleaving pseudo-metric associated to $\mathcal{Y}_{\epsilon,\eta}$ (see (5)).

For fixed ϵ, the data $(\Phi, \mathcal{F})$ with $\Phi = \{\Phi_{\epsilon,\eta}\}_{0<\eta<\epsilon}$, $\mathcal{F} = \{\mathcal{F}_{\epsilon,\eta}\}_{0<\eta<\epsilon}$ is called TPC ϵ-*approximating data* for (X, d). Thus, (X, d) is TPC-approximable if ϵ-TPC-approximating data for (X, d) exists for each $\epsilon > 0$. It can certainly happen that multiple choices of such data exist for the same (X, d). When one of the two parameters ϵ or η are clear from the context we omit it from the notation. For instance, for fixed ϵ, we write Φ_η for the relevant quasi-isometric embeddings , $\bar{d}_{\mathrm{int}}^{\,\eta}$ for the respective pseudo-metrics and so forth.

This definition is the TPC version of Definition **??**. We will also use in the paper the notion of *TPC retract approximability*. TPC retract approximability is defined just as above without any modifications for point i. above but by using $\bar{d}_{\text{r-int}}$ at point ii. (we emphasize that the *only* change is at point ii. in Definition 3.1.2.1). Pairs $(\Phi, \mathcal{F})$ as above but ensuring retract approximability (in other words, using $\bar{d}_{\text{r-int}}$ instead of $\bar{d}_{\text{int}}$ for condition ii in the definition) will be called TPC *retract ϵ-approximating data.*

Point c. in Remark 3.1.2.4 below explains more precisely how TPC approximability, as just defined, fits with the notion of categorical metric approximability introduced in Definition **??**.

REMARK 3.1.2.2. a. Recall that an (A, B)-quasi-isometric embedding

$$(X, d) \to (Y, d')$$

between two metric spaces is a map $\Phi : X \to Y$ such that:

$$\frac{1}{A}\, d(x, y) - B \leq d'(\Phi(x), \Phi(y)) \leq A\ d(x, y) + B\ .$$

Thus, the first point of the Definition 3.1.2.1 shows that, when ϵ is fixed, the metrics $\bar{d}_{\text{int}}^{\,\eta}$ defined on the objects of $\mathcal{Y}_\eta$, when restricted to X, get closer and closer to a (pseudo)-metric equivalent to d, when $\eta \to 0$. More precisely, we can put for each $x, y \in X$

$$\hat{d}(x, y) = \limsup_{\eta \to 0} \bar{d}_{\text{int}}^{\,\eta}(x, y)\ .$$

This gives a pseudo-metric defined on X and point i. of the Definition implies that this $\hat{d}$ is equivalent to d, the original metric on X.

b. A (A, η)-quasi-isometric embedding is also a (A, η')-quasi-isometric embedding whenever $\eta' \geq \eta$. As a result, to verify the property in the definition, one only needs to show the existence of $\mathcal{Y}_\eta$, Φ_η, and of the families $\mathcal{F}_\epsilon = \{\mathcal{F}_{\epsilon,\eta}\}$ whenever η is small enough.

c. This definition will be applied to the case when (X, d) is a space of Lagrangian submanifolds endowed with the spectral metric $d = d_\gamma$ and the categories $\mathcal{Y}_\eta$ are TPC refinements of derived Fukaya categories (we assume again ϵ fixed here). The role of η is to keep track of the size of the perturbations needed to define these Fukaya categories. While this parameter is irrelevant in the non-filtered case, for purposes of approximations it is crucial to keep track of it because any choice of perturbations needed to define the Fukaya category determines some η which constraints what the approximating accuracy ϵ can be. Controlling systematically the various TPC refinements of Fukaya categories relative to the choices of perturbation data of varying size requires significant elaboration which is contained in §3.1.5. In particular, the metric $\hat{d}$ above is a version of $\widehat{d_{\text{int}}}$ from Lemma 3.1.5.3. In this setting of families of filtered A_∞-categories it is operationally useful to use a more refined and precise variant of the definition above. This is formulated in Definition 3.1.5.7 and its retract analogue is in Definition 3.1.5.8. See also Remark 3.1.5.9 for a comparison between these notions.

d. The constant A will be equal to 2 in the Lagrangian topology application. The application to Lagrangian topology was determinant for our choice of the shift invariant metric $\bar{d}_{\text{int}}$ in Definition 3.1.2.1 (as opposed to d_{int} which could as well have been used, in principle). The reason is that the spectral pseudo-metric is shift invariant and thus the limit $\hat{d}$ from point a above needs to also be shift invariant (see also Remark 3.1.2.4).

e. In certain contexts, when choices of perturbations are not required, and ϵ is fixed, one presumably can replace the family $\mathcal{Y}_\eta$ by a single *PC* (or TPC) $\mathcal{Y}_0$ - in other words include $\eta = 0$ - in the definition above.

f. In Definition 3.1.2.1 we have not imposed any relation tying the categories $\mathcal{Y}_{\epsilon,\eta}$, when $\eta \to 0$, nor did we assume any relation among the respective families $\mathcal{F}_{\epsilon,\eta}$. In practice, our constructions in the case of Fukaya categories produce categories that satisfy a property more precise than the one in Definition 3.1.2.1, in the sense that, for fixed $\epsilon > 0$, the categories $\mathcal{Y}_{\epsilon,\eta}$, with $\eta \to 0$, are related by certain comparison functors and the corresponding families $\mathcal{F}_{\epsilon,\eta}$ correspond one to the other through these functors. This more refined property is more technical to state and is formulated in Definition 3.1.5.7. Further below we will also discuss the dependence of our constructions relative to ϵ.

We will also make use of a simplified notion of TPC-approximability that reformulates point ii of Definition 3.1.2.1.

DEFINITION 3.1.2.3. Fix a triangulated persistence category $\mathcal{C}$ and $\mathcal{X} \subset \mathrm{Obj}(\mathcal{C})$. For $\epsilon > 0$, we say that $\mathcal{X}$ is *ϵ-approximable in* $\mathcal{C}$ if there exists a finite family $\mathcal{F}_\epsilon \subset \mathrm{Obj}(\mathcal{C})$, called an *$\epsilon$-approximating family* for $\mathcal{X}$ in $\mathcal{C}$, such that

$$d_{\text{int}}\left(x, \mathrm{Obj}(\langle \mathcal{F}_\epsilon \rangle^\Delta)\right) < \epsilon, \forall x \in \mathcal{X}.$$

The set $\mathcal{X}$ is called approximable in $\mathcal{C}$ if it is ϵ-approximable for each $\epsilon > 0$. We say that $\mathcal{X}$ is *retract ϵ-approximable in* $\mathcal{C}$ if the same property as above holds but with $d_{\text{r-int}}$ in the place of d_{int}.

REMARK 3.1.2.4. a. Recall from §3.1.1.4 that both d_{int} and $d_{\text{r-int}}$ have shift-invariant versions. Thus, one could use these shift invariant metrics, $\bar{d}_{\text{int}}$ and $\bar{d}_{\text{r-int}}$, instead of d_{int} and, respectively, $d_{\text{r-int}}$ in the definition above. However, this has no impact on the notion defined. Indeed, the set $\mathrm{Obj}(\langle \mathcal{S} \rangle^\Delta)$ where $\langle \mathcal{S} \rangle^\Delta$ is the smallest sub-TPC of $\mathcal{C}$ containing $\mathcal{S} \subset \mathrm{Obj}(\mathcal{C})$ is shift and translation invariant. Thus, $d_{\text{int}}(x, \mathrm{Obj}(\langle \mathcal{F}_\epsilon \rangle^\Delta)) = \bar{d}_{\text{int}}(x, \mathrm{Obj}(\langle \mathcal{F}_\epsilon \rangle^\Delta))$ for all $x \in Y$, and similarly for $d_{\text{r-int}}$. As a result, the regular and shift invariant versions of approximability in the sense of Definition 3.1.2.3 coincide and the same is true for retract approximability.

b. Notice that, with the terminology in Definition 3.1.2.3, point ii in Definition 3.1.2.1 is equivalent to the fact that $\mathcal{X} = \Phi_\eta(X)$ is ϵ-approximable in $\mathcal{Y}_\eta$ and similarly for retract approximability (ϵ is fixed here).

c. The notion of ϵ-approximability in Definition 3.1.2.3 obviously implies metric categorical ϵ-approximability of $\mathcal{X}$ in $\mathcal{C}$, as introduced in Definition **??** (the difference between the two definitions is that in 3.1.2.3 the metric on the objects of the category $\mathcal{C}$ is fixed to be the interleaving metric associated with the persistence structure on $\mathcal{C}$).

d. Assume, as above, that $\mathcal{C}$ is a TPC. The following statement is an exercise in manipulating exact triangles in the triangulated category $\mathcal{C}^0$:

$$\text{For } x, y \in \mathrm{Obj}(\mathcal{C}) \text{ if } \bar{d}_{\text{r-int}}(x, y) < \epsilon \text{ , then } \exists\, k_x \in \mathrm{Obj}(\mathcal{C}), \text{ such that } \bar{d}_{\text{int}}(x \oplus k_x, y) < 2\epsilon \text{ .} \quad (6)$$

It is immediate to see that we also have for all $x, y, k \in \mathrm{Obj}(\mathcal{C})$, $\bar{d}_{\text{r-int}}(x, y) \leq \bar{d}_{\text{int}}(x \oplus k, y)$. From (6) we deduce that if $\mathcal{X}$ is retract ϵ-approximable in $\mathcal{C}$, in the sense of Definition 3.1.2.3, then it also is retract categorically ϵ-approximable in the sense of §**??**. In other words, if $\mathcal{X}$ is retract ϵ-approximable in $\mathcal{C}$, then for each $x \in \mathcal{X}$ there exist $k_x \in \mathcal{C}$ and $y \in \mathrm{Obj}(\langle \mathcal{F}_\epsilon \rangle^\Delta))$ such that $d_{\text{int}}(x \oplus k_x, y) < 2\epsilon$.

e. Idempotent completion in TPCs is a subtle matter, much more so than its counterpart in triangulated categories. This topic is studied by Miller in [**Mil25**] but we will not appeal in this paper to the results there.

f. In [**BCZ24b**] were introduced a class of so-called fragmentation pseudo-metrics $d^{\mathcal{F}}(-,-)$ on the objects of a TPC that depend on the choice of a family $\mathcal{F} \subset \mathrm{Obj}(\mathcal{C})$. These pseudo-metrics are closely related to approximability. Given $\mathcal{C}$ as above, $d^{\mathcal{F}_\epsilon}(0, Y) \leq \epsilon/4$ implies that $\mathcal{F}_\epsilon$ is ϵ-approximating. Conversely, if $\mathcal{F}_\epsilon$ is ϵ-approximating, then $d^{\mathcal{F}_\epsilon}(0, Y) \leq \epsilon$. Understanding the behaviour of these fragmentation pseudo-metrics when the family $\mathcal{F}$ changes was one of the main motivations leading to the definition of approximability in this paper.

3.1.3. Conventions for PC's and filtered A_∞ categories.

3.1.3.1. *Filtered A_∞-categories.* Most of the PC's in this paper arise as persistence homological categories of filtered A_∞-categories. Unless otherwise stated we will always assume such categories, as well as A_∞-functors between them, to be strictly unital and endowed with a shift and translation functors. See §1 for the precise definitions.

Let $\mathcal{A}$ be a filtered A_∞-category. We will sometimes abbreviate $\mathcal{A}(X, Y) := \hom_{\mathcal{A}}(X, Y)$. For $X, Y \in \mathrm{Obj}(\mathcal{A})$, $\alpha \in \mathbb{R}$, we denote by $\hom^\alpha_{\mathcal{A}}(X, Y) \subset \hom_{\mathcal{A}}(X, Y)$, or sometimes by $\mathcal{A}^\alpha(X, Y)$ for short, the α-filtration level of $\hom_{\mathcal{A}}(X, Y)$. Whenever we need to combine these with (cohomological) grading we will write $\hom^\alpha_{\mathcal{A}}(X, Y)^i$ (or $\mathcal{A}^\alpha(X, Y)^i$) for the degree-$i$ component of $\hom^\alpha_{\mathcal{A}}(X, Y)$. The notation in case of homological grading is similar, by making the degree i a subscript.

Next, we denote by $\mathcal{A}^0$ the 0-level category of $\mathcal{A}$, namely the (unfiltered) A_∞-category with the same objects as $\mathcal{A}$ and $\hom_{\mathcal{A}^0}(X, Y) = \hom^0_{\mathcal{A}}(X, Y)$.

Passing to homology, we write $PH(\mathcal{A})$ for the persistence homological category of $\mathcal{A}$. The shift and translation functors induce respective functors on $PH(\mathcal{A})$.

As in the case of PC's, we say that an object $A \in \mathcal{A}$ is r-acyclic if A is r-acyclic in the persistence category $PH(\mathcal{A})$, and similarly for acyclic objects (without reference to a specific r). The full A_∞-subcategory of acyclic objects will be denoted by $A\mathcal{A}$.

Finally, we import the interleaving (and r-interleaving) pseudo-metrics also to the realm of A_∞-categories $\mathcal{A}$, by setting both of them to be equal to the respective distances (as defined in §3.1.1.4) in the persistence homological category $PH(\mathcal{A})$.

3.1.3.2. *Various completions.* Let $\mathcal{C}$ be a PC. Given subsets $S_1, S_2, S_3 \subset \mathrm{Obj}(\mathcal{C})$ we define their completions with respect to shifts, translations and 0-isomorphisms, respectively, as follows:

$$\begin{aligned} &S_1^{\Sigma} := \{\Sigma^r(A) \mid A \in S_1, \, r \in \mathbb{R}\}, \quad S_2^T := \{T^j(A) \mid A \in S_2, \, j \in \mathbb{Z}\}, \\ &S_3^{(\mathrm{iso},0)} := \{A \in \mathrm{Obj}(\mathcal{C}) \mid A \text{ is isomorphic in } \mathcal{C}^0 \text{ to an object from } S_3\}. \end{aligned} \tag{7}$$

We will also need the completion of a subset $S \subset \mathrm{Obj}(\mathcal{C})$ with respect to several of the above procedures, for example $S^{\Sigma,T} := (S^{\Sigma})^T$ etc.

Another important completion is with respect to the interleaving distance. Given a subset $S_4 \subset \mathrm{Obj}(\mathcal{C})$ we define

$$S_4^{\mathrm{int}} := \{A \in \mathrm{Obj}(\mathcal{C}) \mid d_{\mathrm{int}}(A, B) < \infty \text{ for some } B \in S_4\}.$$

To simplify the notation, for $S \subset \mathrm{Obj}\mathcal{C}$ we will write $S^c := (S^T)^{\mathrm{int}}$. Note that S^c is automatically complete with respect to both shifts and translations. One can also define $S^{c,\text{r-int}}$ in a similar way to S^c, but with d_{int} replaced by $d_{\text{r-int}}$.

If $\mathcal{D} \subset \mathcal{C}$ is a full sub-PC we define its various completions, e.g. $\mathcal{D}^{\Sigma}$, $\mathcal{D}^T$, $\mathcal{D}^{(\mathrm{iso},0)}$, $\mathcal{D}^c$ etc. by taking the full sub-PC's of $\mathcal{C}$ with the respective completed sets of objects $\mathrm{Obj}(\mathcal{D})^{\Sigma}$, $\mathrm{Obj}(\mathcal{D})^T$, $\mathrm{Obj}(\mathcal{D})^{(\mathrm{iso},0)}$, $\mathrm{Obj}(\mathcal{D})^c$ etc. In the case of A_∞-categories $\mathcal{A}$, we define analogous completions, by completing the subsets of objects using the respective structures in $PH(\mathcal{A})$.

3.1.4. Persistence derived categories associated with a filtered A_∞-category. We will present below three variants of TPC's that can be viewed each as a generalization of the derived category of $\mathcal{A}$ to the realm of persistence categories. However before we go into this we need a quick preparation about A_∞-functors in the filtered context.

3.1.4.1. *Filtered functors and functors with linear deviation.* Let $\mathcal{A}$, $\mathcal{B}$ be two filtered A_∞-categories. A filtered A_∞-functor $\mathcal{F} : \mathcal{A} \longrightarrow \mathcal{B}$ is an A_∞-functor whose action on morphisms (in all orders) preserves the filtration levels. More specifically, for every $d \geq 1$, $X_0, \ldots, X_d \in \mathrm{Obj}(\mathcal{A})$ and $\alpha \in \mathbb{R}$ the d'th order term $\mathcal{F}_d : \mathcal{A}(X_0, \ldots, X_d) \longrightarrow \mathcal{B}(\mathcal{F}X_0, \mathcal{F}X_d)$ of $\mathcal{F}$ satisfies:

$$\mathcal{F}_d\big(\mathcal{A}^{\alpha}(X_0, \ldots, X_d)\big) \subset \mathcal{B}^{\alpha}(\mathcal{F}X_0, \mathcal{F}X_d).$$

Unless otherwise stated, in what follows we will implicitly assume our A_∞-functors to be strictly unital.

Motivated by examples coming from Fukaya categories we will also need the concept of A_∞-functors with linear deviation. These are defined as follows. Given a tuple $\vec{X} = (X_0, \ldots, X_d)$ of objects from $\mathcal{A}$ we define its reduced tuple $\vec{X}_R := (X_{i_0}, \ldots, X_{i_{d_R}})$ by ommitting from $\vec{X}$ subsequent (in the cyclic order) objects that are equal up to a shift. The objects forming $\vec{X}_R$ are well defined only up to shifts, but the length of $\vec{X}_R$, $0 \leq d_R \leq d$, is well defined. We call it the reduced length of $\vec{X}$ and denote it by d_R or $d_R(\vec{X})$ whenever we want to emphasize its dependence on $\vec{X}$. Note that in case every two consecutive objects in $\vec{X}$ are different (up to shifts) then $d_R(\vec{X}) = d$. At the other extreme, if all the objects in $\vec{X}$ are equal up to shifts, then $d_R(\vec{X}) = 0$.

An A_∞-functor $\mathcal{F} : \mathcal{A} \longrightarrow \mathcal{B}$ is said to have linear deviation rate $s \geq 0$ if for every $d \geq 1$, every tuple of objects $\vec{X} = (X_0, \ldots, X_d)$ from $\mathrm{Obj}(\mathcal{A})$ and every $\alpha \in \mathbb{R}$ we have:

$$\mathcal{F}_d\big(\mathcal{A}^\alpha(X_0, \ldots, X_d)\big) \subset \mathcal{B}^{\alpha + d_R(\vec{X})s}(\mathcal{F}X_0, \mathcal{F}X_d).$$

We will often refer to such functors as LD (Linear Deviation) functors.

3.1.4.2. *Filtered modules.* Let $\mathcal{A}$ be a filtered A_∞-category. Denote by $F\mathrm{mod}_{\mathcal{A}}$ the category of filtered strictly unital (left) $\mathcal{A}$-modules. This is a filtered A_∞-category. The Yoneda embedding

$$\mathcal{Y} : \mathcal{A} \longrightarrow F\mathrm{mod}_{\mathcal{A}}$$

is a filtered strictly unital A_∞-functor which is homologically full and faithful. Denote by $\mathcal{Y}(\mathcal{A}) \subset F\mathrm{mod}_{\mathcal{A}}$ the image category of $\mathcal{A}$ under $\mathcal{Y}$. Denote by $\mathcal{Y}(\mathcal{A})^\Delta \subset F\mathrm{mod}_{\mathcal{A}}$ the triangulated completion of $\mathcal{Y}(\mathcal{A})^\Delta$, namely the minimal full subcategory of $F\mathrm{mod}_{\mathcal{A}}$ which is closed under mapping cones over morphisms with filtration level ≤ 0 and quasi-isomorphisms of filtration level ≤ 0 (these are maps of non-positive filtration level that induce an isomorphism in the 0-level homological category). Note that due to our assumptions on the filtered A_∞ category $\mathcal{A}$, the category $\mathcal{Y}(\mathcal{A})$ is closed under shifts and translations, and therefore the same holds for $\mathcal{Y}(\mathcal{A})^\Delta$. The construction of $\mathcal{Y}(\mathcal{A})^\Delta$ can be explicitly carried out by iteratively taking mapping cones over morphisms of filtration level ≤ 0 and adding at each stage 0-quasi-isomorphic modules. The resulting A_∞-category $\mathcal{Y}(\mathcal{A})^\Delta$ is filtered and strictly unital A_∞ and carries a translation and a shift functor. The persistence-homological category of $\mathcal{Y}(\mathcal{A})^\Delta$

$$PD(\mathcal{A}) := PH(\mathcal{Y}(\mathcal{A})^\Delta)$$

is a TPC which we call the *persistence derived* category of $\mathcal{A}$.

Let $\mathcal{A}$, $\mathcal{B}$ be two filtered A_∞-categories. Let $\mathcal{F} : \mathcal{A} \longrightarrow \mathcal{B}$ be an A_∞-functor with a given linear deviation rate (see §3.1.4.1 and §1.6). Then the push-forward operation from §1.8 gives rise to a *filtered* A_∞-functor $\mathcal{F}_* : F\mathrm{mod}_{\mathcal{A}} \longrightarrow F\mathrm{mod}_{\mathcal{B}}$. More importantly for us, it sends $\mathcal{Y}(\mathcal{A})^\Delta$ to $\mathcal{Y}(\mathcal{B})^\Delta$ hence it gives rise to a filtered A_∞-functor

$$\mathcal{F}_* : \mathcal{Y}(\mathcal{A})^\Delta \longrightarrow \mathcal{Y}(\mathcal{B})^\Delta.$$

Passing to homology we obtain a TPC functor

$$PD(\mathcal{F}) : PD(\mathcal{A}) \longrightarrow PD(\mathcal{B}).$$

3.1.4.3. *A larger derived category.* As in the previous sections, let $\mathcal{A}$ be a filtered A_∞-category. We will now construct a somewhat larger version of the category $PD(\mathcal{A})$ from §3.1.4.2 by adding also acyclic modules.

Denote by $AF\mathrm{mod}_{\mathcal{A}} \subset F\mathrm{mod}_{\mathcal{A}}$ the full A_∞-subcategory of acyclic modules (see §3.1.3.1 for the definition of the acyclic subcategory). Consider now the minimal A_∞ full subcategory of $F\mathrm{mod}_{\mathcal{A}}$ which contains both the image of the filtered Yoneda embedding $\mathcal{Y}(\mathcal{A})$ as well as $AF\mathrm{mod}_{\mathcal{A}}$ and is closed under mapping cones over morphisms with filtration level ≤ 0 and quasi-isomorphisms of filtration level ≤ 0. Denote this filtered A_∞-category by

$\langle \mathcal{Y}(\mathcal{A}), AF\text{mod}_{\mathcal{A}} \rangle^{\Delta}$. Define now the complete persistence derived category of $\mathcal{A}$ to be the persistence homology category of the latter, namely:

$$PD^c(\mathcal{A}) := PH\Big(\langle \mathcal{Y}(\mathcal{A}), AF\text{mod}_{\mathcal{A}} \rangle^{\Delta}\Big).$$

The discussion from §3.1.4.2 on functors carries over to this case. More specifically, if $\mathcal{A}$, $\mathcal{B}$ are filtered A_∞-categories and $\mathcal{F} : \mathcal{A} \longrightarrow \mathcal{B}$ is an A_∞-functor with linear deviation then we obtain a TPC functor:

$$PD^c(\mathcal{F}) : PD^c(\mathcal{A}) \longrightarrow PD^c(\mathcal{B})$$

induced by $\mathcal{F}$.

There is an alternative description of $PD^c(\mathcal{A})$, by completing $PD(\mathcal{A})$ with respect to the interleaving distance, or even by applying the same type of completion to $\mathcal{Y}(\mathcal{A})^{\Delta}$ and then passing to persistence homology. The result turns out to be the same:

Lemma 3.1.4.1. $PD^c(\mathcal{A}) = \big(PD(\mathcal{A})\big)^c = PH\Big((\mathcal{Y}(\mathcal{A})^{\Delta})^c\Big)$.

3.1.4.4. *r-isomorphisms.* In the context of TPC's there is an important notion of r-isomorphism that leads to interesting measurements and can also be used to arrive at the same completion $PD^c(\mathcal{A})$ described in §3.1.4.3.

Let $\mathcal{C}$ be a TPC, $A, B \in \text{Obj}(\mathcal{C})$, $\phi : A \longrightarrow B$ a morphism in $\hom_{\mathcal{C}^0}(A, B)$ and $r \geq 0$. We say that ϕ is an r-isomorphism if ϕ can be completed to an exact triangle in the (triangulated) category $\mathcal{C}^0$

$$A \xrightarrow{\phi} B \longrightarrow K \longrightarrow TA$$

with $K \in \text{Obj}(\mathcal{C})$ r-acyclic (see §3.1.1.3). For example, the standard morphism $\eta_r^A : \Sigma^r A \longrightarrow A$ is always an r-isomorphism (this is in fact one of the axioms defining TPC's), but of course there are many other examples. We refer the reader to [**BCZ24b**] for more details on the concept of r-isomorphisms.

Completing with respect to r-isomorphisms is somewhat subtle because of the following. The existence of an r-isomorphism $\phi : A \longrightarrow B$ does not imply that there exists an r-isomorphism $B \longrightarrow A$, hence this does not lead to a symmetric relation on pairs of objects (A, B). Nor is this "relation" transitive (if $\phi : A \longrightarrow B$ is an r_1-isomorphism and r_2-isomorphism $\psi : B \longrightarrow C$ is an r_2-isomorphism then $\psi \circ \phi : A \longrightarrow C$ is in general only an $(r_1 + r_2)$-isomorphism). However by the results of [**BCZ24b**], if $\phi : A \longrightarrow B$ is an r-isomorphism then there exists a $2r$-isomorphism $\Sigma^r B \longrightarrow A$. This leads to the following definition: two objects $A, B \in \text{Obj}(\mathcal{C})$ are called *almost isomorphic* if there is an $r \geq 0$ and $s \in \mathbb{R}$ and an r-isomorphism $\phi : A \longrightarrow \Sigma^s B$. By the results of [**BCZ24b**], being "almost isomorphic" is an equivalence relation. Moreover, we have:

Lemma 3.1.4.2. *Let $\mathcal{C}$ be a TPC and Let $A, B \in$ Obj$(\mathcal{C})$. Then A is almost isomorphic to B if and only if $d_{int}(A, B) < \infty$.*

It follows that completing $PD(\mathcal{A})$ with respect to almost isomorphisms gives precisely the same category as $PD^c(\mathcal{A})$.

To end this discussion, let us also mention the following useful fact: completing a TPC with respect to the interleaving distance always yields a TPC. More precisely

LEMMA 3.1.4.3. *Let $\mathcal{C}$ be a TPC and $\mathcal{D} \subset \mathcal{C}$ a full sub-TPC of $\mathcal{C}$. Then $\mathcal{D}^c \subset \mathcal{C}$ is also a sub-TPC.*

3.1.4.5. *Filtered twisted complexes.* Let $\mathcal{A}$ be a filtered A_∞-category. There is another variant of a TPC associated with $\mathcal{A}$ which is based on twisted complexes (see [**Sei08**, Chapter I, Section 3(l)] for unfiltered A_∞-categories and [**BCZ24b**, Section 2.5] for the case of filtered dg-categories. The case of filtered A_∞-categories is treated in detail in §1.9.

Roughly speaking, one defines first the additive enlargement $\mathcal{A}^\oplus$ of $\mathcal{A}$ as in [**Sei08**] (where it is denoted by $\Sigma\mathcal{A}$, but in our case Σ is already used for other purposes), and defines a filtration structure on it by an obvious extension of the filtrations on $\mathcal{A}$. Recall that, by assumption, $\mathcal{A}$ is closed under shifts and translations. Next one forms a new filtered A_∞-category $FTw\mathcal{A}$ whose objects are all twisted complexes (X, q_X) over $\mathcal{A}$ with $X \in \mathrm{Obj}(\mathcal{A}^\oplus)$ and differential $q_X \in \hom^0_{\mathcal{A}^\oplus}(X, X)$ lying at filtration level ≤ 0. The morphisms in this category come from $\mathcal{A}^\oplus$ and are thus filtered. One then defines the A_∞-operations on $FTw\mathcal{A}$ in the standard way and the fact that $\mathcal{A}$ itself is filtered implies that $FTw\mathcal{A}$ is filtered too. Finally note that since $\mathcal{A}$ is strictly unital the same holds for $FTw\mathcal{A}$.

The filtered A_∞-category $\mathcal{A}$ embeds into $Tw\mathcal{A}$ in an obvious way, the embedding being a filtered A_∞-functor which is full and faithful (on the chain level). Moreover, $Tw\mathcal{A}$ is pre-triangulated in the filtered sense (which in particular means that it is closed under formation of filtered mapping cones). We denote by $PH(FTw\mathcal{A})$ the persistence homological category of $FTw\mathcal{A}$. Standard arguments show that $PH(FTw\mathcal{A})$ is a TPC with translation and shift functors induced from those of $\mathcal{A}$.

An important property of $PH(FTw\mathcal{A})$ is the following. Let $\mathcal{A}$, $\mathcal{B}$ be two filtered A_∞-categories. Let $\mathcal{F} : \mathcal{A} \longrightarrow \mathcal{B}$ be a filtered A_∞-functor. Then there is a canonical extension of $\mathcal{F}$ to a filtered, strictly unital, A_∞-functor $Tw\mathcal{F} : FTw\mathcal{A} \longrightarrow FTw\mathcal{B}$. Moreover, $Tw\mathcal{F}$ induces a homological functor

$$PH(Tw\mathcal{F}) : PH(FTw\mathcal{A}) \longrightarrow PH(FTw\mathcal{B})$$

which is a TPC functor.

3.1.5. Systems of categories with increasing accuracies. In what follows we will deal with families of TPC's that should be thought of as approximations of a (currently not yet defined) limit TPC. The TPC's occurring in the family are parametrized by a space controlling the accuracy-level of their approximation. This situation naturally occurs when dealing with Fukaya categories that can be parametrized by different choices of perturbation data. The setting and definitions below were conceived with the example of Fukaya categories in mind. Nevertheless we believe that a general axiomatic framework may prove useful in other cases too.

We will axiomatize these notions in three different settings below, starting with the most general one - a system of PC's.

3.1.5.1. *The case of PC's.* Let $(\mathcal{P}, \preceq)$ be a directed set (i.e. a preordered set such that $\forall\, p', p'' \in \mathcal{P},\ \exists\, q \in \mathcal{P}$ with $p', p'' \preceq q$), together with a function $\nu : \mathcal{P} \longrightarrow \mathbb{R}_{\geq 0}$ that has the following properties:

(1) For every $p \preceq q$ we have $\nu(p) \geq \nu(q)$ (i.e. ν is decreasing with respect to $\preceq$).
(2) For every $\delta > 0$ there exists $p \in \mathcal{P}$ such that $\nu(p) \leq \delta$.

We will refer to $(\mathcal{P}, \preceq, \nu)$ as the parameter space for our system of PC's. For brevity we will omit $\preceq$ and ν from the notation and simply write $\mathcal{P}$ for the triple. We will refer to $\nu(p)$ as the norm or size of the parameter $p \in \mathcal{P}$.

Consider now a family $\widehat{\mathcal{C}} = \{\mathcal{C}_p\}_{p \in \mathcal{P}}$ of PC's $\mathcal{C}_p$ parametrized by $p \in \mathcal{P}$. We will denote the shift and translation functors of all the $\mathcal{C}_p$'s by the same notation, Σ and T, respectively. Assume further that for every $p \preceq q$ we are given two sets $\mathcal{J}_{p,q} = \mathcal{J}_{p,q}(\widehat{\mathcal{C}})$, $\mathcal{J}_{q,p} = \mathcal{J}_{q,p}(\widehat{\mathcal{C}})$, of PC functors $\mathcal{C}_p \longrightarrow \mathcal{C}_q$, $\mathcal{C}_q \longrightarrow \mathcal{C}_p$, respectively, with the following properties:

(1) $\mathcal{J}_{p,p} = \{\mathrm{id}_{\mathcal{C}_p}\}$ for every $p \in \mathcal{P}$.
(2) If $p \preceq q$ then all the functors in $\mathcal{J}_{p,q}$ are mutually 0-isomorphic. More specifically, for every two functors $\mathcal{F}, \mathcal{G} \in \mathcal{J}_{p,q}$ there is a natural isomorphism $\mathcal{F} \cong \mathcal{G}$ of persistence level 0 (in other words, $\mathcal{F}$ and $\mathcal{G}$ are isomorphic in the 0-level category $\text{pc-fun}^0(\mathcal{C}_p, \mathcal{C}_q)$ of the PC of PC-functors $\text{pc-fun}(\mathcal{C}_p, \mathcal{C}_q)$). Here and in what follows we will abbreviate this by writing $\mathcal{F} \cong_0 \mathcal{G}$. Furthermore, we require the same thing also for any two functors in $\mathcal{J}_{q,p}$.
(3) If $p \preceq q \preceq r$ then for every $\mathcal{F} \in \mathcal{J}_{p,q}$, $\mathcal{G} \in \mathcal{J}_{q,r}$, $\mathcal{H} \in \mathcal{J}_{p,r}$ we have $\mathcal{G} \circ \mathcal{F} \cong_0 \mathcal{H}$.
(4) If $p \preceq q$ then for every $\mathcal{F} \in \mathcal{J}_{p,q}$ and $\mathcal{H} \in \mathcal{J}_{q,p}$ we have
$$d_{\mathrm{int}}(\mathcal{H} \circ \mathcal{F}, \mathrm{id}_{\mathcal{C}_p}),\ d_{\mathrm{int}}(\mathcal{F} \circ \mathcal{H}, \mathrm{id}_{\mathcal{C}_q}) \leq C|\nu(p) - \nu(q)|,$$
for some constant C that depends only on the entire family $\widehat{\mathcal{C}}$ but neither on p, q nor on $\mathcal{F}, \mathcal{G}$. Here the two d_{int}'s stand for the interleaving distances on the PC's of PC functors $\mathcal{C}_p \longrightarrow \mathcal{C}_p$ and $\mathcal{C}_q \longrightarrow \mathcal{C}_q$.

In what follows we will denote by $\widehat{\mathcal{C}}$ the entire data above, namely both the family of categories $\{\mathcal{C}_p\}_{p \in \mathcal{P}}$ as well the sets of functors $\mathcal{J}_{p,q}$ and refer to $\widehat{\mathcal{C}}$ as a *system of PC's with increasing accuracy*, or sometimes a *system of PC's* for short. We will call the functors in $\mathcal{J}_{p,q}$ comparison functors and typically denote such a functor by $\mathcal{H}_{p,q} : \mathcal{C}_p \longrightarrow \mathcal{C}_q$.

Remark 3.1.5.1. (1) Given a system $\widehat{\mathcal{C}}$ of PC's with increasing accuracy, all the ∞-level categories $\mathcal{C}_p^{\infty}$, corresponding to the PC's $\mathcal{C}_p$ in the system $\widehat{\mathcal{C}}$, are mutually equivalent. From the persistence viewpoint, as p "grows" (with respect to $\preceq$) the comparison between $\mathcal{C}_p$ and $\mathcal{C}_q$ for $p \preceq q$ becomes more and more precise.
(2) A single PC $\mathcal{C}$ makes a special case of the definition above. We can view it as a system of PC's parametrized by a 1-point parameter space $\mathcal{P} = \{*\}$. In this case we set the norm function ν to be trivial, $\nu(*) = 0$.

3.1.5.2. *Functors of systems of PC's.* Next we discuss functors between systems of PC's with increasing accuracy. Let $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{D}}$ be two systems of PC's with increasing accuracy. Whenever we need to distinguish between the additional structures of each of the systems $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{D}}$ we will use $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{D}}$ subscripts or superscripts, depending on notational convenience (for example, the parameter space of $\widehat{\mathcal{C}}$ will be denoted $\mathcal{P}_{\widehat{\mathcal{C}}}$, the comparison functors $\mathcal{H}^{\widehat{\mathcal{C}}}_{p,q}$, etc. but we will denote the preorders and size of parameters in both systems by $\preceq$ and ν respectively).

A functor $\widehat{\mathcal{F}} : \widehat{\mathcal{C}} \longrightarrow \widehat{\mathcal{D}}$ consists of the following structures:

(1) A map $\widehat{\mathcal{F}} : \mathcal{P}_{\widehat{\mathcal{C}}} \longrightarrow \mathcal{P}_{\widehat{\mathcal{D}}}$ between the parameter spaces of $\widehat{\mathcal{C}}$ and of $\widehat{\mathcal{D}}$ (which for simplicity we have denoted by $\widehat{\mathcal{F}}$ too).

(2) A family of PC functors $\big\{\mathcal{F}_p : \mathcal{C}_p \longrightarrow \mathcal{D}_{\widehat{\mathcal{F}}(p)}\big\}_{p \in \mathcal{P}_{\widehat{\mathcal{C}}}}$, parametrized by $\mathcal{P}_{\widehat{\mathcal{C}}}$.

These two structures are required to have the following additional properties. The map $\widehat{\mathcal{F}}$ from (1) above should satisfy that for every $p, q \in \mathcal{P}_{\widehat{\mathcal{C}}}$ with $p \preceq q$ we have $\widehat{\mathcal{F}}(p) \preceq \widehat{\mathcal{F}}(q)$. Moreover, we require that $\nu(\widehat{\mathcal{F}}(p)) \leq \nu(p)$ for every $p \in \mathcal{P}_{\widehat{\mathcal{C}}}$. The functors $\mathcal{F}_p$ are required to satisfy the following two conditions. First, for every $p \preceq q$ in $\mathcal{P}_{\widehat{\mathcal{C}}}$ and any choices of comparison functors $\mathcal{H}^{\widehat{\mathcal{C}}}_{p,q}$, $\mathcal{H}^{\widehat{\mathcal{D}}}_{\widehat{\mathcal{F}}(p),\widehat{\mathcal{F}}(q)}$ the following diagram:

$$\begin{tikzcd} \mathcal{C}_p \arrow[r, "\mathcal{F}_p"] \arrow[d, "\mathcal{H}^{\widehat{\mathcal{C}}}_{p,q}"'] & \mathcal{D}_{\widehat{\mathcal{F}}(p)} \arrow[d, "\mathcal{H}^{\widehat{\mathcal{D}}}_{\widehat{\mathcal{F}}(p),\widehat{\mathcal{F}}(q)}"] \\ \mathcal{C}_q \arrow[r, "\mathcal{F}_q"] & \mathcal{D}_{\widehat{\mathcal{F}}(q)} \end{tikzcd} \tag{8}$$

commutes up to natural isomorphisms in the 0-level category $\text{pc-fun}^0(\mathcal{C}_p, \mathcal{D}_{\widehat{\mathcal{F}}(q)})$ of the PC of PC-functors $\text{pc-fun}(\mathcal{C}_p, \mathcal{D}_{\widehat{\mathcal{F}}(q)})$. Similarly, we also require that for every $p \preceq q$ in $\mathcal{P}_{\widehat{\mathcal{C}}}$ and every choice of comparison functors $\mathcal{H}^{\widehat{\mathcal{C}}}_{q,p}$, $\mathcal{H}^{\widehat{\mathcal{D}}}_{\widehat{\mathcal{F}}(q),\widehat{\mathcal{F}}(p)}$ the diagram:

$$\begin{tikzcd} \mathcal{C}_p \arrow[r, "\mathcal{F}_p"] & \mathcal{D}_{\widehat{\mathcal{F}}(p)} \\ \mathcal{C}_q \arrow[r, "\mathcal{F}_q"] \arrow[u, "\mathcal{H}^{\widehat{\mathcal{C}}}_{p,q}"] & \mathcal{D}_{\widehat{\mathcal{F}}(q)} \arrow[u, "\mathcal{H}^{\widehat{\mathcal{D}}}_{\widehat{\mathcal{F}}(p),\widehat{\mathcal{F}}(q)}"'] \end{tikzcd} \tag{9}$$

commutes up to natural isomorphisms in the 0-level category $\text{pc-fun}^0(\mathcal{C}_q, \mathcal{D}_{\widehat{\mathcal{F}}(p)})$ of the PC of PC-functors $\text{pc-fun}(\mathcal{C}_q, \mathcal{D}_{\widehat{\mathcal{F}}(p)})$.

It remains to define natural transformations between functors of systems of PC's. Let $\widehat{\mathcal{C}}, \widehat{\mathcal{D}}$ be two systems of PC's and $\widehat{\mathcal{F}}, \widehat{\mathcal{G}} : \widehat{\mathcal{C}} \longrightarrow \widehat{\mathcal{D}}$ two functors having the same action on the parameter spaces (i.e. the two maps $\widehat{\mathcal{F}}, \widehat{\mathcal{G}} : \mathcal{P}_{\widehat{\mathcal{C}}} \longrightarrow \mathcal{P}_{\widehat{\mathcal{D}}}$ coincide). A natural transformation $\widehat{\theta} : \widehat{\mathcal{F}} \longrightarrow \widehat{\mathcal{G}}$ of shift $s \in \mathbb{R}$, consists of a family of persistence natural transformations $\theta_p : \mathcal{F}_p \longrightarrow \mathcal{G}_p$ for every $p \in \mathcal{P}$ each of which of shift s. In addition the θ_p are required to have the following weak compatibility with the comparison functors. For every $p \preceq q$ and every $X \in \mathrm{Obj}(\mathcal{C}_p)$ there exist 0-isomorphisms

$$\tau_{p,q} : \mathcal{H}_{\widehat{\mathcal{F}}(p),\widehat{\mathcal{F}}(q)}(\mathcal{F}_p(X)) \longrightarrow \mathcal{F}_q(\mathcal{H}_{p,q}(X)), \quad \sigma_{p,q} : \mathcal{H}_{\widehat{\mathcal{F}}(p),\widehat{\mathcal{F}}(q)}(\mathcal{G}_p(X)) \longrightarrow \mathcal{G}_q(\mathcal{H}_{p,q}(X)),$$

such that

$$\sigma_{p,q} \circ \mathcal{H}_{\widehat{\mathcal{F}}(p),\widehat{\mathcal{F}}(q)}(\theta_p(X)) = \theta_q(\mathcal{H}_{p,q}(X)) \circ \tau_{p,q}$$

in $\hom_{\mathcal{D}_{\widehat{\mathcal{F}}(q)}}\big(\mathcal{H}_{\widehat{\mathcal{F}}(p),\widehat{\mathcal{F}}(q)}(\mathcal{F}_p(X)), \mathcal{G}_q(\mathcal{H}_{p,q}(X))\big)$. Finally, we require a similar compatibility also with the comparison functors $\mathcal{H}_{q,p}$.

3.1.5.3. *The case of TPC's.* A system of TPC's with increasing accuracy is simply a system $\widehat{\mathcal{C}}$ of PC's as in §3.1.5.1 with the following additional assumptions. Each of the categories $\mathcal{C}_p$, $p \in \mathcal{P}$, in the system is assumed to be a TPC. Moreover, all the functors in the collections $\mathcal{J}_{p,q}$ and $\mathcal{J}_{q,p}$ are assumed to be TPC functors. The notion of functors of PC's (and natural transformations between them) from §3.1.5.2 extends to the setting of systems of TPC's in a straightforward way, by just requiring the functors $\mathcal{F}_p$ to be TPC functors.

Note also that point (1) of Remark 3.1.5.1 carries over to systems $\widehat{\mathcal{C}} = \{\mathcal{C}_p\}_{p \in \mathcal{P}}$ of TPC's, namely the ∞-levels $\mathcal{C}_p^\infty$, $p \in \mathcal{P}$, are mutually equivalent in the triangulated sense.

3.1.5.4. *Systems of filtered A_∞-categories.* There is also a notion of system of filtered A_∞-categories with increasing accuracy. The definition is analogous to that given for a system of PC's in §3.1.5.1 above, but with one significant change regarding the comparison functors between the qth-category and the pth one when $p \preceq q$.

In the A_∞-case a system of categories with increasing accuracy consists of the following. A family $\widehat{\mathcal{A}} = \{\mathcal{A}_p\}_{p \in \mathcal{P}}$ of filtered A_∞-categories $\mathcal{A}_p$, parametrized by $p \in \mathcal{P}$. Note that we do not assume these A_∞-categories to be pretriangulated. The parameter space $\mathcal{P}$ is endowed with the same structures ν and $\preceq$ as in §3.1.5.3. For $p, q \in \mathcal{P}$ with $p \preceq q$ we have two collections $\mathcal{J}_{p,q}$ and $\mathcal{J}_{q,p}$ of comparison A_∞-functors with the following properties. The functors $\mathcal{H}_{p,q} \in \mathcal{J}_{p,q}$ from the first collection are filtered A_∞-functors. The functors $\mathcal{H}_{q,p} \in \mathcal{J}_{q,p}$ from the second collection are assumed to be A_∞-functors $\mathcal{H}_{q,p} : \mathcal{A}_q \longrightarrow \mathcal{A}_p$ with linear deviation rate $\leq C|\nu(p) - \nu(q)|$, for some constant C that depends only on $\widehat{\mathcal{A}}$ (and not on p, q). Moreover, the functors from $\mathcal{J}_{p,q}$ and $\mathcal{J}_{q,p}$ are assumed to satisfy the A_∞-analogs of the properties listed in §3.1.5.3 on page 82, with the following obvious modifications. Instead of 0-isomorphisms $\cong_0$ between two functors mentioned at points (2) and (3) on page 82 we will now require that we have 0-isomorphisms between the respective functors in the persistence homology category of the filtered A_∞-functors $PH(F\mathrm{fun}_{A_\infty}(\mathcal{A}_p, \mathcal{A}_q))$, and similarly for the comparison functors in $PH(\mathrm{fun}^{\mathrm{LD}}(\mathcal{A}_q, \mathcal{A}_p))$. The assumptions in point (4) on page 82 will be now replaced by requiring $d_{\mathrm{int}}(\mathcal{F} \circ \mathcal{H}, \mathrm{id}) \leq C|\nu(p) - \nu(q)|$ for the interleaving distance on $PH(\mathrm{fun}_{A_\infty}(\mathcal{A}_p, \mathcal{A}_p))$, and similarly for $d_{\mathrm{int}}(\mathcal{H} \circ \mathcal{F}, \mathrm{id})$.

The notion of functors between systems of PC's (as well as natural transformations between them) has an A_∞-analog, following the preceding principle.

3.1.5.5. *Modules and Yoneda embeddings of systems.* Let $\widehat{\mathcal{A}} = \{\mathcal{A}_p\}_{p \in \mathcal{P}}$ be a system of filtered A_∞-categories. Denote by $F\mathrm{mod}_{\mathcal{A}_p}$ the A_∞-category of filtered $\mathcal{A}_p$-modules. Recall that this is a filtered A_∞-category (in fact a filtered dg-category). These categories fit into a system

$$F\mathrm{mod}(\widehat{\mathcal{A}}) := \{F\mathrm{mod}_{\mathcal{A}_p}\}_{p\in\mathcal{P}}$$

of filtered A_∞-categories, where for $p \preceq q$ the comparison functors

$$(\mathcal{H}_{p,q})_* : F\mathrm{mod}_{\mathcal{A}_p} \longrightarrow F\mathrm{mod}_{\mathcal{A}_q}$$

are just the push-forward functors induced by the comparison functors $\mathcal{H}_{p,q}$ of the system $\widehat{\mathcal{A}}$, and similarly for $(\mathcal{H}_{q,p})_*$. Since the comparison functors $\mathcal{H}_{p,q}$, for $p \preceq q$, are filtered the same holds for $(\mathcal{H}_{p,q})_*$. As for the functors in the opposite direction, interestingly the pushforward functors $(\mathcal{H}_{q,p})_*$ behave somewhat better than the $\mathcal{H}_{q,p}$ in the following sense. While $\mathcal{H}_{q,p}$ are not really filtered functors but only LD functors, the induced comparison functors $(\mathcal{H}_{q,p})_*$ are genuinely filtered. This is a very nice feature of the filtered push-forward construction (see §1.8). However, note that it is still the case that $(\mathcal{H}_{p,q})_* \circ (\mathcal{H}_{q,p})_*$ is only r-quasi-isomorphic to id, for $r = C|\nu(p) - \nu(q)|$, and the same for $(\mathcal{H}_{q,p})_* \circ (\mathcal{H}_{p,q})_*$.

We now turn to Yoneda embeddings of systems. Let $\widehat{\mathcal{A}}$ be a system of filtered A_∞-categories as above. Denote by

$$\mathcal{Y}_p : \mathcal{A}_p \longrightarrow F\mathrm{mod}_{\mathcal{A}_p}$$

the filtered Yoneda embedding (see §1.3.1), and let $(\mathcal{Y}_p(\mathcal{A}_p))^{(\text{q-iso},0)}$ be the minimal full subcategory of $F\mathrm{mod}_{\mathcal{A}_p}$ which contains the objects $\mathcal{Y}_p(\mathrm{Obj}(\mathcal{A}_p))$ and is also closed with respect to 0-quasi-isomorphisms. Note that the comparison functors $(\mathcal{H}_{p,q})_*$ and $(\mathcal{H}_{q,p})_*$ preserve these subcategories. The main point here is that for every $A \in \mathrm{Obj}(\mathcal{A}_p)$ and $p \preceq q$ the modules $(\mathcal{H}_{p,q})_*\mathcal{Y}_p(A)$ and $\mathcal{Y}_q(\mathcal{H}_{p,q}(A))$ are 0-quasi-isomorphic. Similarly, for every $B \in \mathrm{Obj}(\mathcal{A}_q)$ the modules $(\mathcal{H}_{q,p})_*(\mathcal{Y}_q(B))$ and $\Sigma^r\mathcal{Y}_p(\mathcal{H}_{q,p}(B))$ are 0-quasi-isomorphic, where $r \leq C|\nu(p) - \nu(q)|$ is the deviation rate of the LD functor $\mathcal{H}_{q,p}$. Recall that our categories $\mathcal{A}_p$ are endowed with a shift functor Σ, hence the categories $\mathcal{Y}(A_p)$ and $(\mathcal{Y}_p(\mathcal{A}_p))^{(\text{q-iso},0)}$ are both closed under shifts.

It follows that the categories $(\mathcal{Y}_p(\mathcal{A}_p))^{(\text{q-iso},0)}$, $p \in \mathcal{P}$, together with the restrictions of the push-forward comparison functors $(\mathcal{H}_{p,q})_*$, $(\mathcal{H}_{q,p})_*$ to them, form a system of filtered A_∞-categories (which can be viewed as a subsystem of $F\mathrm{mod}(\widehat{\mathcal{A}})$). We denote this system by $\mathcal{Y}(\widehat{\mathcal{A}})$ and call it the Yoneda system of $\widehat{\mathcal{A}}$.

3.1.5.6. *Systems of PC's associated with systems of A_∞-categories.* Given a system of filtered A_∞-categories $\widehat{\mathcal{A}} = \{\mathcal{A}_p\}_{p\in\mathcal{P}}$ one can obtain a system of PC's by passing to persistence homology. However, there is a small subtlety in this procedure. The corresponding system $\{PH(\mathcal{A}_p)\}_{p\in\mathcal{P}}$ of persistence homological categories is not really a system of PC's, because for $p \preceq q$ the A_∞-functors from $\mathcal{J}_{q,p}$ are not filtered but only LD functors, hence do not induce PC functors $PH(\mathcal{A}_q) \longrightarrow PH(\mathcal{A}_p)$. However this problem can be rectified by passing to the Yoneda system introduced earlier in §3.1.5.5. More precisely, consider the Yoneda system $\mathcal{Y}(\widehat{\mathcal{A}})$ of $\widehat{\mathcal{A}}$. Recall from §3.1.5.5 that this is (genuine) system of filtered A_∞-categories. Define

$$PH(\widehat{\mathcal{A}}) := \{PH((\mathcal{Y}_p(\mathcal{A}_p))^{(\text{q-iso},0)})\}_{p\in\mathcal{P}}$$

to be system consisting of the persistence homology categories of $(\mathcal{Y}_p(\mathcal{A}_p))^{(\text{q-iso},0)}$ forming the Yoneda system $\mathcal{Y}(\widehat{\mathcal{A}})$. The comparison functors $\mathcal{H}_{p,q}$ are defined to be the persistence homology functors induced by the $(\mathcal{H}_{p,q})_*$, i.e. $\mathcal{H}_{p,q} := PH((\mathcal{H}_{p,q})_*)$ and similarly for $\mathcal{H}_{q,p}$.

The assignment $\widehat{\mathcal{A}} \longmapsto PH(\widehat{\mathcal{A}})$ extends to a functor $\mathrm{SYS}_{FA_\infty} \longrightarrow \mathrm{SYS}_{\mathrm{PC}}$ between the category of systems of filtered A_∞-categories and the category of systems of PC's. This functor assigns to a functor $\widehat{\mathcal{F}} : \widehat{\mathcal{A}} \longrightarrow \widehat{\mathcal{B}}$ between systems of filtered A_∞-categories a functor $PH(\widehat{\mathcal{F}}) : PH(\widehat{\mathcal{A}}) \longrightarrow PH(\widehat{\mathcal{B}})$ between the corresponding systems of PC's and this correspondence behaves as expected on natural transformations between such functors.

Finally, we remark that if $\widehat{\mathcal{B}} = \{\mathcal{B}_p\}_{p\in\mathcal{P}}$ is a systems of pre-triangulated filtered A_∞-categories then the corresponding system $PH(\widehat{\mathcal{B}})$ of PC's is in fact a system of TPC's. For the rest of this paper, the most important example in this context will be the following. Let $\widehat{\mathcal{A}} = \{\mathcal{A}_p\}_{p\in\mathcal{P}}$ be a system of filtered A_∞-categories. Consider the Yoneda system $\mathcal{Y}(\widehat{\mathcal{A}})$ as defined in §3.1.5.5. By applying the construction from §3.1.4.2 to each category that participates in the latter system we obtain a new system $\mathcal{Y}(\widehat{\mathcal{A}})^\Delta = \{\mathcal{Y}(\mathcal{A}_p)^\Delta\}_{p\in\mathcal{P}}$ of filtered pre-triangulated categories. The comparison functors for this system are induced from those of the Yoneda system (see §3.1.5.5). This works since filtered and LD A_∞-functors preserve triangulated completions as defined in §3.1.4.2. We now pass to persistence homology and define the following system of TPC's:

$$PD(\widehat{\mathcal{A}}) := \{PH(\mathcal{Y}(\mathcal{A}_p)^\Delta)\}_{p\in\mathcal{P}}. \tag{10}$$

We call this system the *persistence derived system of* $\widehat{\mathcal{A}}$.

3.1.5.7. *Systems with a coherent base of objects.* Let $\widehat{\mathcal{C}} = \{\mathcal{C}_p\}_{p\in\mathcal{P}}$ be a system of PC's with increasing accuracy. A coherent base of objects $\{mathscr\}B$ for $\widehat{\mathcal{C}}$ is a family of subsets $\mathcal{B} = \{\mathcal{B}_p\}_{p\in\mathcal{P}}$ of objects $\mathcal{B}_p \subset \mathrm{Obj}(\mathcal{C}_p)$ for every $p \in \mathcal{P}$, satisfying the following properties:

(1) For every $p \preceq q$, every two comparison functors $\mathcal{H}'_{p,q}, \mathcal{H}''_{p,q} \in \mathcal{J}_{p,q}$ and every $A \in \mathcal{B}_p$ we have $\mathcal{H}'_{p,q}(A) = \mathcal{H}''_{p,q}(A)$ and $\mathcal{H}'_{p,q}(A) \in \mathcal{B}_q$. We require the analogous properties to hold also for every two comparison functors $\mathcal{H}'_{q,p}, \mathcal{H}''_{q,p} \in \mathcal{J}_{q,p}$ and every $B \in \mathcal{B}_q$.

(2) For every $p \preceq q \preceq r$, every two comparison functors $\mathcal{H}_{p,q} \in \mathcal{J}_{p,q}$, $\mathcal{H}_{q,r} \in \mathcal{J}_{q,r}$ and every $A \in \mathcal{B}_p$ we have $\mathcal{H}_{q,r}(\mathcal{H}_{p,q}(A)) = \mathcal{H}_{p,r}(A)$. We require the analogous properties to hold also for all comparison functors going in the other direction, namely for $\mathcal{H}_{r,q}$, $\mathcal{H}_{q,p}$, $\mathcal{H}_{r,p}$ and every object $B \in \mathcal{B}_r$.

(3) For every $p \preceq q$, every choice of comparison functors $\mathcal{H}_{p,q}$, $\mathcal{H}_{q,p}$ and every $A \in \mathcal{B}_p$, $B \in \mathcal{B}_q$ we have $\mathcal{H}_{q,p}(\mathcal{H}_{p,q}(A)) = A$, $\mathcal{H}_{p,q}(\mathcal{H}_{q,p}(B)) = B$.

It follows from the definition above that for every $p', p'' \in \mathcal{P}$ we have a canonical bijection $\mathcal{B}_{p'} \longrightarrow \mathcal{B}_{p''}$, namely $\mathcal{H}_{q,p''} \circ \mathcal{H}_{p',q}$ where $q \in \mathcal{P}$ is any element with $q \succeq p', p''$.

One can define functors between systems of PC's with a coherent base of objects in a straightforward way.

The concept of systems with a coherent base of objects (as well functors between such) carries over without any modification also to the case of systems of filtered A_∞-categories.

A special case of a system of categories (filtered A_∞ or PC's) with a coherent base of objects that will appear later is when $\mathcal{B}_p = \mathrm{Obj}(\mathcal{C}_p)$ for every $p \in \mathcal{P}$. We will refer to such a situation as a *coherent full base of objects*. In the following we will encounter an even simpler situation, where all the categories $\mathcal{C}_p$, $p \in \mathcal{P}$, have the same set of objects (i.e. $\mathrm{Obj}(\mathcal{C}_{p'}) = \mathrm{Obj}(\mathcal{C}_{p''})$ for every $p', p'' \in \mathcal{P}$) and all the comparison functors act like the identity on objects, namely $\mathcal{H}_{p,q}(X) = X$ for every $p \preceq q$, every comparison functor $\mathcal{H}_{p,q}$, and $X \in \mathrm{Obj}(\mathcal{C}_p)$. In this case we will view the coherent base $\mathcal{B}$ as a single set rather than a family, and set $\mathcal{B} = \mathrm{Obj}(\mathcal{C}_p)$ for any $p \in \mathcal{P}$. We will call this type of systems by the name *system of categories (PC's or filtered A_∞-categories) with a fixed full base of objects*.

REMARK 3.1.5.2. Consider a system of filtered A_∞-categories $\widehat{\mathcal{A}} = \{\mathcal{A}_p\}_{p\in\mathcal{P}}$ with a fixed full base of objects $\mathcal{B}$. Let $F\mathrm{mod}(\widehat{\mathcal{A}})$ be the corresponding system of filtered modules over the $\mathcal{A}_p$'s and $\mathcal{Y}(\widehat{\mathcal{A}})$ be the Yoneda system of $\widehat{\mathcal{A}}$ (see §3.1.5.5). Unlike $\widehat{\mathcal{A}}$ these other two systems do not have a fixed (or even coherent) full base of objects. The reason is that the comparison functors for $F\mathrm{mod}(\widehat{\mathcal{A}})$ and $\mathcal{Y}(\widehat{\mathcal{A}})$ are the "module push-forward" functors $\mathcal{H}_*$ induced from the comparison functors $\mathcal{H}$ of $\widehat{\mathcal{A}}$ and these do not satisfy the conditions required for a coherent base of objects. More specifically, for $p \preceq q \preceq r$, the functors $(\mathcal{H}_{q,r})_* \circ (\mathcal{H}_{p,q})_*$ and $(\mathcal{H}_{q,r})_*$ do not act in the same way on filtered modules (though the images of a module under these two functors are 0-quasi-isomorphic). Similarly, the push forward $(\mathcal{H}_{p,q})_*\mathcal{Y}(X)$ of a Yoneda module is not a Yoneda module but only 0-quasi-isomorphic to a Yoneda module.

3.1.5.8. *Homotopy systems.* Let $\widehat{\mathcal{A}} = \{\mathcal{A}_p\}_{p\in\mathcal{P}}$ be a system of filtered A_∞-categories with a coherent full base of objects. We say that the system $\widehat{\mathcal{A}}$ is a *homotopy system* if the following conditions hold:

(1) For every $p \preceq q$ in $\mathcal{P}$, and every two comparison functors $\mathcal{H}'_{p,q}, \mathcal{H}''_{p,q} \in \mathcal{J}_{p,q}$ there is a homotopy
$$T \in \hom^0_{F\mathrm{fun}(\mathcal{A}_p,\mathcal{A}_q)}(\mathcal{H}'_{p,q}, \mathcal{H}''_{p,q})$$
between $\mathcal{H}'_{p,q}$ and $\mathcal{H}''_{p,q}$ that does not shift filtrations. From now on we will call such T's 0-homotopies and say that $\mathcal{H}'_{p,q}$ and $\mathcal{H}''_{p,q}$ are 0-homotopic, and the superscript in $\hom^0$ stands for pre-natural transformations of filtration level 0 (i.e. they preserve filtrations). We refer to [**Sei08**, Section (1h)] for the definition of homotopy between A_∞-functors in the unfiltered case.

(2) For every $p \preceq q$ in $\mathcal{P}$, every two comparison functors $\mathcal{H}'_{q,p}, \mathcal{H}''_{q,p} \in \mathcal{J}_{q,p}$ are 0-homotopic as LD-functors with deviation rate $C|\nu(p) - \nu(q)|$, where C is the constant associated with the family $\widehat{\mathcal{A}}$ that appears in the definition of systems of filtered A_∞-categories in §3.1.5.4. Specifically, this means that there exists a homotopy T between the LD-functors $(\mathcal{H}'_{q,p}, C|\nu(p) - \nu(q)|)$ and $(\mathcal{H}''_{q,p}, C|\nu(p) - \nu(q)|)$ of shift 0, namely its dth order T_d shifts filtrations by $\leq dC|\nu(p) - \nu(q)|$. See §1.6.

(3) For every $p \preceq q \preceq r$ in $\mathcal{P}$ and every choices of comparison functors $\mathcal{H}_{p,q}$, $\mathcal{H}_{q,r}$, $\mathcal{H}_{p,r}$ we have that $\mathcal{H}_{q,r} \circ \mathcal{H}_{p,q}$ and $\mathcal{H}_{p,r}$ are 0-homotopic.

3.1.5.9. *Invariants of systems.* Systems of categories $\widehat{\mathcal{C}} = \{\mathcal{C}_p\}_{p \in \mathcal{P}}$ with increasing accuracy call for considering their limits (or rather colimits) when the parameter $p \in \mathcal{P}$ goes to "infinity" (or equivalently when the accuracy-levels $\nu(p)$ converges to 0). In other words, it would be desirable to be able to define a colimit category $\underrightarrow{\mathrm{colim}}_{p \in \mathcal{P}} \mathcal{C}_p$ which will have the same structures (e.g. PC or TPC structures, filtered A_∞-structure etc.) as the categories $\mathcal{C}_p$ forming the family $\widehat{\mathcal{C}}$. This seems a non-trivial foundational problem and we will not attempt to solve it in this paper. Instead, we will show that some algebraic invariants and measurements associated with the categories $\mathcal{C}_p$ do admit colimits, even without having a colimit category. We will concentrate on three such invariants (each of which is defined in a different settings): the interleaving (pseudo)-distance on the set of objects of a PC, the Grothendieck groups K_0 of triangulated categories and Hochschild homologies of A_∞-categories.

We begin with a useful equivalence relation on the totality of objects of a system of categories. Let $\widehat{\mathcal{C}} = \{\mathcal{C}_p\}_{p \in \mathcal{P}}$ be a system of PC's with increasing accuracy. Consider

$$\mathrm{Obj}^{\mathrm{tot}}(\widehat{\mathcal{C}}) := \coprod_{p \in \mathcal{P}} \mathrm{Obj}(\mathcal{C}_p)$$

and denote elements of this set by (A, p) with $p \in \mathcal{P}$ and $A \in \mathcal{C}_p$. Define an equivalence relation $\sim_{\widehat{\mathcal{C}}}$ on $\mathrm{Obj}^{\mathrm{tot}}(\widehat{\mathcal{C}})$ as follows: $(A', p') \sim_{\widehat{\mathcal{C}}} (A'', p'')$ if there exists $q \in \mathcal{P}$ with $p', p'' \preceq q$ such that $\mathcal{H}_{p',q}(A') \cong_0 \mathcal{H}_{p'',q}(A'')$ for some comparison functors $\mathcal{H}_{p',q} \in \mathcal{J}_{p',q}$ and $\mathcal{H}_{p'',q} \in \mathcal{J}_{p'',q}$. Here $\cong_0$ means an isomorphism in $\mathcal{C}^0_q$. (Since all the functors in each of $\mathcal{J}_{p',q}$, $\mathcal{J}_{p'',q}$, are mutually 0-isomorphic, requiring that $\mathcal{H}_{p',q}(A') \cong_0 \mathcal{H}_{p'',q}(A'')$ for some $\mathcal{H}_{p',q} \in \mathcal{J}_{p',q}$, $\mathcal{H}_{p'',q} \in \mathcal{J}_{p'',q}$, is the same as to ask that the same property holds for *all* $\mathcal{H}_{p',q} \in \mathcal{J}_{p',q}$, $\mathcal{H}_{p'',q} \in \mathcal{J}_{p'',q}$.) Clearly if $A', A'' \in \mathrm{Obj}(\mathcal{C}_p)$ are 0-isomorphic then $(A', p) \sim_{\widehat{\mathcal{C}}} (A'', p)$, but in general the equivalence relation $\sim_{\widehat{\mathcal{C}}}$ may relate more objects belonging to categories parametrized by different p's. Denote by $\widetilde{\mathrm{Obj}}(\widehat{\mathcal{C}})$ the equivalence classes of $\sim_{\widehat{\mathcal{C}}}$. Given $\mathcal{S} \subset \widetilde{\mathrm{Obj}}(\widehat{\mathcal{C}})$ and $p \in \mathcal{P}$ we write

$$\mathcal{S}_p := \{A \in \mathrm{Obj}(\mathcal{C}_p) \mid [A]_{\widehat{\mathcal{C}}} \in \mathcal{S}\},$$

where $[A]_{\widehat{\mathcal{C}}}$ stands for the equivalence class of A in $\widetilde{\mathrm{Obj}}(\widehat{\mathcal{C}})$. We will use the same notation $\mathcal{S}_p$ also when we deal with just one element $\mathcal{S} \in \widetilde{\mathrm{Obj}}(\widehat{\mathcal{C}})$.

We will define now a pseudo distance on $\widetilde{\mathrm{Obj}}(\widehat{\mathcal{C}})$ induced by the interleaving distances on the $\mathcal{C}_p$'s. Denote by d^p_{int} the interleaving (pseudo) distance on $\mathrm{Obj}(\mathcal{C}_p)$. We define the following measurement on pairs of elements $\widetilde{X}, \widetilde{Y} \in \widetilde{\mathrm{Obj}}(\widehat{\mathcal{C}})$:

$$\widetilde{d}_{\mathrm{int}}(\widetilde{X}, \widetilde{Y}) = \inf\bigl\{d^p_{\mathrm{int}}(X', Y') \mid p \in \mathcal{P}, X' \in \widetilde{X}_p, Y' \in \widetilde{Y}_p\bigr\}, \tag{11}$$

and call it the *limit interleaving distance* (the wording "limit" will be justified shortly, in Lemma 3.1.5.3 below). Note that the term $d^p_{\mathrm{int}}(X', Y')$ that appears in (11) depends only on the 0-isomorphism classes of $X', Y' \in \mathrm{Obj}(\mathcal{C}_p)$. It is straightforward to see that $\widetilde{d}_{\mathrm{int}}$ is a pseudo-distance. Note that due to the inf in (11), $\widetilde{d}_{\mathrm{int}}(\widetilde{X}, \widetilde{Y})$ may vanish even if $\widetilde{X} \neq \widetilde{Y}$. It may also assume the value ∞ in case $\widetilde{X}$ and $\widetilde{Y}$ do not represent isomorphic objects in the persistence ∞-level categories $\mathcal{C}^\infty_p$ for "large" p's. The pseudo-metric $\widetilde{d}_{\mathrm{int}}$ can be viewed as a more robust version of the pseudo-distance $\hat{d}$ from Remark 3.1.2.2. Indeed, as its name suggests, the limit distance can also be defined in terms of the limit of d^p_{int} as $\nu(p) \longrightarrow 0$. More specifically:

LEMMA 3.1.5.3. *Let $\{p_k\}_{k\in\mathbb{N}}$ be a sequence of elements from $\mathcal{P}$ with $\nu(p_k) \longrightarrow 0$ as $k \longrightarrow \infty$. Then for every $\widetilde{X}, \widetilde{Y} \in \widetilde{Obj}(\widehat{\mathcal{C}})$ we have*

$$\lim_{k\to\infty} d^{p_k}_{int}(X'_k, Y'_k) = \widetilde{d}_{int}(\widetilde{X}, \widetilde{Y}),$$

where X'_k, Y'_k are any sequences with $X'_k \in \widetilde{X}_{p_k}$, $Y'_k \in \widetilde{Y}_{p_k}$.

PROOF. This follows from elementary arguments, using the properties of the comparison functors listed on page 82 together with Lemma 3.1.1.2. ☐

REMARK 3.1.5.4. Let $\widehat{\mathcal{F}} : \widehat{\mathcal{C}} \longrightarrow \widehat{\mathcal{D}}$ be a functor between two systems of PC's (see §3.1.5.2). It follows directly from the definitions that $\widehat{\mathcal{F}}$ induces a well defined map $\widetilde{\mathcal{F}} : \widetilde{\mathrm{Obj}}(\widehat{\mathcal{C}}) \longrightarrow \widetilde{\mathrm{Obj}}(\widehat{\mathcal{D}})$ that satisfies $\widetilde{\mathcal{F}}([A]_{\widehat{\mathcal{C}}}) = [\mathcal{F}_p(A)]_{\widehat{\mathcal{D}}}$ for every $A \in \mathrm{Obj}(\mathcal{C}_p)$, $p \in \mathcal{P}$. Moreover, $\widetilde{\mathcal{F}}$ is non-expanding with respect to the limiting interleaving distances on $\widetilde{\mathrm{Obj}}(\widehat{\mathcal{C}})$ and $\widetilde{\mathrm{Obj}}(\widehat{\mathcal{D}})$ in the sense that $\widetilde{d}^{\widehat{\mathcal{D}}}_{\mathrm{int}}(\widetilde{\mathcal{F}}(\widetilde{X}), \widetilde{\mathcal{F}}(\widetilde{Y})) \leq \widetilde{d}^{\widehat{\mathcal{C}}}_{\mathrm{int}}(\widetilde{X}, \widetilde{Y})$ for every $\widetilde{X}, \widetilde{Y} \in \widetilde{\mathrm{Obj}}(\widehat{\mathcal{C}})$.

We now turn to K-groups. Let $\widehat{\mathcal{C}} = \{\mathcal{C}_p\}_{p\in\mathcal{P}}$ be a system of TPC's (see §3.1.5.3). Recall that the 0-persistence level $\mathcal{C}^0_p$ of each $\mathcal{C}_p$ is a triangulated category. Denote by $K_0(\mathcal{C}^0_p)$ the Grothendieck (or K-) group of $\mathcal{C}^0_p$. Let $p \preceq q$. Recall that all the comparison functors $\mathcal{H}_{p,q} : \mathcal{C}_p \longrightarrow \mathcal{C}_q$ are TPC-functors, hence they restrict to triangulated functors $\mathcal{H}^0_{p,q} : \mathcal{C}^0_p \longrightarrow \mathcal{C}^0_q$. We thus obtain an induced homomorphism

$$\mathcal{H}^K_{p,q} : K_0(\mathcal{C}^0_p) \longrightarrow K_0(\mathcal{C}^0_q). \tag{12}$$

This homomorphism turns out to be independent of the specific choice of the comparison functor $\mathcal{H}_{p,q}$. To see this, recall that for $p \preceq q$ every two comparison functors $\mathcal{H}'_{p,q}, \mathcal{H}''_{p,q} \in \mathcal{J}_{p,q}$ are 0-isomorphic, and that if two objects $X, Y \in \mathrm{Obj}(\mathcal{E})$ in a triangulated category $\mathcal{E}$ are isomorphic, then their classes in $K_0(\mathcal{E})$ are equal.

A similar argument shows that the K-group homomorphisms from (12) satisfy $\mathcal{H}^K_{q,r} \circ \mathcal{H}^K_{p,q} = \mathcal{H}^K_{p,r}$ for every $p \preceq q \preceq r$, and clearly $\mathcal{H}^K_{p,p} = \mathrm{id}$ for every p. It follows that $\{K_0(\mathcal{C}^0_p)\}_{p\in\mathcal{P}}$ together with the maps $\mathcal{H}^K_{p,q}$ form a directed system of abelian groups. We define the K_0-group of $\widehat{\mathcal{C}}$ to be the colimit of this system:

$$K_0(\widehat{\mathcal{C}}^0) := \varinjlim_{p\in\mathcal{P}} K_0(\mathcal{C}^0_p). \tag{13}$$

The assignment $\widehat{\mathcal{C}} \longmapsto K_0(\widehat{\mathcal{C}}^0)$ is functorial in the sense that TPC-functors $\widehat{\mathcal{F}} : \widehat{\mathcal{C}} \longrightarrow \widehat{\mathcal{D}}$ between systems of TPC's (see §3.1.5.2, §3.1.5.3) induce homomorphisms $\widehat{\mathcal{F}}^K : K_0(\widehat{\mathcal{C}}^0) \longrightarrow K_0(\widehat{\mathcal{D}}^0)$.

REMARK 3.1.5.5. Denote by $\Lambda^P_{\mathbb{Z}}$ the ring of Novikov polynomials (i.e. finite Novikov series) with coefficients in $\mathbb{Z}$. Its elements can be written as finite sums $\sum n_k t^{a_k}$ with $n_k \in \mathbb{Z}$, $a_k \in \mathbb{R}$

(t stands for the formal Novikov variable). As explained in [BCZ24a], the Grothendieck group $K_0(\mathcal{C}^0)$ of the 0-persistence level category $\mathcal{C}^0$ of a TPC $\mathcal{C}$ is a $\Lambda^P_{\mathbb{Z}}$-module, where the action of t^a on the class $[X]$ of an object $X \in \mathrm{Obj}(\mathcal{C}^0)$ is defined by $t^a[X] := [\Sigma^a X]$ (here Σ is the shift functor of $\mathcal{C}$). It is straightforward to check that the homomorphisms $\mathcal{H}^K_{p,q}$ are $\Lambda^P_{\mathbb{Z}}$-linear and therefore the colimit $K_0(\widehat{\mathcal{C}})$ inherits the structure of a $\Lambda^P_{\mathbb{Z}}$-module. Similarly, the homomorphisms $\widehat{\mathcal{F}}^K$ induced by functors $\widehat{\mathcal{F}}$ of TPC-systems are maps of $\Lambda^P_{\mathbb{Z}}$-modules.

Finally, we discuss persistence Hochschild invariants of systems. Let $\widehat{\mathcal{A}} = \{\mathcal{A}_p\}_{p \in \mathcal{P}}$ be a systems of A_∞-categories with increasing accuracy, and with a coherent full base of objects (see §3.1.5.7). Assume further that $\widehat{\mathcal{A}}$ is a homotopy system as defined in §3.1.5.8.

For every $p \in \mathcal{P}$ we have the persistence Hochschild homology $PHH(\mathcal{A}_p, \mathcal{A}_p)$ of $\mathcal{A}_p$ (with coefficients in the diagonal bimodule of $\mathcal{A}_p$). Let $p, q \in \mathcal{P}$ with $p \preceq q$. Each comparison functor $\mathcal{H}_{p,q} : \mathcal{A}_p \longrightarrow \mathcal{A}_q$ induces a map of persistence modules

$$\mathcal{H}^{\mathrm{PHH}}_{p,q} : PHH(\mathcal{A}_p, \mathcal{A}_p) \longrightarrow PHH(\mathcal{A}_q, \mathcal{A}_q).$$

Since every two comparison functors from $\mathcal{J}_{p,q}$ are homotopic it follows from Proposition .1.6.1 that $\mathcal{H}^{\mathrm{PHH}}_{p,q}$ is independent of the specific choice of $\mathcal{H}_{p,q}$. Moreover, for every $p \preceq q \preceq r$ we have that $\mathcal{H}^{\mathrm{PHH}}_{q,r} \circ \mathcal{H}^{\mathrm{PHH}}_{p,q} = \mathcal{H}^{\mathrm{PHH}}_{p,r}$, and $\mathcal{H}^{\mathrm{PHH}}_{p,p} = \mathrm{id}$ for every $p \in \mathcal{P}$. In other words, $\{PHH(\mathcal{A}_p, \mathcal{A}_p)\}_{p \in \mathcal{P}}$ together with the maps $\mathcal{H}^{\mathrm{PHH}}_{p,q}$, $p \preceq q$, forms a directed system of persistence modules, parametrized by $\mathcal{P}$. We define the persistence Hochschild homology of $\widehat{\mathcal{A}}$ to be the colimit of this system:

$$PHH(\widehat{\mathcal{A}}) := \varinjlim_{p \in \mathcal{P}} PHH(\mathcal{A}_p, \mathcal{A}_p). \tag{14}$$

Note that since colimits of directed systems of persistence modules are persistence modules, $PHH(\widehat{\mathcal{A}})$ is a persistence module too.

Remark 3.1.5.6. Let $\mathcal{A}$ be a strictly unital A_∞-category and denote its derived category by $D(\mathcal{A})$. There is a canonical map

$$\kappa : K_0(D(\mathcal{A})) \longrightarrow HH_*(\mathcal{A}, \mathcal{A}) \tag{15}$$

which satisfies $\kappa([X]) = [e_X]$ for every $X \in \mathrm{Obj}(\mathcal{A})$. Here $[X] \in K_0(D(\mathcal{A}))$ stands for the K_0-class of X, $e_X \in \hom_{\mathcal{A}}(X, X)$ is the unit and $[e_X] \in HH(\mathcal{A}, \mathcal{A})$ is its Hochschild homology class. The map κ is the composition of two maps, $\kappa = j^{-1} \circ \kappa'$, where $\kappa' : K_0(D(\mathcal{A})) \longrightarrow HH(\mathcal{Y}(\mathcal{A})^\Delta, \mathcal{Y}(\mathcal{A})^\Delta)$ and $j : HH(\mathcal{A}, \mathcal{A}) \longrightarrow HH(\mathcal{Y}(\mathcal{A})^\Delta, \mathcal{Y}(\mathcal{A})^\Delta)$. The map κ' is defined by $\kappa([X]) = [e_X]$ for every $X \in \mathrm{Obj}(D(\mathcal{A}))$ and its well-definedness follows from simple considerations (e.g. by using [Sei08, Proposition 3.8]). The map j is the one induced in homology by the chain level inclusion of the Hochschild complex of $\mathcal{A}$ into that of $\mathcal{Y}(\mathcal{A})^\Delta$. A version of Morita invariance for Hochschild homology states that j is an isomorphism, hence we can define $\kappa = j^{-1} \circ \kappa'$. (For Morita invariance of Hochschild homology see e.g. [Toe11] in the case when $\mathcal{A}$ is a dg-category, and the discussion in [Sei12, Section 5]; the case of A_∞-categories is treated in [She20]. See also [BC17, Section 5.4] for the significance of the map (15) in some geometric situations.)

We expect the same considerations to continue to hold also in our persistence setting. More specifically, let $\widehat{\mathcal{A}} = \{\mathcal{A}_p\}_{p \in \mathcal{P}}$ be a system of filtered A_∞-categories. We expect to have for every $p \in \mathcal{P}$ a homomorphism $\kappa_p : K_0(PD(\mathcal{A}_p)^0) \longrightarrow PHH^0(\mathcal{A}_p, \mathcal{A}_p)$, where the 0-superscript on the left stands for the 0-persistence level category of $PD(\mathcal{A}_p)$ and the one on the right denotes the 0-persistence level of the persistence module $PHH(\mathcal{A}_p, \mathcal{A}_p)$. The maps κ_p are then expected to be compatible with the respective directed systems parametrized by $p \in \mathcal{P}$ and to induce one map

$$\widehat{\kappa} : K_0(PD(\widehat{\mathcal{A}})^0) \longrightarrow PHH^0(\widehat{\mathcal{A}}).$$

3.1.5.10. *Approximability revisited.* Let $\widehat{\mathcal{C}}$ be a system of TPC's with increasing accuracy, as defined above. We will use here the equivalence relation $\sim_{\widehat{\mathcal{C}}}$ on $\mathrm{Obj}^{\mathrm{tot}}(\widehat{\mathcal{C}}) = \coprod_{p \in \mathcal{P}} \mathrm{Obj}(\mathcal{C}_p)$ as defined in §3.1.5.9 on page 88 and the notation from that section.

Definition 3.1.5.7. Let $\mathcal{L} \subset \widetilde{\mathrm{Obj}}(\widehat{\mathcal{C}})$ and let $\mathcal{F}_1, \ldots, \mathcal{F}_l \in \widetilde{\mathrm{Obj}}(\widehat{\mathcal{C}})$ be a finite collection (of equivalence classes of objects), and let $\epsilon > 0$. We say that $\mathcal{L}$ is ϵ-approximable by $\{\mathcal{F}_1, \ldots, \mathcal{F}_l\}$ if there exists $\delta > 0$ and $c_\epsilon > 0$ such that for every $p \in \mathcal{P}$ with $\nu(p) < \delta$ we have

$$d_{\mathrm{int}}\Big(\mathcal{L}_p, \mathrm{Obj}\langle\{F_1, \ldots, F_l\}_p\rangle^{\Delta}\Big) < \epsilon + c_\epsilon \nu(p).$$

Let $\mathcal{F} \subset \widetilde{\mathrm{Obj}}(\widehat{\mathcal{C}})$. We say that $\mathcal{L}$ is approximable by $\mathcal{F}$ if for every $\epsilon > 0$ there exists a finite collection $\mathcal{F}_1, \ldots, \mathcal{F}_l \in \mathcal{F}$ which ϵ-approximates $\mathcal{L}$ and the constants c_ϵ are bounded, independently of ϵ.

Recall that $\langle \mathcal{O} \rangle^{\Delta}$ stands for the smallest sub-TPC of $\mathcal{C}_p$ containing the set of objects $\mathcal{O}$. Sometimes we will use the wording $\{\mathcal{F}_1, \ldots, \mathcal{F}_l\}$ *ϵ-approximates* $\mathcal{L}$ and $\mathcal{F}$ *approximates* $\mathcal{L}$.

Similarly we also define retract approximability as follows.

Definition 3.1.5.8. Let $\widehat{\mathcal{C}}$, $\mathcal{L}$, $\mathcal{F}_1, \ldots, \mathcal{F}_l$ be as in Definition 3.1.5.7 and let $\epsilon > 0$. We say that $\mathcal{L}$ is ϵ-retract-approximable by $\{\mathcal{F}_1, \ldots, \mathcal{F}_l\}$ if there exists $\delta > 0$, $c_\epsilon > 0$, such that for every $p \in \mathcal{P}$ with $\nu(p) < \delta$ we have

$$d_{\text{r-int}}\Big(\mathcal{L}_p, \mathrm{Obj}\langle\{F_1, \ldots, F_l\}_p\rangle^{\Delta}\Big) < \epsilon + c_\epsilon \nu(p).$$

Let $\mathcal{F} \subset \widetilde{\mathrm{Obj}}(\widehat{\mathcal{C}})$. We say that $\mathcal{L}$ is retract-approximable by $\mathcal{F}$ if for every $\epsilon > 0$ there exists a finite collection $\mathcal{F}_1, \ldots, \mathcal{F}_l \in \mathcal{F}$ which ϵ-retract-approximates $\mathcal{L}$ and the constants c_ϵ are uniformly bounded.

Remark 3.1.5.9. It is useful at this point to relate our two notions of approximability, in Definitions 3.1.2.1 and 3.1.5.7. In our geometric applications, the set $\mathcal{L}$ will be a set of Lagrangian submanifolds $\mathcal{L}ag(M)$ and the category $\mathcal{C}_p$ will be a certain TPC associated with a filtered Fukaya category defined using perturbation data that is identified by the parameter p. The inclusion $\mathcal{L}ag(M) \subset \mathcal{C}_p$ is the Yoneda embedding. The set $\mathcal{L}ag(M)$ is endowed with the spectral metric d_γ (that will be recalled later) and this metric is closely related to the stabilized interleaving metric $\bar{d}_{\mathrm{int}}$ from (5) restricted to $\mathcal{L}ag(M)$. In view of this, we will see that ϵ -

approximability in the sense of Definition 3.1.5.7 implies ϵ'-approximability for all $\epsilon' > \epsilon$ in the sense of Definition 3.1.2.1. The reason for this discrepancy in the parameters ϵ, ϵ' is related to the presence of the term $c_\epsilon \nu(p)$ in Definition 3.1.5.7. Given that these properties are supposed to be satisfied for all $\epsilon, \epsilon' > 0$ and $\nu(p)$ can be taken as small as desired, this discrepancy is essentially irrelevant. However, for a fixed ϵ, including the term $c_\epsilon \nu(p)$ in the upper bound in Definition 3.1.5.7 is useful for questions having to do with minimizing the cardinality of the set $\{\mathcal{F}_i\}$ for that fixed ϵ. A similar remark applies to retract-approximability.

3.1.5.11. *Stabilizing functors.* Let $\widehat{\mathcal{C}} = \{\mathcal{C}_p\}_{p \in \mathcal{P}}$ be a system of PC's with increasing accuracy and let $\widehat{\mathcal{F}} : \widehat{\mathcal{C}} \longrightarrow \widehat{\mathcal{C}}$ be an endo-functor of such systems (see §3.1.5.2). Let $\mathcal{B} \subset \mathrm{Obj}^{\mathrm{tot}}(\widehat{\mathcal{C}})$. We say that $\widehat{\mathcal{F}}$ is $\mathcal{B}$-*stabilizing* if the following holds for every $p \in \mathcal{P}$:

(1) For every $A \in \mathcal{B}_p$ we have $(A, p) \sim_{\widehat{\mathcal{C}}} (\mathcal{F}_p(A), \widehat{\mathcal{F}}(p))$, where $\sim_{\widehat{\mathcal{C}}}$ is the equivalence relation from §3.1.5.9, page 88.
(2) For every $A', A'' \in \mathcal{B}_p$ there exists $q \in \mathcal{P}$ with $p, \widehat{\mathcal{F}}(p) \preceq q$ and two 0-isomorphisms $\sigma_{A'} : \mathcal{H}_{\widehat{\mathcal{F}}(p),q} \mathcal{F}_p A' \longrightarrow \mathcal{H}_{p,q} A'$, $\sigma_{A''} : \mathcal{H}_{\widehat{\mathcal{F}}(p),q} \mathcal{F}_p A'' \longrightarrow \mathcal{H}_{p,q} A''$ such that for every $u \in \hom_{\mathcal{C}_p}(A', A'')$ we have:
$$\sigma_{A''} \circ \mathcal{H}_{\widehat{\mathcal{F}}(p),q} \circ \mathcal{F}_p(u) = \mathcal{H}_{p,q}(u) \circ \sigma_{A'}. \tag{16}$$

REMARK 3.1.5.10. Condition (1) above is equivalent to requiring that the map $\widetilde{\mathcal{F}} : \widetilde{\mathrm{Obj}}(\widehat{\mathcal{C}}) \longrightarrow \widetilde{\mathrm{Obj}}(\widehat{\mathcal{C}})$ (see Remark 3.1.5.4) stabilizes $\mathcal{B}$ (i.e. $\widetilde{\mathcal{F}}$ sends $\mathcal{B}$ to itself and restricts to the identity map $\mathcal{B} \longrightarrow \mathcal{B}$).

Condition (2) can be rephrased as follows. By a slight abuse of notation denote for every $p \in \mathcal{P}$ by $\mathcal{B}_p \subset \mathcal{C}_p$ the full persistence subcategory with objects $\mathcal{B}_p$. We require that given $p \in \mathcal{P}$, for large enough (with respect to $\preceq$) $q \in \mathcal{P}$ the two restricted PC functors
$$\mathcal{H}_{\widehat{\mathcal{F}}(p),q} \circ \mathcal{F}_p|_{\mathcal{B}_p},\ \mathcal{H}_{p,q}|_{\mathcal{B}_p} : \mathcal{B}_p \longrightarrow \mathcal{B}_q$$
are naturally 0-isomorphic in the category of persistence functors $\mathcal{B}_p \longrightarrow \mathcal{B}_q$.

In the presence of approximating families in TPC's, stabilizing functors have a remarkable metric property.

PROPOSITION 3.1.5.11. *Let $\widehat{\mathcal{C}}$ be a system of TPC's, $\mathcal{L} \subset \widetilde{Obj}(\widehat{\mathcal{C}})$ and $\mathcal{F}_1, \ldots, \mathcal{F}_l \in \widetilde{Obj}(\widehat{\mathcal{C}})$ an ϵ-approximating family for $\mathcal{L}$, as in Definition 3.1.5.7. Let $\widehat{\Phi} : \widehat{\mathcal{C}} \longrightarrow \widehat{\mathcal{C}}$ be a TPC endo-functor of systems and assume that $\widehat{\Phi}$ is $\{\mathcal{F}_1, \ldots, \mathcal{F}_l\}$-stabilizing. Then for every $\widetilde{L} \in \mathcal{L}$ we have $\widetilde{d}_{int}(\widetilde{L}, \widetilde{\Phi}(\widetilde{L})) \leq \epsilon$.*

3.1.6. Filtered and persistence derived Fukaya categories. Let (X, ω) be a symplectic manifold of one of the following types:

(1) X is closed.
(2) (X, ω) is symplectically convex at infinity.
(3) $(X, \omega = d\lambda)$ is a Liouville manifold with a prescribed primitive λ of the symplectic structure ω, and such that X is symplectically convex at infinity with respect to these structure).

(4) $(X, \omega = d\lambda)$ is a compact Liouville domain.

In what follows we will work with several variants of Fukaya categories of X, and we will specify below in each case the class of admissible Lagrangians for this purpose. In each of these cases we will use the following convention. Unless otherwise stated, whenever X is a manifold with boundary, the closed Lagrangian submanifolds $\overline{L} \subset X$ will be implicitly assumed to lie in the *interior* of X.

3.1.6.1. *The exact case.* Here we assume that X is Liouville (with a given Liouville form λ). In order to obtain a graded theory we add the following assumption: $2c_1(X) = 0$, where $c_1(X)$ stands for the 1'st Chern class of the tangent bundle of X, viewed as a complex vector bundle by endowing X with any ω-compatible almost complex structure. We also fix a nowhere vanishing quadratic complex n-form (where $n = \dim_{\mathbb{C}} X$), namely a nowhere vanishing section Θ of the bundle $\Omega^n(X,J)^{\otimes 2}$.

Denote by $\mathcal{L}\mathrm{ag}^{(\mathrm{ex})}$ the collection of closed, graded, marked, exact Lagrangian submanifold $L = (\overline{L}, h_L, \theta_L)$. Here $\overline{L} \subset X$ is a closed λ-exact Lagrangian submanifold, which we call the *underlying Lagrangian of* L, $h_L : \overline{L} \longrightarrow \mathbb{R}$ is a primitive of $\lambda|_{\overline{L}}$ and $\theta_L : \overline{L} \longrightarrow \mathbb{R}$ is a grading of $\overline{L}$, see e.g.[**Sei00**], [**BCZ24b**, Section 3.2.2.2] for more details. Sometimes we will write $\mathcal{L}\mathrm{ag}^{(\mathrm{ex})}(X)$ or $\mathcal{L}\mathrm{ag}^{(\mathrm{ex})}_{\lambda}(X)$, instead of simply $\mathcal{L}\mathrm{ag}^{(\mathrm{ex})}$, in order to keep track of the additional data X or λ.

We can now form the Fukaya category of exact Lagrangians in X. We will follow here a variant of the standard construction from [**Sei08**] due to Ambrosioni [**Amb25**] (see also [**BCZ24b**] for a somewhat simpler implementation in a special case) which yields a filtered A_∞-category. The objects of our Fukaya category are the elements of $\mathcal{L}\mathrm{ag}^{(\mathrm{ex})}$. To define the morphisms and higher structures we need perturbation data. The space of admissible perturbation data $\mathcal{P}$ is described in detail in [**Amb25**]. Once we fix $p \in \mathcal{P}$ we can define the Fukaya category $\mathcal{F}uk(\mathcal{L}\mathrm{ag}^{(\mathrm{ex})}; p)$ as in [**Amb25**]. (Sometimes we will write $\mathcal{F}uk(X, \mathcal{L}\mathrm{ag}^{(\mathrm{ex})}; p)$ to emphasize the ambient manifold X.) For $L_0, L_1 \in \mathcal{L}\mathrm{ag}^{(\mathrm{ex})}$ we will sometime denote $\hom_{\mathcal{F}uk(\mathcal{L}\mathrm{ag}^{(\mathrm{ex})};p)}(L_0, L_1)$ by $CF(L_0, L_1; p)$ or by $\hom(L_0, L_1; p)$. We use here $\mathbb{Z}_2$ for the coefficients in these hom's. The grading on $CF(L_0, L_1; p)$ is defined using the given grading functions $\theta_{L_0}, \theta_{L_1}$ with which L_0, L_1 are endowed, respectively.

3.1.6.2. *Filtered Fukaya categories in the exact case.* Given two geometrically distinct elements $L_0, L_1 \in \mathcal{L}\mathrm{ag}^{(\mathrm{ex})}$ (i.e. $\overline{L}_0 \neq \overline{L}_1$) and $p \in \mathcal{P}$, denote by $H_p^{L_0,L_1} : [0,1] \times X \longrightarrow \mathbb{R}$ the Hamiltonian function prescribed by p for the Floer datum of (L_0, L_1). Let $x \in CF(L_0, L_1; p)$ be a generator (i.e. an $H_p^{L_0,L_1}$-Hamiltonian chord with endpoints on L_0, L_1). Its action is defined by

$$\mathcal{A}(x) := \int_0^1 H_p^{L_0,L_1}(t, x(t))dt - \int_0^1 \lambda(\dot{x}(t))dt + h_{L_1}(x(1)) - h_{L_0}(x(0)). \tag{17}$$

We use the action to filter the chain complexes $CF(L_0, L_1; p)$, $L_0, L_1 \in \mathcal{L}\mathrm{ag}^{(\mathrm{ex})}$. The grading on $CF(L_0, L_1; p)$ follows the standard recipe [**Sei00**], using the grading functions $\theta_{L_0}, \theta_{L_1}$.

In case $\overline{L}_0 = \overline{L}_1$ the Floer complex $CF(L_0, L_1; p)$ coincides (up to a shift in grading) with the Morse complex of $\overline{L} := \overline{L}_0 = \overline{L}_1$ and we set the action level $\mathcal{A}(x)$ of all the (non-zero) elements $x \in CF(L_0, L_1; p)$ to be $h_{L_1} - h_{L_0}$ (note that the latter is a constant since $\overline{L}_0 = \overline{L}_1$). The (cohomological) grading of a generator $x \in CF(L_0, L_1; p)$ is defined as $|x| = (n - \mathrm{ind}_f(x)) + \kappa$, where $f : \overline{L} \longrightarrow \mathbb{R}$ is the Morse function on $\overline{L}$ presribed by p, $\mathrm{ind}_f(x)$ is the Morse index of the critical point x, $n := \dim \overline{L}$, and $\kappa = \theta_{L_1} - \theta_{L_0}$ (which is again an integer).

The work of [**Amb25**] introduces a special class $\mathcal{P}$ of perturbation data such that for every $p \in \mathcal{P}$, the Fukaya category $\mathcal{F}uk(\mathcal{L}\mathrm{ag}^{(\mathrm{ex})}; p)$ is a genuine filtered and strictly unital A_∞-category. We endow the category $\mathcal{F}uk(\mathcal{L}\mathrm{ag}^{(\mathrm{ex})}; p)$ with translation and shift functors as follows. Let $L = (\overline{L}, h_L, \theta_L) \in \mathcal{L}\mathrm{ag}^{(\mathrm{ex})}$. We define the translation and shift functors, respectively, on objects by:

$$TL := (\overline{L}, h_L, \theta_L + 1), \quad \Sigma^r L := (\overline{L}, h_L + r, \theta_L), \ \forall r \in \mathbb{R}. \tag{18}$$

These extend to maps on $\hom_{\mathcal{F}uk(\mathcal{L}\mathrm{ag}^{(\mathrm{ex})}; p)}$'s in the obvious way and furthermore to filtered A_∞-functors $\mathcal{F}uk(\mathcal{L}\mathrm{ag}^{(\mathrm{ex})}; p) \longrightarrow \mathcal{F}uk(\mathcal{L}\mathrm{ag}^{(\mathrm{ex})}; p)$ with trivial higher order terms. With these definitions we have natural isomorphisms:

$$\begin{aligned} &CF(T^k L_0, T^l L_1; p)^i \cong CF(L_0, L_1; p)^{i-k+l}, \ \forall\, k, l \in \mathbb{Z}, \\ &CF^\alpha(\Sigma^r L_0, \Sigma^s L_1; p) \cong CF^{\alpha + r - s}(L_0, L_1; p), \ \forall\, \alpha, r, s \in \mathbb{R}. \end{aligned} \tag{19}$$

The superscripts $i, i - k + l$ in the first line stand for degrees and the superscripts $\alpha, \alpha + r - s$ in the second line are action levels. In the first line we have used cohomological grading conventions. In homological grading conventions the isomorphism from the first line takes the following form: $CF(T^k L_0, T^l L_1; p)_j \cong CF(L_0, L_1; p)_{j+k-l}, \ \forall\, k, l \in \mathbb{Z}$. In case no confusion between grading and action levels may arise we will sometimes also write $CF^i(L_0, L_1; p)$ instead of $CF(L_0, L_1; p)^i$ and similarly in homology.

Before we proceed we remark that the filtered Fukaya category $\mathcal{F}uk(\mathcal{L}\mathrm{ag}^{(\mathrm{ex})}; p)$ depends on the choice of λ in two ways. Firstly, the set of objects $\mathcal{L}\mathrm{ag}^{(\mathrm{ex})}$ depends on λ and secondly the filtrations on the hom's depend on λ. Whenever we need to emphasize the dependence on λ we will write $\mathcal{L}\mathrm{ag}^{(\mathrm{ex})}_\lambda$ for the set of objects and denote this category by $\mathcal{F}uk(\mathcal{L}\mathrm{ag}^{(\mathrm{ex})}_\lambda, \lambda; p)$. (The second λ in the notation indicates that the action is measured with respect to λ.) Note that if $\lambda' = \lambda + dG$, where $G : X \longrightarrow \mathbb{R}$ is a smooth function, compactly supported in the interior of X, then there is a canonical isomorphism of filtered A_∞-categories

$$\mathcal{F}uk(\mathcal{L}\mathrm{ag}^{(\mathrm{ex})}_\lambda, \lambda; p) \longrightarrow \mathcal{F}uk(\mathcal{L}\mathrm{ag}^{(\mathrm{ex})}_{\lambda'}, \lambda'; p). \tag{20}$$

Its action on objects is given by $(\overline{L}, h, \theta) \longmapsto (\overline{L}, h + G|_{\overline{L}}, \theta)$. Its action on morphisms is induced by the identity map and it has zero higher order terms. We will often identify all the categories $\mathcal{F}uk(\mathcal{L}\mathrm{ag}^{(\mathrm{ex})}_{\lambda'}, \lambda'; p)$ with λ' as above.

However if λ' differs form λ by a non-exact closed form then the collection $\mathcal{L}\mathrm{ag}^{(\mathrm{ex})}$ entirely changes, so in that case $\mathcal{F}uk(\mathcal{L}\mathrm{ag}^{(\mathrm{ex})}_\lambda, \lambda; p)$ and $\mathcal{F}uk(\mathcal{L}\mathrm{ag}^{(\mathrm{ex})}_{\lambda'}, \lambda'; p)$ have completely different objects and there is no general way to compare these categories.

We will discuss further important properties of the categories $\mathcal{F}uk(\mathcal{L}\text{ag}^{(\text{ex})};p)$ in §3.1.6.6 below.

3.1.6.3. *The monotone case.* In case X is closed, or symplectically convex at infinity we will consider monotone Lagrangian submanifolds. (Note that the existence of monotone Lagrangians in X is not obvious, there are global obstructions, e.g. that the ambient manifold X is itself monotone.)

To obtain a graded Floer theory we follow Seidel's approach for graded Lagrangians submanifolds [**Sei00**]. Fix $N \in \mathbb{Z}_{\geq 2}$ for which the image of $2c_1(X)$ in $H^2(X;\mathbb{Z}_N)$ vanishes (we assume that such an N does exist). Fix an N-fold Maslov covering $Gr^N_{\text{Lag}}(X,\omega) \longrightarrow Gr_{\text{Lag}}(X,\omega)$ of the Lagrangian Grassmannian bundle of $(T(X),\omega)$.

Denote by Λ the Novikov field with coefficients in $\mathbb{Z}_2$, namely:

$$\Lambda = \Big\{\sum_{k=0}^{\infty} a_k T^{\lambda_k} \mid a_k \in \mathbb{Z}_2, \lim_{k\to\infty} \lambda_k = \infty\Big\}, \tag{21}$$

and let $\Lambda_0 \subset \Lambda$ be the positive Novikov ring:

$$\Lambda_0 = \Big\{\sum_{k=0}^{\infty} a_k T^{\lambda_k} \mid a_k \in \mathbb{Z}_2, \lambda_k \geq 0, \lim_{k\to\infty} \lambda_k = \infty\Big\}. \tag{22}$$

Denote by $\mathcal{L}\text{ag}^{(\text{mon},\mathbf{d})}$ the collection of closed, graded, monotone Lagrangian submanifolds whose Maslov-2 pseudo-holomorphic disk count (through a generic point and for a generic J that tames ω; see for instance [**BCS21**] §3.5 page 66 for a definition) is $\mathbf{d} \in \Lambda_0$. We write these elements as triples $L = (\overline{L}, a_L, s_L)$, where:

- $\overline{L} \subset X$ is a monotone Lagrangian submanifold with Maslov-2 disk count $\mathbf{d}$.
- s_L is a $Gr^N_{\text{Lag}}(X,\omega)$-grading of L. More specifically, $s_L : \overline{L} \longrightarrow Gr^N_{\text{Lag}}(X,\omega)$ is a lift of the canonical section $\overline{L} \longrightarrow Gr_{\text{Lag}}(X,\omega)$. See [**Sei00**] for more details.
- $a_L \in \mathbb{R}$ is a real number that will be used for action filtration purposes in §3.1.6.4 below.

3.1.6.4. *Filtered Fukaya categories in the monotone case.* Let $L_0 = (\overline{L}_0, a_{L_0}, s_{L_0})$, $L_1 = (\overline{L}_1, a_{L_1}, s_{L_1})$ be two geometrically distinct elements in $\mathcal{L}\text{ag}^{(\text{mon},\mathbf{d})}$. Let $p \in \mathcal{P}$, and $H^{L_0,L_1}_p : [0,1]\times X \longrightarrow \mathbb{R}$ the Hamiltonian function prescribed by p for the Floer datum of (L_0,L_1). Recall that the Floer complex $CF(L_0,L_1;p)$ is a free Λ-module generated by the H^{L_0,L_1}_p-Hamiltonian chords x with endpoints on $\overline{L}_0, \overline{L}_1$.

Let $0 \neq P(T) \in \Lambda$ and x be an H^{L_0,L_1}_p-Hamiltonian chord as above. We define the action of $P(T)x$ to be:

$$\mathcal{A}\big(P(T)x\big) := -\lambda_0 + \int_0^1 H^{L_0,L_1}_p(t,x(t))dt + a_{L_1} - a_{L_0}, \tag{23}$$

where $\lambda_0 \in \mathbb{R}$ is the minimal exponent that appears in the formal power series of $P(T) \in \Lambda$, i.e. $P(T) = a_0T^{\lambda_0} + \sum_{i=1}^{\infty} a_iT^{\lambda_i}$ with $a_0 \neq 0$ and $\lambda_i > \lambda_0$ for every $i \geq 1$. We now extend $\mathcal{A}$ to

$CF(L_0, L_1; p)$ as follows. For a non-trivial element $c = P_1(T)x_1 + \cdots + P_l(T)x_l \in CF(L_0, L_1; p)$ we define

$$\mathcal{A}(c) = \max\{\mathcal{A}(P_k(T)x_k) \mid 1 \leq k \leq l\},$$

and finally we set $\mathcal{A}(0) = -\infty$. The filtration on $CF(L_0, L_1; p)$ is then defined by:

$$CF^{\alpha}(L_0, L_1; p) := \{c \in CF(L_0, L_1; p) \mid \mathcal{A}(c) < \alpha\}.$$

Standard arguments show that the Floer differential μ_1 preserves this filtration. It is important to note however, that the filtration levels $CF^{\alpha}(L_0, L_1; p)$ of $CF(L_0, L_1; p)$ are *not* Λ-modules but rather Λ_0-modules.

The grading on $CF(L_0, L_1; p)$, when $\overline{L}_0 \neq \overline{L}_1$, is defined using the N-fold maslov covering $Gr^N_{\text{Lag}}(X, \omega)$ and the gradings s_{L_0}, s_{L_1} following the recipe from [**Sei00**]. In contrast to the case described in §3.1.6.2, here we only obtain a $\mathbb{Z}_N$-grading.

As in the exact case, following [**Amb25**] there is a class of perturbation data $\mathcal{P}$ such that for every $p \in \mathcal{P}$ the Fukaya category $\mathcal{F}uk(\mathcal{L}\text{ag}^{(\text{mon},\mathbf{d})}; p)$ is a genuine filtered and strictly unital A_∞-category. The total category is Λ-linear, however when viewed as a filtered A_∞-category, $\mathcal{F}uk(\mathcal{L}\text{ag}^{(\text{mon},\mathbf{d})}; p)$ is only Λ_0-linear.

Similarly to the exact case, here too we have translation and shift functors. For the translation functor, recall from [**Sei00**] that the N-fold Maslov covering $Gr^N_{\text{Lag}}(X, \omega) \longrightarrow Gr_{\text{Lag}}(X, \omega)$ comes with a $\mathbb{Z}_N$-action $\rho : \mathbb{Z}_N \longrightarrow \text{Aut}\Big(Gr^N_{\text{Lag}}(X, \omega) \longrightarrow Gr_{\text{Lag}}(X, \omega)\Big)$. We now define for $L = (\overline{L}, a_L, s_L) \in \mathcal{L}\text{ag}^{(\text{mon},\mathbf{d})}$:

$$TL := (\overline{L}, a_L, \rho(1) \circ s_L), \quad \Sigma^r L := (\overline{L}, a_L + r, s_L), \ \forall r \in \mathbb{R}. \tag{24}$$

Both T and Σ^r extend to filtered A_∞-functors $\mathcal{F}uk(\mathcal{L}\text{ag}^{(\text{mon},\mathbf{d})}; p) \longrightarrow \mathcal{F}uk(\mathcal{L}\text{ag}^{(\text{mon},\mathbf{d})}; p)$ and the identities from (19) continue to hold (with the grading being taken modulo N).

3.1.6.5. *The weakly exact case.* Assume that X is closed, or symplectically convex at infinity. we will consider in this case also weakly exact closed Lagrangian submanifolds $\overline{L} \subset X$. By weakly exact we mean that $\langle [\omega], A \rangle = 0$ for all classes $A \in \text{image}\,(\pi_2(X, L) \longrightarrow H_2(X, L))$. A necessary condition for such Lagrangians to exist is that X is symplectically aspherical, i.e. $\langle [\omega], A \rangle = 0$ for all spherical classes $A \in \text{image}\,(\pi_2(X) \longrightarrow H_2(X))$. Also note that if (X, ω) is exact then every exact Lagrangian is automatically weakly exact, regardless of the chosen primitive λ of ω.)

Denote by $\mathcal{L}\text{ag}^{(\text{wex})}$ the collection of triples $L = (\overline{L}, a_L, \theta_L)$, where $\overline{L} \subset X$ is a closed weakly exact Lagrangian submanifold, $a_L \in \mathbb{R}$ and $\theta_L : \overline{L} \longrightarrow \mathbb{R}$ is a grading of $\overline{L}$ as in §3.1.6.1. Similarly to the monotone case, we will work here with coefficients in the Novikov ring Λ and its positive version Λ_0. The Floer complexes $CF(L_0, L_1; p)$ are defined exactly as in the monotone case §3.1.6.4. For the action filtration we follow the recipe from the monotone case, while for the grading we use the recipe described in the exact case. More specifically, the action filtration is defined by (23) and the grading is defined as in §3.1.6.2. The defintion of the translation functor is the same as in (18) and the shift functors Σ^r are defined as in (24). The identities from (19) continue to hold.

3.1.6.6. *Systems of Fukaya categories and their TPC's.* Let (X, ω) be a symplectic manifold as at the beginning of §3.1.6 and let $\mathcal{L}\mathrm{ag}$ be a subset of one of the collections $\mathcal{L}\mathrm{ag}^{(\mathrm{ex})}$, $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}$ or $\mathcal{L}\mathrm{ag}^{(\mathrm{wex})}$ (depending on the type of X). We assume that $\mathcal{L}\mathrm{ag}$ is closed under all translations and all shifts.

Let $\mathcal{P}$ be the space of perturbation data, as in §3.1.6.2 and §3.1.6.4, and denote for every $p \in \mathcal{P}$ by $\mathcal{F}uk(\mathcal{L}\mathrm{ag}; p)$ the full A_∞-subcategory (of the respective Fukaya categories $\mathcal{F}uk(\mathcal{L}\mathrm{ag}^{(\mathrm{ex})}; p)$, $\mathcal{F}uk(\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}; p)$ or $\mathcal{F}uk(\mathcal{L}\mathrm{ag}^{(\mathrm{wex})}; p)$) with the objects $\mathcal{L}\mathrm{ag}$. The categories $\mathcal{F}uk(\mathcal{L}\mathrm{ag}; p)$ inherit the structure of filtered A_∞-categories from their ambient categories.

We will further restrict the space of perturbation data $\mathcal{P}$ to satisfy the following condition. Fix a constant $B > 0$, and consider only perturbation data $p \in \mathcal{P}$ such that for every pair of geometrically distinct objects $L_0, L_1 \in \mathcal{L}\mathrm{ag}$ with (i.e. $\overline{L}_0 \neq \overline{L}_1$) we have

$$H_p^{L_0,L_1}(t,x) \leq B, \ \forall \, t \in [0,1], x \in X. \tag{25}$$

Recall from [**Amb25**] that for every $p \in \mathcal{P}$ and L_0, L_1 as above, the function $H_p^{L_0,L_1}$ is strictly positive. As we will see later, the particular choice of the constant B will not play an important role for our considerations (intuitively, it should be a small number), and its main purpose is to have a uniform bound on all the Hamiltonian functions that appear in the perturbation data. By a slight abuse of notation we will denote the space of perturbation data satisfying the additional condition (25) also by $\mathcal{P}$.

Define now $\nu : \mathcal{P} \longrightarrow \mathbb{R}_{>0}$ by:

$$\nu(p) := \sup \Big\{ H_p^{L_0,L_1}(t,x) \ \Big| \ L_0, L_1 \in \mathcal{L}\mathrm{ag} \text{ are geometrically distinct, } (t,x) \in [0,1] \times X \Big\}. \tag{26}$$

Next, define a preorder on $\mathcal{P}$. For $p, q \in \mathcal{P}$ we define $p \preceq q$ if for every two geometrically distinct $L_0, L_1 \in \mathcal{L}\mathrm{ag}$ we have

$$H_q^{L_0,L_1}(t,x) \leq H_p^{L_0,L_1}(t,x), \ \forall \, (t,x) \in [0,1] \times X.$$

Note that when endowed with $\preceq$, the set $\mathcal{P}$ becomes directed (i.e. in addition to being a preorder we have that $\forall \, p', p'' \in \mathcal{P}$, $\exists \, q \in \mathcal{P}$ with $p', p'' \preceq q$).

Let $p, q \in \mathcal{P}$ with $p \preceq q$. From [**Amb25**], there exists a distinguished class of filtered, strictly unital, continuation A_∞-functors $\mathcal{H}_{p,q} : \mathcal{F}uk(\mathcal{L}\mathrm{ag}; p) \longrightarrow \mathcal{F}uk(\mathcal{L}\mathrm{ag}; q)$. Different functors in this class are 0-homotopic Moreover, we also have a distinguished class of strictly unital continuation A_∞-functors $\mathcal{H}_{q,p} : \mathcal{F}uk(\mathcal{L}\mathrm{ag}; q) \longrightarrow \mathcal{F}uk(\mathcal{L}\mathrm{ag}; p)$ with linear deviation rate $\leq 2(\nu(p) - \nu(q))$ (see §1.6 for the precise definition). Here too, any two functors in this class are 0-homotopic (the homotopy being a homotopy of LD-functors of a given deviation rate; see §1.6.1).

By [**Amb25**] the functors $\mathcal{H}_{p,q}$ staisfy the conditions listed in §3.1.5.4, hence the family of categories $\{\mathcal{F}uk(\mathcal{L}\mathrm{ag}; p)\}_{p \in \mathcal{P}}$ together with the comparison functors $\mathcal{H}_{p,q}$, $\mathcal{H}_{q,p}$, $p \preceq q$, become a system of filtered A_∞-categories which we denote by $\widehat{\mathcal{F}uk}(\mathcal{L}\mathrm{ag})$.

The system $\widehat{\mathcal{F}uk}(\mathcal{L}\mathrm{ag})$ has in fact two additional important properties. The first is that all the categories $\mathcal{F}uk(\mathcal{L}\mathrm{ag}; p)$ have the same set of objects $\mathcal{L}\mathrm{ag}$ and the comparison functors act like the identity on this set. In the terminology of §3.1.5.7, the system $\widehat{\mathcal{F}uk}(\mathcal{L}\mathrm{ag})$ has a fixed full base of objects. The second property is that $\widehat{\mathcal{F}uk}(\mathcal{L}\mathrm{ag})$ is in fact a homotopy system, according to the definitions from §3.1.5.8. This follows immediately from the discussion above on the homotopy properties of the comparision functors.

Note however that although $\widehat{\mathcal{F}uk}(\mathcal{L}\mathrm{ag})$ has a fixed full base of objects, this does not hold anymore for the associated system of filtered modules $F\mathrm{mod}(\widehat{\mathcal{F}uk}(\mathcal{L}\mathrm{ag}))$ or the Yoneda system $\mathcal{Y}(\widehat{\mathcal{F}uk}(\mathcal{L}\mathrm{ag}))$ (in particular, they cannot even be considered as homotopy systems). This has nothing to do with Fukaya categories and the reason is purely algebraic - see Remark 3.1.5.2. Nevertheless, the fact that $\widehat{\mathcal{F}uk}(\mathcal{L}\mathrm{ag})$ is a homotopy system is still useful, as it allows to define some of the limit invariants introduced in §3.1.5.9, e.g. the persistence Hochschild homology $PHH(\widehat{\mathcal{F}uk}(\mathcal{L}\mathrm{ag}))$ of the system $\widehat{\mathcal{F}uk}(\mathcal{L}\mathrm{ag})$ as defined in (14).

Returning to $\widehat{\mathcal{F}uk}(\mathcal{L}\mathrm{ag})$, we will turn it into a genuine system of TPC's with increasing accuracy. There are two ways to do this, each leading to a slightly different system. The first way is to use the persistence derived Fukaya category using filtered modules as described in §3.1.4.2, and the second one is to use the construction from §3.1.4.3.

We begin with the first implementation. We use the parameter space $\mathcal{P}$, the size function $\nu : \mathcal{P} \longrightarrow \mathbb{R}_{>0}$ and the pre-order $\preceq$ discussed above. The categories in the system of TPC's will be the persistence derived categories $PD(\mathcal{F}uk(\mathcal{L}\mathrm{ag}; p))$, $p \in \mathcal{P}$. The comparison functors for $p \preceq q$ are defined to be

$$\mathfrak{H}_{p,q} := PD(\mathcal{H}_{p,q}) : PD(\mathcal{F}uk(\mathcal{L}\mathrm{ag}; p)) \longrightarrow PD(\mathcal{F}uk(\mathcal{L}\mathrm{ag}; q)),$$

where we use here the notation and construction from §3.1.4.2. For $p \preceq q$ we define $\mathfrak{H}_{q,p} : PD(\mathcal{F}uk(\mathcal{L}\mathrm{ag}; q)) \longrightarrow PD(\mathcal{F}uk(\mathcal{L}\mathrm{ag}; p))$ in the same way. Note that although $\mathcal{H}_{q,p}$ has linear deviation, its induced functor on the persistence derived categories is still a TPC-functor. See §3.1.4.2 for more details. We will denote this system of TPC's by $PD(\widehat{\mathcal{F}uk}(\mathcal{L}\mathrm{ag}))$.

The second implementation is very similar, only that it appeals to the construction from §3.1.4.3. The categories in the system are now $PD^c(\mathcal{F}uk(\mathcal{L}\mathrm{ag}; p))$, $p \in \mathcal{P}$, and the comparison functors are $\mathfrak{H}^c_{p,q} := PD^c(\mathcal{H}_{p,q})$ and similarly for $\mathfrak{H}^c_{q,p}$. We denote this system of TPC's by $PD^c(\widehat{\mathcal{F}uk}(\mathcal{L}\mathrm{ag}))$.

Finally, we remark that although $\widehat{\mathcal{F}uk}(\mathcal{L}\mathrm{ag})$ is a system with a fixed full base of objects (and even a homotopy system), neither $PD(\widehat{\mathcal{F}uk}(\mathcal{L}\mathrm{ag}))$ nor $PD^c(\widehat{\mathcal{F}uk}(\mathcal{L}\mathrm{ag}))$ has a coherent full base of objects.

3.1.6.7. *Functors associated with symplectomorphisms.* Let (X, ω) be a symplectic manifold of one of the types listed at the beginning of §3.1.6 and $\mathcal{L}\mathrm{ag}$ be a subset of the collection of Lagrangians in X as at the beginning of §3.1.6.6. Denote by $\mathcal{P}$ the space of admissible perturbation data parametrizing the system of filtered Fukaya categories with objects $\mathcal{L}\mathrm{ag}$.

Let $\phi : X \longrightarrow X$ be a symplectic diffeomorphism compactly supported in the interior of X. We will define below an endo-functor of each of the systems of TPC's $PD(\widehat{\mathcal{F}uk}(\mathcal{L}\mathrm{ag}))$ and $PD^c(\widehat{\mathcal{F}uk}(\mathcal{L}\mathrm{ag}))$, that is induced by ϕ.

In order to define such a functor we need several additional assumptions on ϕ. First of all we assume that ϕ preserves the collection of underlying Lagrangians $\overline{L}$ with $L \in \mathcal{L}\mathrm{ag}$. The other assumptions on ϕ depend on the collection of Lagrangians (exact, monotone or weakly exact) which $\mathcal{L}\mathrm{ag}$ is associated with.

We begin with the case of exact Lagrangians. Here we first need to keep in mind the Liouville form λ. Let $\mathcal{L}\mathrm{ag}_\lambda \subset \mathcal{L}\mathrm{ag}_\lambda^{(\mathrm{ex})}$ be a subset which is closed under translations and shifts. We will make the following additional assumptions: ϕ is assumed to be exact, namely $\phi^*\lambda = \lambda + dF$ for some smooth function $F : X \longrightarrow \mathbb{R}$ which is compactly supported inside the interior of X. We also assume that ϕ preserves the homotopy class of the quadratic complex n-form Θ (among non-vanishing forms of this type, and after indentifying $(T(X), J)$ with $(T(X), \phi^*J)$). We also assume that ϕ is graded (see [Sei00] for the definition of a graded symplectomorphism) and denote its action on graded Lagrangians by $(\overline{L}, \theta) \longmapsto (\phi(\overline{L}), \phi_*(\theta))$.

Under the above assumptions we obtain a bijective map,

$$\mathrm{Obj}(\mathcal{F}uk(\mathcal{L}\mathrm{ag}, \lambda; p)) \longrightarrow \mathrm{Obj}(\mathcal{F}uk(\mathcal{L}\mathrm{ag}, \lambda'; q)), \quad (\overline{L}, h, \theta) \longmapsto (\phi(\overline{L}), h \circ \phi^{-1}|_{\phi(\overline{L})}, \phi_*(\theta)), \tag{27}$$

where $\lambda' := (\phi^{-1})^*\lambda$. Here p, q are any two choices of perturbation data (indeed, below we will need to work with $p \neq q$).

By the assumptions on ϕ we have $\lambda' = \lambda - d(F \circ \phi^{-1})$, hence after composing the preceding map with the canonical isomorphism (20) we obtain a bijection

$$\begin{aligned} &\phi : \mathrm{Obj}(\mathcal{F}uk(\mathcal{L}\mathrm{ag}, \lambda; p)) \longrightarrow \mathrm{Obj}(\mathcal{F}uk(\mathcal{L}\mathrm{ag}, \lambda; q)), \\ &(\overline{L}, h, \theta) \longmapsto (\phi(\overline{L}), h \circ \phi^{-1}|_{\phi(\overline{L})} - F|_{\overline{L}} \circ \phi^{-1}|_{\phi(\overline{L})}, \phi_*(\theta)), \end{aligned} \tag{28}$$

which by abuse of notation we still denote by ϕ.

Next we define a map $\phi_* : \mathcal{P} \longrightarrow \mathcal{P}$ by pushing forward the structures in the perturbation datum. Specifically, let $p \in \mathcal{P}$ be a perturbation data. Recall that p assigns to each tuple of Lagrangians in $\mathcal{L}\mathrm{ag}$ some auxiliary structures. In the simplest case, when this tuple $\vec{L} = (L_0, \ldots, L_d)$ has no consecutive entries (in the cyclic sense) with the same underlying Lagrangians, the perturbation p assigns to $\vec{L}$ a tuple (K^p, J^p) with two components: the first is a family $K^p = \{K^p_S \in \Omega^1(S; C^\infty_0(X))\}_{S \in \mathcal{S}}$ of 1-forms parametrized by the space $\mathcal{S}$ of boundary-punctured disks (with any number ≥ 2 of punctures). Each member of the family K^p is a 1-form K^p_S on the punctured disk S with values in the space of compactly supported smooth functions on X (which should be viewed as Hamiltonian functions). The second component J^p is a family $\{J^p_S\}_{S \in \mathcal{S}}$ of (domain dependent) almost complex structure, or more precisely each J^p_S is itelsf an S-parametrized family $\{J^p_{S,z}\}_{z \in S}$ of ω-compatible almost complex structures on X which are also compatible with the symplectic convexity of (X, ω) at infinity (in case X is not closed). The push forward $\phi_*(p)$ of p by ϕ assigns to $\phi(\vec{L}) := (\phi(L_0), \ldots, \phi(L_d))$ the perturbation data $\phi_*(p) = (\phi_* K^p, \phi_* J^p)$, where

$$(\phi_* K^p)_S(\xi) := K^p_S(\xi) \circ \phi^{-1},\ \forall \xi \in T(S), \quad (\phi_* J^p)_{S,z} := D\phi \circ J_{S,z} \circ D\phi^{-1},\ \forall S \in \mathcal{S}, z \in S. \tag{29}$$

This completes the definition of $\phi_*(p)$ in the case when $\vec{L}$ does not contain consecutive entries whose underlying Lagrangians coincide.

When the tuple of Lagrangians does contain consecutive entries with the same underlying Lagrangians the perturbation data p are defined over the space clusters (see [**Amb25**], and also [**BCZ24b**, Section 3.3] where these are called decorated clusters of punctured disks). In essence these combine punctured disks and some trees attached to some of the boundary arcs of the punctured disks. Over each punctured disk in the cluster the perturbation data are as before (namely, they consits of a pair (K^p, J^p) as described above). Over each tree the perturbation data consists of a Morse data, which is a collection of paramter-depending Morse functions and Riemannian metrics assigned to each edge of the graph (and depending on a parameter varying along each edge). The push forward by ϕ of the Morse data (in the tree part of a cluster) is simply done by composing the Morse functions with ϕ^{-1} and similarly for the Riemannian metrics. Finally, over each punctured disk in a cluster we use the same recipe as in (29).

The space of admissible perturbation data $\mathcal{P}$ is closed under the action of ϕ, namely for every $p \in \mathcal{P}$ we have $\phi_*(p) \in \mathcal{P}$. This follows from the definition of $\mathcal{P}$ in [**Amb25**]. Moreover, it follows from (26) that for every $p \in \mathcal{P}$ we have $\nu(\phi_*(p)) = \nu(p)$ and that for every $p \preceq q$ we have $\phi_*(p) \preceq \phi_*(q)$.

The diffeomorphism ϕ also induces obvious chain isomorphisms, defined for every $L_0, L_1 \in \mathcal{L}\mathrm{ag}_\lambda$ and $p \in \mathcal{P}$:

$$\phi^{CF} : CF(L_0, L_1; \lambda; p) \longrightarrow CF(\phi(L_0), \phi(L_1); \lambda; \phi_*(p)). \tag{30}$$

Note that we are using here the same Liouville form λ both in the domain and target of ϕ^{CF}, as we have done also in (28). Since $(\phi^{-1})^*\lambda = \lambda - d(F \circ \phi^{-1})$ it follows that the chain map ϕ^{CF} preserves action filtrations.

The maps ϕ^{CF} extend to a filtered A_∞-functor (in fact an A_∞-isomorphism)

$$\phi_p^{\mathcal{F}uk} : \mathcal{F}uk(\mathcal{L}\mathrm{ag}, \lambda; p) \longrightarrow \mathcal{F}uk(\mathcal{L}\mathrm{ag}, \lambda; \phi_*(p)) \tag{31}$$

whose first order terms are ϕ^{CF} and with vanishing higher order terms. These functors fit together to a functor of systems of filtered A_∞-categories,

$$\widehat{\phi}^{\mathcal{F}uk} : \widehat{\mathcal{F}uk}(\mathcal{L}\mathrm{ag}, \lambda) \longrightarrow \widehat{\mathcal{F}uk}(\mathcal{L}\mathrm{ag}, \lambda),$$

as defined at the end of §3.1.5.4 (see also §3.1.5.2).

Passing to the persistence derived level, we obtain by §3.1.4 TPC functors:

$$PD(\phi_p^{\mathcal{F}uk}) : PD(\mathcal{F}uk(\mathcal{L}\mathrm{ag}, \lambda, p)) \longrightarrow PD(\mathcal{F}uk(\mathcal{L}\mathrm{ag}, \lambda, \phi_*(p))). \tag{32}$$

The collection of functors $PD(\phi_p^{\mathcal{F}uk})$, $p \in \mathcal{P}$, and the push forward map $\phi_* : \mathcal{P} \longrightarrow \mathcal{P}$, form together one functor of systems of TPC's, according to the definitions from §3.1.5.3:

$$PD(\widehat{\phi}^{\mathcal{F}uk}) : PD(\widehat{\mathcal{F}uk}(\mathcal{L}\mathrm{ag}, \lambda)) \longrightarrow PD(\widehat{\mathcal{F}uk}(\mathcal{L}\mathrm{ag}, \lambda)). \tag{33}$$

The compatibility with respect to the comparison functors, as required by diagrams (8) and (9), follows by standard arguments.

There is also a similar functor $PD^c(\widehat{\phi}^{\mathcal{F}uk}) : PD^c(\widehat{\mathcal{F}uk}(\mathcal{L}\mathrm{ag}, \lambda)) \longrightarrow PD^c(\widehat{\mathcal{F}uk}(\mathcal{L}\mathrm{ag}, \lambda))$, defined analogously to (33), in which we just replace PD by PD^c everywhere in (32).

The definition of $\phi_p^{\mathcal{F}uk}$, $\widehat{\phi}^{\mathcal{F}uk}$ and their persistence derived versions $PD(\phi_p^{\mathcal{F}uk})$, $PD(\widehat{\phi}^{\mathcal{F}uk})$ in the monotone and the weakly exact cases is very similar to the above. In these cases, the action on the perturbation data $p \mapsto \phi_*(p)$ is precisely the same as in the exact case described above. The action of ϕ on the grading in the weakly exact case is the same as in the exact case, while in the montone case it follows the receipe from [**Sei00**, Section 2.b]. Finally, the action of ϕ on the middle parameter $a_L \in \mathbb{R}$ of an object $L = (\overline{L}, a_L, -)$ is the identity in both the monotone and the weakly exact cases.

3.1.7. Ungraded setting. Sometimes it will be more convenient to consider Fukaya categories without any grading. This can be done in each of the settings described in §3.1.6.1 - §3.1.6.5, by simply omitting the grading component from the objects $L \in \mathcal{L}\text{ag}$ and viewing the hom's as ungraded chain complexes. In the ungraded context, whenever we deal with triangulated structures we just set the translation functor to be id. Note also that all the considerations from §3.1.6.6 carry over to the ungraded setting in a straightforward way.

3.2. Nearby Approximability

The aim of this section is to prove an approximability result, Theorem 3.2.2.1, which is formulated in §3.2.2. In the last subsection, §3.2.7, the first point of Theorem **??** is seen to easily follow from Theorem 3.2.2.1.

3.2.1. Setting. We start by providing some necessary notation and background.

Let (N^n, g) be a closed n-manifold with Riemannian metric g. We denote by D_r^*N the closed r-disk cotangent bundle of N, with respect to the metric corresponding to g (under the identification $T^*N \cong TN$, induced by g)

$$D_r^*(N) \subset T^*N \ .$$

We endow T^*N with the canonical symplectic form $\omega = d\lambda$ with λ the Liouville 1-form on T^*N.

We consider the class $\mathcal{L}\text{ag}^{(\text{ex})}(D_r^*N)$ of all closed, *exact* Lagrangian submanifolds in the interior of D_r^*N, see §3.1.6.1. For $r = 1$, we simplify the notation by omitting the subscript r. There are filtered Fukaya categories with objects the elements in $\mathcal{L}\text{ag}^{(\text{ex})}(D_r^*N)$ denoted by $\mathcal{F}uk(\mathcal{L}\text{ag}^{(\text{ex})}(D_r^*N); p)$ where $p \in \mathcal{P}$ is a choice of perturbations, as described in §3.1.6.2. The categories associated with various perturbation choices are related through comparison functors, as described in §3.1.6.6. To simplify the various constructions below we denote $\mathcal{A}_p(r) = \mathcal{F}uk(\mathcal{L}\text{ag}^{(\text{ex})}(D_r^*N); p)$. We omit p from the notation, when this parameter is fixed and no confusion is possible and in case $r = 1$ we write $\mathcal{A}_p$ instead of $\mathcal{A}_p(r)$.

The considerations below are independent of r and thus we fix $r = 1$. The Yoneda functor embeds $\mathcal{A}_p$ inside the category of filtered A_∞ modules, $F\text{mod}_{\mathcal{A}_p}$ (see 3.1.4.2),

$$\mathcal{Y}_p : \mathcal{A}_p \hookrightarrow F\text{mod}_{\mathcal{A}_p} \ .$$

For a point $x \in N$, let $\overline{F}_x$ be the cotangent fiber through x,

$$\overline{F}_x = \pi^{-1}(x)$$

where $\pi : T^*N \to N$ is the projection of the cotangent bundle. Such a fiber also admits markings and we denote by $F_x = (\overline{F}_x, h_{F_x}, \theta_{F_x})$ a marked fiber, with underlying Lagrangian submanifold $\overline{F}_x$ (h_{F_x} is a constant function in this case). Each F_x defines a filtered A_∞-module still denoted

$$F_x \in F\mathrm{mod}_{\mathcal{A}_p} \ .$$

For this it is necessary to define the perturbations required for defining μ_d operations involving tuples of closed Lagrangians and one copy of $\overline{F}_x$. Denoting marked Lagrangians and the associated Yoneda modules by the same symbols simplifies notation and it is generally non-problematic, however it is important to keep track of the precise context, namely of the underlying A_∞-category. With these conventions, we have $F_x(L) = CF(L, F_x; p)$ as filtered chain complexes.

We denote by $\mathcal{A}_p^F$ the filtered full sub-category of $F\mathrm{mod}_{\mathcal{A}_r}$ that contains the image of $\mathcal{Y}_p$, and all the modules F_x, $x \in N$ (with all possible markings on the fibers $\overline{F}_x$). As before, in case we want to render explicit both the perturbation parameter p and the radius of the disk bundle D_r^*N we write $\mathcal{A}_p^F(r)$.

The next step is to complete each category $\mathcal{A}_p^F$ with respect to mapping cones over morphisms with filtration level ≤ 0 and quasi-isomorphisms of filtration level ≤ 0 and denote the result by $[\mathcal{A}_p^F]^\Delta$. Finally, we let $\mathcal{C}_p$ be the associated TPC, $\mathcal{C}_p = PH([\mathcal{A}_p^F]^\Delta)$. The categories $\mathcal{A}_p^F$ fit into a system of filtered categories for varying p, as in §3.1.5.4. Thus we have functors $\mathcal{H}_{p,q} : \mathcal{A}_p^F \to \mathcal{A}_q^F$ defined using the push-forward construction of filtered modules. More specifically, recall that for $p \preceq q$ we have a continuation functor $\mathcal{H}_{p,q}^{\mathrm{cont}} : \mathcal{A}_p \longrightarrow \mathcal{A}_q$. The functor $\mathcal{H}_{p,q}$ is defined as $\mathcal{H}_{p,q} := (\mathcal{H}_{p,q}^{\mathrm{cont}})_*$, the induced push-forward of modules. The functors $\mathcal{H}_{p,q}^{\mathrm{cont}}$ preserve filtration whenever $p \preceq q$. We also have functors $\mathcal{H}_{q,p}^{\mathrm{cont}}$, however these are A_∞-functors with a possibly non-vanishing linear deviation (see §3.1.5.4).

Summing up, we obtain comparison functors, still denoted $\mathcal{H}_{p,q} : \mathcal{C}_p \to \mathcal{C}_q$. The categories $[\mathcal{A}_p^F]^\Delta$ are pre-triangulated and, as in §3.1.6.6, as a result the system $\widehat{\mathcal{C}}(D^*N) = (\mathcal{C}_p, \mathcal{H}_{p,q})$ forms a system of TPCs with increasing accuracy in the sense of §3.1.5.

3.2.2. The main statement, and the idea of the proof. Using the system $\widehat{\mathcal{C}}(D^*(N))$ of TPCs we can now formulate the main result of the section.

To proceed, recall the equivalence relation in §3.1.5.9 defined on

$$\mathrm{Obj}^{\mathrm{tot}}(\widehat{\mathcal{C}}(D^*N)) := \coprod_{p \in \mathcal{P}} \mathrm{Obj}(\mathcal{C}_p) \ .$$

The set of associated equivalence classes is denoted by $\widetilde{\mathrm{Obj}}(\widehat{\mathcal{C}}(D^*N))$. All exact, marked, Lagrangians $L \in \mathcal{L}ag^{(ex)}(D^*N)$ correspond to well-defined equivalence classes in $\widetilde{\mathrm{Obj}}(\widehat{\mathcal{C}}(D^*N))$ because $\mathcal{H}_{p,q}$ takes Yoneda modules to modules 0-equivalent to Yoneda modules whenever $p \preceq q$. We denote by $\mathcal{L}(D^*N) \subset \widetilde{\mathrm{Obj}}(\widehat{\mathcal{C}}(D^*N))$ the set of equivalence classes of all such, closed, exact, marked L's. We formulate a similar property for a family of marked fibers $\mathcal{F} = \{F_{x_1}, \ldots, F_{x_i}, \ldots\}$ of D^*N. We say that the family $\mathcal{F}$ satisfies property $(*)$ if:

$(*)$ *for all $p \preceq q$, for every marked fiber $F_x \in \mathcal{F}$ we have $\mathcal{H}_{p,q}(F_x) \cong_0 F_x$, where the F_x on the right-hand side of $\cong_0$ stands for the module corresponding to F_x in Obj$(\mathcal{C}_q)$.*

If this property is satisfied by the family $\mathcal{F}$, then each marked fiber $F_x \in \mathcal{F}$ corresponds to a well-defined equivalence class $\mathfrak{F}_x \in \widetilde{\mathrm{Obj}}(\widehat{\mathcal{C}}(D^*N))$.

THEOREM 3.2.2.1. *Fix $\epsilon > 0$. It is possible to construct the system $\widehat{\mathcal{C}}(D^*(N))$ such that there is a finite family of marked fibers $\mathcal{F}_\epsilon = \{F_{x_1}, \ldots, F_{x_l}\}$ that satisfies property $(*)$ and, moreover, the set $\mathcal{L}(D^*N)$ is ϵ-approximable in the sense of Definition 3.1.5.7 by $\{\mathfrak{F}_{x_1}, \ldots, \mathfrak{F}_{x_l}\}$.*

In other words, for some $\beta = \beta(\epsilon) > 0$ we have that for each $L \in \mathcal{L}ag^{(ex)}(D^*N)$ and each p with $\nu(p) \leq \beta$ there is an iterated cone C inside $\mathcal{C}_p$ constructed by using the marked fibers $(\overline{F}_{x_i}, \alpha_{x_i})$, for some constants $\alpha_{x_i} \in \mathbb{R}$ (and some choice of grading), such that C is at interleaving distance at most ϵ from L in $\mathcal{C}_p$. Of course, we identify here a Lagrangian L with its corresponding Yoneda module. Moreover, fitting the categories $\mathcal{C}_p$ in the system $\widehat{\mathcal{C}}(D^*N)$ renders the relevant construction independent of the choice of p, as long as $\nu(p)$ is small enough.

We will see in §3.2.7 that this result immediately implies approximability in the sense of Definition 3.1.2.1 and thus point i of Theorem **??** is established.

REMARK 3.2.2.2. The dependence of the statement on ϵ is subtle. It is possible to construct the system $\widehat{\mathcal{C}}(D^*N)$ such that each category $\mathcal{C}_p$ is independent of ϵ (for $\nu(p)$ small) and all the disk bundle fibers, and in particular all the elements of $\mathcal{F}_\epsilon$, for all ϵ, are represented as modules in $\mathcal{C}_p$. However, the system $\widehat{\mathcal{C}}(D^*N)$ also depends on the functors $\mathcal{H}_{p,q}$ and these functors need to be so that they satisfy property $(*)$ relative to the family of fibers $\mathcal{F}_\epsilon$. In our argument, this property follows from a geometric construction that is the central part of the proof and that requires a fixed ϵ as starting point. As a result, when ϵ changes, apriori, the system $\widehat{\mathcal{C}}(D^*N)$ changes too, along with the family $\mathcal{F}_\epsilon$ whose number of elements increases when ϵ tends to 0. See also Remark 3.2.7.5.

3.2.2.1. *Outline of the proof of Theorem 3.2.2.1.* The proof proceeds roughly as follows. We first establish the existence of some special Morse functions that have small variation and large gradient away from their critical points. More precisely, denote by $B_\eta(x)$ the disk in N (with respect to the metric g) of radius η and center at $x \in N$. For any values $K, \delta > 0$ and sufficiently small $\eta > 0$ there exists a Morse function $f : N \to [0, K]$ such that $||df_x|| \geq \delta$ for all points $x \in N \backslash (\cup_{z \in Crit(f)} B_\eta(z))$. We then consider a Lefschetz fibration structure on T^*N, $p : T^*N \to \mathbb{C}$, as constructed by Giroux [**Gir25**], such that the real part of the projection p coincides with a Morse function f with the properties above, with $1 > \delta > 0$ fixed and K small. We then use a method introduced in [**BC17**] to surger each exact Lagrangian with a certain number of (compactifications) of the fibers through $Crit(f)$ and use a Hamiltonian isotopy to push the result away from D^*N. This part of the construction takes place in the total space E of a modified Lefschetz fibration that is constructed along the approach in [**BC17**]. Algebraically, this process expresses the Lagrangian as an iterated cone of fibers and an s-acyclic term, where s is at most equal to the Hofer energy needed to generate the previous Hamiltonian displacement. The key point is to control this energy. This is where the fact that K is small (and δ is relatively big) comes into play: through some precise estimates that appear in Giroux's construction we show that this energy can be related to the area of a certain planar region corresponding to the projection on $\mathbb{C}$ of D^*N, and, as a result, it is bounded uniformly, independently of L, by a constant times K.

The steps required for the proof of Theorem 3.2.2.1 are each contained in a subsection below, as follows. In §3.2.3 we outline the construction of the special Morse functions with small variation but big differential away from the critical points. This is a soft result, but not completely trivial. In §3.2.4 we review the main ingredients in Giroux's construction. Next, in §3.2.5 we recall the approach from [**BC17**] and adjust it to Giroux's set-up. Most of the proof - in §3.2.3, §3.2.4, §3.2.5 and §3.2.6 up to §3.2.6.3 - is entirely geometric and serves to establish the statement in Proposition 3.2.6.3 which, essentially, claims that:

Given $\epsilon > 0$ there exists a Lefschetz fibration $p : E \to \mathbb{C}$ that contains D^*N and such that there are exact Lagrangian spheres $\hat{S}_1, \ldots, \hat{S}_l$ in E, each of them intersecting D^*N along a (different) fiber of D^*N, with the property that, for any exact, closed Lagrangian $L \subset D^*N$, the iterated Dehn twist $\tau_{\hat{S}_l}(\tau_{\hat{S}_{l-1}}(\ldots \tau_{\hat{S}_1}(L))..)$ can be disjoined from D^*N by a Hamiltonian diffeomorphism of Hofer norm at most ϵ.

The last part of the proof of Theorem 3.2.2.1, in §3.2.6.3, translates this geometric result in an appropriate system $\widehat{\mathcal{C}}(D^*N)$.

3.2.3. Special Morse functions.

PROPOSITION 3.2.3.1. *Let (N^n, g) be a smooth, closed riemannian manifold . For any $K > 0$, $1 > \delta > 0$ and $\eta > 0$ there exists a Morse function $\varphi = \varphi_{N,K,\delta} : N \to [0, K]$ with the property that $||d\varphi(x)|| \geq \delta$ for all $x \in N \backslash (\cup_{z \in Crit(\varphi_{N,K,\delta})} B_{\eta_z}(z))$, where $0 < \eta_z \leq \eta$, $\forall z \in Crit(\varphi_{N,K,\delta})$.*

Notice that we assume here that the radius η_z of the ball $B_{\eta_z}(z)$ around the critical points z of f depends on the critical point. We also assume that all these balls B_{η_z} are pairwise disjoint.

PROOF. To start the proof we fix a Morse function $f_0 : N \to \mathbb{R}$ as well as some small quantity $\alpha > 0$. We assume that the function f_0 and the metric g are standard inside $B_\alpha(z)$ in the neighbourhoods $B_\alpha(z)$ for each $z \in Crit(f_0)$. These two conditions are not essential for the argument, they just simplify some of the calculations in the proof. With these assumptions, the gradient (calculated with respect to $\mathtt{g}$) ∇f_0 is standard, of the form $k(\ldots, \pm x_i \partial/\partial x_i, \ldots)$ with respect to coordinates $(x_1, \ldots, x_i, \ldots, x_n)$ inside each $B_\alpha(z)$; k is a constant that depends on z. We also assume that f_0 has m critical points $z_0, \ldots, z_{m-1}$ and $f_0(z_j) = jB$ for some positive constant B. After possibly multiplying f_0 with a large positive constant we may assume that $||df_0|| \geq \delta$ outside the union of all the balls $B_\alpha(z)$. We denote by $V = (m-1)B$ the maximal value of f_0. We let $U_z(\mu) = f_0^{-1}([f_0(z) - \mu, f_0(z) + \mu])$ and $U'_z(\mu) = f_0^{-1}([f_0(z) - \mu/2, f_0(z) + \mu/2])$ and assume that $B_\alpha(z)$ is contained in $U'_z(\mu)$. We fix $\mu > 0$ so that all these sets $U_z(\mu)$ are pairwise disjoint and to simplify notation below we denote $U_z = U_z(\mu)$, $U'_z = U'_z(\mu)$.

We will modify f_0 with two aims: diminish the variation V and decrease the size of the neighbourhoods $B_z(\alpha)$, in both cases as much as needed. Of course, the price to pay is that we will add many new additional critical points to those of f_0. The argument uses induction on the dimension of N. Thus, we assume the Proposition established for all manifolds of dimension at most $n-1$ (it is an easy exercise to prove the result in dimension 1).

The modifications of the function f_0 are of two types.

The first is a change inside a small neighbourhood of each critical point z of f_0. We will later refer to this step as "shrinking". To simplify notation we fix such a critical point and its critical value $v = f(z)$. We consider $\eta < \alpha/2$ and a ball $B_\eta(z)$ as well as the larger ball $B_{2\eta}$. We then consider a diffeomorphism $\psi_z : U'_z \to U_z$ with the following properties:

- $\psi_z(f_0^{-1}(v \pm \mu/2)) = f_0^{-1}(v \pm \mu)$.
- ψ_z preserves $f^{-1}(v)$.
- ψ_z maps $B_{2\eta}(z)$ to $B_\alpha(z)$ while keeping z fixed and on $B_\eta(z)$ is acts by $x \to rx$ where $r > 1$ is given by $r^2 = \frac{\delta}{\eta k}$ (this obviously assumes η small enough, see also the next point).
- we assume η small enough such that that $2r\eta = 2\sqrt{\frac{\delta\eta}{k}} < \alpha$.
- ψ_z transports $f_0^{-1}(v - \mu/2)$ along the flow of $-\nabla f_0$, mapping it to $f_0^{-1}(v - \mu)$, and it transports $f_0^{-1}(v + \mu/2)$ along the flow of $+\nabla f_0$, sending it to $f_0^{-1}(v + \mu)$.

It is easy to construct such a diffeomorphism, see Figure 1.

We now define a new function f_1 as follows: on the set U'_z it is given by $f_0 \circ \psi_z$; on the set $f_0^{-1}(-\infty, v - \mu/2]$ it is equal to $f_0 - \mu/2$; on the set $f_0^{-1}[v + \mu/2, \infty)$ it is equal to $f_0 + \mu/2$. Notice that this function is continuous, of total variation $V + \mu$, but not smooth along $f_0^{-1}(v \pm \mu/2)$. However, it can be smoothed along these two hypersurfaces, and by this smoothing - still denoted f_1 - does not add any new critical points and $||df_1(x)|| \geq \delta$ for all points x outside $B_{2\eta}(z)$.

The total variation of the perturbed function f_1 is smaller than $V + 2\mu$, by taking the smoothing sufficiently small. We now discuss the role of two of the properties of ψ_z. The role of the third property is to ensure that $||df_1(x)|| \leq \delta$ for $x \in B_\eta(z)$ (this property will be important for us later in the proof of the proposition). This is easily seen as $||\nabla f_0(x)|| = k||x||$ inside

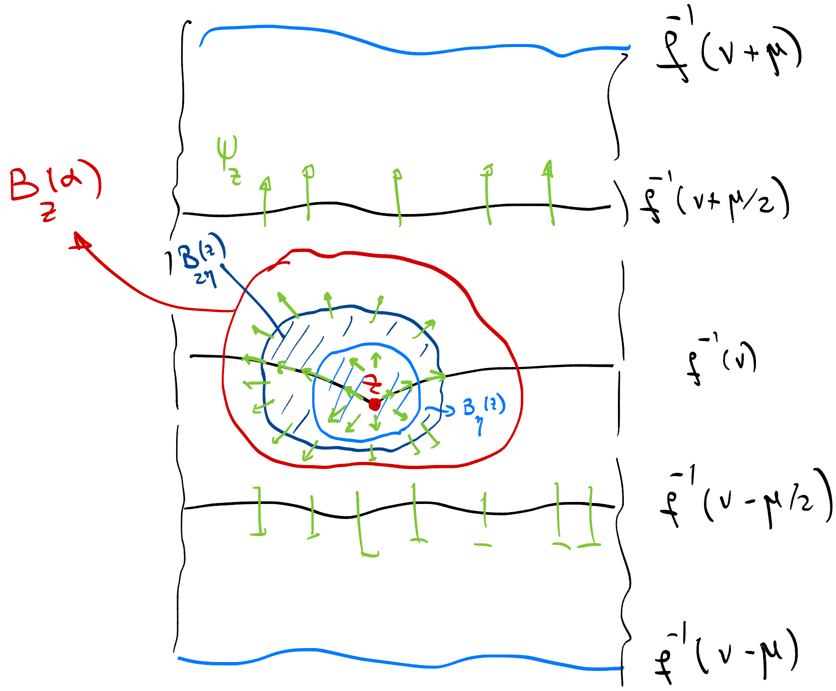


FIGURE 1. The diffeomorphism ψ_z.

$B_\alpha(z)$ and thus $||\nabla f_1(x)|| = rk||rx|| = r^2k||x|| = \delta\frac{||x||}{\eta}$ for $x \in B_\eta(z)$. The role of the fourth property is to allow for enough "room" so that $||df_1(x)|| \geq \delta$ for the points $x \in B_{2\eta}(z)\backslash B_\eta(z)$.

We apply this procedure for each critical point of f_0 and we call the resulting function still f_1. Its variation is $< V + 2m\mu$. It is useful to add an appropriate constant to the function f_1 so that its minimum is 0. With this convention, we notice that the critical values of the function f_1 are (as close as desired, by taking the smoothings small enough) to, in order, 0, $B+\mu/2$, $2B+3\mu/2$ and so forth. In particular, these critical values are separated by $\approx B+\mu$. To fix ideas we will assume also $\mu << B$.

We denote $C = V + 2m\mu$ and notice that this constant is independent of η. Therefore, if further arguments require us to reduce η, this does not affect this upper bound for the variation of f_1, nor for the difference between successive critical values which remains $\approx B + \mu$. We denote by V_1 the maximal value of f_1. To summarize, the function f_1 is Morse and satisfies:

$$||df_1(x)|| \geq \delta,\ \forall x \notin \cup_{z\in \mathrm{Crit}(f_1)} B_\eta(z) \ \text{ and } \ \min(f_1) = 0, \ \ \max(f_1) = V_1 < C\ . \tag{34}$$

Moreover, as explained before we also have:

$$||df_1(x)|| \leq \delta \ \ \forall x \in \cup_{z\in \mathrm{Crit}(f_1)} B_\eta(z)\ , B \leq f(z_{i+1}) - f(z_i) \leq B + 3\mu/2\ . \tag{35}$$

An important consequence of property (35) is that the variation of f_1 over each of the balls $B_\eta(z)$ is bounded from above by $2\eta\delta$, and thus it can be reduced as needed by making η small.

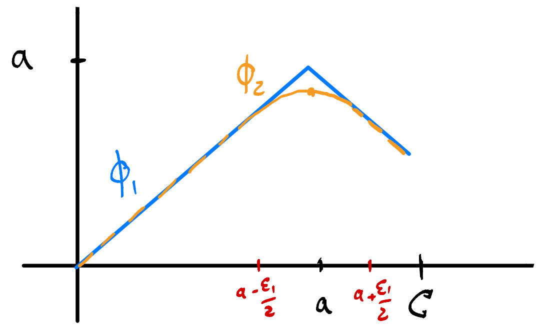


FIGURE 2. The graphs of the functions ϕ_1 and ϕ_2.

The second step is a "folding" procedure around regular hypersurfaces of the form $f_1^{-1}(a)$. This procedure produces a function of small variation, but adds many new additional critical points.

We first explain the prototype of this folding construction, and its impact on the variation of the function. The description of folding is as follows. Pick a value $a \in [C/2, C)$. To describe the folding we assume that a is a regular value of f_1 and that $W_1 = f_1^{-1}(a)$ does not intersect any of the balls $B_\eta(z)$.

We consider the following function $\phi_1 : [0, C] \to [0, a]$ defined by

$$\phi_1(t) = t \ \text{ when } t \leq a \ \ , \ \phi_1(t) = a - t \ \text{ when } t \geq a \ .$$

Clearly, this function is continuous but not smooth. Nonetheless, we define $f_2' = \phi_1 \circ f_1$. We notice that the variation of f_2' is a and that outside the balls $B_\eta(z)$ we still have $||df_2'|| \geq \delta$ except along W_1, where f_2' is not differentiable.

Our aim is to perturb the function f_2' to obtain a Morse function f_2 such that, for a fixed $\xi > 0$ we have:

$$||df_2(x)|| \geq \delta, \ \forall x \notin \cup_{z \in \mathrm{Crit}(f_2)} B_{\eta_z}(z) \ , \ \min(f_2) = 0, \ \max(f_2) = V_2 < \xi + a \ . \tag{36}$$

First, for $\epsilon > 0$ we let $K_{\epsilon,a} = f_1^{-1}([a-\epsilon, a+\epsilon])$. We pick $\epsilon_1 > 0$ small enough such that $K_{\epsilon_1,a} \cap B_\eta(z) = \emptyset$ for all critical points z of f_1. The function f_2 has the following form:

$$f_2 = \phi_2 \circ f_1 + (\beta \circ f_1) \cdot f_{W_1} \ . \tag{37}$$

Here $\phi_2 : [0, C] \to [0, a]$ is smooth, it coincides with ϕ_1 outside the interval $(a - \epsilon_1/2, a + \epsilon_1/2)$, ϕ_2 is Morse with a unique maximum at a, and $\phi_2 \leq \phi_1$ - see Figure 2.

The function β is a smooth bump function $\beta : \mathbb{R} \to [0, 1]$, which vanishes outside $(a - \epsilon_1, a + \epsilon_1)$ is increasing on $(a - \epsilon_1, a)$, decreasing on $(a, a + \epsilon_1)$ and $\beta(a) = 1$. The function $f_{W_1} : W_1 \to \mathbb{R}$ is obtained through the inductive hypothesis, by applying the Proposition to W_1. More precisely, we fix on W_1 the metric induced from g and construct the function f_{W_1} such that it is Morse, with minimal value equal to 0, and with maximal value $C_1 < \xi$, and with $||df_{W_1}(z)|| \geq 2\delta$, outside the union of balls $\hat{B}_{\eta_z}(z)$, $\eta_z \leq \eta$, for each critical point z of

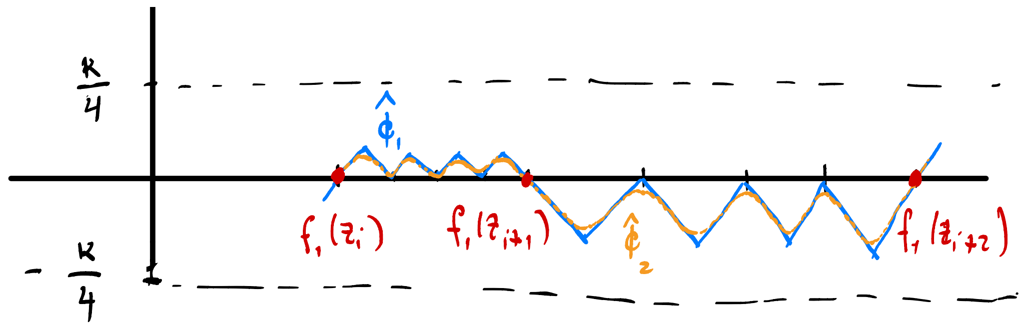


FIGURE 3. The graphs of the functions $\hat{\phi}_1$ and $\hat{\phi}_2$.

f_{W_1}. The notation $\hat{B}_-(-)$ indicates that the respective balls are in W_1. It is easy to see that by adjusting conveniently the profiles of the functions ϕ_2 and β we may ensure the desired properties for f_2. Notice that the critical points of f_2 are of two types, critical points of f_1, and critical points of f_{W_1}, with their Morse index raised by 1 when viewed as critical points of f_2.

We will now apply this folding construction for not just one hypersurface such as W_1 before but for more hypersurfaces at the same time. First, recall the constant K in the statement of the proposition and fix a natural number l such that $\frac{B+3\mu/2}{2^l} < K/4$.

We consider a function f_0 as in the shrinking procedure and apply that construction for $\eta < \frac{B}{2^{l+1}\delta}$. We denote by f_1 the resulting function.

Consider the interval $J_i = [f_1(z_i), f_1(z_{i+1})]$, $0 \leq i < m-1$, and divide it into 2^l equal pieces denoted $I_{k,i} = (f_1(z_i) + \frac{kS_i}{2^l}, f_1(z_i) + \frac{(k+1)S_i}{2^l})$ where $S_i = f_1(z_{i+1}) - f_1(z_i)$. Consider a continuous function: $\varphi_i : J_i \to [0, \infty)$ with the following properties:

- φ_i is smooth on each of the intervals $I_{k,i}$ for $k = 0, 1, \ldots 2^l - 1$, with constant slope equal to $+1$ on $I_{k,i}$ when k is even and with slope equal to -1 when k is odd.
- We let $a_{k,i} = \frac{kS_i}{2^l}$, $0 < k \leq 2^l - 1$. The minimal value of φ_i is 0 and is attained at the points $a_{k,i}$ with k even. The maximal value of φ_i is $\frac{S_i}{2^l}$ and is attained at the points $a_{k,i}$ with k odd.

Given that $S_i \geq B$ we have $S_i/2^l \geq B/2^l > 2\eta\delta$. This implies by (35) that all the values $a_{k,i}$ are regular values for f_1 and that the hypersurfaces $W_{k,i} = f^{-1}(a_{k,i})$ do not intersect $B_\eta(z_i)$ and $B_\eta(z_{i+1})$. We piece together the functions φ_i to a new continuous function $\hat{\phi}_1$ defined as follows: $\hat{\phi}_1$ is equal to φ_i on each interval J_i for i even and is equal to $-\varphi_i$ on the intervals J_i for i odd. This means that the function $\hat{\phi}_1$ is smooth at each value $f(z_i)$, with slope alternating between $+1$ and -1, depending on the parity of i. Moreover, the hypersurfaces $W_{k,i}$ do not intersect any of the balls $B_\eta(z)$, $z \in Crit(f_1)$. Finally, the variation of $\hat{\phi}_1$ is at most $2 \max \frac{S_i}{2^l} \leq 2\frac{B+3\mu/2}{2^l} < K/2$, see Figure 3.

Next, we consider a Morse smoothing $\hat{f}_2$ of $\hat{\phi}_1 \circ f_1$ that coincides with $\hat{\phi}_1 \circ f_1$ except on the sets $K_{\epsilon_2, a_{k,i}} = f_1^{-1}([a_{k,i} - \epsilon_2, a_{k,i} + \epsilon_2])$. More precisely, $\hat{f}_2$ is given by a formula similar to (37) with some straightforward modifications: the place of ϕ_2 is taken by a Morse smoothing $\hat{\phi}_2$ of $\hat{\phi}_1$ with maxima at $a_{k,i}$ for ki odd, and minima at $a_{k,i}$ for ki even; on each $K_{\epsilon_2, a_{k,i}}$ the place of the function f_{W_1} is taken by a corresponding function $f_{W_{a_{k,i}}} : W_{a_{k,i}} \to \mathbb{R}$ of variation smaller than $\frac{S_i}{2^{l+1}}$; the bump function $\beta : \mathbb{R} \to [-1, 1]$ is supported in the union of the intervals $(a_{k,i} - \epsilon_2, a_{k,i} + \epsilon_2)$, it is monotone on each of the intervals $(a_{k,i} - \epsilon_2, a_{k,i})$ and $(a_{k,i}, a_{k,i} + \epsilon_2)$, it attains its maximum equal to 1 at the points $a_{k,i}$ for ki odd and its minimum, equal to -1, at the points $a_{k,i}$ for ki even $(0 < k \leq 2^l - 1,\ 0 \leq i < m - 1)$.

As a result of all these choices the function $\hat{f}_2$ has a variation bounded from above by K and the function $\varphi_{N,K,\delta}$ in the statement is defined by adding a constant to $\hat{f}_2$ such that the resulting function, denoted $\varphi_{N,K,\delta}$, is positive, with minimal value equal to 0 and with a maximum value smaller than K. This function satisfies the other properties claimed in the statement. □

REMARK 3.2.3.2. In the arguments appearing later in the paper it is useful to assume, in addition to the properties in Proposition 3.2.3.1, that the norm $||d\varphi_{N,K,\delta}|| \leq 1$ over all of N. This is not so easy to achieve while keeping the metric g fixed. However, it will be enough for us that there is a metric $\bar{\mathsf{g}}$ on N, conformal to g, with $\bar{\mathsf{g}} = \alpha \mathsf{g}$, with $\alpha : N \to [1, \infty)$ such that the rest of the properties claimed in the proposition remain true, relative to $\bar{\mathsf{g}}$, and in addition $||d\varphi_{N,K,\delta}||_{\bar{\mathsf{g}}} \leq 1$. This is very easy to see, by adjusting the metric g, outside the neighbourhoods $B_{\eta_z}(z)$, at those points $x \in N$ where the norm $||d\varphi_{N,K,\delta}(x)||$ (in the metric g) is greater than 1.

3.2.4. Review of Giroux's construction. We will review here the parts of [Gir25] that will be relevant later on in the proof of Theorem 3.2.2.1. The type of Lefschetz fibration considered (Definition 1 in [Gir25]) is a Liouville manifold $(W, d\lambda)$ endowed with a smooth map

$$h \colon W \to \mathbb{C}$$

such that h has the following properties:

1. The singularities of h are of the form
$$h(z) = h(0) + \sum_j z_j^2$$
in coordinates $(z_1, \ldots, z_n)$ centered at the singularity and in which $\omega = d\lambda$ is a positive $(1,1)$ form at 0.
2. The distribution $\ker dh \subset TW$ consists of symplectic subspaces and the singular connection provided by its symplectic orthogonal complement is complete.
3. The manifold W is exhausted by Liouville domains $(W_k, \lambda|_{W_k})$ such that for every $w \in \mathbb{C}$ and for all sufficiently large $k \geq k_w$, the fiber $F_w = h^{-1}(w)$ intersects ∂W_k transversely along a positive contact submanifold of ∂W_k, and the Liouville field on F_w dual to $\lambda|_{F_w}$ is complete.

The main result in Giroux's paper is the following.

THEOREM 3.2.4.1. *[Giroux [Gir25]] Let N be a closed manifold, $\varphi : N \to \mathbb{R}$ a Morse function and ν an adapted gradient of φ satisfying the Morse-Smale condition. Then φ extends to a Lefschetz fibration (in the sense above) $h = f + ig \colon T^*N \to \mathbb{C}$ whose imaginary part is the function:*

$$g \colon T^*N \to \mathbb{R}\ ,\ \ (q,p) \mapsto\ \ g(q,p) := \langle p, \nu(q)\rangle$$

and whose real part f coincides with φ on the 0-section and is homogenous of degree 1 in the variable p near infinity.

A vector field ν is called adapted to φ if $d\varphi(\nu) > 0$ away from the critical points of φ and near each critical point a there exists a local coordinate chart that expresses φ in Morse form, $\varphi(x) = \varphi(a) + \sum_j \epsilon_j x_j^2$, and ν in linear form, $\nu(x) = 2\sum_j \epsilon_j x_j \partial_{x_j}$, $\epsilon_j \in \{-1, +1\}$.

In our case, it is convenient to assume that ν is an actual gradient vector field of φ with respect to a riemannian metric g on M,

$$\nu = grad_{\mathsf{g}}(\varphi)\ . \tag{38}$$

We will also assume that each critical point of φ is unique on its critical level.

A few properties of Giroux's construction are important in our proof and we will review them now.

i. The function $f : T^*N \to \mathbb{R}$ has the form $f = f_0 + f_1$ with:

$$f_0(q,p) = \varphi(q) - \frac{1}{2}\nabla^2\varphi(q)(p,p)\ . \tag{39}$$

Here $\nabla^2\varphi(q)$ is the covariant second derivative of φ at q viewed as bilinear map on T^*N. The function $f_1 : T^*N \to \mathbb{R}$ is supported away from the 0 section $N \subset T^*N$. The precise construction of this perturbative term f_1 requires much of the effort in [**Gir25**]. Our notation is somewhat different from Giroux's and, to facilitate the correspondence between the two papers, we indicate the differences here - our function f_0 is denoted f^0 in [**Gir25**] and the function f_1 here has the form:

$$f_1 = \tau_1 f^1 - (1 - \tau_0) f^0$$

where the notation on the right side of the equality is that in [**Gir25**]. Namely, τ_i are functions only depending on the radial distance r from the 0-section such that τ_1 vanishes for small values of r and equals a large positive constant C for r large enough, while τ_0 equals 1 for small r and vanishes away from the 0-section. The function f^1 is of the form $f^1 = rf^\infty$ and f^∞ is a well chosen function defined on the sphere cotangent bundle $f^\infty : ST^*N \to \mathbb{R}$.

ii. The only critical points of h are along the 0 section and they coincide with the critical points of φ.

iii. For each critical point $a \in Crit(\varphi)$, the vertical thimble along the vertical straight half-line originating at $h(a)$ and pointing upwards coincides with the unstable manifold of the Hamiltonian vector field X^f at a.

iv. The Poisson bracket $\{f, g\}$ is positive at each point $x \in T^*(N)\backslash Crit(\varphi)$

Our arguments require a quantitative complement to property iv which we state next:

v. Assume that $d\varphi(\nu) \geq \delta$ outside $\cup_{a\in Crit(\varphi)} B_\epsilon(a)$, for some small ϵ such that the balls $B_\epsilon(a)$, $B_\epsilon(b) \subset N$ are disjoint for all $a \neq b \in Crit(\varphi)$. With this assumption, the construction of f can be made such that

$$\{f, g\}(x) \geq \delta, \ \forall x \in T^*N\backslash \cup_{a\in Crit(\varphi)} B'_\epsilon(a)$$

where we denote by $B'_\epsilon(-)$ the respective balls of radius ϵ in T^*N.

A reformulation of property v will play an important role in our argument.

Corollary 3.2.4.2. *Under the assumptions in property* v *above, we have that $dh(X^f)$ is purely imaginary and $Im(dh(X^f(x))) \geq \delta$ for each $x \in T^*N$ outside any of the balls $B'_\epsilon(a)$, $a \in Crit(h)$.*

Properties i, ii, iii, and iv appear explicitly in [**Gir25**]. Property v follows easily from the proof of property iv as given in §E [**Gir25**]. To give a few more details, here are our conventions: $\omega(Y, X^f) = df(Y)$, $\{f, g\} = \omega(X^f, X^g) = dg(X^f) = -df(X^g)$. The arguments in [**Gir25**] make use of the vector field $\tilde{\nu} = -X^g$ that is easily seen to be a lift of ν to T^*N. It is shown that the quantity $df(\tilde{\nu}) = \{f, g\}$ is positive at all points different from the critical points of h, which is the claim at iv. The proof of this positivity depends on the construction of the perturbative term f_1 and appeals to a choice of the constant $C > 0$ that has appeared at point i above. When this constant is taken sufficiently big, $df(\tilde{\nu})$ is seen to be non-negative and to only vanish at the critical points of h. By possibly taking C even bigger, the same argument shows that property v is also true.

Starting from property v the statement in the corollary is immediate. Indeed, $\{f, g\} = dg(X^f) = d(p_2 \circ h)(X^f) = dp_2 \circ (dh(X^f)) = dh(X^f)$ where $p_2 : \mathbb{C} \to \mathbb{R}$ is the projection on the second coordinate. We have used the fact that for each point x, $dh(X^f(x))$ is a vertical vector in $\mathbb{C}$. This is true because $X^f(x)$ is tangent to the level hypersurface $f^{-1}(c) = (p_1 \circ h)^{-1}(c) = h^{-1}(\{(c, y) : \ y \in \mathbb{R}\})$ with $c = f(x)$ and $p_1 : \mathbb{C} \to \mathbb{R}$ the projection on the first coordinate.

3.2.5. Disjunction through Dehn twists. In this section we consider Giroux's Lefschetz fibration from [**Gir25**], as recalled in §3.2.4, and apply to it the construction in [**BC17**], in particular the arguments in Section 4.4 in that paper. The methods in [**BC17**] as reflected in Proposition 2.3.1 (see also §2.3 [**BC16**] for more details on the relevant construction), together with the doubling of singularities in §4.4.2 [**BC17**] (see Figure 17 there) applied to the Lefschetz fibration $h \colon T^*N \to \mathbb{C}$ from §3.2.4 lead to a new Lefschetz fibration $\bar{h} : E \to \mathbb{C}$ which is schematically represented in Figure 4. Here are the main relevant properties of $\bar{h}$. We assume the setting in Theorem 3.2.4.1. In particular, we have the Morse function $\varphi : N \to \mathbb{R}$

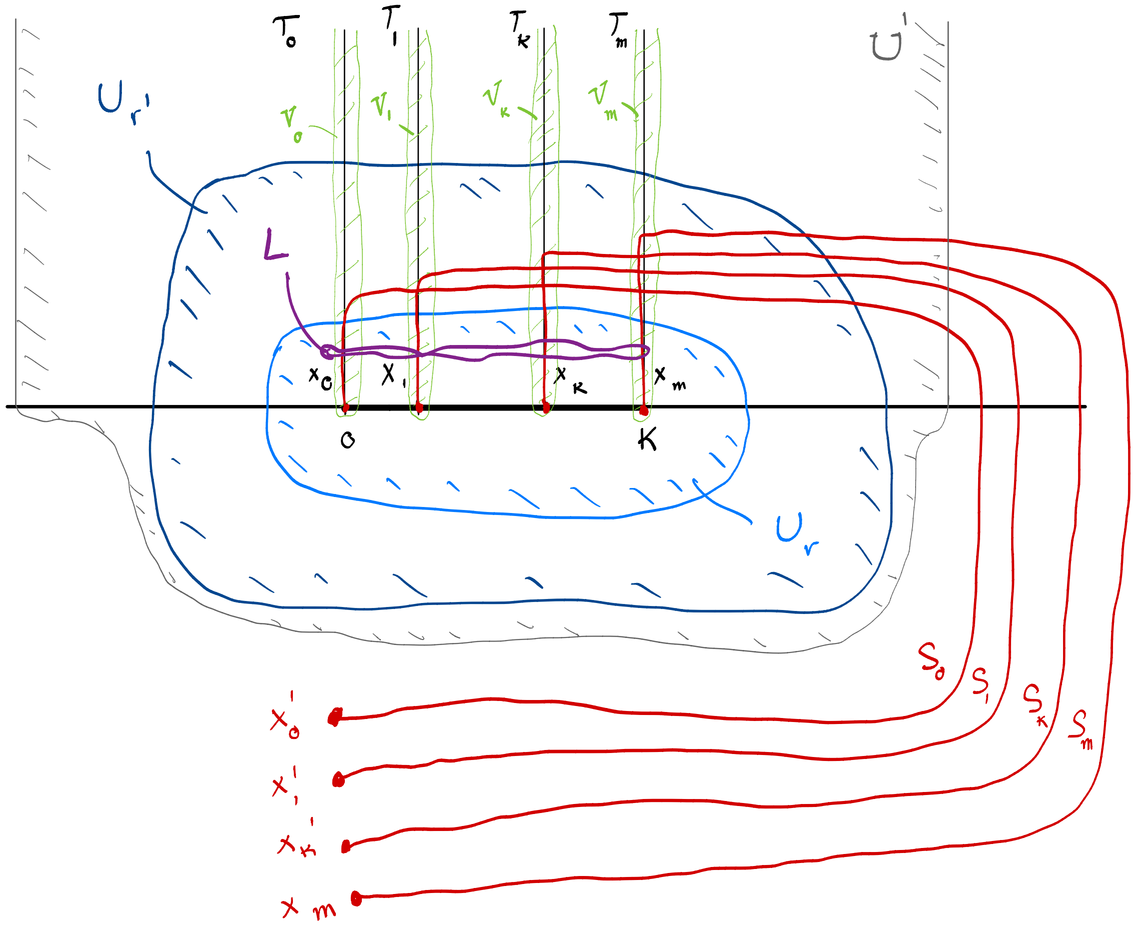


FIGURE 4. The fibration $\bar{h} : E \to \mathbb{C}$.

with a unique critical point on each critical level and that is such that the pair (φ, g) is Morse-Smale where g is a fixed Riemannian metric on N and the properties i.-v. from §3.2.4 are satisfied. Furthermore, we have:

a. The Morse function $\varphi : N \to \mathbb{R}$ has values in $[0, K]$, having 0 as minimal value and K for maximal value.
b. We put $U_r = h(D_r^* N)$ where the disk cotangent bundle $D_r^* N$ is defined relative to the metric g, and similarly $U_{r'} = h(D_{r'}^* N)$ - both these sets are represented in Figure 4.
c. Fix a constant $R > 0$ big enough such that $U_{r'} \subset [-R, R] \times \mathbb{R}$. There is a neighbourhood $U' \subset \mathbb{C}$ of $U_{r'} \cup ([-R, R] \times [0, \infty))$ such that h and $\bar{h}$ agree on $h^{-1}(U')$.
d. The fibration $\bar{h}$ has one additional critical point x'_i for each critical point x_i of h. For each pair x_i, x'_i of such critical points, x_i and x'_i are related through a matching cycle $S_i \subset E$ (which is a Lagrangian sphere) whose image onto $\mathbb{C}$ is like in Figure 4.
e. The vertical thimbles pointing up and originating at the critical points x_i are denoted by τ_i. They are unstable manifolds of the Hamiltonian vector field $X^{\bar{f}}$ at each critical point x_i with $\bar{f} = Re(\bar{h})$ (recall from iii. that $\bar{f} = f$ over U').

f. Fix some $\delta > 0$ and assume that $h = f + ig$ satisfies $\{f, g\}(x) \geq \delta$ at all points x outside small balls $B'_\epsilon(x_i)$, $x_i \in Crit(\varphi)$, as at point v. in §3.2.4. Under this assumption we fix small neighbourhoods $V_i \subset E$ for each of the thimbles τ_i such that $V_i \supset B'_\epsilon(x_i)$. The projection of the V_i's onto $\mathbb{C}$ is as in the picture.

In our arguments the constants $\delta > 0$, $K > 0$ are fixed in advance with $\delta < 1$ and the function $\varphi = \varphi_{N,K,\delta}$ is given by Proposition 3.2.3.1. The key property that will be used further below in the proof is provided by Corollary 3.2.4.2:

$$d\bar{h}(X^{\bar{f}}(x)) \in i[\delta, +\infty) \subset i\mathbb{R} \subset \mathbb{C} \quad , \forall\; x \in \bar{h}^{-1}(U')\backslash(\cup_i V_i)\ . \tag{40}$$

We will also assume that, after possibly a conformal change of metric, as in Remark 3.2.3.2, passing from the arbitrary metric g, to a conformally equivalent metric $\bar{\mathsf{g}} = \alpha\mathsf{g}$ with $\alpha : N \to [1, \infty)$, we have $||d\varphi|| \leq 1$. We will use this metric $\bar{\mathsf{g}}$ throughout from this point on and discuss this modification further below in §3.2.6.4. This means that $||\nu|| \leq 1$ - see (38) - and further implies by the description of g in Theorem 3.2.4.1 that:

$$U_r \subset \mathbb{R} \times [-r, r], \quad \forall r \in \mathbb{R} \tag{41}$$

The description of f_0 from the point i, in §3.2.4 implies that, for r sufficiently small, we have

$$U_r \subset [-C_\varphi r, K + C_\varphi r] \times [-r, r] \tag{42}$$

where C_φ is a positive constant depending on the C^2 norm of φ.

In this context, the main result that we need from [**BC17**] is next.

PROPOSITION 3.2.5.1 ([**BC17**]). *For any Lagrangian $L \subset D^*_r N$, and any arbitrarily small neighbourhoods N_i of the matching cycles S_i, there exist models of the Dehn twists τ_{S_i} that are supported in N_i and such that $\tau_{S_m} \circ \ldots \circ \tau_{S_2} \circ \tau_{S_1} \circ \tau_{S_0}(L)$ does not intersect any of the thimbles τ_i.*

3.2.6. Proof of Theorem 3.2.2.1. The proof of the theorem will be completed in four steps. In §3.2.6.1 we start from the setting in §3.2.5 and use (40), (42) and Proposition 3.2.5.1 to show that, if r is sufficiently small, the iterated Dehn twist

$$\tau_{m,\ldots,1,0}L = \tau_{S_m} \circ \ldots \circ \tau_{S_2} \circ \tau_{S_1} \circ \tau_{S_0}(L)$$

can be disjoined from $D^*_r(N)$ through a Hamiltonian isotopy of energy at most $8Kr/\delta$. In §3.2.6.2 we reformulate the disjunction result in Lemma 3.2.6.1 in terms of Lagrangian spheres that restrict to fibers $F_{x_i} \subset D^*_r(N)$. In §3.2.6.3 we establish an approximability type result in Proposition 3.2.6.4 that still involves the constants K, r, δ. Finally, the proof concludes in §3.2.6.4 where we get rid of the restriction of r being sufficiently small, showing that the conformal change of metric g does not affect the argument and that the constants K, δ in Proposition 3.2.6.4 can be picked as needed to deduce the statement of Theorem 3.2.2.1.

3.2.6.1. *Disjunction with controlled energy.* The aim in this subsection is to prove the following Lemma. To state it we fix the constants $0 < \delta < 1$, $K > 0$ and the function $\varphi = \varphi_{N,K,\delta}$ given by Proposition 3.2.3.1, just as discussed in §3.2.5. We also assume that points a. - f. there are satisfied as well as the inclusions in (40) and (42).

LEMMA 3.2.6.1. *For r sufficiently small, for any (marked) exact Lagrangian $L \subset D_r^*N$ the iterated Dehn twist $\tau_{m,\ldots,1,0}L$ can be disjoined from $D_{2r}^*(N)$ inside E with less than $8Kr/\delta$ energy.*

PROOF. To simplify notation denote $L' = \tau_{m,\ldots,1,0}L$. Recall that the constant C_φ from (42) only depends on φ. We start by choosing $r > 0$ such that $rC_\phi < K/4$. We also pick the constant r' such that $r' = 2r$. Therefore, we deduce from (42) that

$$U_r \subset [-K/4, 5K/4] \times [-r, r]$$

and, similarly

$$U_{2r} \subset [-K/2, 3K/2] \times [-2r, 2r] \ .$$

Assume for a moment that

$$L' \subset [\bar{h}^{-1}(U') \backslash \ (\cup_i V_i)] \cup (\cup_i N_i') \tag{43}$$

where $N_i' \supset N_i$ are neighbourhoods of the matching cycles S_i, possibly slightly larger than N_i. In that case, inspecting Figure 4 and using (40), we see that the Hamiltonian flow $-X^{\bar{f}}$ moves L' outside of $W_{K,r} = \bar{h}^{-1}([-K/2, 3K/2] \times [-2r, \infty))$ in time less than $4r/\delta$. At the same time the variation of $\bar{f}$ over $W_{K,r}$ is at most $2K$. It follows that the Hofer displacement energy of L' from $W_{K,r}$, and, in particular, from $D_{2r}^*(N) \subset W_{K,r}$ is less than $8Kr/\delta$.

We now intend to show that assumption (43) is satisfied by some Lagrangian $L'' = \psi(L')$ for some Hamiltonian diffeomorphism ψ that will be described next. To this purpose, we notice that the closures of the neighbourhoods V_i of the thimbles τ_i are included in the interior of $W_{K,r}$. Around each thimble τ_i we can find a Hamiltonian isotopy ψ_i supported in a slightly larger neighbourhood $V_i' \subset W_{K,r}$ that is "repelling" away from the thimble τ_i and such that ψ_i leaves the thimble τ_i invariant (as a set) and moves the intersection $L' \cap V_i$ outside of V_i , without creating any new intersections with V_i. For different i's these Hamiltonian isotopies have disjoint support and thus they can be incorporated in a single isotopy whose time-one map, ψ, has the property that $L'' = \psi(L')$ satisfies (43).

We next apply to L'' the estimate for the displacement Hofer energy relative to $W_{K,r}$. Thus, there exists a Hamiltonian isotopy ϕ of energy less than $8Kr/\delta$ such that $\phi(L'') \cap W_{K,r} = \emptyset$. As the support of ψ is inside $W_{K,r}$ we deduce that $\psi^{-1}\phi\psi(L') \cap W_{K,r} = \emptyset$ and, by the bi-invariance of the Hofer norm, we also have $||\psi^{-1}\phi\psi||_H = ||\phi||_H < 8Kr/\delta$ which concludes the proof of the lemma.

□

REMARK 3.2.6.2. Notice that, as in Proposition 3.2.5.1, the support of the Dehn twists can be assumed to be contained in neighbourhoods of the $\hat{S}_i$ that are as small as desired.

3.2.6.2. *Replacing matching cycles by Lagrangian spheres that restrict to fibers.* In this subsection we rewrite the disjunction energy estimate from Lemma 3.2.6.1 in terms of Dehn twists where the matching cycles S_i are replaced by perturbations $\hat{S}_i$ such that the intersection of $\hat{S}_i$ with $D_r^*(N)$ coincides with the cotangent fiber F_{x_i}.

We assume that r is sufficiently small such that for each fiber $\overline{F}_{x_i} \subset D_r^*(N)$ (we recall $\{x_i\}_i = Crit(\varphi)$) the image $\bar{h}(\overline{F}_{x_i})$ is included in V_i, see (39), and recall the expression of $g = Im(h)$ from Theorem 3.2.4.1. In particular, all these images are disjoint. We then consider a Hamiltonian diffeomorphism η that is supported in $D^*_{\frac{3}{2}r}(N)$ and that deforms $S_i \cap D_r^*(N)$ to F_{x_i}. We will assume that the "bends" of S_i in Figure 4 take place outside of $D^*_{\frac{3}{2}r}(N)$. We denote $\hat{S}_i = \eta(S_i)$. We may also assume that η is constructed such that $\hat{S}_i \cap D_r^*(N) = \overline{F}_{x_i}$ and $\eta(D_r^*(N)) = D_r^*(N)$. Constructing such an η is a simple exercise by comparing the local expressions of the fiber x_i and that of the thimble τ_i. Obviously, the submanifolds $\hat{S}_i$ are also Lagrangian spheres and we pick models for the Dehn twists relative to $\hat{S}_i$ such that $\tau_{\hat{S}_i} = \eta \circ \tau_{S_i} \circ \eta^{-1}$. This means, in particular, that for any Lagrangian $L \subset D_r^*(N)$ we can write:

$$\tau_{\hat{S}_i}(L) = \eta(\tau_{S_i}(\eta^{-1}(L))) \ .$$

By iterating this relation we deduce that the iterated Dehn twist

$$\hat{\tau}_{m,\ldots,1,0}(L) = \tau_{\hat{S}_m} \circ \ldots \circ \tau_{\hat{S}_1} \circ \tau_{\hat{S}_0}(L)$$

satisfies

$$\hat{\tau}_{m,\ldots,1,0}(L) = \eta(\tau_{m,\ldots,1,0}(\eta^{-1}(L)) \ .$$

We know from the proof of Lemma 3.2.6.1 that there exists a Hamiltonian diffeomorphism ϕ' that, for any Lagrangian L'' (in our class), displaces $\tau_{m,\ldots,1,0}(L'')$ from $W_{K,r}$ and is of energy less than $8Kr/\delta$. For a fixed Lagrangian L we apply this property to $L'' = \eta^{-1}(L)$ and, using the fact that the support of η is included in $W_{K,r}$, we deduce that $\eta \circ \phi' \circ \eta^{-1}$ displaces $\hat{\tau}_{m,\ldots,1,0}(L)$ from $W_{K,r}$. The Hofer norm of $\eta \circ \phi' \circ \eta^{-1}$ equals that of ϕ' and is thus smaller than $8Kr/\delta$. In summary:

PROPOSITION 3.2.6.3. *With the notation above, and for r sufficiently small, the conclusion of Lemma 3.2.6.1 also applies to the iterated Dehn twist $\hat{\tau}_{m,\ldots,1,0}(L)$ with each Dehn twist $\tau_{\hat{S}_i}$ having support in a neighbourhood of $\hat{S}_i$ that can be assumed as small as desired.*

3.2.6.3. *Translating geometry into algebra.* In this section, we interpret the geometric disjunction result in Proposition 3.2.6.3 in homological terms in a Fukaya TPC system $\widehat{\mathcal{C}}(D_r^*(N))$. The purpose here is to translate homologically the statement in Proposition 3.2.6.3. Two systems of related Fukaya categories will be important for us here, with components associated with the perturbation choice $p \in \mathcal{P}$ as below:

$$\mathcal{F}uk(\mathcal{L}ag^{(ex)}(D_r^*N), p) \hookrightarrow \mathcal{F}uk(\mathcal{L}ag^{(ex)}E, p) \ . \tag{44}$$

The category on the left is the one appearing in §3.2.1 and the one on the right is constructed just as in §3.1.6.2. Its objects are marked exact, closed Lagrangian submanifolds in E, the total space E of the Lefschetz fibration from §3.2.5. An important point has to do with the choice of perturbations p: they are first picked on $D^*_r N$ and then extended to E. This is why there is an inclusion of A_∞-categories relating the two sides. These categories fit into systems of filtered A_∞-categories as in §3.1.5.4. Further, as in §3.1.6.6, we obtain the system of TPCs $\widehat{PD}(\mathcal{F}uk(\mathcal{L}ag^{(ex)}(E)))$. We also have the system of TPCs from §3.2.1, $\widehat{\mathcal{C}}(D^*_r N) = (\mathcal{C}_p(r), \mathcal{H}_{p,q})$. Notice that we include in the notation the radius r of the relevant disk bundle. The categories $\mathcal{C}_p(r)$ are homotopy categories of filtered modules over the filtered Fukaya category $\mathcal{A}_p(r) = \mathcal{F}uk(\mathcal{L}\mathrm{ag}^{(\mathrm{ex})}(D^*_r N); p)$ and we have an inclusion of TPCs:

$$J_{p,r} : \mathcal{C}_p(r) \hookrightarrow H(F\mathrm{mod}_{\mathcal{A}_p(r)}) \ .$$

Because the perturbations on E extend the perturbations on $D^*_r N$, the inclusion (44) induces pull-back TPC functors:

$$\xi_p : PD(\mathcal{F}uk(\mathcal{L}ag^{(ex)}(E), p)) \to H(F\mathrm{mod}_{\mathcal{A}_p(r)}) \ .$$

These functors commute with the comparison functors $\mathcal{H}_{p,q}$ if we choose the perturbations used to define the comparison $\mathcal{H}_{p,q}$ functors for $D^*_r N$ by restriction of the perturbations used to define the corresponding functors for E. Notice also that, because the Lagrangian spheres $\hat{S}_i$ from §3.2.6.2 intersect $D^*_r N$ along the fibers $\overline{F}_{x_i}$ it follows that the module F_{x_i} corresponding to the fiber $\overline{F}_{x_i}$ (see §3.2.1) is the pull-back of the Yoneda module $\mathcal{Y}(\hat{S}_i)$. In particular, this module belongs to the image of $J_{p,r}$. In E, the comparison functors $\mathcal{H}_{p,q}$ preserve the (Yoneda) modules associated with $\hat{S}_i$ for $p \preceq q$ and thus we deduce that, with these choices of perturbations, the family $\mathcal{F} = \{F_{x_i}\}_i$ satisfies property $(*)$ from the statement of Theorem 3.2.2.1. The next result brings us very close to the statement of this theorem .

Proposition 3.2.6.4. *Assuming the setting and notation above, for r sufficiently small and for a choice of perturbations p with $\nu(p)$ sufficiently small, any exact, marked Lagrangian $L \subset D^*_r(N)$ satisfies in $\mathcal{C}_p(r)$:*

$$d_{int}\left(L, Obj\ \langle \{F_{x_0}, \dots, F_{x_m}\} \rangle^{\Delta}\right) \leq 48Kr/\delta \ .$$

Proof. To simplify notation denote in this proof $c = 8Kr/\delta$, and

$$\mathcal{D}_p(E) := PD(\mathcal{F}uk(\mathcal{L}ag^{(ex)}(E), p)) \ . \tag{45}$$

We first notice that to prove the claim it is enough to show:

Lemma 3.2.6.5. *There are strict exact, weighted triangles in $\mathcal{D}_p(E)$ of the form:*

$$\Delta_i \ : \ Z_i \longrightarrow X_i \longrightarrow X_{i+1} \ , \ 0 \leq i \leq m-1 \tag{46}$$

such that X_0 is the Yoneda module of a Lagrangian disjoint from $D^*_r N$, $X_m = L$, and with the Z_i of the form $\Sigma^{p(i)} \hat{S}_{x_{q(i)}}$ for some integers $p(i), q(i)$, or possibly $Z_i = 0$, and of total weight not more than $3c$.

Indeed, in that case, by the construction in Lemma 2.87 in [**BCZ24b**] we obtain that there is a similar sequence of triangles

$$\Delta_i' \ : \ Z_i' \longrightarrow X_i' \longrightarrow X_{i+1}'$$

with Z_j' again of the form $\Sigma^{l(j)}\hat{S}_{x_{s(j)}}$ or $= 0$, with each Δ_i' exact in $(\mathcal{D}_p(E))^0$, with, $X_0' = X_0$, and with the last term X_m' carrying a $6c$-isomorphism $X_m' \to L$. This means by Lemma 2.85 [**BCZ24b**] that $d_{int}^{\mathcal{D}_p(E)}(L, X_m') \leq 6c$. We now pull-back the triangles Δ_i' to $H(F\mathrm{mod}_{\mathcal{A}_p(r)})$. This produces triangles:

$$\Delta_i'' \ : \ Z_i'' \longrightarrow X_i'' \longrightarrow X_{i+1}''$$

that are exact in $(H(F\mathrm{mod}_{\mathcal{A}_p(r)}))^0$, with $X_0'' = \xi_p(X_0') = 0$, with Z_j'' of the form $\Sigma^{l(j)}F_{x_{s(j)}}$ (or $= 0$) and with $d_{\mathrm{int}}(X_m'', L) \leq 6c$. Given that $\mathcal{C}_p(r)$ contains the triangular completion of all the fiber modules F_{x_i} we deduce that $X_i'' \in \mathrm{Obj}(\mathcal{C}_p(r))$ for all i and this ends the proof of the Proposition 3.2.6.4 up to showing the Lemma 3.2.6.5. □

REMARK 3.2.6.6. It is useful to emphasize that the pull-back ξ_p is not necessarily full and faithful. In particular, $\xi_p : HF(\hat{S}_{x_i}, \hat{S}_{x_j}) \to \hom_{\mathcal{C}_p(r)}(F_{x_i}, F_{x_j})$ is not necessarily either injective nor surjective.

PROOF OF LEMMA 3.2.6.5. To construct the desired sequence of strict exact triangles we first fix the notation $L_{m+1} = L$, $L_{m-i} = \tau_{\hat{S}_i}(L_{m-i+1})$ so that $L_0 = \hat{\tau}_{m,\ldots,1,0}(L)$. The statement of Proposition 3.2.6.3 shows that L_0 can be disjoined from $D_r^*(N)$ by a Hamiltonian isotopy Φ of Hofer norm $||\Phi||_H < c = \frac{8Kr}{\delta}$. By taking p such that $\nu(p) \ll c - ||\Phi||_H$ we deduce, in TPC terminology, that there is a weighted strict exact triangle in $\mathcal{D}_p(E)$:

$$0 \longrightarrow L_0' \longrightarrow L_0 \longrightarrow 0 \tag{47}$$

of weight $2c$, with L_0' the Yoneda module of a Lagrangian $L_0' \subset E$ disjoint from $D_r^*(N)$. This will be taken as the first triangle in our sequence (in other words we take $X_0 = L_0'$).

To construct the next triangles, we consider the homological interpretation of the Dehn twist, as in Seidel's work [**Sei08**], with the exception that we want to express the output in a persistence context. Namely, we claim that in $\mathcal{D}_p(E)$ we have weighted strict exact triangles

$$\hat{S}_{x_{m-i}} \otimes HF(\hat{S}_{x_{m-i}}, L_i) \longrightarrow L_{i-1} \longrightarrow L_i \ . \tag{48}$$

Here the Floer homology group $HF(\hat{S}_{x_{m-i}}, L_i)$ is a persistence module and the tensor product is a tensor product in the persistence realm. Moreover, to avoid complicating the notation, we identify the Lagrangians L_i with their associated filtered Yoneda modules.

The triangle (48) is a persistence refinement of Seidel's classical Dehn-twist exact triangle. This refinement is a strict exact triangle in the terminology of TPC's, of a weight denoted by ϵ_i (see [**BCZ24b**]) and, additionally, ϵ_i depends on the size of the support of the Dehn twist $\tau_{\hat{S}_i}$, and can thus be made arbitrarily small. This feature of (48) can be seen most easily through the cobordism approach to the Dehn twist due to Mak-Wu [**MW18**] which, combined with the Lagrangian cobordism results in [**BCS21**], provides an upper bound for the weight of the triangle in terms of the shadow of the cobordism constructed in [**MW18**]. This shadow can be made as close to 0 as needed by diminishing the neighbourhood of $\hat{S}_i$ that contains the support of $\tau_{\hat{S}_i}$.

To proceed, we pick the Dehn twists $\tau_{\hat{S}_i}$ with sufficiently small support such that the weights ϵ_i of the triangles (48) sum up to less than c. We now notice that the persistence module $HF(\hat{S}_{x_{m-i}}, L_i)$ can be viewed as the homology of a direct sum S of (translates of) elementary filtered complexes $E_2(a,b) = \mathbf{k}(a, b : da = 0, db = a)$ and $E_1(c) = \mathbf{k}(c : dc = 0)$ with the generators x in both cases filtered with values $v(x) \in \mathbb{R}$, and $v(b) > v(a)$. The generators have degrees 0 for a and c, and 1 for b. The first type of filtered module can be written as a cone

$$E_2(a,b) = Cone\,(E_1(b)[-1] \to E_1(a))$$

over the obvious isomorphism, $b \to a$. Making use of this decomposition of $HF(\hat{S}_{x_{m-i}}, L_i)$, the triangle (48) can be refined to a sequence of exact triangles in $(\mathcal{D}_p(E))^0$ of the form

$$\Delta_{i,j} \quad : \quad \hat{S}_{x_i} \otimes E_1(d_j) \to L_{i,j} \to L_{i,j+1} \tag{49}$$

for $0 \le j \le m(i)$ where $L_{i,0} = L_i$ and d_j is one of the generators of type a, b, c that appear in the sum S, followed by a last strict exact triangle of weight ϵ_i of the form

$$0 \longrightarrow L_{i,m(i)} \longrightarrow L_{i+1}\ .$$

This refinement is an immediate consequence of the octahedral axiom.

Notice that $\hat{S}_{x_i} \otimes E_1(x)$ is isomorphic to $\Sigma^{v(x)}\hat{S}_{x_i}$. Therefore, by splicing together all these exact triangles, starting with (47) and following in order with $\Delta_{0,j}$, $0 \le j \le m(0)$ followed by $\Delta_{1,j}$, $0 \le j \le m(1)$ and so forth, we obtain a sequence of exact triangles of the form required and of total weight not more than $3c$ which concludes the proof of the Lemma. □

3.2.6.4. *Conclusion of the proof.* Theorem 3.2.2.1 is formulated in terms of the unit disk bundle D^*N with respect to some metric g but Proposition 3.2.6.4 is established for a disk bundle $D^*_r N$ with r small and with respect to a metric $\bar{\mathsf{g}}$ that is conformally equivalent to g, as at the end of §3.2.5.

To show Theorem 3.2.2.1 it suffices to show that for *any* r, and *any* metric g and any $L \subset D^*_r(N)$ we have

$$d_{int}(L, \mathrm{Obj}\langle \mathcal{F}_\epsilon \rangle^\Delta)) < \epsilon + c_\epsilon \nu(p)$$

where $\mathcal{F}_\epsilon$ is a finite family of (modules of) fibers depending on ϵ and on r. The interleaving distance $d_{int}(-)$ is defined on the triangulated persistence category $\mathcal{C}_p(r)$ for the perturbation parameter p such that $\nu(p)$ is small enough. Moreover, these perturbation parameters, and the respective TPCs, are supposed to fit into a coherent family of TPCs as in Definition 3.1.5.7 and the assumption $(*)$ from Theorem 3.2.2.1 should be satisfied.

We fix $\epsilon > 0$ and the metric g. We keep the constant $0 < \delta < 1$ as in the previous subsections. We intend to use Proposition 3.2.6.4. We need to address the following points:

i. The proposition applies to the metric $\bar{\mathsf{g}} = \alpha\mathsf{g}$ with $\alpha : N \to [1, \infty)$ (chosen such that we have $||d\varphi|| \le 1$, see Remark 3.2.3.2) and not directly to g.

ii. The proposition applies to only values of r that are sufficiently small.

For i. we denote by $||-||_{\mathsf{g}}$ and respectively by $||-||_{\bar{\mathsf{g}}}$ the norms with respect to the two metrics and notice that for $a \in T^*(N)$ we have $||a||_{\bar{\mathsf{g}}} = \frac{1}{\sqrt{\alpha}}||a||_{\mathsf{g}}$. As a result $D^*_{r,\mathsf{g}}(N) \subset D^*_{r,\bar{\mathsf{g}}}(N)$ (where we add the metrics in the notation for the respective disk bundles). Therefore, the statement for the metric $\bar{\mathsf{g}}$ implies automatically the corresponding result for the metric g.

It remains to discuss point ii. We want to show the statement for an arbitrary value $R > 0$. We pick $K > 0$ such that $48KR/\delta \leq \epsilon$ and a Morse function $\varphi = \varphi_{N,K,\delta}$, as in Proposition 3.2.3.1. This determines the family $\mathcal{F}_\epsilon$ of fibers F_{x_i}, one for each critical point x_i of φ. We also consider the metric $\bar{\mathsf{g}} = \alpha\mathsf{g}$ obtained from g by rescaling as in Remark 3.2.3.2 with $\alpha : N \to [0, \infty)$ such that with respect to $\bar{\mathsf{g}}$ we have $||d\varphi|| \leq 1$.

In case R is small enough, the statement of Proposition 3.2.6.4 directly applies and there is nothing further to prove. Assuming this is not the case, consider a sufficiently small r such that the estimate in the statement of Proposition 3.2.6.4

$$d_{int}(L,\ \mathrm{Obj}\langle\{F_{x_0}, \ldots, F_{x_m}\}\rangle^{\Delta}) \leq 48Kr/\delta \leq \epsilon\frac{r}{R} \tag{50}$$

is valid in $\mathcal{C}_p(r)$ - the TPC associated with the exact, marked Lagrangians in $D^*_{r,\bar{\mathsf{g}}}(N)$ with respect to perturbation data p with $\nu(p)$ sufficiently small.

We consider the following rescaling map $\psi_{R,r} : T^*N \to T^*N$, $\psi_{R,r}(q,p) = (q, \frac{r}{R}p)$. Obviously, $\psi_{R,r}$ maps symplectomorphically $(D^*_{R,\bar{\mathsf{g}}}(N), \frac{r}{R}\omega)$ to $(D^*_{r,\bar{\mathsf{g}}}(N), \omega)$ (where ω is the standard form). The map $\psi_{R,r}$ preserves the fibers F_{x_i}. Using the rescaling $\psi_{R,r}$, the estimate (50) applies as well to $(D^*_{R,\bar{\mathsf{g}}}(N), \frac{r}{R}\omega)$. We emphasize that this unit disk bundle is viewed as symplectic manifold with the non-canonical symplectic form $\frac{r}{R}\omega$. There is a corresponding TPC that we will denote by $\mathcal{C}_{\bar{p}}(R; \frac{r}{R}\omega)$ where the perturbation data $\bar{p}$ is also obtained by pull-back through the rescaling map. In this TPC we have the inequality:

$$d_{int}(L,\ \mathrm{Obj}\langle\{F_{x_0}, \ldots, F_{x_m}\}\rangle^{\Delta}) \leq \epsilon\frac{r}{R}\ . \tag{51}$$

The TPC that interests us is $\mathcal{C}_q(R)$, associated with the symplectic manifold $(D^*_{R,\bar{\mathsf{g}}}(N), \omega)$ and with perturbation data q such that $\nu(q)$ is sufficiently small. It is easy to see that there is an isomorphism of categories (but not of TPC's):

$$\Phi : \mathcal{C}_{\bar{p}}\left(R; \frac{r}{R}\omega\right) \longrightarrow \mathcal{C}_{\bar{p}}(R)$$

such that, on geometric objects, Φ rescales the relevant primitives,

$$\Phi((L, f_L)) = \left(L, \frac{R}{r}f_L\right)$$

and, for morphisms, it identifies $\hom^{\alpha}_{\mathcal{C}_{\bar{p}}(R;\frac{r}{R}\omega)}(X,Y)$ with $\hom^{\alpha\frac{R}{r}}_{\mathcal{C}_{\bar{p}}(R)}(\Phi(X), \Phi(Y))$. In brief, Φ preserves all algebraic structures except that it rescales all action values and shifts by $\frac{R}{r}$. Consequently, the interleaving distance is also rescaled by $\frac{R}{r}$. Hence, the inequality (51) is also rescaled by the same constant when passing to $\mathcal{C}_{\bar{p}}(R)$. Thus we get:

$$d_{int}(L,\ \mathrm{Obj}\langle\{F_{x_0},\ldots,F_{x_m}\}\rangle^{\Delta})\leq\epsilon$$

in $\mathcal{C}_{\bar{p}}(R)$. We now consider the system of TPCs $\mathcal{C}_{\bar{p}}(R)$ for varying p and notice that, by pushing-forward also the comparison functors from $\widehat{\mathcal{C}}(D^*_{r,\bar{\mathrm{g}}}(N))$, these categories fit into a system $\widehat{\mathcal{C}}(D^*_R N)$ such that property $(*)$ is satisfied relative to the same family $\mathcal{F}_\epsilon$. To conclude, the claim of Theorem 3.2.2.1 is satisfied in $\widehat{\mathcal{C}}(D^*_R N)$ for $R=1$.

Remark 3.2.6.7. The inequalities that we obtain in the proof of this theorem are not optimal in the sense that we do not try to minimize the number of fibers $F_{x_0},\ldots,F_{x_m}$ in the argument and we do not try to optimize the other choices in the construction. As a result, at the end of the proof we obtain an inequality claiming that, with our choices and $\nu(p)$ small enough, the relevant interleaving distance is not larger than ϵ, while with optimal choices, the expected inequality would be that the interleaving distance has $\epsilon+c_\epsilon\nu(p)$ as upper bound (for some universal constant c_ϵ and for $\nu(p)$ small enough).

3.2.7. Theorem 3.2.2.1 implies Theorem ?? i. and more. Theorem **??** i. claims that the (pseudo)-metric space $(\mathcal{L}\mathrm{ag}^{(ex)}(D^*N),d_\gamma)$ is TPC-approximable in the sense of Definition 3.1.2.1. Here d_γ is the spectral metric whose definition is recalled below, in Remark 3.2.7.1.

To show this statement we need to show that for any $\epsilon>0$ there exists ϵ-TPC-approximating data $\Phi=\{\Phi\}_\eta$, $\mathcal{F}=\{\mathcal{F}_{\epsilon,\eta}\}$ as in Definition 3.1.2.1. By Remark 3.1.2.2 b it is enough to show this for $\eta<\epsilon$ sufficiently small. Of course, we will deduce the existence of this data out of Theorem 3.2.2.1. We will actually produce two choices of such data, each with its own advantages.

Remark 3.2.7.1. For completeness we provide a simple description of the spectral pseudo-metric d_γ defined on $\mathcal{L}ag^{(ex)}(D^*N)$. For $L,L'\in\mathcal{L}ag^{(ex)}(D^*N)$ we consider the persistence module $HF(L,L';H^{L,L'})$ where $H^{L,L'}:[0,1]\times D^*N\to\mathbb{R}$ is an admissible Hamiltonian function in the sense of standard Floer theory (see also §3.1.6.2). We define

$$v(L,L';H^{L,L'})=b_{max}(HF(L,L';H^{L,L'}))-b_{min}(HF(L,L';H^{L,L'}))\ .$$

Here, if $\mathcal{M}$ is a persistence module with a finite number of bars such that $\mathcal{M}=\oplus_{i\in I}[a_i,b_i)\oplus_{j\in J}[c_j,\infty)$ (with $a_i,b_i,c_j\in\mathbb{R}$, $a_i<b_i$) we put $b_{max}(\mathcal{M})=\max_{j\in J}c_j$ and $b_{min}(\mathcal{M})=\min_{j\in J}c_j$. The definition of the spectral metric is:

$$d_\gamma(L,L')=\limsup_{||H^{L,L'}||_{C^0}\to 0} v(L,L';H^{L,L'})\ .$$

It is non-trivial but well-known that this is finite and that it satisfies the properties of a pseudo-metric. The objects in $\mathcal{L}ag^{(ex)}(D^*N)$ are marked Lagrangians, they come with fixed primitives and gradings. The spectral metric does not "see" the choices of grading and primitive in the sense that, with the notation of the paper, $d_\gamma(T^r\Sigma^sL,L')=d_\gamma(L,L')$ for all $r\in\mathbb{Z}$, $s\in\mathbb{R}$

(here T indicates a shift in grading and Σ one in "action"). Thus, this pseudo metric descends to the actual space of exact, closed Lagrangian submanifolds in D^*N without any additional structure and, remarkably, it is non-degenerate on this space.

3.2.7.1. *Local ϵ-TPC-approximating data.* Our aim here is to use Theorem 3.2.2.1 to show that we can obtain approximating data $(\Phi, \mathcal{F})$ for $(\mathcal{L}\mathrm{ag}^{(ex)}(D^*N), d_\gamma)$ in the sense of Definition 3.1.2.1 where the categories $\mathcal{Y}_\eta$ are of the form $\mathcal{C}_p$ and thus complete the proof of Theorem **??** i.

There are a few differences between the type of approximability used in the statement of Theorem 3.2.2.1 (namely Definition 3.1.5.7) and that of the Definition 3.1.2.1 used in Theorem **??**.

First, Theorem 3.2.2.1 makes use of the interleaving pseudo-metric on $\mathcal{C}_p^\epsilon$ and not of the shift invariant pseudo metric. However, by Remark 3.1.2.4, the inequality claimed in Theorem 3.2.2.1 remains true with respect to the shift invariant interleaving pseudo-metric $\bar{d}_{int}^{\mathcal{C}_p^\epsilon}$, as defined in equation (5).

A second difference is that Definition 3.1.5.7 contains an inequality of the form $d_{\mathrm{int}}(-,-) < \epsilon + c_\epsilon \nu(p)$ while in Definition 3.1.2.1 the term $c_\epsilon \nu(p)$ does not appear. To address this point, pick some positive $\epsilon' < \epsilon$ and recall from Remark 3.2.2.2 that the system of TPCs, $\widehat{\mathcal{C}}(D^*N) = \{\mathcal{C}_p\}$ in Theorem 3.2.2.1 can be defined for this ϵ' through the construction of the Lefschetz fibration $\bar{h} : E \to \mathbb{C}$, as in §3.2.5 (see also Remark 3.2.7.5). To avoid confusion we will include ϵ' in the notation, and thus write $\mathcal{C}_p^{\epsilon'}$ and $\widehat{\mathcal{C}}^{\epsilon'}(D^*N)$. We denote the action of the Yoneda embedding on objects by

$$\mathcal{Y}_{\epsilon',p} : \mathcal{L}ag^{(ex)}(D^*N) \to \mathrm{Obj}(\mathcal{C}_p^{\epsilon'}) \ .$$

Theorem 3.2.2.1 shows that for this ϵ', there is a finite family of fibers $\{F_{x_1}, \ldots F_{x_l}\}$ such that for any δ sufficiently small the set $\mathcal{L}ag^{(ex)}(D^*N)$ is ϵ'-approximable, in the sense of Definition 3.1.5.7, by the family of modules $\mathcal{F}_{x_i}$ corresponding to the F_{x_i} in each triangulated persistence category $\mathcal{C}_p^{\epsilon'}$ with $\nu(p) \le \delta$. In this case, if we put:

$$\Phi_{\epsilon,\eta} = \mathcal{Y}_{\epsilon',p_\eta}$$

for p_η depending on η and picked such that $\nu(p_\eta) < \eta/4$, and we denote $\mathcal{F}_{\epsilon,\eta} = \{F_{x_1}, \ldots, F_{x_l}\}$, then the system

$$(\Phi, \mathcal{F}) = (\Phi_{\epsilon,\eta}, \mathcal{F}_{\epsilon,\eta})$$

satisfies the second point in Definition 3.1.2.1 relative to ϵ for η small enough (such that $\eta/4 < (\epsilon - \epsilon')/c_{\epsilon'}$).

A third and last difference is that the first point in Definition 3.1.2.1 is not present in Definition 3.1.5.7. This has to do with the comparison between the spectral metric on the domain of the maps Φ and the interleaving pseudo-metric defined on the target of these maps. Thus, to show that $(\Phi, \mathcal{F})$ as above are a choice of TPC ϵ-approximating data for

$$(\mathcal{L}\mathrm{ag}^{(ex)}(D^*N), d_\gamma) \ ,$$

in the terminology from §3.1.2.2 it remains to show that $\mathcal{Y}_{\epsilon',p}$ also satisfies the first point in Definition 3.1.2.1, namely that $\mathcal{Y}_{\epsilon',p}$ is a (A,η)-quasi-isometric embedding where the pseudo-metric on $\mathrm{Obj}(\mathfrak{C}_p^{\epsilon'})$ is the interleaving metric, the pseudo-metric on the domain is d_γ, and the constant $A>0$ can be picked independent of all other choices.

We will see that the statement is true for $A=2$. We start by revisiting the definition of the interleaving distance from equation (3). An equivalent definition is

$$\begin{aligned} d_{int}(X,Y) = \inf\{r \geq 0 \mid \exists\ \phi \in \hom^r(X,Y), \psi \in \hom^r(Y,X) \\ \text{such that } \psi\circ\phi = i_{0,2r}(id_X),\ \phi\circ\psi = i_{0,2r}(id_Y)\}\ , \end{aligned}$$

where we recall that $X,Y\in \mathrm{Obj}(\mathcal{C})$, $\mathcal{C}$ is a persistence category (see §3.1.1.2) and

$$i_{\alpha,\beta} : \hom^\alpha(X,Y) \to \hom^\beta(X,Y)$$

are the persistence structural maps. An alternative definition of an interleaving type pseudo-metric is the following:

$$\begin{aligned} D_{int}(X,Y) = \inf\{a+b \mid \exists\ \phi\in\hom^a(X,Y), a\geq 0, \psi\in\hom^b(Y,X), b\geq 0 \\ \text{such that } \psi\circ\phi = i_{0,a+b}(id_X),\ \phi\circ\psi = i_{0,a+b}(id_Y)\} \end{aligned} \tag{52}$$

It is immediate to see that:

$$\frac{1}{2}D_{int}(X,Y) \leq d_{int}(X,Y) \leq D_{int}(X,Y)\ . \tag{53}$$

It is obviously also possible to stabilize the distance D_{int}, just as in formula (5), thus putting

$$\bar{D}_{int}(X,Y) = \inf_{r,s\in\mathbb{R}} D_{int}(\Sigma^r X, \Sigma^s Y)$$

and we again have $\frac{1}{2}\bar{D}_{int} \leq \bar{d}_{int} \leq \bar{D}_{int}$. The corresponding interleaving type pseudo-metric, $\bar{D}^{\mathfrak{C}_p^\epsilon}$, is defined on each of the categories $\mathfrak{C}_p^\epsilon$ and, for all $L, L' \in \mathcal{L}ag^{(ex)}(D^*N)$ we can consider the limit:

$$\hat{D}_{int}(L,L') = \limsup_{\nu(p)\to 0} \bar{D}_{int}^{\mathfrak{C}_p^\epsilon}(L,L')\ .$$

where the limit is taken in the system of categories $\hat{\mathfrak{C}}_p^\epsilon(D^*N)$. This is again a pseudo-metric defined on $\mathcal{L}\mathrm{ag}^{(ex)}(D^*N)$. We claim that we have:

$$\hat{D}_{int}(L,L') = d_\gamma(L,L')\ \ \forall\ L,L'\in\mathcal{L}\mathrm{ag}^{(ex)}(D^*N) \tag{54}$$

This identity is well-known in the subject, the argument can be found in 3.4.1.2.(i) in [**BCZ24b**] (see also Remark 3.26 there).

Taking into account $\nu(p_\eta) < \eta/4$ and the definition of $\nu(p)$ from (26) it is immediate to see that (54) implies that the maps $\mathcal{Y}_{\epsilon',p_\eta}$ are $(2,\eta)$-quasi-isometric embeddings when η is sufficiently small and thus finishes the proof of Theorem **??** i.

REMARK 3.2.7.2. It is clear from the argument above that the interleaving type pseudo-metric D_{int} from Definition 52 is more directly tied to the spectral metric compared to the pseudo-metric d_{int} from Definition 3. On the other hand, d_{int}, when applied to the homotopy category of filtered chain complexes, coincides with the bottleneck distance from persistence theory - as can be seen through the isometry theorem as in, for instance, §2.2 [**PRSZ20**] - and thus is, in a way, the more “standard” notion.

3.2.7.2. *Ambient ϵ-TPC-approximating data.* We will see here that the proof of Theorem 3.2.2.1 also provides a different sort of ϵ-approximating data (in the terminology from §3.1.2.2) for the space $(\mathcal{L}\mathrm{ag}^{(ex)}(D^*_rN), d_\gamma)$. It is important to note from the outset that, by contrast to §3.2.7.1, we obtain *retract* approximating data in this case - see Corollary 3.2.7.4.

For a fixed $\epsilon > 0$ consider some positive $\epsilon' < \epsilon$. Recall from (45) the persistence derived Fukaya categories

$$\mathcal{D}_p(E) = PD(\mathcal{F}uk(\mathcal{L}ag^{(ex)}(E), p))$$

as well as the Lagrangian spheres $\hat{S}_{x_i} \in \mathcal{L}ag^{(ex)}(E)$ that appear in Lemma 3.2.6.5 that can be constructed for ϵ' (for some small r). The corresponding Yoneda embeddings yield maps:

$$\mathcal{Y}'_{\epsilon',p} : \mathcal{L}ag^{(ex)}(D^*_rN) \to \mathrm{Obj}(\mathcal{D}_p(E)) \ .$$

We define the family $\mathcal{F}'_\epsilon = \{\ldots, \hat{S}_{x_i}, \ldots\}$ where here $\hat{S}_{x_i}$ represents the Yoneda module of the respective Lagrangian sphere (these are preserved by the comparison functors $\mathcal{H}_{p,q}$, $p \preceq q$). We put $\Phi'_{\epsilon,\eta} = \mathcal{Y}'_{\epsilon',p_\eta}$ where $\nu(p_\eta) \leq \eta/4$, just as in §3.2.7.1. Here the system of perturbations p is defined relative to $\mathcal{L}\mathrm{ag}^{(ex)}(E)$ and thus $\nu(-)$ is calculated for Hamiltonians defined over E. The proof that Φ'_η is a $(2,\eta)$-quasi-isometric embedding is the same as in §3.2.7.1 because the identity (54) remains true even if the interleaving type distance $\hat{D}_{int}$ is computed in the larger category $\mathcal{D}_p(E)$ instead of $\mathcal{C}_p(r)$ (this is due again to the argument in 3.4.1.2 (i) in [**BCZ24b**]).

It remains to show that $\mathcal{F}'_\epsilon$ is a retract ϵ-approximating family for $\mathcal{L}\mathrm{ag}^{(ex)}(D^*_rN)$ in each $\mathcal{D}_{p_\eta}(E)$ for η sufficiently small.

To show this we proceed as in the proof of Proposition 3.2.6.4. From the proof of this proposition we know that there is a sequence (3.2.6.3) of exact triangles in $\mathcal{D}_p(E)$:

$$\Delta'_i \ : \ Z'_i \longrightarrow X'_i \longrightarrow X'_{i+1} \ , \ 0 \leq i < m \tag{55}$$

with Z'_j of the form $\Sigma^{l(j)}\hat{S}_{x_{s(j)}}$ or $= 0$, with each Δ'_i exact in $(\mathcal{D}_p(E))^0$, with X'_0 the Yoneda module of a Lagrangian disjoint from D^*_rN, and such that there is a $6c$-isomorphism

$$f : X'_m \to L \ .$$

We represent all iterated cones in (55) by attachment of cones in the pre-triangulated category of filtered A_∞-modules (this is possible because the sequence (55) is in $(\mathcal{D}_p(E))^0$) and we consider the obvious compositions $u_j : X'_0 \to X'_j$. The sequence (55) induces a new iterated sequence of cones (in the same category):

$$\tilde{\Delta}_i \ : \ Z'_i \longrightarrow \tilde{X}_i \longrightarrow \tilde{X}_{i+1} \ , \ 0 \leq i < m \tag{56}$$

with $\tilde{X}_i = \text{Cone}\,(u_i)$. We first remark that because X'_0 is a Yoneda module all $\tilde{X}_i$'s are objects of $\mathcal{D}_p(E)$. We now consider the cone attachment

$$X'_0 \xrightarrow{u_m} X'_m \longrightarrow \tilde{X}_m \ .$$

Because X'_0 is the Yoneda module of a Lagrangian that does not intersect $D^*_r N$ it follows that the composition $f \circ u_m$ is 0-chain homotopic to the null map. As a result, the map f induces another $6c$-isomorphism $\tilde{f} : \tilde{X}_m \to L \oplus TX'_0$. This means that

$$d^{\mathcal{D}_p(E)}_{\text{r-int}}(L, \tilde{X}_m) \leq 6c$$

and, moreover, in $(\mathcal{D}_p(E))^0$ we have that $\tilde{X}_0 \cong 0$.

As a result, similarly to Proposition 3.2.6.4, we obtain:

Corollary 3.2.7.3. *With the notation above, for r sufficiently small and for a choice of perturbations p with $\nu(p)$ sufficiently small, any exact, marked Lagrangian $L \subset D^*_r(N)$ satisfies in $\mathcal{D}_p(E)$):*

$$d_{\textit{r-int}}(\ L\ , \textit{Obj}\ \langle \{\hat{S}_{x_0}, \ldots, \hat{S}_{x_m}\}\rangle^{\Delta}\ ) \leq 48Kr/\delta\ .$$

The arguments in the proof of Theorem 3.2.2.1 that are found in §3.2.6.4 can then be pursued in this somewhat different setting to get rid of the dependence on r, K, δ, and they lead to a conclusion that parallels the similar statement for $(\Phi, \mathcal{F})$:

Corollary 3.2.7.4. *In the setting above and for a fixed $\epsilon > 0$ we have:*

a. There exists a Lefschetz fibration $\bar{h} : E \to \mathbb{C}$ as before such that the family $\mathcal{F}'_\epsilon$ of the Yoneda modules of the Lagrangian spheres $\{\hat{S}_{x_0}, \ldots, \hat{S}_{x_m}\}$ is a retract ϵ-approximating family for $(\mathcal{L}ag^{(ex)}(D^*_r N), d_\gamma)$ in $\mathcal{D}_p(E)$ for each p with $\nu(p)$ sufficiently small. Moreover, $(\Phi', \mathcal{F}'_\epsilon)$ is TPC retract ϵ-approximating for $(\mathcal{L}ag^{(ex)}(D^*_r N), d_\gamma)$.

b. For each $L \in \mathcal{L}ag^{(ex)}(D^*_r N)$ there exists $X_L \subset \mathcal{L}ag^{(ex)}(E)$ disjoint from $D^*_r N$ such that

$$d_{\textit{int}}(L \oplus X_L, \textit{Obj}\ \langle \{\hat{S}_{x_0}, \ldots, \hat{S}_{x_m}\}\rangle^{\Delta}) < \epsilon + c_\epsilon \nu(p)$$

in the category $\mathcal{D}_p(E)$ and for $\nu(p)$ small enough.

While the result above is established for small r, given that we can construct the Lefschetz fibration as appropriate, the same result can be inferred for all values of r.

Remark 3.2.7.5. a. The advantages of the approximating data $(\Phi, \mathcal{F}_\epsilon)$ from §3.2.7.1 are that, first, it is more intrinsic relative to the cotangent disk bundle D^*N and, second, it does not involve retracts. To expand on the first point, fix some $\epsilon > 0$. The category $\mathcal{C}_p$, and the object level part of the Yoneda embedding $\mathcal{Y}_p : \mathcal{L}ag^{(ex)}(D^*N) \to \text{Obj}(\mathcal{C}_p)$ that form the family Φ are constructed - as in §3.2.1 - for some admissible choice of perturbation data p, independently of ϵ and any other auxiliary construction. The dependence on ϵ only appears when we fix a set of fibers $\mathcal{F}_\epsilon$ and require that the associated modules be preserved by the comparison maps $\mathcal{H}_{p,q}$, $p \preceq q$ - this is our assumption $(*)$. The proof of Theorem 3.2.2.1 shows that it is possible to find comparison maps preserving the finite family of modules associated to $\mathcal{F}_\epsilon$ but it is not clear whether there is a choice of comparison maps that preserves each fiber module F_x, $\forall x \in N$ (see also Remark 3.2.2.2).

b. The approximating data $(\Phi', \mathcal{F}'_\epsilon)$ described in the current subsection is highly dependent on the choice of Lefschetz fibration $\bar{h} : E \to \mathbb{C}$ and also is only retract approximating. On the other hand, the category

$$\mathcal{D}_p(E) = PD\mathcal{F}uk(\mathcal{L}\text{ag}^{(ex)}(E), p)$$

has the property that all its objects can be written as finite iterated cones of Yoneda modules of closed Lagrangians, and as a result each $\hom_{\mathcal{D}_p(E)}(X, Y)$ is a finite type persistence module (in the sense that the associated bar-code contains only a finite number of bars) for all $X, Y \in \mathrm{Obj}(\mathcal{D}_p(E))$. Moreover, the retract ϵ-approximating family $\mathcal{F}'_\epsilon$ consists of the Yoneda modules of the Lagrangian spheres $\hat{S}_{x_i} \subset E$. By contrast, this may not be true for the elements of the family $\mathcal{F}_\epsilon$ which are defined as modules over the category $\mathcal{F}uk(\mathcal{L}\text{ag}^{(ex)}(D^*N), p)$ corresponding to fibers. Moreover, for two fibers F_{x_1}, F_{x_2}, $\hom_{\mathcal{C}_p}(F_{x_1}, F_{x_2})$ is not identified with some variant of Floer homology $HF(F_{x_1}, F_{x_2})$ and, indeed, it is not necessarily of finite type.

3.3. Abouzaid's splitting principle in the filtered case and approximability

In this section we prove the second and third points in Theorem **??**. To this aim, we establish a persistence version of the split-generation criterion due to Abouzaid [**Abo10**] (for the closed monotone case see [**She16**]). The key point is that the energy cost of reaching the unit in the quantum homology of the ambient symplectic manifold through the open-closed map can be directly interpreted in terms of retract approximability, as defined in §3.1.2.2. We provide two examples of computations, completing the proof of Theorem **??**. Along the way, we adjust to this persistence setting the various ingredients in Abouzaid's result, namely the open-closed and closed open maps and their relation to a persistence version of Hochschild homology.

3.3.1. The results. Let (X, ω) be a monotone symplectic manifold. We work in the setting outlined in §3.1.6.3. We recall that given a Novikov element $\mathbf{d} \in \Lambda_0$, then $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}$ is the collection of closed, graded and monotone Lagrangians $L = (\overline{L}, a_L, s_L)$ with Maslov-2 disk count equal to $\mathbf{d}$. We fix a choice of $\mathbf{d} \in \Lambda_0$ for the rest of this chapter. Recall from §3.1.6.6 that we associated with $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}$ a homotopy system of A_∞-categories (see §3.1.5.8 for the definition) $\widehat{\mathcal{F}uk}(\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})})$ indexed by the space $\mathcal{P}$ of perturbation data, such that each filtered and strictly unital A_∞-category $\mathcal{F}uk(\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}; p)$, $p \in \mathcal{P}$, has $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}$ as set of objects. We will often abbreviate and write the system as $\widehat{\mathcal{A}} := \widehat{\mathcal{F}uk}(\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})})$ and each component as

$$\mathcal{A}_p := \mathcal{F}uk(\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}; p)\ .$$

We omit the manifold X from the notation as it is also fixed for the remainder of the section.

A subset $\mathcal{B} \subset \mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}$ induces a full A_∞-subcategory $\mathcal{B}_p$ of $\mathcal{A}_p$ for any perturbation datum $p \in \mathcal{P}$. This ultimately leads to a subsystem $\widehat{\mathcal{B}}$ of $\widehat{\mathcal{A}}$.

The following geometric criterion for retract approximability, in the sense of Definition 3.1.5.8 with $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})} \subset \mathrm{Obj}(\widehat{\mathcal{A}})$, is the main result of the section.

THEOREM 3.3.1.1. *Let $\epsilon > 0$. If there is a finite subset $\mathcal{F}_\epsilon \subset \mathcal{L}ag^{(mon,\mathbf{d})}$ such that the colimit persistence open-closed map*

$$\widehat{OC}\colon PHH_*(\widehat{\mathcal{F}_\epsilon}) \to QH^{*+n}(X, \Lambda)_{\mathbf{d}}$$

satisfies

$$i_{0,\epsilon}(u_{\mathbf{d}}) \in image\left(PHH^\epsilon(\widehat{\mathcal{F}_\epsilon}) \to QH^\epsilon(X, \Lambda)_{\mathbf{d}}\right)$$

where $u_{\mathbf{d}} \in QH^0(X, \Lambda)_{\mathbf{d}}$ is the projection of the quantum cohomology unit $u \in QH(X, \Lambda)$ to $QH(X, \Lambda)_{\mathbf{d}}$ and $i_{0,\epsilon}$ is part of the structure maps of the quantum cohomology persistence module, then $\mathcal{L}ag^{(mon,\mathbf{d})}$ is $\frac{\epsilon}{2}$-retract-approximable by $\mathcal{F}_\epsilon$ in $PD(\widehat{\mathcal{F}uk}(\mathcal{L}ag^{(mon,\mathbf{d})}))$.

The open closed map OC, the sense in which we take a colimit, and the various other notions appearing in the statement of Theorem 3.3.1.1 and its proof, among them, the persistence Hochschild homology $PHH(-)$, will be made explicit below. We now restate the points (ii) and (iii) of Theorem **??** as corollaries of Theorem 3.3.1.1.

COROLLARY 3.3.1.2. *The classes $\mathcal{L}ag^{(mon,\mathbf{0})}(S^2)$ and $\mathcal{L}ag^{(wex)}(\mathbb{T}^2)$ are TPC retract approximable.*

The proof of Theorem 3.3.1.1 is contained in §3.3.2. The proof of Corollary 3.3.1.2 occupies §3.3.3. The proof of the last two points of Theorem **??** follows directly from this Corollary and is concluded in §3.3.4.

3.3.2. Proof of Theorem 3.3.1.1. We split the proof of Theorem 3.3.1.1 into several steps. §3.3.2.1 contains a persistence version of Abouzaid's algebraic split-generation criterion. The main result, in Proposition 3.3.2.1, shows that retract approximability is naturally present in that context. §3.3.2.2 recalls some of the main technical ingredients in the definition of the filtered Fukaya categories $\mathcal{F}uk(\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})};p)$ following [**Amb25**] in sufficient detail so that, in the following sections 3.3.2.3, 3.3.2.4, 3.3.2.5 we are able to set-up in the persistence setting the main geometric constructions needed for the split-generation criterion. Finally, in §3.3.2.6 we put the constructions together and prove Theorem 3.3.1.1.

3.3.2.1. *Retract approximability.* Let $\mathcal{A}$ be a filtered A_∞-category over Λ, such as one of the filtered Fukaya categories $\mathcal{A}_p$ from the beginning of §3.3, and let $\mathcal{B} \subset \mathrm{Obj}(\mathcal{A})$. We regard $\mathcal{B}$ as a full A_∞-subcategory of $\mathcal{A}$. In contrast to our general conventions, we do not assume here that $\mathcal{B}$ is closed under shifts or translations. We define a filtered $(\mathcal{A},\mathcal{A})$-bimodule $\overline{\mathcal{B}}$, which will be called the $\mathcal{B}$-bar bimodule. For two objects, A and B, of $\mathcal{A}$, we set, with the notation in Appendix .1,

$$\overline{\mathcal{B}}(A,B) := \mathcal{Y}^r(A) \otimes_{\mathcal{B}} \mathcal{Y}^l(B) = \bigoplus_{d\geq 0} \bigoplus_{L_0,\dots,L_d\in \mathrm{Ob}(\mathcal{B})} \mathcal{A}(A,L_0,\dots,L_d,B)\ .$$

Here $\mathcal{Y}^l$ and $\mathcal{Y}^r$ denote the left and right filtered Yoneda embeddings, defined in Appendix .1.3.1, and we implicitly view $\mathcal{Y}^r(A)$ and $\mathcal{Y}^l(B)$ as A_∞-modules over $\mathcal{B}$. The tensor product $\otimes_{\mathcal{B}}$ is the usual derived tensor product over $\mathcal{B}$. Note that $\mathcal{A}(A,B)$ is not a summand in $\overline{\mathcal{B}}(A,B)$. We will often write a pure tensor $\gamma_1\otimes a_1\otimes\cdots\otimes a_d\otimes\gamma_2 \in \mathcal{A}(A,L_0,\dots,L_d,B)$ as $\vec{\gamma}$. The bar differential $\mu^{\mathrm{bar}}_{0|1|0}$ (which we will also denote by d_{bar}) is defined via

$$\begin{aligned}\mu^{\mathrm{bar}}_{0|1|0}(\vec{\gamma}) :=& \sum_{i=0}^{d} \mu_{1+i}(\gamma_1,a_1,\dots,a_i)\otimes a_{i+1}\otimes\cdots\otimes a_d\otimes\gamma_2\\ &+\sum_{i\leq j}\gamma_1\otimes a_1\otimes\cdots\otimes a_{i-1}\otimes\mu_{j-i}(a_i,\dots,a_j)\otimes a_{j+1}\otimes\cdots\otimes a_d\otimes\gamma_2\\ &+\sum_{i=1}^{d+1}\gamma_1\otimes a_1\otimes\cdots\otimes a_{i-1}\otimes\mu_{d-i+2}(a_i,\dots,a_d,\gamma_2)\end{aligned}$$

while the higher operations $\mu^{\mathrm{bar}}_{l|1|r}$ are defined via

- for $l\geq 1$ and $r=0$:

$$\mu^{\mathrm{bar}}_{l|1|0}(x_1,\dots,x_l,\vec{\gamma}) := \sum_{i=0}^{d}\mu_{l+i+1}(x_1,\dots,x_l,\gamma_1,a_1,\dots,a_i)\otimes a_{i+1}\otimes\cdots\otimes a_d\otimes\gamma_2$$

- for $l = 0$ and $r \geq 1$:

$$\mu^{\mathrm{bar}}_{0|1|r}(\vec{\gamma}, y_1, \ldots, y_r) := \sum_{i=1}^{d+1} \gamma_1 \otimes a_1 \otimes \cdots \otimes a_{i-1} \otimes \mu_{d+r-i+2}(a_i, \ldots, a_d, \gamma_2, y_1, \ldots, y_r)$$

- for $l, r \geq 1$: $\mu^{\mathrm{bar}}_{l|1|r} = 0$

It is straightforward to see that $\overline{\mathcal{B}}$ is filtered, since $\mathcal{A}$ is. There is an obvious length filtration

$$F^N\overline{\mathcal{B}}(A, B) = \bigoplus_{d=0}^{N} \bigoplus_{L_0,\ldots,L_d \in \mathrm{Ob}(\mathcal{B})} \mathcal{A}(A, L_0, \ldots, L_d, B), \quad N \geq 0, \tag{57}$$

which induces A_∞-bimodules $F^N\mathcal{B}$. The A_∞-structure maps on $\mathcal{A}$ induce a filtered morphism of $(\mathcal{A}, \mathcal{A})$-bimodules

$$\mu := \mu^{\overline{\mathcal{B}}} \colon \overline{\mathcal{B}} \to \Delta_{\mathcal{A}}$$

in an obvious way, where $\Delta_{\mathcal{A}}$ stands for the diagonal bimodule of $\mathcal{A}$.

Let now $K \in \mathrm{Obj}(\mathcal{A})$ and consider the filtered chain map

$$\mu^{\overline{\mathcal{B}}}(K) := \mu^{\overline{\mathcal{B}}}_{0|1|0} \colon \overline{\mathcal{B}}(K, K) \to \mathcal{A}(K, K)\ . \tag{58}$$

Denote by

$$[\mu^{\overline{\mathcal{B}}}(K)] \colon H(\overline{\mathcal{B}}(K, K)) \to H(\mathcal{A}(K, K))$$

the morphism of persistence modules induced by $\mu^{\overline{\mathcal{B}}}(K)$.

In the context above, given an object K of $\mathcal{A}$, we denote its strict unit by $e_K \in \mathcal{A}^{\leq 0}(K, K)$ and by $[e_K] \in \hom^0_{PD(\mathcal{A})}(K, K)$ its homology class in the TPC $PD(\mathcal{A})$.

Finally, before stating the main result, we need one more notion. Let V and W be persistence modules and $f \colon V \to W$ a persistence morphism. We will make use of a measurement that estimates the energy gap separating an element in W from the image of f. More precisely, for $w \in W_r$, we put:

$$R(w, f) := \inf \left\{ s \geq r \mid \exists v \in V_s,\ f_s(v) = i^W_{r,s}(w) \right\} \tag{59}$$

where $i^W_{r,s} \colon W_r \to W_s$ denotes the persistence structure map for W. If $i^W_{r,s}(w) \notin \mathrm{image}(f_s)$ for all $s \geq r$ we set $R(w, f) = \infty$.

Below and in what follows we denote by $\mathcal{F}\mathbf{Ch}$ the filtered dg-category of filtered chain complexes over Λ, and by $H^0(\mathcal{F}\mathbf{Ch})$ its persistence homological category in cohomological degree 0, which is a TPC (see [**BCZ24b**, Section 2.5.2]). We recall that in our conventions, a filtered chain complex over Λ is a chain complex over Λ, filtered by an increasing sequence of Λ_0-subcomplexes indexed by the real line (see Appendix .1.1).

PROPOSITION 3.3.2.1. *Let K be an object of $\mathcal{A}$, $\mathcal{B}$ be a finite full A_∞-subcategory of $\mathcal{A}$ and $\alpha \in \mathbb{R}_{\geq 0}$. The following statements are equivalent:*

(1) The estimate $R([e_K], [\mu^{\overline{\mathcal{B}}}(K)]) \leq \alpha$ holds.

(2) The map $\mu^{\overline{\mathcal{B}}}(K) \in hom_{(H^0(\mathcal{F}\mathbf{Ch}))^0}(\overline{\mathcal{B}}(K, K), \mathcal{A}(K, K))$ is an α-isomorphism in $H^0(\mathcal{F}\mathbf{Ch})$.

(3) *The object K is $\frac{\alpha}{2}$-retract-approximable in $PD(\mathcal{A})$ by the family Obj($\mathcal{B}$) in the sense of Definition 3.1.2.3.*

Moreover, each of the above statements holds for all objects K of $\mathcal{A}$ if and only if the cone of $\mu^{\overline{\mathcal{B}}}(K)$ seen as a morphism of $(\mathcal{A},\mathcal{A})$-bimodules is α-acyclic in $H^0(Fbimod_{\mathcal{A},\mathcal{B}})$.

The notion of r-isomorphism for $r \geq 0$ is introduced in [**BCZ24b**] and is recalled in §3.1.4.4, and r-acyclicity is recalled in §3.1.1.3.

Remark 3.3.2.2. Note that by [**BCZ24b**, Lemma 2.85], if one of the statements in Proposition 3.3.2.1 holds, then the interleaving distance in $H^0(\mathcal{F}\mathbf{Ch})$ between $\overline{\mathcal{B}}(K,K)$ and $\mathcal{A}(K,K)$ is $\leq \alpha$.

Before proving Proposition 3.3.2.1, we start with two simple auxiliary results that will be used in the proof. Let $\mathcal{C}$ be a PC, and consider objects R and X of $\mathcal{C}$. The measurement $d_{\text{r-int}}(R,X)$ has been defined in (4). We define a similar measurement $d'_{\text{r-int}}$ via

$$d'_{\text{r-int}}(R,X) := \frac{1}{2}\inf\Big\{r_1+r_2 \,\Big|\, r_1,r_2\geq 0,\ \exists\,\varphi:\Sigma^{r_1}R\longrightarrow X,\ \exists\,\psi:\Sigma^{r_2}X\longrightarrow R \\ \text{such that } \psi\circ\Sigma^{r_2}\varphi=\eta^R_{r_1+r_2}\Big\}.$$

Of course we have

$$d'_{\text{r-int}}(R,X)\leq d_{\text{r-int}}(R,X)\leq 2d'_{\text{r-int}}(R,X).$$

In the proof of Proposition 3.3.2.1 it is more convenient to use $d'_{\text{r-int}}$ instead of $d_{\text{r-int}}$. This is possible because their shift-invariant versions agree:

Lemma 3.3.2.3. *For any objects R and X of $\mathcal{C}$ we have*

$$\bar{d}_{\textit{r-int}}(R,X)=\bar{d}'_{\textit{r-int}}(R,X)$$

where $\bar{d}'_{\textit{r-int}}$ is defined as in (5) but with $\bar{d}_{\textit{r-int}}$ replaced by $d'_{\textit{r-int}}(R,X)$.

Proof. Assume $d'_{\text{r-int}}(R,X)<r$. Let $\varphi:\Sigma^{r_1}R\longrightarrow X$ and $\psi:\Sigma^{r_2}X\longrightarrow R$ such that $\psi\circ\Sigma^{r_2}\varphi=\eta^R_{r_1+r_2}$ and $r_1+r_2=2r$. Assume (without loss of generality) that $r_1\geq r_2$ and define $\alpha := \frac{r_1-r_2}{2}$. Note that $r-\alpha=r_2$ Then $\Sigma^{-\alpha}\varphi\colon\Sigma^r R\to\Sigma^{-\alpha}X$ and $\psi\colon\Sigma^{-\alpha+r}X\to R$ show that $\bar{d}_{\text{r-int}}(R,X)<r$. The opposite inequality is obvious. □

Proof of Proposition 3.3.2.1. We start by assuming (1) and proving (3). We construct a TPC version of Diagram A.4 in [**Abo10**, Appendix A]. We refer to this diagram as the (filtered) Abouzaid's algebraic split generation diagram.

Let K be an object of $\mathcal{A}$ and $\vec{L}=(L_0,\ldots,L_d)$ a tuple of objects of the full subcategory $\mathcal{B}$ and $N\geq 1$ such that the unit $[e_K]$ lies in the image of $\mu^{\overline{\mathcal{B}}}=\mu^{\overline{\mathcal{B}}}(K)$ (we omit K here to ease notation) restricted to $F^N\overline{\mathcal{B}}^{\leq\alpha}(K,K)$. Recall that here F^N indicates the Nth term in the length filtration of $\overline{\mathcal{B}}$ (as in 57), while the exponent $\leq\alpha$ stands for the real filtration level.

Consider the filtered twisted complex

$$T_{\vec{L}}K := T_{L_0}\cdots T_{L_d}K$$

obtained by composing twistings by objects from the tuple $\vec{L}$, as explained in §1.9.3 and consider the inclusion

$$i_{\vec{L},K}\colon K \to T_{\vec{L}}K$$

which is a composition of morphisms of twisted complexes induced by strict units (see §1.9.3). Let $N \geq 1$. We define i_K^N to be the direct sum over all the maps $i_{\vec{L},K}$ where $\vec{L}$ varies over all tuples of objects of $\mathcal{B}$ of length $\leq N+1$. We define the filtered twisted complex

$$U_K^N := \mathrm{Cone}(i_K^N)$$

and its image

$$\mathcal{U}_K^N := \widetilde{\mathcal{Y}}_{U_K^N}$$

under the filtered extended Yoneda embedding $\widetilde{\mathcal{Y}}\colon FTw\mathcal{A} \to F\mathrm{mod}_{\mathcal{A}}$ constructed in §1.9.2. By construction, the underlying vector space of the chain complex of $\mathcal{U}_K^N$ applied to an object Q of $\mathcal{A}$ has the form

$$\bigoplus_{d\leq N} \bigoplus_{\vec{L}=(L_0,\ldots,L_d)\subset\mathcal{B}} \mathcal{A}(Q,L_0)\otimes\mathcal{A}(\vec{L})\otimes\mathcal{A}(L_d,K).$$

It is an iterated filtered mapping cone built with objects in $\mathcal{B}$.

We have the following result.

LEMMA 3.3.2.4. *There is a commutative persistence diagram*

$$\begin{tikzcd} H_*(F^N\overline{\mathcal{B}}(K,K)) \arrow[r, "\overline{\lambda}"] \arrow[d, "{[\mu^{\overline{\mathcal{B}}}]}"'] & \hom_{PD(\mathcal{A})}(K,U_K^N) \arrow[d, "\xi\circ -"] \\ \hom_{PH(\mathcal{A})}(K,K) \arrow[r, "\lambda"] & \hom_{PD(\mathcal{A})}(K,K) \end{tikzcd} \tag{60}$$

where $PH(\mathcal{A})$ denotes the persistence homological category of $\mathcal{A}$ (§3.1.3.1) and $PD(\mathcal{A})$ its persistence derived category constructed using A_∞-modules (§3.1.4.2).

PROOF OF LEMMA 3.3.2.4. We define a chain level version of Diagram (60).

$$\begin{tikzcd} F^N\overline{\mathcal{B}}(K,K) \arrow[r, "\overline{\lambda}"] \arrow[d, "\mu^{\overline{\mathcal{B}}}"'] & \mathrm{mod}_{\mathcal{A}}(\mathcal{Y}_K^l,\mathcal{U}_K^N) \arrow[d, "{\mu_2^{\mathrm{mod}}(-,\xi)}"] \\ \mathcal{A}(K,K) \arrow[r, "\lambda"] & \mathrm{mod}_{\mathcal{A}}(\mathcal{Y}_K^l,\mathcal{Y}_K^l) \end{tikzcd} \tag{61}$$

which will be commutative up to a filtered chain homotopy $H\colon F^N\overline{\mathcal{B}}(K,K) \to \mathrm{mod}_{\mathcal{A}}(\mathcal{Y}_K^l,\mathcal{Y}_K^l)$. We now define the maps appearing in Diagram (61) and the chain homotopy H. The map λ is a variation of the Yoneda embedding and is defined in §1.3.2. The map μ_2^{mod} is the composition in the filtered dg-category of filtered A_∞-modules over $\mathcal{A}$. We define $\overline{\lambda}, \xi$, and H below.

Let $\vec{\gamma} = \gamma_1 \otimes a_1 \otimes \cdots \otimes a_d \otimes \gamma_2 \in F^N\overline{\mathcal{B}}(K,K)$.

i. We define

$$\overline{\lambda}(\vec{\gamma}) =: \overline{\lambda}_{\vec{\gamma}} \in \mathrm{mod}_{\mathcal{A}}(\mathcal{Y}^l_K, \mathcal{U}^N_K)$$

to be the morphism of A_∞-modules

$$\overline{\lambda}_{\vec{\gamma}} = \sum_{j=0}^{d} \mathcal{Y}_{j+1}(\gamma_1, a_1, \ldots, a_j) \otimes a_{j+1} \otimes \cdots \otimes a_d \otimes \gamma_2$$

where $\mathcal{Y}_{j+1}$ is the $(j+1)$-component of the (left) Yoneda embedding $\mathcal{Y}$ (see §.1.3.1). More explicitly, we can write the component

$$(\overline{\lambda}_{\vec{\gamma}})_{l|1} \colon \mathcal{A}(X_0, X_1) \otimes \cdots \otimes \mathcal{A}(X_l, K) \to \mathcal{U}^N_K(X_0)$$

as

$$(\overline{\lambda}_{\vec{\gamma}})_{l|1}(x_1, \ldots, x_l, y) = \sum_{j=0}^{d} \mu_{j+l+2}(x_1, \ldots, x_l, y, \gamma_1, a_1, \ldots, a_j) \otimes a_{j+1} \otimes \cdots \otimes a_d \otimes \gamma_2.$$

ii. The map $\xi \in \mathrm{mod}_{\mathcal{A}}(\mathcal{U}^N_K, \mathcal{Y}^l_K)$ is the full contraction map, that is,

$$\xi_{l|1} \colon \mathcal{A}(X_0, X_1) \otimes \cdots \otimes \mathcal{A}(X_{l-1}, X_l) \otimes \mathcal{U}^N_K(X_l) \to \mathcal{A}(X_0, K)$$

contracts an input coming from a summand of length $d \leq N$ in $\mathcal{U}^N_{\mathcal{K}}$ via μ_{l+d+1}.

iii. The map $H \colon F^N\overline{\mathcal{B}}(K,K) \to \mathrm{mod}_{\mathcal{A}}(\mathcal{Y}^l_K, \mathcal{Y}^l_K)$ is also induced by a full contraction, but in the following way: we define

$$H(\vec{\gamma}) =: H_{\vec{\gamma}} \in \mathrm{mod}_{\mathcal{A}}(\mathcal{Y}^l_K, \mathcal{Y}^l_K)$$

to be the morphism of A_∞-modules with $l|1$ component given by

$$(H_{\vec{\gamma}})_{l|1}(x_1, \ldots, x_l, y) := \mu_{l+d+3}(x_1, \ldots, x_l, y, \gamma_1, a_1 \ldots, a_d, \gamma_2).$$

It is straightforward to see that ξ is a filtered morphism of A_∞-modules (that is, it lies in $\mathrm{mod}_{\mathcal{A}}(\mathcal{U}^N_K, \mathcal{Y}^l_K)^{\leq 0}$), and that all the five maps appearing in Diagram (61) are filtered chain maps.

We prove that the diagram commutes up to H, that is, that

$$\lambda \circ \mu^{\overline{\mathcal{B}}}(\vec{\gamma}) + \mu_2^{\mathrm{mod}}(\overline{\lambda}_{\vec{\gamma}}, \xi) = \mu_1^{\mathrm{mod}}(H_{\vec{\gamma}}) + H_{\mu^{bar}_{0|1|0}(\vec{\gamma})}$$

for all $\vec{\gamma} \in F^N\overline{\mathcal{B}}(K,K)$. Let $l \geq 0$ and $\vec{X} = (X_0, \ldots, X_l)$ be a tuple of objects of $\mathcal{A}$. Consider $x_1 \otimes \cdots \otimes x_l \in \mathcal{A}(\vec{X})$, $y \in \mathcal{A}(X_l, K)$ and $\vec{\gamma} \in F^N\overline{\mathcal{B}}(K,K)$ as above. We compute the four terms in the equation above, applied to a composable element $x_1 \otimes \cdots \otimes x_l \otimes y$ as above: we have

$$\mu_2^{\mathrm{mod}}\left(\overline{\lambda}_{\vec{\gamma}}, \xi\right)_{l|1}(x_1, \ldots, x_l, y) = \sum_{j=0}^{l}\sum_{i=0}^{d} \mu_{j+d-i+2}\big(x_1, \ldots, x_j, \mu_{l-j+i+2}(x_{j+1}, \ldots, x_l, y, \gamma_1, a_1, \ldots, a_i), a_{i+1}, \ldots, a_d, \gamma_2\big)$$

and

$$\left(\lambda \circ \mu^{\overline{\mathcal{B}}}(\vec{\gamma})\right)_{l|1}(x_1,\dots,x_l,y) := \mu_{l+2}(x_1,\dots,x_l,y,\mu_{d+2}(\gamma_1,a_1,\dots,a_d,\gamma_2))$$

and

$$\begin{aligned}\left(H_{\mu^{\mathrm{bar}}_{0|1|0}(\vec{\gamma})}\right)_{l|1}(x_1,\dots,x_l,y) &= \sum_{i=0}^{d} \mu_{l+d-i+3}(x_1,\dots,x_l,y,\mu_{1+i}(\gamma_1,a_1,\dots,a_i),a_{i+1},\dots,a_d,\gamma_2)\\ &+\sum_{i\le j}\mu_{l+i+d-j+3}(x_1,\dots,x_l,y,\gamma_1,a_1,\dots,a_{i-1},\mu_{j-i+1}(a_i,\dots,a_j),\\ &\qquad a_{j+1},\dots,a_d,\gamma_2)\\ &+\sum_{i=1}^{d+1}\mu_{l+i+2}(x_1,\dots,x_l,y,\gamma_1,a_1,\dots,a_{i-1},\mu_{d-i+2}(a_i,\dots,a_d,\gamma_2))\end{aligned}$$

and

$$\begin{aligned}\mu_1^{\mathrm{mod}}(H_{\vec{\gamma}})_{l|1}(x_1,\dots,x_l,y) &= \sum_{i=0}^{l}\mu_{i+1}(x_1,\dots,x_i,(H_{\vec{\gamma}})_{l-i|1}(x_{i+1},\dots,x_l,y))\\ &+\sum_{i=0}^{l}(H_{\vec{\gamma}})_{i|1}(x_1,\dots,x_i,\mu_{l-i+1}(x_{i+1},\dots,x_l,y))\\ &+\sum_{i=0}^{l}\sum_{j=i+1}^{l}(H_{\vec{\gamma}})_{l-j+i|1}(x_1,\dots,x_i,\mu_{j-i}(x_{i+1},\dots,x_j),x_{j+1},\dots,x_l,y)\\ &=\sum_{i=0}^{l}\mu_{i+1}(x_1,\dots,x_i,\mu_{l-i+d+3}(x_{i+1},\dots,x_l,y,\gamma_1,a_1\dots,a_d,\gamma_2))\\ &+\sum_{i=0}^{l}\mu_{i+d+3}(x_1,\dots,x_i,\mu_{l-i+1}(x_{i+1},\dots,x_l,y),\gamma_1,a_1\dots,a_d,\gamma_2)\\ &+\sum_{i=0}^{l}\sum_{j=i+1}^{l}\mu_{l-j+i+d+3}(x_1,\dots,x_i,\mu_{j-i}(x_{i+1},\dots,x_j),x_{j+1},\dots,x_l,y,\\ &\qquad\gamma_1,a_1\dots,a_d,\gamma_2)\end{aligned}$$

The sum of the above terms is the $(d+l+2)$-th term in the A_∞-equations for $\mathcal{A}$, hence it vanishes. This proves the claim $\square$

Consider the identity $id_K \in \hom^0_{PD(\mathcal{A})}(K,K)$. It is straightforward to see that, in the notation of Lemma 3.3.2.4

$$\bar{d}'_{\text{r-int}}(K,U_K^N) \le \frac{1}{2}R(id_K,\xi\circ -).$$

The fact that (1) implies (3) is then a consequence of the following simple result.

LEMMA 3.3.2.5. *Consider a commutative square of persistence modules and maps*

$$\begin{tikzcd} V' \arrow[r, "h^V"] \arrow[d, "f'"'] & V \arrow[d, "f"] \\ W' \arrow[r, "h^W"] & W \end{tikzcd} \tag{62}$$

Then for any $w \in W_r$ we have

$$R(w,f) \leq \inf\left\{R(w',f') \mid r' \geq r,\ w' \in W'_{r'} \text{ such that } h^W_{r'}(w') = i^W_{r,r'}(w)\right\}.$$

Indeed, since Diagram (60) commutes and $\lambda([e_K]) = id_K$, we get

$$\bar{d'}_{\text{r-int}}(K, U_K^N) \leq \frac{1}{2}R(id_K, \xi \circ -) \leq \frac{1}{2}R\left([e_K], [\mu^{\overline{\mathcal{B}}}(K)]\right) \leq \frac{\alpha}{2}$$

which is point (3) in the statement of the proposition.

We prove that statement (1) implies (2). Consider the map $\mu^{\overline{\mathcal{B}}} \colon \overline{\mathcal{B}} \to \Delta_{\mathcal{A}}$ as a map of $(\mathcal{A},\mathcal{A})$-bimodules. Consider the $(\mathcal{A},\mathcal{A})$-bimodule $C_{\mathcal{B}} := \mathrm{Cone}(\mu^{\overline{\mathcal{B}}})$. Denote the A_∞-bimodule maps of $C_{\mathcal{B}}$ as $\mu^{C_{\mathcal{B}}}_{l|1|r}$ for $l, r \geq 0$. Let X_0, X_1 and X_2 be objects of $\mathcal{A}$ and consider $\vec{a} = a_1 \otimes \cdots \otimes a_d \in C_{\mathcal{B}}(X_0, X_1)$ and $\vec{b} = b_1 \otimes \cdots \otimes b_n \in C_{\mathcal{B}}(X_1, X_2)$. We define

$$\vec{a} \star \vec{b} := \sum_{j=0}^{n-1}\sum_{k=0}^{d-1} a_1 \otimes \cdots \otimes x_a \otimes \mu_{d-k+j}(a_{k+1}, \ldots, a_d, b_1, \ldots, b_j) \otimes b_{j+1} \otimes \cdots \otimes b_n \in C_{\mathcal{B}}(X_0, X_2).$$

It is straightforward to see that $\star$ satisfies the Leibniz rule with respect to $\mu^{C_{\mathcal{B}}}_{0|1|0}$.

We consider $C_{\mathcal{B}}(K,K)$, which is the cone of $\mu^{\overline{\mathcal{B}}}(K) \colon \overline{\mathcal{B}}(K,K) \to \mathcal{A}(K,K)$. Since $R(e_K, [\mu^{\overline{\mathcal{B}}}]) \leq \alpha$, there exists a cycle $\vec{h} \in \overline{\mathcal{B}}^{\leq \alpha}(K,K)$ such that $\mu^{\overline{\mathcal{B}}}(\vec{h})$ is a representative of the unit of the object K in $\mathcal{A}^{\leq 0}(K,K)$. More explicitly, we write

$$\mu^{\overline{\mathcal{B}}}(\vec{h}) = e_K + da_K$$

for some $a_K \in \mathcal{A}^{\leq 0}(K,K)$. We assume without loss of generality that $\vec{h} = h_1 \otimes \cdots \otimes h_n$, $n \geq 1$ is a pure tensor. We define a map $H \colon C_{\mathcal{B}}(K,K) \to C_{\mathcal{B}}(K,K)$ and prove it is a contracting homotopy. For a pure element $\vec{x} = x_1 \otimes \cdots \otimes x_d \in \mathrm{Cone}(\mu^{\overline{\mathcal{B}}})$, where $d \geq 1$, we define

$$H(\vec{x}) := \vec{x} \star (\vec{h} + a_K)$$

Then

$$\begin{aligned} \mu^{C_{\mathcal{B}}}_{0|1|0}(H(\vec{x}) + H(\mu^{C_{\mathcal{B}}}_{0|1|0}\vec{x}) =& \mu^{C_{\mathcal{B}}}_{0|1|0}(\vec{x} \star (\vec{h} + a_K)) + \mu^{C_{\mathcal{B}}}_{0|1|0}(\vec{x}) \star (\vec{h} + a_K) \\ =& \vec{x} \star \mu^{C_{\mathcal{B}}}_{0|1|0}(\vec{h} + a_K) = \vec{x} \star e_K = \vec{x} \end{aligned}$$

that is, H is indeed a contracting homotopy. It follows that $C_{\mathcal{B}}(K,K)$ is α-acyclic in $H^0(\mathcal{F}\mathbf{Ch})$, or, equivalently, that $\mu^{\mathcal{B}}(K)$ is an α-isomorphism, i.e. statement (2) in Proposition 3.3.2.1.

The fact that (2) implies (1) is quite straightforward: by Proposition 2.28(i) in [BCZ24b], there exists a right α-inverse

$$\psi \in \hom_{(H^0(\mathcal{F}\mathbf{Ch}))^0}(\Sigma^\alpha \mathcal{A}(K,K), \overline{\mathcal{B}}(K,K))$$

of $\mu^{\overline{\mathcal{B}}}$, that is a map such that $\mu^{\overline{\mathcal{B}}} \circ \psi = \eta_\alpha^{\mathcal{A}(K,K)}$, which in turn equals the identity seen as a map with shift α, so that the conclusion follows.

We prove that (3) implies (1). Suppose there is an iterated cone C_K over $\mathcal{B}$ in $PD(\mathcal{A})^0$ such that $\bar{d}_{\text{r-int}}(K, C_K) \le \alpha$. Pick maps $\varphi \in \hom^0_{PD(\mathcal{A})}(\Sigma^{r_1}K, C_K)$ and $\psi \in \hom_{PD(\mathcal{A})^0}(\Sigma^{r_2}C_K, K)$ such that

$$r_1 + r_2 = \alpha, \quad \text{and} \quad \psi \circ \Sigma^{r_2}\varphi = \eta_\alpha^K.$$

We represent C_K by a filtered twisted complex

$$\left(\bigoplus_{i=1}^n \Sigma^{\beta_i} X_i[d_i], (q_{ij})_{ij} \right)$$

where X_i is an element of $\mathcal{B}$ for all i. Let $f \in FTw^{\le 0}(\Sigma^{r_1}K, C_K)$ be a representative of φ and let $g \in FTw^{\le 0}(\Sigma^{r_2}C_K, K)$ be a representative of ψ. Then f can be written as a matrix $(f_1, \dots, f_n)$ where $f_i \in FTW^{\le r_1 - \beta_i}(K, X_i)[d_i]$. Similarly, g can be written as a matrix $(g_1, \dots, g_n)^T$ where $g_i \in FTW^{\le r_2 + \beta_i}(X_i, K)[-d_i]$. Since our A_∞-category $\mathcal{A}$ is strictly unital, we have that

$$e_K = \mu_2^{FTw}(f, g) = \sum_{i=1}^n \sum_{j=1}^n \sum_{i<i_1<\ldots<i_k<j} \mu_{2+k}(f_i, q_{ii_1}, \dots, q_{i_k j}, g_j).$$

Note that every tensor $f_i \otimes q_{ii_1} \otimes \ldots \otimes q_{i_k j} \otimes g_j$ is an element of $\overline{\mathcal{B}}^{\le \alpha}(K,K)$. This proves (1).

This completes the proof of Proposition 3.3.2.1. $\square$

3.3.2.2. *Perturbation data leading to filtered A_∞-Fukaya categories.* We recall here the choice of perturbation data introduced in [Amb25].

The construction of the Fukaya category as it appears in Seidel's book, [Sei08], does not produce, in general, a filtered category. Indeed, let p be a perturbation datum in the sense of Seidel. Let $\vec{L} := (L_0, \dots, L_d)$ be a tuple of Lagrangians in $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}$ such that any two subsequent Lagrangians (in cyclic order) are geometrically different. Let $\vec{\gamma} = (\gamma_1, \dots, \gamma_d) \in CF(\vec{L}; p)$ be a tuple made of generators of the Floer complexes (these are defined with respect to Hamiltonian perturbations prescribed by p) and let $\gamma_+ \in CF(L_0, L_d; p)$ be another generator. Let u be a perturbed J-holomorphic polygon, defined on some punctured disk S with d entries and one exit, that contributes to the γ_+-coefficient in the expression of $\mu_d(\gamma_1, \dots, \gamma_d)$. By definition, this u satisfies a perturbed pseudoholomorphic equation with a perturbation term corresponding to a 1-form K^p (see Section 8 in [Sei08]). Standard computations give us that the energy $E(u)$ of u satisfies

$$0 \leq E(u) \leq \omega(u) + \sum_{i=1}^{d} \int_0^1 H_p^{L_{i-1},L_i} \circ \gamma_i \, dt - \int_0^1 H_p^{L_0,L_d} \circ \gamma_+ \, dt + \int_S R^p \circ u$$

where $R^p \in \Omega^2(S, C^\infty(M))$ is the so-called curvature form of the perturbation datum K^p induced by p on the punctured disk S. In conformal coordinates (s,t) on S we can locally write

$$K^p = F ds + G dt$$

and the curvature term has then the form

$$R^p = \left(\partial_s G - \partial_t F + \omega\left(X^F, X^G\right)\right) ds \wedge dt$$

where X^F, X^G are the Hamiltonian vector fields induced by F and G. Writing $\mathbb{A}_p$ for the action functional defined with respect to p, we can rewrite the action-energy estimate above as

$$\mathbb{A}_p(T^{\omega(u)}\gamma_+) \leq \sum_{i=1}^{d} \mathbb{A}_p(\gamma_i) + C^p(u)$$

where $C^p(u) := \int_S R^p \circ u$, is called the *curvature term* of the polygon u associated with the choice of the perturbation data p. We formalize the conclusion of this computation as follows.

Negative curvature requirement:

$$\begin{gathered}\text{To construct filtration preserving maps that are defined by means of counts}\\ \text{of perturbed } J- \text{ holomorphic curves it is enough to show that we can pick}\\ \text{perturbation data such that the curvature terms admit a strictly negative}\\ \text{constant as an } \textit{apriori} \text{ uniform upper bound.}\end{gathered} \tag{63}$$

The requirement for a *stricly negative* upper bound is due to the fact that, for transversality reasons, we might need to introduce further, arbitrarily small perturbations.

We now outline, following [**Amb25**], the construction of the perturbation data $\mathcal{P}$ satisfying (63). This construction depends on a parameter $\frac{1}{2} < \delta < 1$ that we now fix.

We describe the choices made for $p \in \mathcal{P}$ and for a tuple $\vec{L}$ of cyclically different geometric Lagrangians as above. To start, consider a couple of different Lagrangians L_0, L_1 in $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}$. The Hamiltonian perturbation function $H_p^{L_0,L_1}$ assigned to this couple, and part of the perturbation data p, is subject to the condition

$$\delta\nu(p) < H_p^{L_0,L_1}(t,x) < \nu(p)$$

where ν is defined in equation (26). Consider now a triple $\vec{L} = (L_0, L_1, L_2)$ of Lagrangians in $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}$. Let S be the unique equivalence class of disks with three punctures, equipped with strip-like ends (two entries and one exit, see [**Sei08**, Section (8d)], [**Amb25**, Section

2.1]). The Hamiltonian perturbation $K_p^S \in \Omega^1(S, C_c^\infty(X \times [0,1]))$ on S is required to have a controlled behaviour on the strip-like ends near punctures, and moreover it essentially vanishes away from the strip-like ends. More precisely:

- on the ith negative strip-like end, $i = 1, 2$, K_p^S restricts to a monotone homotopy from the Hamiltonian Floer datum to the zero map, that is
$$K_p^S = (1 + \beta_p^{L_{i-1},L_i}(s+1))H^{L_{i-1},L_i}dt$$
with strip-like ends coordinate $(t, s) \in [0,1] \times (-\infty, 0]$, where $\beta_p^{L_{i-1},L_i} : \mathbb{R} \to [0,1]$ is a smooth, increasing and surjective function with derivative supported in $[0,1]$;
- on the positive strip-like end the Hamiltonian part K_p^S restricts to a monotone homotopy from the zero map to the Hamiltonian Floer datum, that is
$$K_p^S = \beta_p^{L_0,L_2}(s)H^{L_0,L_2}dt$$
with strip-like ends coordinate $(t, s) \in [0,1] \times [0, \infty)$, where $\beta_p^{L_0,L_2} : \mathbb{R} \to [0,1]$ is a smooth, increasing and surjective function with derivative supported in $[0,1]$;
- K_p^S vanishes away from strip-like ends.

Assume for a moment tranversality for K_p^S as above and consider a K_p^S-Floer polygon u joining Floer generators $\gamma_1 \in CF(L_0, L_1; p)$ and $\gamma_2 \in CF(L_1, L_2; p)$ to $\gamma_+ \in CF(L_0, L_2; p)$. Then the curvature term $C^p(S, u)$ of u is supported in the strip-like ends only and its a simple computation to see that
$$C^p(S, u) < \nu(p)(1 - 2\delta) < 0$$
as, by assumption, $\frac{1}{2} < \delta < 1$. It follows that the map μ_2 on $\vec{L}$ defined via K_p^S is filtered. Perturbation data for longer tuples of cyclically different Lagrangians is obtained by piecing together lower order choices (see [**Amb25**, Section 3.2]). Transversality is achieved by the recipe in [**Sei08**]. Given that $\nu(p)(1 - 2\delta)$ is negative (and not only non-positive), we can achieve both transversality and filtration preserving maps μ_d by considering small enough additional perturbations (see [**Amb25**, p. 479]). Extending this perturbation system to arbitrary tuples of Lagrangian (and not only cyclically different ones) is taken care by using clusters type moduli spaces mixing Floer polygons and Morse-pearly trajectories (as, for instance, in the pearl homology construction in [**BC09b**], or in [**CL06**], [**She16**]). Full details of this construction appear in [**Amb25**]. We remark that we work with the quantum model of $CF(L, L)$ in order to achieve (strict) units at vanishing filtration levels (cfr. [**Amb25**, Section 3.3]).

REMARK 3.3.2.6. In the construction outlined above it is possible to require, in addition, that if $L_0 \pitchfork L_1$ intersect transversally, then $H_p^{L_0,L_1} = H_p^{L_1,L_0}$ are constant. This is not a necessary assumption, but it makes computations simpler. Notice, however, that we cannot require Hamiltonian Floer data for transversally intersecting Lagrangians to vanish identically.

3.3.2.3. *The coproduct.* We recall that given an A_∞-category $\mathcal{A}$ we denote by $\Delta_{\mathcal{A}}$ its diagonal A_∞-bimodule. We fix a finite family $\mathcal{B}$ of Lagrangians in $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}$ such that any two distinct elements in $\mathcal{B}$ intersect transversally. We recall that $\mathcal{B}$ induces a finite full A_∞-subcategory $\mathcal{B}_p$ of $\mathcal{A}_p = \mathcal{F}uk(\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}; p)$ for any choice of perturbation datum $p \in \mathcal{P}$, as explained on page 126. We also fix an object $K \in \mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}$. The aim here is to refine the perturbative choices described in the last section and show the next result.

PROPOSITION 3.3.2.7. *For any $p \in \mathcal{P}$ and any object $K \in \mathcal{L}ag^{(mon,\mathbf{d})}$ there is a non-empty space of perturbation data $\mathcal{P}(p, \mathcal{B}, K)$ such that for each $q \in \mathcal{P}(p, \mathcal{B}, K)$ there exists a morphism of $(\mathcal{B}_p, \mathcal{B}_p)$-bimodules*

$$\Delta = \Delta_q^{\mathcal{B},K} \colon \Delta_{\mathcal{B}_p} \to \mathcal{Y}_p^l(K) \otimes \mathcal{Y}_p^r(K)$$

which shifts filtration by $\leq 2\nu(p)$ and extends the usual coproduct map. Moreover, in the case wher $\mathcal{B} = \{L\}$ consists of a single Lagrangian, $\Delta_q^{\mathcal{B}}$ is filtered.

We will often drop q, $\mathcal{B}$ and K from the notation of the coproduct and simply write it as Δ when there is no risk of confusion. The proof of Proposition 3.3.2.7 will be sketched in the remaining of this subsection.

REMARK 3.3.2.8. a. The perturbations in the class $\mathcal{P}(p, \mathcal{B}, K)$ are specific to the definition of the coproduct map Δ.

b. As it will be apparent from the proof, one could refine the choice of the Floer and perturbation data for elements in the finite family $\mathcal{B}$ and get a map of A_∞-bimodules Δ which shifts filtration by an arbitrarily small amount. Since this is not particularly useful for the aim of this paper, we do not explain the details here.

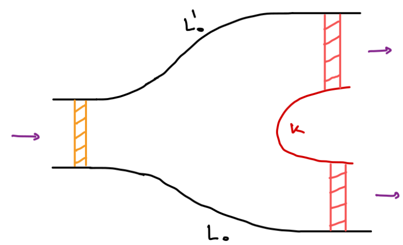
c. We remark that the fact that Δ shifts filtration by $\leq 2\nu(p)$ is equivalent to saying that the map $\eta_{2\nu(p)} \circ \Delta \colon \Delta_{\mathcal{B}_p} \to \Sigma^{-2\nu(p)}(\mathcal{Y}_p^l(K) \otimes \mathcal{Y}_p^r(K))$ is filtered. Here $\Sigma^{-2\nu(p)}(\mathcal{Y}_p^l(K) \otimes \mathcal{Y}_p^r(K))$ is the filtered bimodule with $(\Sigma^{-2\nu(p)}(\mathcal{Y}_p^l(K) \otimes \mathcal{Y}_p^r(K)))^{\leq \alpha} = (\mathcal{Y}_p^l(K) \otimes \mathcal{Y}_p^r(K))^{\leq \alpha + 2\nu(p)}$ and $\eta_{2\nu(p)}$ is the structural map from §3.1.1.2 (see also [**BCZ24b**, Section 2.2.3]).

PROOF. The construction of Δ depends on the construction of certain moduli spaces whose definition is well understood (see [**Abo10**], [**She16**]). Therefore, we will only sketch the construction and only emphasize the adjustments required to ensure that Δ has a controlled shift in action filtration. We sketch in Figure 5 some configurations of clusters contributing to the definition of Δ.

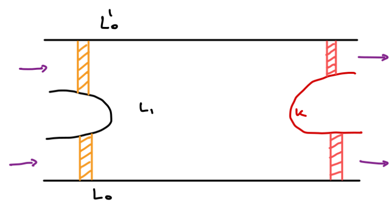
Let $\epsilon > 0$ and fix a regular filtered perturbation datum $p \in \mathcal{P}$ of size $\nu(p) = \epsilon$. Δ will consists of a family of linear maps

$$\Delta_{l|1|r} \colon CF(L_0, \ldots, L_l) \otimes CF(L_l, L_0') \otimes CF(L_0', \ldots, L_r') \to CF(L_0, K) \otimes CF(K, L_r')$$

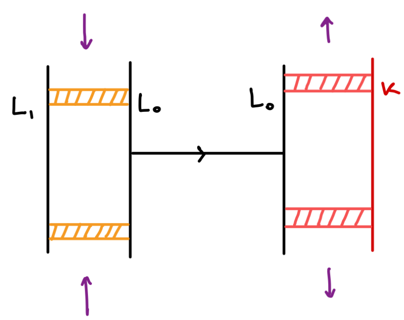
for any $l, r \geq 0$ and any two tuples $\vec{L} = (L_0, \ldots, L_l)$ and $\vec{L'} = (L_0', \ldots, L_r')$ of Lagrangians in $\mathcal{B}$, where all the Floer chain complexes are defined with respect to p. The source spaces for



(A) A possible configuration counted for $\Delta_{0|1|0}\colon CF(L_0, L_0') \to CF(L_0, K) \otimes CF(K, L_0')$.



(B) A possible configuration counted for $\Delta_{1|1|0}\colon CF(L_0, L_1) \otimes CF(L_1, L_0') \to CF(L_0, K) \otimes CF(K, L_0')$.



(C) A possible configuration counted for $\Delta_{1|1|0}\colon CF(L_0, L_1) \otimes CF(L_1, L_0) \to CF(L_0, K) \otimes CF(K, L_0)$. The horizontal line segment represents a finite Morse flowline on L_0.

FIGURE 5. Examples of configurations contributing to the coproduct $\Delta_q^{\mathcal{B},K}\colon \Delta_{\mathcal{B}_p} \to \mathcal{Y}_p^l(K) \otimes \mathcal{Y}_p^r(K)$. The colored strips indicate the support of the non-trivial parts of Hamiltonian perturbation data.

$\Delta_{l|1|r}$ are parametrized by moduli spaces of clusters (following the terminology in [**Amb25**]) $\mathcal{R}_C^{l,r,2}(\vec{L}, \vec{L}', K)$ labelled by $\vec{L}$, $\vec{L}'$ and K defined as follows:

(1) We start with the moduli space $\mathcal{R}^{l,r}(\vec{L}, \vec{L}', K) := \mathcal{R}^{l+r+3}(\vec{L} \cup \vec{L}' \cup K)$ of disks with $l + r + 3$ marked points, labelled by our fixed Lagrangians as follows: we choose a

marked point and label by K the first arc going clockwise from this point and then keep on labelling clockwise successive arcs first with $\vec{L}$, and then with $\vec{L'}$.

(2) Over $\mathcal{R}^{l,r}(\vec{L},\vec{L'},K)$ we have a bundle

$$\pi^{l,r}(\vec{L},\vec{L'},K)\colon \mathcal{S}^{l,r}(\vec{L},\vec{L'},K) \to \mathcal{R}^{l,r}(\vec{L},\vec{L'},K)$$

such that given $r \in \mathcal{R}^{l,r}(\vec{L},\vec{L'},K)$ its preimage is the standard unit disks with labels and prescribed marked points but with the marked points where two arcs corresponding to geometrically different Lagrangians meet replaced by a puncture.

(3) We endow $\mathcal{R}^{l,r}(\vec{L},\vec{L'},K)$ with strip-like ends in the usual manner: near marked points adjacent to the arc labelled by K, we choose positive strip-like ends, while on the remaining $l+r+1$ marked points we choose negative ones (this choice will be required to be consistent with gluing and breaking of disks, as usual).

(4) We consider the compactification $\overline{\mathcal{R}^{l,r}(\vec{L},\vec{L'},K)}$ of $\mathcal{R}^{l,r}(\vec{L},\vec{L'},K)$ which has the structure of a manifold with corners of dimension $l+r+1$, and add collar neighbourhoods to some boundary components as [**Amb25**, p. 14] to get the needed moduli space $\mathcal{R}^{l,r}_C(\vec{L},\vec{L'},K)$

(5) Over this moduli space we define the bundle

$$\pi^{l,r}_C(\vec{L},\vec{L'},K)\colon \mathcal{S}^{l,r}_C(\vec{L},\vec{L'},K) \to \mathcal{R}^{l,r}_C(\vec{L},\vec{L'},K)$$

where fibers are cluster of marked disks with punctures with $l+r+1$ entries and two (adjacent) exits.

Similarly to the case of filtered A_∞-categories (sketched in §3.3.2.2 and worked out in [**Amb25**]), there is a space $\mathcal{P}^\Delta(p,\mathcal{B},K)$ of universal choices of perturbation data on the family of bundles of clusters $\pi^{l,r}_C(\vec{L},\vec{L'},K)$ (with fixed K), and an associated compatibility requirement with the fixed choice of perturbation datum $p \in \mathcal{P}$.

Given a choice of $q \in \mathcal{P}^\Delta(p,\mathcal{B},K)$, of tuples $\vec{L}$ and $\vec{L'}$ as above, and of generators

$$\gamma_0 \otimes \cdots \otimes \gamma_l \otimes \gamma \otimes \gamma'_0 \otimes \cdots \otimes \gamma'_r \in CF(L_0,\ldots,L_l) \otimes CF(L_l,L'_0) \otimes CF(L'_0,\ldots,L'_r)$$

and

$$\gamma_1^+ \otimes \gamma_2^+ \in CF(L_0,K) \otimes CF(K,L'_r)$$

we have the moduli spaces

$$\mathcal{M}^{l+r+1}_{\vec{L},\vec{L'},K}(\gamma_0,\ldots,\gamma_l;\gamma;\gamma'_1,\ldots,\gamma_r;\gamma_1^+,\gamma_2^+;q)$$

of Morse-Floer clusters u defined for the perturbation datum q, joining such generators. One then defines $\Delta_{l|1|r}$ by counting rigid clusters u in these moduli spaces, weighted by their symplectic area (by multiplication with $T^{\omega(u)}$, with T the Novikov ring variable). Standard compactness and transversality arguments then show that for a generic choice of $q \in \mathcal{P}^\Delta(p,\mathcal{B},K)$ these maps fit together into a morphism of A_∞-bimodules

$$\Delta\colon \Delta_{\mathcal{B}_p} \to \mathcal{Y}^l_p(K) \otimes \mathcal{Y}^r_p(K)$$

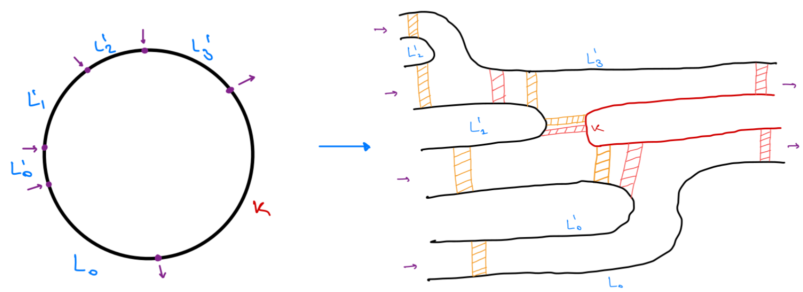


FIGURE 6. A curve contributing to $\Delta_{0|1|3}\colon CF(L_0, L_0', L_1', L_2', L_3') \to CF(L_0, K) \otimes CF(K, L_3')$ whose source space lies near a corner point in $\overline{\mathcal{R}_C^{0,3}(\vec{L}, \vec{L}', K)}$. The orange strips represent the region where perturbation data contribute negatively to the curvature, while the red strips where it contributes positively. The total curvature is then bounded from above by $2\nu(p)$.

which, of course, depends on the choice of the perturbation datum q. Most importantly, the filtered properties of Δ strongly depend on the choice of q: for an arbitrary choice of q there is no reason to expect that the associated morphism Δ has a controlled shift in filtration. However, the same idea introduced in [**Amb25**] (and outlined in §3.3.2.2) for the case of the μ_d-maps can be replicated here to control the shift of the map Δ. Notice that the term $\Delta_{0|1|0}\colon CF(L, L) \to CF(L, K) \otimes CF(K, L)$ counts Floer curves with a Morse input (recall that we defined $CF(L, L)$ using the pearly model) and two Floer outputs (as $K \notin \mathcal{B}$ by assumption), which do not pose any problem from a filtered point of view. On the other hand, we encounter some novel problems relative to the constructions described previously when dealing with two different Lagrangians $L_0 \neq L_0'$ and the map $\Delta_{0|1|0}\colon CF(L_0, L_0') \to CF(L_0, K) \otimes CF(K, L_0')$. In this case, the methods from [**Amb25**] give a map of shift $\leq 2\nu(p) - \delta\nu(p) \leq 2\nu(p)$. Since the perturbation data for higher order components $\Delta_{l|1|r}$ of Δ are constructed –following [**Amb25**]– inductively, this shifts propagates. It is important to remark that given the structure of the moduli spaces defining $\Delta_{l|1|r}$ this shift does not grow proportionally to the order, but remains $\leq 2\nu(p)$. Indeed near corners of $\overline{\mathcal{R}_C^{l,r}(\vec{L}, \vec{L}', K)}$, the inductively constructed perturbation data come from gluing of perturbation data associated with configurations with ≤ 3 marked points, of which at most one is a configuration with one entry and two exit (that is, on which $\Delta_{0|1|0}$ is modeled on) and contributes positively to the curvature (and the contribution is bounded above by $2\nu(p)$ as explained above); all the other configuration involved in the gluing have only one output, and do not contribute to the curvature (see Figure 6). □

REMARK 3.3.2.9. It is clear from the proof of Proposition 3.3.2.7 that the shift $\leq 2\nu(p)$ of the map Δ is not sharp for our choices of perturbations (indeed, $2\nu(p) - \delta\nu(p)$ can be used as an upper bound). However, we decided to stick to $2\nu(p)$ for notational ease. Since in Theorem 3.3.1.1 we work with a system of A_∞-categories, this difference is irrelevant.

3.3.2.4. *Ambient quantum cohomology.* The material briefly recalled here is well-known (see [**She16**] for a possible reference). Let (X, ω) be a closed and monotone symplectic manifold. We consider a Morse function $f\colon X \to \mathbb{R}$ with a unique maximum $u \in X$ and a Riemannian Metric g on X such that the pair (f, g) is Morse-Smale. We define the quantum cochain complex $CQ^*(f, g; \Lambda)$ with Λ coefficients associated with the pair (f, g) as its Morse cochain complex over Λ, that is coindexed by the Morse index. The differential d counts negative gradient flow lines connecting critical points with index difference equal to one. In the following, we will denote $CQ(f, g; \Lambda)$ by $CQ(X; \Lambda)$ when the specific choice of Morse-Smale data is not important for our discussion. We filter this complex by setting every generator at zero action level and using the standard filtration on Λ. The homology of this complex is denoted by $QH(X; \Lambda)$ and is called the quantum cohomology of X. Note that neither the vector space nor the persistence structure of $QH(X; \Lambda)$ depends on the choice of the Morse-Smale pair. We endow $\mathrm{QH}(X, \Lambda)$ with the standard $\mathbb{R}/2\mathbb{Z}$ grading, with the Novikov variable having degree 0.

The vector space $QH(X; \Lambda)$ is endowed with a product which deforms the usual Morse product, denoted by $*$ and called the quantum product. This product is defined at the chain level by counting (and weighting in the Novikov coefficient by the symplectic area of) J-holomorphic spheres with three marked points, which lie in stable and unstable manifolds of perturbations of the Morse function f^M, much like in the definition of the A_∞-maps for tuples of identical Lagrangians in the definition of the filtered Fukaya category. The quantum product with the first Chern class $c_1(X)$ defines an endomorphism of $QH(X; \Lambda)$ and hence we have a decomposition

$$QH^*(X; \Lambda) = \bigoplus_{\mathbf{d}} QH^*(X; \Lambda)_{\mathbf{d}}$$

where the index corresponds to eigenvalues $\mathbf{d} \in \Lambda$ of this endomorphism, and $QH(X, \Lambda)_{\mathbf{d}}$ is the eigenspace associated with the eigenvalue $\mathbf{d} \in \Lambda$. It is well known that the Fukaya category associated with the class $\mathcal{L}\mathrm{ag}^{(\mathrm{mon}, \mathbf{d})}$ is non-trivial if and only if $\mathbf{d}$ is an eigenvalue of the map above. Moreover, all the eigenvalues $\mathbf{d}$ in fact lie in Λ_0.

3.3.2.5. *The persistence open-closed and closed-open maps.* In this section we first define a persistence version of the open-closed map OC relating Fukaya categories with ambient quantum cohomology. This follows standard constructions in the subject (cfr. [**Abo10, She16**]) therefore we only give details sufficient to justify the persistence aspects.

Let $\mathcal{B} \subset \mathcal{L}\mathrm{ag}^{(\mathrm{mon}, \mathbf{d})}$ be a finite subset. Let $\epsilon > 0$ and fix a regular filtered perturbation datum $p \in \mathcal{P}$ of size $\nu(p) = \epsilon$.

We now outline the construction of the persistence open-closed map

$$OC = OC^{\mathcal{B}}_p \colon PHH(\mathcal{B}_p) \to QH(X, \Lambda) \tag{64}$$

where $\mathcal{B}_p$ is the full subcategory of $\mathcal{F}uk(\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}; p)$ induced by $\mathcal{B}$, and $PHH(\mathcal{B}_p)$ stands for the Hochschild homology of $\mathcal{B}_p$ with coefficients in the diagonal bimodule of $\mathcal{B}_p$ (see Appendix .1.4).

Recall that OC shifts degree up by $n \mod 2$. At the chain level, OC consists of a family of linear maps

$$OC_d \colon CF(L_0, L_1) \otimes \cdots \otimes CF(L_{d-1}, L_d) \otimes CF(L_d, L_0) \to CQ(X, \Lambda)$$

for each tuple $\vec{L} = (L_0, \ldots, L_d)$ of Lagrangians in $\mathcal{B}$, where all the Floer chain complexes are defined with respect to the perturbation data p. The construction of these maps is based on moduli spaces and corresponding choices of perturbation data that follows the scheme used to define the filtered A_∞ operations μ_d in §3.3.2.2 with a few modifications that we now outline.

- The perturbed J-holomorphic polygons u used to define OC_d only have negative strip-like ends near the punctures (by contrast, in the μ_d case, there is one positive strip-like end).
- There is an interior marked point x_u in the interior of the domain of u.
- There is a fixed Morse smale pair (f^X, g^X) on X as in the definition $QH(X, \Lambda)$, and an ω-compatible almost complex structure J on X as needed to define the quantum product on $QH(X, \Lambda)$. The equation satisfied by u is J-holomorphic (without Hamiltonian perturbations) in a small neighbourhood of x_u.
- Given

  $$\gamma_1 \otimes \cdots \otimes \gamma_{d+1} \in CF(L_0, L_1) \otimes \cdots \otimes CF(L_d, L_0)$$

  and a critical point x of f^X, there are moduli spaces

  $$\mathcal{M}^{d+1,1}_{\vec{L}}(\gamma_1, \ldots, \gamma_{d+1}; x; q)$$

  defined just like in the definition of the μ_d's except that the output condition is replaced by the requirement that $u(x_u)$ is carried by a trajectory of $-\nabla_{g^X} f^X$ to x.
- OC_d is defined by counting rigid curves as in the moduli space above. Of course, these moduli spaces are defined with respect to regular perturbation data that is compatible in the obvious sense with p.

In this case (because there are no positive-strip-like ends), the arguments showing that the A_∞ operations from §3.3.2.2 are filtered apply directly. As a result the OC map described above induces a morphism of persistence modules as in (64). Moreover, we also have the following result whose proof is completely analogous to [She16, Corollary 2.11].

Lemma 3.3.2.10. *For any $\mathcal{B}$ and any $p \in \mathcal{P}$, image$(OC^{\mathcal{B}}_p) \subset QH(X, \Lambda)_{\mathbf{d}}$, where $\mathbf{d} \in \Lambda_0$ is the parameter prescribing our class of monotone Lagrangians.*

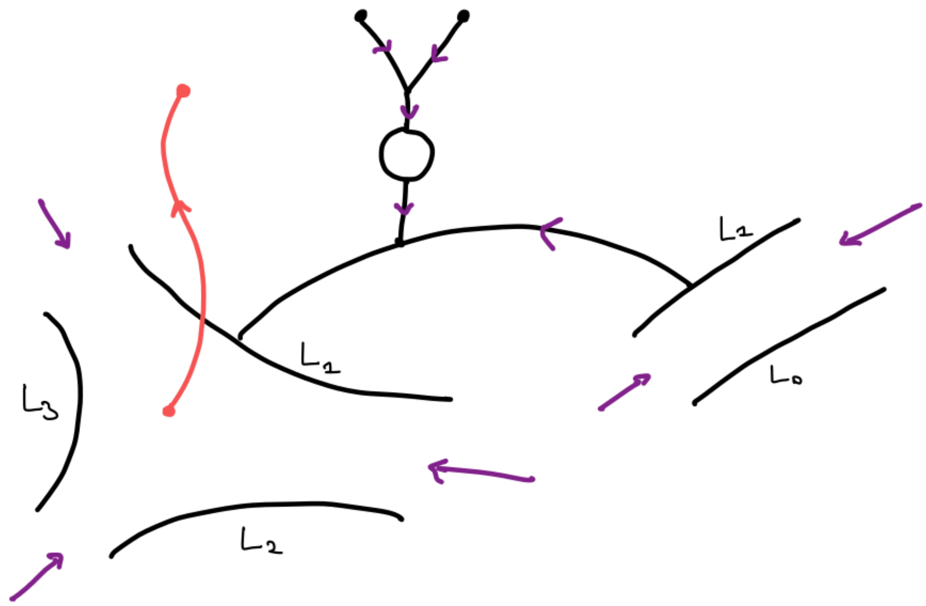


FIGURE 7. A curve counted in the definition of OC_7 for the tuple $\vec{L} = (L_0, L_1, L_1, L_1, L_3, L_2, L_1, L_0)$.

We end this section by briefly recalling from [**BC09b**] the definition of the linear part of the closed-open map. Let $p \in \mathcal{P}$ and consider the ω-compatible almost complex structure J_p^K-prescribed by p for $CF(K,K)$. Then $CO_p^K \colon CQ(X,\Lambda) \to CF(K,K)$ is defined by counting rigid J_p^K-pearly trajectories starting at a critical point of f_p^X and ending at a critical point of f_p^K (see [**BC09a**, Figure 3] with $x = e_K$). Since there is no Hamiltonian perturbation involved, it is straightforward to see that CO_p^K induces a persistence map

$$CO = CO_p^K \colon QH(X,\Lambda) \to HF(K,K). \tag{65}$$

Recall (see [**She16**, Proposition 2.9]) that CO restricted to $QH(X,\Lambda)_{\mathbf{d}'}$ vanishes unless $\mathbf{d}' = \mathbf{d}$. We recall that CO is unital in the sense that it sends the projection to $QH(X,\Lambda)_{\mathbf{d}}$ of the unit to the unit in $HF(K,K)$.

REMARK 3.3.2.11. It is possible to define a version of the richer, more general, (non-linear) closed-open map, that is a persistence map

$$CO_p^{\mathcal{B}} \colon QH^*(X;\Lambda) \to PHH^*(\mathcal{B}_p),$$

where $PHH^*(\mathcal{B}_p)$ denotes the persistence Hochschild cohomology of the A_∞-category $\mathcal{B}_p$, but this goes beyond the scope of this paper.

3.3.2.6. *Abouzaid retract-approximability criterion for a fixed choice of perturbations.* Consider a finite family $\mathcal{B} \subset \mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}$ of Lagrangians and an element $\mathcal{K} \in \mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})} \setminus \mathcal{B}$. We recall the class $\mathcal{P}(\mathcal{B}) \subset \mathcal{P}$ of perturbations adapted to $\mathcal{B}$ defined in §3.3.2.3. Let $p \in \mathcal{P}(\mathcal{B})$. The coproduct $\Delta_p \colon \Delta_{\mathcal{B}_p} \to \mathcal{Y}^l_p(K) \otimes \mathcal{Y}^r_p(K)$ constructed as in §3.3.2.3 (where we often dropped the subscript p from the notation) induces a filtered chain map, which we still denote by Δ_p,

$$\Delta_p \colon CC_*(\mathcal{B}_p) \to \Sigma^{-2\nu(p)} CC_*(\mathcal{B}_p, \mathcal{Y}^l(K) \otimes \mathcal{Y}^r(K))$$

(see [**Gan12**, Equation 2.178]) defined by

$$\Delta_p(\gamma_1 \otimes \cdots \otimes \gamma_{d+1}) := \sum_{i=0}^{d+1} \sum_{j=i}^{d+1} (\Delta_p)_{d-j|1|i}(\gamma_{j+1}, \ldots, \gamma_{d+1}, \gamma_1, \ldots, \gamma_i) \otimes \gamma_{i+1} \otimes \cdots \otimes \gamma_j.$$

Consider the filtered chain isomorphism $\varphi : CC_*(\mathcal{B}_p, \mathcal{Y}^l(K) \otimes \mathcal{Y}^r(K)) \cong \overline{\mathcal{B}_p}(K, K)$ given by rotating pure tensors once to the left. It is defined on a pure tensor

$$a \otimes b \otimes \gamma_1 \otimes \cdots \otimes \gamma_n \in CC_*(\mathcal{B}_p, \mathcal{Y}^l(K) \otimes \mathcal{Y}^r(K))$$

as

$$\varphi(a \otimes b \otimes \gamma_1 \otimes \cdots \otimes \gamma_n) := b \otimes \gamma_1 \otimes \cdots \otimes \gamma_n \otimes a.$$

In the following we will consider the composition

$$(\Sigma^{-2\nu(p)} \varphi) \circ \Delta_p \colon CC_*(\mathcal{B}_p) \to \Sigma^{-2\nu(p)} \overline{\mathcal{B}_p}(K, K),$$

and, by a slight abuse of notation, still denote this by Δ_p.

Proposition 3.3.2.12. *Let $p \in \mathcal{P}$. Then there is a commutative diagram of persistence modules and maps*

$$\begin{tikzcd} PHH(\mathcal{B}_p) \arrow[r, "\Delta_p"] \arrow[d, "OC_p^{\mathcal{B}}"'] & \Sigma^{-2\nu(p)} H(\overline{\mathcal{B}_p}(K, K)) \arrow[d, "\Sigma^{-2\nu(p)} \mu^{\overline{\mathcal{B}_p}}"] \\ QH(X, \Lambda) \arrow[r, "\eta_{2\nu(p)} \circ CO_p^K"] & \Sigma^{-2\nu(p)} HF(K, K) \end{tikzcd} \tag{66}$$

where $\eta_{2\nu(p)} \colon HF(K, K) \to \Sigma^{-2\nu(p)} HF(K, K)$ is the standard map induced by the structural maps of the persistence module $HF(K, K)$ as in [**BCZ24b**, Section 2.2.4].

Consider the measurement $R(-, -)$ introduced in (59) and denote by $u_{\mathbf{d}}$ the projection of the unit in $QH(X, \Lambda)$ to $QH(X, \Lambda)_{\mathbf{d}}$.

Corollary 3.3.2.13. *Let $p \in \mathcal{P}$. If $R(u_{\mathbf{d}}, OC_p^{\mathcal{B}}) \leq \alpha$, then $\mathcal{L}ag^{(mon, \mathbf{d})}$ is $\frac{\alpha}{2} + \nu(p)$-retract approximable in $PD(\mathcal{A}_p)$ by the family $\mathcal{B}$.*

The proof of the Corollary will make use of the following simple result.

Lemma 3.3.2.14. *Let $f \colon V \to W$ be a map of persistence modules and consider $w \in W_r$. Given $\delta > 0$ we have*

$$R(w, f) \leq R(i_{r, r+\delta}(w), \Sigma^{-\delta} f) + \delta.$$

PROOF. Consider $\Sigma^{-\delta}f\colon \Sigma^{-\delta}V \to \Sigma^{-\delta}W$ and $i^W_{r,r+\delta}(w) \in (\Sigma^{-\delta}W)_r$. Let $s \geq r$ such that there is $v \in (\Sigma^{-\delta}V)_s$ with $(\Sigma^{-\delta}f)_s(v) = i^{\Sigma^{-\delta}W}_{r,s}(i^W_{r,r+\delta}(w))$. Since $i^{\Sigma^{-\delta}W}_{r,s} = i^W_{r+\delta,s+\delta}$ and $(\Sigma^{-\delta}V)_s = V_{s+\delta}$ the above is equivalent to $f_{s+\delta}(v) = i^W_{r,s+\delta}(w)$. In particular, $s+\delta \geq R(w,f)$. $\square$

PROOF OF COROLLARY 3.3.2.13. Assume that $R(u, OC^{\mathcal{B}}_p) \leq \alpha$. By Lemma 3.3.2.5, Proposition 3.3.2.12, Lemma 3.3.2.14 and the fact that CO is unital, we directly get $R([e_K],[\mu^{\overline{\mathcal{B}}}]) \leq \alpha + \nu(p)$ for all $K \in \mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}$. Using Proposition 3.3.2.1 this implies the claim. $\square$

PROOF OF PROPOSITION 3.3.2.12. The proof is essentially a combination of the proofs of Theorem 8.1.1 in [BC09a] and Lemma 2.15 in [She16] while keeping track of the changes in filtration.

We outline the construction of a chain homotopy $H\colon CC_*(\mathcal{B}_p) \to \Sigma^{-2\nu(p)}CF(K,K)$ that makes Diagram (66) commutative. Given a tuple $\vec{L} := (L_0,\dots,L_d)$ of elements chosen from the family $\mathcal{B}$, we intend to define

$$H\colon CF(L_0,\dots,L_d,L_0) \to \Sigma^{-2\nu(p)}CF(K,K).$$

This again consists of three steps: picking the correct moduli space (this part is already present in the literature); adjust the spaces of perturbations conveniently so that the construction is compatible with filtrations (following the methods from [Amb25]); ensure that the energy bounds are indeed as required so that the resulting chain homotopy preserves filtrations.

The source spaces for this H are parametrized by a moduli space denoted $\mathcal{R}^{d,A}_C(\vec{L},K)$ (the superscript A indicates that *annuli* appear in the domain) labelled along the boundary by $\vec{L}$ and K and defined as follows:

(1) We consider the moduli space $\mathcal{R}^{d,A}(\vec{L},K)$ of annular configurations consisting of an annulus $S^1 \times [1,\rho] \subset \mathbb{C}$ of unit internal radius and with outside radius $\rho \in (1,\infty)$. We fix marked points $z_0 = 1$, $z_1 = -\rho$ as well as $d-1$ other marked points $z_2,\dots,z_d$, $|z_i| = \rho,\ 1 \leq i \leq d$ along the exterior boundary of the annulus, ordered in clockwise direction starting from z_1. We label the arcs on the exterior circle clockwise by $\vec{L}$ starting from x_1, and we label the internal circle by K.

(2) Over $\mathcal{R}^{d,A}(\vec{L},K)$ we have a bundle
$$\pi^{d,A}(\vec{L},K)\colon \mathcal{S}^{d,A}(\vec{L},K) \to \mathcal{R}^{d,A}(\vec{L},K)$$
such that the fiber of a point in $\mathcal{R}^{d,A}(\vec{L},K)$ is a boundary-punctured annulus with the interior circle of radius one, the exterior circle of radius some $\rho \in (1,\infty)$ and such that the marked points on the exterior circle that are adjacent to two geometrically distinct Lagrangian labels are replaced with punctures.

(3) We endow $\pi^{d,A}(\vec{L},K)$ with a compatible choice of strip-like ends, all of negative type, associated with each of the punctures at the previous point.

(4) We consider the compactification $\overline{\mathcal{R}^{d,A}_C(\vec{L},K)}$ of $\mathcal{R}^{d,A}_C(\vec{L},K)$. This moduli space of clusters mixes flow lines, disks, and polygons, just as in the definition of the operations

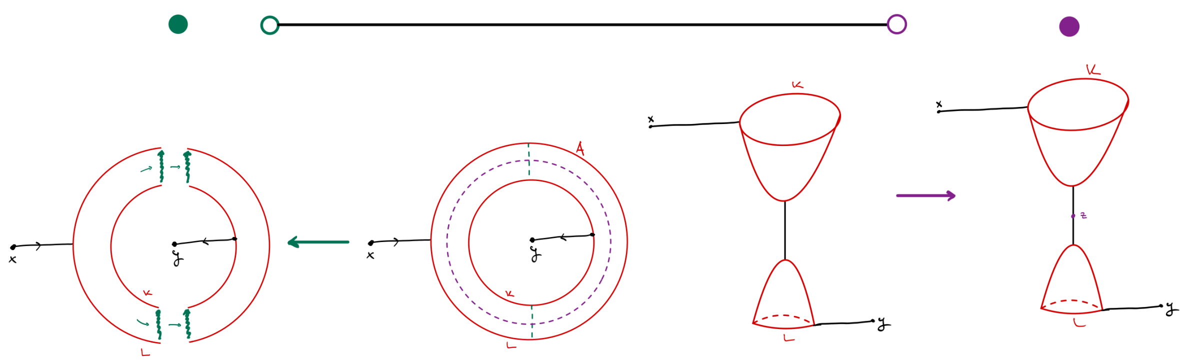


FIGURE 8. A schematic description of the interior of $\overline{\mathcal{R}_C^{1,A}(L,K)}$, isomorphic to a closed interval, and the fibers over it. In the middle, given Lagrangians L and K we see configurations contributing to the y-summand (where $y \in \mathrm{Crit}(f_K)$) of $H(x)$ (where $x \in \mathrm{Crit}(f_L)$). On the left (green dot) we see the configuration contributing to $\mu_2 \circ \Delta$, while on the right (purple dot) a configuration contributing to $OC \circ CO$ (breaking at a critical point z of f_X).

μ_d from §3.3.2.2, but also has one annular component. The boundary of $\overline{\mathcal{R}_C^{d,A}(\vec{L},K)}$ in the case $d=1$ is drawn in Figure 8.

(5) There are incidence conditions for each of the marked points and punctures. The punctures correspond to generators of the respective Floer complexes $CF(L_i, L_{i+1}; p)$ (with p the perturbation data); the marked points on the exterior circle (they are so that the adjacent edges have the same label, L_j, for some j) are mapped to critical points of the fixed Morse function on L_j that is part of p; the marked point z_o is mapped to a critical point of the fixed Morse function on K, which is part of the perturbation data p. Perturbation data for all of this is picked such that it is compatible with $p \in \mathcal{P}$ and are constructed as in [**Amb25**].

The construction sketched above appeared before in the literature - see [**She16**] as well as [**BC09a**] - and the structure of the compactification $\overline{\mathcal{R}_C^{d,A}(\vec{L},K)}$ implies that it is a chain homotopy as desired. From the point of view of the current paper the only additional point of interest is that no further modifications are needed to ensure that this homotopy is filtration preserving as a map to $\Sigma^{-2\nu(p)}CF(K,K)$ (that is, it shifts filtration by $\leq 2\nu(p)$ as a map to $CF(K,K)$).

□

REMARK 3.3.2.15. a. The statement of Proposition 3.3.2.12 contains an intrinsic limitation in the sense that for a fixed perturbation p it allows to discuss retract-approximability with accuracy bounded below by a constant times $\nu(p)$. Gaining adequate control on the structures associated with a p that varies so that the accuracy level gets below some fixed ϵ was the main motivation behind the notion of systems of categories with increasing accuracy in §3.1.5.

b. We could refine the statement of Proposition 3.3.2.12 and avoid the shift $\Sigma^{-2\nu(p)}$ in the bottom right of Diagram (66). This can be done by choosing more elaborate perturbation data for the maps appearing in the diagram. This would have the result of allowing for a slightly better approximation accuracy given a fixed perturbation datum p, but the remark above would continue to apply.

b. Note that when $\mathcal{B}$ consists of a single Lagrangian L, then $R(u_{\mathbf{d}}, OC_p^{\mathcal{B}}) \leq \alpha$ implies that $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}$ is $\frac{\alpha}{2}$-retract approximated in $PD(\mathcal{A}_p)$ by the family $\mathcal{B}$. That is, there is no $\nu(p)$ term appearing in this case, in contrast to the general statement in Corollary 3.3.2.13. The reason is that, in this case, the map Δ is filtered (see Proposition 3.3.2.7); the proof uses a version of Proposition 3.3.2.6 where no shift functor $\Sigma^{-2\nu(p)}$ appears. The only place where this refinement will be used is Proposition 3.3.3.1.

3.3.2.7. *Colimit of open-closed maps.* In this section we prove that the open-closed map behaves well with respect to continuation functors and define the colimit open-closed map that appears in the statement of Theorem 3.3.1.1.

Consider the system

$$\widehat{\mathcal{F}uk}(\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}) = \left\{ \left\{ \mathcal{F}uk(\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}; p) \right\}_{p \in \mathcal{P}}, \{\mathcal{H}_{p,q}, \mathcal{H}_{q,p}\}_{p \preceq q} \right\}$$

of filtered Fukaya categories associated with $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}$. As explained in §3.1.6.6, this is in fact a homotopy system in the sense of §3.1.5.8. As a result of that, via Proposition .1.6.1, we get a directed system

$$\left\{ \left\{ PHH(\mathcal{F}uk(\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}; p)) \right\}_{p \in \mathcal{P}}, \{\mathcal{H}_{p,q}^{PHH}\}_{p \preceq q} \right\}$$

of persistence modules, where $\mathcal{H}_{p,q}^{PHH}$ is the persistence map induced in persistence Hochschild homology by the (family of) continuation functors $\mathcal{H}_{p,q}$ ($\mathcal{H}_{p,q}^{PHH}$ is uniquely defined by Proposition .1.6.1). In particular, the colimit

$$PHH(\widehat{\mathcal{F}uk}) := \underrightarrow{\mathrm{colim}}_{p \in \mathcal{P}} PHH(\mathcal{F}uk(\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}; p))$$

is well-defined and a persistence module, as explained in §3.1.5.9. Moreover, since the system $\widehat{\mathcal{F}uk}(\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})})$ has, in particular, a fixed full base of objects (see §3.1.5.7), the following is true: given a subfamily $\mathcal{B} \subset \mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}$ we get a colimit

$$PHH(\widehat{\mathcal{B}}) := \underrightarrow{\mathrm{colim}}_{p \in \mathcal{P}} PHH(\mathcal{B}_p)$$

where $\mathcal{B}_p$ is the full A_∞-subcategory of $\mathcal{F}uk(\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}; p)$ induced by $\mathcal{B}$ and $\widehat{\mathcal{B}}$ is the subsystem of A_∞-categories of $\widehat{\mathcal{F}uk}$ induced by $\{\mathcal{B}_p\}_{p\in\mathcal{P}}$.

For any $N \geq 1$, we denote by $F^N PHH(\mathcal{B}_p)$ the persistence homology of $F^N CC(\mathcal{B})$, that is, the N-th level in the length filtration of the Hochschild chain complex (see §1.4).
Let $p \preceq q$ and consider the continuation A_∞-functor $\mathcal{H}_{q,p}\colon \mathcal{B}_q \to \mathcal{B}_p$ restricted to $\mathcal{B}_q$. We recall (see §3.1.6.6, [**Amb25**]) that these functors have linear deviation rate $\leq 2(\nu(p)-\nu(q))$. We write $s(q,p) := 2(\nu(p)-\nu(q))$. In particular, $\mathcal{H}_{q,p}$ does *not* induce a map in persistence Hochschild homology, as the shift in filtration at the chain level blows up as $N \to \infty$. To overcome this problem, we use the length filtration on $CC(\mathcal{B}_q)$. Note that, at the chain level, $\mathcal{H}_{q,p}$ induces for any $N \geq 1$ a chain map $F^N CC(\mathcal{B}_q) \to F^N CC(\mathcal{B}_q)$ of shift $\leq N s(q,p)$. In particular, by precomposing and postcomposing with structural maps η associated with the shift functor (see §3.1.1), $\mathcal{H}_{q,p}$ induces a persistence map

$$\mathcal{H}_{q,p}^{F^N PHH}\colon \Sigma^{-2N\nu(q)} F^N PHH(\mathcal{B}_q) \to \Sigma^{-2N\nu(p)} F^N PHH(\mathcal{B}_p).$$

The next lemma lemma follows directly from the properties of the continuation functors.

LEMMA 3.3.2.16. *Let $p \in \mathcal{P}$. The family*

$$\left\{\mathcal{H}_{q,p}^{F^N PHH}\colon \Sigma^{-2N\nu(q)} F^N PHH(\mathcal{B}_q) \to \Sigma^{-2N\nu(p)} F^N PHH(\mathcal{B}_p)\right\}_{q\in\mathcal{P},\ p\preceq q}$$

defines a homomorphism of directed systems of persistence modules.

Exploiting again the fact that the length filtration is preserved by maps induced on Hochschild chains by A_∞-functors, we define

$$F^N PHH(\widehat{\mathcal{B}}) := \varinjlim_{p\in\mathcal{P}} F^N PHH(\mathcal{B}_p)$$

for any $N \geq 1$.

As a result of Lemma 3.3.2.16 and of the fact that $\nu(q) \to 0$ as $q \to \infty$, we get a unique map

$$H_{\infty,p}^{F^N PHH}\colon F^N PHH(\widehat{\mathcal{B}}) \to \Sigma^{-2N\nu(p)} F^N PHH(\mathcal{B}_p)$$

for any $p \in \mathcal{P}$, such that the diagram

$$\begin{tikzcd} \Sigma^{-2N\nu(q)} PHH(\mathcal{B}_q) \arrow[rr] \arrow[dr, "\mathcal{H}_{q,p}^{F^N PHH}"'] & & F^N PHH(\widehat{\mathcal{B}}) \arrow[dl, "\mathcal{H}_{\infty,p}^{F^N PHH}"] \\ & \Sigma^{-2N\nu(p)} F^N PHH(\mathcal{B}_p) & \end{tikzcd} \tag{67}$$

commutes for all $q \in \mathcal{P}$ with $p \preceq q$.

So far, what we proved are intrinsic properties of homotopy systems, with no geometric considerations. We now move to geometry and bring in the open-closed maps.

We consider the quantum cohomology $QH(X,\Lambda)$ as a trivial directed system of persistence modules parametrized by $\mathcal{P}$.

LEMMA 3.3.2.17. *The family of persistence maps*

$$\left\{OC_p^{\mathcal{B}}\colon PHH(\mathcal{B}_p) \to QH(X,\Lambda)\right\}_{p\in\mathcal{P}}$$

defines a homomorphism of directed systems of persistence modules.

PROOF. To show that $\{OC_p^{\mathcal{B}}\}_p$ is a homomorphism of directed systems, we have to prove that $OC_q^{\mathcal{B}} \circ \mathcal{H}_{p,q}^{PHH} = OC_p^{\mathcal{B}}$. To this aim, one defines a chain homotopy between the two maps at the chain level, which are filtered chain maps, and shows that it is filtered. This is very similar to the proof of Proposition 3.3.2.12 and is omitted here. □

As a result of Lemma 3.3.2.17, we obtain a unique persistence morphism

$$\widehat{OC}^{\mathcal{B}} := \varinjlim_{p\in\mathcal{P}} OC_p^{\mathcal{B}}\colon PHH(\widehat{\mathcal{B}}) \to QH(X,\Lambda)$$

such that the diagram

$$\begin{tikzcd}
PHH(\mathcal{B}_p) \arrow[rr] \arrow[dr, "OC_p^{\mathcal{B}}"'] & & PHH(\widehat{\mathcal{B}}) \arrow[dl, "\widehat{OC}^{\mathcal{B}}"] \\
& QH(X,\Lambda) &
\end{tikzcd} \tag{68}$$

commutes for all $p \in \mathcal{P}$. The morphism $\widehat{OC}^{\mathcal{B}}$ is called the *colimit open-closed map* for $\mathcal{B}$ and is the persistence map appearing in the statement of Theorem 3.3.1.1.

In fact, we will prove more.

LEMMA 3.3.2.18. *Let $p \in \mathcal{P}$. The persistence diagram*

$$\begin{tikzcd}
\Sigma^{-2N\nu(p)} F^N PHH(\mathcal{B}_p) \arrow[d, "\Sigma^{-2N\nu(p)} OC_p^{\mathcal{B}}"'] & F^N PHH(\widehat{\mathcal{B}}) \arrow[l, "\Sigma^{-2N\nu(p)} \mathcal{H}_{\infty,p}^{F^N PHH}"] \arrow[d, "\widehat{OC}^{\mathcal{B}}"] \\
\Sigma^{-2N\nu(p)} QH(X,\Lambda) & QH(X,\Lambda) \arrow[l]
\end{tikzcd} \tag{69}$$

commutes, where the bottom map is induced by the structural maps of the persistence module $QH(X,\Lambda)$.

Proof. Let p, q in $\mathcal{P}$ such that $p \preceq q$. Consider the persistence diagram

$$\begin{tikzcd} \Sigma^{-2N\nu(p)} F^N PHH(\mathcal{B}_p) \arrow[d, "OC_p^{\mathcal{B}}"'] & & F^N PHH(\mathcal{B}_q) \arrow[ll, "\mathcal{H}_{q,p}^{F^N PHH}"'] \arrow[d, "OC_q^{\mathcal{B}}"] \\ \Sigma^{-2N\nu(p)} QH(X,\Lambda) & & QH(X,\Lambda) \arrow[ll] \end{tikzcd} \tag{70}$$

This diagram commutes, and the proof goes along the same lines as the proof of Lemma 3.3.2.17, with the slight complication that one has to keep track of the shift in filtration. For varying q such that $p \preceq q$, the above diagram induce a commutative diagram of direct systems of persistence modules. Note that the colimit of $\Sigma^{Ns(q,p)} QH(X,\Lambda)$ over this family is equal to $\Sigma^{-2N\nu(p)} QH(X,\Lambda)$. This proves the claim. $\square$

3.3.2.8. *Proof of Theorem 3.3.1.1.* It is known that the infinity-level open closed map provides an isomorphism $PHH^{\infty}(\mathcal{B}_p) \cong QH^{\infty}(X,\Lambda)$ as vector spaces [**Gan12**], and that the unit $u \in QH^0(X,\Lambda)$ persists to $QH^{\infty}(X,\Lambda)$. Any element in $PHH^{\infty}(\mathcal{B}_p)$ can be represented by a Hochschild chain of length $\leq N_p$ for some $N_p \geq 0$. Moreover, we can choose N_p such that the spectral invariants of $F^{N_p} PHH(\mathcal{B}_p)$ are optimal, i.e. they agree with those of $PHH(\mathcal{B}_p)$ for for the homology classes that persist to infinity in both persistence modules. We claim that N_p is independent of $p \in \mathcal{P}$. This follows from the following fact: by forgetting filtrations, the functors $\mathcal{H}_{q,p}$ induce a unique map $PHH^{\infty}(\mathcal{B}_q) \to PHH^{\infty}(\mathcal{B}_p)$ which is an isomorphism (since, forgetting filtrations, continuation funtors are quasi-equivalence of A_{∞}-categories) and, hence, preserve the length filtration (note that, of course, all of this holds for the maps induced by $\mathcal{H}_{p,q}$ on PHH^{∞}). We write $N_{\mathcal{B}} := N_p$.

At this point, the proof of Theorem 3.3.1.1 becomes a matter of book-keeping.

Let $\mathcal{F}^{\epsilon} \subset \mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}$ as in the statement. We work with the notation above in the case $\mathcal{B} = \mathcal{F}^{\epsilon}$. Consider the constant $N_{\mathcal{F}^{\epsilon}}$ introduced above. Let now $a \in PHH^{\epsilon}(\widehat{\mathcal{F}^{\epsilon}})$ be such that $\widehat{OC}(a) = i_{o,\epsilon}(u_{\mathbf{d}})$, where $u_{\mathbf{d}} \in QH^0(X,\Lambda)_{\mathbf{d}}$ is the projection of the cohomological unit in $QH(X,\Lambda)$. From the above it follows that a can be seen as an element in $F^{N_{\mathcal{F}^{\epsilon}}} PHH(\widehat{\mathcal{F}^{\epsilon}})$. From this and the commutativity of Diagram (69) we get

$$R(u, OC_p^{\mathcal{F}^{\epsilon}}) \leq \epsilon + 2N_{\mathcal{F}^{\epsilon}}\nu(p)$$

for all $p \in \mathcal{P}$. Using Proposition 3.3.2.12, we conclude that $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})}$ is $\frac{\epsilon}{2}$-retract approximable by $\mathcal{F}^{\epsilon}$ in the system $\widehat{\mathcal{F}uk}(\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{d})})$ in the sense of Definition 3.1.5.8. $\square$

Remark 3.3.2.19. The complex formalism in the proof of Theorem 3.3.1.1 is due to the lack of a concept of colimit of homotopy system of filtered A_{∞}-categories. Indeed, the open-closed map appearing in the statement of Theorem 3.3.1.1 should be thought as the open-closed map associated with a limit Fukaya category (see the discussion at the beginning of §3.1.5.9).

3.3.3. Proof of Corollary 3.3.1.2. We split the proofs of the two geometric situations into subsections. The basic idea is the same: cut the ambient manifold into more and more equally spaced straight slices. We will keep on writing Fukaya categories as $\mathcal{A}_p$ and systems of Fukaya categories as $\widehat{\mathcal{A}}$ as indicated at the beginning of Section 3.3.

3.3.3.1. *Approximability of equators on the 2-sphere.* In what follows we will refer to a closed monotone Lagrangian $L \subset S^2$ as an "equator". Note that these Lagrangians are precisely the embedded curves in S^2 which separate it into two domeains of equal area.

The first Chern class of S^2 vanishes in $H^2(S^2;\mathbb{Z}_2)$, so the eigenspace of the quantum homology $QH(S^2,\Lambda)$ associated to the zero eigenvalue is the whole space.
In this subsection, we will only consider (without loss of generality) perturbation data $p \in \mathcal{P}$ satisfying the following two conditions:

(1) Let $L \in \mathcal{L}\text{ag}^{(\text{mon},\mathbf{0})}(S^2)$ be an equator on S^2 and consider the Morse function $f_p^L \colon L \to \mathbb{R}$ in the p-Floer datum of L. We assume that f_p^L has a unique maximum $e_L \in L$ and a unique minimum $\text{pt}_L \in L$. We will always assume that both e_L and pt_L are neither the north nor south poles of S^2.
(2) Consider the Morse function $f_p^{S^2} \colon S^2 \to \mathbb{R}$ in the p-Floer datum for S^2. We assume that $f_p^{S^2}$ has a unique maximum $u_{S^2} \in S^2$ and a unique minimum $\text{pt}_{S^2} \in S^2$ and no other critical points. We further assume that both u_{S^2} and pt_{S^2} differ from both the north and south poles of S^2,

We will abuse notation and write this "restriction" of $\mathcal{P}$ again as $\mathcal{P}$ and consider the homotopy system of A_∞-categories $\widehat{\mathcal{A}} = \widehat{\mathcal{F}uk}(\mathcal{L}\text{ag}^{(\text{mon},\mathbf{0})}(S^2))$ as parametrized by this restricted $\mathcal{P}$.

Proposition 3.3.3.1. *Let $L \in \mathcal{L}ag^{(mon,\mathbf{0})}(S^2)$ be an equator on S^2. Then $\mathcal{L}ag^{(mon,\mathbf{0})}(S^2)$ is $\frac{1}{4}$-retract approximable by L.*

Proof. Let $p \in \mathcal{P}$. It is easy to see that at the chain level we have

$$OC_p^L(\text{pt}_L \otimes \text{pt}_L) = T^{\frac{1}{2}} u_{S^2}.$$

Moreover, since d_{CC} is cyclic, $\text{pt}_L \otimes \text{pt}_L$ is a cycle in $CC_*(L,L)$. Note that the persistence Hochschild homology $PHH_*(L_p)$ does not depend on $p \in \mathcal{P}$. The claim then follows by Theorem 3.3.1.1. □

Remark 3.3.3.2. a. We actually proved that for a single equator the constant $\delta > 0$ appearing in Definition 3.1.5.8 can be taken to be arbitrarily big. This is due to the fact that for the self-Floer homology of L we do not need any Hamiltonian perturbation.

b. The linear component $HF(L,L) \to QH(S^2)$ of the open-closed map does not hit the unit in our setting: indeed $OC(e_L) = 0$ by a degree argument while $OC(\text{pt}_L) = \text{pt}_{S^2} + T^{\frac{1}{2}} u_{S^2}$. Note however, that if we would have worked with coefficients in the Novikov field over the real numbers, the situation would have been different, as $T^{-1\frac{1}{2}}\text{pt}_{S^2} + u_{S^2}$ is the projection of the unit in the zero eigensummand of quantum homology of $\mathcal{C}P^1$ over this coefficient field.
Note that as $\text{pt}_{S^2} + T^{\frac{1}{2}} u_{S^2}$ and u_{S^2} are non-homologous cycles in $CQ(S^2)$, it follows that $[\text{pt}_L]$ and $[\text{pt}_L \otimes \text{pt}_L]$ generate $HH(L)$.

c. As we know that $OC^L \colon HH(L) \to QH(S^2)$ is an isomorphism, it follows from the computation above that $e_L \in PHH^\infty(L,L)$ has to be a boundary. In fact, it equals $T^{-\frac{1}{2}} d_{CC}(a_L \otimes$

$a_L \otimes a_L)$, where $a_L = \mathrm{pt}_L + T^{\frac{1}{4}} e_L$; indeed notice that $a_L \in CF(L, L)$ is a cycle satisfying

$$\mu_k(a_L, \dots, a_L) = T^{\frac{1}{2}} e_L \text{ for any } k \geq 2.$$

Hence, $[e_L]$ has boundary depth $\frac{1}{2}$ in $PHH(L)$. Note that $[e_L \otimes \mathrm{pt}_L]$ also has boundary depth equal to $\frac{1}{2}$. Indeed

$$d_{CC}(a_L^{\otimes 4}) = T^{\frac{1}{2}}(e_L \otimes \mathrm{pt}_L + \mathrm{pt}_L \otimes e_L)$$

and $\mathrm{pt}_L \otimes e_L = d_{CC}(\mathrm{pt}_L \otimes e_L \otimes e_L)$.

We will now show that we can approximate the Fukaya category of S^2 with better accuracy by considering larger approximating families. Fix an equator $S^1 \subset S^2$ as a reference and consider the set $\mathcal{E} \subset \mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{0})}(S^2)$ great circles which pass trough the north $n \in S^2$ and south pole $s \in S^2$. Of course, any two Lagrangians in $\mathcal{E}$ intersect transversally. Recall that in our setting, given a perturbation datum $p \in \mathcal{P}$, the Hamiltonian part of the p-Floer datum associated to a pair (L_0, L_1) of transversally intersecting Lagrangians is assumed to be constant (see page 135), and this constant is $< \nu(p)$, the size of p, by definition. Moreover, we require $H_p^{L_0,L_1} = H_p^{L_1,L_0}$. Hence the Floer complexes $CF(L_0, L_1; p)$ and $CF(L_1, L_0; p)$ are generated by the (finitely many) intersection points in $L_0 \cap L_1$, and these lie at the same filtration level. Suppose $L_1 \in \mathcal{E}$ is obtained by rotating another equator $L_0 \in \mathcal{E}$ by an angle smaller than π. Of course, $L_0 \cap L_1$ consists of the north and south poles only: we work with the convention that in the complex $CF(L_0, L_1; p)$ the north pole $n_0 \in CF(L_0, L_1; p)$ has grading 1 and the south pole $s_0 \in CF(L_0, L_1; p)$ has grading 0, while in the complex $CF(L_1, L_0; p)$ this is viceversa (and the poles are denoted by n_1 and s_1 in this complex).

The relevant notion of approximability for the next result is the retract version of Definition 3.1.2.3.

Proposition 3.3.3.3. *Let $p \in \mathcal{P}$ and $N \geq 2$. Consider a finite family of great circles $\mathcal{E}(N) = \{L_1, \dots, L_N\} \subset \mathcal{E}$ on S^2, all passing through the north and south poles and such that L_i lies at angle difference $\frac{\pi}{2N}$ from L_{i+1}. Then $\mathcal{L}ag^{(mon,\mathbf{0})}(S^2)$ is retract-approximable in $\mathcal{A}_p$ by the family $\mathcal{E}(N)$ with accuracy $\frac{1}{4N} + 2\nu(p)$.*

Remark 3.3.3.4. Proof. We will number our equators so that looking from the north pole, they are ordered clockwise starting from L_1. We drop p from the notation and write $C_i := CF(L_i, L_{i+1})$ and $C_i' := CF(L_{i+1}, L_i)$ for any $i = 1, \dots, N$[1]. For any i, we denote by $n_i, s_i \in C_i$ and $n_i', s_i' \in C_i'$ the north and south poles respectively. To simplify the computations in the remainings of the proof, we impose the following additional restrictions on the functions $f_p^{L_i}$ in the p-Floer datum of each equator L_i: we require that flowing from the north pole to the south pole, one encounters first pt_i and then e_i, before getting to the south pole. We also require that the critical points u_{S^2} and pt_{S^2} of the Morse function $f_p^{S^2}$ in the p-Floer datum

[1]With the convention $L_{N+1} = L_1$.

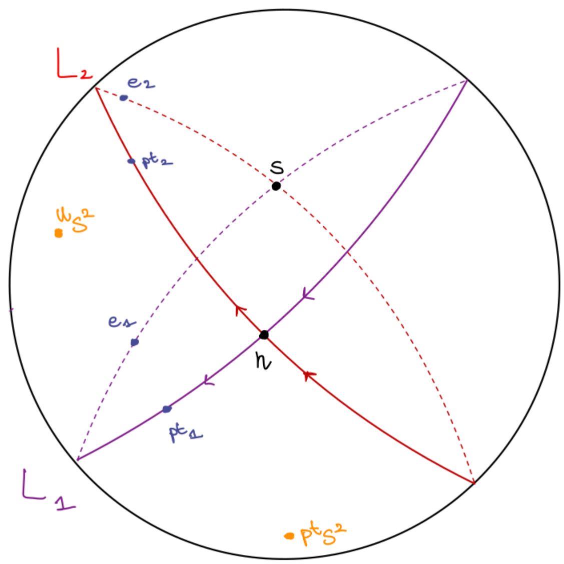


FIGURE 9. The geometric situation described in Proposition 3.3.3.3 in the case $N = 2$.

of the ambient manifold $f_p^{S^2}\colon S^2 \to \mathbb{R}$ both lie in the area between L_1 and L_2, but not on L_1 or L_2.

We claim that

$$\vec{\gamma} := \sum_{i=1}^{N} n_i \otimes s_i' \in \bigoplus_{i=1}^{N} C_i \otimes C_i' \subset CC_*(\mathcal{E}(N))$$

is a Hochschild cycle, and moreover

$$OC_p^{\mathcal{E}(N)}\left(\sum_{i=1}^{N} n_i \otimes s_i'\right) = T^{\frac{1}{2N}} u_M.$$

This would prove the statement as

$$\mathbb{A}_{CC}\left(\sum_{i=1}^{N} n_i \otimes s_i'\right) = \mathbb{A}_{CC}(n_i \otimes s_i') = \mathbb{A}(n_i) + \mathbb{A}(s_i') \leq 2\nu(p)$$

by definition of the filtration on the Hochschild complex.

We first prove that $\vec{\gamma}$ is a Hochschild cycle. Let $i \in \{1, \dots, N\}$. We compute

$$\mu_2(n_i, s_i') = T^{\frac{1}{2N}} e_i \text{ and } \mu_2(s_i', n_i) = T^{\frac{1}{2N}} e_{i+1}$$

by just counting slices of the sphere as in Figure 9. By definition of the Hochschild differential we then have

$$d_{CC}(n_i \otimes s_i') = T^{\frac{1}{2N}}(e_i + e_{i+1}),$$

hence

$$d_{CC}(\vec{\gamma}) = T^{\frac{1}{2N}} \sum_{i=1}^{N} (e_i + e_{i+1}) = 0$$

as we work with $\mathbb{Z}_2$ coefficients. This shows that $\vec{\gamma}$ is an Hochschild cycle.

The computation of $OC_p^{\mathcal{B}}(\vec{\gamma})$ is easy in this setting. Because of the choice of $f_p^{S^2}$ we have

$$OC_p^{\mathcal{B}}(\vec{\gamma}) = OC_p^{\mathcal{B}}(n_1 \otimes s_1') = T^{\frac{1}{2N}} u_{S^2}$$

by degree reasons. □

Another way to prove the above proposition is to show that the element $\vec{\gamma}$ is homologous, in $CC(\mathcal{E}(N))$, to $T^{-\frac{N-1}{2N}} \mathrm{pt}_{L_i} \otimes \mathrm{pt}_{L_i} \in CF(L_i, L_i) \otimes CF(L_i, L_i)$ for any i.

3.3.3.2. *Approximability of non-contractible circles on the* 2*-torus.* Consider the standard symplectic 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with unit volume. We are interested in the class of non-contractible Lagrangians $\mathcal{L}\mathrm{ag}^{(\mathrm{wex})}(\mathbb{T}^2)$. We will use coordinates (x, y) on $\mathbb{T}^2$ and denote by (n, m) the lattice coordinates on $\mathbb{Z}^2$. We recall that, for us, the weakly exact case is a particular instance of the monotone setting.

We introduce the following notation:

$$L_y := \{x = 0\} \subset \mathbb{T}^2 \text{ and } L_x^\theta := \{y = \theta\} \subset \mathbb{T}^2 \text{ for any } \theta \in S^1$$

and denote the unique intersection point between L_y and L_x^θ by $a^\theta \in L_x^\theta \pitchfork L_y$. We denote $L_x := L_x^0$ and $a := a^0$. In this section, we will work with the family

$$\mathcal{N} := \{L_y\} \cup \{L_x^\theta \mid \theta \in S^1\} \subset \mathcal{L}\mathrm{ag}^{(\mathrm{wex})}(\mathbb{T}^2).$$

We will write a^θ seen as the generator of Floer complexes as

$$a_{xy}^\theta \in CF(L_x^\theta, L_y) \text{ and } a_{yx}^\theta \in CF(L_y, L_x^\theta)$$

for any choice of perturbation datum $p \in \mathcal{P}$, which we omit from the notation. In this section, we will only consider (without loss of generality) perturbation data $p \in \mathcal{P}$ satisfying the following:

(1) Consider the Morse function $f_p^{\mathbb{T}^2} \colon \mathbb{T}^2 \to \mathbb{R}$ in the p-Floer datum for $\mathbb{T}^2$. We assume that $f_p^{\mathbb{T}^2}$ has exactly four critical points: a unique maximum $u_{\mathbb{T}^2} \in \mathbb{T}^2$, two saddle points $s_1, s_2 \in \mathbb{T}^2$ and a unique minimum $\mathrm{pt}_{\mathbb{T}^2} \in \mathbb{T}^2$. We further assume that all these critical points do not lie on the Lagrangian L_y introduced above.
(2) Let $L \in \mathcal{L}\mathrm{ag}^{(\mathrm{wex})}(\mathbb{T}^2)$ and consider the Morse function $f_p^L \colon L \to \mathbb{R}$ in the p-Floer datum of L. We assume that f_p^L has a unique maximum $e_L \in L$ and a unique minimum $\mathrm{pt}_L \in L$. We will always assume that both e_L and pt_L do not lie on the Lagrangian L_y introduced above. In the special cases where L is one of the Lagrangians L_y or L_x^θ in $\mathcal{N}$, we denote the critical points as e_y, pt_y or e_x^θ, pt_y^θ.

REMARK 3.3.3.5. The role of x and y are intercheangeable. Moreover, we could work with the countable subfamily $\mathcal{N}_c$ containing L_y and L_x^θ for all $\theta \in \mathbb{Q}/\mathbb{Z}$.

As for the case of the 2-sphere in the subsection above, we first prove a warm up low-accuracy retract-approximation result, before moving to a more general case. In the following proposition we will work with the Lagrangian L_x, but the same results hold for all L_x^θ.

PROPOSITION 3.3.3.6. *Let $p \in \mathcal{P}$. Then $\mathcal{L}ag^{(wex)}(\mathbb{T}^2)$ is retract approximable in $\mathcal{A}_p$ by $\mathcal{B}_{xy} := \{L_x, L_y\}$ with accuracy $\frac{1}{2} + 3\nu(p)$.*

PROOF. Consider the Hochschild chain $\vec{a} = a_{xy} \otimes a_{yx} \otimes a_{xy} \otimes a_{yx} \in CC_*(\mathcal{B}_{xy})$. We show it is a cycle. For this, we compute all possible cyclic partial contractions.

(1) First, both $\mu_2(a_{xy}, a_{yx}) \in CF(L_x, L_x)$ and $\mu_2(a_{yx}, a_{xy}) \in CF(L_y, L_y)$ vanish, as the only allowed contributions are Morse trajectories to pt_x and pt_y respectively, which come in pairs.
(2) The terms $\mu_3(a_{xy}, a_{yx}, a_{xy}) \in CF(L_x, L_y)$ and $\mu_3(a_{yx}, a_{xy}, a_{yx})$ vanish as well. For instance,
$$\mu_3(a_{xy}, a_{yx}, a_{xy}) = 2 \sum_{n,m>0} T^{nm} a_{xy}$$
as we can see the "two opposite" a_{xy} in the fundamental square of $\mathbb{T}^2$ as exits in two different ways (for each lattice-polygon (n, m)).
(3) The terms $\mu_4(a_{xy}, a_{yx}, a_{xy}, a_{yx}) \in CF(L_x, L_x)$ and $\mu_4(a_{yx}, a_{xy}, a_{yx}, a_{x,y}) \in CF(L_y, L_y)$ both vanish. For instance
$$\mu_4(a_{xy}, a_{yx}, a_{xy}, a_{yx}) = 2 \sum_{n,m>0} nT^{nm} e_x$$
since in a lattice polygon of width n we can have e_x as an exit in $2n$ different ways.

It follows that $\vec{a} \in CC_*(B_{xy})$ is a cycle. We compute
$$OC^{\mathcal{B}_{xy}}(\vec{a}) = \sum_{n,m>0} nmT^{nm} u_{\mathbb{T}^2}$$
since in every (n, m) lattice polygon we can see u as an exit in nm different position (one for every fundamental square embedded in the polygon). Since we are working over $\mathbb{Z}_2$ it follows that
$$OC^{\mathcal{B}_{xy}}(\vec{a}) = \sum_{n>0} n^2 T^{n^2} u_{\mathbb{T}^2} = \sum_{n\geq 0} T^{(2n+1)^2} u_{\mathbb{T}^2}.$$
Note that the smallest Novikov exponent in this expression is 1. Since intersection points lie at filtration level $\leq \nu(p)$ by definition, the claim follows. $\square$

REMARK 3.3.3.7. More formally, given the Novikov element
$$f(T) = \sum_{n\geq 0} T^{(2n+1)^2}$$
we have
$$f^{-1}(T) = T^{-1} \sum_{n\geq 0} g_n T^n$$

where $g_0 = 1$ and
$$g_n = \sum_{e\in E\setminus\{0\},\ e\le n} g_{n-e}$$
where $E = \{(2n+1)^2 - 1\ :\ n \ge 0\}$. Indeed, writing $\frac{f(T)}{T} = \sum_{n\ge 0} f_n T^n$ where $f_n = 1$ if and only if $n \in E$, we have that $f_0 \cdot g_0 = 1$ while the coefficient of T^n for $n \ge 1$ in $f \cdot f^{-1}$ is
$$\sum_{j=1}^{n} f_j g_{n-j} = \sum_{j\in E,\ j\le n} g_{n-j} = g_n + \sum_{j\in E\setminus\{0\},\ j\le n} g_{n-j} = 0$$
It follows directly from our definition of the action filtration on Λ, that $f^{-1} \in \Lambda$ lies at filtration level ≤ 1.
Using the formalism of theta-functions we can write
$$\sum_{n\ge 0} T^{(2n+1)^2} = \sum_{n\in\mathbb{Z}} T^{(4n+1)^2} = \theta_{\frac{1}{4},0}(16,0).$$
But also notice that
$$\sum_{n\ge 0} n^2 T^{n^2} = T\theta'_{0,0}(1,0)$$

PROPOSITION 3.3.3.8. *Let $p \in \mathcal{P}$ and and $N \ge 1$. Consider the family $\mathcal{N}(N) = \{L_y, L_1, \ldots, L_N\} \subset \mathcal{N}$, where*
$$L_j = L_x^{\frac{j-1}{N}}$$
for $j = 1, \ldots, N$. $\mathcal{L}ag^{(wex)}(\mathbb{T}^2)$ is retract approximable in $\mathcal{A}_p$ by $\mathcal{N}(N)$ with accuracy $\frac{1}{2N} + 3\nu(p)$.

PROOF. We assume that the critical point $u_{\mathbb{T}^2}$ doesn't lie on any Lagrangian in $\mathcal{N}(N)$. We denote by $e_j \in CF(L_j, L_j)$ and $\mathrm{pt}_j \in CF(L_j, L_j)$ the maximum and the minimum of the Morse function $f_p^{L_j}$ for any $j = 1, \ldots, N$. Moreover, for any j, we denote by $a^j_{xy} \in CF(L_j, L_y)$ and $a^j_{yx} \in CF(L_y, L_j)$ the intersection point a seen as a generator of the Floer complexes. We denote by $i \in \{1, \ldots, N\}$ the unique index such that $u_{\mathbb{T}^2}$ lies in the area between L_i and L_{i+1}. Similarly, we denote by l the unique index such that $e_y \in CF(L_y, L_y)$ lies in the L_y-segment between L_l and L_{l+1}. Figure 10 provides a schematic depiction of the geometric setup.

For any $j = 1, \ldots, N$, consider the Hochschild chain[2]
$$\gamma^j := a^j_{xy} \otimes a^{j+1}_{yx} \otimes a^{j+1}_{xy} \otimes a^j_{yx} \in CC_*(\mathcal{N}(N)).$$
Given a real number $A \in \mathbb{R}$, let[3]
$$q^h_A(T) := \sum_{n,m>0} n(T^{n(m-1+A)} + T^{n(m-A)}) \text{ and } \tilde{q}^h_{\frac{1}{N}}(T) :` = \sum_{n,m>0} T^{n(m-1+A)} + T^{n(m-A)}$$

[2]Here $N + 1 = 1$.

[3]Here h stands for *horizontal* and v for *vertical.*

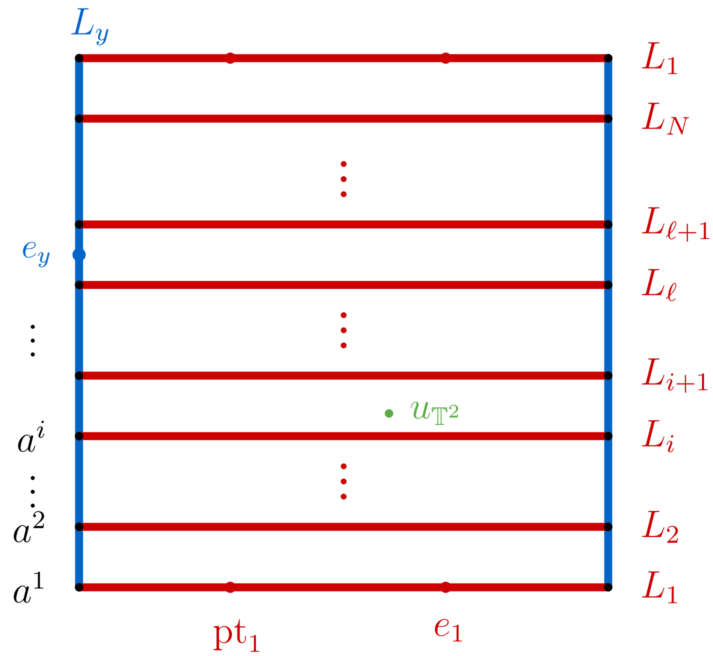


FIGURE 10. The fundamental domain of $\mathbb{T}^2$ with the approximating family of Lagrangians $\mathcal{N}(N)$ discussed in Proposition 3.3.3.8.

as well as

$$q^v_A(T) := \sum_{n,m>0} m(T^{n(m-A)} + T^{n(m+A)}) \text{ and } \tilde{q}^v_A(T) := \sum_{n,m>0} T^{n(m-A)} + T^{n(m+A)}$$

be elements of the Novikov field Λ. We compute the Hochschild differential of γ^j by computing all possible contractions.

(1) We have

$$\mu_4(a^j_{xy}, a^{j+1}_{yx}, a^{j+1}_{xy}, a^j_{yx}) = q^h_{\frac{1}{N}}(T)e_j \text{ and } \mu_4(a^{j+1}_{xy}, a^j_{yx}, a^j_{xy}, a^{j+1}_{yx}) = q^h_{\frac{1}{N}}(T)e_{j+1},$$

while

$$\mu_4(a^j_{yx}, a^j_{xy}, a^{j+1}_{yx}, a^{j+1}_{xy}) = \mu_4(a^{j+1}_{yx}, a^{j+1}_{xy}, a^j_{yx}, a^j_{xy}) = \begin{cases} q^v_{\frac{1}{N}}(T)e_y, & \text{if } j \neq l \\ q^v_{1-\frac{1}{N}}(T)e_y, & \text{if } j = l. \end{cases}$$

(2) We have

$$\mu_3(a^j_{xy}, a^{j+1}_{yx}, a^{j+1}_{xy}) = \tilde{q}^h_{\frac{1}{N}}(T)a^j_{xy} \text{ and } \mu_3(a^{j+1}_{yx}, a^{j+1}_{xy}, a^j_{yx}) = \tilde{q}^h_{\frac{1}{N}}(T)a^j_{yx}$$

while

$$\mu_3(a^j_{yx}, a^j_{xy}, a^{j+1}_{yx}) = \tilde{q}^h_{\frac{1}{N}}(T)a^{j+1}_{yx} \text{ and } \mu_3(a^{j+1}_{xy}, a^j_{yx}, a^j_{xy}) = \tilde{q}^h_{\frac{1}{N}}(T)a^{j+1}_{xy}$$

We now compute the relevant μ_3's. It follows that

$$\begin{aligned} d_{CC}(\gamma^j) &= q^h_{\frac{1}{N}}(T)(e_j + e_{j+1}) + 2q^v_{c_j}(T)e_y + \tilde{q}^h_{\frac{1}{N}}(T)(2a^j_{xy} \otimes a^j_{yx} + a^{j+1}_{xy} \otimes a^{j+1}_{yx} + a^{j+1}_{yx} \otimes a^{j+1}_{xy}) \\ &= q^h_{\frac{1}{N}}(T)(e_j + e_{j+1}) + \tilde{q}^h_{\frac{1}{N}}(T)(a^{j+1}_{xy} \otimes a^{j+1}_{yx} + a^{j+1}_{yx} \otimes a^{j+1}_{xy}) \end{aligned}$$

where $c_j = \frac{1}{N}$ if $j \neq l$ and $c_j = 1 - \frac{1}{N}$ otherwise. Hence

$$d_{CC}\left(\sum_{j=1}^N \gamma^j\right) = \tilde{q}^h_{\frac{1}{N}}(T)\sum_{j=1}^N (a^j_{xy} \otimes a^j_{yx} + a^j_{yx} \otimes a^j_{xy})$$

as by summing over j the summands of the form e_j cancel out in pairs. In particular, $\sum_{j=1}^N \gamma^j$ is not a cycle. However, we claim that

$$\vec{\gamma} = \sum_{j=1}^N \gamma^j + \tilde{q}^h_{\frac{1}{N}}(T)\ e_j \otimes a^j_{xy} \otimes a^j_{yx} \in CC_*(\mathcal{N}(N))$$

is a cycle. Indeed:

$$d_{CC}(e_j \otimes a^j_{xy} \otimes a^j_{yx}) = \mu_2(e_j, a^j_{xy}) \otimes a^j_{yx} + \mu_2(a^j_{yx}, e_j) \otimes a^j_{xy} = a^j_{xy} \otimes a^j_{yx} + a^j_{yx} \otimes a^j_{xy},$$

since e_j is a strict unit and the product of the point with itself vanishes.

We now compute $OC^{\mathcal{N}(N)}(\vec{\gamma}) \in CQ(\mathbb{T}^2)$. Recall that $i \in \{1, \ldots, N\}$ is defined as the only index such that the critical point $u = u_{\mathbb{T}^2}$ of the Morse function $f_p^{\mathbb{T}^2}$ used to define $QH_*(M)$ lies in the area between L_i and L_{i+1}. Let $j \in \{1, \ldots, N\}$, then

$$OC(\gamma^j) = \begin{cases} \sum_{n,m>0} nm(T^{n(m-\frac{1}{N})} + T^{n(m+\frac{1}{N})})u, & \text{if } j \neq i \\ \sum_{n,m>0} nm(T^{n(m-1+\frac{1}{N})} + T^{n(m+1-\frac{1}{N})})u, & \text{if } j = i \end{cases}$$

On the other hand, for all $j \in \{1, \ldots, N\}$ it is easy to see that we have

$$OC(e_j \otimes a^j_{xy} \otimes a^j_{yx}) = 0.$$

If N is odd, then we have

$$OC(\vec{\gamma}) = OC(\gamma^i) = \sum_{k \geq 0} \sum_{\substack{d = \pm 1\ (2N) \\ d | 2k+1}} T^{\frac{2k+1}{N}}$$

while if N is even we have, for some $j \neq i$:

$$OC(\vec{\gamma}) = OC(\gamma^i) + OC(\gamma^j) = \sum_{k \geq 0} \sum_{\substack{d = \pm 1, \pm 1 + N\ (2N) \\ d | 2k+1}} T^{\frac{2k+1}{N}}$$

In all cases the lowest order term is $T^{\frac{1}{N}} u_{\mathbb{T}^2}$. The claim follows.

$\square$

In Proposition 3.3.3.8 we worked with only one Lagrangian in the y direction. We can generalize the result to families containing $N+M$ circles, N in the x direction and M in the y direction, to get retract-approximation of order $\frac{1}{NM}$. Below we prove this for $N=M$.

PROPOSITION 3.3.3.9. *Let $p \in \mathcal{P}$ and $N \geq 1$. Consider the family*

$$\mathcal{N}(N) = \{L_y^1, \dots, L_y^N, L_x^1, \dots, L_x^N\} \subset \mathcal{N},$$

where

$$L_y^j := L_y^{\frac{j-1}{N}} \text{ and } L_x^j = L_x^{\frac{j-1}{N}}$$

for $j = 1, \dots, N$. Then $\mathcal{N}(N)$ retract-approximates $\mathcal{L}^{we}(\mathbb{T}^2)$ with accuracy $\frac{1}{2N^2} + 3\nu(p)$ in $\mathcal{A}_p$.

PROOF. Essentially, the proof is the same as Proposition 3.3.3.8. We denote by $a^{j,k}$ the unique intersection point in $L_x^j \cap L_y^k$ and write it as $a_{xy}^{j,k}$ when seen as a generator of $CF(L_x^j, L_y^k)$ and as $a_{y,x}^{j,k}$ when seen as a generator of $CF(L_y^k, L_x^j)$. Let i_x and i_y the only indices such that the critical point $u = u_{\mathbb{T}^2}$ of the Morse function defining $QH(\mathbb{T}^2)$ lies in the square bounded by the Lagrangians $L_x^{i_x}$, $L_x^{i_x+1}$, $L_y^{i_y}$ and $L_y^{i_y+1}$. The key fact is that

$$\begin{aligned}
OC(a_{xy}^{i_x,i_y} \otimes a_{yx}^{i_y,i_x+1} \otimes a_{xy}^{i_x+1,i_y+1} \otimes a_{yx}^{i_y+1,i_x}) &= \sum_{n,m>0} nm(T^{(n-1+\frac{1}{N})(m-1+\frac{1}{N}} + T^{(n+1-\frac{1}{N})(m+1-\frac{1}{N}})u \\
&= \sum_{n>0} n^2(T^{(n-1+\frac{1}{N})^2} + T^{(n+1-\frac{1}{N})^2})u \\
&= \sum_{n\geq 0} T^{(2n+\frac{1}{N})^2} + T^{(2n+2-\frac{1}{N})^2} u \\
&= \sum_{n\in\mathbb{Z}} T^{(2n+\frac{1}{N})^2} u = \theta_{\frac{1}{2N},0}(4,0)u
\end{aligned}$$

In particular, the smallest exponent in this serie is $\frac{1}{N^2}$. One finishes the proof by showing that summing all possible cyclic combinations of tensors of intersection point plus some deformations involving the units in the Lagrangian Floer complex as in the proof of Proposition 3.3.3.8 one gets a Hochschild cycle. $\square$

3.3.3.3. *End of proof of Corollary 3.3.1.2.* The proof of Corollary 3.3.1.2 is now a matter of packing the results of the last two sections.

PROOF OF COROLLARY 3.3.1.2. We begin with the case of the sphere. Let $\varepsilon_0 > 0$ and consider the family $\mathcal{E} \subset \mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{0})}(S^2)$ of straight equators passing through north and south pole introduced above. Let $N_0 \geq 1$ such that $\frac{1}{4N_0} < \varepsilon_0$ and choose a subset $\mathcal{E}(N_0) \subset \mathcal{E}$ of cardinality N_0 as in Proposition 3.3.3.3. Let $\delta_0 := \frac{1}{2}\left(\varepsilon_0 - \frac{1}{4N_0}\right)$. Then for any perturbation datum p as in the proof of Proposition 3.3.3.3 of size $\nu(p) < \delta_0$, the family $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{0})}(S^2)$ is ε_0-retract approximated by $\mathcal{E}(N_0)$ in $PD(\mathcal{F}uk(\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{0})}(S^2);p))$. It follows that $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{0})}(S^2)$ is retract-approximable by $\mathcal{E}$ in the sense of Definition 3.1.5.8.
Let $\varepsilon_1 > 0$ and consider the family $\mathcal{N} \subset \mathcal{L}\mathrm{ag}^{(\mathrm{wex})}(\mathbb{T}^2)$ introduced above. Let $N_1 \geq 1$ such that $\frac{1}{2N_1} < \varepsilon_1$ and choose a family $\mathcal{N}(N_1)$ of cardinality N_1+1 and containing L_y as in

Proposition 3.3.3.8. Let $\delta_1 := \frac{1}{3}(\varepsilon_1 - \frac{1}{2N_1})$. Then for any perturbation datum as in te proof of Proposition 3.3.3.8 of size $\nu(p) < \delta_1$, the family $\mathcal{L}\mathrm{ag}^{(\mathrm{wex})}(\mathbb{T}^2)$ is ε_1-retract-approximable by $\mathcal{N}(N_1)$ in $PD(\mathcal{F}uk(\mathcal{L}\mathrm{ag}^{(\mathrm{wex})}(\mathbb{T}^2); p))$. It follows that $\mathcal{L}\mathrm{ag}^{(\mathrm{wex})}(\mathbb{T}^2)$ is retract-approximable by $\mathcal{N}$ in the sense of Definition 3.1.5.8. $\square$

3.3.4. Proof of Theorem ?? ii and iii. Corollary 3.3.1.2 claims retract approximability in the sense of Definition 3.1.5.8 for the classes of Lagrangian submanifolds, $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{0})}(S^2)$ and $\mathcal{L}\mathrm{ag}^{(\mathrm{wex})}(\mathbb{T}^2)$, that appear at the points ii and iii of Theorem **??**. To conclude the proof of this theorem we need to remark that retract approximability in the sense of Definition 3.1.5.8 implies retract approximability in the sense of Definition 3.1.2.1. This argument has already been discussed in the proof of the point i of Theorem **??**, in §3.2.7.1 but we will revisit it here.

First, notice that the approximating data $(\Phi, \mathcal{F})$ is clear from the proof of Corollary 3.3.1.2. For instance, for $\mathcal{L}\mathrm{ag}^{(mon,\mathbf{0})}(S^2)$ the maps

$$\Phi_\eta : \mathcal{L}\mathrm{ag}^{(mon,\mathbf{0})}(S^2) \longrightarrow \mathrm{Obj}(PD(\mathcal{F}uk(\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{0})}(S^2); p)))$$

are Yoneda embeddings (with $\nu(p) < \eta$, small enough) and $\mathcal{F}_\epsilon$ is a family of sufficiently many great circles going through the north and south poles on S^2, as in Proposition 3.3.3.3.

The only point that remains to be clarified is the relation between the spectral metric on the domain of Φ and the interleaving metric on the target of the maps Φ. This relationships is analogous to the one described in §3.2.7.1 but there are some adjustments. First, we need to provide a definition of the spectral metric d_γ that applies to monotone Lagrangians as well as to weakly-exact Lagrangians, and not only to exact Lagrangians as in §3.2.7.1. This is well-known in the literature but, for completeness, we recall this definition here. Consider two Lagrangians that are Hamiltonian istotopic L, L' (in one of the classes considered in Theorem **??**) and a Hamiltonian $H : [0,1] \times M \to \mathbb{R}$ such that the Floer homology $HF(L, L'; H)$ is defined. Because L and L' are Hamiltonian isotopic the PSS map $QH(L) \to HF(L, L'; H)$ is well defined and we let $\gamma([L]; H) \in \mathbb{R}$ be the spectral invariant in (the persistence module) $HF(L, L'; H)$ of the PSS-image of the fundamental class $[L] \in QH(L)$. We then put, see for instance [**KS21**]:

$$d_\gamma(L, L') = \limsup_{||H||_H \to 0} \gamma([L]; H) + \gamma([L]; \overline{H})$$

where $\overline{H}$ is the inverse Hamiltonian $\overline{H}(t,x) = -H(1-t,x)$. For exact Lagrangians this definition is equivalent to the one in §3.2.7.1. Moreover, the arguments in §3.2.7.1 - and in particular, the identity (54) and the arguments in [**BCZ24b**] justifying it - remain true in this context and they show that the maps Φ restricted to a Hamiltonian isotopy class are quasi-isometric embeddings. Therefore, this completes the argument in the case of $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{0})}(S^2)$ given that all the Lagrangians in this space are Hamiltonian isotopic to the standard equator.

Finally, we consider the space $\mathcal{L}\mathrm{ag}^{(\mathrm{wex})}(\mathbb{T}^2)$. The spectral distance, given as above, is not defined for two L and L' that are not Hamiltonian isotopic. However, the interleaving type distance $\hat{D}_{int}(-,-)$ from §3.2.7.1 is defined on all of $\mathcal{L}\mathrm{ag}^{(\mathrm{wex})}(\mathbb{T}^2)$ and, as explained before, it coincides with the spectral distance on each Hamiltonian isotopy class. Thus by putting

$$d_\gamma(L,L') := \hat{D}_{int}(L,L'), \forall \ \ L,L' \in \mathcal{L}\mathrm{ag}^{(\mathrm{wex})}(\mathbb{T}^2)$$

the proof of Theorem **??** is complete.

3.4. Corollaries of approximability and weighted complexity

The aim of this section is to prove Corollaries **??**, **??**, and **??**, and to discuss some notions of complexity that are then used to deduce Corollary **??**. This type of complexity has interest in itself and we start the section with a discussion of this topic, that is purely algebraic in nature, in §3.4.1. We continue with the proof of Corollary **??** in §3.4.2. In §3.4.4 and §3.4.5, respectively, we prove the other two corollaries.

3.4.1. Complexity in TPCs. Most of this subsection is purely algebraic and can be read independently of any considerations related to symplectic topology, Fukaya categories and so forth. In §3.4.1.1 we introduce the notions of complexity we are interested in, the most important one being that of *weighted cone-length*. We also state the main result of the subsection, Proposition 3.4.1.3, providing a lower bound for weighted cone-length in terms of a count of bars in a certain barcode. The proof of this proposition occupies the rest of the subsection. In §3.4.1.2 we establish some algebraic properties that lead to the inequality in Corollary 3.4.1.9 that reduces the statement of the proposition to some properties of weighted cone-length in the homotopy category of filtered chain complexes. These properties are established in §3.4.1.3. In §3.4.1.4 the various arguments are put together to show Proposition 3.4.1.3. In §3.4.1.5 we introduce a notion of weighted categorical entropy which is a weighted analogue of a similar notion introduced by Dimitrov-Haiden-Katzarkov-Kontsevich [**DHKK14**] and, as a corollary of Proposition 3.4.1.3, we estimate this weighted entropy from below by the barcode entropy as defined by Çineli-Ginzburg-Gurel [**cGG24**]. Finally, in §3.4.1.6 we return to Lagrangians in cotangent bundles and make some first comments concerning their complexity.

The setting of the subsection is that of a triangulated persistence category $\mathcal{C}$ endowed with the interleaving pseudo-metric d_{int} as in equation (3).

3.4.1.1. *Weighted generating rank and cone-length.* In the setting above consider $X \subset \text{Obj}(\mathcal{C})$ and define the *ϵ-generating rank* of X by:

$$g_{\mathcal{C}}(X;\epsilon) = \inf\big\{\#(\mathcal{F}_\epsilon) \mid \mathcal{F}_\epsilon \subset \text{Obj}(\mathcal{C}),\ \forall x \in X,\ d_{int}\big(x, \text{Obj}(\langle \mathcal{F}_\epsilon \rangle^{\Delta})\big) \leq \epsilon\big\}. \tag{71}$$

Here $\#(\mathcal{S})$ indicates the cardinality of the set $\mathcal{S}$, and $\langle \mathcal{S} \rangle^{\Delta}$ is defined in §3.1.2.1.

Finiteness of the ϵ-generating rank of $X \subset \text{Obj}(\mathcal{C})$ is equivalent to ϵ-approximability of X in $\mathcal{C}$, as seen by inspecting Definition 3.1.2.3.

A more important complexity measurement, also related to approximability, is a weighted version of the classical notion of cone-length of a topological space in homotopy theory [**Gan67**], [**Cor94**], [**Cor95**]. We fix some preliminary notation. Let $\mathcal{C}'$ be a triangulated category and let $\mathcal{F} \subset \text{Obj}(\mathcal{C}')$. Consider a sequence $\eta = (\Delta_0, \ldots, \Delta_m)$ of exact triangles in $\mathcal{C}'$ of the form:

$$\Delta_i \quad : \quad F_i \longrightarrow A_i \longrightarrow A_{i+1} \longrightarrow TF_i \ \ , \ 0 \leq i \leq m \tag{72}$$

with $F_i \in \mathcal{F}$ and with $A_{m+1} = A$. We symbolically denote such a sequence by

$$\eta : A_0 \overset{\mathcal{F}}{\rightsquigarrow} A$$

and we let $\#(\eta) = m + 1$ be the number of triangles in the sequence. We call the ordered family $(F_0, \ldots, F_m)$ the linearization of η and denote it by $\ell(\eta)$. By analogy with classical topology, we will refer to each exact triangle as in (72) as a *cone attachment over* F_i. Assume now that $\mathcal{C}$ is a TPC. In this case we will use the notation

$$\eta : A_0 \overset{\mathcal{F}^{\Sigma,T}}{\rightsquigarrow} A$$

to refer to a sequence similar to (72) where each triangle Δ_i is now *strict exact in* $\mathcal{C}$. More specifically this means that each triangle Δ_i is now replaced by a strict exact triangle (in the sense of [**BCZ24b**, Definition 2.42])

$$\Delta_i' \quad : \quad F_i \longrightarrow A_i \longrightarrow A_{i+1} \longrightarrow \Sigma^{-w_i} T F_i,$$

for some weight $w_i \geq 0$, $i = 0, \ldots, m$.

We emphasize that in this case the F_i's are taken to be elements in the set $\mathcal{F}^{\Sigma,T} = \{T^i \Sigma^\alpha F \mid F \in \mathcal{F}, i \in \mathbb{Z}, \alpha \in \mathbb{R}\}$. The weight $w(\eta)$ of the decomposition η, is defined as the sum of the weights of each of the triangles Δ_i' in η, $w(\eta) = w_0 + \cdots + w_m$ (see [**BCZ24b**] for the basics on TPCs). If η is of weight 0, as it will be the case in most of what follows, then each of the strict exact triangles in η is an exact triangle in the usual triangulated category $\mathcal{C}^0$.

Definition 3.4.1.1. Fix $\mathcal{C}$ a TPC, as above, a family $\mathcal{F} \subset \mathrm{Obj}(\mathcal{C})$, and also $\epsilon \in [0, \infty)$. For any two objects A, B of $\mathcal{C}$ the *ϵ-weight cone-length of A relative to B with linearization in $\mathcal{F}$* is given by:

$$N_{\mathcal{C}}(A, B; \mathcal{F}, \epsilon) := \inf \left\{ \#(\eta) \;\; | \;\; \eta : B \overset{\mathcal{F}^{\Sigma,T}}{\rightsquigarrow} A', \quad w(\eta) = 0, \quad d_{\mathrm{int}}(A, A') \leq \epsilon \right\}. \qquad (73)$$

In case the category $\mathcal{C}$ is clear from the context we write instead $N(A, B; \mathcal{F}, \epsilon)$. Moreover, we write $N(A; \mathcal{F}, \epsilon)$ when $B = 0$,

$$N(A; \mathcal{F}, \epsilon) := N(A, 0; \mathcal{F}, \epsilon) \; .$$

In short, $N(A, B; \mathcal{F}, \epsilon)$ tells us how many iterated exact triangles in the (usual) triangulated category $\mathcal{C}^0$ are needed to get ϵ-close to A in the interleaving pseudo-distance, starting from B, by attaching cones over objects of the form $T^i \Sigma^\alpha F$, $F \in \mathcal{F}$, $\alpha \in \mathbb{R}$, $i \in \mathbb{Z}$.

The relation between weighted cone-length and approximability is that if $\mathcal{F}$ is finite and $N_{\mathcal{C}}(A; \mathcal{F}, \epsilon) < \infty, \forall \, A \in \mathrm{Obj}(\mathcal{C})$, then $\mathcal{F}$ is an ϵ-approximating family in the sense of Definition 3.1.2.3, and $g_{\mathcal{C}}(X; \epsilon) \leq \#(\mathcal{F})$.

Remark 3.4.1.2. a. The definition of $N(A, B; \mathcal{F}, \epsilon)$ above is, in some sense, the simplest possible as it does not require any TPC machinery. Indeed, η in (73) consists of exact triangles in the usual sense, in $\mathcal{C}^0$, and $d_{\mathrm{int}}(-)$ is the interleaving pseudo-metric which is already defined in a persistence category, see (3). Another measurement, more natural from the TPC perspective, can be defined by

$$N'(A,B;\mathcal{F},\epsilon)=\inf\{\ \#(\eta)\ \mid\ \eta : B \overset{\mathcal{F}^{\Sigma,T}}{\rightsquigarrow} A\ \ ,\ \ w(\eta)\leq\epsilon\}\ .$$

This does not bring any essential new information because basic TPC algebra - see proof of Lemma 2.87 in [BCZ24b] - implies:

$$N(A;\mathcal{F},2\epsilon)\leq N'(A;\mathcal{F},\epsilon)\leq N(A;\mathcal{F},\frac{\epsilon}{4})\ .$$

b. Weighted cone-length satisfies some simple splitting inequalities associated with refinement of decompositions. These have generally a simpler expression for $N'(-,-)$ rather than for $N(-,-)$. For instance, we have

$$N'(A,A'';\mathcal{F},\epsilon'+\epsilon'')\leq N'(A,A';\mathcal{F},\epsilon')+N'(A',A'';\mathcal{F},\epsilon'')$$

which leads to the definition of fragmentation metrics on $\mathrm{Obj}(\mathcal{C})$ (see [BCZ24b]).

c. The definition of cone-length above is very flexible. By changing the family of objects over which cones are attached one obtains several other variants that are also of interest. One such choice, leading to smaller values for the resulting cone-length, is to replace $\mathcal{F}^{\Sigma,T}$ by

$$\mathcal{F}^{\otimes}=\{F\otimes V\mid F\in\mathcal{F},\quad V\ \text{ is a finite dimensional, graded, filtered vector space}\}$$

with the convention that $F\otimes\mathbf{k}[a]=\Sigma^{v(a)}T^{|a|}F$ where $\mathbf{k}[a]$ is the 1-dimensional vector space generated by a with degree $|a|$ and filtration level $v(a)$. Notice that $\mathcal{F}^{\Sigma,T}\subset\mathcal{F}^{\otimes}$. The resulting notion of cone length is such that, at each stage, one may attach a cone over a finite sum of copies of shifts and translates of elements of $\mathcal{F}$. We will denote this notion of cone-length by $N(A,B;\mathcal{F}^{\otimes},\epsilon)$.

d. The numbers $N(A,B;\mathcal{F},\epsilon)$ are decreasing in ϵ and, by taking $\epsilon\to\infty$, we obtain at the limit variants of the same notions that are often more familiar from standard topology and homological algebra.

Under certain constraints, there exists a useful lower bound for $N_{\mathcal{C}}(L;\mathcal{F},\epsilon)$ which is well defined whenever $\mathcal{F}$ is finite. The inspiration for much of the algebraic considerations related to the next result is found in Dimitrov-Haiden-Katzarkov-Kontsevich [DHKK14].

PROPOSITION 3.4.1.3. *Let $\mathcal{C}$ be a TPC which is the homological category of a pre-triangulated category of filtered A_∞-modules over a strictly unital filtered A_∞-category and such that* $\hom_{\mathcal{C}}(X,Y)$ *is of finite type for all $X,Y\in Obj(\mathcal{C})$. Assume that $\mathcal{F}\subset Obj(\mathcal{C})$ is finite and consists of Yoneda modules. Under these assumptions, there exists a constant $k(\mathcal{F})$ depending on the family $\mathcal{F}$ (and on $\mathcal{C}$) such that for every $L\in Obj(\mathcal{C})$ we have:*

$$N(L;\mathcal{F},\epsilon)\geq k(\mathcal{F})\sum_{F\in\mathcal{F}}\#\left(\mathcal{B}^{2\epsilon}_{\hom_{\mathcal{C}}(F,L)}\right)$$

where $\mathcal{B}^{\delta}_{V}$ is the barcode consisting of the bars of length greater than δ in the persistence module V.

Recall that a persistence module is of finite type if its barcode contains finitely many bars. The units of a filtered A_∞-category are always assumed to be in filtration 0 (see §1.3.1 for our conventions).

The conditions on $\mathcal{C}$ in the statement are satisfied for some of the TPCs of Fukaya type that are among the main examples of interest in the paper as well as for the homotopy category $H^0(\mathcal{F}\mathbf{Ch}^{fg})$ of finitely generated filtered chain complexes over the field $\mathbf{k}$ (our ground field). We denote by $H^0(\mathcal{F}\mathbf{Ch})$ the corresponding category without the finite generation condition. Both are TPCs as shown in [**BCZ24b**].

The proof of the proposition appears in §3.4.1.4 and is based on the results in the next two subsections that have some interest in themselves.

3.4.1.2. *Weighted retracts.* We fix here a triangulated persistence category $\mathcal{C}$ and we refer again to [**BCZ24b**] for the basic definitions and notation relevant to TPCs.

Similarly to (73) we define the ϵ-weight *retract* cone-length of A relative to B by:

$$N^r_{\mathcal{C}}(A, B; \mathcal{F}, \epsilon) = \inf \left\{ \#(\eta) \;\;|\;\; \eta : B \overset{\mathcal{F}^{\Sigma,T}}{\rightsquigarrow} A', \;\; w(\eta) = 0, \;\; d_{\text{r-int}}(A, A') \leq \epsilon \right\} \tag{74}$$

(we recall that $d_{\text{r-int}}$ is defined in (4)). In what follows it is useful to use the following terminology: given two objects $A, \bar{A}$ in $\mathcal{C}$ we will say that A is *an ϵ-retract of* $\bar{A}$ if $d_{\text{r-int}}(A, \bar{A}) < \epsilon$. It is obvious that

$$N^r_{\mathcal{C}}(A, B; \mathcal{F}, \epsilon) \leq N_{\mathcal{C}}(A, B; \mathcal{F}, \epsilon) \; .$$

The point of the definition is that the retract cone-length is, in practice, easier to estimate as we will see below. As before, if $B = 0$ we omit it from the notation and we skip $\mathcal{C}$ if it is clear from context. The proof of Proposition 3.4.1.3 will make use **of** some algebraic properties of $N^r(-;-,\epsilon)$ and, indeed, the argument shows a stronger inequality than the one in the statement of the Proposition, namely:

$$N^r(L; \mathcal{F}, \epsilon) \geq k(\mathcal{F}) \sum_{F \in \mathcal{F}} \# \left(\mathcal{B}^{2\epsilon}_{\hom_{\mathcal{C}}(F,L)} \right) \tag{75}$$

REMARK 3.4.1.4. a. There is yet another variant of weighted retract cone-length which is defined by:

$$\tilde{N}^r(A, B; \mathcal{F}, \epsilon) = \inf \left\{ N'(\bar{A}, B; \mathcal{F}, \epsilon) \mid d_{\text{r-int}}(A, \bar{A}) = 0 \right\} .$$

This variant is particularly relevant if $\mathcal{C}^0$ is split complete in which case the condition

$$d_{\text{r-int}}(A, \bar{A}) = 0$$

means that A is isomorphic to a direct summand of $\bar{A}$.

b. It is easy to see using Remark 3.4.1.2 a. that:

$$N^r(A, B; \mathcal{F}, 2\epsilon) \leq \tilde{N}^r(A, B; \mathcal{F}, \epsilon) \; .$$

c. The definition in (74) and the point a. of the remark, are reminiscent of the relation between the Lusternik-Schnirelmann (LS) category and cone-length in topology, Theorem 1.1

in [**Cor94**]: the LS category of a topological space X is the smallest integer n such that X is a homotopy factor of an n-cone.

For the triangulated persistence category $\mathcal{C}$ and $\mathcal{F} \subset \mathrm{Obj}(\mathcal{C})$, assume that $\mathcal{F}$ is finite. Let $G_{\mathcal{F}} = \oplus_{F \in \mathcal{F}} F$. We will identify the family $\{G_{\mathcal{F}}\}$ with the single object $G_{\mathcal{F}}$. One useful property of the weighted retract cone-length is:

LEMMA 3.4.1.5. *With the notation above we have*

$$N^r(A, B; G_{\mathcal{F}}, \epsilon) \leq N^r(A, B; \mathcal{F}, \epsilon) \leq N^r(A, B; G_{\mathcal{F}}, \epsilon) \cdot \#(\mathcal{F}).$$

The proof of the lemma is a simple exercise in manipulation of exact triangles in triangulated categories, in this case in $\mathcal{C}_0$. The interest of the statement is that the cone-decomposition giving $N^r(A, B; G_{\mathcal{F}}, \epsilon)$ is a sequence of exact triangles as in (72) but such that each F_i is replaced by a possible shift and/or translate of the single object $G_{\mathcal{F}}$. As a result, it makes sense to consider the measurement $N^r_{\mathcal{C}}(A, B; G, \epsilon)$ for any three objects A, B, G in $\mathcal{C}$.

Both variants of weighted retract cone-length, N^r as well as $\tilde{N}^r$, satisfy the additive splitting inequality in Remark 3.4.1.2 b but also a different, multiplicative one which requires some assumptions on $\mathcal{C}$. Without weights, this type of inequality was first noticed and used by Dimitrov-Haiden-Katzarkov-Kontsevich [**DHKK14**]. The weighted version will be instrumental here in proving Proposition 3.4.1.3.

LEMMA 3.4.1.6. *Assume that $\mathcal{C}$ is the persistence homological category associated with a pre-triangulated, filtered A_∞-category. For any $A, G, G' \in Obj(\mathcal{C})$ we have:*

$$N^r_{\mathcal{C}}(A; G, \epsilon) \leq N^r_{\mathcal{C}}(A; G', \epsilon') N^r_{\mathcal{C}}(G'; G, \epsilon'')$$

whenever $\epsilon \geq \epsilon' + N^r_{\mathcal{C}}(A; G', \epsilon')\epsilon''$.

The same formula remains true for $\tilde{N}^r(-; -, -)$ under a different assumption on $\mathcal{C}$, namely that $\mathcal{C}^0$ is split complete.

REMARK 3.4.1.7. a. The relation tying $\epsilon, \epsilon', \epsilon''$ in the lemma is complicated but in this paper we will only need to use the case when $\epsilon'' = 0$ and $\epsilon = \epsilon'$.

b. The weighted cone length $N_{\mathcal{C}}(-; -, \epsilon)$ also satisfies a multiplicative inequality of the type in Lemma 3.4.1.6 but the precise formula is more complicated to state and thus we skip it here as it is not necessary later in the paper.

c. The main examples of interest in this paper of categories $\mathcal{C}$ as in the first part of the Lemma are the homological category of filtered modules over a filtered A_∞-category as well as, the simplest example, the homotopy category of filtered chain complexes.

PROOF OF LEMMA 3.4.1.6. The statement for $N^r_{\mathcal{C}}(-)$ is an immediate consequence of Lemma 6.12 in [**BCS21**]. We recall the statement of this lemma reformulated for filtered modules. Assume that $\mathcal{M}$ is an iterated cone of filtered A_∞-modules (over 0-shift maps):

$$\mathcal{M} = \mathcal{C}one(\mathcal{K}_s \to \mathcal{C}one(\mathcal{K}_{s-1} \to \ldots \to \mathcal{C}one(\mathcal{N} \to \mathcal{C}one(\mathcal{K}_{i-1} \to \ldots \to \\ \to \mathcal{C}one(\mathcal{K}_2 \to \mathcal{K}_1) \ldots)) \ .$$

Lemma 6.12 from [**BCS21**] states that if $\mathcal{N}$ is an r-retract of another filtered module $\mathcal{N}'$, then $\mathcal{M}$ is an r-retract of a filtered module $\mathcal{M}'$ which is an iterated cone of the same form as $\mathcal{M}$ but with $\mathcal{N}'$ replacing $\mathcal{N}$ in the decomposition above. The proof of this result is based on the fact that in a category such as those in the statement of 3.4.1.6 a homotopy commutative square has the property that the map induced between the cones over the two horizontal maps in the square can be expressed explicitly by a formula involving the two vertical maps and the commuting homotopy.

This result is applied in our context as follows. We start with a decomposition in $\mathcal{C}^0$ $\eta : 0 \overset{(G')^{\Sigma,T}}{\rightsquigarrow} \bar{A}$ and we assume that A is an ϵ'-retract of $\bar{A}$. We also assume that there is a second decomposition, also in $\mathcal{C}^0$, $\eta' : 0 \overset{G^{\Sigma,T}}{\rightsquigarrow} \bar{G}'$ and we assume that G' is an ϵ''-retract of $\bar{G}'$. By applying the algebraic result mentioned above to each of the terms in the linearization $\ell(\eta)$ of η we transform η into a new decomposition $\bar{\eta} : 0 \overset{(\bar{G}')^{\Sigma,T}}{\rightsquigarrow} \bar{A}'$, still in $\mathcal{C}^0$, such that A is now an $(\epsilon' + \#\ell(\eta)\epsilon'')$-retract of $\bar{A}'$. Finally, by refining each term in the linearization of $\bar{\eta}$ by using the decomposition η' we obtain the final decomposition $\bar{\eta}' : 0 \overset{G^{\Sigma}}{\rightsquigarrow} \bar{A}'$ which shows the claim (see Proposition 2.55 in [**BCZ24b**] for the precise description of the refinement process).

The argument for $\tilde{N}^r(-;--)$, under the assumption that $\mathcal{C}^0$ is split-complete, is similar but simpler as it does not require Lemma 6.12 from [**BCS21**], and we leave it as an exercise. $\square$

One more property of weighted retract cone-length is needed.

Lemma 3.4.1.8. *Assume that $\mathcal{C}$ satisfies the hypotheses in the statement of Proposition 3.4.1.3. Let any $A, B \in Obj(\mathcal{C})$ and assume that $X \in Obj(\mathcal{C})$ is a Yoneda module. For any $\epsilon \geq 0$ we have:*

$$N^r_{\mathcal{C}}(A; B, \epsilon) \geq N^r_{H^0(\mathcal{F}\mathbf{Ch})}(\hom_{\mathcal{C}}(X, A); \hom_{\mathcal{C}}(X, B), \epsilon)\ .$$

Proof. Denote by $\mathcal{D}$ the pre-triangulated A_∞-category of modules such that $\mathcal{C} = H^0\mathcal{D}$. Given that the underlying A_∞-category is strictly unital, and that X is a Yoneda module, we see that the functor $\hom_{\mathcal{D}}(X, -)$ transforms exact triangles in $\mathcal{D}$ into exact triangles in $[H^0(\mathcal{F}\mathbf{Ch})]^0$. This follows from the identification $\mathcal{M}(X) \cong \hom_{\mathcal{D}}(X, \mathcal{M})$, for each module $\mathcal{M}$, that is part of the properties of the Yoneda embedding functor - see §1.3.1 for the persistence setting - and the explicit formula giving the cone of a module morphism $\phi : \mathcal{M} \to \mathcal{N}$, §3e in [**Sei08**]. By applying $\hom_{\mathcal{D}}(X, -)$ this formula reduces to the usual formula giving the cone of the induced chain morphism $\phi_X : \hom(X, \mathcal{M}) \to \hom(X, \mathcal{N})$. All these constructions respect filtrations and all the cones are taken over shift 0 morphisms. As a result, exact triangles in $\mathcal{C}^0$ are represented by cone-decompositions in $\mathcal{D}$, these are sent by $\hom_{\mathcal{D}}(X, -)$ to corresponding cone-decomposition in $\mathcal{F}\mathbf{Ch}^0$ that, in turn, give exact triangles in $[H^0(\mathcal{F}\mathbf{Ch})]^0$. The inequality in the statement is an immediate consequence of this observation. $\square$

We proceed under the assumptions in the statement of Proposition 3.4.1.3 and we also assume, as there, that $\mathcal{F} \subset \text{Obj}(\mathcal{C})$ is finite and consists of Yoneda modules. In particular, we can consider $G_{\mathcal{F}} = \oplus_{F\in\mathcal{F}} F$. In this case, Lemma 3.4.1.8 implies that:

$$N^r_{\mathcal{C}}(A; G_{\mathcal{F}}, \epsilon) \geq N^r_{H^0(\mathcal{F}\mathbf{Ch})}(\hom_{\mathcal{C}}(G_{\mathcal{F}}, A); \hom_{\mathcal{C}}(G_{\mathcal{F}}, G_{\mathcal{F}}), \epsilon). \tag{76}$$

Before we go on we temporarily introduce some simplified notation that will be used from now on till the end of §3.4.1.4. Namely, we will use $\mathbf{k}$ to denote two different (albeit closely related) things. Recall that throughout the paper $\mathbf{k}$ stands for the ground field for our (unfiltered) categories and chain complexes, but we will also denote by the same $\mathbf{k}$ the filtered chain complex with a single generator in degree 0 and of filtration level 0. When we want to specify that generator, say x, we will denote this chain complex also by $\mathbf{k}\langle x\rangle$.

Getting back to our considerations and continuing from (76) we deduce from Lemma 3.4.1.6 that

$$\begin{aligned} N^r_{H^0(\mathcal{F}\mathbf{Ch})}(\hom_{\mathcal{C}}(G_{\mathcal{F}}, A); \hom_{\mathcal{C}}(G_{\mathcal{F}}, G_{\mathcal{F}}), \epsilon) \cdot N^r_{H^0(\mathcal{F}\mathbf{Ch})}(\hom_{\mathcal{C}}(G_{\mathcal{F}}, G_{\mathcal{F}}); \mathbf{k}, 0) \geq \\ \geq N^r_{H^0(\mathcal{F}\mathbf{Ch})}(\hom_{\mathcal{C}}(G_{\mathcal{F}}, A); \mathbf{k}, \epsilon) \end{aligned}$$

Combining this with Lemma 3.4.1.5 we deduce the next inequality.

COROLLARY 3.4.1.9. *Under the assumptions of Proposition 3.4.1.3 and for*

$$k(\mathcal{F}) := 1/N^r_{H^0(\mathcal{F}\mathbf{Ch})}(\hom_{\mathcal{C}}(G_{\mathcal{F}}, G_{\mathcal{F}}); \mathbf{k}, 0)$$

we have for each object A of $\mathcal{C}$ and all $\epsilon \geq 0$:

$$N^r_{\mathcal{C}}(A; \mathcal{F}, \epsilon) \geq k(\mathcal{F}) N^r_{H^0(\mathcal{F}\mathbf{Ch})}(\hom_{\mathcal{C}}(G_{\mathcal{F}}, A); \mathbf{k}, \epsilon)\ .$$

3.4.1.3. *Weighted cone-length for filtered chain complexes.* The aim of this subsection is to examine the various notions of weighted cone-length that we introduced before in the very basic but important example when the triangulated persistence category $\mathcal{C}$ is the homotopy category of filtered, finite dimensional chain complexes, $H^0(\mathcal{F}\mathbf{Ch})^{fg}$, over the base field $\mathbf{k}$.

EXAMPLE 3.4.1.10. The following statements hold in the category $H^0(\mathcal{F}\mathbf{Ch})^{fg}$.

a) Each object V in $H^0(\mathcal{F}\mathbf{Ch})^{fg}$ can be written as a finite direct sum, unique up to permutation, of translations of elementary filtered chain complexes of two types

$$E_2(a, b) = \mathbf{k}(a, b : da = 0, db = a), \text{ and } \ E_1(c) = \mathbf{k}(c : dc = 0).$$

For each of the generators x of these chain complexes we denote by $v(x) \in \mathbb{R}$ its filtration level. For $E_2(a, b)$ we assume $v(b) \geq v(a)$ with degrees $|a| = 0$, $|b| = 1$. For $E_1(c)$ we assume $v(c) = 0$ and $|c| = 0$. Note that this notation differs form other conventions common in the literature. For example, in [BCZ24b, Section 2.5.2] $E_2(a, b)$ denotes a chain complex with two generators as above but a, b stand for the filtration levels of these two generators rather than for the generators themselves (and similarly for $E_1(c)$). In our notation the barcode of the persistence homology $H(E_2(a, b))$ is $[v(a), v(b))$ while in [BCZ24b] this barcode is $[a, b)$.

b) The translation T acts by shifting degrees in the obvious way: $|T^k x| = |x| + k$ and the shift functor changes the filtration levels by $v(\Sigma^\alpha x) = v(x) + \alpha$.

c) Recall that $\mathbf{k}\langle x\rangle$ also stands for the 1-dimensional filtered chain complex over $\mathbf{k}$ with one generator x, with $|x| = 0$, $v(x) = 0$. We then have:

$$E_2(a,b) = \mathcal{C}one(\ \Sigma^{v(b)}\mathbf{k}\langle b\rangle \ \rightarrow\ \Sigma^{v(a)}\mathbf{k}\langle a\rangle\ )$$

where the cone is taken over the map $b \rightarrow a$. As for $E_1(c)$, we have:

$$E_1(c) \cong \Sigma^{v(c)}\mathbf{k}\langle c\rangle\ .$$

As a result, for $\mathcal{C} = H^0(\mathcal{F}\mathbf{Ch})^{fg}$ and $X = \mathcal{O}b(H^0(\mathcal{F}\mathbf{Ch})^{fg})$ we have $g(X; 0) = 1$ (see (71)) with $\mathcal{F}_0 = \mathbf{k}\langle x\rangle$. Thus $H^0(\mathcal{F}\mathbf{Ch})^{fg}$ is 0-approximable with generating rank equal to 1.

The next result is also elementary but less immediate. We fix some additional notation. For a persistence module K, we denote by $\mathcal{B}_K$ its barcode, by $\mathcal{B}^{\delta}_K$ the barcode consisting of only the bars of length $> \delta$ in K, and by K^{∞} the ∞-limit of the persistence module K (it is a vector space of dimension equal to the number of semi-infinite bars in K).

Lemma 3.4.1.11. *Let V be a filtered, finite dimensional chain complex. Then:*

$$N(V,0;\mathbf{k},\epsilon) = N^r(V,0;\mathbf{k},\epsilon) = 2\#\left(\mathcal{B}^{2\epsilon}_{H(V)}\right) - \dim_{\mathbf{k}}(H(V)^{\infty})\ . \tag{77}$$

Proof. For a filtered chain complex V in our class and $\delta \geq 0$, let V_δ be the chain complex obtained by writing V as a direct sum of elementary terms, as at point a) in Example 3.4.1.10, and eliminating from the sum all the terms $E_2(a,b)$ with $v(b) - v(a) \leq \delta$. Up to a filtered chain homotopy equivalence of V there is an obvious inclusion of chain complexes $i_{V,\delta} : V_\delta \rightarrow V$ as well as a projection $j_{V,\delta} : V \rightarrow V_\delta$. It is easy to see that:

i. V and $V_{2\delta}$ are δ interleaved. This happens because the map $\eta^V_{2\delta} : \Sigma^\delta V \rightarrow \Sigma_{-\delta}V$ is easily seen to be chain homotopic to the composition $i_V \circ j_V$ (because $\eta^{E_2(a,b)}_{2\delta}$ is null-homotopic if $v(b) - v(a) \leq 2\delta$).
ii. The dimension of $V_{2\delta}$ is given by $2\#(\ \mathcal{B}^{2\delta}_{H(V)}\ ) - \dim_{\mathbf{k}}(H(V)^{\infty})$. This is because a pair of generators a, b in a term of type $E_2(a,b)$ (in the direct sum writing of $V_{2\delta}$) are associated with a finite bar of the form $[v(a), v(b))$ and each generator of type c in a term $E_1(c)$ corresponds to one semi-infinite bar, and survives to a generator in $H(V)^{\infty}$.
iii. There exists an obvious cone decomposition $\eta : 0 \overset{\langle\mathbf{k}\rangle^{\Sigma,T}}{\rightsquigarrow} V_{2\delta}$ with a number of cones equal to the dimension of $V_{2\delta}$.

As a result of these three points we conclude that $N(V,0;\mathbf{k},\epsilon) \leq 2\#\left(\mathcal{B}^{2\epsilon}_{H(V)}\right) - \dim_{\mathbf{k}}(H(V)^{\infty})$.

To conclude the proof of (77) we need to show that, if V is an ϵ-retract of W, then W is at least a $\dim_{\mathbf{k}}(V_{2\epsilon})$-cone for a decomposition $\eta' : 0 \rightsquigarrow W$ with a linearization with terms of the form $\Sigma^\alpha T^i\mathbf{k}$ (these are the elements of $\langle\mathbf{k}\rangle^{\Sigma,T}$).

To show this we first notice that for any W in our class of filtered chain complexes we have that any cone decomposition $\eta : 0 \rightsquigarrow W$ in $(H^0(\mathcal{F}\mathbf{Ch})^{fg})^0$ (this is the 0-level category of the TPC $H^0(\mathcal{F}\mathbf{Ch})^{fg}$) with linearization terms in $\langle\mathbf{k}\rangle^{\Sigma,T}$ has at least $\dim_{\mathbf{k}}(W_0)$ terms, where W_0 is obtained from W by omitting all terms $E_2(a,b)$ such that $v(a) = v(b)$. Indeed, any such decomposition is the image of a decomposition $\bar{\eta} : 0 \rightsquigarrow W'$ in the underlying category of filtered chain complexes, $\mathcal{F}\mathbf{Ch}$, and there is a filtered quasi-isomorphism (with 0-shift) $\phi : W \to W'$. The number of cones in $\bar{\eta}$ is at least $\dim_{\mathbf{k}}(W')$ for dimension reasons. On the other hand, $\phi|_{W_0}$ is injective (this can be seen by noting that ϕ induces an identification $\mathcal{B}_W \cong \mathcal{B}_{W'}$ and $\mathcal{B}_W = \mathcal{B}_{W_0}$). Therefore, it remains to show that if V is an ϵ-retract of W, then $\dim_{\mathbf{k}}(W_0) \geq \dim_{\mathbf{k}}(V_{2\epsilon})$. If V is an ϵ-retract of W it is immediate that it also is an ϵ-retract of W_0. It is a simple exercise to show that in this case $V_{2\epsilon}$ injects into W_0, which concludes the proof of the lemma. □

Remark 3.4.1.12. An argument similar to the proof of Lemma 3.4.1.11 shows:

$$N(0, V; \mathbf{k}, \epsilon) = 2\#\left(\mathcal{B}^{2\epsilon}_{H(V)}\right) - \dim_{\mathbf{k}}(H(V)^{\infty})\ . \tag{78}$$

As a result of this identity and (77), and of the triangular inequality in Remark 3.4.1.2 b) we obtain rough upper bounds for $N(V, V'; \mathbf{k}, \epsilon)$, $N^r(V, V'; \mathbf{k}, \epsilon)$ in terms of the barcodes of $H(V)$ and of $H(V')$.

3.4.1.4. *Proof of Proposition 3.4.1.3.* Using the preparatory material from the last two sub-subsections we can finalize the proof of Proposition 3.4.1.3.

We start with two other simple algebraic remarks concerning filtered chain complexes. For this let V be a finitely generated, filtered chain complex. First, for any $\delta \geq 0$, we obviously have the inequality

$$\#\left(\mathcal{B}^{\delta}_{H(V)}\right) - \dim_{\mathbf{k}}(H(V)^{\infty}) \geq 0$$

which we rewrite as $2\,\#(\mathcal{B}^{\delta}_{H(V)}) - \dim_{\mathbf{k}}(H(V)^{\infty}) \geq \#(\mathcal{B}^{\delta}_{H(V)})$. Secondly, if V' is another chain complex in our class we have

$$\#\left(\mathcal{B}^{\delta}_{H(V\oplus V')}\right) = \#\left(\mathcal{B}^{\delta}_{H(V)}\right) + \#\left(\mathcal{B}^{\delta}_{H(V')}\right)\ .$$

Combining these relations with Corollary 3.4.1.9 and Lemma 3.4.1.11 and recalling $G_{\mathcal{F}} = \oplus_{F\in\mathcal{F}} F$ we deduce:

$$\begin{aligned} N_{\mathcal{C}}(L; \mathcal{F}, \epsilon) \geq N^r_{\mathcal{C}}(L; \mathcal{F}, \epsilon) &\geq k(\mathcal{F}) N^r_{H^0(\mathcal{F}\mathbf{Ch})}(\hom_{\mathcal{C}}(G_{\mathcal{F}}, L); \mathbf{k}, \epsilon) \geq \\ &\geq k(\mathcal{F}) \cdot \#\left(\mathcal{B}^{2\epsilon}_{\hom_{\mathcal{C}}(G_{\mathcal{F}}, L)}\right) = k(\mathcal{F}) \sum_{F\in\mathcal{F}} \#\left(\mathcal{B}^{2\epsilon}_{\hom_{\mathcal{C}}(F, L)}\right) \end{aligned}$$

which shows the inequality (75) and also concludes the proof of the proposition. □

3.4.1.5. *Weighted categorical entropy.* We introduce here a weighted notion of categorical entropy that is the weighted analogue of a notion introduced by Dimitrov-Haiden-Katzarkov-Kontsevich in [**DHKK14**].

Consider a triangulated persistence category $\mathcal{C}$ and fix $X \subset \mathrm{Obj}(\mathcal{C})$ as well as a family $\mathcal{F} \subset \mathcal{O}b(\mathcal{C})$, as in §3.4.1.1.

DEFINITION 3.4.1.13. Given a (set theoretic) map $\Psi : X \to X$ and one object $A \in X$, the ϵ-weighted categorical entropy of Ψ at A, relative to $\mathcal{F} \subset \mathcal{C}$, is defined by

$$h_{\mathcal{F}}(\Phi; A, \epsilon) = \limsup_{n\to\infty} \frac{\log(N_{\mathcal{C}}(\Psi^n A; \mathcal{F}, \epsilon))}{n}$$

Notice that this definition is only of interest when $\mathcal{F}$ is an ϵ-approximating family for X in $\mathcal{C}$ (as in Definition 3.1.2.3) which we will assume from now on, otherwise there is no way to ensure that $N_{\mathcal{C}}(\Psi^n A; \mathcal{F}, \epsilon)$ is finite. In other words, TPC approximability is needed for this definition to make sense.

REMARK 3.4.1.14. a. Of course, a similar notion can be defined using $N^r_{\mathcal{C}}(-; --)$ leading to a variant denoted by $h^r_{\mathcal{F}}(\Psi; A, \epsilon)$.

b. The numbers $h_{\mathcal{F}}(\Psi; A, \epsilon)$, $h^r_{\mathcal{F}}(\Psi; A, \epsilon)$ are decreasing functions in ϵ. It is useful to allow for $\epsilon = \infty$. In that case, the weight constraints disappear and we obtain purely (triangulated) categorical notions. In particular, when $\mathcal{F}$ is finite, the categorical entropy introduced in [**DHKK14**] is given by $h_{G_{\mathcal{F}}}(\Psi; A, \infty)$ with $G_{\mathcal{F}} = \oplus_{F\in\mathcal{F}} F$.

c. By contrast to the un-weighted case, the weighted entropy definition depends on the choice of the family $\mathcal{F}$.

d. There are also notions of *slow* entropy that are defined by formulae of a similar type, in analogy with the standard corresponding dynamical systems definitions.

We will now see that Proposition 3.4.1.3 implies that weighted categorical entropy admits as lower bound a version of the barcode entropy introduced in [**cGG24**].

To make this relationship precise we define the notion of barcode entropy that we will use here in the setting of the TPC, $\mathcal{C}$. We have as before the map $\Psi : X \to X$ and we pick two objects $A \in X$, $B \in \mathrm{Obj}(\mathcal{C})$. We now consider the ϵ-barcode entropy of Ψ at A, relative to B:

$$\hbar(\Psi; A, B; \epsilon) = \limsup_{n\to\infty} \frac{\log\left(\#(\mathcal{B}^{\epsilon}_{\hom_{\mathcal{C}}(B,\Psi^n A)})\right)}{n} \tag{79}$$

COROLLARY 3.4.1.15. *Assume that the assumptions in Proposition 3.4.1.3 are satisfied. In particular, $\mathcal{F}$ is finite and all $\hom_{\mathcal{C}}(-,-)$ are finite type persistence modules. We have the inequalities:*

$$h_{\mathcal{F}}(\Psi; A, \epsilon) \geq h^r_{\mathcal{F}}(\Psi; A, \epsilon) \geq h^r_{G_{\mathcal{F}}}(\Psi; A, \epsilon) \geq \hbar(\Psi; A, G_{\mathcal{F}}; 2\epsilon)$$

where $G_{\mathcal{F}} = \oplus_{F\in\mathcal{F}} F$.

This follows directly from Proposition 3.4.1.3, see also (75).

3.4.1.6. *Weighted complexity of Lagrangians.* We return here to the geometric context from §3.2.1 and discuss some implications of the nearby approximability Theorem 3.2.2.1 from the point of view of the complexity measurements introduced earlier in this section.

We start with some definitions of complexity that are natural from the point of view of TPC-approximability - see Definition 3.1.2.1 as well as Remarks 3.1.2.2 and 3.1.2.4.

Definition 3.4.1.16. Assume (X,d) is TPC-approximable and that, for a fixed $\epsilon>0$, the quasi-isometric embeddings

$$\Phi=\{\Phi_\eta\}\ ,\ \Phi_\eta:(X,d)\to(\mathrm{Obj}(\mathcal{Y}_\eta),\bar{d}^{\,\eta}_{\mathrm{int}})\ .$$

that are part of the ϵ-TPC approximating data are fixed.

i. The ϵ-generating rank of (X,d) relative to Φ is defined by

$$g_\Phi(X;\epsilon)=\limsup_{\eta\to 0}\ g_{\mathcal{Y}_\eta}(\Phi_\eta(X);\epsilon)$$

See (71) for the definition of $g_{\mathcal{Y}_\eta}$.

ii. At this point fix also a corresponding family $\mathcal{F}_{\epsilon,\eta}\subset\mathrm{Obj}(\mathcal{Y}_\eta)$ as in Definition 3.1.2.1. For some $\epsilon'\geq\epsilon$, the ϵ'-cone-length of an element $a\in X$ relative to the family $\mathcal{F}_\epsilon=\{\mathcal{F}_{\epsilon,\eta}\}_{0<\eta<\epsilon}$ is given by:

$$N_\Phi(a;\mathcal{F}_\epsilon,\epsilon')=\limsup_{\eta\to 0}\ N_{\mathcal{Y}_\eta}(\Phi_\eta(a);\mathcal{F}_{\epsilon,\eta},\epsilon')\ .$$

Notice that for $\epsilon_1\geq\epsilon_2\geq\epsilon$ we have $N_\Phi(a;\mathcal{F}_\epsilon,\epsilon_1)\leq N_\Phi(a;\mathcal{F}_\epsilon,\epsilon_2)$. These notions have properties very similar to those of the corresponding measurements from §3.4.1.1. In particular, given a map $\Psi:X\to X$ and assuming that (X,d) is TPC-approximable we define the ϵ'-weighted categorical entropy of Ψ at $a\in X$, relative to the approximating data (Φ, $\mathcal{F}_\epsilon=\{\mathcal{F}_{\epsilon,\eta}\}$), for $\epsilon'\geq\epsilon$ by

$$h_{\Phi,\mathcal{F}_\epsilon}(\Psi;a,\epsilon')=\limsup_{n\to\infty}\frac{\log N_\Phi(\Psi^n a;\mathcal{F}_\epsilon,\epsilon')}{n}\ . \tag{80}$$

We also have similar measurements for the corresponding retract notions that will be denoted by $N^r_\Phi(a;\mathcal{F}_\epsilon,\epsilon')$, $h^r_{\Phi,\mathcal{F}_\epsilon}(\Psi;a,\epsilon')$ respectively.

Remark 3.4.1.17. Assume that the metric space (X,d) is totally bounded and that it is TPC-approximable (respectively, retract-approximable) with a system of (A,η)-quasi-isometric embeddings $\Phi=\{\Phi_\eta\}$ and a corresponding family $\{\mathcal{F}_{\epsilon,\eta}\}$ as above. Then, for each fixed ϵ, there exists $K>0$ depending on (X,d), Φ and $\{\mathcal{F}_{\epsilon,\eta}\}$ such that

$$N_\Phi(a;\mathcal{F}_\epsilon,2\epsilon)\leq K$$

(respectively, $N^r_\Phi(a;\mathcal{F}_\epsilon,2\epsilon)\leq K$) for all $a\in X$. This is a simple exercise: the bound K is given by considering a finite ϵ_1-net of (X,d), $x_1,\ldots,x_k\subset X$ such that $\epsilon_1/A<\epsilon$ and taking

$$K=\max_i N_{\mathcal{Y}_{\eta'}}(\Phi_{\eta'}(x_i);\mathcal{F}_{\epsilon,\eta'},\epsilon_1)\ \ (\text{respectively } K=\max_i N^r_{\mathcal{Y}_{\eta'}}(\Phi_{\eta'}(x_i);\mathcal{F}_{\epsilon,\eta'},\epsilon)\ )$$

for some $\eta'<\epsilon$. In particular, if (X,d) is totally bounded, for all $x\in X$ and any map $\Psi:X\to X$ the entropies $h_{\Phi,\mathcal{F}_\epsilon}(\Psi;a,2\epsilon)$ and $h^r_{\Phi,\mathcal{F}_\epsilon}(\Psi;a,2\epsilon)$ vanish. Thus, to show that (X,d) is not totally bounded, it is enough to find one $\epsilon>0$ and a sequence $a_k\in X$ such that $N_\phi(a_k;\mathcal{F}_\epsilon,2\epsilon)\to\infty$ in k.

We now turn to geometry. We have shown in §3.2 that the metric space

$$\mathcal{L}(N) = (\mathcal{L}\mathrm{ag}^{(ex)}(D^*N), d_\gamma)$$

is approximable in the sense of Definition 3.1.2.1. In §3.2.7.1 and §3.2.7.2 we produced two types of TPC ϵ- approximating data for this space: $(\Phi, \mathcal{F}_\epsilon)$ in the first case and $(\Phi', \mathcal{F}'_\epsilon)$ in the second, which is only *retract* approximating. We referred to the first type as *local* and to the second as *ambient*.

To improve readability we recall the basic features of these two choices of approximating data. The family $\Phi = \{\Phi_\eta\}$ of quasi-isometric embeddings consists of Yoneda embeddings - see §3.2.7:

$$\Phi_\eta = \mathcal{Y}_{\epsilon',p} : \mathcal{L}\mathrm{ag}^{(ex)}(D^*N) \to \mathrm{Obj}(\mathcal{C}_p) \ .$$

Here $\mathcal{C}_p$ are persistence homology categories of filtered A_∞-modules over the filtered Fukaya category $\mathcal{F}uk(\mathcal{L}\mathrm{ag}^{(ex)}(D^*N), p)$ constructed for the choice of perturbations p, with $\nu(p) < \eta$ and $\epsilon' < \epsilon$. The family $\mathcal{F}_\epsilon$ consists of modules corresponding to a finite family $\{F_0, \ldots, F_{x_m}\}$ of fibers of D^*N where $\{x_0, x_1, \ldots, x_m\}$ are the critical points of an auxiliary Morse function $\varphi_{N,K,\delta} : N \to [0, K]$. The family Φ' is very similar except that the place of $\mathcal{C}_p$ is taken by the persistence derived category

$$\mathcal{D}_p(E) = PD(\mathcal{F}uk(\mathcal{L}ag^{(ex)}(E), p))$$

that is constructed using the auxiliary Lefschetz fibration $\bar{h} : E \to \mathbb{C}$ (see §3.2.5) that itself depends on $\varphi_{N,K,\delta}$. The family $\mathcal{F}'_\epsilon$ consists of the Yoneda modules of Lagrangian spheres $\hat{S}_{x_i}$, one for each critical point of $\varphi_{N,K,\delta}$ and such that $\hat{S}_{x_i} \cap D^*N = F_{x_i}$.

In essence, one can view the approximability results in $\mathcal{C}_p$ as obtained by pull-back from those in $\mathcal{D}_p(E)$. Notice though that $(\Phi', \mathcal{F}'_\epsilon)$ is only retract ϵ-approximating, as indicated in Corollary 3.2.7.4.

REMARK 3.4.1.18. a. Recall that the comparison functors $\mathcal{H}_{p,q} : \mathcal{C}_p \to \mathcal{C}_q$ are TPC-functors for $p \preceq q$. They also preserve the generating families $\mathcal{F}_\epsilon$ in the sense that they send the module corresponding to a fibre F_{x_i} in the p-category $\mathcal{C}_p$ to a module 0-isomorphic to the module associated to the same F_{x_i} in $\mathcal{C}_q$. As a result, with ϵ fixed, our approximating families $\mathcal{F}_\epsilon$ are no longer dependent of η. Moreover, for each $L \in \mathcal{L}\mathrm{ag}^{(ex)}(D^*N)$ we have

$$N_{\mathcal{C}_p}(L; \mathcal{F}_\epsilon, \epsilon) \geq N_{\mathcal{C}_q}(L; \mathcal{F}_\epsilon, \epsilon)$$

as soon as $p \preceq q$. The same remark also applies to the system of categories $\{\mathcal{D}_p(E)\}_p$ and the family $\mathcal{F}'_\epsilon$, and leads to the same inequality, with $\mathcal{D}_-(E)$ in the place of $\mathcal{C}_-$ and as well with $N^r(-; -\epsilon)$ replacing $N(-; -\epsilon)$.

b. In the setting above we also have the inequality:

$$N_{\mathcal{C}_q}(L; \mathcal{F}_\epsilon, \epsilon) \geq N_{\mathcal{C}_p}\left(L; \mathcal{F}_\epsilon, \epsilon + c|\eta(p) - \eta(q)|\right)$$

also under the assumption $p \preceq q$ and for c a constant independent of L.

Given that the number of elements in $\mathcal{F}_\epsilon$ and in $\mathcal{F}'_\epsilon$ is equal to the number of critical points of $\varphi_{N,K,\delta}$ we deduce:

$$g_\Phi(\mathcal{L}(N);\epsilon) \leq \#\ \mathrm{Crit}(\varphi_{N,K,\delta})$$

as well as the same inequality for Φ'. When $\epsilon \to 0$ the number of elements in $\mathcal{F}_\epsilon$ and $\mathcal{F}'_\epsilon$ goes to ∞ at a rate that can be estimated through the construction in §3.2.3.

We now discuss in the same context and with the same notation the ϵ-weight cone-length $N_\Phi(L;\mathcal{F}_\epsilon,\epsilon)$ for $L \in \mathcal{L}\mathrm{ag}^{(ex)}(D^*N)$.

An upper bound for this cone-length can be obtained by counting all the exact triangles $\Delta_{i,j}$ from (49) that give the decomposition of L. As a result, $m(i) \leq 2\#(\mathcal{B}_{HF(\hat{S}_{x_{m-i}},L_i)})$ and thus:

$$N_\Phi(L;\mathcal{F}_\epsilon,\epsilon) \leq \sum_{i=0}^{m} 2\#(\mathcal{B}_{HF(\hat{S}_{x_{m-i}},L_i)})\ . \tag{81}$$

The Lagrangians $L_i \subset E$ are related through Dehn twists - as discussed in §3.2.6.3: $L_{m+1} = L$, $L_{m-i} = \tau_{\hat{S}_i}(L_{m-i+1})$. As a result, we can obtain iteratively an upper bound for $\#(\mathcal{B}_{HF(\hat{S}_{x_{m-i}},L_i)})$, and, moreover it is easy to see that there exists a constant $k'(\mathcal{F}_\epsilon)$, independent of L, as well as $\epsilon'' > 0$, $\epsilon'' \ll \epsilon$, depending again on $\mathcal{F}_\epsilon$ and not on L, such that

$$N_\Phi(L;\mathcal{F}_\epsilon,\epsilon) \leq k'(\mathcal{F}_\epsilon)\ \max_{S\in\mathcal{F}_\epsilon} \left(\#(\mathcal{B}^{\epsilon''}_{HF(S,L)})\right) \tag{82}$$

These estimates are, of course, first established for $N_{\mathcal{Y}_\eta}(\Phi_\eta(a);\mathcal{F}_{\epsilon,\eta},\epsilon)$ in $\mathcal{C}^\epsilon_p$ for $\nu(p)$ sufficiently small. By making $\nu(p) \to 0$ we see that they apply also to $N_\Phi(L;\mathcal{F}_\epsilon,\epsilon)$, as stated above. The estimates (81) and (82) remain obviously true also for $\Phi', \mathcal{F}'_\epsilon$ by using $N^r_{\Phi'}(-)$ in that case.

In the opposite direction, from Proposition 3.4.1.3 together with the stronger form in (75) we deduce a lower bound for weighted cone-length and entropy relative to the data $(\Phi',\mathcal{F}'_\epsilon)$:

Corollary 3.4.1.19. *For each $L \in \mathcal{L}ag^{(ex)}(D^*N)$ and $\epsilon' \geq \epsilon$ we have:*

$$N^r_{\Phi'}(L;\mathcal{F}'_\epsilon,\epsilon') \geq k(\mathcal{F}'_\epsilon) \sum_{S\in\mathcal{F}'_\epsilon} \#(\mathcal{B}^{2\epsilon'}_{HF(S,L)})$$

for a constant $k(\mathcal{F}'_\epsilon)$ only depending on the family $\mathcal{F}'_\epsilon$ and we also have:

$$h^r_{\Phi',\mathcal{F}'_\epsilon}(\Psi;L,\epsilon') \geq \hbar(\Psi;L,G_{\mathcal{F}'_\epsilon};2\epsilon')$$

for $G_{\mathcal{F}'_\epsilon} = \oplus_{S\in\mathcal{F}'_\epsilon} S$.

Corollary 3.4.1.19 is stated for the ambient ϵ-TPC -approximating data $(\Phi',\mathcal{F}'_\epsilon)$. This is essential for two reasons: first, the category $\mathcal{D}_p(E)$ has the property that $\hom_{\mathcal{D}_p(E)}(A,B)$ is a persistence module of finite type $\forall A,B \in \mathrm{Obj}(\mathcal{D}_p(E))$, as noted in Remark 3.2.7.5, and, secondly, the family $\mathcal{F}'_\epsilon$ consists of Yoneda modules. Thus, we may apply Proposition 3.4.1.3 for each $\mathcal{D}_p(E)$ and by then letting $\nu(p) \to 0$ the statement follows. In contrast, it is not clear whether a similar statement is valid for the local approximating data $(\Phi, \mathcal{F}_\epsilon)$. Indeed, even the constant $k(-)$, introduced in Corollary 3.4.1.9, is not *apriori* well-defined for $\mathcal{F}_\epsilon$.

REMARK 3.4.1.20. In the setting above, by combining Corollary 3.4.1.19 and (82) we deduce

$$\hbar(\Psi; L, G_{\mathcal{F}'_\epsilon}; \epsilon'') \geq h^r_{\Phi', \mathcal{F}'_\epsilon}(\Psi; L, \epsilon) \geq \hbar(\Psi; L, G_{\mathcal{F}'_\epsilon}; 2\epsilon) \text{ for } 0 < \epsilon'' \ll \epsilon \tag{83}$$

which shows that, in this situation ($M = D^*N$ etc), weighted categorical entropy is very closely related to the barcode entropy from [**cGG24**]. In [**DHKK14**] the authors obtain an upper bound for (the unweighted) categorical entropy by an algebraic approach that is more general than the one used here so that, in principle, one expects an inequality such as (83) to remain true in wider generality.

3.4.2. Non-vanishing weighted categorical entropy and Corollary ??. The aim of this subsection is to produce a class of examples with non-vanishing ϵ-weighted categorical entropy in the sense of (80) but with vanishing "classical" categorical entropy. We formulate this more precisely in the next statement.

PROPOSITION 3.4.2.1. *Let (N, g) be a closed hyperbolic manifold. Consider the metric space $(\mathcal{L}ag^{(ex)}(D^*N), d_\gamma)$ and, for some small ϵ, fix the ambient ϵ-TPC - approximating data $(\Phi', \mathcal{F}'_\epsilon)$ as recalled in §3.4.1.6. There exists a Lagrangian $L \in \mathcal{L}ag^{(ex)}(D^*N)$ Hamiltonian isotopic to the zero section and a Hamiltonian diffeomorphism $\Psi : D^*N \to D^*N$ with support inside D^*N, such that*

$$h^r_{\Phi', \mathcal{F}'_\epsilon}(\Psi; L, 2\epsilon) > h^r_{\Phi', \mathcal{F}'_\epsilon}(\Psi; L, \infty) = 0$$

where the relevant ϵ-weighted entropy is defined in (80).

The result means that filtered categorical algebra is sensitive to the dynamics in a cotangent bundle in ways in which the non-filtered version is not. Moreover, in view of Remark 3.4.1.17, this proposition implies one part of Corollary **??** from the introduction, namely that the metric space $(\mathcal{L}\mathrm{ag}^{(ex)}(D^*N), d_\gamma)$ is not totally bounded. The second part of the Corollary that refers to $M = S^2$ and with $\mathcal{L}\mathrm{ag}(S^2)$ consisting of equators is addressed in §3.4.3.

The vanishing of the non-weighted categorical entropy, $h^r_{\Phi', \mathcal{F}'_\epsilon}(\Psi; L, \infty)$, in the statement is immediate because all $\Psi^n L$ are Hamiltonian isotopic to the zero section and thus, without weight constraints, they admit cone-decompositions with the same number of terms.

Till now our arguments did not depend on the ground field $\mathbf{k}$, however from this point on we will assume in this subsection that $\mathbf{k} = \mathbb{Z}_2$.

The proof of Proposition 3.4.2.1 makes essential use of Corollary 3.4.1.15 and is contained in the next five sub-subsections. The proposition is stated for the unit disk bundle but in the proof we will perform a series of constructions involving several nested disk bundles that,

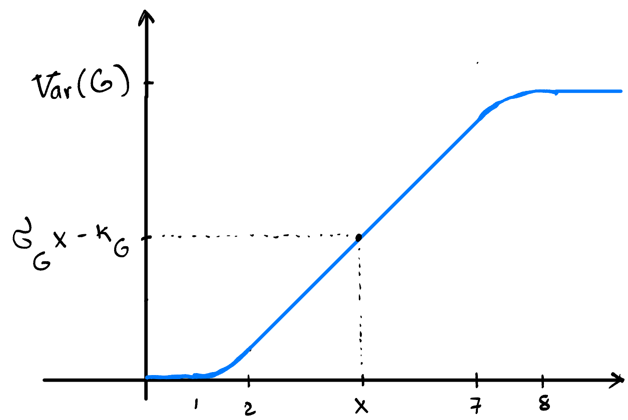


FIGURE 11. The graph of the function η.

through appropriate rescaling, we will assume are all included in E with $\bar{h} : E \to \mathbb{C}$, the Lefschetz fibration from §3.2. The largest r needed is $r = 10$. Most of the argument takes place inside this disk bundle and is described in the first four subsections. We return to E in 3.4.2.5 to conclude.

3.4.2.1. *The choice of* Ψ *and* L. We will consider an auxiliary Hamiltonian flow ϕ generated by a smooth Hamiltonian $G : T^*(N) \to \mathbb{R}$ which is defined by

$$G(q, p) = \eta(||p||) \tag{84}$$

where $||-||$ is the norm with respect to the metric induced by g and $\eta : [0, \infty) \to [0, \infty)$ is smooth with the properties that: $\eta(x) = \sigma_G x - k_G$ for $2 \leq x \leq 7$ with $0 < k_G < 2$ and $1 \leq \sigma_G < \frac{3}{2}$ fixed constants chosen in a way that will be further discussed later below; η is non-decreasing; η is constant equal to 0 for $0 \leq x \leq 1$ and constant equal to some value $= \mathrm{Var}(G) < 10$ for $x \geq 8$; $\eta''(x) > 0$ for $x \in (1, 2)$ and $\eta''(x) < 0$ for $x \in (7, 8)$ see Figure 11.

We notice that $\eta'(x) \leq \frac{3}{2}, \forall x \in [0, \infty)$. The Hamiltonian diffeomorphism $\Gamma = \phi_1^G$ generated by G is supported inside $D_8^*(N)$. The Hamiltonian diffeomorphism in the statement is the inverse of Γ:

$$\Psi = \Gamma^{-1} .$$

We now describe the Lagrangian L. This Lagrangian submanifold is obtained in two steps. The first is to consider a function $\varphi : N \to [0, \frac{1}{2}]$ as the one constructed in Proposition 3.2.3.1 but for $\delta < 10$ and define a Lagrangian submanifold L' by

$$L' = \text{graph } (d\varphi) .$$

We denote by x_i, $1 \leq i \leq m$, the critical points of φ and take this function φ in such a way that $L' \subset \mathrm{Int}(D_{10}^*(N))$ and such that $L' \cap D_9^*(N)$ is a disjoint union:

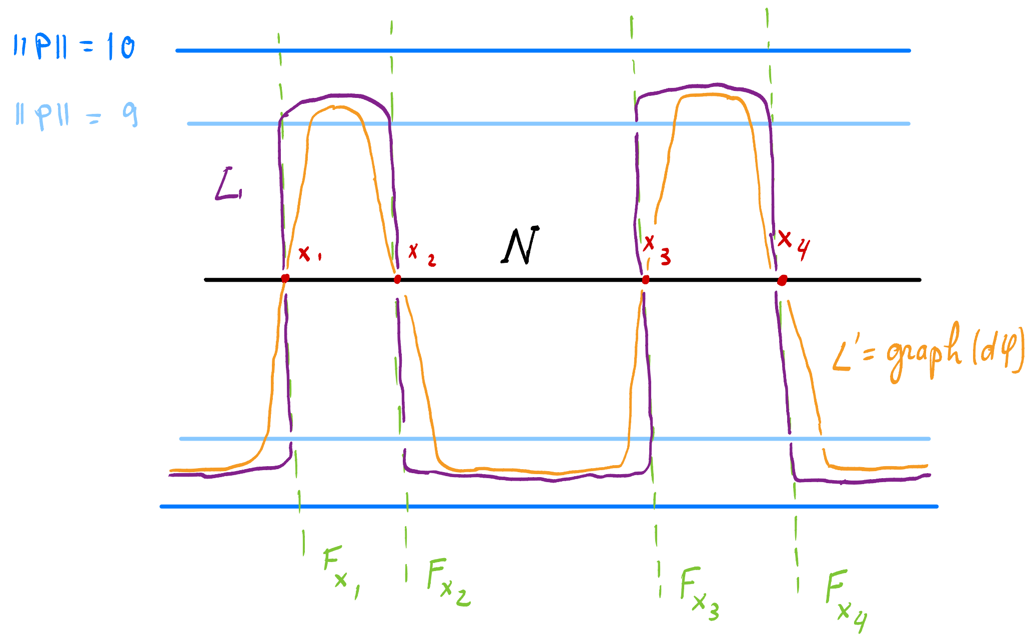


FIGURE 12. The shape of the Lagrangian L.

$$L' \cap D_9^*(N) = \coprod_{i=1}^{m} Z_i$$

with each $Z_i \subset W_i$ where W_i is a small tubular neighbourhood of F_{x_i} (this is the fiber of T^*N over the critical point x_i of φ). Finally, L is obtained by modifying L' by a Hamiltonian isotopy in such a way that (see Figure 12):

$$L \subset \mathrm{Int}(D_{10}^*(N)) \ , \ L \cap D_9^*(N) = \coprod_{i=1}^{m} F_{x_i} \cap D_9^*(N) \ .$$

REMARK 3.4.2.2. The constants $8, 9, 10$ might need to be replaced by $8 < k < k'$ because in the construction in the proof of Proposition 3.2.3.1 we do not directly obtain an upper bound for $||d\varphi||$ while keeping g fixed (see also Remark 3.2.3.2). However, the choice $k = 9$, $k' = 10$ is simply a choice of convenience and has no real incidence on the proof.

We will fix some additional notation for the Lagrangian L. We let

$$L_i = F_{x_i} \cap D_9^*(N) \tag{85}$$

so that $L \cap D_9^*(N) = \coprod_i L_i$. We also consider a primitive $f_L : L \to \mathbb{R}$ of $\lambda|_L$ which, up to a shift, is a small perturbation of the primitive φ of $\lambda|_{L'}$. In view of this, we may assume that f_L has values inside the interval $[-1, 1]$. Of course, f_L is constant on each L_i, with a constant

k_i on each such component, such that each of these constants belongs to $[-1, 1]$ and we may assume also that $k_1 = 0$.

3.4.2.2. *Floer homology.* In this subsection we establish some properties of certain Floer homology groups that will be essential for the proof. All the Floer homologies considered here are defined in $T^*(N)$ for an almost complex structure that is convex outside of $D^*_8(N)$. The maximum principle implies that the Floer trajectories contributing to the relevant differentials are contained in $D^*_{10}(N)$. The ground field is $\mathbf{k} = \mathbb{Z}_2$.

We consider two exact Lagrangian manifolds K, K' with primitives f_K and $f_{K'}$, respectively and with fixed choices of gradings, as in the rest of the paper. In our arguments these Lagrangians K, K' are either compact, included in the interior of $D^*_{10}(N)$, or they coincide with fibers F_x, for $x \in N$. We also consider a Hamiltonian $H : T^*(N) \to \mathbb{R}$ and an almost complex structure $J = \{J_t\}_{0 \le t \le 1}$ on $T^*(N)$ that is autonomous and coincides with a convex almost complex structure J^∞ outside (the interior of) $D^*_8(N)$. We assume that the Hamiltonian H is constant in a neighbourhood of $\partial D^*_{10}(N)$ and that K and $\phi_1^{-H}(K')$ intersect transversely. We denote by $\mathcal{P}(K, K'; H)$ the Hamiltonian orbits $\gamma : [0, 1] \to D^*_{10}(N)$ associated with the Hamiltonian flow ϕ_t^H of X^H with $\gamma(0) \in K$, $\gamma(1) \in K'$. There is an action functional on the path-space

$$\mathcal{P}(K, K') = \{\gamma : [0, 1] \to D^*_{10}(N) \mid \ \gamma \text{ smooth},\ \ \gamma(0) \in K,\ \ \gamma(1) \in K' \ \}$$

$$\mathcal{A}_{K,K';H}(\gamma) = \int_0^1 H(\gamma(t))dt - \int_0^1 \lambda(\dot\gamma(t))dt + f_{K'}(\gamma(1)) - f_K(\gamma(0)) \tag{86}$$

whose critical points are the elements in $\mathcal{P}(K, K'; H)$. As a vector space, the Floer complex $CF(K, K'; H, J)$, when it is defined (thus assuming regularity) is spanned by $\mathcal{P}(K, K'; H)$. The differential of the Floer complex counts solutions $u : \mathbb{R} \times [0, 1] \to D^*_{10}(N)$ of Floer's equation:

$$\frac{\partial u}{\partial s} + J_t \frac{\partial u}{\partial t} + \nabla_{\rho_t} H(u) = 0 \tag{87}$$

where the gradient is taken with respect to the Riemannian metric $\rho_t = \omega(-, J_t)$ and $u(\mathbb{R} \times \{0\}) \subset K$, $u(\mathbb{R}\times\{1\}) \subset K'$. We denote by $CF^a(K, K'; H, J)$ the subcomplex of $CF(K, K'; H, J)$ which is spanned by the Hamiltonian chords x such that $\mathcal{A}_{K,K';H}(x) \le a$.

In our application we will have $H = nG$ (with G from (84))for successively larger values of $n \in \mathbb{N}$. In particular, H is constant outside $D^*_8(N)$. This type of Hamiltonian of the form $H = h(||p||)$, with h smooth, has been extensively studied in the literature (see [**BPS03**][**MS11**]). The basic situation of interest for us is when both K and K' are fibers. For convenience, we introduce a special shortened notation

$$C(x, y; n) := CF(F_x, F_y; nG, J)$$

where F_x and F_y are two different fibers of T^*N. On both fibers F_x and F_y we take as primitives the identically zero functions. Regularity can be achieved in this case because, in all our examples, all Floer trajectories will be included inside $D^*_8(N)$ and we have the freedom to pick $\{J_t\}$ as desired there while keeping it equal to J^∞ outside of $D^*_8(N)$. We emphasize that in the Floer complexes we take into account all the homotopy classes of paths. Denote by

$$\mathcal{G}_{x,y} = \{ \ \bar{\gamma} \ \ | \ \ \bar{\gamma} : [0, \ell] \to N \text{ is a geodesic }, \ ||\dot{\bar{\gamma}}(t)|| = 1 \quad \bar{\gamma}(0) = x, \ \bar{\gamma}(\ell) = y \ \}$$

the geodesic arcs in N, parametrized with unit speed length, that go from x to y. We denote the length of such a geodesic arc $\bar{\gamma}$ by $\ell(\bar{\gamma})$. The associated length spectrum is:

$$\mathcal{L}_{x,y} = \{ \ \ell(\bar{\gamma}) \mid \bar{\gamma} \in \mathcal{G}_{x,y}\} \subset [0, \infty) \ .$$

Because (N, g) is hyperbolic each homotopy class of paths (with fixed ends) contains a unique geodesic arc in $\mathcal{G}_{x,y}$ and $\mathcal{L}_{x,y}$ is discrete. In particular, $\mathcal{L}_{x,y}$ is countable. It follows, that the set of quotients $Q\mathcal{L}_{x,y} = \{\frac{\ell}{n} \mid \ell \in \mathcal{L}_{x,y} \ , \ n \in \mathbb{N}^*\}$ is also countable. We now assume:

$$\sigma_G \notin Q\mathcal{L}_{x,y} \ . \tag{88}$$

This is obviously a generic assumption because the set $Q\mathcal{L}_{x,y}$ is of zero Lebesgue measure.

LEMMA 3.4.2.3. *Under the assumption above the complexes $CF(x, y; n)$ are well defined and, for any $\delta > 0$, the number of bars of length $\geq \delta$ in the persistence module $HC(x, y; n)$ increases with n at least as fast as $\frac{e^{hn}}{hn}$ where $h = \frac{5}{8} h_{top}$ with h_{top} the topological entropy of the geodesic flow.*

REMARK 3.4.2.4. i. We thank Viktor Ginzburg who suggested to the third author the method to estimate the numbers of bars in $HC(x, y; n)$ as in the proof below.

ii. The quotient $\frac{5}{8}$ in the statement is an artefact of the choice of η, it can be brought as close as needed to 1 by diminishing appropriately the length of the transition regions $(1, 2)$ and $(7, 8)$ in the definition of η.

PROOF. We will use here some of the calculations in [**BPS03**], see Equation (16) and Lemma 5.3.2 there, adapted to Lagrangian Floer homology. These calculations imply:

i. The complex $C(x, y; n)$ is well defined and finite dimensional if $n\sigma_G \notin \mathcal{L}_{x,y}$ which is the case due to (88). The only contributions to the differential of $C(x, y; n)$ is provided by solutions to (87) (with $H = nG$) that are contained in $D^*_8(N)$.
ii. The generators γ of $C(x, y; n)$ are in bijection with geodesic segments $\bar{\gamma}$ in N that join the point x to the point y in the sense that if $\gamma(t) = (q(t), p(t))$, then $q(t)$ is a geodesic from x to y parametrized such that $||\dot{q}(t)|| = \ell$; $\bar{\gamma}$ is the same geodesic, re-parametrized with unit speed, and of length $\ell(\bar{\gamma}) = \ell$.
iii. For each such generator, $\gamma(t) = (q(t), p(t))$, we have that

$$\frac{p(t)}{r} = \frac{\dot{q}(t)}{||\dot{q}(t)||} \quad \text{with} \ \ r > 0 \ \ \text{such that} \quad n\eta'(r) = \ell \ .$$

In particular, $||p(t)||$ is constant and is equal to r and $\ell \leq n\sigma_G$.
iv. The action of the generator γ as above is:

$$\mathcal{A}_n(\gamma) := \mathcal{A}_{F_x, F_y; nG}(\gamma) = n\eta(r) - r\ell$$

As a result of these remarks, each geodesic arc $\bar{\gamma}$ from x to y of length $\ell(\bar{\gamma}) < n\sigma_G$ gives rise to a generator of $C(x, y; n)$ given by

$$\gamma(t) = (q(t), \frac{r\dot{q}(t)}{\ell(\bar{\gamma})}) \quad \text{for each } r \text{ such that } \eta'(r) = \frac{\ell(\bar{\gamma})}{n} \tag{89}$$

where, as above, $q(t)$ is the geodesic $\bar{\gamma}$ reparametrized such that $||\dot{q}(t)|| = \ell(\bar{\gamma})$ (we identify here T^*N and TN using g). In view of the properties of the function η, for a given value $\ell < n\sigma_G$, there are exactly two such values $r_n(\ell), r'_n(\ell)$ with one value, say, $r_n(\ell) \in (1, 2)$ and the other $r'_n(\ell) \in (7, 8)$.

Fix a geodesic arc $\bar{\gamma}$. Denote by γ_n and γ'_n the two corresponding Hamiltonian chords associated to $\bar{\gamma}$ and to the values $r_n = r_n(\ell(\bar{\gamma}))$ and $r'_n = r'_n(\ell(\bar{\gamma}))$, respectively, through formula (89). They appear as generators of $C(x, y; n)$ as soon as $\sigma_G > \frac{\ell(\bar{\gamma})}{n}$, in particular for all $n > \ell(\bar{\gamma})$ (because $\sigma_G \geq 1$).

We now consider the action values of the two relevant generators. We have:

$$\begin{aligned} \mathcal{A}_n(\gamma'_n) - \mathcal{A}_n(\gamma_n) = \; & n(\eta(r'_n) - \eta(r_n)) - \ell(\bar{\gamma})(r'_n - r_n) \; \geq \\ & \geq n(\eta(7) - \eta(2)) - 7\ell(\bar{\gamma}) \geq 5n - 7\ell(\bar{\gamma}) \end{aligned}$$

with the last inequality resulting from $\eta(7) - \eta(2) = 5\sigma_G$ and $\sigma_G \geq 1$.

Fix $\delta > 0$ as in the statement. We deduce:

$$\mathcal{A}_n(\gamma'_n) - \mathcal{A}_n(\gamma_n) \geq \delta \;\text{ if }\; n \;\geq\; \frac{7}{5}\,\ell(\bar{\gamma}) + \frac{\delta}{5}\,. \tag{90}$$

In summary, for $n \geq \ell(\bar{\gamma})$, γ_n and γ'_n appear as generators in $CF(x, y; n)$ and when $n \geq \frac{7}{5}\ell(\bar{\gamma}) + \frac{\delta}{5}$ their actions are separated by no less than δ.

The standard Floer homology $HC^\infty(x, y; n)$ vanishes (forgetting the persistence structure is indicated by the superscript ∞) and the Floer differential in $C(x, y; n)$ is very simple. Indeed, Floer trajectories can only join Hamiltonian chords that project to paths in N in the same homotopy class. Given that each geodesic in $\mathcal{G}_{x,y}$ is unique in its homotopy class, it follows that when $n > \frac{7}{5}\ell(\bar{\gamma}) + \frac{\delta}{5}$, we have $d(\gamma'_n) = \gamma_n$ (recall that we work over $\mathbb{Z}_2$ and that $HC^\infty(x, y; n)$ vanishes) and the couple (γ_n, γ'_n) defines a bar of length at least δ in the persistence module $HC(x, y; n)$. Thus, for n big enough, the number of bars of length at least δ in $HC(x, y; n)$ is at least the number of geodesic arcs in $\mathcal{G}_{x,y}$ of length $\leq \frac{5}{8}n < \frac{5}{7}n(1 - \frac{\delta}{5n})$. By classical results, such as in [**PP90**] Chapter 9, the number of geodesics in $\mathcal{G}_{x,y}$ of length less than T increases as fast as

$$\frac{e^{Th_{top}}}{Th_{top}}\,.$$

By replacing T with $\frac{5}{8}n$ we obtain the result. □

We next turn our attention to the filtered complex

$$C(x, L; n) := CF(F_x, L; nG, J)$$

where L is the Lagrangian fixed in §3.4.2.1. Recall that L has the property $L \cap D^*_9(N) = \cup_i L_i$ with $L_i = F_{x_i} \cap D^*_9(N)$ (see (85)).

We assume that $x \neq x_i$ for all the critical points x_i of φ and we add two additional assumptions relative to F_x,L, and G:

$$\sigma_G \notin Q\mathcal{L}_{x,x_i}, \quad \forall x_i \tag{91}$$

As mentioned after (88), this is a generic assumption on the choice of σ_G. Note that:

$$L \cap F_x = \{z_x\} \subset D^*_{10}(N) \backslash D^*_9(N) \ . \tag{92}$$

To make sense of this recall from §3.4.2.1 that L is a small deformation of L' which is the graph of $d\varphi$. It follows that, under the assumption that $x \neq x_i$, $\forall i$, we have that F_x intersects L in a single point - denoted by z_x - which lies outside of $D^*_9(N)$.

Given that G is constant outside of $D^*_8(N)$ we deduce that the underlying vector space of $C(x, L; n)$ can be written as:

$$C(x, L; n) = \oplus_i \Sigma^{k_i} C(x, x_i; n) \oplus \mathbb{Z}_2\langle z_x \rangle \ . \tag{93}$$

Moreover, for the same reason, the complexes $C(x, x_i, n)$ are finite dimensional over $\mathbb{Z}_2$. The shift Σ^{k_i} appears here because the value of the primitive f_L on the component L_i is not 0, but rather $k_i \in [-1, 1]$. The action of the point z_x is $f_L(z_x)$, see (86). Denote by $d^{x_i;n}$ the differential of the complex $C(x, x_i; n)$ and let D^n be the differential of the complex $C(x, L; n)$. Finally, recall that for a filtered chain complex C we denote by C^a the filtration $\leq a$ subcomplex.

Lemma 3.4.2.5. *With the notation above there exists a constant $\xi > 0$ that depends on (N, g), on L, on x, and on the almost complex structure J (but not on n) such that*

$$D^n = \oplus_i d^{x_i;n} + \bar{D}^n$$

with the property that

$$\bar{D}^n(C^\alpha(x, L; n)) \ \subset \ C^{\alpha-\xi}(x, L; n) \ , \ \forall \ \alpha \ .$$

Proof. The Floer strips that contribute to the differential D^n are of two types: those that stay inside $D^*_8(N)$ - in this case they appear in one of the differentials $d^{x_i;n}$, and strips u, called below of *second type*, such that $\mathrm{Im}(u) \not\subset D^*_8(N)$. Recall that the Lagrangian L coincides with a union of fibers inside $D^*_9(N)$. Using the maximum principle, this implies that any strip u of the second type has to reach the non-fiber-like region of L. In other words, there exists a point $(s_0, 0) \in \mathbb{R} \times \{0\}$ such that $u(s_0, 0) \notin D^*_9(N)$. The aim of the proof is to show that there is a constant δ, as in the statement, such that the energy

$$E(u) = \int ||\frac{\partial u}{\partial s}||^2 \ dsdt$$

satisfies $E(u) \geq \delta$ for any strip u of the second type.

The argument is based on standard monotonicity results such as in Sikorav [**Sik94**]. The first step is to fix the constant ξ. Let $K = D^*_9 N \backslash \mathrm{Int}(D^*_{8.5}(N))$. The results in [**Sik94**] - see Proposition 4.7.2 there - imply that there exist positive constants r_0 and C that depend only on J, (N, g), and L with the following property. For each point $p \in L \cap K$, if $v : \Sigma \to K$ is a J-holomorphic curve with Σ compact with boundary, such that:

i. There is some point $x \in \partial\Sigma$ with $v(x) = p$,
ii. $v(\Sigma) \subset B_r(p) \subset K$ with $r \leq r_0$ (where $B_r(a)$ is the ball in $T^*(N)$ of radius r with center a),
iii. $v(\partial\Sigma) \subset \partial B_r(p) \cup L$,

then the symplectic area $A(v)$ of v satisfies:

$$A(v) \geq Cr^2 \ .$$

Now pick some $r' < r_0$ such that for each point in $z \in L \cap (D^*_{8.8}(N) \backslash \mathrm{Int}(D^*_{8.6}(N)))$ we have

$$B_{r'}(z) \subset K \backslash F_x. \tag{94}$$

We put:

$$\xi = C(r')^2/2 \ .$$

With this notation we return to a Floer strip of the second type $u : \mathbb{R} \times [0, 1] \to D^*_{10}(N)$. We want to show $E(u) \geq \xi$. We know that there is some s_0 with the property that the point $u(s_0, 0)$ is in $L \backslash D^*_9(N)$. Moreover, at least one of the asymptotic limits of u is a Hamiltonian chord of nG (for some n) and is therefore inside $D^*_8(N)$. It follows that there is also some point $s_1 \in \mathbb{R}$ such that $p = u(s_1, 0) \in L \cap (D^*_{8.8}(N) \backslash \mathrm{Int}(D^*_{8.6}(N)))$.

Denote by u_r the restriction of u to the set $S_r = u^{-1}(B_r(p)) \subset \mathbb{R} \times [0, 1]$ for $0 < r \leq r'$. By making use of Sard's theorem, for a generic choice of r the set S_r is a compact surface with boundary. We pick such a regular $r_1 > \frac{r'}{\sqrt{2}}$. We notice that $u_1 : S_{r_1} \to B_1(p)$ has the property that $u_1(\partial S_{r_1}) \subset \partial B_{r_1}(p) \cup L$. We also have, of course, $u_1(S_{r_1}) \subset B_{r_1}(p) \subset K$. Recall that G is constant outside of $D^*_8(N)$ and thus $\nabla(nG)$ vanishes on K. As a result u_1 is J-holomorphic and thus we obtain:

$$E(u) \geq A(u_1) \geq Cr_1^2 \geq \xi$$

which concludes the proof. □

REMARK 3.4.2.6. The constant ξ in the lemma depends very lightly on the point x. Indeed, if we assume that F_x does not intersect a fixed neighbourhood U of $L \cap D^*_9(N)$, then we may replace condition (94) by $B_{r'}(z) \subset K \cap U$ and under this assumption ξ is independent of x and only depends on L, J, (N, g), and U.

3.4.2.3. *An auxiliary algebraic result.* We will next use the following simple statement.

LEMMA 3.4.2.7. *Suppose that C is a filtered, finite dimensional chain complex over $\mathbb{Z}_2$ that is $\mathbb{Z}_2$-graded. Fix $\delta > 0$. If the differential D of C can be written as*

$$D = d + D'$$

with $d^2 = 0$ and D' such that $D'(C^\alpha) \subset C^{\alpha - \delta}$ for all $\alpha \in \mathbb{R}$, then for any $\epsilon < \delta$, we have

$$\#(\ \mathcal{B}^{\epsilon}_{H(C))}\ ) \geq \#(\ \mathcal{B}^{\epsilon}_{H(C,d)}\ )\ .$$

Proof. The result is most likely known by experts but we give a proof for completeness. We denote by C_0 the complex (C, d). It is easy to see that it is enough to show the statement when C_0 is acyclic which we will now assume. We write C_0 as a direct sum of terms $E_2(a, b)$ with filtrations given by a valuation v with values in $\mathbb{R}$ (see §3.4.1.3). There are two types of terms of this form, those such that $v(a) \leq v(b) \leq v(a) + \epsilon$ - they will be denoted by $E_2^s(a, b)$, and those with $v(b) > v(a) + \epsilon$, denoted by $E_2^l(a, b)$. The number n_l of terms of type E_2^l equals the number of bars in $\mathcal{B}_{HC_0}$ of length greater than ϵ. We now consider one E_2^s term, $E_2^s(a_0, b_0)$, such that $v(b_0) - v(a_0) \leq \epsilon$ is minimal among all terms E_2^s. We let C_1 be the subspace of C generated by the generators of C different from $\{a_0, b_0\}$ so that as d-chain complexes we have $C_0 = E_2^s(a_0, b_0) \oplus C_1$. We consider the element

$$a_0' = a_0 + D'(b_0) \ .$$

It has the property $a_0' = D(b_0)$. We notice that $v(a_0') = v(a_0)$ because $\epsilon < \delta$ and thus $v(D'(b_0)) < v(a_0)$. Therefore, we also have $D'(b_0) \in C_1$. Let

$$E_2(a_0', b_0) = \mathbb{Z}_2\langle\ a_0',\ b_0\ :\ D(b_0) = a_0', D(a_0') = 0\ \rangle\ .$$

Then $E_2(a_0', b_0)$ is a D-subcomplex of C. We have the projection $p_1 : C \to C_1$ defined on generators as the identity on C_1 and that sends b_0 to 0, and a_0 to $D'(b_0)$. This projection has as kernel $E_2(a_0', b_0)$ and is filtration preserving in the sense that it sends C^α to C_1^α, for all $\alpha \in \mathbb{R}$. The differential D induces a differential D_1 on C_1 which is the unique linear map making p_1 a chain map. Moreover, D_1 can be written:

$$D_1 = d + D_1'$$

where $D_1' = p_1 \circ D'$. It is easy to see that D_1' also drops the filtration by at least δ, just as D' in the statement. Assume for the moment that the map p_1 admits a section $j_1 : (C_1, D_1) \to (C, D)$ which is a filtration preserving chain map. In that case, we can reapply iteratively the same procedure to (C_1, D_1) to successively eliminate all the $E_2^s(a, b)$'s. We are left at the end with a complex (C_k, D_k) whose dimension is $2n_l$ and whose differential $D_k = d + D_k'$ drops filtration by more than ϵ (d drops differential by more than ϵ on C_k, and D_k' by at least $\delta > \epsilon$). As a result, all the bars in HC_k are longer than ϵ and there are at least n_l of them. By our iterative construction, (C_k, D_k) is a filtration preserving retract of (C, D) and therefore the number of bars of length more than ϵ in HC is at least n_l, as claimed.

Thus, to finish the proof we are left to construct the section j_1. We will construct a linear map $j_1 : C_1 \to C$ that is filtered and has the property that $p_1 \circ j_1 = id$ and, additionally, $\mathrm{Im}(j_1)$ is a subcomplex of C. This implies that j_1 is also a chain map. As a vector space, we have the splitting $C = E_2(a_0', b_0) \oplus C_1$. Let $x \in C_1$. There are two possibilities. Assume that, relative to this splitting, $D(x) = w \in C_1$. In that case, we put $j_1(x) = x$. On the other hand, assume that $D(x) = a_0' + w$ with $w \in C_1$. In this case we put $j_1(x) = x + b_0$. We notice $v(x) \geq v(b_0)$. Moreover, $D(x + b_0) \in C_1$ and, with this definition, j_1 is different from the identity only in degree $|a_0| + 1$. We also have that j_1 is filtration preserving in the stronger sense that $v(j_1(y)) = v(y)$ for all $y \in C_1$. It is clear that $p_1 \circ j_1 = id$. We now want to remark that $\bar{C}_1 = \mathrm{Im}(j_1)$ is closed with respect to D. First, the construction above means that for each $x \in \bar{C}_1$, $D(x)$ can not have the form $a'_0 + w$, $w \in \bar{C}_1$. Indeed, all $x \in \bar{C}_1$ in degree $|a_0|+1$ satisfy $D(x) \in C_1$ and C_1 and $\bar{C}_1$ coincide in degree $|a_0|$. Now assume that for some $x \in \bar{C}_1$, $D(x) = b_0 + \zeta$, $\zeta \in \bar{C}_1$. Then $0 = D^2(x) = a'_0 + D(\zeta)$ which leads to a contradiction. Thus $\bar{C}_1$ is a D-subcomplex in C and this concludes the proof. $\square$

Remark 3.4.2.8. We have used at certain points in the proof the fact that we work over $\mathbb{Z}_2$ but the result remains true with a similar proof over any field $\mathbf{k}$.

3.4.2.4. *The main step in the proof of Proposition 3.4.2.1.* The aim of this subsection is to show:

Lemma 3.4.2.9. *For ϵ sufficiently small and for any ϵ-approximating family $\mathcal{F}_\epsilon$ for $\mathcal{L}ag^{(ex)}(D^*_{10}N)$ in $\mathcal{C}_p(10)$, with $\nu(p)$ sufficiently small - as in Theorem 3.2.2.1, thus with $\mathcal{F}_\epsilon$ consisting of fibers - there is a Hamiltonian isotopy Ψ with support in $D^*_{10}(N)$, a Lagrangian L, both as in §3.4.2.1, and one fiber, $F_x \in \mathcal{F}_\epsilon$ such that the number of bars:*

$$\#(\mathcal{B}^{\epsilon}_{HF(F_x,\Psi^n L)})$$

grows in n at least as fast as $\frac{e^{hn}}{hn}$, where h is the constant in Lemma 3.4.2.3.

Proof. We start the argument by choosing a Lagrangian L as in §3.4.2.1. We also fix a small neighbourhood U of $L \cap D^*_9(N)$. We pick δ to be the constant given by Lemma 3.4.2.5 under the assumption that $F_x \cap U = \emptyset$. As noted in Remark 3.4.2.6 this constant is then independent of x. We pick $\epsilon < \delta$ and let $\mathcal{F}_\epsilon$ be an ϵ-approximating family for $\mathcal{C}_p(10)$. If ϵ is small enough there is at least one element F_x of $\mathcal{F}_\epsilon$ that does not intersect U (this follows from the fact that when $\epsilon \to 0$ the union $\cup_{F \in \mathcal{F}_\epsilon} F$ becomes dense in $D^*_{10}(N)$). At this point we can pick the Hamiltonian G. We pick G as in §3.4.2.1 subject to the assumption (91) which is generic. This means that the complex $CF(F_x, L; nG, J)$ is defined for each n, it is finite dimensional over $\mathbb{Z}_2$, and that it satisfies the statement in Lemma 3.4.2.5. This means that

$$CF(F_x, L; nG, J) = \oplus \Sigma^{k_i} CF(F_x, F_{x_i}; nG, J) \oplus \mathbb{Z}_2\langle z_x \rangle$$

as filtered vector spaces, with F_{x_i} as in (85) and that the differential D^n of $CF(F_x, L; nG, J)$ satisfies:

$$D^n = \oplus_i d^{x_i;n} + \bar{D}^n$$

with $\bar{D}^n$ dropping filtration by at least ξ and with $d^{x_i;n}$ the differential in $CF(F_x, F_{x_i}; nG, J)$. In this situation we can apply Lemma 3.4.2.7 to deduce :

$$\#(\mathcal{B}^{\epsilon}_{HF(F_x,L;nG,J)}) \geq \sum_i \#(\mathcal{B}^{\epsilon}_{HF(F_x,F_{x_i};nG,J)}) \ .$$

We use Lemma 3.4.2.3 to conclude that $\#(\mathcal{B}^{\epsilon}_{HF(F_x,L;nG,J)})$ increases in n as fast as $\frac{e^{hn}}{nh}$.

The argument concludes by noting that by the naturality of Floer's equation, there is a filtered chain isomorphism:

$$CF(F_x, \Psi^n L; 0, J_n) \to CF(F_x, L; nG, J) \ .$$

Here $\Psi = \Theta^{-1}$ where $\Theta = \phi_1^G$ is induced by the Hamiltonian flow of G. The almost complex structure J_n is such that $\Theta_*^n J_n = J$. Because G is constant outside $D_8^*(N)$, the almost complex structures J_n coincide with J there and thus the complexes $CF(F_x, \Psi^n L; 0, J_n)$ are well defined because $||-||^2$ is J_n- convex outside $D_8^*(N)$. The persistence module $HF(F_x, \Psi^n L; 0, J^*)$ is in fact independent of the almost complex structure J^*, as long as it coincides with J outside $D_8^*(N)$.

□

3.4.2.5. *End of the proof of Proposition 3.4.2.1.* The constructions in the subsections above can be appropriately rescaled so that they take place in the unit disk bundle D^*N instead of $D_{10}^*(N)$ and, further, this disk bundle is included in the total space of the Lefschetz fibration $\bar{h} : E \to \mathbb{C}$ that was instrumental in the proof of Theorem 3.2.2.1. Moreover, we may assume as in §3.2.6.2 that the TPC ϵ-approximating family $\mathcal{F}'_\epsilon$ consists of Lagrangian spheres $\hat{S}_{x_i}$ that each intersects D^*N in a fiber F_{x_i} of D^*N. The collection of these fibers is precisely the ϵ-TPC-approximating family $\mathcal{F}_\epsilon$, see §3.4.1.6 for a brief recap of the context.

We now recall the conclusion of Lemma 3.4.2.9: the number of bars $\#(\mathcal{B}^{\epsilon}_{HF(F_x,\Psi^n L)})$ grows in n at least as fast as $\frac{e^{hn}}{hn}$. It is easy to see that the constructions leading to this result (that all take place in $D^*(N)$) can be adjusted so that we have the same rate of growth for $\#(\mathcal{B}^{4\epsilon}_{HF(F_x,\Psi^n L)})$ which is what we will assume from now on. Recall that F_x is one of the fibers in the approximating family $\mathcal{F}_\epsilon$ and $\Psi^n L \subset \mathrm{Int}(D^*N)$. Therefore, $F_x \cap \Psi^n L = \hat{S}_x \cap \Psi^n L \subset \mathrm{Int}(D^*N)$. Due to the convexity of ∂D^*N we deduce that the Floer complexes $CF(F_x, \Psi^n L; 0, J_n)$ and $CF(\hat{S}_x, \Psi^n L; 0, J_n)$ coincide. As a result

$$\#(\mathcal{B}^{4\epsilon}_{HF(\hat{S}_x,\Psi^n L)})$$

also grows in n at least as fast as $\frac{e^{hn}}{hn}$.

Both $\hat{S}_x$ and $\Psi^n L$ are Yoneda modules so that $HF(\hat{S}_x, \Psi^n L) = \hom_{\mathcal{D}_p(E)}(\hat{S}_x, \Psi^n L)$. As a result $\hbar(\Psi; L, \hat{S}_x, 4\epsilon)$, as defined in (79) does not vanish. Given that $G_{\mathcal{F}'_\epsilon} = \oplus_{S \in \mathcal{F}'_\epsilon} S$ we also have $\hbar(\Psi; L, G_{\mathcal{F}'_\epsilon}, 4\epsilon) > 0$. At this point we can apply Corollary 3.4.1.19 and we deduce

$$h^r_{\Phi', \mathcal{F}'_\epsilon}(\Psi; L, 2\epsilon) > 0$$

which is the desired statement. □

3.4.3. Complexity of equators on S^2. The aim of this subsection is to complete the proof of Corollary **??** by showing that the space of equators on S^2 endowed with the spectral metric is not totally bounded. We will use the notation in §3.3. In particular, the set of equators on the 2-sphere is denoted by $\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{0})}(S^2)$. Here the notation reflects the fact that equators are monotone, and $\mathbf{0}$ indicates that the $\mathbb{Z}_2$-number of Maslov -2 J-holomorphic disks with boundaries on an equator L, and passing through a fixed point $x \in L$, vanishes. Recall from Proposition 3.3.3.3 that this class of Lagrangians is retract approximable with an $\frac{1}{4N} + 2\nu(p)$-approximating family of the form $L_1, \ldots, L_N$ where L_i is a great circle passing through the north and south poles of S^2 and is at angle $\frac{\pi}{2N}$ from L_{i-1}. We fix L_1 such that it passes through the point $(1,0,0)$. We assume that the metric on S^2 is such that its area is equal to 1.

Notice that in this case, by contrast to the case of $M = D^*N$, the approximating data $(\Phi, \mathcal{F})$ is simpler in the sense that the categories $\mathcal{Y}_{\epsilon,\eta}$ do not depend on ϵ, the only dependence is on p (we may assume $\eta = \nu(p)$) and for each N, the family $\mathcal{E}(N) = \{L_1, \ldots, L_N\} \subset \mathcal{E}$ where $\mathcal{E}$ is the set of all great circles passing through the two poles.

PROPOSITION 3.4.3.1. *There exists a Hamiltonian diffeomorphism $\Psi\colon S^2 \to S^2$, $L \in \mathcal{E}$, $N \in \mathbb{N}$, and $\epsilon' \geq 2(\frac{1}{4N} + 2\nu(p))$, such that:*

$$N^r(\Psi^k L, \mathcal{E}(N), \epsilon') \overset{k\to\infty}{\longrightarrow} \infty \ .$$

In view of this proposition and of Remark 3.4.1.17 it follows that the space $(\mathcal{L}\mathrm{ag}^{(\mathrm{mon},\mathbf{0})}(S^2), d_\gamma)$ is not totally bounded.

PROOF OF PROPOSITION 3.4.3.1. We start by identifying Ψ: this is the Dehn twist along the horizontal equator on the sphere with support inside the annulus

$$U = \{(x,y,z) \in S^2 \mid -a < z < a\}$$

with $a > 0$ picked in such a way that the area of U is smaller than $\frac{1}{4}$. Next we pick L to be the great circle on S^2 that passes through the north and south poles and also through the point $(0,1,0)$. Note that L intersects transversely L_1, the first element in the family $\mathcal{E}(N)$.

The idea of the proof is simple: we would like to use an analogue of the first inequality in Corollary 3.4.1.19, for a fixed ϵ' and some N (with $\mathcal{E}(N)$ in the place of $\mathcal{F}'_\epsilon$), and then show that the number of bars of length larger than $2\epsilon'$ in $HF(L_1, \Psi^k L)$ goes to infinity with k. The key issue to be resolved is that we work here in the monotone context and over the Novikov field Λ, as in §3.1.6.3. As a result, the algebraic results in Proposition 3.4.1.3 and (75) need to be adjusted to this context. This can be achieved by using the results of Usher-Zhang [**UZ16**] as we will briefly review next.

Assume that C is a finite dimensional, filtered chain-complex defined over the Novikov field Λ. The homology $H(C)$ is a persistence module in the classical sense over the field $\mathbb{Z}_2$. However, counting bars in the usual way does not make sense because if one bar of type $[a,b)$ appears, all translates of type $T^\alpha[a,b)$ also appear for all $\alpha \in \mathbb{R}$. The tools required to deal with this problem are introduced in [**UZ16**] together with a specific terminology. First, the methods in [**UZ16**] apply to rings of Novikov type but more general than our choice here, that are denoted there by $\Lambda^{\mathcal{K},\Gamma}$ where $\mathcal{K}$ is the ground field ($\mathbb{Z}_2$ in our case), and Γ is an additive subgroup of $\mathbb{R}$. In our case, $\Gamma = \mathbb{R}$. Moreover, [**UZ16**] applies to so-called *non-Archimedean normed vector spaces that are orthogonalizable* which means vector spaces C over Λ endowed with a filtration function $\ell : C \to \mathbb{R} \cup \{-\infty\}$ with the properties typical for action functionals together with a property of admitting special "orthogonal" bases. Such complexes are called of Floer-type in Definition 4.1 in [**UZ16**] and, in particular, all Floer complexes in our paper fit into this class. All Floer-type complexes will always be finite dimensional over Λ in our case. To such a Floer-type chain complex C [**UZ16**] assigns a bar-code $\bar{\mathcal{B}}_{HC}$, called there the *concise* bar-code of C, that consists of intervals of non-negative length, possibly infinite and that all have 0 as lower bound (this is, of course, different from the case considered earlier in this section; the fact that the lower bound of the bars is always 0 reflects the issue mentioned above having to do with the action of Λ). It is shown that such a bar-code determines C up to filtered chain-homotopy equivalence (which explains why we use $H(C)$ in the notation). The construction of this barcode parallels the standard construction over a non filtered field: it goes through a writing of the differential of the complex C, $d : C \to C$, in a special basis over Λ such that the matrix of the differential is upper triangular and only contains 0's and 1's, with at most a single 1 on each row and column. We denote by $\bar{\mathcal{B}}^{\delta}_{H(C)}$ the barcode formed by eliminating from $\bar{\mathcal{B}}_{H(C)}$ all the bars of length $\leq \delta$, just as in §3.4.1.3. The construction of the barcode $\bar{\mathcal{B}}_{H(C)}$ easily shows that the analogue of Lemma 3.4.1.11 remains true in this context with the place of V being taken by a Floer-type finite dimensional complex over Λ and with $\bar{\mathcal{B}}$ in the place of $\mathcal{B}$. As result Proposition 3.4.1.3 and (75) remain also true with the modification that the base A_∞-category is a Fukaya filtered category over Λ and that $\bar{\mathcal{B}}$ takes the place of $\mathcal{B}$.

In brief, to prove the proposition we need to show that

$$\# \left(\bar{\mathcal{B}}^{2\epsilon'}_{HF(L_1, \Psi^k L)}\right) \stackrel{k \to \infty}{\longrightarrow} \infty \ . \tag{95}$$

At this point we pick the value $\epsilon' = \frac{1}{32}$. We will consider the complex $CF(L_1, \Psi^k L)$ defined over Λ. The generators of this complex are the intersection points $L_1 \cap \Psi^k L$. The number of these intersection points is $2k + 2$: two of them are the north and south poles and there are two others for each successive application of Ψ. Floer trajectories are holomorphic and by using the open mapping theorem it is easy to see that if a Floer trajectory starts at a point $\in L_1 \cap \Psi^k L \cap U$ then it can not remain entirely inside U and thus it fills a connected region of $S^2 \backslash (U \cup L_1 \cup L)$. As a result each such Floer trajectory has area at least $\frac{3}{32}$. At the same time the Floer homology with $\mathbb{Z}_2$ coefficients $HF(L_1, \Psi^k L; \mathbb{Z}_2)$ is well defined (this is obtained by making $T = 1$) and is isomorphic to $HF(L_1, L; \mathbb{Z}_2)$ which is of dimension 2 over $\mathbb{Z}_2$ and is generated by the north and the south poles. It follows that $\bar{\mathcal{B}}_{HF(L_1, \Psi^k L)}$ contains at least k bars and each of them is of length more than $\frac{2}{32} = 2\epsilon'$. This concludes the proof. $\square$

REMARK 3.4.3.2. It is easy to reformulate Proposition 3.4.3.1 in terms of the non-vanishing of an entropy type measurement that one may call ϵ-weighted *slow* (categorical) entropy which is defined just as in formula (80) except that $\log(n)$ is used at the denominator, in the place of n. In these terms, what we have shown in the proof above is that the slow ϵ-weighted retract entropy of Ψ with respect to L, both chosen as in the proof, is at least 1 as soon as $\epsilon \leq \frac{1}{32}$. Because Ψ is a Hamiltonian diffeomorphism, it is easy to see that for $\epsilon = \infty$ the slow entropy vanishes. Relations between entropy and Floer theoretic calculations have been initiated in [**FS05**]. See [**Daw23**] [**Daw25**] for some other results involving bar-code entropy estimates.

3.4.4. Quasi-rigidity and Corollary ??. We start by restating Corollary **??** in a more precise way.

COROLLARY 3.4.4.1. *Fix a closed Riemannian manifold* (N, g)*, its unit cotangent bundle* D^*N *as well as an* $\epsilon > 0$*. Fix also a family* $\mathcal{F} = \{F_0, \ldots, F_l\}$ *of fibres of* D^*N *that* ϵ*-approximates* $\mathcal{L}ag^{(ex)}(D^*N)$*, as constructed in Theorem 3.2.2.1 (see Definition 3.1.5.7). Consider an exact symplectomorphism* $\phi : D^*N \to D^*N$ *with support in the interior of* D^*N *and let*

$$\chi(\phi; \mathcal{F}) = \max_{F \in \mathcal{F}} \{\max h_{\phi(F)} - \min h_{\phi(F)}\} \ .$$

For any $L \in \mathcal{L}ag^{(ex)}(D^*N)$*, we have*

$$d_\gamma(L, \phi(L)) \leq 4(\epsilon + N(L; \mathcal{F}, \epsilon)\chi(\phi; \mathcal{F})) \ .$$

We recall that for an exact Lagrangian $L \subset D^*N$ we denote by $h_L : L \to \mathbb{R}$ a choice of a primitive on L for the exact 1-form λ. This notation is used for $L \in \mathcal{L}\mathrm{ag}^{(ex)}(D^*N)$, for fibers F of D^*N, as well as for Lagrangians such as $\phi(F)$ which coincide with the fiber F outside the support of ϕ. Notice also that $Var(h_L) = \max h_L - \min h_L$ is independent of the choice of primitive h_L.

REMARK 3.4.4.2. a. We refer to the property in the Corollary 3.4.4.1 as *quasi-rigidity* because its meaning is that the size of the action of ϕ on the whole of $\mathcal{L}\mathrm{ag}^{(ex)}(D^*N)$ is constrained by that on the finite family of fibers $\mathcal{F}$. In particular, notice that if $\phi(F) = F$ for all the fibers $F \in \mathcal{F}$, then, for such an F, $Var(h_{\phi(F)}) = 0$ and thus $\chi(\phi; \mathcal{F}) = 0$. Therefore, in this case we have

$$d_\gamma(L, \phi(L)) \leq 4\epsilon \quad , \forall \ L \in \mathcal{L}\mathrm{ag}^{(ex)}(D^*N) \ .$$

b. In both Corollaries 3.4.5.1 and 3.4.4.1 there is a factor two on the right side of the respective stated inequalities that could be avoided by using in the definition of TPC-approximability the pseudo-metric D_{int} from (52) instead of d_{int}.

c. It is likely that in the statement of the corollary it is possible to replace $N(L; \mathcal{F}, \epsilon)$ by $\#(\mathcal{F})$, the number of elements in the ϵ-approximating family $\mathcal{F}$.

PROOF OF COROLLARY 3.4.4.1. The argument makes use of the Lefschetz fibration $\bar{h} : E \to \mathbb{C}$, the Lagrangian spheres $\hat{S}_{x_i} \subset E$ and of the setting leading to the definitions of the local and ambient approximability data in §3.2.7.1 and §3.2.7.2, respectively. The construction of $\bar{h}$ and its properties are in §3.2.4 and §3.2.5.

Given $L \in \mathcal{L}\mathrm{ag}^{(ex)}(D^*N)$ there exists an iterated cone C_L in the category $\mathcal{D}_p(D^*N)$ (see §3.2.7.2) as given in equation (56) :

$$\Delta_i \ : \ Z_i \longrightarrow X_i \longrightarrow X_{i+1} \ , \ 0 \leq i < m \ , \tag{96}$$

with Z_j of the form $\Sigma^{l(j)} \hat{S}_{x_{s(j)}}$ or $= 0$, with each Δ_i exact in $(\mathcal{D}_p(E))^0$. Here $X_0 = 0$, $C_L = X_m$, and there is a Lagrangian X_L disjoint from D^*N such that $d_{\mathrm{int}}(L \oplus X_L, C_L) < \epsilon + c_\epsilon \nu(p)$ - see Corollary 3.2.7.4. We will also assume that this decomposition is of minimal length, thus $m = N(L; \mathcal{F}, \epsilon)$.

The exact symplectomorphism ϕ from the statement naturally extends to E by the identity in the exterior of D^*N. We denote this extension still by ϕ. Notice that $\phi(X_L) = X_L$.

With these conventions, recall from §3.1.6.7, see (32) and (33), that the symplectic diffeomorphism ϕ induces a TPC functor $\bar{\phi} := PD(\phi) : \mathcal{D}_p(D^*N) \to \mathcal{D}_{\phi_* p}(D^*N)$ (we recall that the space of perturbations $\mathcal{P}$ is closed under the action of ϕ).

Denote $L' = L \oplus X_L$. Notice that $d_{\text{int}}(L, \phi(L)) = d_{\text{int}}(L \oplus X_L, \phi(L) \oplus X_L)$ and the interleaving distance $d_{\text{int}}(X, \phi(L))$ is the same in D^*N and in E (for a choice of perturbation data p on E that extends the choice on D^*N). Let $p, \phi_* p \preceq q$. In $\mathcal{D}_q(D^*N)$ we have the inequality:

$$d_{\text{int}}(\phi(L'), L') \leq d_{\text{int}}(\phi(L'), \mathcal{H}_{\phi_* p,q}(\bar{\phi}(C_L))) + d_{\text{int}}(\mathcal{H}_{\phi_* p,q}(\bar{\phi}(C_L)), \mathcal{H}_{p,q}(C_L)) + d_{\text{int}}(\mathcal{H}_{p,q}(C_L), L') \; .$$

At the same time we also have

$$d_{\text{int}}(\phi(L'), \bar{\phi}(C_L)) \leq d_{\text{int}}(L', C_L) < \epsilon + c_\epsilon \nu(p)$$

in $\mathcal{D}_{\phi_* p}(D^*N)$ because $\bar{\phi}$ is a persistence functor and therefore non-dilating with respect to d_{int} (see Lemma 3.1.1.2 i). The functors $\mathcal{H}_{p,q}$, $\mathcal{H}_{\phi_* p,q}$ too are non-dilating with respect to the respective interleaving pseudo-metrics. Thus, in $\mathcal{D}_q(D^*N)$ we have

$$d_{\text{int}}(\phi(L'), \mathcal{H}_{\phi_* p,q}(\bar{\phi}(C_L))) < \epsilon + c_\epsilon \nu(p), \;\; d_{\text{int}}(\mathcal{H}_{p,q}(C_L), L') < \epsilon + c_\epsilon \nu(p) \; .$$

Starting from this point we first prove the statement under the additional assumption:

$$\phi(F) = F \; , \; \forall \; F \in \mathcal{F} \; . \tag{97}$$

Under this assumption we first notice that the statement follows if we can show that:

$$d_{\text{int}}(\mathcal{H}_{\phi_* p,q}(\bar{\phi}(C_L), \mathcal{H}_{p,q}(C_L)) = 0 \tag{98}$$

for some q such that $p, \phi_* p \preceq q$ by taking into account the relationships among d_{int}, D_{int}, and d_γ as in equations (52), (53) and (54) and taking p such that $\nu(p) \to 0$.

We now proceed to justify (98) under the additional assumption (97). The iterated cone C_L belongs to the category

$$\mathcal{D}_p(E) = PD(\mathcal{F}uk(\mathcal{L}ag^{(ex)}(E), p)) \; .$$

Recall that there is another model for this category in terms of twisted complexes as described in §3.1.4.5 that we will denote by $\mathcal{D}^{Tw}_p(E)$. This TPC is the persistence homological category of the filtered twisted complexes over the filtered A_∞-category with objects the exact, closed Lagrangians in E. With the notation in §3.1.4.5:

$$\mathcal{D}^{Tw}_p(E) = PH\left(FTw\left(\mathcal{F}uk(\mathcal{L}ag^{(ex)}(E), p)\right)\right) .$$

The Yoneda embedding extends to a 0-equivalence of TPCs:

$$\mathcal{Y}_p : \mathcal{D}^{Tw}_p(E) \to \mathcal{D}_p(E)$$

which is compatible (up to 0-isomorphism) with the comparison functors $\mathcal{H}_{p,q}$, when $p \preceq q$. In particular, the iterated cone C_L is 0-isomorphic to a filtered twisted complex

$$\bar{C}_{L;p} = (\oplus_i Z'_i, A_{\bar{C}_{L,p}})$$

over the shifted Lagrangian spheres $Z'_i = \Sigma^{s_i}\hat{S}_{x_{j(i)}}$ that appear in the decomposition (96). Such a twisted complex is characterized by an upper triangular $m \times m$ matrix, $A_{\bar{C}_{L;p}}$, with elements in $CF(\hat{S}_{x_j}, \hat{S}_{x_s}; p)$ (up to obvious shifts), as described in [**Sei08**] (see also the filtered λ-lemma in [**BCS21**] and §1.3.2). Here $CF(\hat{S}_{x_i}, \hat{S}_{x_j}; p)$ is the filtered Floer chain complex of $\hat{S}_{x_i}$ and $\hat{S}_{x_j}$ computed in E with respect to the perturbation data p. To emphasize the dependence on the perturbation data we included p in the notation.

It is easy to track the effect of $\bar{\phi}$ by using twisted complexes through the push-forward of twisted complexes: this transports the Lagrangians involved by ϕ, it pushes-forward the perturbation data $p \to \phi_* p$ and sends the matrix $A_{\bar{C}_{L;p}}$ to a matrix $\phi_*(A_{\bar{C}_{L;p}})$, element to element, by the obvious transformation:

$$CF(\hat{S}_{x_j}, \hat{S}_{x_s}; p) \xrightarrow{\phi_*} CF(\phi(\hat{S}_{x_j}), \phi(\hat{S}_{x_s}); \phi_* p) = CF(\hat{S}_{x_j}, \hat{S}_{x_s}; \phi_* p) \tag{99}$$

(where we use our assumption that ϕ keeps each $\hat{S}_{x_i}$ invariant). We denote the resulting twisted complex by $\phi_*(\bar{C}_{L,p})$. It has the form:

$$\phi_*(\bar{C}_{L,p}) = (\oplus_i Z'_i, A'_{\phi_* p}) \in FTw\left(\mathcal{F}uk(\mathcal{L}ag^{(ex)}(E), \phi_* p)\right)$$

where $A'_{\phi_* p} = \phi_*(A_{\bar{C}_{L,p}})$.

At this point we need to be more precise about the various choices of Floer data for the pairs $(\hat{S}_{x_i}, \hat{S}_{x_j})$. Due to the properties of the systems of categories involved here, as described in §3.1.5 and in §3.1.6, any such choice that is part of our perturbation data family $\mathcal{P}$ is acceptable for this argument. Similarly, we need to discuss the two relevant continuation functors $\mathcal{H}_{p,q}$ and $\mathcal{H}_{\phi_* p,q}$. It is useful to revisit Figure 4 a part of which is reproduced in Figure 13 with minor adjustments.

Recall that U in the picture contains the projection onto $\mathbb{C}$ of the disk bundle $D^*N \subset E$ and that the spheres $\hat{S}_{x_i}$ can be assumed to coincide with the spheres S_{x_i} in the picture outside of D^*N (this requires slightly modifying some of the U, U' in ways irrelevant for the rest of the argument). We may also assume that each two spheres $\hat{S}_{x_i}$, $\hat{S}_{x_j}$ interesect transversely. The Floer data for all the pairs $(\hat{S}_{x_i}, \hat{S}_{x_j})$ is chosen such that the Hamiltonian $H^{\hat{S}_{x_i}, \hat{S}_{x_j}}$ is constant outside D^*N. Thus the generators of the respective Floer complexes are the intersection points between the relevant spheres. The Morse functions on each of the spheres $\hat{S}_{x_i}$ are picked to have exactly two critical points, a maximum and a minimum, both outside $\bar{h}^{-1}(U')$. The almost complex structure $J^{i,j} = J^{\hat{S}_{x_i}, \hat{S}_{x_j}}$ is picked such that $\bar{h}$ is $(J^{i,j}, i)$-holomorphic over $W = \bar{h}^{-1}(U'' \backslash U)$. Here, the set U'' represents a neighbourhood of D^*N that is smaller than U', we could take it as the projection of $D^*_2 N$, and such that all the intersection points of the Lagrangian spheres $\hat{S}_{x_i}$ are outside U''. Further choices of almost complex structures on E used to define higher operations as well as the continuation maps $\mathcal{H}_{p,q}$ and $\mathcal{H}_{\phi_* p,q}$ are from the same class, in the sense that they make the projection $\bar{h}$ holomorphic.

With these choices a simple application of the open mapping theorem shows that there are no Floer strips (or even polygons) that have as inputs intersection points of paris of

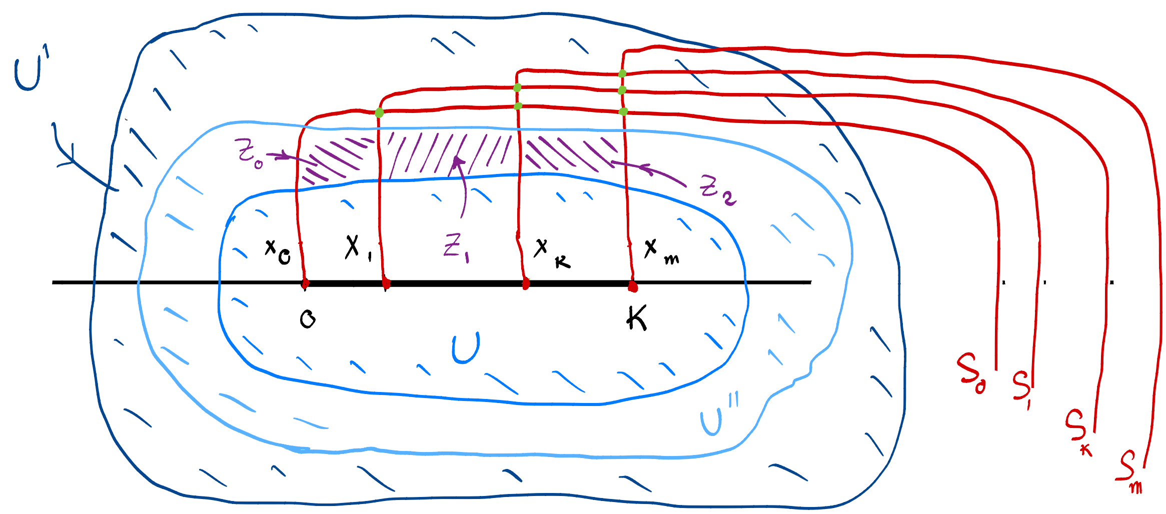


Figure 13. The Lagrangian spheres and their intersections.

distinct $\hat{S}_{x_i}$, have output at another such intersection point, have boundaries on (some of) these spheres, and that intersect the region $U''\backslash U$. For instance, if one such curve would intersect the region Z_1 in the picture it is easy to see that it also would have to extend to the regions Z_0 and Z_2 and, by pursuing this argument, one reaches a contradiction.

Recall that the support of ϕ is inside $D^*N \subset \bar{h}^{-1}(U)$. This means that in fact $\bar{C}_{L,p}$ coincides with $\phi_*(\bar{C}_{L,p})$ - the two matrixes $A'_{\phi_* p}$ and $A_{\bar{C}_{L,p}}$ coincide. Moreover, again because the Floer type strips or polygons can not cross the region $U''\backslash U$, is follows that, with obvious choices of perturbation data, $\mathcal{H}_{p,q}$ and $\mathcal{H}_{\phi_* p,q}$ act in the same way on $\bar{C}_{L,p}$ and $\phi_*(\bar{C}_{L,p})$. Therefore,

$$d_{\text{int}}(\mathcal{H}_{p,q}(\bar{C}_{L,p}), \mathcal{H}_{\phi_* p,q}(\phi_*(\bar{C}_{L,p}))) = 0$$

which implies (98) and concludes the proof under the additional assumption (97).

We now drop this assumption (97) and consider the general case. Our aim is to show

$$d_{\text{int}}(\mathcal{H}_{\phi_* p,q}(\bar{\phi}(C_L), \mathcal{H}_{p,q}(C_L)) \leq 2m\chi(\phi; \mathcal{F}) \ . \tag{100}$$

which is sufficient to end the proof of the Corollary. Denote $\bar{S}_i = \phi(\hat{S}_{x_i})$. These are exact Lagrangians spheres with the property that $\bar{S}_i$ coincides with $\hat{S}_{x_i}$ outside D^*N and are so that the $\bar{S}_i$'s are pairwise disjoint inside D^*N. As a result, the same argument used under the assumption (97) also shows in the current more general context that the matrix $A'_{\phi_* p}$ that defines the twisted complex

$$\phi_*(\bar{C}_{L,p}) = (\oplus_i \phi(Z'_i), A'_{\phi_* p}) \in FTw\left(\mathcal{F}uk(\mathcal{L}ag^{(ex)}(E), \phi_* p)\right)$$

is identified as a linear map - without taking into account filtrations - to $A_{\bar{C}_{L,p}}$ under the transformation $\hat{S}_{x_i} \to \bar{S}_i$ (that is akin to a basis change). However, the elements of the $m \times m$ matrix $A'_{\phi_* p}$ have potentially different filtration levels compared to $A_{\bar{C}_L, p}$. To understand this change notice that the filtered complex $CF(\bar{S}_i, \bar{S}_j)$ is isomorphic to $\Sigma^{k_{i,j}} CF(\hat{S}_{x_i}, \hat{S}_{x_j})$ where $|k_{i,j}| \leq Var(h_{\phi(F_{x_i})}) + Var(h_{\phi(F_{x_j})}) \leq 2\chi(\phi; \mathcal{F})$. These shifts cumulate and given that we have m terms in the twisted complexes under consideration we deduce (100) which concludes the proof. □

3.4.5. Lagrangian Gromov width and Corollary ??. We start by recalling the definition of the Gromov width ([**BC07**], [**BC06**]) that is relevant here. For this consider a symplectic manifold (M, ω), a Lagrangian submanifold $L \subset M$ and a closed subset $K \subset M$. The *Gromov width of L relative to K* is defined by:

$$\mathcal{W}(L; K) = \sup\Big\{\pi \frac{r^2}{2} \mid \exists\, j : (B_r, \omega_0) \to (M, \omega),\ j \text{ is an embedding}, \\ j^*\omega = \omega_0,\ \ j^{-1}(L) = B_r \cap \mathbb{R}^n,\ j(B_r) \cap K = \emptyset\Big\}$$

where $(B_r, \omega_0) \subset (\mathbb{C}^n, \omega_0)$ is the standard symplectic r-ball in $\mathbb{C}^n$, centred at the origin.

The aim of this subsection is to show Corollary **??**. We reformulate here this corollary in a more precise way.

Corollary 3.4.5.1. *Assume that $(\mathcal{L}ag(M), d_\gamma)$ is TPC retract ϵ-approximable in the sense of Definition 3.1.5.8 in the system $\widehat{\mathcal{C}}$ of Fukaya type TPCs, by the familiy $\mathcal{F}_\epsilon$ such that each element of $\mathcal{F}_\epsilon$ is a Yoneda module. For each $L \in \mathcal{L}ag(M)$ we have*

$$\mathcal{W}(L; \cup_{F \in \mathcal{F}_\epsilon} F) \leq 2\epsilon\ .$$

Remark 3.4.5.2. a. Corollary 3.4.5.1 does not directly apply to the case of $M = D^*N$ because the ϵ-generating families $\mathcal{F}_\epsilon$ consist of modules $\mathcal{F}_{x_i}$ that correspond to fibers F_{x_i} but that are not Yoneda modules in the categories $\mathcal{C}_p$ (these are part of the local approximating data $(\Phi, \mathcal{F}_\epsilon)$ - see §3.2.7.1). However, the Corollary does apply to the ambient approximating data $(\Phi', \mathcal{F}'_\epsilon)$ from §3.2.7.2 because, in this case, the elements of $\mathcal{F}'_\epsilon$ are represented by Lagrangian spheres $\hat{S}_{x_i} \subset E$. Therefore, recalling $F_{x_i} = \hat{S}_{x_i} \cap D^*N$, we deduce that the conclusion of the Corollary remains valid in the case of $M = D^*N,\ \mathcal{F}_\epsilon = \{F_{x_1}, \ldots, F_{x_l}\}$.

b. The inequality in the statement immediately implies that, when $\epsilon \to 0$, the family $\mathcal{F}_\epsilon$ becomes dense in M. Indeed, assume that a family $\mathcal{F}_\epsilon$ avoids some ball $B \subset M$. Take L that passes through the center x of this ball and also pick an embedding j, as in the definition of width, with $j(0) = x$ and such that $j(B_r) \subset B$. We deduce in this case that $\epsilon \geq \pi\frac{r^2}{2}$ and therefore ϵ is bounded away from zero under the hypothesis that $\mathcal{F}_\epsilon$ does not intersect B.

c. In this remarks consider the case $M = D^*N$. A fiber is determined by its base point, and the disk bundle D^*N is determined by the Riemannian metric g. In view of this, the Corollary indicates that there are some symplectic invariants associated with pairs $(\mathcal{S}, \mathsf{g})$ where $\mathcal{S}$ is a configuration of k distinct points in the base. To see this, for any finite family $\mathcal{S} \in N$ denote by $\mathcal{F}_{\mathcal{S}}$ the corresponding family of fibers inside D^*N (the metric defining the disk bundle being g). Now define the *approximability precision* of $\mathcal{S}$ by:

$$\epsilon_{\mathcal{S}} = \inf\{\ \epsilon > 0 \mid \text{there exists } \epsilon - \text{approximating data } (\Phi, \mathcal{F}_{\mathcal{S}}) \text{ as in } \S 3.2.7.1\ \}\ .$$

This is non-zero, by point b, and it is finite. Indeed, for $\mathcal{S}$ consisting of a single point, it follows from the arguments in [**BC21**] (which are partially a precursor to the proof of Theorem 3.2.2.1) that $\epsilon_{\mathcal{S}}$ is already finite. This quantity is, obviously, independent of the point in question and is an upper bound for all $\epsilon_{\mathcal{S}}$, for arbitrary finite $\mathcal{S}$. However, for $\mathcal{S}$ with more than a single element this number generally depends on the placement of the points in N. For instance, by taking $\mathcal{S} \subset D$ with $D \subset N$ a small disk, $\epsilon_{\mathcal{S}}$ certainly will stay away from 0 (again by point b), independently of the number of elements in $\mathcal{S}$. But, if the same number of fibers are more uniformly distributed along N, then, by increasing the number of elements of $\mathcal{S}$ one can make choices such that $\epsilon_{\mathcal{S}} \to 0$ because, by Theorem 3.2.2.1, ϵ-approximating data exists for all ϵ. We will not pursue this topic here but it obviously warrants more study.

Proof of Corollary 3.4.5.1. The proof follows directly from Corollary 6.13 i. in [**BCS21**]. We only need to relate the notation and conventions in that paper to the context here. The first remark is that [**BCS21**] works in the setting of weakly-filterered A_∞ categories. The setting in the current paper is that of filtered A_∞ categories and thus the results and arguments in [**BCS21**] apply here too. Denote the TPCs in the system $\widehat{\mathcal{C}}$ by $\mathcal{C}_p(M)$. Take $L \in \mathcal{L}\mathrm{ag}(M)$. Recall from Remark 3.1.2.4 that there is no difference whether we use $\bar{d}_{\mathrm{int}}$ or d_{int} in the definition of retract approximability. Moreover, from the description of $\bar{d}_{\mathrm{int}}$ in Lemma 4.2.3, we deduce that there exists an element $a \in \mathrm{Obj}(\mathcal{C}_p(M))$, a cone decomposition $\eta : 0 \rightsquigarrow a$ with linearization in $\mathcal{F}_\epsilon$, and a couple of maps $\alpha \in \hom^{r_1}(L, a)$, $\beta \in \hom^{r_2}(a, L)$ such that $\beta \circ \alpha = i_{0, r_1 + r_2}(id_L)$ with $r_1 + r_2 < 2\epsilon + c_\epsilon \nu(p)$ where $p \in \mathcal{P}$ is a choice of approriate perturbation data. By inspecting the definition of the quantity ρ in (6.16) in [**BCS21**] we see that $\rho(\alpha) < 2\epsilon + c_\epsilon \nu(p)$ (in brief, this $\rho(\alpha)$ infimizes $r_1 + r_2$ such that there is a morphism β as before). Moreover, this means that, for $\nu(p)$ small enough, we have, with the notation in [**BCS21**], $w_p((\alpha, a, \eta)) < 2\epsilon + c_\epsilon \nu(p)$ (this w_p in (6.19) [**BCS21**] is a measurement assigned to triples (α, a, η) that infimizes $\rho(\alpha)$ over all choices of $\alpha : L \to a$; it also depends on the perturbation data p). From Corollary 6.13. i [**BCS21**] we deduce $2\epsilon \geq \frac{1}{2}\delta(L; \cup_{F \in \mathcal{F}_\epsilon} F)$. The quantity $\delta(-;-)$ defined on p. 85 in [**BCS21**] is the double of the Gromov width as defined here, at the beginning of this section. This concludes the proof of the corollary. □

Remark 3.4.5.3. a. As mentioned in the introduction, there are some purely geometric implications of Corollaries 3.4.4.1 and 3.4.5.1 that are independent of approximability as well as of the machinery used for their proofs. The first statement can be read as saying that for each $\epsilon > 0$ there exists a finite family of Lagrangians in the class providing approximations (fibers for T^*N, equators for S^2 etc) such that the Gromov width of any $L \in \mathcal{L}\mathrm{ag}^{(ex)}(M)$ relative to this family is less than ϵ. Similarly, focusing on the case of $M = D^*N$, for any $\epsilon > 0$ there exists a finite family of fibers $\mathcal{F}$ such that any exact symplectic diffeormorphism satisfying $\phi(F) = F$ for all $F \in \mathcal{F}$ only moves each $L \in \mathcal{L}\mathrm{ag}^{(ex)}(D^*N)$ less than ϵ in the spectral metric.

b. Some of these geometric consequences can also be obtained by more direct means. Indeed, it was noticed by Egor Shelukhin that for any $\epsilon > 0$ one can prove by methods unrelated to Fukaya category techniques that there exists a finite family $\mathcal{F}$ of fibers of D^*N such that any Hamiltonian diffeomorphism ϕ induced by a Hamiltonian isotopy with support inside D^*N and disjoint from the fibers in $\mathcal{F}$, has spectral norm smaller than ϵ. The proof uses the same type of Morse functions φ as constructed in Proposition 3.2.3.1 and a result of Usher [**Ush10**]. This result also implies that the Gromov width of the complement of the fibers inside D^*M can be bounded above by ϵ. This Gromov width result was also noticed by Mark Gudiev by means of a direct displacement argument using the Morse functions φ as in Proposition 3.2.3.1.

3.4.5.1. *Gromov width, cone length and sizes of approximating families.* Let (M, ω) be a symplectic manifold of one of the types described in §3.1.6 and let $\mathcal{L}\mathrm{ag}$ be a subset of the collection of Lagrangians in M, as in §3.1.6.6. Recall that we have the system of filtered A_∞-categories $\widehat{\mathcal{F}uk}(\mathcal{L}\mathrm{ag})$ and its filtered derived system of TPC's $PD(\widehat{\mathcal{F}uk}(\mathcal{L}\mathrm{ag}))$.

Let $\mathcal{S}$ be a collection of subsets of M and $\mathcal{L} \subset \mathcal{L}\mathrm{ag}$ a collection of Lagrangians from $\mathcal{L}\mathrm{ag}$. For $w > 0$ define the w-density index of $\mathcal{S}$ relative $\mathcal{L}$ to be:

$$I(\mathcal{S}, \mathcal{L}; w) := \inf\big\{k \geq 1 \mid \exists\, S_1, \ldots, S_k \in \mathcal{S} \text{ such that } \mathcal{W}(L; S_1 \cup \cdots \cup S_k) \leq w,\ \forall L \in \mathcal{L}\big\}. \quad (101)$$

Here and elsewhere in the paper we use the convention that $\inf \emptyset = \infty$, $\sup \emptyset = -\infty$. Therefore if $L \subset S_1 \cup \cdots \cup S_k$ for some $L \in \mathcal{L}\mathrm{ag}$, $S_1, \ldots, S_k \in \mathcal{S}$, then we have $\mathcal{W}(L; S_1 \cup \cdots \cup S_k) = -\infty$ hence $\mathcal{W}(L; S_1 \cup \cdots \cup S_k) \leq w$ is automatically satisfied in (101).

For a subset $\mathcal{K} \subset \mathcal{L}\mathrm{ag}$ we denote by

$$\widetilde{\mathcal{K}} \subset \widetilde{\mathrm{Obj}}(PD(\widehat{\mathcal{F}uk}(\mathcal{L}\mathrm{ag})))$$

the collection of equivalence classes (see §3.1.5.10) of the objects $\mathcal{Y}_p(K) \in \mathrm{Obj}(PD(\mathcal{F}uk(\mathcal{L}\mathrm{ag}; p)))$, $K \in \mathcal{K}$, $p \in \mathcal{P}$, where $\mathcal{Y}_p : \mathcal{F}uk(\mathcal{L}\mathrm{ag}; p) \longrightarrow F\mathrm{mod}_{\mathcal{F}uk(\mathcal{L}\mathrm{ag};p)}$ is the Yoneda embedding (see §1.3.1 and §3.1.5.5).

The following is an immediate consequence of Corollary 3.4.5.1.

COROLLARY 3.4.5.4. *Let $\mathcal{L}, \mathcal{F} \subset \mathcal{L}ag$ be two collections of Lagrangians. Let $w_0 > 0$ and $\mathcal{F}^0 \subset \mathcal{F}$ be a finite subset such that $\widetilde{\mathcal{L}}$ is w_0-retract-approximable by $\widetilde{\mathcal{F}^0}$ in $PD(\widehat{\mathcal{F}uk}(\mathcal{L}ag))$. Then*

$$|\mathcal{F}^0| \geq I(\mathcal{F}, \mathcal{L}; w_0). \quad (102)$$

Moreover, there exists $L \in \mathcal{L}$ such that $N^r(L; \widetilde{\mathcal{F}^0}, w_0) \geq I(\mathcal{F}, \mathcal{L}; w_0)$.

REMARKS 3.4.5.5.
(1) Point (a) of Remark 3.4.5.2 applies here as well. Namely, the conclusion of Corollary 3.4.5.4 continues to hold for $M = D^*N$ with $\mathcal{L} = \mathcal{L}\text{ag}$ being the collection of all closed exact Lagrangians in D^*N and with $\mathcal{F}$ being any collection of cotangent fibers.
(2) If $\mathcal{S}' \subset \mathcal{S}''$, $\mathcal{L}' \supset \mathcal{L}''$, $w' \leq w''$, then

$$I(\mathcal{S}', \mathcal{L}'; w') \geq I(\mathcal{S}'', \mathcal{L}''; w''). \tag{103}$$

In particular, for every $\mathcal{S}$ and $\mathcal{L}$ we have $I(\mathcal{S}, \mathcal{L}; w) \geq I(\mathcal{S}, \emptyset; w)$. Note that the latter invariant depends on the collection $\mathcal{S}$ itself and does not involve any Lagrangian submanifolds.
(3) The inequality (102) may become useful if we fix a collection $\mathcal{F}$ that (retract) approximates $\mathcal{L}$ and want to bound from below the size of any possible finite family $\mathcal{F}^0 \subset \mathcal{F}$ that retract-w_0-approximates $\mathcal{L}$. Note that there is no particular assumption on the collection of Lagrangian $\mathcal{F}$ in Corollary 3.4.5.4, besides containing $\mathcal{F}^0$ (and being a subset of $\mathcal{L}\text{ag}$ of course). In particular we do not even need assume that $\mathcal{F}$ retract approximates $\mathcal{L}$. But in view of (103) the most effective bounds from below in (102) on $|\mathcal{F}^0|$ are obtained when $\mathcal{F}$ is minimal as possible and in applications it makes sense to assume that $\mathcal{F}$ retract approximates $\mathcal{L}$.

Here is a simple example showing how Corollary 3.4.5.4 can be used in practice. Consider $M = S^2$ endowed with its standard symplectic structure ω normalized such that $\text{Area}(S^2, \omega) = 1$. Let $\mathcal{L}\text{ag}$ be the collection of all closed monotone Lagrangian in S^2 (in this case, all the embedded closed curves which separate S^2 into two domains of area $\frac{1}{2}$). Let $\mathcal{F} \subset \mathcal{L}\text{ag}$ be the collection of all great circles passing through the north and south poles of S^2. By Proposition 3.3.3.3, $\widetilde{\mathcal{F}}$ retract-approximates $\widetilde{\mathcal{L}\text{ag}}$ in $PD(\widehat{\mathcal{F}uk}(\mathcal{L}\text{ag}))$.

We claim that $I(\mathcal{F}, \emptyset; w) \geq \lceil \frac{1}{4w} \rceil$. Indeed, if $F_1, \ldots, F_k \in \mathcal{F}$ then at least one of the connected components of $S^2 \setminus (F_1 \cup \cdots \cup F_k)$ must have area $\geq \frac{1}{2k}$ hence $\mathcal{W}(\emptyset; F_1 \cup \cdots \cup F_k) \geq \frac{1}{4k}$. It follows that $I(\mathcal{F}, \emptyset; w) \geq \lceil \frac{1}{4w} \rceil$. (In fact, it is not hard to see that we have equality here.)

It follows from Corollary 3.4.5.4 that for every $\mathcal{L} \subset \mathcal{L}\text{ag}$, every $w_0 > 0$, and every finite collection $\mathcal{F}^0 \subset \mathcal{F}$ which w_0-retract-approximates $\mathcal{L}$ we have:

$$|\mathcal{F}^0| \geq I(\mathcal{F}, \mathcal{L}; w_0) \geq I(\mathcal{F}, \emptyset; w_0) \geq \left\lceil \frac{1}{4w_0} \right\rceil.$$

.1. Filtered A_∞-categories

.1.1. Filtered A_∞-categories, A_∞-functors, A_∞-natural transformations and shift A_∞-functors. Filtered A_∞-categories have already appeared in the literature, see e.g. [**BCZ24b**, Section 3.2] and [**BCS21**, Chapter 2]. We will therefore go over this subject very briefly and cover below mainly aspects of the theory that have not been addressed in the literature.

Fix a field $\mathbf{k}$ and $\mathbf{k}_0 \subset \mathbf{k}$ a subring. For most of the applications in this paper we will either have $\mathbf{k}_0 = \mathbf{k} = \mathbb{Z}_2$, or $\mathbf{k} = \Lambda$ (the universal Novikov field with coefficients in $\mathbb{Z}_2$) and $\mathbf{k}_0 = \Lambda_0 \subset \Lambda$ the positive Novikov ring (see (21) and (22) in §3.1.6.3).

In the followings, we say that (C, d) is a filtered chain complex over $\mathbf{k}$ if (C, d) is chain complex over $\mathbf{k}$ and comes endowed with an increasing filtration parametrized by the real line of $\mathbf{k}_0$-modules that are preserved by d.

Let $\mathcal{A}$ be an A_∞-category over $\mathbf{k}$. For $X_0, X_1 \in \mathrm{Obj}(\mathcal{A})$ we abbreviate:

$$\mathcal{A}(X_0, X_1) := \hom_{\mathcal{A}}(X_0, X_1),$$

and more generally, for a tuple of objects $\vec{X} := (X_0, \ldots, X_d)$, $d \geq 1$, we write

$$\mathcal{A}(\vec{X}) := \mathcal{A}(X_0, X_1) \otimes_{\mathbf{k}} \cdots \otimes_{\mathbf{k}} \mathcal{A}(X_{d-1}, X_d)$$

and denote by $\mu_d : \mathcal{A}(\vec{X}) \longrightarrow \mathcal{A}(X_0, X_d)$ the A_∞-operations.

A filtered A_∞-category (over $(\mathbf{k}, \mathbf{k}_0)$) is an A_∞-category $\mathcal{A}$ over $\mathbf{k}$ such that the spaces of morphisms $\mathcal{A}(X, Y) = \hom_{\mathcal{A}}(X, Y)$ between every two objects X, Y are filtered chain complexes over $\mathbf{k}$ For every $\alpha \in \mathbb{R}$ we denote by $\mathcal{A}^\alpha(X, Y) \subset \mathcal{A}(X, Y)$ the $\mathbf{k}_0$-submodule which is the filtration level parametrized by α. More generally, for $\alpha \in \mathbb{R}$ and a tuple $\vec{X} = (X_0, \ldots, X_d)$ of objects of $\mathcal{A}$ we write

$$\mathcal{A}^\alpha(\vec{X}) \subset \mathcal{A}(\vec{X})$$

for the $\mathbf{k}_0$-submodule generated by the images of all the $\mathbf{k}_0$-modules

$$\mathcal{A}^{\alpha_1}(X_0, X_1) \otimes_{\mathbf{k}_0} \cdots \otimes_{\mathbf{k}_0} \mathcal{A}^{\alpha_d}(X_{d-1}, X_d), \text{ with } \alpha_1 + \cdots + \alpha_d \leq \alpha,$$

under the obvious ($\mathbf{k}_0$-linear) map

$$\mathcal{A}(X_0, X_1) \otimes_{\mathbf{k}_0} \cdots \otimes_{\mathbf{k}_0} \mathcal{A}(X_{d-1}, X_d) \longrightarrow \mathcal{A}(\vec{X}).$$

We then require that for every $d \geq 1$, every $\vec{X}$ and every $\alpha \in \mathbb{R}$, we have:

$$\mu_d(\mathcal{A}^\alpha(\vec{X})) \subset \mathcal{A}^\alpha(X_0, X_d).$$

A filtered A_∞-category $\mathcal{A}$ is called strictly unital (in the filtered sense) if it is strictly unital as an A_∞-category without filtrations and moreover for every $X \in \mathrm{Obj}(\mathcal{A})$ the strict unit $e_X \in \mathcal{A}(X, X)$ lies in filtration level 0, i.e. $e_X \in \mathcal{A}^0(X, X)$. Unless otherwise stated, all the filtered A_∞-categories $\mathcal{A}$ in this paper are implicitly assumed to be strictly unital.

An example of filtered A_∞-category is the dg-category $\mathcal{F}\mathbf{Ch}$ of filtered chain complexes over $\mathbf{k}$.

The concept of A_∞-functors has a filtered counterpart. An A_∞-functor $\mathcal{F} : \mathcal{A} \longrightarrow \mathcal{B}$ between two A_∞-categories is said to be filtered if for every $d \geq 1$ and every tuple of objects $\vec{X}$ its d-th order term $\mathcal{F}_d$ preserves filtrations in the sense that

$$\mathcal{F}_d(\mathcal{A}^\alpha(\vec{X})) \subset \mathcal{B}^\alpha(\mathcal{F}X_0, \mathcal{F}X_d), \; \forall \alpha \in \mathbb{R}.$$

All the filtered A_∞-functors in this paper will be implicitly assumed to be strictly unital.

One can define pre-natural transformations between filtered A_∞-functors as follows. Let $\mathcal{F}, \mathcal{G} : \mathcal{A} \longrightarrow \mathcal{B}$ be two filtered A_∞-functors. A pre-natural transformation $T : \mathcal{F} \longrightarrow \mathcal{G}$ of shift s is a pre-natural transformation (in the usual A_∞-sense) such that its d-th order term T_d shifts filtrations by not more than s, namely we have a family of elements $T^0 \in \mathcal{B}^s(\mathcal{F}X, \mathcal{G}X)$ for every $X \in \mathrm{Obj}(\mathcal{A})$ and moreover, for every $d \geq 0$ and every tuple of objects $\vec{X} = (X_0, \ldots, X_d)$ we have:

$$T_d(\mathcal{A}^\alpha(\vec{X})) \subset \mathcal{B}^{\alpha+s}(\mathcal{F}X_0, \mathcal{G}X_d), \ \forall d \geq 1, \ \alpha \in \mathbb{R}.$$

We will always endow our A_∞-categories $\mathcal{A}$ with a shift functor. This is not a single functor but in fact a system of filtered A_∞-functors $\{\Sigma^r : \mathcal{A} \longrightarrow \mathcal{A}\}_{r \in \mathbb{R}}$, $r \in \mathbb{R}$, all having trivial higher order terms ($\Sigma^r_d = 0$, $\forall d \geq 2$), and together with filtered natural transformations $\eta_{r,s} : \Sigma^r \longrightarrow \Sigma^s$ of shift $s - r$, for every $r, s \in \mathbb{R}$, and such that:

(1) $\Sigma^0 = \mathrm{id}$.
(2) $\Sigma^r \circ \Sigma^s = \Sigma^{r+s}$ for all $r, s \in \mathbb{R}$.
(3) The higher order terms $(\eta_{r,s})_d, d \geq 1$ of $\eta_{r,s}$ vanish.
(4) For every $X \in \mathrm{Obj}(\mathcal{A})$ the 0-term of $\eta_{r,s}$ assigned to X is a cycle $(\eta^X_{r,s})_0 \in \mathcal{A}^{s-r}(\Sigma^r X, \Sigma^s X)$. Moreover, we have $(\eta^X_{0,0})_0 = e_X \in \mathcal{A}^0(X, X)$.
(5) For every $X \in \mathrm{Obj}(\mathcal{A})$ and every $t \in \mathbb{R}$ we have $\mu_2\big((\eta^X_{r,s})_0, (\eta^X_{s,t})_0\big) = (\eta^X_{r,t})_0$.
(6) For every $X \in \mathrm{Obj}(\mathcal{A})$ and every $t \in \mathbb{R}$ we have $\Sigma^t_1\big((\eta^X_{r,s})_0\big) = (\eta^X_{r+t,s+t})_0$.

It follows that each $\eta_{r,s}$ is an isomorphism and its inverse is $\eta_{s,r}$. By composing from the left and from the right with $\eta^X_{0,r}$ and $\eta^Y_{s,0}$ respectively we obtain isomorphisms

$$\mathcal{A}^\alpha(\Sigma^r X, \Sigma^s Y) \cong \mathcal{A}^{\alpha+r-s}(X, Y), \ \forall\, \alpha \in \mathbb{R}$$

that are compatible with the filtration $\mathcal{A}^{\beta'}(-,-) \subset \mathcal{A}^{\beta''}(-,-)$, $\beta' \leq \beta''$.

Remark .1.1.1. The conditions (1)-(6) may look excessively strong in the A_∞-context and can probably be relaxed to give a weaker yet meaningful definition of shift functors. A more appropriate name for Σ as above might be a *strict* shift functor, but we will not use this term in this paper.

From now on all filtered A_∞-functors will be assumed to be compatible with the shift functor in the sense that $\mathcal{F} \circ \Sigma^r = \Sigma^r \circ \mathcal{F}$ for every $r \in \mathbb{R}$ and $(\eta^{\mathcal{F}X}_{r,s})_0 = \mathcal{F}_1(\eta^X_{r,s})_0$ for every $r, s \in \mathbb{R}$.

.1.2. Filtered A_∞-modules and bimodules. Let $\mathcal{A}$ be a filtered A_∞-category. A filtered left $\mathcal{A}$-module $\mathcal{M}$ is an $\mathcal{A}$-module with the following additional structures and properties. For every $X \in \mathrm{Obj}(\mathcal{A})$ the chain complex $(\mathcal{M}(X), \mu^{\mathcal{M}}_1)$ (of vector spaces over $\mathbf{k}$) is filtered by an increasing filtration of subcomplexes of $\mathbf{k}_0$-modules which is parametrized by $\mathbb{R}$. For $\alpha \in \mathbb{R}$ we denote by $\mathcal{M}^\alpha(X) \subset \mathcal{M}(X)$ the α-level of this filtration. For $\alpha \in \mathbb{R}$ and a tuple of objects $\vec{X} = (X_0, \ldots, X_k)$ denote by

$$(\mathcal{A}(\vec{X}) \otimes \mathcal{M}(X_k))^\alpha \subset \mathcal{A}(\vec{X}) \otimes_{\mathbf{k}} \mathcal{M}(X_k) \tag{104}$$

the $\mathbf{k}_0$-submodule generated by the images of all the $\mathbf{k}_0$-submodules

$$\mathcal{A}^{\alpha'_1}(X_0, X_1) \otimes_{\mathbf{k}_0} \cdots \otimes_{\mathbf{k}_0} \mathcal{A}^{\alpha'_k}(X_{k-1}, X_k) \otimes_{\mathbf{k}_0} \mathcal{M}^{\alpha''}(X_k), \text{ with } \alpha'_1 + \cdots + \alpha'_k + \alpha'' \leq \alpha,$$

under the obvious map

$$\mathcal{A}(X_0, X_1) \otimes_{\mathbf{k}_0} \cdots \otimes_{\mathbf{k}_0} \mathcal{A}(X_{k-1}, X_k) \otimes_{\mathbf{k}_0} \mathcal{M}(X_k) \longrightarrow \mathcal{A}(\vec{X}) \otimes_{\mathbf{k}} \mathcal{M}(X_k).$$

We require that the higher operations $\mu^{\mathcal{M}}_{k|1}$, $k \geq 1$, preserves filtrations. Namely, for every tuple of objects $\vec{X} = (X_0, \ldots, X_k)$ in $\mathcal{A}$ and every $\alpha \in \mathbb{R}$ we have:

$$\mu^{\mathcal{M}}_{k|1}\big((\mathcal{A}(\vec{X}) \otimes \mathcal{M}(X_k))^{\alpha}\big) \subset \mathcal{M}^{\alpha}(X_0).$$

For uniformity we will denote the differential $\mu^{\mathcal{M}}_1$ also by $\mu^{\mathcal{M}}_{0|1}$.
Alternatively, one can view filtered $\mathcal{A}$-modules as filtered A_∞-functors $\mathcal{A} \to \mathcal{F}\mathbf{Ch}^{\mathrm{op}}$.

Pre-homomorphisms between filtered modules are defined as follows. Let $\mathcal{M}$, $\mathcal{N}$ be filtered A_∞-modules. A module pre-homomorphism $t : \mathcal{M} \longrightarrow \mathcal{N}$ of shift $\leq s \in \mathbb{R}$ is a pre-homomorphism such that for every $k \geq 0$, every tuple $\vec{X}$, and $\alpha \in \mathbb{R}$ we have

$$t_{k|1}((\mathcal{A}(\vec{X}) \otimes \mathcal{M}(X_k))^{\alpha}) \subset \mathcal{N}^{\alpha+s}(X_0).$$

Filtered A_∞-modules and their pre-homomorphisms form a filtered dg-category $F\mathrm{mod}_{\mathcal{A}}$, where $\hom^s_{F\mathrm{mod}_{\mathcal{A}}}(\mathcal{M}, \mathcal{N})$ consists of all pre-homomorphisms $t : \mathcal{M} \longrightarrow \mathcal{N}$ of shift $\leq s$. Note that $F\mathrm{mod}_{\mathcal{A}}$ comes with a natural shift functor Σ. Σ. Its action on objects $\mathcal{M} \in \mathrm{Obj}(F\mathrm{mod}_{\mathcal{A}})$ is given by:

$$(\Sigma^r \mathcal{M})^{\alpha}(X) := \mathcal{M}^{\alpha - r}(X), \ \forall r \in \mathbb{R}, \alpha \in \mathbb{R}, X \in \mathrm{Obj}(\mathcal{A}).$$

Unless otherwise stated, all the modules in this paper will be assumed to be strictly unital. We will also assume all the filtered modules to be compatible with the shift functor in the obvious sense, i.e. $\mathcal{M}(\Sigma^r X) = (\Sigma^r \mathcal{M})(X)$ for all objects $X \in \mathrm{Obj}(\dashv)$ and all $r \in \mathbb{R}$.

Right $\mathcal{A}$-modules can be defined in a completely analogous way to left modules. To distinguish the two, whenever confusion may arise we will denote the category of filtered left $\mathcal{A}$-modules by $F\mathrm{mod}^l_{\mathcal{A}}$ and the right modules by $F\mathrm{mod}^r_{\mathcal{A}}$.

Given two filtered A_∞-categories $\mathcal{A}, \mathcal{B}$ we also have the filtered $(\mathcal{A}, \mathcal{B})$ modules. They are defined by similar means to the above. Filtered bimodules too will be assumed in this paper to be strictly unital and compatible with the shift functor. They form a filtered dg-category which we denote by $F\mathrm{bimod}_{\mathcal{A},\mathcal{B}}$.

.1.3. The filtered Yoneda embedding.

.1.3.1. *Definition.* Let $\mathcal{A}$ be a filtered A_∞-category with structure maps $(\mu_d)_{d \geq 1}$. Recall that we always assume that our A_∞-categories are strictly unital. We upgrade the Yoneda embedding (see [**Sei08**, Section (11)]) to a filtered version, that is, we define a filtered A_∞-functor

$$\mathcal{Y} \colon \mathcal{A} \to F\mathrm{mod}_{\mathcal{A}}.$$

We will work with left A_∞-modules, but the same can be analogously developed for right ones. Whenever confusion may arise we will denote by $\mathcal{Y}^l \colon \mathcal{A} \to F\mathrm{mod}^l_{\mathcal{A}}$ the left Yoneda embedding and by $\mathcal{Y}^r\colon \mathcal{A} \to F\text{mod}^r_{\mathcal{A}}$. Sometimes we will need to specicify the A_∞-category in the notation and we will write the Yoneda embedding as $\mathcal{Y}_{\mathcal{A}}$.

Let L be an object of $\mathcal{A}$. We define $\mathcal{Y}(L)$ via

$$\mathcal{Y}(L)(L') := \mathcal{A}(L', L)$$

for any object L' of $\mathcal{A}$ and endow it with the A_∞-structure given by $\mu^{\mathcal{Y}(L)}_{l|1} := \mu_{l+1}$ for $l \geq 0$. It is straightforward to see that $\mathcal{Y}(L)$ indeed defines a filtered A_∞-module.
We now define higher operations of the functor $\mathcal{Y}$. To ease the notation, we will often write $\mathcal{Y}_L$ instead of $\mathcal{Y}(L)$ in the rest of this subsection. Consider two objects L_0 and L_1 of $\mathcal{A}$ and an element $c \in \mathcal{A}^{\leq\alpha}(L_0, L_1)$ for some $\alpha \in \mathbb{R}$. We define its image $\mathcal{Y}_1(c) \in F\text{mod}_{\mathcal{A}}(\mathcal{Y}_{L_0}, \mathcal{Y}_{L_1})$ as the pre-morphism of modules with components $\mathcal{Y}_1(c)_{l|1}\colon \mathcal{A}(\vec{X}) \otimes \mathcal{Y}_{L_0}(X_l) \to \mathcal{Y}_{L_1}(X_0)$ given by

$$\mathcal{Y}_1(c)_{l|1}(x_1, \ldots, x_l, y) := \mu_{l+1}(x_1, \ldots, x_l, y, c).$$

We define the higher components $\mathcal{Y}_d$ of the Yoneda embedding in a similar manner using contraction maps, that is, given a tuple $\vec{L} = (L_0, \ldots, L_d)$ and an element $\vec{c} \in \mathcal{A}(\vec{L})$, we define the pre-morphism of modules $\mathcal{Y}_d(\vec{c}) \in F\text{mod}_{\mathcal{A}}(\mathcal{Y}_{L_0}, \mathcal{Y}_{L_d})$ via

$$\mathcal{Y}_d(\vec{c})_{l|1}(x_1, \ldots, x_l, y) := \mu_{l+d+1}(x_1, \ldots, x_l, y, \vec{c}).$$

Notice that $\mathcal{Y}$ is a filtered A_∞-functor, since the maps μ_d, $d \geq 1$, are filtered by assumption.

.1.3.2. *The λ-map.* Consider an object L of $\mathcal{A}$ and a filtered A_∞-module $\mathcal{M}$. We have the following map, usually simply called the λ-map:

$$\lambda\colon \mathcal{M}(L) \to F\text{mod}_{\mathcal{A}}(\mathcal{Y}(L), \mathcal{M})$$

defined for $c \in \mathcal{M}(L)$ via

$$\lambda(c)_{l|1}(x_1, \ldots, x_l, y) := \mu^{\mathcal{M}}_{l+1|1}(x_1, \ldots, x_l, y, c).$$

It is clear that λ is well-defined (i.e. that it lands in the category of *filtered* modules), simply because $\mathcal{A}$ is a filtered A_∞-category. Similarly to [**Sei08**, Lemma 2.12] and [**BCS21**, Proposition 2.5.1] we have the following result. This result is usually proved using spectral sequence, but strict unitality of $\mathcal{A}$ allows us for a more explicit proof, which we present here.

Proposition .1.3.1. *The map λ is a filtered quasi-isomorphism, with filtered quasi-inverse via filtered chain-homotopies. In particular, it induces an isomorphism in persistence homology.*

Proof. The fact that λ is filtered is obvious, as $\mathcal{A}$ is filtered. We will prove the rest of the statement by explicitly constructing a quasi-inverse and the chain homotopies. First, we define a candidate quasi-inverse

$$\theta\colon F\text{mod}_{\mathcal{A}}(\mathcal{Y}(L), \mathcal{M}) \to \mathcal{M}(L)$$

as follows: given a pre-module morphism $\varphi \in F\mathrm{mod}_{\mathcal{A}}(\mathcal{Y}(L),\mathcal{M})$, we set

$$\theta(\varphi) := \varphi_{0|1}(e_L)$$

where we recall that $e_L \in \mathcal{A}(L,L)$ stands for the (given) choice of strict unit for L. Clearly θ is a chain map. It is easy to see that given $c \in \mathcal{M}(L)$, $\theta(\lambda(c)) = c$, exactly because e_L is a *strict* unit, so no chain homotopy is needed in this direction. Moreover, since $e_L \in \mathcal{A}^{\leq 0}(L,L)$ by our definition of filtered A_∞-category, we get that θ is a filtered chain map, i.e. it sends pre-morphisms with shift α to $\mathcal{M}^{\leq\alpha}(L)$ for any $\alpha \in \mathbb{R}$.
We now define a candidate chain homotopy for the other composition: we define

$$H\colon F\mathrm{mod}_{\mathcal{A}}(\mathcal{Y}(L),\mathcal{M}) \to F\mathrm{mod}_{\mathcal{A}}(\mathcal{Y}(L),\mathcal{M})$$

by sending $\varphi \in F\mathrm{mod}_{\mathcal{A}}(\mathcal{Y}(L),\mathcal{M})$ to the pre-morphism of modules $H(\varphi)$ with components

$$H(\varphi)_{l|1}(x_1,\dots,x_l,y) := \varphi_{l+1|1}(x_1,\dots,x_l,y,e_L)$$

for any $l \geq 0$. Again, since strict units lie at vanishing filtration level, the map H preserves $F\mathrm{mod}_{\mathcal{A}}(\mathcal{Y}(L),\mathcal{M})^{\leq\alpha}$ for any $\alpha \in \mathbb{R}$. We prove that

$$\lambda \circ \theta + id_{F\mathrm{mod}_{\mathcal{A}}(\mathcal{Y}(L),\mathcal{M})} = \mu_1^{\mathrm{mod}} \circ H + H \circ \mu_1^{\mathrm{mod}},$$

that is, that θ is a right quasi-inverse of λ via the chain-homotopy H. Let $\varphi \in F\mathrm{mod}_{\mathcal{A}}(\mathcal{Y}(L),\mathcal{M})$ and $l \geq 0$, we compute:

(1) first, the term

$$\begin{aligned}\mu_1^{\mathrm{mod}}(H(\varphi))_{l|1}(x_1,\dots,x_l,y) = &\sum_{i=0}^{l} \mu_{i|1}^{\mathcal{M}}(x_1,\dots,x_i,\varphi_{l-i+1|1}(x_{i+1},\dots,x_l,y,e_L)) \\ &+ \sum_{i=0}^{l} \varphi_{i+1|1}(x_1,\dots,x_i,\mu_{l-i+1}(x_{i+1},\dots,x_l,y),e_L) \\ &+ \sum_{j=0}^{l-1}\sum_{i=1}^{l-j} \varphi_{l-i+2|1}(x_1,\dots,x_j,\mu_i(x_{j+1},\dots,x_{j+i}),x_{j+i+1},\dots,x_l,y,e_L)\end{aligned}$$

(2) then the term

$$\begin{aligned}
H(\mu_1^{\mathrm{mod}}\varphi)_{l|1}(x_1,\dots,x_l,y) = \ & \mu_1^{\mathrm{mod}}\varphi_{l+1|1}(x_1,\dots,x_l,y,e_L)\\
= & \sum_{i=0}^{l}\mu^{\mathcal{M}}_{i|1}(x_1,\dots,x_i,\varphi_{l-i+1|1}(x_{i+1},\dots,x_l,y,e_L)+\mu^{\mathcal{M}}_{l+1|1}(x_1,\dots,x_l,y,\varphi_1(e_L))\\
& +\sum_{i=0}^{l}\varphi_{i|1}(x_1,\dots,x_i,\mu_{l-i+2}(x_{i+1},\dots,x_l,y,e_L))\\
& +\sum_{j=0}^{l-1}\sum_{i=1}^{l-j}\varphi_{l-i+2|1}(x_1,\dots,x_j,\mu(x_{j+1},\dots,x_{j+i}),x_{j+i+1},\dots,x_l,y,e_L)\\
& +\sum_{j=0}^{l}\varphi_{j+1|1}(x_1,\dots,x_j,\mu_{l-j+1}(x_{j+1},\dots,x_l,y),e_L)
\end{aligned}$$

and note that the second big sum after the last equality equals $\varphi_{l|1}(x_1,\dots,x_l,y)$ since e_L is a strict unit.

(3) finally, we compute

$$(\lambda\circ\theta+id)(\varphi)_{l|1}(x_1,\dots,x_l,y)=\mu^{\mathcal{M}}_{l+1|1}(x_1,\dots,x_l,y,\varphi_1(e_L))+\varphi_{l|1}(x_1,\dots,x_l,y). \tag{105}$$

We see that summing all the contributions above we get the expected equality at the $l|1$ level. This proves the proposition. □

The following result is an easy corollary of the above. We denote by $\mathcal{Y}(\mathcal{A})=\mathrm{image}(\mathcal{Y})$ the subcategory of $F\mathrm{mod}_{\mathcal{A}}$ consisting of Yoneda modules.

COROLLARY .1.3.2. *The induced persistence functor $H(\mathcal{Y})\colon H(\mathcal{A})\to H(image(\mathcal{Y}))$ is an equivalence of persistence categories.*

.1.4. Persistence Hochschild homology. Let $\mathcal{A}$ be an A_∞-category and $\mathcal{M}$ an $(\mathcal{A},\mathcal{A})$-bimodule. Denote by $CC(\mathcal{A},\mathcal{M})$ the Hochschild chain complex of $\mathcal{A}$ with coefficients in $\mathcal{M}$ (a.k.a. the cyclic bar complex). Recall that this chain complex decomposes into a direct sum $CC(\mathcal{A},\mathcal{M})=\oplus_{d\geq 0}CC(\mathcal{A},\mathcal{M};d)$, with

$$CC(\mathcal{A},\mathcal{M};d)=\bigoplus_{\vec{X}}\mathcal{M}(X_d,X_0)\otimes_{\mathbf{k}}\mathcal{A}(\vec{X}),$$

where the direct sum runs over all tuples $\vec{X}=(X_0,\dots,X_d)$ of objects in $\mathcal{A}$. For $d=0$ we set $\mathcal{A}(\vec{X})=\mathbf{k}$ so that $CC(\mathcal{A},\mathcal{M};0):=\bigoplus_{X\in\mathrm{Obj}(\mathcal{A})}\mathcal{M}(X,X)$. We endow $CC(\mathcal{A},\mathcal{M})$ with the Hochschild differential d_{CC} (see e.g. [**Gan12, She20, RS17**]). Its homology is denoted by $HH(\mathcal{A},\mathcal{M})$ and is called the Hochschild homology of $\mathcal{A}$ with coefficients in $\mathcal{M}$.

In case $\mathcal{M}=\Delta_{\mathcal{A}}$ is the diagonal $(\mathcal{A},\mathcal{A})$-bimodule we write $CC(\mathcal{A},\mathcal{A})$ and $HH(\mathcal{A},\mathcal{A})$ for the Hochschild chain complex and homology respectively, with coefficients in $\Delta_{\mathcal{A}}$. Sometimes we even abbreviate these to $CC(\mathcal{A})$ and $HH(\mathcal{A})$.

Assume now that $\mathcal{A}$ is a filtered A_∞-category and $\mathcal{M}$ is a filtered $(\mathcal{A},\mathcal{A})$-bimodule. For every $\alpha \in \mathbb{R}$, $d \geq 0$, and $\vec{X} = (X_0, \ldots, X_d)$ denote by

$$(\mathcal{M}(X_d, X_0) \otimes \mathcal{A}(\vec{X}))^\alpha \subset \mathcal{M}(X_d, X_0) \otimes_{\mathbf{k}} \mathcal{A}(\vec{X})$$

the $\mathbf{k}_0$-submodule generated by the images of all the submodules

$$\mathcal{M}^{\alpha'}(X_d, X_0) \otimes_{\mathbf{k}_0} \mathcal{A}^{\alpha''_1}(X_0, X_1) \otimes_{\mathbf{k}_0} \cdots \otimes_{\mathbf{k}_0} \mathcal{A}^{\alpha''_d}(X_{d-1}, X_d), \text{ with } \alpha' + \alpha''_1 + \cdots + \alpha''_d \leq \alpha,$$

under the obvious map

$$\mathcal{M}(X_d, X_0) \otimes_{\mathbf{k}_0} \mathcal{A}(X_0, X_1) \otimes_{\mathbf{k}_0} \cdots \otimes_{\mathbf{k}_0} \mathcal{A}(X_{d-1}, X_d) \longrightarrow \mathcal{M}(X_d, X_0) \otimes_{\mathbf{k}} \mathcal{A}(\vec{X}).$$

Denote

$$CC^\alpha(\mathcal{A}, \mathcal{M}; d) := \bigoplus_{\vec{X}} (\mathcal{M}(X_d, X_0) \otimes \mathcal{A}(\vec{X}))^\alpha$$

which is a $\mathbf{k}_0$-submodule of $CC(\mathcal{A}, \mathcal{M}; d)$. Finally, denote by

$$CC^\alpha(\mathcal{A}, \mathcal{M}) := \bigoplus_{d \geq 0} CC^\alpha(\mathcal{A}, \mathcal{M}; d).$$

This gives us an increasing filtration of $CC(\mathcal{A}, \mathcal{M})$ by $\mathbf{k}_0$-submodules $CC^\alpha(\mathcal{A}, \mathcal{M}) \subset CC(\mathcal{A}, \mathcal{M})$, $\alpha \in \mathbb{R}$. Since $\mathcal{A}$, $\mathcal{M}$ are filtered the Hochschild differential preserves this filtration, i.e.

$$d_{CC}(CC^\alpha(\mathcal{A}, \mathcal{M})) \subset CC^\alpha(\mathcal{A}, \mathcal{M})$$

so $(CC(\mathcal{A}, \mathcal{M}), d_{CC})$ becomes a filtered chain complex. We denote by $PHH(\mathcal{A}, \mathcal{M})$ its persistence homology.

The Hochschild complex $CC(\mathcal{A}, \mathcal{M})$ admits also a discrete filtration, traditionally called the *length* filtration, $(F^N CC(\mathcal{A}, \mathcal{M}))_{N \geq 0}$ defined for $N \geq 0$ as

$$F^N CC(\mathcal{A}, \mathcal{M}) := \bigoplus_{d \leq N} CC(\mathcal{A}, \mathcal{M}; d).$$

It is easy to see that d_{CC} preserves this filtration too, and hence induces a differential on each $F^N CC(\mathcal{A}, \mathcal{M})$. We denote by $F^H H(\mathcal{A}, \mathcal{M})$ the resulting homology. The real filtration defined above is obviously compatible with the length filtrations. We thus get a sequence $F^N PHH(\mathcal{A}, \mathcal{M})$, $N \geq 0$, of persistence modules.

Note that if $\mathcal{A}$ is filtered then the diagonal bimodule $\Delta_{\mathcal{A}}$ is also filtered and we denote the corresponding filtered chain complex and its persistence homology by $CC(\mathcal{A}, \mathcal{A})$ and $PHH(\mathcal{A}, \mathcal{A})$ respectively.

We briefly describe the grading formalism for Hochschild homology used in our geometric application in §3.3. Let (X, ω) be a monotone symplectic manifold of real dimension $2n$ and $\mathcal{A}$ be a Fukaya category build from Lagrangians in (X, ω). Floer complexes are defined over the Novikov field $\mathbf{k} = \Lambda$ and are $\mathbb{R}/2\mathbb{Z}$ graded, with the Novikov variable having degree 0. In particular, the following has to be understood $\mod 2$. The dth order A_∞-map μ_d of $\mathcal{A}$ has cohomological degree $2 - d$, hence homological degree $(d-2) + (d-1)n$. This leads to the following grading on the Hochschild complex $CC_*(\mathcal{A}, \mathcal{A})$ of $\mathcal{A}$: given $\vec{x} := x_1 \otimes \cdots \otimes x_k \in \mathcal{A}(X_0, \ldots, X_k, X_0)$ we set

$$\deg(\vec{x}) := \sum_{i=1}^{k} \deg(x_i) + k - 1 - n.$$

It is easy to see that d_{CC} has degree -1.

.1.5. Shifted categories. Let $\mathcal{A}$ be a filtered A_∞-category with a shift functor Σ, and let $r \geq 0$. Define the normalized r-shift $S^r\mathcal{A}$ of $\mathcal{A}$ to be the filtered A_∞-category with the same objects as $\mathcal{A}$ and morphism spaces $(S^r\mathcal{A})(L, L')$ between two objects L and L':

$$(S^r\mathcal{A})^\alpha(L, L') := \begin{cases} \mathcal{A}^\alpha(L, L'), & \text{if } L = \Sigma^l L' \text{ for some } l \in \mathbb{R}, \\ \mathcal{A}^{\alpha - r}(L, L'), & \text{otherwise.} \end{cases}$$

The A_∞-operations $\mu_d^{\mathcal{A}}$ on $\mathcal{A}$ naturally induce maps $\mu_d^{S^r\mathcal{A}}$ on $S^r\mathcal{A}$. The shift and translation functors Σ and T carry over to $S^r\mathcal{A}$ in the obvious way.

There is an obvious filtered A_∞-functor $\eta_r^{\mathcal{A}} : S^r\mathcal{A} \longrightarrow \mathcal{A}$ which is the identity on objects, is induced by the identity on morphisms, and has trivial higher operations.

.1.6. Functors with Linear Deviation. For readability, we repeat here the definition of A_∞-functors with linear deviation (LD functors, for short).

Let $\mathcal{A}$ be a filtered A_∞-category. Given a tuple $\vec{X} = (X_0, \ldots, X_d)$ of objects from $\mathcal{A}$ we define its reduced tuple $\vec{X}_R := (X_{i_0}, \ldots, X_{i_{d_R}})$ by omitting from $\vec{X}$ subsequent (in the cyclic order) objects that are equal up to a shift. The objects forming $\vec{X}_R$ are well defined only up to shifts, but the length of $\vec{X}_R$ (or rather, its length-1) $0 \leq d_R \leq d$ is well defined. We call it the reduced length of $\vec{X}$ and denote it by d_R or $d_R(\vec{X})$ whenever we want to emphasize its dependence on $\vec{X}$. Note that in case every two consecutive objects in $\vec{X}$ are different (up to shifts) then $d_R(\vec{X}) = d$. At the other extremity, if all the objects in $\vec{X}$ are equal up to shifts, then $d_R(\vec{X}) = 0$.

Let $\mathcal{A}$, $\mathcal{B}$ be two filtered A_∞-categories. Let $s \geq 0$. An A_∞-functor $\mathcal{F} : \mathcal{A} \longrightarrow \mathcal{B}$ is said to have linear deviation rate s if for every $d \geq 1$, every tuple of objects $\vec{X} = (X_0, \ldots, X_d)$ from $\mathrm{Obj}(\mathcal{A})$ and every $\alpha \in \mathbb{R}$ we have:

$$\mathcal{F}_d\big(\mathcal{A}^\alpha(X_0, \ldots, X_d)\big) \subset \mathcal{B}^{\alpha + d_R(\vec{X})s}(\mathcal{F}X_0, \mathcal{F}X_d).$$

We will refer to such functors as LD-functors (LD stands for Linear Deviation).

Let $\mathcal{F}, \mathcal{G} : \mathcal{A} \longrightarrow \mathcal{B}$ be two LD-functors, both with deviation rate $\leq s$. A pre-natural transformation $T : \mathcal{F} \longrightarrow \mathcal{G}$ is said to have linear deviation rate s and shift $\leq r$ if for every $d \geq 0$ it's d-th order component T_d shifts filtration levels by $\leq r + ds$.

LD-functors $\mathcal{F} : \mathcal{A} \longrightarrow \mathcal{B}$ with a given deviation rate $s \geq 0$, and their pre-natural transformations form a filtered A_∞-category $\mathrm{fun}^{\mathrm{LD};s}(\mathcal{A}, \mathcal{B})$. To emphasize the given deviation rate we will sometimes denote the objects of these category by $(\mathcal{F}, s)$. Note that for every $s' \leq s''$

we have an obvious (faithful but not full) inclusion $\mathrm{fun}^{\mathrm{LD};s'}(\mathcal{A},\mathcal{B}) \subset \mathrm{fun}^{\mathrm{LD};s''}(\mathcal{A},\mathcal{B})$ of filtered A_∞-categories.

.1.6.1. *Homotopy between LD-functors.* We also have the notion of homotopy between LD-functors, which is an adaptation of the notion from [**Sei08**, Section (1h)] to the filtered and LD setting. Let $\mathcal{F}, \mathcal{G} : \mathcal{A} \longrightarrow \mathcal{B}$ be two LD-functors with deviation $\leq s$ and assume that both functors *act in the same way on objects.* Consider the pre-natural transformation $D = \{D_d\}_{d\geq 0}$ between $\mathcal{F}$ and $\mathcal{G}$ defined by $D_0 := 0$ and $D_d := \mathcal{F}_d - \mathcal{G}_d$. We say that $\mathcal{F}$ and $\mathcal{G}$ are r-homotopic, as LD-functors with deviation $\leq s$ if there exists a pre-natural transformation $T \in hom^r_{\mathrm{fun}^{\mathrm{LD};s}(\mathcal{A},\mathcal{B})}\big((\mathcal{F}, s), (\mathcal{G}, s)\big)$ with vanishing 0-term, $T^0 = 0$, and with deviation rate $\leq s$ and shift $\leq r$, such that $D = \mu_1(T)$.

Recall that filtered functors are a special case of LD-functors, and homotopies between them (as defined above; i.e. with deviation $s = 0$) are interesting too. In this paper we will need homotopies both between filtered functors as well as between LD-functors.

Next we briefly discuss the relation between homotopy of functors and persistence Hochschild homology. Recall from §.1.4, that if $\mathcal{A}$ is a filtered A_∞-category, then its Hochschild chain complex $CC(\mathcal{A},\mathcal{A})$ inherits a filtration from $\mathcal{A}$ and consequently the homology of the latter becomes a persistence module which we call the persistence Hochschild homology $PHH(\mathcal{A},\mathcal{A})$ of $\mathcal{A}$. Note that its ∞-limit $PHH^\infty(\mathcal{A},\mathcal{A})$ coincides with the usual Hochschild homology $HH(\mathcal{A},\mathcal{A})$.

PROPOSITION .1.6.1. *Let $\mathcal{F}, \mathcal{G} : \mathcal{A} \longrightarrow \mathcal{B}$ be two filtered functors. If $\mathcal{F}, \mathcal{G}$ are 0-homotopic then the chain maps induced by $\mathcal{F}$, $\mathcal{G}$ on the filtered Hochschild chain complexes $\mathcal{F}_{CC}, \mathcal{G}_{CC} : CC(\mathcal{A},\mathcal{A}) \longrightarrow CC(\mathcal{B},\mathcal{B})$ are 0-chain homotopic. In particular the induced maps on the persistence Hochschild homologies $\mathcal{F}_{PHH}, \mathcal{G}_{PHH} : PHH(\mathcal{A},\mathcal{A}) \longrightarrow PHH(\mathcal{B},\mathcal{B})$ coincide as maps of persistence modules.*

.1.6.2. *From LD-functors to filtered functors.* Let $\mathcal{F} : \mathcal{A} \longrightarrow \mathcal{B}$ be an LD A_∞-functor with deviation rate $\leq r$. Using §.1.5, define a new A_∞-functor

$$\eta^r\mathcal{F} : S^r\mathcal{A} \longrightarrow \mathcal{B}, \quad \eta^r\mathcal{F} := \mathcal{F} \circ \eta_r^{\mathcal{A}},$$

which we call the normalized r-shift of $\mathcal{F}$. It is straightforward to see that $\eta^r\mathcal{F}$ is a *filtered* A_∞-functor.

.1.7. Filtered bimodules associated with functors. Given a filtered A_∞-functor $\mathcal{F} : \mathcal{A} \longrightarrow \mathcal{B}$, define the $(\mathcal{B},\mathcal{A})$-bimodule

$$\underline{\mathcal{F}} := (\mathrm{id}_{\mathcal{B}} \otimes \mathcal{F})^* \Delta_{\mathcal{B}},$$

where $\Delta_{\mathcal{B}}$ stands for the diagonal bimodule of $\mathcal{B}$. In other words, given $Y \in \mathrm{Obj}(\mathcal{B})$, $X \in \mathrm{Obj}(\mathcal{A})$, we have

$$\underline{\mathcal{F}}(Y, X) := \mathcal{B}(Y, \mathcal{F}(X)).$$

On bimodule-composable elements, the structure maps $\mu^{\underline{\mathcal{F}}}_{l|1|r}$, for $l, r \geq 0$, of $\underline{\mathcal{F}}$ are given by

$$\mu^{\underline{\mathcal{F}}}_{l|1|r}(b_1, \ldots, b_l, y, a_1, \ldots, a_r) :=$$
$$\sum_{j=1}^{r} \sum_{\substack{s_1+\cdots+s_j=r \\ 1\leq s_i}} \mu^{\mathcal{B}}_{l+j+1}(b_1, \ldots, b_l, y, \mathcal{F}_{s_1}(a_1, \ldots, a_{s_1}), \ldots, \mathcal{F}_{s_j}(a_{r-s_j}, \ldots, a_r)).$$

Note that since $\mathcal{F}$ is filtered, the bimodule $\underline{\mathcal{F}}$ is filtered too.

The assignment $\mathcal{F} \longmapsto \underline{\mathcal{F}}$ extends to a filtered A_∞-functor $\mathcal{U} : F\mathrm{fun}(\mathcal{A}, \mathcal{B}) \longrightarrow F\mathrm{bimod}_{\mathcal{B},\mathcal{A}}$. Its 1st order term $\mathcal{U}_1$ has the following description. Let $\mathcal{F}, \mathcal{G}; \mathcal{A} \longrightarrow \mathcal{B}$ be two filtered A_∞-functors and $T : \mathcal{F} \longrightarrow \mathcal{G}$ a pre-natural transformation between them, of some given shift α (i.e. $T \in \hom^{\alpha}_{F\mathrm{fun}(\mathcal{A},\mathcal{B})}(\mathcal{F}, \mathcal{G})$). Define the $(\mathcal{B}, \mathcal{A})$-bimodule pre-homomorphism

$$\mathcal{U}_1(T) \colon \underline{\mathcal{F}} \longrightarrow \underline{\mathcal{G}},$$
$$\mathcal{U}_1(T)_{l|1|r}(b_1, \ldots, b_l, y, a_1, \ldots, a_r) :=$$
$$\sum_{j=1}^{r} \sum_{i=1}^{j} \sum_{\substack{s_1+\cdots+s_j=r \\ 1\leq s_k, \forall k\neq i}} \mu^{\mathcal{B}}_{l+1+j}\big(b_1, \ldots, b_l, y, (\mathcal{F}_{s_1}, \ldots, \mathcal{F}_{s_{i-1}}, T_{s_i}, \mathcal{G}_{s_{i+1}}, \ldots, \mathcal{G}_{s_j})(a_1, \ldots, a_r)\big).$$

It is straightforward to verify that $\mathcal{U}_1(T)$ is bimodule pre-homomorphism of the same shift α. Given composable natural transformations $T_1, \ldots, T_d$ between A_∞-functors from $\mathcal{A}$ to $\mathcal{B}$, the pre-homomorphism of $(\mathcal{B}, \mathcal{A})$-bimodules $\mathcal{U}_d(T_1, \ldots, T_d)$ is defined by a similar formula.

LEMMA .1.7.1 (Theorem 8.2.1.2 in [**LH03**]). *The component $\mathcal{U}_1$ of the functor $\mathcal{U}$ induces a quasi-isomorphism.*

.1.8. Pull back and push forward of filtered A_∞-modules. Let $\mathcal{A}$, $\mathcal{B}$ be filtered A_∞-categories and $\mathcal{F} : \mathcal{A} \longrightarrow \mathcal{B}$ a filtered A_∞-functor. Let $\mathcal{M}$ be a filtered $\mathcal{B}$-module. Then the pull back $\mathcal{F}^*\mathcal{M}$ can be endowed with the structure of a filtered $\mathcal{A}$-module in a straightforward way. However, if instead of assuming that $\mathcal{F}$ is filtered we only assume it to be an LD-functor then the pull back $\mathcal{F}^*\mathcal{M}$ is in general not filtered, but rather an LD-module (a concept which will not be used in this paper).

At the same time, there is a way to define push-forward of filtered modules by LD-functors and still get filtered modules. We present this construction next.

Let $\mathcal{A}, \mathcal{B}$ be filtered A_∞-categories and $\mathcal{F} : \mathcal{A} \longrightarrow \mathcal{B}$ an LD-functor with deviation rate $\leq r$. Let $\mathcal{M}$ be a filtered $\mathcal{A}$-module. The push-forward $(\mathcal{F}, r)_*\mathcal{M}$ of $\mathcal{M}$ by $\mathcal{F}$ is defined to be the filtered $\mathcal{B}$-module:

$$(\mathcal{F}, r)_*\mathcal{M} := \underline{\eta^r \mathcal{F}} \otimes_{S^r\mathcal{A}} (\eta^{\mathcal{A}}_r)^*\mathcal{M}. \tag{106}$$

Note that the value of the parameter r affects the filtration structure on the push-forward module, and therefore we include it in the notation. In other words, if $r < r'$ then $(\mathcal{F}, r')_*\mathcal{M}$ has a different filtration structure than $(\mathcal{F}, r)_*\mathcal{M}$. Of course, if one forgets the filtrations, then the push-forward of $\mathcal{M}$ depends only on $\mathcal{F}$, and gives the same result for all r's.

Given a pre-homomorphism $f : \mathcal{M}_0 \longrightarrow \mathcal{M}_1$ beetween filtered $\mathcal{A}$-modules, define its push-forward $(\mathcal{F}, r)_* f : (\mathcal{F}, r)_*\mathcal{M}_0 \longrightarrow (\mathcal{F}, r)_*\mathcal{M}_1$ as follows. The higher order terms $((\mathcal{F}, r)_* f)_{l|1}$, $l \geq 1$, are defined to be 0, and the linear order term is:

$$((\mathcal{F}, r)_* f)_{0|1}(b \otimes a_1 \otimes \cdots \otimes a_d \otimes m) = \sum_{i=1}^{d} b \otimes a_1 \otimes \cdots \otimes a_i \otimes f_{d-i|1}(a_{i+1}, \ldots, a_d, m).$$

The construction above extends to a filtered A_∞-functor

$$PF : \mathrm{fun}^{\mathrm{LD};r}(\mathcal{A}, \mathcal{B}) \longrightarrow F\mathrm{fun}(F\mathrm{mod}_{\mathcal{A}}, F\mathrm{mod}_{\mathcal{B}}),$$

whose action on the object $(\mathcal{F}, r)$ is the filtered functor $(\mathcal{F}, r)_* : F\mathrm{mod}_{\mathcal{A}} \longrightarrow F\mathrm{mod}_{\mathcal{B}}$. To define the action of PF on morphisms, let $\mathcal{F}, \mathcal{G} \in \mathrm{Obj}(\mathrm{fun}^{\mathrm{LD};r}(\mathcal{A}, \mathcal{B}))$, and let $T\colon \mathcal{F} \to \mathcal{G}$ be a pre-natural transformation (in the category $\mathrm{fun}^{\mathrm{LD};r}(\mathcal{A}, \mathcal{B})$ of LD-functors with deviation rate $\leq r$) of shift $\rho \geq 0$. The image $PF_1(T)$ of T under the push-forward functor PF is defined for an object $\mathcal{M} \in \mathrm{Obj}(F\mathrm{mod}_{\mathcal{A}})$ as the pre-module homomorphism $PF_1(T) \in \hom_{F\mathrm{mod}_{\mathcal{B}}}(\mathcal{F}_*\mathcal{M}, \mathcal{G}_*\mathcal{M})$ given by:

$$PF(T)_{l|1}(b_1, \ldots, b_l, y \otimes \vec{a} \otimes m) := \sum_{j=0}^{d} \mathcal{U}_1(T)_{l|1|j}(b_1, \ldots, b_l, y, a_1, \ldots, b_j) \otimes a_{j+1} \otimes \cdots \otimes a_d \otimes m$$

$$= \sum_{j=0}^{d} \sum_{l=1}^{j} \sum_{i=1}^{l} \sum_{\substack{s_1+\cdots+s_l=j \\ 1 \leq s_k,\, \forall k \neq i}} \mu^{\mathcal{B}}_{l+1+j}\big(b_1, \ldots, b_l, y, (\mathcal{F}_{s_1}, \ldots, \mathcal{F}_{s_{i-1}}, T_{s_i}, \mathcal{G}_{s_{i+1}}, \ldots, \mathcal{G}_{s_l})(a_1, \ldots, a_j)\big)$$
$$\otimes a_{j+1} \otimes \cdots \otimes a_d \otimes m.$$

The image of a composable tuple $T_1, \ldots, T_d$ of A_∞-pre-natural transformations under PF_d is defined in a very similar manner. It is straightforward to verify that PF is a filtered (strictly unital) A_∞-functor.

The next proposition shows that, up to shifts, the push forward of the Yoneda module of an objects is the same as the Yoneda module of the image of this object by the given functor. We will denote here by $\mathcal{Y}$ the filtered Yoneda embedding, both for $\mathcal{A}$ and for $\mathcal{B}$.

PROPOSITION .1.8.1. *Let $\mathcal{F} : \mathcal{A} \longrightarrow \mathcal{B}$ be an LD-functor with deviation rate $\leq r$. For every $X \in Obj(\mathcal{A})$ the push forward $(\mathcal{F}, r)_*\mathcal{Y}(X)$ of the Yoneda module $\mathcal{Y}(X)$ is 0-quasi-isomorphic to $\Sigma^r\mathcal{Y}(\mathcal{F}X)$.*

To simplify the notation, when the deviation rate of a functor is clear from the context, we will sometimes omit the r from the pair $(\mathcal{F}, r)$ and simply write $\mathcal{F}_*\mathcal{M}$.

.1.9. Filtered twisted complexes. In this section, we introduce filtered twisted complexes. The constructions developed here are upgrades of [**Sei08**, Chapter 3] from the unfiltered to the filtered setting, and of [**BCZ24b**, Section 2.5.1] from the dg to the A_∞ case.

.1.9.1. *The filtered A_∞-categories of filtered twisted complexes.* Let $\mathcal{A}$ be a filtered A_∞-category, without a choice of shift functor. We will write $\mathbf{k}_0^r[d]$ for interval persistence module over $\mathbf{k}_0$ with a unique generator in degree $d \in \mathbb{Z}$ placed at filtration level $r \in \mathbb{R}$ and given an object L of $\mathcal{A}$ write $\Sigma^r L[d]$ for the formal tensor product $\mathbf{k}_0^r[d] \otimes L$.

DEFINITION .1.9.1. We define the (standard) shift completion of $\mathcal{A}$ as the category $\mathcal{A}^\Sigma$ with objects

$$\mathrm{Ob}(\mathcal{A}^\Sigma) := \{\Sigma^r L[d] \ : \ L \in \mathrm{Ob}(\mathcal{A}),\ d \in \mathbb{Z},\ r \in \mathbb{R}\},$$

morphism spaces

$$\mathcal{A}^\Sigma(\Sigma^r L_0[d_0], \Sigma^s L_1[d_1]) := \Sigma^{-(r-s)}\mathcal{A}(L_0, L_1)[d_1 - d_0]$$

that is, $\mathcal{A}(\Sigma^r L_0[d_0], \Sigma^s L_1[d_1])$ is the chain complex $\mathcal{A}(L_0, L_1)$ shifted up in degree by $d_1 - d_0$ and in filtration by $r - s$, and with the same A_∞-operations of $\mathcal{A}$.

REMARK .1.9.2. (1) It is easy to see that $\mathcal{A}^\Sigma$ is a filtered A_∞-category.
(2) On the A_∞-category $\mathcal{A}^\Sigma$ there is an immediate choice of shift A_∞-functor (in the sense of [**BCZ24b**, Section 3.2.1]), which on objects is $\Sigma^r(L) := \Sigma^r L$.
(3) In some texts the prefix Σ denotes the additive enlargement of an A_∞-category (e.g. [**Sei08**, Section (3k)]), which is not the case here.

DEFINITION .1.9.3. We define the filtered additive enlargement $\mathcal{A}^\oplus$ of $\mathcal{A}$ as the additive enlargement of $\mathcal{A}^\Sigma$ in the sense of [**Sei08**, Section (3k)], that is, as the A_∞-category with objects given by formal sums

$$\overline{L} = \bigoplus_{i=1}^{n} \Sigma^{r_i} L_i[d_i],$$

where $n \in \mathbb{N}$ and $\Sigma^{r_i} L_i[d_i]$ is an object of $\mathcal{A}^\Sigma$ for any i; and morphisms spaces between objects $\overline{L} = \bigoplus_{i=1}^{n} \Sigma^{r_i} L_i[d_i]$ and $\overline{L}' = \bigoplus_{i=1}^{m} \Sigma^{r'_i} L'_i[d'_i]$ given by $n \times m$ matrices

$$f = (f_{ij}), \text{ where } f_{ij} \in \mathcal{A}^\Sigma\left(\Sigma^{r_i} L_i[d_i], \Sigma^{r'_j} L'_j[d'_j]\right)$$

with usual matrix grading. Given an object $\overline{L} = \bigoplus_{i=1}^{n} \Sigma^{r_i} L_i[d_i]$ we will write $|\overline{L}| = n$ and call it the length of $\overline{L}$. The A_∞-maps $\mu_d^\oplus$, $d \geq 1$, are defined by extending μ_d matrixwise, that is, given $\overline{L}_0, \dots, \overline{L}_d$ objects of $\mathcal{A}^\oplus$ and matrices $f^1, \dots, f^d$ with $f^i \in \mathcal{A}^\oplus(\overline{L}_{i-1}, \overline{L}_i)$ we set for $i = 1, \dots, |\overline{L}_0|$ and $j = 1, \dots, |\overline{L}_d|$:

$$\mu_d^\oplus\left(f^1, \dots, f^d\right)_{ij} := \sum_{i_1=1}^{|\overline{L}_1|} \cdots \sum_{i_{d-1}=1}^{|\overline{L}_{d-1}|} \mu_d\left(f^1_{ii_1}, \dots, f^d_{i_{d-1}j}\right).$$

Moreover, we filter morphism spaces in $\mathcal{A}^\oplus$ by setting the filtration level of a matrix to be the maximum of the filtration level of its entries, that is:

$$(\mathcal{A}^\oplus)^{\leq\alpha}(\overline{L}, \overline{L}') := \left\{ f = (f_{ij}) \in \mathcal{A}^\oplus(\overline{L}, \overline{L}') : \ f_{ij} \in (\mathcal{A}^\Sigma)^{\leq\alpha}\left(\Sigma^{r_i} L_i[d_i], \Sigma^{r'_j} L'_j[d'_j]\right) \text{ for any } i, j \right\}.$$

REMARK .1.9.4. (1) We will not write all the sums in the definition of $\mu_d^{\oplus}$, but rather use the shorthand notation $\sum_{i_1,\dots,i_{d-1}}$.
(2) It is straightforward to see that $\mathcal{A}^{\oplus}$ is a filtered A_∞-category.

DEFINITION .1.9.5. A filtered (one-sided) twisted complex of $\mathcal{A}^{\Sigma}$ is a pair

$$(\overline{L}, q = q_{\overline{L}}) := \left(\bigoplus_{i=1}^{n} \Sigma^{r_i} L_i[d_i], (q_{ij})\right)$$

for some $n \in \mathbb{N}$ such that:

(1) $\overline{L}$ is an object of $\mathcal{A}^{\oplus}$, that is, for any $i = 1, \dots, n$, $\Sigma^{r_i} L_i[d_i]$ is an object of $\mathcal{A}^{\Sigma}$,
(2) q is a morphism in $\mathcal{A}^{\oplus}(\overline{L}, \overline{L})$ lying at degree 1 and vanishing filtration level.
(3) the matrix q is strictly upper triangular, that is $q_{ij} = 0$ for any $i \geq j$,
(4) the matrix q satisfies

$$\sum_{d\geq 1} \mu_d^{\oplus}(q, \dots, q) = 0,$$

to which we will refer as the Maurer-Cartan identity of q.

The matrix $q = q_{\overline{L}}$ is called the differential of $\overline{L}$ and will be often dropped from the notation.

REMARK .1.9.6. As q is strictly upper triangular, any entry $\mu_d^{\oplus}(q, \dots, q)_{i,j}$, $i, j = 1, \dots, |\overline{L}|$, can be written as

$$\sum_{i<i_1<\dots<i_{d-1}<j} \mu_d(q_{ii_1}, q_{i_1i_2}, \dots, q_{i_{d-1}j}).$$

Moreover, the sum in the Maurer-Cartan identity of q is a finite sum: indeed, for any $d \geq |\overline{L}|$ we have $\mu_d^{\oplus}(q, \dots, q) = 0$ since the longest possible term is

$$\mu_{|L|-1}(q_{12}, q_{23}, \dots, q_{|\overline{L}|-1|\overline{L}|})$$

in the matrix $\mu_{|L|-1}^{\oplus}(q, \dots, q)$, which has only one non-zero entry in position $(1, |\overline{L}|)$.

Given a twisted complex over a filtered A_∞-category in the sense of [**Sei08**, Section (3l) and Remark 3.26], it might be the case that the differential is not filtration-preserving (i.e. Condition 4 above does not hold). The analogue of Lemma 2.96 in [**BCZ24b**] holds: there are shifts r_i turning this twisted complex in a filtered one.

DEFINITION .1.9.7. We define the filtered pre-triangulated completion $FTw(\mathcal{A})$ of $\mathcal{A}$ as follows:

(1) The objects of $FTw(\mathcal{A})$ are filtered one-sided twisted complexes over $\mathcal{A}$,
(2) Given two filtered twisted complexes $\overline{L}$ and $\overline{L}'$, the morphism space $FTw(\mathcal{A})(\overline{L}, \overline{L}')$ is defined as $\mathcal{A}^{\oplus}(\overline{L}, \overline{L}')$, with the same filtration,
(3) The A_∞-operations are deformed by the differentiasl in the following way: given any $d \geq 1$ and twisted complexes

$$\left(\overline{L_i} = \bigoplus_{k=1}^{n_i} \Sigma^{r_{i,k}} L_{i,k}[d_{i,k}], q_i\right)$$

for any $i = 0, \dots, d$ and morphisms $f^i \in FTw(\mathcal{A})(\overline{L_{i-1}}, \overline{L_i})$ for any $i = 1, \dots, d$ we define $\mu_d^{Tw}(f^1, \dots, f^d)$ as the $|\overline{L}_0| \times |\overline{L}_d|$ matrix

$$\mu_d^{Tw}(f^1, \dots, f^d) := \sum_{k_0, \dots, k_d \geq 0} \mu^{\oplus}_{d+k_0+\cdots+k_d}\left(q_0^{\otimes k_0}, f^1, q_1^{\otimes k_1}, \dots, q_{d-1}^{\otimes k_{d-1}}, f^d, q_d^{\otimes k_d}\right)$$

Moreover, given $N \geq 1$ we define $FTw^N\mathcal{A}$ to be the full A_∞-subcategory of $FTw\mathcal{A}$ with filtered twisted complexes $\overline{L}$ of length $|\overline{L}| \leq N$ as objects.

REMARK .1.9.8. a. The fact that $FTw(\mathcal{A})$ and $FTw^N(\mathcal{A})$ are a A_∞-category is a consequence of differentials satisfying the Maurer-Cartan identity above.

b. There is an obvious filtered full and faithful A_∞-functors $\mathcal{I}^N_{\mathcal{A}} \colon \mathcal{A} \to FTw^N(\mathcal{A})$ for all $N \geq 1$, as well as $\mathcal{I}_{\mathcal{A}} \colon \mathcal{A} \to FTw(\mathcal{A})$.

c. The shift functor Σ on $\mathcal{A}^\Sigma$ induces a shift functor, still denoted by Σ, on $FTw(\mathcal{A})$. It is defined on an object $\overline{L} = \bigoplus_{i=1}^n \Sigma^{r_i} L_i[d_i]$ by

$$\Sigma^r \overline{L} = \bigoplus_{i=1}^n \Sigma^{r_i + r} L_i[d_i]$$

and viewing each entry q_{ij} of the differential q of $\overline{L}$ as a morphism

$$q_{ij} \in FTw(\mathcal{A})(\Sigma^{r_i+r} L_i[d_i], \Sigma^{r_j+r} L_j[d_j]) \cong FTw(\mathcal{A})(\Sigma^{r_i} L_i[d_i], \Sigma^{r_j} L_j[d_j]).$$

d. The sum in the definition of μ_d^{Tw} is a finite sum, for basically the same reason that the Maurer-Cartan identity consists of a finite sum. For instance, in $\mu_1^{Tw} f$ for $f \in FTw(\mathcal{A})(\overline{L}, \overline{L}')$ we sum elements of the form $\mu_{1+k_0+k_1}(q^{\otimes k_0}, f, q'^{\otimes k_1})$, whose (i,j)-entry is

$$\sum_{i_1, \dots, i_{k_0}=1}^{|\overline{L}|} \sum_{i_{k_0+1}, \dots, i_{k_0+k_1}=1}^{|\overline{L}'|} \mu_{1+k_0+k_1}\left(q_{ii_1}, \dots, q_{i_{k_0-1}i_{k_0}}, f_{i_{k_0}i_{k_0+1}}, q'_{i_{k_0+1}i_{k_0+2}}, \dots, q'_{i_{k_0+k_1}j}\right)$$

so that, by lower triangularity of the differentials, $\mu_{1+k_0+k_1}(q^{\otimes k_0}, f, q'^{\otimes k_1}) = 0$ whenever $k_0 \geq |\overline{L}|$ or $k_1 \geq |\overline{L}'|$. In particular, the longest possible summand is

$$\mu_{|\overline{L}|+|\overline{L}'|-1}\left(q_{12}, \dots, q_{|\overline{L}|-1|\overline{L}|}, f_{|\overline{L}|1}, q'_{12}, \dots, q'_{|\overline{L}'|-1|\overline{L}'|}\right)$$

in the $(1, |\overline{L}'|)$-entry. It is easy to see that the same holds in $\mu_d^{Tw}(f^1, \dots, f^d)$ whenever $k_i \geq |\overline{L_i}|$ for some i.

DEFINITION .1.9.9. Let $\overline{L}$ and $\overline{L}'$ be two filtered twisted complexes in $FTw(\mathcal{A})$ as above, $f \in FTw(\mathcal{A})(\overline{L}, \overline{L}')$ be a degree zero morphism such that $\mu_1^{FTw(\mathcal{A})} f = 0$, and $\lambda \in \mathbb{R}$ such that $\lambda \geq \mathbb{A}(f)$. We define the λ-filtered mapping cone of f as

$$\mathrm{Cone}^\lambda(f) := \left(\overline{L}' \oplus \Sigma^\lambda \overline{L}[1],\ q_{co}\right) \text{ where } q_{co} := \begin{pmatrix} q' & f \\ 0 & \Sigma^\lambda q[1] \end{pmatrix}.$$

LEMMA .1.9.10. *$Cone^\lambda(f)$ is a filtered twisted complex.*

Remark .1.9.11. Note that the subcategories $FTw^N(\mathcal{A})$ are not pre-triangulated, as the cone construction doesn't preserve the length filtration.

Proposition .1.9.12. *Let $\mathcal{A}$ be a filtered A_∞-category. Then $H^0(FTw(\mathcal{A}))$ is a TPC.*

Remark .1.9.13. Filtered twisted complexes might be described by allowing tensoring with graded filtered vector spaces (not persistence modules!) which are not one dimensional. Given a finitely and freely generated filtered graded vector $V := \bigoplus_{i\in I} \Lambda \cdot \gamma_i$ over Λ and an object L of $\mathcal{A}$, we can define the tensor $V \otimes L$ simply as the direct sum (or in other words, 0-filtered mapping cone of the zero maps) of the elements $\Sigma^{|\gamma_i|}L[\deg(\gamma_i)] = \Lambda \cdot \gamma_i \otimes L$, where $\deg(\gamma_i)$ denotes the degree of γ_i in V. We will often denote $\Sigma^{|\gamma_i|}L[\deg(\gamma_i)]$ simply as $L \cdot \gamma_i$. If V is a filtered chain complex, i.e. it carries a differential d_V, then the tensor product inherits a deformed differential as follows: write $d_V\gamma_i = \sum_j \alpha^i_j \gamma_j$ for any $i \in I$, then the differential on $V \otimes L$ is the matrix with (i,j) entry equal to $\alpha^i_j e_L$, where $e_L \in \mathcal{A}(L,L)$ is the strict unit. This definition extends to the whole category $FTw(\mathcal{A})$ (i.e. it's not only well-defined for images of objects of $\mathcal{A}$) in an obvious way. Another possibility (cfr. [**Sei08**, Section (3l)]) is to define the category FTw as having as objects tensors of the form $\bigoplus V_i \otimes L_i$, where V_i is a finite dimensional filtered vector space and L_i is an object of $\mathcal{A}$, together with an upper triangular differential q. In this case morphisms between $\bigoplus_i V_i \otimes L_i$ and $\bigoplus_j W_j \otimes L'_j$ are elements of

$$\bigoplus_{i,j} \mathrm{Lin}(V_i, W_j) \otimes \mathcal{A}(L_i, L'_j)$$

and the A_∞-maps are modified by considering compositions of linear maps (see [**Sei08**, Equation (3.17)]). It is striaghtforward to see that the two descriptions are equivalent.

.1.9.2. *The filtered extended Yoneda embedding.* We extend the Yoneda embedding $\mathcal{Y}_\mathcal{A} \colon \mathcal{A} \to F\mathrm{mod}_\mathcal{A}$ introduced above to a filtered A_∞-functor

$$\widetilde{\mathcal{Y}_\mathcal{A}} \colon FTw\mathcal{A} \to F\mathrm{mod}_\mathcal{A}$$

via

$$\widetilde{\mathcal{Y}_\mathcal{A}} := \mathcal{I}^*_\mathcal{A} \circ \mathcal{Y}_{FTw\mathcal{A}}$$

where $\mathcal{I}^*_\mathcal{A}$ is the pullback of the A_∞-functor $\mathcal{I}_\mathcal{A} \colon \mathcal{A} \to FTw(\mathcal{A})$ introduced in Remark .1.9.8b., $\mathcal{Y}_{FTw\mathcal{A}}$ is the filtered Yoneda embedding for the filtered A_∞-category $FTw\mathcal{A}$ and $\circ$ is the usual composition of A_∞-functors. As pullbacks of filtered A_∞-functors preserve filtered A_∞-modules and are themselves filtered, the functor $\widetilde{\mathcal{Y}_\mathcal{A}}$ is indeed a filtered A_∞-functor. Note that this is just an immediate extension to the filtered world of the definition of the extended Yoneda embedding in [**Sei08**, Section (3s)].

Proposition .1.9.14. *The induced functor $H(\widetilde{\mathcal{Y}_\mathcal{A}}) \colon PD(\mathcal{A}) \to H(F\mathrm{mod}_\mathcal{A})$ gives a TPC equivalence between $PD(\mathcal{A})$ and the image of $H(\widetilde{\mathcal{Y}_\mathcal{A}})$.*

Proof. The fact that $H(\widetilde{\mathcal{Y}_\mathcal{A}})$ gives an equivalence of persistence categories is clear. We need to prove that it is a TPC functor, that is, it is compatible with the shift functors and that it is triangulated at the 0-level. The fact that it is compatible with shift functors is a direct consequence of the fact that the shift functor on $FTw\mathcal{A}$ is compatible with modules by construction. The fact that the 0-level is triangulated follows from the Yoneda embedding being an A_∞-functor at the chain level. $\square$

.1.9.3. *Filtered twistings.* Let X and Y be objects of $\mathcal{A}$, and consider the filtered twisted complex $Y \otimes \mathcal{A}(X,Y)$ with differential

$$e_Y \otimes \mu_1 \in FTw(Y \otimes \mathcal{A}(X,Y), Y \otimes \mathcal{A}(X,Y)) = \mathcal{A}(Y,Y) \otimes \mathrm{Lin}(\mathcal{A}(X,Y), \mathcal{A}(X,Y))$$

where $e_Y \in \mathcal{A}(Y,Y)$ is the strict unit of Y and $\mathrm{Lin}(-,-)$ denotes the space of linear maps between two vector spaces. In other words, given a basis $\vec{a} = (a_1,\dots,a_n)$ of $\mathcal{A}(Y,X)$, the twisted complex above is $\bigoplus_{i=1}^n \Sigma^{|a_i|} Y[\deg(a_i)]$ with differential given by the matrix $q_{ij} = \alpha_{ij} e_Y$, where $e_Y \in \mathcal{A}(Y,Y)$ is the strict unit for Y and the $\alpha_{ij} \in \mathbf{k}$ are the coeffients in $\mu_1(a_i) = \sum_{j=1}^n \alpha_{ij} a_j$. We define the filtered morphism of filtered twisted complexes

$$\xi \colon Y \otimes \mathcal{A}(Y,X) \to X$$

to be the map corresponding to the identity under the canonical filtered isomorphism of chain complexes

$$FTw(Y \otimes \mathcal{A}(Y,X), X) \cong \mathrm{Lin}(\mathcal{A}(Y,X), \mathcal{A}(Y,X)).$$

Definition .1.9.15. We define the twisting $T_Y X$ of X by Y as the twisted complex given by the cone of ξ, i.e. $T_Y X = \mathrm{Cone}(\xi)$

We can write

$$T_Y X = X \oplus \bigoplus_{i=1}^n \Sigma^{|a_i|} Y[\deg(a_i)+1]$$

with differential given by the matrix $\begin{pmatrix} 0 & \vec{a} \\ \vec{0}^T & (q_{ij}) \end{pmatrix}$. Note that the strict unit $e_X \in \mathcal{A}(X,X)$ induces a morphism of twisted complexes $i \colon X \to T_Y X$ as a vector $(e_X, 0, \dots, 0)$.

We are interested in describing the image of $T_Y X$ under the extended filtered Yoneda embedding $\widetilde{\mathcal{Y}_{\mathcal{A}}}$ introduced in §1.9.2.

Lemma .1.9.16. *We have that*

$$\tilde{\mathcal{Y}}_{T_Y X} \cong Cone(\phi)$$

where $\phi \colon \mathcal{Y}_Y \otimes \mathcal{A}(Y,X) \to \mathcal{Y}_X$ *is the filtered morphism of modules given by*

$$\phi_{l|1}(x_1,\dots,x_l, a \otimes b) := \mu_{s+2}(x_1,\dots,a,b)$$

and $\mathcal{Y}_Y \otimes \mathcal{A}(Y,X)$ *has a module structure as defined in* [Sei08, Section (3c)].

Proof. It is easy to see, that as an element of $FTw(Y \otimes \mathcal{A}(Y,X), X) = \mathcal{A}(Y,X) \otimes \mathrm{Lin}(A(Y,X), \Lambda)$, the morphism ξ corresponds to the element

$$\sum_{i=1}^n a_i \otimes \psi_{a_i}$$

where $a_i \in \mathcal{A}(Y, X)$ is a basis element as above, and $\psi_{a_i}\colon A(Y, X) \to \Lambda$ is the map sending a_i to 1 and all the other basis elements to 0. Under the Yoneda embedding, i.e. as a morphism of A_∞-modules $\mathcal{Y}_Y \otimes \mathcal{A}(Y, X) \to \mathcal{Y}_X$, ξ becomes the map

$$\begin{aligned}\phi_{l|1}(x_1, \dots, x_l, b \otimes c) &= \sum_{i=1}^{n} \psi_{a_i}(c) \lambda(a_i)_{l|1}(x_1, \dots, x_l, b) \\ &= \sum_{i=1}^{n} \psi_{a_i}(c) \mu_{l+2}(x_1, \dots, x_l, b, a_i) = \mu_{l+2}(x_1, \dots, x_l, b, c).\end{aligned}$$

This ends the proof. □

We can easily extend this construction to the case where X is itself a filtered twisted complex over $\mathcal{A}$. In this case, under the extended Yoneda embedding, X corresponds to a filtered A_∞-module $\mathcal{M}$ and $T_Y X$ corresponds to the 0-cone of the full contraction map

$$\phi\colon \mathcal{Y}_Y \otimes \mathcal{M}(Y) \to \mathcal{M}$$

(cfr. [**Sei08**, Section (5a)]). We will write the above cone as

$$\mathcal{T}_Y \mathcal{M} := \mathrm{Cone}(\phi).$$

The analogous of the lemma above holds in this case too.

LEMMA .1.9.17. *Let Y be an object of $\mathcal{A}$ and X be a filtered twisted complex over $\mathcal{A}$. Then there is a filtered quasi-isomorphism of A_∞-modules*

$$\widetilde{\mathcal{Y}}_{T_Y X} \to \mathcal{T}_Y \widetilde{\mathcal{Y}}_X.$$

.1.9.4. *Functoriality of filtered twisted complexes.* Let $\mathcal{A}$ and $\mathcal{B}$ be filtered A_∞-categories. Recall that, for any $s \geq 0$, we defined the filtered A_∞-category $\mathrm{fun}^{\mathrm{LD};s}(\mathcal{A}, \mathcal{B})$ of A_∞-functors with linear deviation rate s in §.1.6.
We introduce the following notations: for $N \geq 1$ and $s \geq 0$ we write

$$FTw^N \mathcal{Q}_s = \mathrm{fun}^{\mathrm{LD};s}(FTw^N \mathcal{A}, FTw^N \mathcal{B}) \tag{107}$$

as well as

$$FTw\mathcal{Q}_s = \mathrm{fun}^{\mathrm{LD};s}(FTw\mathcal{A}, FTw\mathcal{B})$$

Recall that $FTw^N \mathcal{A}$ is the full A_∞-subcategory ()of the A_∞-category $FTw\mathcal{A}$ of filtered twisted complexes) admitting only twisted complexes of length $\leq N$ as objects (see §.1.9). In this section we define filtered A_∞-functors

$$FTw^N\colon \mathrm{fun}^{\mathrm{LD};s}(\mathcal{A}, \mathcal{B}) \to FTw^N \mathcal{Q}_{Ns}$$

for any $s \geq 0$ and $N \geq 1$. Note that in the target category, the deviation is Ns. Unfortunately, these functors do not extend to a functor FTw because the deviation of functors in the image depends on the length of the twisted complexes it is applied to, as will be apparent from the discussion below. The best we can do is to define functors

$$\overline{FTw}\colon \mathrm{fun}^{\mathrm{LD};s}(\mathcal{A}, \mathcal{B}) \to \overline{FTw\mathcal{Q}_s}$$

where $\overline{FTw\mathcal{Q}_s}$ is the A_∞-category of A_∞-functors from $FTw\mathcal{A}$ to $FTw\mathcal{B}$ such that, for any $N \geq 1$, when restricted to the subcategory $FTw^N\mathcal{A}$ they have linear deviation rate Ns.

Let $s \geq 0$ and $(\mathcal{F}, s)$ be an A_∞-functor with linear deviation s. Then $\mathcal{F}$ immediately extends to an A_∞-functor $\mathcal{F}^\Sigma \colon \mathcal{A}^\Sigma \to \mathcal{B}^\Sigma$ with deviation s between the shift completions of $\mathcal{A}$ and $\mathcal{B}$. We define the extension $(\mathcal{F}, s)^\oplus \colon \mathcal{A}^\oplus \to \mathcal{B}^\oplus$ to the filtered additive enlargments as follows (we drop the s from the notation, but the induced functor depends on this choice):

(1) on an object $\overline{L} = \bigoplus_{i=1}^n \Sigma^{r_i} L_i[d_i]$, $\mathcal{F}^\oplus$ acts as

$$\mathcal{F}^\oplus\left(\bigoplus_{i=1}^n \Sigma^{r_i} L_i[d_i]\right) := \bigoplus_{i=1}^n \Sigma^{r_i-(i-1)s}\mathcal{F}(L_i)[d_i] \tag{108}$$

(2) given objects $\overline{L}_0, \ldots, \overline{L}_d$ of $\mathcal{A}^\oplus$ and morphisms $f^i \in \mathcal{A}^\oplus(\overline{L_{i-1}}, \overline{L_i})$ for any $i = 1, \ldots, d$, we set $\mathcal{F}^\oplus_d(f^1, \ldots, f^d) \in \mathcal{B}^\oplus(\mathcal{F}\overline{L}_0, \mathcal{F}\overline{L}_d)$ to be the matrix with (i,j)-entry equal to

$$\sum_{i_1=1}^{|\overline{L}_1|} \cdots \sum_{i_{d-1}=1}^{|\overline{L}_{d-1}|} \mathcal{F}_d\left(f^1_{ii_1}, \ldots, f^d_{i_{d-1}j}\right)$$

for $i = 1, \ldots, |\overline{L_0}|$ and $j = 1, \ldots, |\overline{L_d}|$.

We can then easily extend $\mathcal{F}$ to an A_∞-functor $FTw\mathcal{F} \colon FTw\mathcal{A} \to FTw\mathcal{B}$ by setting

$$FTw\mathcal{F}(\overline{L}) := \mathcal{F}^\oplus(\overline{L}) \text{ and } q_{Tw\mathcal{F}(\overline{L})} := \sum_{d\geq 1} \mathcal{F}^\oplus_d(q_{\overline{L}}, \ldots, q_{\overline{L}})$$

for any object $(\overline{L}, q_{\overline{L}})$ of $FTw\mathcal{A}$, and, given objects $(\overline{L}_0, q_0), \ldots, (\overline{L}_d, q_d)$ of $FTw\mathcal{A}$ and morphisms $f^i \in FTw\mathcal{A}(\overline{L_{i-1}}, \overline{L_i})$ for any $i = 1, \ldots, d$, we set

$$(FTw\mathcal{F})_d(f^1, \ldots, f^d) := \sum_{k_0,\ldots,k_d\geq 0} \mathcal{F}^\oplus_{d+k_0+\cdots+k_d}\left(q_0^{\otimes k_0}, f^1, q_1^{\otimes k_1}, \ldots, q_{d-1}^{\otimes k_{d-1}}, f^d, q_d^{\otimes k_d}\right)$$

REMARK .1.9.18. (1) The sums in the definition of $q_{Tw\mathcal{F}(\overline{L})}$ and $FTw\mathcal{F}_d$ are finite.

(2) The s-shifts in the image of $\overline{L}$ in (108) is essential in order for the image $q_{Tw\mathcal{F}(\overline{L})}$ of $q_{\overline{L}}$ to be at filtration level ≤ 0, as we prove in the next lemma. Moreover, we remark here again that the induced functors $\mathcal{F}^\oplus$ and $FTw\mathcal{F}$ obviously depend on the choice of deviation s, although this fact is not explicitly reflected by the notation.

LEMMA .1.9.19. *Let $(\mathcal{F}, s)$ be an object of $fun^{LD;s}(\mathcal{A}, \mathcal{B})$. Let $(\overline{L}, q_{\overline{L}})$ be a filtered twisted complex. Then its image $(FTw\mathcal{F}(\overline{L}), q_{Tw\mathcal{F}(\overline{L})})$ under $FTw\mathcal{F}$ is a filtered twisted complex as well.*

REMARK .1.9.20. The fact that the image $FTw\mathcal{F}(\overline{L})$ preserves the structure of a filtered twisted complex although $\mathcal{F}$ is not in general a filtered A_∞-functor may seem a bit counter-intuitive, but might be taken as an hint that functors with linear deviation behave well with respect to filtered homological constructions and are hence the right setting.

PROOF. It is easy to see that $q_{Tw\mathcal{F}(\overline{L})}$ satisfies the Maurer-Cartan equation. It remains to check that it lies at non-positive filtration level. We write q for $q_{\overline{L}}$. Let $d \in \{1, \dots, |\overline{L}| - 1\}$ and consider the (i,j)-entry of $\mathcal{F}_d^{\oplus}(q, \dots, q)$, which equals

$$\sum_{i<i_1<\dots<i_{d-1}<j} \mathcal{F}_d(q_{ii_1}, \dots, q_{i_{d-1}j}).$$

Assume that $\mathcal{F}_d(q_{ii_1}, \dots, q_{i_{d-1}j})$ is non-zero for some $i_1, \dots, i_{d-1}$; in particular, $j - i \geq d$, as otherwise, by triangularity of q, this term would be zero. Note that $\mathcal{F}_d(q_{ii_1}, \dots, q_{i_{d-1}j})$ lies in

$$(\mathcal{B}^\Sigma)^{\leq ds}(\Sigma^{r_i}\mathcal{F}(L_i)[d_i], \Sigma^{r_j}\mathcal{F}(L_j)[d_j]) \subset (\mathcal{B}^\Sigma)^{\leq (j-i)s}(\Sigma^{r_i}\mathcal{F}(L_i)[d_i], \Sigma^{r_j}\mathcal{F}(L_j)[d_j])$$

which is isomorphic to

$$(\mathcal{B}^\Sigma)^{\leq 0}(\Sigma^{r_i-(i-1)s}\mathcal{F}(L_i)[d_i], \Sigma^{r_j-(j-1)s}\mathcal{F}(L_j)[d_j]).$$

By our definition of the action of $FTw\mathcal{F}$ on objects of $FTw\mathcal{A}$, it follows directly that $q_{Tw\mathcal{F}(\overline{L})}$ lies at vanishing filtration level. $\square$

REMARK .1.9.21. Since the induced functor $FTw\mathcal{F}$ preserves lengths of twisted complexes by definition, it restricts to a functor $Tw^N\mathcal{A} \to Tw^N\mathcal{B}$. We denote this functor by $FTw^N\mathcal{F}$, i.e. highlighting the N, because of the following proposition.

PROPOSITION .1.9.22. *Let $N \geq 1$ and let $(\mathcal{F}, s)$ be an object of $fun^{LD;s}(\mathcal{A}, \mathcal{B})$. Then the induced functor*

$$Tw^N\mathcal{F} \colon FTw^N\mathcal{A} \to FTw^N\mathcal{B}$$

is an A_∞-functor with deviation rate $N \cdot s$, that is, an object of $FTw^N Q_{Ns}$ (see (107)).

PROOF. Consider objects $(\overline{L^0}, q_0), \dots, (\overline{L^d}, q_d)$ of $FTw^N\mathcal{A}$ of length $n^0, \dots, n^d \leq N$ respectively, and morphisms $f^i \in FTw\mathcal{A}^{\alpha_i}(\overline{L_{i-1}}, \overline{L_i})$ for any $i = 1, \dots, d$. The "most shifted" (in general) non-zero contribution to the matrix

$$FTw\mathcal{F}_d(f^1, \dots, f^d) \in FTw\mathcal{B}(Tw\mathcal{F}\overline{L_0}, Tw\mathcal{F}\overline{L_d})$$

is the $(1, n^d)$ entry of the summand

$$\mathcal{F}^{\oplus}_{d+(n^0-1)+\dots+(n^d-1)}(q_0^{n^0-1}, f^1, \dots, f^d, q_d^{n^d-1}).$$

This term lies at filtration level

$$\leq \sum_{i=1}^d \alpha_i + \sum_{j=0}^d (n^j - 1)s + ds = \sum_{i=1}^d \alpha_i + \sum_{j=0}^d n^j s - s$$

in $\mathcal{B}^\Sigma(\Sigma^{r_1}\mathcal{F}(L^0_1), \Sigma^{r_{n^d}}\mathcal{F}(L^d_{n^d}))$, i.e. at level

$$\leq \sum_{i=1}^d \alpha_i + \sum_{j=0}^{d-1} n^j s$$

in $\mathcal{B}^\Sigma(\Sigma^{r_1^0}\mathcal{F}(L^0_1), \Sigma^{r^d_{n^d}-(n^d-1)s}\mathcal{F}(L^d_{n^d}))$. As by assumption $n^0, \dots, n^d \leq N$, we have $\sum_{j=0}^{d-1} n^j s \leq dNs$ and the claim follows.

$\square$

REMARK .1.9.23. The fact that $FTw\mathcal{F}$ acts on twisted complexes shifting summand by a quantity depending on the order is needed in order to have induced functors with shift that is sublinear on the order. The fact that the shift depend on the order seems counterintuitive, but it is in line with upper triangularity of differentials, which of course depends on the order of the summands.

So far, we defined the A_∞-functor FTw just at the level of objects. We now define the first order term FTw_1. Let $(\mathcal{F}, s)$ and $(\mathcal{G}, s)$ be objects of $\mathrm{fun}^{\mathrm{LD};s}(\mathcal{A}, \mathcal{B})$ and consider an A_∞-pre-natural transformation T from $(\mathcal{F}, s)$ to $(\mathcal{G}, s)$ with shift ρ (see §.1.6 for the definition). We define the 'A_∞-pre-natural transformation'[4] FTw_1T with dth term, $d \geq 0$,

$$(FTw_1T)_d(f_1, \ldots, f_d) := \sum_{k_0,\ldots,k_d \geq 0} T^{\oplus}_{d+k_0+\cdots+k_d}\left(q_0^{\otimes k_0}, f^1, q_1^{\otimes k_1}, \ldots, q_{d-1}^{\otimes k_{d-1}}, f^d, q_d^{\otimes k_d}\right)$$

where $T^{\oplus}$ is the matrixwise extension of T, defined analogously to the case of functors above. As in the case of functors, we can restrict to subcategories of twisted complexes of length $\leq N$ and define $FTw_1^N T$ for any $N \geq 1$. Taking filtrations into account, we have the following result.

LEMMA .1.9.24. *Let $(\mathcal{F}, s)$ and $(\mathcal{G}, s)$ be objects of $fun^{LD;s}(\mathcal{A}, \mathcal{B})$ and consider an A_∞-pre-natural transformation $T\colon (\mathcal{F}, s) \to (\mathcal{G}, s)$ of shift ρ between them. Let $N \geq 1$. Then $FTw_1^N T$ is an A_∞-pre-natural transformation between $FTw^N\mathcal{F}$ and $FTw^N\mathcal{G}$ of shift $\leq \rho$, that is*

$$FTw_1^N T \in FTw^N Q_{Ns}(FTw^N\mathcal{F}, FTw^N\mathcal{G})^{\rho}.$$

REMARK .1.9.25. In other words, the above lemma tells us that FTw_1 sends A_∞-pre-natural transformations with linear deviation s and shift $\leq \rho$ to A_∞-pre-natural transformations with linear deviation Ns and shift ρ.

We define the higher order terms FTw_d, $d \geq 2$, to be zero. In particular, we proved the following result, which was announced at the beginning of this section.

COROLLARY .1.9.26. *For any $N \geq 1$ and $s \geq 0$, the A_∞-functor*

$$FTw^N\colon fun^{LD;s}(\mathcal{A}, \mathcal{B}) \to fun^{LD;Ns}(FTw^N\mathcal{A}, FTw^N\mathcal{B}) = FTw^N\mathcal{Q}_{Ns}$$

is filtered.

[4]The quotation marks are there as this is formally a natural transformation between functors that are not well-defined.

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