Reduced GLT sequences and Applications

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Numerical Linear Algebra Days - Due giorni di Albra Lineare Numerica
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Spectral Symbol

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$\kappa : D \subset \mathbb{R}^M \to \mathbb{C}$ is a spectral symbol for $\{A_n\}_n \sim_\lambda \kappa$ if

$$\lim_{n\to\infty} \frac{1}{d_n} \sum_{k=1}^{d_n} F\left(\lambda_k\left(A_n\right)\right) = \frac{1}{\mu(D)} \int_D F\left(\kappa(\boldsymbol{x})\right) d\boldsymbol{x}, \quad \forall F \in C_c(\mathbb{C})$$

where $\infty > \mu(D) > 0$ and $A_n \in \mathbb{C}^{d_n \times d_n}$, $d_n \to \infty$.

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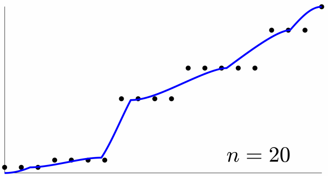
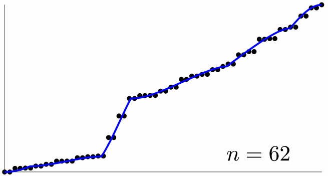
$$\{A_6, A_{20}, A_{62}, \dots\} \equiv \{A_n\}_n \sim_\lambda \kappa$$





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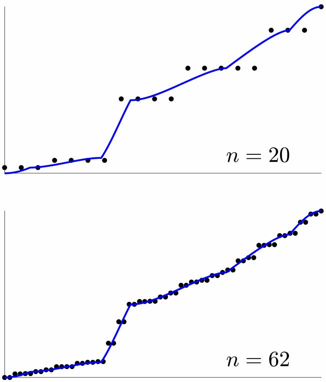
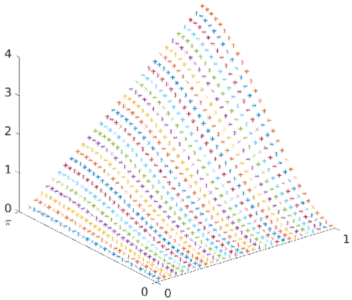


$\{A_n\}_n \sim_\lambda \kappa$ when the plot of $\lambda_i(A_n)$ converges to the plot of κ over a same domain D



Multilevel Toeplitz

Given a real function f in $L^1([-\pi,\pi]^q)$, its associated Toeplitz sequence is

$$T_{\boldsymbol{n}}(f) = [f_{\boldsymbol{i}-\boldsymbol{j}}]_{\boldsymbol{i},\boldsymbol{j}=1}^{\boldsymbol{n}} \qquad f_{\boldsymbol{k}} = \frac{1}{(2\pi)^q}\int_{-\pi}^{\pi} f(\boldsymbol{\theta}) e^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\theta}} \mathrm{d}\boldsymbol{\theta} \qquad \{T_{\boldsymbol{n}}(f)\}_n \sim_\lambda f$$

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$$\begin{pmatrix}
\begin{array}{|cccc|}\hline f_{0,0} & f_{0,-1} & \cdots & f_{0,-n+1} \\ f_{0,1} & f_{0,0} & \ddots & \vdots \\ \vdots & \ddots & \ddots & f_{0,-1} \\ f_{0,n-1} & \cdots & f_{0,1} & f_{0,0} \\ \hline \end{array} &
\begin{array}{|cccc|}\hline f_{-1,0} & f_{-1,-1} & \cdots & f_{-1,-n+1} \\ f_{-1,1} & f_{-1,0} & \ddots & \vdots \\ \vdots & \ddots & \ddots & f_{-1,-1} \\ f_{-1,n-1} & \cdots & f_{-1,1} & f_{-1,0} \\ \hline \end{array} &
\cdots\cdots &
\begin{array}{|cccc|}\hline f_{-n+1,0} & f_{-n+1,-1} & \cdots & f_{-n+1,-n+1} \\ f_{-n+1,1} & f_{-n+1,0} & \ddots & \vdots \\ \vdots & \ddots & \ddots & f_{-n+1,-1} \\ f_{-n+1,n-1} & \cdots & f_{-n+1,1} & f_{-n+1,0} \\ \hline \end{array} \\
\begin{array}{|cccc|}\hline f_{1,0} & f_{1,-1} & \cdots & f_{1,-n+1} \\ f_{1,1} & f_{1,0} & \ddots & \vdots \\ \vdots & \ddots & \ddots & f_{1,-1} \\ f_{1,n-1} & \cdots & f_{1,1} & f_{1,0} \\ \hline \end{array} &
\begin{array}{|cccc|}\hline f_{0,0} & f_{0,-1} & \cdots & f_{0,-n+1} \\ f_{0,1} & f_{0,0} & \ddots & \vdots \\ \vdots & \ddots & \ddots & f_{0,-1} \\ f_{0,n-1} & \cdots & f_{0,1} & f_{0,0} \\ \hline \end{array} &
\ddots & \vdots \\
\vdots & \ddots & \ddots &
\begin{array}{|cccc|}\hline f_{-1,0} & f_{-1,-1} & \cdots & f_{-1,-n+1} \\ f_{-1,1} & f_{-1,0} & \ddots & \vdots \\ \vdots & \ddots & \ddots & f_{-1,-1} \\ f_{-1,n-1} & \cdots & f_{-1,1} & f_{-1,0} \\ \hline \end{array} \\
\begin{array}{|cccc|}\hline f_{n-1,0} & f_{n-1,-1} & \cdots & f_{n-1,-n+1} \\ f_{n-1,1} & f_{n-1,0} & \ddots & \vdots \\ \vdots & \ddots & \ddots & f_{n-1,-1} \\ f_{n-1,n-1} & \cdots & f_{n-1,1} & f_{n-1,0} \\ \hline \end{array} &
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\begin{array}{|cccc|}\hline f_{1,0} & f_{1,-1} & \cdots & f_{1,-n+1} \\ f_{1,1} & f_{1,0} & \ddots & \vdots \\ \vdots & \ddots & \ddots & f_{1,-1} \\ f_{1,n-1} & \cdots & f_{1,1} & f_{1,0} \\ \hline \end{array} &
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\end{pmatrix}$$

2-Dimensional Laplacian

Take the 2-Dimensional Laplace problem with Dirichlet boundary conditions

$$\Delta u(x, y) = u_{xx}(x, y) + u_{yy}(x, y) = f(x, y) \qquad (x, y) \in [0, 1]^2$$

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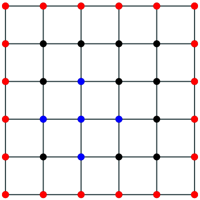
We discretize it over the grid

$$\left\{ (x_i, y_j) : x_i = ih, y_j = jh, i,j = 1,\ldots,n, h = \frac{1}{n+1} \right\}$$

using a classical second order Finite Difference Method, so that

$$(\Delta u)_{i,j} := \Delta u(x_i, y_j) \sim \frac{u_{i,j+1} + u_{i,j-1} + u_{i+1,j} + u_{i-1,j} - 4u_{i,j}}{h^2}$$



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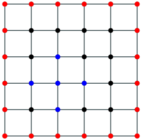

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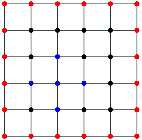

	(1,1)	(1,2)	(1,3)	(1,4)	(2,1)	(2,2)	(2,3)	(2,4)	(3,1)	(3,2)	(3,3)	(3,4)	(4,1)	(4,2)	(4,3)	(4,4)
(1,1)	4	−1			−1											
(1,2)	−1	4	−1			−1										
(1,3)		−1	4	−1			−1									
(1,4)			−1	4				−1								
(2,1)	−1				4	−1			−1							
(2,2)		−1			−1	4	−1			−1						
(2,3)			−1			−1	4	−1			−1					
(2,4)				−1			−1	4				−1				
(3,1)					−1				4	−1			−1			
(3,2)						−1			−1	4	−1			−1		
(3,3)							−1			−1	4	−1			−1	
(3,4)								−1			−1	4				−1
(4,1)									−1				4	−1		
(4,2)										−1			−1	4	−1	
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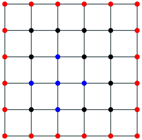

	(1,1)	(1,2)	(1,3)	(1,4)	(2,1)	(2,2)	(2,3)	(2,4)	(3,1)	(3,2)	(3,3)	(3,4)	(4,1)	(4,2)	(4,3)	(4,4)
(1,1)	4	−1			−1											
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(2,1)	−1				4	−1			−1							
(2,2)		−1			−1	4	−1			−1						
(2,3)			−1			−1	4	−1			−1					
(2,4)				−1			−1	4				−1				
(3,1)					−1				4	−1			−1			
(3,2)						−1			−1	4	−1			−1		
(3,3)							−1			−1	4	−1			−1	
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(4,1)									−1				4	−1		
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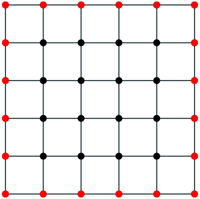
(4,3)											−1			−1	4	−1
(4,4)												−1			−1	4

$$\{T_{\boldsymbol{n}}\}_n \sim_\lambda 4 - e^{\mathrm{i}\theta_1} - e^{-\mathrm{i}\theta_1} - e^{\mathrm{i}\theta_2} - e^{-\mathrm{i}\theta_2} = 4\sin^2(\theta_1/2) + 4\sin^2(\theta_2/2)$$

Domain Change

$$\Delta u(x,y) = u_{xx}(x,y) + u_{yy}(x,y) = f(x,y) \qquad (x,y) \in [0,1]^2$$



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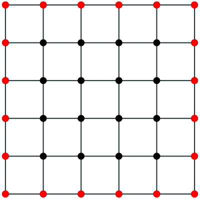
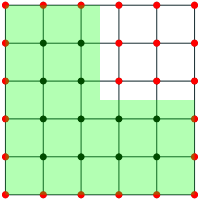
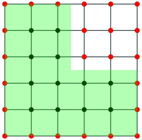


$$\Delta u(x, y) = u_{xx}(x, y) + u_{yy}(x, y) = f(x, y) \qquad (x, y) \in L$$



Domain Change



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	(1,1)	(1,2)	(1,3)	(1,4)	(2,1)	(2,2)	(2,3)	(2,4)	(3,1)	(3,2)	(3,3)	(3,4)	(4,1)	(4,2)	(4,3)	(4,4)
(1,1)	4	−1			−1											
(1,2)	−1	4	−1			−1										
(1,3)		−1	4	−1			−1									
(1,4)			−1	4				−1								
(2,1)	−1				4	−1			−1							
(2,2)		−1			−1	4	−1			−1						
(2,3)			−1			−1	4	−1			−1					
(2,4)				−1			−1	4				−1				
(3,1)					−1				4	−1			−1			
(3,2)						−1			−1	4	−1			−1		
(3,3)							−1			−1	4	−1			−1	
(3,4)								−1			−1	4				−1
(4,1)									−1				4	−1		
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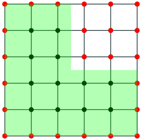

	(1,1)	(1,2)	(1,3)	(1,4)	(2,1)	(2,2)	(2,3)	(2,4)	(3,1)	(3,2)	(3,3)	(3,4)	(4,1)	(4,2)	(4,3)	(4,4)
(1,1)	4	−1			−1											
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(2,1)	−1				4	−1			−1							
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(2,3)			−1			−1	4	−1			−1					
(2,4)				−1			−1	4				−1				
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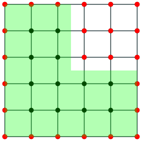

	(1,1)	(1,2)	(1,3)	(1,4)	(2,1)	(2,2)	(2,3)	(2,4)	(3,1)	(3,2)	(4,1)	(4,2)
(1,1)	4	−1			−1							
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(1,4)			−1	4				−1				
(2,1)	−1				4	−1			−1			
(2,2)		−1			−1	4	−1			−1		
(2,3)			−1			−1	4	−1				
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(3,2)						−1			−1	4		−1
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(4,2)										−1	−1	4

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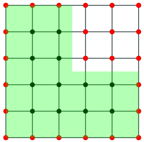


	$(1,1)$	$(1,2)$	$(1,3)$	$(1,4)$	$(2,1)$	$(2,2)$	$(2,3)$	$(2,4)$	$(3,1)$	$(3,2)$	$(4,1)$	$(4,2)$
$(1,1)$	4	-1			-1							
$(1,2)$	-1	4	-1			-1						
$(1,3)$		-1	4	-1			-1					
$(1,4)$			-1	4				-1				
$(2,1)$	-1				4	-1			-1			
$(2,2)$		-1			-1	4	-1			-1		
$(2,3)$			-1			-1	4	-1				
$(2,4)$				-1			-1	4				
$(3,1)$					-1				4	-1	-1	
$(3,2)$						-1			-1	4		-1
$(4,1)$									-1		4	-1
$(4,2)$										-1	-1	4

The reduced matrix $T_{\boldsymbol{n}}^L$ is not a multilevel Toeplitz but has the same symbol

$$\{T_{\boldsymbol{n}}^L\}_n \sim_\lambda 4\sin^2(\theta_1/2) + 4\sin^2(\theta_2/2)$$

The same holds for any $\Omega \subseteq [0,1]^2$ with $\mu(\Omega) > 0$, $\mu(\delta\Omega) = 0$ ($\iff$ χ_Ω R.I.)

Intermediate Step

$$T_n = \begin{array}{c|cccc|cccc|cccc|cccc|}
 & (1,1) & (1,2) & (1,3) & (1,4) & (2,1) & (2,2) & (2,3) & (2,4) & (3,1) & (3,2) & (3,3) & (3,4) & (4,1) & (4,2) & (4,3) & (4,4) \\
\hline
(1,1) & 4 & -1 & & & -1 & & & & & & & & & & & \\
(1,2) & -1 & 4 & -1 & & & -1 & & & & & & & & & & \\
(1,3) & & -1 & 4 & -1 & & & -1 & & & & & & & & & \\
(1,4) & & & -1 & 4 & & & & -1 & & & & & & & & \\
\hline
(2,1) & -1 & & & & 4 & -1 & & & -1 & & & & & & & \\
(2,2) & & -1 & & & -1 & 4 & -1 & & & -1 & & & & & & \\
(2,3) & & & -1 & & & -1 & 4 & -1 & & & -1 & & & & & \\
(2,4) & & & & -1 & & & -1 & 4 & & & & -1 & & & & \\
\hline
(3,1) & & & & & -1 & & & & 4 & -1 & & & -1 & & & \\
(3,2) & & & & & & -1 & & & -1 & 4 & -1 & & & -1 & & \\
(3,3) & & & & & & & -1 & & & -1 & 4 & -1 & & & -1 & \\
(3,4) & & & & & & & & -1 & & & -1 & 4 & & & & -1 \\
\hline
(4,1) & & & & & & & & & -1 & & & & 4 & -1 & & \\
(4,2) & & & & & & & & & & -1 & & & -1 & 4 & -1 & \\
(4,3) & & & & & & & & & & & -1 & & & -1 & 4 & -1 \\
(4,4) & & & & & & & & & & & & -1 & & & -1 & 4 \\
\hline
\end{array}$$

Intermediate Step

$$
\widetilde{T}_n =
\begin{array}{c|cccc|cccc|cccc|cccc|}
 & (1,1) & (1,2) & (1,3) & (1,4) & (2,1) & (2,2) & (2,3) & (2,4) & (3,1) & (3,2) & \color{red}{(3,3)} & \color{red}{(3,4)} & (4,1) & (4,2) & \color{red}{(4,3)} & \color{red}{(4,4)} \\
\hline
(1,1) & 4 & -1 & & & -1 & & & & & & & & & & & \\
(1,2) & -1 & 4 & -1 & & & -1 & & & & & & & & & & \\
(1,3) & & -1 & 4 & -1 & & & -1 & & & & & & & & & \\
(1,4) & & & -1 & 4 & & & & -1 & & & & & & & & \\
\hline
(2,1) & -1 & & & & 4 & -1 & & & -1 & & & & & & & \\
(2,2) & & -1 & & & -1 & 4 & -1 & & & -1 & & & & & & \\
(2,3) & & & -1 & & & -1 & 4 & -1 & & & \color{red}{0} & & & & & \\
(2,4) & & & & -1 & & & -1 & 4 & & & & \color{red}{0} & & & & \\
\hline
(3,1) & & & & & -1 & & & & 4 & -1 & & & -1 & & & \\
(3,2) & & & & & & -1 & & & -1 & 4 & \color{red}{0} & & & 0 & & \\
\color{red}{(3,3)} & & & & & & & \color{red}{0} & & & \color{red}{0} & \color{red}{0} & \color{red}{0} & & & \color{red}{0} & \\
\color{red}{(3,4)} & & & & & & & & \color{red}{0} & & & \color{red}{0} & \color{red}{0} & & & & \color{red}{0} \\
\hline
(4,1) & & & & & & & & & -1 & & & & 4 & -1 & & \\
(4,2) & & & & & & & & & & -1 & & & -1 & 4 & \color{red}{0} & \\
\color{red}{(4,3)} & & & & & & & & & & & \color{red}{0} & & & \color{red}{0} & \color{red}{0} & \color{red}{0} \\
\color{red}{(4,4)} & & & & & & & & & & & & \color{red}{0} & & & \color{red}{0} & \color{red}{0} \\
\hline
\end{array}
$$

Intermediate Step

$$
\widetilde{T}_{\boldsymbol{n}} =
\begin{array}{c|cccc|cccc|cccc|cccc|}
 & (1,1) & (1,2) & (1,3) & (1,4) & (2,1) & (2,2) & (2,3) & (2,4) & (3,1) & (3,2) & (3,3) & (3,4) & (4,1) & (4,2) & (4,3) & (4,4) \\
\hline
(1,1) & 4 & -1 & & & -1 & & & & & & & & & & & \\
(1,2) & -1 & 4 & -1 & & & -1 & & & & & & & & & & \\
(1,3) & & -1 & 4 & -1 & & & -1 & & & & & & & & & \\
(1,4) & & & -1 & 4 & & & & -1 & & & & & & & & \\
\hline
(2,1) & -1 & & & & 4 & -1 & & & -1 & & & & & & & \\
(2,2) & & -1 & & & -1 & 4 & -1 & & & -1 & & & & & & \\
(2,3) & & & -1 & & & -1 & 4 & -1 & & & 0 & & & & & \\
(2,4) & & & & -1 & & & -1 & 4 & & & & 0 & & & & \\
\hline
(3,1) & & & & & -1 & & & & 4 & -1 & & & -1 & & & \\
(3,2) & & & & & & -1 & & & -1 & 4 & 0 & & & 0 & & \\
(3,3) & & & & & & & 0 & & & 0 & 0 & 0 & & & 0 & \\
(3,4) & & & & & & & & 0 & & & 0 & 0 & & & & 0 \\
\hline
(4,1) & & & & & & & & & -1 & & & & 4 & -1 & & \\
(4,2) & & & & & & & & & & -1 & & & -1 & 4 & 0 & \\
(4,3) & & & & & & & & & & & 0 & & & 0 & 0 & 0 \\
(4,4) & & & & & & & & & & & & 0 & & & 0 & 0 \\
\hline
\end{array}
$$

$$
\widetilde{T}_{\boldsymbol{n}} = D_{\boldsymbol{n}}(\chi_L) T_{\boldsymbol{n}} D_{\boldsymbol{n}}(\chi_L) \qquad D_{\boldsymbol{n}}(\chi_L) = \operatorname{diag}\left(\chi_L\left(\frac{i}{n+1}, \frac{j}{n+1}\right)\right)_{i,j=1,\ldots,n}
$$

$$
\{\widetilde{T}_{\boldsymbol{n}}\}_n \sim_\lambda \ ?
$$

Intermediate Step

Multilevel GLT Theory

For any $a(\boldsymbol{x}) : [0,1]^p \to \mathbb{C}$ Riemann Integrable, its Sampling Diagonal matrix is the p-multilevel matrix defined as

$$D_{\boldsymbol{n}}(a) := \operatorname{diag}\left(a\left(\frac{i_1}{n+1}, \dots, \frac{i_p}{n+1} \right) \right)_{i_j=1,\dots,n}$$

If $A_{\boldsymbol{n}}^{(j)}$ are p-level matrices, then

$$\left\{ A_{\boldsymbol{n}}^{(1)} A_{\boldsymbol{n}}^{(2)} \dots A_{\boldsymbol{n}}^{(q)} \right\} \sim_\lambda \kappa_1 \kappa_2 \dots \kappa_q$$

$$A_{\boldsymbol{n}}^{(j)} = T_{\boldsymbol{n}}(f_j) \implies \kappa_j(\boldsymbol{x}, \boldsymbol{\theta}) = f_j(\boldsymbol{\theta}) \qquad A_{\boldsymbol{n}}^{(j)} = D_{\boldsymbol{n}}(a_j) \implies \kappa_j(\boldsymbol{x}, \boldsymbol{\theta}) = a_j(\boldsymbol{x})$$

where all κ_j have domain on $[0,1]^p \times [-\pi, \pi]^p$.

Intermediate Step

$$
\widetilde{T}_{\boldsymbol{n}} =
\begin{array}{c|cccc|cccc|cccc|cccc|}
 & (1,1) & (1,2) & (1,3) & (1,4) & (2,1) & (2,2) & (2,3) & (2,4) & (3,1) & (3,2) & (3,3) & (3,4) & (4,1) & (4,2) & (4,3) & (4,4) \\
\hline
(1,1) & 4 & -1 & & & -1 & & & & & & & & & & & \\
(1,2) & -1 & 4 & -1 & & & -1 & & & & & & & & & & \\
(1,3) & & -1 & 4 & -1 & & & -1 & & & & & & & & & \\
(1,4) & & & -1 & 4 & & & & -1 & & & & & & & & \\
\hline
(2,1) & -1 & & & & 4 & -1 & & & -1 & & & & & & & \\
(2,2) & & -1 & & & -1 & 4 & -1 & & & -1 & & & & & & \\
(2,3) & & & -1 & & & -1 & 4 & -1 & & & 0 & & & & & \\
(2,4) & & & & -1 & & & -1 & 4 & & & & 0 & & & & \\
\hline
(3,1) & & & & & -1 & & & & 4 & -1 & & & -1 & & & \\
(3,2) & & & & & & -1 & & & -1 & 4 & 0 & & & 0 & & \\
(3,3) & & & & & & & 0 & & & 0 & 0 & 0 & & & 0 & \\
(3,4) & & & & & & & & 0 & & & 0 & 0 & & & & 0 \\
\hline
(4,1) & & & & & & & & & -1 & & & & 4 & -1 & & \\
(4,2) & & & & & & & & & & -1 & & & -1 & 4 & 0 & \\
(4,3) & & & & & & & & & & & 0 & & & 0 & 0 & 0 \\
(4,4) & & & & & & & & & & & & 0 & & & 0 & 0 \\
\hline
\end{array}
$$

$$\widetilde{T}_{\boldsymbol{n}} = D_{\boldsymbol{n}}(\chi_L)\, T_{\boldsymbol{n}} D_{\boldsymbol{n}}(\chi_L) \qquad D_{\boldsymbol{n}}(\chi_L) = \mathrm{diag}\left(\chi_L(x_i, y_j)\right)$$

$$\{\widetilde{T}_{\boldsymbol{n}}\}_n \sim_\lambda \ ?$$

Intermediate Step

$$
\widetilde{T}_{\boldsymbol{n}} =
\begin{array}{c|cccc|cccc|cccc|cccc|}
 & (1,1) & (1,2) & (1,3) & (1,4) & (2,1) & (2,2) & (2,3) & (2,4) & (3,1) & (3,2) & (3,3) & (3,4) & (4,1) & (4,2) & (4,3) & (4,4) \\
\hline
(1,1) & 4 & -1 & & & -1 & & & & & & & & & & & \\
(1,2) & -1 & 4 & -1 & & & -1 & & & & & & & & & & \\
(1,3) & & -1 & 4 & -1 & & & -1 & & & & & & & & & \\
(1,4) & & & -1 & 4 & & & & -1 & & & & & & & & \\
\hline
(2,1) & -1 & & & & 4 & -1 & & & -1 & & & & & & & \\
(2,2) & & -1 & & & -1 & 4 & -1 & & & -1 & & & & & & \\
(2,3) & & & -1 & & & -1 & 4 & -1 & & & 0 & & & & & \\
(2,4) & & & & -1 & & & -1 & 4 & & & & 0 & & & & \\
\hline
(3,1) & & & & & -1 & & & & 4 & -1 & & & -1 & & & \\
(3,2) & & & & & & -1 & & & -1 & 4 & 0 & & & 0 & & \\
(3,3) & & & & & & & 0 & & & 0 & 0 & 0 & & & 0 & \\
(3,4) & & & & & & & & 0 & & & 0 & 0 & & & & 0 \\
\hline
(4,1) & & & & & & & & & -1 & & & & 4 & -1 & & \\
(4,2) & & & & & & & & & & -1 & & & -1 & 4 & 0 & \\
(4,3) & & & & & & & & & & & 0 & & & 0 & 0 & 0 \\
(4,4) & & & & & & & & & & & & 0 & & & 0 & 0 \\
\hline
\end{array}
$$

$$\widetilde{T}_{\boldsymbol{n}} = D_{\boldsymbol{n}}(\chi_L)\, T_{\boldsymbol{n}} D_{\boldsymbol{n}}(\chi_L) \qquad D_{\boldsymbol{n}}(\chi_L) = \mathrm{diag}\left(\chi_L(x_i, y_j)\right)$$

$$\{\widetilde{T}_{\boldsymbol{n}}\}_n \sim_\lambda \chi_L(\boldsymbol{x})[4\sin^2(\theta_1/2) + 4\sin^2(\theta_2/2)] \text{ on } [0,1]^2 \times [-\pi,\pi]^2$$

Intermediate Step

$$
\widetilde{T}_{\boldsymbol{n}} =
\begin{array}{c|cccc|cccc|cccc|cccc|}
 & (1,1) & (1,2) & (1,3) & (1,4) & (2,1) & (2,2) & (2,3) & (2,4) & (3,1) & (3,2) & (3,3) & (3,4) & (4,1) & (4,2) & (4,3) & (4,4) \\
\hline
(1,1) & 4 & -1 & & & -1 & & & & & & & & & & & \\
(1,2) & -1 & 4 & -1 & & & -1 & & & & & & & & & & \\
(1,3) & & -1 & 4 & -1 & & & -1 & & & & & & & & & \\
(1,4) & & & -1 & 4 & & & & -1 & & & & & & & & \\
\hline
(2,1) & -1 & & & & 4 & -1 & & & -1 & & & & & & & \\
(2,2) & & -1 & & & -1 & 4 & -1 & & & -1 & & & & & & \\
(2,3) & & & -1 & & & -1 & 4 & -1 & & & 0 & & & & & \\
(2,4) & & & & -1 & & & -1 & 4 & & & & 0 & & & & \\
\hline
(3,1) & & & & & -1 & & & & 4 & -1 & & & -1 & & & \\
(3,2) & & & & & & -1 & & & -1 & 4 & 0 & & & 0 & & \\
(3,3) & & & & & & & 0 & & & 0 & 0 & 0 & & & 0 & \\
(3,4) & & & & & & & & 0 & & & 0 & 0 & & & & 0 \\
\hline
(4,1) & & & & & & & & & -1 & & & & 4 & -1 & & \\
(4,2) & & & & & & & & & & -1 & & & -1 & 4 & 0 & \\
(4,3) & & & & & & & & & & & 0 & & & 0 & 0 & 0 \\
(4,4) & & & & & & & & & & & & 0 & & & 0 & 0 \\
\hline
\end{array}
$$

$$\widetilde{T}_{\boldsymbol{n}} = D_{\boldsymbol{n}}(\chi_L) T_{\boldsymbol{n}} D_{\boldsymbol{n}}(\chi_L) \qquad D_{\boldsymbol{n}}(\chi_L) = \mathrm{diag}\left(\chi_L(x_i, y_j)\right)$$

$$\{\widetilde{T}_{\boldsymbol{n}}\}_n \sim_\lambda \chi_L(\boldsymbol{x})[4\sin^2(\theta_1/2) + 4\sin^2(\theta_2/2)] \text{ on } [0,1]^2 \times [-\pi,\pi]^2$$

- The eigenvalues of $\widetilde{T}_{\boldsymbol{n}}$ distribute like those of $T_{\boldsymbol{n}}$ except for about $n^2(1-\mu(L))$ eigenvalues that are close to zero
- The number of zeroed rows/column in $\widetilde{T}_{\boldsymbol{n}}$ is also about $n^2(1-\mu(L))$, so we can remove them both from the symbol and the matrices

Intermediate Step

$$
T_{\boldsymbol{n}}^L = \begin{array}{c|cccc|cccc|cc|cc|}
 & (1,1) & (1,2) & (1,3) & (1,4) & (2,1) & (2,2) & (2,3) & (2,4) & (3,1) & (3,2) & (4,1) & (4,2) \\ \hline
(1,1) & 4 & -1 & & & -1 & & & & & & & \\
(1,2) & -1 & 4 & -1 & & & -1 & & & & & & \\
(1,3) & & -1 & 4 & -1 & & & -1 & & & & & \\
(1,4) & & & -1 & 4 & & & & -1 & & & & \\ \hline
(2,1) & -1 & & & & 4 & -1 & & & -1 & & & \\
(2,2) & & -1 & & & -1 & 4 & -1 & & & -1 & & \\
(2,3) & & & -1 & & & -1 & 4 & -1 & & & & \\
(2,4) & & & & -1 & & & -1 & 4 & & & & \\ \hline
(3,1) & & & & & -1 & & & & 4 & -1 & -1 & \\
(3,2) & & & & & & -1 & & & -1 & 4 & & -1 \\ \hline
(4,1) & & & & & & & & & -1 & & 4 & -1 \\
(4,2) & & & & & & & & & & -1 & -1 & 4 \\ \hline
\end{array}
$$

$$\widetilde{T}_{\boldsymbol{n}} = D_{\boldsymbol{n}}(\chi_L) T_{\boldsymbol{n}} D_{\boldsymbol{n}}(\chi_L) \qquad D_{\boldsymbol{n}}(\chi_L) = \mathrm{diag}\left(\chi_L(x_i, y_j)\right)$$

$$\{\widetilde{T}_{\boldsymbol{n}}\}_n \sim_\lambda \chi_L(\boldsymbol{x})[4\sin^2(\theta_1/2) + 4\sin^2(\theta_2/2)] \text{ on } [0,1]^2 \times [-\pi,\pi]^2$$

- The eigenvalues of $\widetilde{T}_{\boldsymbol{n}}$ distribute like those of $T_{\boldsymbol{n}}$ except for about $n^2(1-\mu(L))$ eigenvalues that are close to zero
- The number of zeroed rows/column in $\widetilde{T}_{\boldsymbol{n}}$ is also about $n^2(1-\mu(L))$, so we can remove them both from the symbol and the matrices

$$T_{\boldsymbol{n}}^L = R_L(T_{\boldsymbol{n}}) \qquad \chi_L(\boldsymbol{x})[4\sin^2(\theta_1/2)+4\sin^2(\theta_2/2)]\Big|_{\boldsymbol{x}\in L} = 4\sin^2(\theta_1/2)+4\sin^2(\theta_2/2)$$

$$\implies \{T_{\boldsymbol{n}}^L\}_n \sim_\lambda 4\sin^2(\theta_1/2) + 4\sin^2(\theta_2/2) \text{ on } L \times [-\pi,\pi]^2$$

Reduced GLT

Multilevel GLT Sequences

The p-level Generalized Locally Toeplitz family $\mathscr{G}_p$ is the $\mathbb{C}^*$-algebra of couples sequences-symbol $(\{A_{\boldsymbol{n}}\}_n, \kappa)$ where $\{A_{\boldsymbol{n}}\}_n$ are p-level matrices, $\kappa : [0,1]^p \times [-\pi,\pi]^p \to \mathbb{C}$ and $\{A_{\boldsymbol{n}}\}_n \sim \kappa$ generated by

$$\{T_{\boldsymbol{n}}(f)\}_n \sim f(\boldsymbol{\theta}) \qquad \{D_{\boldsymbol{n}}(a)\}_n \sim a(\boldsymbol{x}) \qquad \{Z_{\boldsymbol{n}}\}_n \sim 0$$

where $\boldsymbol{x} \in [0,1]^p$, $\boldsymbol{\theta} \in [-\pi,\pi]^p$, $f(\boldsymbol{\theta}) \in L^1([-\pi,\pi]^p)$ and $a(\boldsymbol{x})$ is R.I.

Reduced GLT

Multilevel GLT Sequences

The p-level Generalized Locally Toeplitz family $\mathscr{G}_p$ is the $\mathbb{C}^*$-algebra of couples sequences-symbol $(\{A_{\boldsymbol{n}}\}_n, \kappa)$ where $\{A_{\boldsymbol{n}}\}_n$ are p-level matrices, $\kappa : [0,1]^p \times [-\pi,\pi]^p \to \mathbb{C}$ and $\{A_{\boldsymbol{n}}\}_n \sim \kappa$ generated by

$$\{T_{\boldsymbol{n}}(f)\}_n \sim f(\boldsymbol{\theta}) \qquad \{D_{\boldsymbol{n}}(a)\}_n \sim a(\boldsymbol{x}) \qquad \{Z_{\boldsymbol{n}}\}_n \sim 0$$

where $\boldsymbol{x} \in [0,1]^p$, $\boldsymbol{\theta} \in [-\pi,\pi]^p$, $f(\boldsymbol{\theta}) \in L^1([-\pi,\pi]^p)$ and $a(\boldsymbol{x})$ is R.I.

$\Omega \subseteq [0,1]^p$ is 'regular' if $\chi_\Omega(\boldsymbol{x})$ is R.I. in $[0,1]^p$ $(\mu(\partial\Omega) = 0)$ and $\mu(\Omega) > 0$

$$\{R_\Omega(T_{\boldsymbol{n}}(f))\}_n \sim f(\boldsymbol{\theta}) \qquad \{R_\Omega(D_{\boldsymbol{n}}(a))\}_n \sim a(\boldsymbol{x})\Big|_{\boldsymbol{x}\in\Omega}$$

Reduced GLT

Multilevel GLT Sequences

The p-level Generalized Locally Toeplitz family $\mathscr{G}_p$ is the $\mathbb{C}^*$-algebra of couples sequences-symbol $(\{A_{\boldsymbol{n}}\}_n, \kappa)$ where $\{A_{\boldsymbol{n}}\}_n$ are p-level matrices, $\kappa : [0,1]^p \times [-\pi,\pi]^p \to \mathbb{C}$ and $\{A_{\boldsymbol{n}}\}_n \sim \kappa$ generated by

$$\{T_{\boldsymbol{n}}(f)\}_n \sim f(\boldsymbol{\theta}) \qquad \{D_{\boldsymbol{n}}(a)\}_n \sim a(\boldsymbol{x}) \qquad \{Z_{\boldsymbol{n}}\}_n \sim 0$$

where $\boldsymbol{x} \in [0,1]^p$, $\boldsymbol{\theta} \in [-\pi,\pi]^p$, $f(\boldsymbol{\theta}) \in L^1([-\pi,\pi]^p)$ and $a(\boldsymbol{x})$ is R.I.

$\Omega \subseteq [0,1]^p$ is 'regular' if $\chi_\Omega(\boldsymbol{x})$ is R.I. in $[0,1]^p$ ($\mu(\partial\Omega) = 0$) and $\mu(\Omega) > 0$

$$\{R_\Omega(T_{\boldsymbol{n}}(f))\}_n \sim f(\boldsymbol{\theta}) \qquad \{R_\Omega(D_{\boldsymbol{n}}(a))\}_n \sim a(\boldsymbol{x})\Big|_{\boldsymbol{x}\in\Omega}$$

Reduced GLT Sequences

The p-level Reduced Generalized Locally Toeplitz family $\mathscr{G}_p^\Omega$ relative to the regular domain $\Omega \subseteq [0,1]^p$ is the $\mathbb{C}^*$-algebra of couples sequences-symbol $(\{A_{\boldsymbol{n}}^\Omega\}_n, \kappa^\Omega)$ where $\{A_{\boldsymbol{n}}^\Omega\}_n$ are p-level matrices, $\kappa^\Omega : \Omega \times [-\pi,\pi]^p \to \mathbb{C}$ and $\{A_{\boldsymbol{n}}^\Omega\}_n \sim \kappa^\Omega$ generated by

$$\{R_\Omega(T_{\boldsymbol{n}}(f))\}_n \sim f(\boldsymbol{\theta}) \qquad \{R_\Omega(D_{\boldsymbol{n}}(a))\}_n \sim a(\boldsymbol{x})\Big|_{\boldsymbol{x}\in\Omega} \qquad \{R_\Omega(Z_{\boldsymbol{n}})\}_n \sim 0$$

where $\boldsymbol{x} \in [0,1]^p$, $\boldsymbol{\theta} \in [-\pi,\pi]^p$, $f(\boldsymbol{\theta}) \in L^1([-\pi,\pi]^p)$ and $a(\boldsymbol{x})$ is R.I.

Reduced GLT

Algebraic Relations: Given $\{A_{\boldsymbol{n}}^{\Omega}\}_n \sim \kappa_A^{\Omega}$, $\{B_{\boldsymbol{n}}^{\Omega}\}_n \sim \kappa_B^{\Omega}$, $c \in \mathbb{C}$

- $\{A_{\boldsymbol{n}}^{\Omega} B_{\boldsymbol{n}}^{\Omega}\}_n \sim \kappa_A^{\Omega} \kappa_B^{\Omega}$
- $\{A_{\boldsymbol{n}}^{\Omega} + B_{\boldsymbol{n}}^{\Omega}\}_n \sim \kappa_A^{\Omega} + \kappa_B^{\Omega}$
- $\{c A_{\boldsymbol{n}}^{\Omega}\}_n \sim c \kappa_A^{\Omega}$

Reduced GLT

Algebraic Relations: Given $\{A_{\boldsymbol{n}}^{\Omega}\}_n \sim \kappa_A^{\Omega}$, $\{B_{\boldsymbol{n}}^{\Omega}\}_n \sim \kappa_B^{\Omega}$, $c \in \mathbb{C}$

- $\{A_{\boldsymbol{n}}^{\Omega} B_{\boldsymbol{n}}^{\Omega}\}_n \sim \kappa_A^{\Omega} \kappa_B^{\Omega}$
- $\{A_{\boldsymbol{n}}^{\Omega} + B_{\boldsymbol{n}}^{\Omega}\}_n \sim \kappa_A^{\Omega} + \kappa_B^{\Omega}$
- $\{cA_{\boldsymbol{n}}^{\Omega}\}_n \sim c\kappa_A^{\Omega}$

Conjugation and Inversion: Given $\{A_{\boldsymbol{n}}^{\Omega}\}_n \sim \kappa_A^{\Omega}$

- $\{(A_{\boldsymbol{n}}^{\Omega})^H\}_n \sim \overline{\kappa_A^{\Omega}}$
- $\{(A_{\boldsymbol{n}}^{\Omega})^{\dagger}\}_n \sim (\kappa_A^{\Omega})^{-1}$ when $\kappa_A^{\Omega} \neq 0$ a.e.

... other results about the metric on $\mathscr{G}^{\Omega}$, its closure, $\{f(A_{\boldsymbol{n}}^{\Omega})\}_n$, ...

Reduced GLT

Algebraic Relations: Given $\{A_{\boldsymbol{n}}^{\Omega}\}_n \sim \kappa_A^{\Omega}$, $\{B_{\boldsymbol{n}}^{\Omega}\}_n \sim \kappa_B^{\Omega}$, $c \in \mathbb{C}$

- $\{A_{\boldsymbol{n}}^{\Omega} B_{\boldsymbol{n}}^{\Omega}\}_n \sim \kappa_A^{\Omega} \kappa_B^{\Omega}$
- $\{A_{\boldsymbol{n}}^{\Omega} + B_{\boldsymbol{n}}^{\Omega}\}_n \sim \kappa_A^{\Omega} + \kappa_B^{\Omega}$
- $\{cA_{\boldsymbol{n}}^{\Omega}\}_n \sim c\kappa_A^{\Omega}$

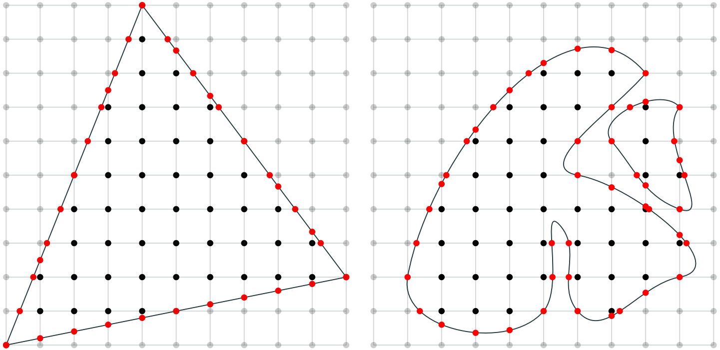
Conjugation and Inversion: Given $\{A_{\boldsymbol{n}}^{\Omega}\}_n \sim \kappa_A^{\Omega}$

- $\{(A_{\boldsymbol{n}}^{\Omega})^H\}_n \sim \overline{\kappa_A^{\Omega}}$
- $\{(A_{\boldsymbol{n}}^{\Omega})^{\dagger}\}_n \sim (\kappa_A^{\Omega})^{-1}$ when $\kappa_A^{\Omega} \neq 0$ a.e.

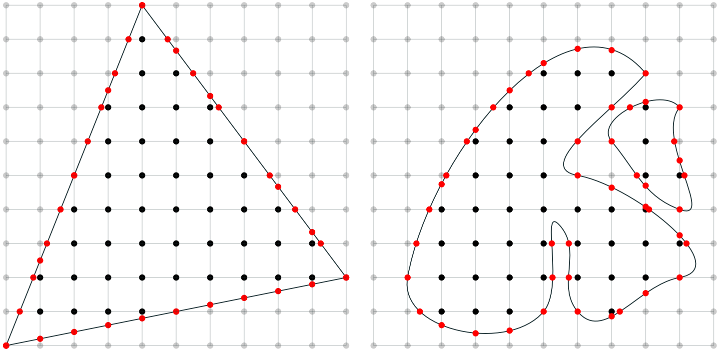
... other results about the metric on $\mathscr{G}^{\Omega}$, its closure, $\{f(A_{\boldsymbol{n}}^{\Omega})\}_n$, ...

What else?

$$-\sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left(a_i \frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{d} b_i \frac{\partial u}{\partial x_i} + cu = f \text{ in } \Omega^\circ$$



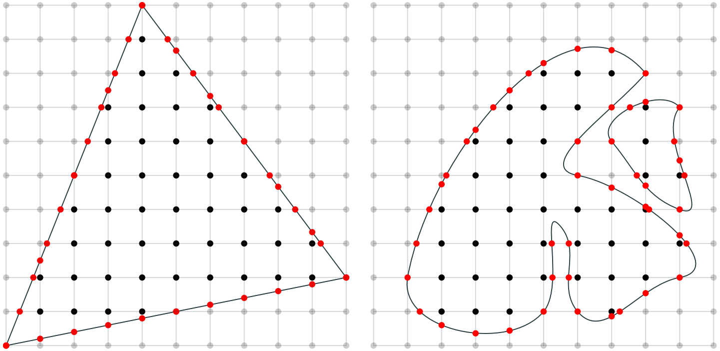
$$-\sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left(a_i \frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{d} b_i \frac{\partial u}{\partial x_i} + cu = f \text{ in } \Omega^\circ$$



Shortley and Weller:

$$\frac{\partial}{\partial x_i}\left(a_i \frac{\partial u}{\partial x_i}\right)\bigg|_{x=x_{\mathbf{j}}} \approx a_i(x_{\mathbf{j}+s_i^+ \mathrm{e}_i/2}) \frac{u(x_{\mathbf{j}+s_i^+ \mathrm{e}_i}) - u(x_{\mathbf{j}})}{\frac{1}{2} s_i^+ (s_i^+ + s_i^-) h_i^2} - a_i(x_{\mathbf{j}-s_i^- \mathrm{e}_i/2}) \frac{u(x_{\mathbf{j}}) - u(x_{\mathbf{j}-s_i^- \mathrm{e}_i})}{\frac{1}{2} s_i^- (s_i^+ + s_i^-) h_i^2}$$

$$-\sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left(a_i \frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{d} b_i \frac{\partial u}{\partial x_i} + cu = f \text{ in } \Omega^\circ$$



Shortley and Weller:

$$\frac{\partial}{\partial x_i}\left(a_i \frac{\partial u}{\partial x_i}\right)\Big|_{x=x_{\mathbf{j}}} \approx a_i(x_{\mathbf{j}+s_i^+ e_i/2}) \frac{u(x_{\mathbf{j}+s_i^+ e_i}) - u(x_{\mathbf{j}})}{\frac{1}{2}s_i^+(s_i^+ + s_i^-)h_i^2} - a_i(x_{\mathbf{j}-s_i^- e_i/2}) \frac{u(x_{\mathbf{j}}) - u(x_{\mathbf{j}-s_i^- e_i})}{\frac{1}{2}s_i^-(s_i^+ + s_i^-)h_i^2}$$

$\rightarrow$ The approximation coincides with the classic second order FD method for points whose stencil does not cross $\partial\Omega$

$\rightarrow$ Usually we have classical methods for 'internal' points and modified relations at the border, so we need **perturbation results**

Perturbation results

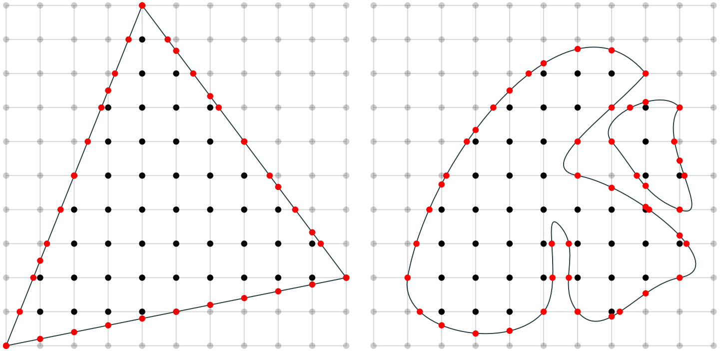
Theorem

Given a regular $\Omega \subseteq [0,1]^d$, let Ξ_n be the regular grid on $[0,1]^d$ and

$$d_n := |\Omega \cap \Xi_n| \qquad d_n^h := |\{\, p \in \Xi_n \mid d(p, \partial\Omega) \le h \,\}|$$

Then for any sequence $h_n \to 0$, $d_n^{h_n} = o(d_n)$

Notice that in a regular grid the points whose stencil crosses $\partial\Omega$ have distance at most $1/(n+1)$ from $\partial\Omega$, so their number is negligible when compared with those in Ω°



Perturbation results

Theorem

Given a regular $\Omega \subseteq [0,1]^d$, let Ξ_n be the regular grid on $[0,1]^d$ and

$$d_n := |\Omega \cap \Xi_n| \qquad d_n^h := |\{\, p \in \Xi_n \mid d(p, \partial\Omega) \leq h \,\}|$$

Then for any sequence $h_n \to 0$, $d_n^{h_n} = o(d_n)$

Notice that in a regular grid the points whose stencil crosses $\partial\Omega$ have distance at most $1/(n+1)$ from $\partial\Omega$, so their number is negligible when compared with those in Ω°

Theorem

Let $\Gamma_n \subseteq [0,1]^d$ (not necessarily regular) and let Ω be regular. Suppose that

$$d_n^{\Omega\triangle\Gamma_n} := |\Xi_n \cap (\Omega\triangle\Gamma_n)| = o(d_n)$$

Given a multilevel sequence $\{A_{\boldsymbol{n}}\}_n$ and a function κ,

$$\{R_\Omega(A_{\boldsymbol{n}})\}_n \sim \kappa \iff \{R_{\Gamma_n}(A_{\boldsymbol{n}})\}_n \sim \kappa$$

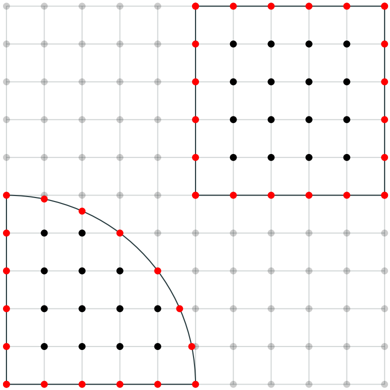
If $B_{\boldsymbol{n}}^\Omega$ are the Shortley and Weller matrices, they coincide with the classical FD matrices $A_{\boldsymbol{n}}$ on the internal grid points Γ_n, with $\{A_{\boldsymbol{n}}\}_n \sim \kappa$ and

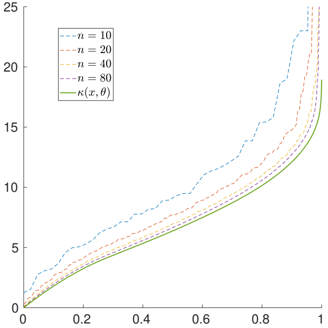
$$\{R_\Omega(A_{\boldsymbol{n}})\}_n \sim \kappa^\Omega \iff \{R_{\Gamma_n}(A_{\boldsymbol{n}})\}_n = \{R_{\Gamma_n}(B_{\boldsymbol{n}}^\Omega)\}_n \sim \kappa^\Omega \iff \{B_{\boldsymbol{n}}^\Omega\}_n \sim \kappa^\Omega$$

Numerical Example

$$-\sum_{i=1}^{d}\frac{\partial}{\partial x_i}\left(a_i\frac{\partial u}{\partial x_i}\right)+\sum_{i=1}^{d}b_i\frac{\partial u}{\partial x_i}+cu=f \text{ in } \Omega^\circ$$

$$\frac{\partial}{\partial x_i}\left(a_i\frac{\partial u}{\partial x_i}\right)\bigg|_{x=x_{\mathbf{j}}}\approx a_i(x_{\mathbf{j}+s_i^+\mathrm{e}_i/2})\frac{u(x_{\mathbf{j}+s_i^+\mathrm{e}_i})-u(x_{\mathbf{j}})}{\frac{1}{2}s_i^+(s_i^++s_i^-)h_i^2}-a_i(x_{\mathbf{j}-s_i^-\mathrm{e}_i/2})\frac{u(x_{\mathbf{j}})-u(x_{\mathbf{j}-s_i^-\mathrm{e}_i})}{\frac{1}{2}s_i^-(s_i^++s_i^-)h_i^2}$$

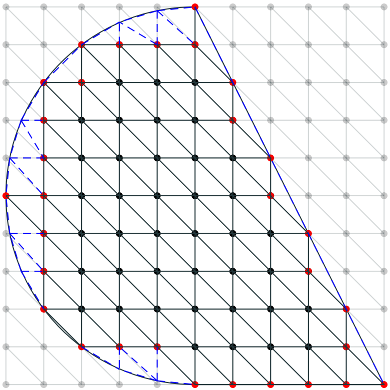




Modified Grid

$$-\sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left(a_i \frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{d} b_i \frac{\partial u}{\partial x_i} + cu = f \text{ in } \Omega^\circ$$

P1 FE Method



We can always modify a small number of points to better approximate the boundary without changing the relative symbol

Concatenation

Theorem

Given regular sets Ω_i and the sequences $\{A_{\boldsymbol{n}}^{\Omega_i}\}_n \sim \kappa^{\Omega_i}$ in $\mathscr{G}_p^{\Omega_i}$, let $\Omega := \coprod_{i=1,\dots,q} \Omega_i$ and $A_{\boldsymbol{n}}^{\Omega} := \oplus_{i=1,\dots,q} A_{\boldsymbol{n}}^{\Omega_i}$ i.e.

$$A_{\boldsymbol{n}}^{\Omega} = \begin{pmatrix} A_{\boldsymbol{n}}^{\Omega_1} & & & \\ & A_{\boldsymbol{n}}^{\Omega_2} & & \\ & & \ddots & \\ & & & A_{\boldsymbol{n}}^{\Omega_q} \end{pmatrix}$$

Then $\{A_{\boldsymbol{n}}^{\Omega}\}_n \sim \kappa^{\Omega}$, where $\kappa^{\Omega} : \Omega \times [-\pi, \pi]^p \to \mathbb{C}$, $\kappa^{\Omega}(\boldsymbol{x}, \boldsymbol{\theta})\big|_{\boldsymbol{x}\in\Omega_i} = \kappa^{\Omega_i}(\boldsymbol{x}, \boldsymbol{\theta})$

Moreover if d_n is the size of $A_{\boldsymbol{n}}^{\Omega}$ and $\mathrm{rk}(K_{\boldsymbol{n}}^{\Omega}) = o(d_n)$, then $\{A_{\boldsymbol{n}}^{\Omega} + K_{\boldsymbol{n}}^{\Omega}\}_n \sim \kappa^{\Omega}$

Concatenation

Theorem

Given regular sets Ω_i and the sequences $\{A_{\boldsymbol{n}}^{\Omega_i}\}_n \sim \kappa^{\Omega_i}$ in $\mathscr{G}_p^{\Omega_i}$, let $\Omega := \coprod_{i=1,\dots,q} \Omega_i$ and $A_{\boldsymbol{n}}^{\Omega} := \oplus_{i=1,\dots,q} A_{\boldsymbol{n}}^{\Omega_i}$ i.e.

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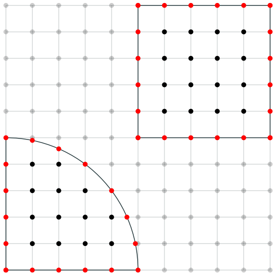
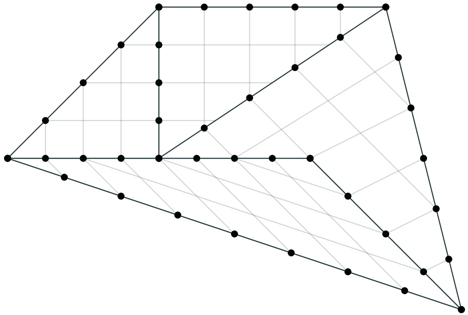
Then $\{A_{\boldsymbol{n}}^{\Omega}\}_n \sim \kappa^{\Omega}$, where $\kappa^{\Omega} : \Omega \times [-\pi, \pi]^p \to \mathbb{C}$, $\kappa^{\Omega}(\boldsymbol{x}, \boldsymbol{\theta})\big|_{\boldsymbol{x}\in\Omega_i} = \kappa^{\Omega_i}(\boldsymbol{x}, \boldsymbol{\theta})$

Moreover if d_n is the size of $A_{\boldsymbol{n}}^{\Omega}$ and $\mathrm{rk}(K_{\boldsymbol{n}}^{\Omega}) = o(d_n)$, then $\{A_{\boldsymbol{n}}^{\Omega} + K_{\boldsymbol{n}}^{\Omega}\}_n \sim \kappa^{\Omega}$

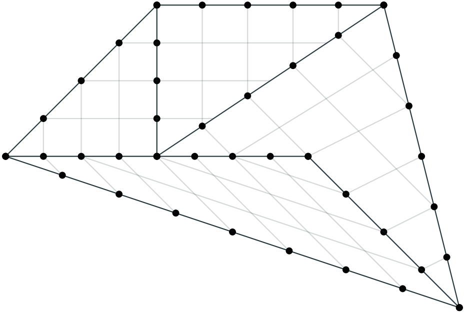


It's an easier way to handle disconnected domains, but there's actually much more to it...

Domain Mix



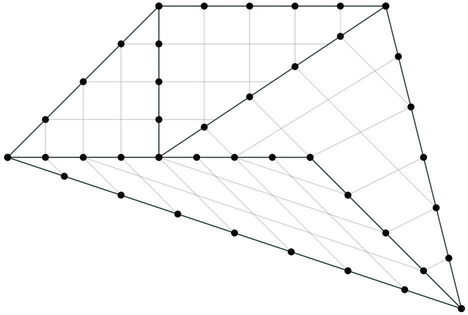
Domain Mix



- Reorder the points according to the different regions

$$A_{\boldsymbol{n}}^{\Omega} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

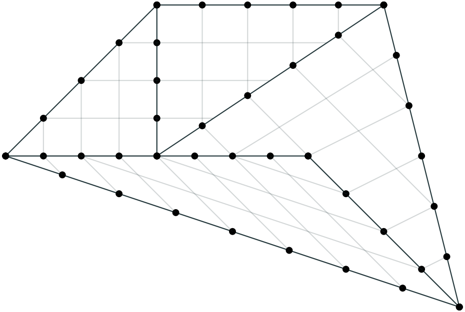
Domain Mix



- Reorder the points according to the different regions
- The points far from the border of each region follow a classic scheme

$$A_{\boldsymbol{n}}^{\Omega} = \begin{pmatrix} A_{\boldsymbol{n}}^{\Omega_1} & * & * & * \\ * & A_{\boldsymbol{n}}^{\Omega_2} & * & * \\ * & * & A_{\boldsymbol{n}}^{\Omega_3} & * \\ * & * & * & A_{\boldsymbol{n}}^{\Omega_4} \end{pmatrix}$$

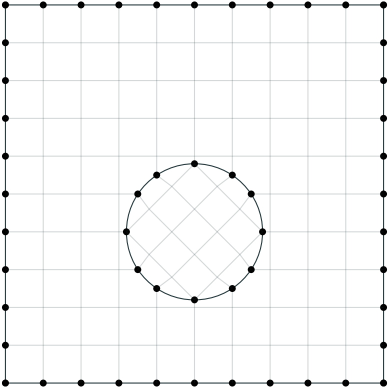
Domain Mix



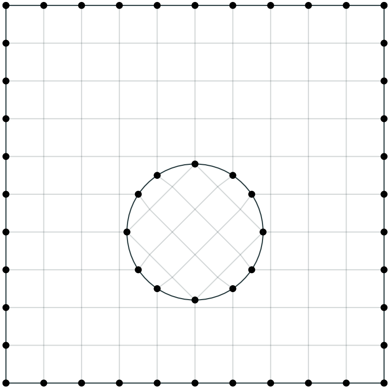
- Reorder the points according to the different regions
- The points far from the border of each region follow a classic scheme
- The number of points near the borders is negligible when compared to the total number of points

$$A_{\boldsymbol{n}}^{\Omega} = \begin{pmatrix} A_{\boldsymbol{n}}^{\Omega_1} & & & \\ & A_{\boldsymbol{n}}^{\Omega_2} & & \\ & & A_{\boldsymbol{n}}^{\Omega_3} & \\ & & & A_{\boldsymbol{n}}^{\Omega_4} \end{pmatrix} + K_{\boldsymbol{n}}^{\Omega}$$

Applications



Applications



- Fictitious Domains
- Interface Problems
- Immersed Boundaries
- Trimmed Geometries
- IgA, Coco-Russo, Shortley and Weller... You tell me

Thank You!

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