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An extension of the theory of GLT sequences: sampling on asymptotically uniform grids

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Abstract

The theory of generalized locally Toeplitz (GLT) sequences is a powerful apparatus for computing the asymptotic singular value and spectral distributions of matrices A_n arising from virtually any kind of numerical discretization of differential equations (DEs). Indeed, when the mesh fineness parameter n tends to infinity, these matrices A_n give rise to a sequence $\{A_n\}_n$, which often turns out to be a GLT sequence. In this paper, we provide an extension of the theory of GLT sequences: we show that any sequence of diagonal sampling matrices constructed from asymptotically uniform samples of an almost everywhere continuous function falls in the class of GLT sequences. We also detail a few representative applications of this result in the context of finite difference discretizations of DEs with discontinuous coefficients.

Keywords: generalized locally Toeplitz sequences, singular value and spectral distributions, discretization of differential equations, finite differences, asymptotically uniform grids

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1 Introduction

When a linear differential equation (DE) is discretized by a linear numerical method, the computation of the numerical solution reduces to solving a linear system $A_n\boldsymbol{u}_n = \boldsymbol{f}_n$, whose size d_n increases with the mesh fineness parameter n. What is often observed in practice is that A_n enjoys an asymptotic spectral distribution in the limit of mesh refinement $n \to \infty$. More precisely, it often turns out that, for a large class of test functions F,

$$\lim_{n\to\infty} \frac{1}{d_n}\sum_{j=1}^{d_n} F(\lambda_j(A_n)) = \frac{1}{\mu_k(D)}\int_D F(\kappa(\boldsymbol{y}))\mathrm{d}\boldsymbol{y},$$

where $\lambda_j(A_n)$, $j = 1,\dots,d_n$, are the eigenvalues of A_n, μ_k is the Lebesgue measure in $\mathbb{R}^k$, and $\kappa : D \subset \mathbb{R}^k \to \mathbb{C}$. In this scenario, the function κ is referred to as the spectral symbol of the sequence $\{A_n\}_n$ and we write $\{A_n\}_n \sim_\lambda \kappa$. We refer the reader to Remark 2.1 for the informal meaning behind the spectral distribution $\{A_n\}_n \sim_\lambda \kappa$ and to [13, Chapter 1] for a list of practical uses of the spectral symbol κ.

The theory of generalized locally Toeplitz (GLT) sequences is a powerful apparatus for computing the spectral symbol κ. Indeed, the sequence of discretization matrices $\{A_n\}_n$ turns out to be a GLT sequence for virtually any kind of DEs and numerical methods. Nowadays, the main references for the theory of GLT sequences and related applications are the books [13, 14] and the review papers [4, 6, 7]. We therefore refer the reader to these works, especially [13], for a comprehensive treatment of the topic. For a more concise introduction to the subject, we recommend the papers [11, 12, 15, 16], whereas for recent advanced theoretical developments, we recommend the works [2, 3, 8, 10].

In the main result of this paper (Theorem 3.1), we provide an extension of the theory of GLT sequences. To explain the novelty of Theorem 3.1, we point out that the three main examples of GLT sequences, also known as the “building blocks” of the theory of GLT sequences, are given by zero-distributed sequences, Toeplitz sequences

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and sequences of diagonal sampling matrices constructed from uniform samples of an almost everywhere (a.e.) continuous function (see Section 2 for the corresponding definitions). While the former two classes of sequences are maximal and cannot be expanded further, the latter class can and should be expanded in order to increase the application potential of the theory of GLT sequences. In Theorem 3.1, we show that the latter class can indeed be expanded to include sequences of diagonal sampling matrices constructed from asymptotically uniform samples of an a.e. continuous function (see Section 3 for the precise meaning of "asymptotically uniform"). As illustrated in Section 4, Theorem 3.1 can be applied whenever it is necessary to compute the spectral symbol of sequences of matrices $\{A_n\}_n$ arising from the discretization of DEs with discontinuous coefficients (see also the discussion in Section 5).

The paper is organized as follows. In Section 2, we overview the basics of the theory of GLT sequences. In Section 3, we state and prove the main result (Theorem 3.1). In Section 4, we detail a few representative applications of the main result in the context of finite difference (FD) discretizations of DEs with discontinuous coefficients. In Section 5, we collect some final remarks, where we also mention further applications of Theorem 3.1.

2 Overview of the theory of GLT sequences

In this section, we overview the basics of the theory of (multilevel) GLT sequences.

2.1 Multi-index notation

A multi-index $\boldsymbol{i}$ of size d, also called a d-index, is a row vector in $\mathbb{Z}^d$; its components are denoted by $i_1, \ldots, i_d$. $\mathbf{0}$, $\mathbf{1}$, $\mathbf{2}$, ... are the vectors of all zeros, all ones, all twos, ... (their size will be clear from the context). For any vector $\boldsymbol{m} \in \mathbb{R}^d$, we set $N(\boldsymbol{m}) = \prod_{j=1}^d m_j$ and we write $\boldsymbol{m} \to \infty$ to indicate that $\min(\boldsymbol{m}) \to \infty$. If $\boldsymbol{h}, \boldsymbol{k} \in \mathbb{R}^d$, an inequality such as $\boldsymbol{h} \le \boldsymbol{k}$ means that $h_j \le k_j$ for all $j = 1, \ldots, d$. If $\boldsymbol{h}, \boldsymbol{k}$ are d-indices such that $\boldsymbol{h} \le \boldsymbol{k}$, the d-index range $\{\boldsymbol{h}, \ldots, \boldsymbol{k}\}$ is the set $\{\boldsymbol{j} \in \mathbb{Z}^d : \boldsymbol{h} \le \boldsymbol{j} \le \boldsymbol{k}\}$. We assume for this set the standard lexicographic ordering:

$$\Big[\ \ldots \ \Big[\ [\ (j_1, \ldots, j_d) \ ]_{j_d = h_d, \ldots, k_d} \ \Big]_{j_{d-1} = h_{d-1}, \ldots, k_{d-1}} \ \ldots \ \Big]_{j_1 = h_1, \ldots, k_1}.$$

For instance, in the case $d = 2$ the ordering is

$$\begin{aligned} &(h_1, h_2),\ (h_1, h_2 + 1),\ \ldots,\ (h_1, k_2),\ (h_1 + 1, h_2),\ (h_1 + 1, h_2 + 1),\ \ldots,\ (h_1 + 1, k_2), \\ &\ldots\ldots\ldots,\ (k_1, h_2),\ (k_1, h_2 + 1),\ \ldots,\ (k_1, k_2). \end{aligned}$$

When a d-index $\boldsymbol{j}$ varies in a d-index range $\{\boldsymbol{h}, \ldots, \boldsymbol{k}\}$ (this is often written as $\boldsymbol{j} = \boldsymbol{h}, \ldots, \boldsymbol{k}$), it is understood that $\boldsymbol{j}$ varies from $\boldsymbol{h}$ to $\boldsymbol{k}$ following the lexicographic ordering. For instance, if $\boldsymbol{m} \in \mathbb{N}^d$ and $\boldsymbol{x} = [x_{\boldsymbol{i}}]_{\boldsymbol{i}=\mathbf{1}}^{\boldsymbol{m}}$, then $\boldsymbol{x}$ is a vector of size $N(\boldsymbol{m})$ whose components $x_{\boldsymbol{i}}$, $\boldsymbol{i} = \mathbf{1}, \ldots, \boldsymbol{m}$, are ordered in accordance with the lexicographic ordering: the first component is $x_{\mathbf{1}} = x_{(1,\ldots,1,1)}$, the second component is $x_{(1,\ldots,1,2)}$, and so on until the last component, which is $x_{\boldsymbol{m}} = x_{(m_1,\ldots,m_d)}$. Similarly, if $X = [x_{\boldsymbol{ij}}]_{\boldsymbol{i},\boldsymbol{j}=\mathbf{1}}^{\boldsymbol{m}}$, then X is an $N(\boldsymbol{m}) \times N(\boldsymbol{m})$ matrix whose components are indexed by a pair of d-indices $\boldsymbol{i}, \boldsymbol{j}$, both varying in $\{\mathbf{1}, \ldots, \boldsymbol{m}\}$ following the lexicographic ordering. If $\boldsymbol{h}, \boldsymbol{k}$ are d-indices with $\boldsymbol{h} \le \boldsymbol{k}$, the notation $\sum_{\boldsymbol{j}=\boldsymbol{h}}^{\boldsymbol{k}}$ indicates the summation over all $\boldsymbol{j}$ in $\{\boldsymbol{h}, \ldots, \boldsymbol{k}\}$. Operations involving d-indices (or general vectors with d components) that have no meaning in the vector space $\mathbb{R}^d$ must always be interpreted in the componentwise sense. For instance, $\boldsymbol{jh} = (j_1 h_1, \ldots, j_d h_d)$, $\boldsymbol{i}/\boldsymbol{j} = (i_1/j_1, \ldots, i_d/j_d)$, etc. If $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d$ with $\boldsymbol{a} \le \boldsymbol{b}$, we denote by $(\boldsymbol{a}, \boldsymbol{b}]$ the d-dimensional rectangle $(a_1, b_1] \times \cdots \times (a_d, b_d]$. Similar meanings have the notations for the open d-dimensional rectangle $(\boldsymbol{a}, \boldsymbol{b})$ and the closed d-dimensional rectangle $[\boldsymbol{a}, \boldsymbol{b}]$.

2.2 Matrix norms

Given $1 \le p \le \infty$, we use the notation $|\cdot|_p$ for both the p-norm of vectors and the associated operator norm for matrices. The 2-norm $|\cdot|_2$ is also known as the Euclidean (or spectral) norm and is preferably denoted by $\|\cdot\|$. The Schatten p-norm of a matrix X is denoted by $\|X\|_p$ and is defined as the p-norm of the vector formed by the singular values of X. The Schatten 2-norm $\|X\|_2$ coincides with the Frobenius norm of X. The Schatten ∞-norm $\|X\|_\infty$ is the largest singular value $\sigma_{\max}(X)$ and coincides with the spectral norm $\|X\|$. The Schatten 1-norm $\|X\|_1$ is the sum of the singular values of X and is also known as the trace-norm of X. For more on Schatten p-norms, see [9].

2.3 Tensor products

If $X \in \mathbb{C}^{m_1 \times m_2}$ and $Y \in \mathbb{C}^{\ell_1 \times \ell_2}$, the tensor (Kronecker) product of X and Y is the $m_1\ell_1 \times m_2\ell_2$ matrix defined by

$$X \otimes Y = \big[x_{ij}Y\big]_{\substack{i=1,\ldots,m_1 \\ j=1,\ldots,m_2}} = \begin{bmatrix} x_{11}Y & \cdots & x_{1m_2}Y \\ \vdots & & \vdots \\ x_{m_11}Y & \cdots & x_{m_1m_2}Y \end{bmatrix}.$$

Here is a list of properties satisfied by tensor products [14, Section 2.5]. In what follows, the conjugate transpose of a matrix X is denoted by X^*.

P1. Associativity: $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$ for all matrices X, Y, Z.
P2. Bilinearity: for each fixed matrix X, the map $Y \mapsto X \otimes Y$ is linear on $\mathbb{C}^{\ell_1 \times \ell_2}$ for every $\ell_1, \ell_2 \in \mathbb{N}$; for each fixed matrix Y, the map $X \mapsto X \otimes Y$ is linear on $\mathbb{C}^{m_1 \times m_2}$ for every $m_1, m_2 \in \mathbb{N}$.
P3. $(X \otimes Y)^* = X^* \otimes Y^*$ and $(X \otimes Y)^T = X^T \otimes Y^T$ for all matrices X, Y.
P4. If $X_r \in \mathbb{C}^{m_r \times m_r}$ for $r = 1, \ldots, d$ and $\boldsymbol{m} = (m_1, \ldots, m_d)$, then

$$(X_1 \otimes X_2 \otimes \cdots \otimes X_d)_{\boldsymbol{ij}} = (X_1)_{i_1j_1}(X_2)_{i_2j_2} \cdots (X_d)_{i_dj_d}, \qquad \boldsymbol{i}, \boldsymbol{j} = \boldsymbol{1}, \ldots, \boldsymbol{m}.$$

2.4 Singular value and spectral distributions of a sequence of matrices

A sequence of matrices is a sequence of the form $\{A_n\}_n$, where n varies in some infinite subset of $\mathbb{N}$ and A_n is a square matrix of size d_n such that $d_n \to \infty$ as $n \to \infty$. Let μ_k be the Lebesgue measure in $\mathbb{R}^k$. Throughout this paper, all the terminology coming from measure theory (such as "measurable set", "measurable function", "a.e.", etc.) always refers to the Lebesgue measure. Let $C_c(\mathbb{R})$ (respectively, $C_c(\mathbb{C})$) be the space of continuous complex-valued functions with bounded support defined on $\mathbb{R}$ (respectively, $\mathbb{C}$). The singular values and eigenvalues of a matrix $A \in \mathbb{C}^{m \times m}$ are denoted by $\sigma_1(A), \ldots, \sigma_m(A)$ and $\lambda_1(A), \ldots, \lambda_m(A)$, respectively.

Definition 2.1. Let $\{A_n\}_n$ be a sequence of matrices, with A_n of size d_n, and let $\kappa : D \subset \mathbb{R}^k \to \mathbb{C}$ be a measurable function defined on a set D with $0 < \mu_k(D) < \infty$.

- We say that $\{A_n\}_n$ has a spectral (or eigenvalue) distribution described by κ, and we write $\{A_n\}_n \sim_\lambda \kappa$, if

$$\lim_{n\to\infty} \frac{1}{d_n} \sum_{i=1}^{d_n} F(\lambda_i(A_n)) = \frac{1}{\mu_k(D)} \int_D F(\kappa(\boldsymbol{y}))\mathrm{d}\boldsymbol{y}, \qquad \forall\, F \in C_c(\mathbb{C}). \tag{2.1}$$

  In this case, κ is called the spectral (or eigenvalue) symbol of $\{A_n\}_n$.
- We say that $\{A_n\}_n$ has a singular value distribution described by κ, and we write $\{A_n\}_n \sim_\sigma \kappa$, if

$$\lim_{n\to\infty} \frac{1}{d_n} \sum_{i=1}^{d_n} F(\sigma_i(A_n)) = \frac{1}{\mu_k(D)} \int_D F(|\kappa(\boldsymbol{y})|)\mathrm{d}\boldsymbol{y}, \qquad \forall\, F \in C_c(\mathbb{R}). \tag{2.2}$$

  In this case, κ is called the singular value symbol of $\{A_n\}_n$.

Remark 2.1. The informal meaning behind the spectral distribution (2.1) is the following [13, p. 46]: assuming that κ is continuous a.e., the eigenvalues of A_n, except possibly for $o(d_n)$ outliers, are approximately equal to the samples of κ over a uniform grid in the domain D (for n large enough). A completely analogous meaning can be given for the singular value distribution (2.2).

2.5 Special sequences of matrices

We now introduce some special sequences of matrices that play a central role in the theory of GLT sequences.

***d*-level matrix-sequences.** A d-level matrix-sequence is a special sequence of matrices of the form $\{A_{\boldsymbol{n}}\}_n$, where

- n varies in some infinite subset of $\mathbb{N}$,
- $\boldsymbol{n} = \boldsymbol{n}(n) \in \mathbb{N}^d$ and $\boldsymbol{n} \to \infty$ (i.e., $\min(\boldsymbol{n}) \to \infty$) as $n \to \infty$,
- $A_{\boldsymbol{n}}$ is a square matrix of size $N(\boldsymbol{n})$.

Zero-distributed sequences. A sequence of matrices $\{Z_n\}_n$ such that $\{Z_n\}_n \sim_\sigma 0$ is referred to as a zero-distributed sequence. In other words, $\{Z_n\}_n$ is zero-distributed if and only if

$$\lim_{n\to\infty} \frac{1}{d_n} \sum_{i=1}^{d_n} F(\sigma_i(Z_n)) = F(0), \qquad \forall\, F \in C_c(\mathbb{R}),$$

where d_n is the size of Z_n. The following property can be found in [13, Chapter 3].

Z1. We have $\{Z_n\}_n \sim_\sigma 0$ if and only if $Z_n = R_n + N_n$ with $\lim_{n\to\infty}(d_n)^{-1}\mathrm{rank}(R_n) = \lim_{n\to\infty} \|N_n\| = 0$, where d_n is the size of Z_n.

Sequences of diagonal sampling matrices. If $\boldsymbol{n} \in \mathbb{N}^d$ and $a : [0,1]^d \to \mathbb{C}$, the $\boldsymbol{n}$th (d-level) diagonal sampling matrix generated by a is the $N(\boldsymbol{n}) \times N(\boldsymbol{n})$ diagonal matrix given by

$$D_{\boldsymbol{n}}(a) = \mathop{\mathrm{diag}}_{\boldsymbol{i}=\boldsymbol{1},\ldots,\boldsymbol{n}} a\Big(\frac{\boldsymbol{i}}{\boldsymbol{n}}\Big).$$

Each d-level matrix-sequence of the form $\{D_{\boldsymbol{n}}(a)\}_n$, with $\boldsymbol{n} = \boldsymbol{n}(n) \to \infty$ as $n \to \infty$, is referred to as a sequence of (d-level) diagonal sampling matrices generated by a.

Toeplitz sequences. If $\boldsymbol{n} \in \mathbb{N}^d$ and $f : [-\pi,\pi]^d \to \mathbb{C}$ is a function in $L^1([-\pi,\pi]^d)$, the $\boldsymbol{n}$th (d-level) Toeplitz matrix generated by f is the $N(\boldsymbol{n}) \times N(\boldsymbol{n})$ matrix given by

$$T_{\boldsymbol{n}}(f) = [f_{\boldsymbol{i}-\boldsymbol{j}}]_{\boldsymbol{i},\boldsymbol{j}=\boldsymbol{1}}^{\boldsymbol{n}},$$

where the numbers $f_{\boldsymbol{k}}$ are the Fourier coefficients of f,

$$f_{\boldsymbol{k}} = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} f(\boldsymbol{\theta})\, \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{\theta}} \mathrm{d}\boldsymbol{\theta}, \qquad \boldsymbol{k} \in \mathbb{Z}^d.$$

Each d-level matrix-sequence of the form $\{T_{\boldsymbol{n}}(f)\}_n$, with $\boldsymbol{n} = \boldsymbol{n}(n) \to \infty$ as $n \to \infty$, is referred to as a (d-level) Toeplitz sequence generated by f. The following properties can be found in [14, Chapter 3].

T1. For every $\boldsymbol{n} \in \mathbb{N}^d$ the map $T_{\boldsymbol{n}}(\cdot) : L^1([-\pi,\pi]^d) \to \mathbb{C}^{N(\boldsymbol{n})\times N(\boldsymbol{n})}$ is linear.

T2. If $f_1, f_2, \ldots, f_d \in L^1([-\pi,\pi])$ and $\boldsymbol{n} \in \mathbb{N}^d$ then

$$T_{\boldsymbol{n}}(f_1 \otimes f_2 \otimes \cdots \otimes f_d) = T_{n_1}(f_1) \otimes T_{n_2}(f_2) \otimes \cdots \otimes T_{n_d}(f_d),$$

where $(f_1 \otimes f_2 \otimes \cdots \otimes f_d)(\boldsymbol{\theta}) = f_1(\theta_1) f_2(\theta_2) \cdots f_d(\theta_d)$ for all $\boldsymbol{\theta} \in [-\pi,\pi]^d$.

Approximating classes of sequences. Let $\{A_n\}_n$ be a matrix-sequence and let $\{\{B_{n,m}\}_n\}_m$ be a sequence of matrix-sequences, with A_n and $B_{n,m}$ of the same size d_n. We say that $\{\{B_{n,m}\}_n\}_m$ is an approximating class of sequences (a.c.s.) for $\{A_n\}_n$, and we write $\{B_{n,m}\}_n \xrightarrow{\text{a.c.s.}} \{A_n\}_n$, if the following condition holds: for every m there exists n_m such that, for $n \geq n_m$,

$$A_n = B_{n,m} + R_{n,m} + N_{n,m}, \qquad \mathrm{rank}(R_{n,m}) \leq c(m) d_n, \qquad \|N_{n,m}\| \leq \omega(m),$$

where n_m, $c(m)$, $\omega(m)$ depend only on m, and $\lim_{m\to\infty} c(m) = \lim_{m\to\infty} \omega(m) = 0$. Roughly speaking, $\{\{B_{n,m}\}_n\}_m$ is an a.c.s. for $\{A_n\}_n$ if, for all sufficiently large m, the sequence $\{B_{n,m}\}_n$ approximates the sequence $\{A_n\}_n$ in the sense that A_n is eventually equal to $B_{n,m}$ plus a small-rank matrix (with respect to the matrix size d_n) plus a small-norm matrix. The convergence notation $\{B_{n,m}\}_n \xrightarrow{\text{a.c.s.}} \{A_n\}_n$ is justified by the fact that the a.c.s. notion is a notion of convergence in the space of matrix-sequences [2, 5, 13]. The following property can be found in [13]. Throughout this paper, we use the convention $1/\infty = 0$.

ACS1. Let $p \in [1,\infty]$ and suppose for every m there exists n_m such that $\|A_n - B_{n,m}\|_p \leq \varepsilon(m,n)(d_n)^{1/p}$ for $n \geq n_m$, where d_n is the size of A_n and $B_{n,m}$, and $\lim_{m\to\infty} \limsup_{n\to\infty} \varepsilon(m,n) = 0$. Then, $\{B_{n,m}\}_n \xrightarrow{\text{a.c.s.}} \{A_n\}_n$.

2.6 GLT sequences

A d-level GLT sequence $\{A_{\boldsymbol{n}}\}_n$ is a special d-level matrix-sequence equipped with a measurable function $\kappa : [0,1]^d \times [-\pi,\pi]^d \to \mathbb{C}$ called symbol (or kernel). We use the notation $\{A_{\boldsymbol{n}}\}_n \sim_{\rm GLT} \kappa$ to indicate that $\{A_{\boldsymbol{n}}\}_n$ is a d-level GLT sequence with symbol κ. The symbol of a d-level GLT sequence is unique in the sense that if $\{A_{\boldsymbol{n}}\}_n \sim_{\rm GLT} \kappa$ and $\{A_{\boldsymbol{n}}\}_n \sim_{\rm GLT} \xi$ then $\kappa = \xi$ a.e. in $[0,1]^d \times [-\pi,\pi]^d$.

GLT1. If $\{A_{\boldsymbol{n}}\}_n \sim_{\rm GLT} \kappa$ then $\{A_{\boldsymbol{n}}\}_n \sim_\sigma \kappa$. If $\{A_{\boldsymbol{n}}\}_n \sim_{\rm GLT} \kappa$ and each $A_{\boldsymbol{n}}$ is Hermitian then $\{A_{\boldsymbol{n}}\}_n \sim_\lambda \kappa$.

GLT2. If $\{A_{\boldsymbol{n}}\}_n \sim_{\rm GLT} \kappa$ and $A_{\boldsymbol{n}} = X_{\boldsymbol{n}} + Y_{\boldsymbol{n}}$, where

- every $X_{\boldsymbol{n}}$ is Hermitian,
- $\|Y_{\boldsymbol{n}}\|_2/\sqrt{N(\boldsymbol{n})} \to 0$ as $n \to \infty$,

then $\{A_{\boldsymbol{n}}\}_n \sim_\lambda \kappa$.

GLT3. Suppose $\boldsymbol{n} = \boldsymbol{n}(n) \in \mathbb{N}^d$ with $\boldsymbol{n} \to \infty$ as $n \to \infty$. We have

- $\{T_{\boldsymbol{n}}(f)\}_n \sim_{\rm GLT} \kappa(\boldsymbol{x},\boldsymbol{\theta}) = f(\boldsymbol{\theta})$ if $f \in L^1([-\pi,\pi]^d)$,
- $\{D_{\boldsymbol{n}}(a)\}_n \sim_{\rm GLT} \kappa(\boldsymbol{x},\boldsymbol{\theta}) = a(\boldsymbol{x})$ if $a : \mathbb{R}^d \to \mathbb{C}$ is continuous a.e.,
- $\{Z_{\boldsymbol{n}}\}_n \sim_{\rm GLT} \kappa(\boldsymbol{x},\boldsymbol{\theta}) = 0$ if and only if $\{Z_{\boldsymbol{n}}\}_n \sim_\sigma 0$.

GLT4. If $\{A_{\boldsymbol{n}}\}_n \sim_{\rm GLT} \kappa$ and $\{B_{\boldsymbol{n}}\}_n \sim_{\rm GLT} \xi$ then

- $\{A_{\boldsymbol{n}}^*\}_n \sim_{\rm GLT} \overline{\kappa}$,
- $\{\alpha A_{\boldsymbol{n}} + \beta B_{\boldsymbol{n}}\}_n \sim_{\rm GLT} \alpha\kappa + \beta\xi$ for all $\alpha, \beta \in \mathbb{C}$,
- $\{A_{\boldsymbol{n}} B_{\boldsymbol{n}}\}_n \sim_{\rm GLT} \kappa\xi$.

GLT5. $\{A_{\boldsymbol{n}}\}_n \sim_{\rm GLT} \kappa$ if and only if there exist d-level GLT sequences $\{B_{\boldsymbol{n},m}\}_n \sim_{\rm GLT} \kappa_m$ such that $\{B_{\boldsymbol{n},m}\}_n \xrightarrow{\text{a.c.s.}} \{A_{\boldsymbol{n}}\}_n$ and $\kappa_m \to \kappa$ in measure.

In **GLT3** (second statement), it is understood that we are considering $a(\boldsymbol{x})$ for $\boldsymbol{x} \in [0,1]^d$, because the domain of the symbol of any d-level GLT sequence is always $[0,1]^d \times [-\pi,\pi]^d$. We intentionally avoided stating the formal definition of GLT sequences for two reasons. First, the definition is rather cumbersome as it requires to expound other related (and complicated) concepts. Second, the knowledge of **GLT1–GLT5** is practically more helpful than the definition. We refer the reader to [14] for the formal definition along with the proofs of **GLT1** and **GLT3–GLT5**. The proof of **GLT2** can be found in [8].

3 Main result

The second statement in **GLT3** only concerns diagonal sampling matrices on the uniform grid $\{\frac{\boldsymbol{i}}{\boldsymbol{n}}\}_{\boldsymbol{i}=\boldsymbol{1},\ldots,\boldsymbol{n}}$. This limitation negatively affects the overall applicability of the theory of GLT sequences, as we shall see in Section 4. In our main result, we extend **GLT3** to the case where general asymptotically uniform grids are used. If $\boldsymbol{n} \in \mathbb{N}^d$, $a : \mathbb{R}^d \to \mathbb{C}$ and $\mathcal{G}_{\boldsymbol{n}} = \{\boldsymbol{x}_{\boldsymbol{i},\boldsymbol{n}}\}_{\boldsymbol{i}=\boldsymbol{1},\ldots,\boldsymbol{n}}$ is a sequence of $N(\boldsymbol{n})$ grid points in $\mathbb{R}^d$, the $\boldsymbol{n}$th (d-level) diagonal sampling matrix generated by a corresponding to the grid $\mathcal{G}_{\boldsymbol{n}}$ is the $N(\boldsymbol{n}) \times N(\boldsymbol{n})$ diagonal matrix given by

$$D_{\boldsymbol{n}}^{\mathcal{G}_{\boldsymbol{n}}}(a) = \operatorname*{diag}_{\boldsymbol{i}=\boldsymbol{1},\ldots,\boldsymbol{n}} a(\boldsymbol{x}_{\boldsymbol{i},\boldsymbol{n}}).$$

Each d-level matrix-sequence of the form $\{D_{\boldsymbol{n}}^{\mathcal{G}_{\boldsymbol{n}}}(a)\}_n$, with $\boldsymbol{n} = \boldsymbol{n}(n) \to \infty$ as $n \to \infty$, is referred to as a sequence of (d-level) diagonal sampling matrices generated by a. We say that the grid $\mathcal{G}_{\boldsymbol{n}}$ is asymptotically uniform (a.u.) in $[0,1]^d$ if

$$\lim_{\boldsymbol{n}\to\infty} \left(\max_{\boldsymbol{i}=\boldsymbol{1},\ldots,\boldsymbol{n}} \left| \boldsymbol{x}_{\boldsymbol{i},\boldsymbol{n}} - \frac{\boldsymbol{i}}{\boldsymbol{n}} \right|_\infty \right) = 0.$$

Note that this condition does not imply that the grid points $\boldsymbol{x}_{\boldsymbol{i},\boldsymbol{n}}$ belong to $[0,1]^d$. In the special case where $\mathcal{G}_{\boldsymbol{n}} = \{\frac{\boldsymbol{i}}{\boldsymbol{n}}\}_{\boldsymbol{i}=\boldsymbol{1},\ldots,\boldsymbol{n}}$, we drop the superscript and we write $D_{\boldsymbol{n}}(a)$ as in **GLT3**. Our main result is the following.

Theorem 3.1. *Let $a : \mathbb{R}^d \to \mathbb{C}$ be continuous a.e., let $\mathcal{G}_{\boldsymbol{n}} = \{\boldsymbol{x}_{\boldsymbol{i},\boldsymbol{n}}\}_{\boldsymbol{i}=\boldsymbol{1},\ldots,\boldsymbol{n}}$ be a.u. in $[0,1]^d$, and let $\boldsymbol{n} = \boldsymbol{n}(n) \in \mathbb{N}^d$ with $\boldsymbol{n} \to \infty$ as $n \to \infty$. Then*

$$\{D_{\boldsymbol{n}}^{\mathcal{G}_{\boldsymbol{n}}}(a)\}_n \sim_{\rm GLT} a(\boldsymbol{x}). \tag{3.1}$$

In the case where $\mathcal{G}_{\boldsymbol{n}} = \{\frac{\boldsymbol{i}}{\boldsymbol{n}}\}_{\boldsymbol{i}=\boldsymbol{1},\ldots,\boldsymbol{n}}$, the result is just **GLT3**. In the case where $\mathcal{G}_{\boldsymbol{n}} = \{\frac{\boldsymbol{i}}{\boldsymbol{n}+\boldsymbol{1}}\}_{\boldsymbol{i}=\boldsymbol{1},\ldots,\boldsymbol{n}}$, the result has been proved by Barbarino [4, Lemma 3.3]. We now prove the result for a general grid $\mathcal{G}_{\boldsymbol{n}} = \{\boldsymbol{x}_{\boldsymbol{i},\boldsymbol{n}}\}_{\boldsymbol{i}=\boldsymbol{1},\ldots,\boldsymbol{n}}$ that is a.u. in $[0,1]^d$. The proof is based on the same ideas as the proof of [4, Lemma 3.3], but it also avoids a somewhat artificial argument used therein by resorting to the more natural Lemma 3.1 below. In what follows, we say that a function $a : \mathbb{R}^d \to \mathbb{R}$ is locally bounded on $\mathbb{R}^d$ if it is bounded on every compact subset of $\mathbb{R}^d$.

Lemma 3.1. *Let $a : \mathbb{R}^d \to \mathbb{R}$ be continuous a.e. and locally bounded on $\mathbb{R}^d$. For each $\boldsymbol{n} \in \mathbb{N}^d$, consider the partition of $(0,1]^d$ given by the d-dimensional rectangles $I_{\boldsymbol{i},\boldsymbol{n}} = \big(\frac{\boldsymbol{i}-\boldsymbol{1}}{\boldsymbol{n}}, \frac{\boldsymbol{i}}{\boldsymbol{n}}\big]$, $\boldsymbol{i} = \boldsymbol{1},\ldots,\boldsymbol{n}$, and let $\mathcal{G}_{\boldsymbol{n}} = \{\boldsymbol{x}_{\boldsymbol{i},\boldsymbol{n}}\}_{\boldsymbol{i}=\boldsymbol{1},\ldots,\boldsymbol{n}}$ be a.u. in $[0,1]^d$. Then,*

$$\lim_{\boldsymbol{n}\to\infty} \sum_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{n}} a(\boldsymbol{x}_{\boldsymbol{i},\boldsymbol{n}})\chi_{I_{\boldsymbol{i},\boldsymbol{n}}}(\boldsymbol{x}) = a(\boldsymbol{x}) \ \textit{ for a.e. } \boldsymbol{x} \in [0,1]^d \tag{3.2}$$

and

$$\lim_{\boldsymbol{n}\to\infty} \frac{1}{N(\boldsymbol{n})} \sum_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{n}} a(\boldsymbol{x}_{\boldsymbol{i},\boldsymbol{n}}) = \int_{[0,1]^d} a(\boldsymbol{x})\mathrm{d}\boldsymbol{x}. \tag{3.3}$$

Proof. Let $R = [-1,2]^d$. The grid $\mathcal{G}_{\boldsymbol{n}}$ is a.u. in $[0,1]^d$ and hence it is eventually contained in R as $\boldsymbol{n} \to \infty$. Without loss of generality, we can assume that $\mathcal{G}_{\boldsymbol{n}} \subset R$ for all $\boldsymbol{n}$. Let $\boldsymbol{x} \in (0,1]^d$ be a continuity point of a and fix $\varepsilon > 0$. Then, there is a $\delta = \delta_{\boldsymbol{x},\varepsilon} > 0$ such that $|a(\boldsymbol{y}) - a(\boldsymbol{x})| \le \varepsilon$ whenever $\boldsymbol{y} \in R$ and $|\boldsymbol{y} - \boldsymbol{x}|_\infty \le \delta$. Since $\mathcal{G}_{\boldsymbol{n}}$ is a.u. in $[0,1]^d$, we can choose $\boldsymbol{n}_\delta$ such that, for $\boldsymbol{n} \ge \boldsymbol{n}_\delta$,

$$\max_{\boldsymbol{i}=\boldsymbol{1},\ldots,\boldsymbol{n}} \left|\boldsymbol{x}_{\boldsymbol{i},\boldsymbol{n}} - \frac{\boldsymbol{i}}{\boldsymbol{n}}\right|_\infty \le \frac{\delta}{2}, \qquad \frac{1}{\min(\boldsymbol{n})} \le \frac{\delta}{2}.$$

For $\boldsymbol{n} \ge \boldsymbol{n}_\delta$, if we call $I_{\boldsymbol{k},\boldsymbol{n}}$ the unique d-dimensional rectangle of the partition $(0,1]^d = \bigcup_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{n}} I_{\boldsymbol{i},\boldsymbol{n}}$ containing $\boldsymbol{x}$, we have

$$|\boldsymbol{x}_{\boldsymbol{k},\boldsymbol{n}} - \boldsymbol{x}|_\infty \le \left|\boldsymbol{x}_{\boldsymbol{k},\boldsymbol{n}} - \frac{\boldsymbol{k}}{\boldsymbol{n}}\right|_\infty + \left|\frac{\boldsymbol{k}}{\boldsymbol{n}} - \boldsymbol{x}\right|_\infty \le \frac{\delta}{2} + \frac{\delta}{2} = \delta$$

and

$$\left|\sum_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{n}} a(\boldsymbol{x}_{\boldsymbol{i},\boldsymbol{n}})\chi_{I_{\boldsymbol{i},\boldsymbol{n}}}(\boldsymbol{x}) - a(\boldsymbol{x})\right| = |a(\boldsymbol{x}_{\boldsymbol{k},\boldsymbol{n}}) - a(\boldsymbol{x})| \le \varepsilon.$$

As a consequence, $\sum_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{n}} a(\boldsymbol{x}_{\boldsymbol{i},\boldsymbol{n}})\chi_{I_{\boldsymbol{i},\boldsymbol{n}}}(\boldsymbol{x}) \to a(\boldsymbol{x})$ whenever $\boldsymbol{x} \in (0,1]^d$ is a continuity point of a. This implies (3.2), because a is continuous a.e. Since

$$\left|\sum_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{n}} a(\boldsymbol{x}_{\boldsymbol{i},\boldsymbol{n}})\chi_{I_{\boldsymbol{i},\boldsymbol{n}}}\right| \le \|a\|_{\infty,R} < \infty, \qquad \frac{1}{N(\boldsymbol{n})} \sum_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{n}} a(\boldsymbol{x}_{\boldsymbol{i},\boldsymbol{n}}) = \int_{[0,1]^d} \left(\sum_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{n}} a(\boldsymbol{x}_{\boldsymbol{i},\boldsymbol{n}})\chi_{I_{\boldsymbol{i},\boldsymbol{n}}}\right),$$

the limit (3.3) follows from (3.2) and the dominated convergence theorem. ∎

Proof of Theorem 3.1. For an arbitrary a.e. continuous function $a : \mathbb{R}^d \to \mathbb{C}$, we can write $a = \alpha_+ - \alpha_- + \mathrm{i}\beta_+ - \mathrm{i}\beta_-$, where $\alpha_\pm, \beta_\pm : \mathbb{R}^d \to [0,\infty)$ are continuous a.e. and non-negative. Hence, due to **GLT4** and the linearity of $D_{\boldsymbol{n}}^{\mathcal{G}_{\boldsymbol{n}}}(a)$ with respect to its argument a, it suffices to prove (3.1) in the case where $a : \mathbb{R}^d \to [0,\infty)$ is continuous a.e. and non-negative. The proof consists of three steps.

Step 1. First, we prove the result in the case where a is continuous on $\mathbb{R}^d$. The grid $\mathcal{G}_{\boldsymbol{n}}$ is a.u. in $[0,1]^d$ and hence we can fix a closed d-dimensional rectangle R that contains both $[0,1]^d$ and $\mathcal{G}_{\boldsymbol{n}}$ for all n. Let ω_a be the modulus of continuity of a over R:

$$\omega_a(\delta) = \sup_{\substack{\boldsymbol{x},\boldsymbol{y}\in R\\ |\boldsymbol{x}-\boldsymbol{y}|_\infty\le\delta}} |a(\boldsymbol{x}) - a(\boldsymbol{y})|, \qquad \delta > 0.$$

Note that

$$\|D_{\boldsymbol{n}}^{\mathcal{G}_{\boldsymbol{n}}}(a) - D_{\boldsymbol{n}}(a)\| = \max_{\boldsymbol{i}=\boldsymbol{1},\ldots,\boldsymbol{n}} \left|a(\boldsymbol{x}_{\boldsymbol{i},\boldsymbol{n}}) - a\Big(\frac{\boldsymbol{i}}{\boldsymbol{n}}\Big)\right| \le \omega_a\left(\max_{\boldsymbol{i}=\boldsymbol{1},\ldots,\boldsymbol{n}} \left|\boldsymbol{x}_{\boldsymbol{i},\boldsymbol{n}} - \frac{\boldsymbol{i}}{\boldsymbol{n}}\right|_\infty\right) \overset{n\to\infty}{\longrightarrow} 0.$$

Hence, $\{D_{\boldsymbol{n}}^{\mathcal{G}_{\boldsymbol{n}}}(a) - D_{\boldsymbol{n}}(a)\}_n \sim_\sigma 0$ by **Z1** and $\{D_{\boldsymbol{n}}^{\mathcal{G}_{\boldsymbol{n}}}(a)\}_n \sim_{\mathrm{GLT}} a(\boldsymbol{x})$ by **GLT3–GLT4**.

Step 2. Now, we prove the result in the case where a is continuous a.e. and locally bounded on $\mathbb{R}^d$. As before, fix a d-dimensional rectangle R that contains both $[0,1]^d$ and $\mathcal{G}_{\boldsymbol{n}}$ for all n. Let a_m be a sequence of continuous functions in $\mathbb{R}^d$ such that $a_m \to a$ in $L^1(R)$. It is clear that $\{D_{\boldsymbol{n}}^{\mathcal{G}_{\boldsymbol{n}}}(a_m)\}_n \sim_{\rm GLT} a_m(\boldsymbol{x})$ by Step 1 and $a_m \to a$ in measure on $[0,1]^d$. We prove that $\{D_{\boldsymbol{n}}^{\mathcal{G}_{\boldsymbol{n}}}(a_m)\}_n \xrightarrow{\text{a.c.s.}} \{D_{\boldsymbol{n}}^{\mathcal{G}_{\boldsymbol{n}}}(a)\}_n$, after which the relation $\{D_{\boldsymbol{n}}^{\mathcal{G}_{\boldsymbol{n}}}(a)\}_n \sim_{\rm GLT} a(\boldsymbol{x})$ follows from **GLT5**. Since a_m is continuous on $\mathbb{R}^d$, $|a_m - a|$ is continuous a.e. and locally bounded on $\mathbb{R}^d$. By Lemma 3.1,

$$\lim_{m\to\infty}\lim_{n\to\infty}\frac{\|D_{\boldsymbol{n}}^{\mathcal{G}_{\boldsymbol{n}}}(a_m)-D_{\boldsymbol{n}}^{\mathcal{G}_{\boldsymbol{n}}}(a)\|_1}{N(\boldsymbol{n})}=\lim_{m\to\infty}\lim_{n\to\infty}\frac{1}{N(\boldsymbol{n})}\sum_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{n}}|a_m(\boldsymbol{x}_{\boldsymbol{i},\boldsymbol{n}})-a(\boldsymbol{x}_{\boldsymbol{i},\boldsymbol{n}})|=\lim_{m\to\infty}\|a_m-a\|_{L^1([0,1]^d)}=0.$$

We can therefore write $\|D_{\boldsymbol{n}}^{\mathcal{G}_{\boldsymbol{n}}}(a_m)-D_{\boldsymbol{n}}^{\mathcal{G}_{\boldsymbol{n}}}(a)\|_1=\varepsilon(m,n)N(\boldsymbol{n})$ with $\lim_{m\to\infty}\lim_{n\to\infty}\varepsilon(m,n)=0$, and **ACS1** implies that $\{D_{\boldsymbol{n}}^{\mathcal{G}_{\boldsymbol{n}}}(a_m)\}_n \xrightarrow{\text{a.c.s.}} \{D_{\boldsymbol{n}}^{\mathcal{G}_{\boldsymbol{n}}}(a)\}_n$.

Step 3. Finally, we prove the result in the case where a is continuous a.e. in $\mathbb{R}^d$. Let $a_m=\min(a,m)$ be the truncation of a at level $m\in\mathbb{N}$. Note that a_m is continuous a.e. and bounded in $\mathbb{R}^d$. We have $\{D_{\boldsymbol{n}}^{\mathcal{G}_{\boldsymbol{n}}}(a_m)\}_n \sim_{\rm GLT} a_m(\boldsymbol{x})$ by Step 2 and $a_m\to a$ in measure on $[0,1]^d$ because, by the continuity of the measure μ_d,

$$\lim_{m\to\infty}\mu_d\big\{\boldsymbol{x}\in[0,1]^d : a(\boldsymbol{x})>m\big\}=\mu_d\left(\bigcap_{m=1}^{\infty}\big\{\boldsymbol{x}\in[0,1]^d : a(\boldsymbol{x})>m\big\}\right)=\mu_d(\emptyset)=0.$$

We prove that $\{D_{\boldsymbol{n}}^{\mathcal{G}_{\boldsymbol{n}}}(a_m)\}_n \xrightarrow{\text{a.c.s.}} \{D_{\boldsymbol{n}}^{\mathcal{G}_{\boldsymbol{n}}}(a)\}_n$, after which the relation $\{D_{\boldsymbol{n}}^{\mathcal{G}_{\boldsymbol{n}}}(a)\}_n \sim_{\rm GLT} a(\boldsymbol{x})$ follows from **GLT5**. Consider a function $F_m\in C_c(\mathbb{R})$ such that $\chi_{[0,m-1]}\le F_m\le\chi_{[-1,m]}$. We have

$$\begin{aligned}\frac{\operatorname{rank}(D_{\boldsymbol{n}}^{\mathcal{G}_{\boldsymbol{n}}}(a_m)-D_{\boldsymbol{n}}^{\mathcal{G}_{\boldsymbol{n}}}(a))}{N(\boldsymbol{n})}&=\frac{\#\{\boldsymbol{i}\in\{\boldsymbol{1},\ldots,\boldsymbol{n}\}: a(\boldsymbol{x}_{\boldsymbol{i},\boldsymbol{n}})>m\}}{N(\boldsymbol{n})}=1-\frac{\#\{\boldsymbol{i}\in\{\boldsymbol{1},\ldots,\boldsymbol{n}\}: a(\boldsymbol{x}_{\boldsymbol{i},\boldsymbol{n}})\le m\}}{N(\boldsymbol{n})}\\&=1-\frac{1}{N(\boldsymbol{n})}\sum_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{n}}\chi_{[-1,m]}(a(\boldsymbol{x}_{\boldsymbol{i},\boldsymbol{n}}))\le 1-\frac{1}{N(\boldsymbol{n})}\sum_{\boldsymbol{i}=\boldsymbol{1}}^{\boldsymbol{n}}F_m(a(\boldsymbol{x}_{\boldsymbol{i},\boldsymbol{n}})).\end{aligned}$$

Since $F_m(a)$ is continuous a.e. and bounded on $\mathbb{R}^d$, passing to the limit in the previous inequality and using Lemma 3.1, we obtain

$$\begin{aligned}\limsup_{n\to\infty}\frac{\operatorname{rank}(D_{\boldsymbol{n}}^{\mathcal{G}_{\boldsymbol{n}}}(a_m)-D_{\boldsymbol{n}}^{\mathcal{G}_{\boldsymbol{n}}}(a))}{N(\boldsymbol{n})}&\le 1-\int_{[0,1]^d}F_m(a(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}\le 1-\int_{[0,1]^d}\chi_{[0,m-1]}(a(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}\\&=1-\mu_d\{\boldsymbol{x}\in[0,1]^d : a(\boldsymbol{x})\le m-1\}=\gamma(m)\xrightarrow{m\to\infty}0.\end{aligned}$$

As a consequence, for every m we can find n_m such that, for $n\ge n_m$, $\operatorname{rank}(D_{\boldsymbol{n}}^{\mathcal{G}_{\boldsymbol{n}}}(a_m)-D_{\boldsymbol{n}}^{\mathcal{G}_{\boldsymbol{n}}}(a))\le c(m)N(\boldsymbol{n})$, where $c(m)=\gamma(m)+\frac{1}{m}$ and $\lim_{m\to\infty}c(m)=0$. Thus, $\{D_{\boldsymbol{n}}^{\mathcal{G}_{\boldsymbol{n}}}(a_m)\}_n \xrightarrow{\text{a.c.s.}} \{D_{\boldsymbol{n}}^{\mathcal{G}_{\boldsymbol{n}}}(a)\}_n$ by definition of a.c.s. $\square$

4 Applications

In this section, we detail a few representative applications of our main result in the context of FD discretizations of DEs with discontinuous coefficients. We focus on diffusion equations for simplicity, and we address the unidimensional case in Section 4.1 and the multidimensional case in Section 4.2.

4.1 FD discretization of diffusion equations: the unidimensional case

Consider the diffusion problem

$$\begin{cases}-(a(x)u'(x))'=f(x), & x\in(0,1),\\ u(0)=u(1)=0.\end{cases}$$

Let $n\in\mathbb{N}$, set $h=\frac{1}{n+1}$ and $x_j=jh$ for all $j\in[0,n+1]$. Using the classical second-order central FD formula, for $j=1,\ldots,n$ we have

$$(a(x)u'(x))'|_{x=x_j}\approx\frac{a(x_{j+\frac12})u'(x_{j+\frac12})-a(x_{j-\frac12})u'(x_{j-\frac12})}{h}\approx a(x_{j+\frac12})\frac{u(x_{j+1})-u(x_j)}{h^2}-a(x_{j-\frac12})\frac{u(x_j)-u(x_{j-1})}{h^2}.$$

We then approximate $u(x_j)$ by u_j for $j = 0, \ldots, n+1$, where $u_0 = u_{n+1} = 0$ and $\boldsymbol{u} = (u_1, \ldots, u_n)^T$ solves

$$-a(x_{j+\frac{1}{2}})u_{j+1} + (a(x_{j+\frac{1}{2}}) + a(x_{j-\frac{1}{2}}))u_j - a(x_{j-\frac{1}{2}})u_{j-1} = h^2 f(x_j), \qquad j = 1, \ldots, n.$$

The matrix A_n of this linear system is the $n \times n$ tridiagonal symmetric matrix given by

$$A_n = \begin{bmatrix} a_{\frac{1}{2}} + a_{\frac{3}{2}} & -a_{\frac{3}{2}} & & & \\ -a_{\frac{3}{2}} & a_{\frac{3}{2}} + a_{\frac{5}{2}} & -a_{\frac{5}{2}} & & \\ & -a_{\frac{5}{2}} & \ddots & \ddots & \\ & & \ddots & \ddots & -a_{n-\frac{1}{2}} \\ & & & -a_{n-\frac{1}{2}} & a_{n-\frac{1}{2}} + a_{n+\frac{1}{2}} \end{bmatrix},$$

where $a_i = a(x_i)$ for all $i \in [0, n+1]$. It is known that $\{A_n\}_n \sim_{\mathrm{GLT},\sigma,\lambda} a(x)(2 - 2\cos\theta)$ in the case where $a : [0,1] \to \mathbb{R}$ is continuous on $[0,1]$; see, e.g., [13, Section 10.5]. Using Theorem 3.1, we now prove that the same result is true under the much weaker assumption that $a : [0,1] \to \mathbb{R}$ is continuous a.e. in $[0,1]$.

Theorem 4.1. *If $a : [0,1] \to \mathbb{R}$ is continuous a.e. in $[0,1]$ then*

$$\{A_n\}_n \sim_{\mathrm{GLT},\sigma,\lambda} a(x)(2 - 2\cos\theta).$$

Proof. Write

$$A_n = D_n^+ K_n^+ + D_n^- K_n^-,$$

where

$$K_n^+ = \begin{bmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \\ & & & & 1 \end{bmatrix} = T_n(1 - \mathrm{e}^{-\mathrm{i}\theta}), \qquad K_n^- = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & \ddots & \ddots & & \\ & & -1 & 1 & \\ & & & -1 & 1 \end{bmatrix} = T_n(1 - \mathrm{e}^{\mathrm{i}\theta}), \tag{4.1}$$

$$D_n^+ = \operatorname*{diag}_{j=1,\ldots,n} a_{j+\frac{1}{2}} = \operatorname*{diag}_{j=1,\ldots,n} a(x_{j+\frac{1}{2}}), \qquad D_n^- = \operatorname*{diag}_{j=1,\ldots,n} a_{j-\frac{1}{2}} = \operatorname*{diag}_{j=1,\ldots,n} a(x_{j-\frac{1}{2}}).$$

It is clear that the grids $\mathcal{G}_n = \{x_{j+\frac{1}{2}}\}_{j=1,\ldots,n}$ and $\mathcal{H}_n = \{x_{j-\frac{1}{2}}\}_{i=1,\ldots,n}$ are a.u. in $[0,1]$. Hence, by Theorem 3.1, $\{D_n^+\}_n \sim_{\mathrm{GLT}} a(x)$ and $\{D_n^-\}_n \sim_{\mathrm{GLT}} a(x)$. We then infer from **GLT3–GLT4** that

$$\{A_n\}_n \sim_{\mathrm{GLT}} a(x)(1 - \mathrm{e}^{-\mathrm{i}\theta}) + a(x)(1 - \mathrm{e}^{\mathrm{i}\theta}) = a(x)(2 - 2\cos\theta), \tag{4.2}$$

and we finally obtain $\{A_n\}_n \sim_{\sigma,\lambda} a(x)(2 - 2\cos\theta)$ by **GLT1** and the symmetry of A_n. $\square$

Remark 4.1. The grids $\mathcal{G}_n$ and $\mathcal{H}_n$ in the proof of Theorem 4.1 are uniform and not just a.u. One may therefore argue that Theorem 4.1 could be proved in some simpler way, without resorting to Theorem 3.1. Actually, this is not the case: invoking Theorem 3.1 in the proof of Theorem 4.1 is as necessary as in other more complicated contexts where one has to deal with pure (non-uniform) a.u. grids. We can avoid invoking Theorem 3.1 in the proof of Theorem 4.1 if (and only if) the function a is continuous on $[0,1]$. In this case, we can show by continuity arguments that $\|D_n^+ - D_n(a)\| \to 0$ and $\|D_n^- - D_n(a)\| \to 0$ as $n \to \infty$, so that $\{D_n^+ - D_n(a)\}_n$ and $\{D_n^- - D_n(a)\}_n$ are zero-distributed, and the GLT relation (4.2) follows from **GLT3–GLT4** and the decomposition

$$A_n = D_n^+ K_n^+ + D_n^- K_n^- = D_n(a)K_n^+ + D_n(a)K_n^- + (D_n^+ - D_n(a))K_n^+ + (D_n^- - D_n(a))K_n^-.$$

4.2 FD discretization of diffusion equations: the multidimensional case

Let $A(\boldsymbol{x})=[a_{\alpha\beta}(\boldsymbol{x})]_{\alpha,\beta=1}^d$ and consider the diffusion problem

$$\begin{cases} -\nabla\cdot A(\boldsymbol{x})\nabla u(\boldsymbol{x})=f(\boldsymbol{x}), & \boldsymbol{x}\in(0,1)^d,\\ u(\boldsymbol{x})=0, & \boldsymbol{x}\in\partial((0,1)^d),\end{cases}$$

$$\iff\quad \begin{cases} -\displaystyle\sum_{\alpha,\beta=1}^d \frac{\partial}{\partial x_\alpha}\Big(a_{\alpha\beta}(\boldsymbol{x})\frac{\partial u}{\partial x_\beta}(\boldsymbol{x})\Big)=f(\boldsymbol{x}), & \boldsymbol{x}\in(0,1)^d,\\ u(\boldsymbol{x})=0, & \boldsymbol{x}\in\partial((0,1)^d).\end{cases}$$

Let $\boldsymbol{n}\in\mathbb{N}^d$, set $\boldsymbol{h}=\frac{\boldsymbol{1}}{\boldsymbol{n}+\boldsymbol{1}}$ and $\boldsymbol{x_j}=\boldsymbol{jh}$ for $\boldsymbol{j}\in[\boldsymbol{0},\boldsymbol{n}+\boldsymbol{1}]$. Let $\boldsymbol{e}_1,\ldots,\boldsymbol{e}_d$ be the vectors of the canonical basis of $\mathbb{R}^d$. Using the classical second-order central FD formula, for $\boldsymbol{j}=\boldsymbol{1},\ldots,\boldsymbol{n}$ and $\alpha,\beta=1,\ldots,d$ with $\alpha\neq\beta$ we have

$$\begin{aligned}
\frac{\partial}{\partial x_\alpha}\Big(a_{\alpha\alpha}(\boldsymbol{x})\frac{\partial u}{\partial x_\alpha}(\boldsymbol{x})\Big)\Big|_{\boldsymbol{x}=\boldsymbol{x_j}} &\approx \frac{a_{\alpha\alpha}(\boldsymbol{x}_{\boldsymbol{j}+\frac12\boldsymbol{e}_\alpha})\frac{\partial u}{\partial x_\alpha}(\boldsymbol{x}_{\boldsymbol{j}+\frac12\boldsymbol{e}_\alpha})-a_{\alpha\alpha}(\boldsymbol{x}_{\boldsymbol{j}-\frac12\boldsymbol{e}_\alpha})\frac{\partial u}{\partial x_\alpha}(\boldsymbol{x}_{\boldsymbol{j}-\frac12\boldsymbol{e}_\alpha})}{h_\alpha}\\
&\approx a_{\alpha\alpha}(\boldsymbol{x}_{\boldsymbol{j}+\frac12\boldsymbol{e}_\alpha})\frac{u(\boldsymbol{x}_{\boldsymbol{j}+\boldsymbol{e}_\alpha})-u(\boldsymbol{x_j})}{h_\alpha^2}-a_{\alpha\alpha}(\boldsymbol{x}_{\boldsymbol{j}-\frac12\boldsymbol{e}_\alpha})\frac{u(\boldsymbol{x_j})-u(\boldsymbol{x}_{\boldsymbol{j}-\boldsymbol{e}_\alpha})}{h_\alpha^2},\\
\frac{\partial}{\partial x_\alpha}\Big(a_{\alpha\beta}(\boldsymbol{x})\frac{\partial u}{\partial x_\beta}(\boldsymbol{x})\Big)\Big|_{\boldsymbol{x}=\boldsymbol{x_j}} &\approx \frac{a_{\alpha\beta}(\boldsymbol{x}_{\boldsymbol{j}+\boldsymbol{e}_\alpha})\frac{\partial u}{\partial x_\beta}(\boldsymbol{x}_{\boldsymbol{j}+\boldsymbol{e}_\alpha})-a_{\alpha\beta}(\boldsymbol{x}_{\boldsymbol{j}-\boldsymbol{e}_\alpha})\frac{\partial u}{\partial x_\beta}(\boldsymbol{x}_{\boldsymbol{j}-\boldsymbol{e}_\alpha})}{2h_\alpha}\\
&\approx a_{\alpha\beta}(\boldsymbol{x}_{\boldsymbol{j}+\boldsymbol{e}_\alpha})\frac{u(\boldsymbol{x}_{\boldsymbol{j}+\boldsymbol{e}_\alpha+\boldsymbol{e}_\beta})-u(\boldsymbol{x}_{\boldsymbol{j}+\boldsymbol{e}_\alpha-\boldsymbol{e}_\beta})}{4h_\alpha h_\beta}\\
&\qquad -a_{\alpha\beta}(\boldsymbol{x}_{\boldsymbol{j}-\boldsymbol{e}_\alpha})\frac{u(\boldsymbol{x}_{\boldsymbol{j}-\boldsymbol{e}_\alpha+\boldsymbol{e}_\beta})-u(\boldsymbol{x}_{\boldsymbol{j}-\boldsymbol{e}_\alpha-\boldsymbol{e}_\beta})}{4h_\alpha h_\beta}.
\end{aligned}$$

We then approximate $u(\boldsymbol{x_j})$ by $u_{\boldsymbol{j}}$ for $\boldsymbol{j}=\boldsymbol{0},\ldots,\boldsymbol{n}+\boldsymbol{1}$, where $u_{\boldsymbol{j}}=0$ if $\boldsymbol{j}\notin\{\boldsymbol{1},\ldots,\boldsymbol{n}\}$ and $\boldsymbol{u}=(u_{\boldsymbol{1}},\ldots,u_{\boldsymbol{n}})^T$ solves

$$\begin{aligned}
&-\sum_{\alpha=1}^d\Big[a_{\alpha\alpha}(\boldsymbol{x}_{\boldsymbol{j}+\frac12\boldsymbol{e}_\alpha})\frac{u_{\boldsymbol{j}+\boldsymbol{e}_\alpha}-u_{\boldsymbol{j}}}{h_\alpha^2}-a_{\alpha\alpha}(\boldsymbol{x}_{\boldsymbol{j}-\frac12\boldsymbol{e}_\alpha})\frac{u_{\boldsymbol{j}}-u_{\boldsymbol{j}-\boldsymbol{e}_\alpha}}{h_\alpha^2}\Big]\\
&-\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^d\Big[a_{\alpha\beta}(\boldsymbol{x}_{\boldsymbol{j}+\boldsymbol{e}_\alpha})\frac{u_{\boldsymbol{j}+\boldsymbol{e}_\alpha+\boldsymbol{e}_\beta}-u_{\boldsymbol{j}+\boldsymbol{e}_\alpha-\boldsymbol{e}_\beta}}{4h_\alpha h_\beta}-a_{\alpha\beta}(\boldsymbol{x}_{\boldsymbol{j}-\boldsymbol{e}_\alpha})\frac{u_{\boldsymbol{j}-\boldsymbol{e}_\alpha+\boldsymbol{e}_\beta}-u_{\boldsymbol{j}-\boldsymbol{e}_\alpha-\boldsymbol{e}_\beta}}{4h_\alpha h_\beta}\Big]=f(\boldsymbol{x_j}),\qquad \boldsymbol{j}=\boldsymbol{1},\ldots,\boldsymbol{n}.
\end{aligned}$$

The matrix $A_{\boldsymbol{n}}$ of this linear system is an $N(\boldsymbol{n})\times N(\boldsymbol{n})$ matrix that can be decomposed as follows:

$$A_{\boldsymbol{n}}=\sum_{\alpha=1}^d\frac{1}{h_\alpha^2}\Big[D^+_{\boldsymbol{n},\alpha\alpha}K^+_{\boldsymbol{n},\alpha\alpha}+D^-_{\boldsymbol{n},\alpha\alpha}K^-_{\boldsymbol{n},\alpha\alpha}\Big]+\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^d\frac{1}{h_\alpha h_\beta}\Big[D^+_{\boldsymbol{n},\alpha\beta}K^+_{\boldsymbol{n},\alpha\beta}+D^-_{\boldsymbol{n},\alpha\beta}K^-_{\boldsymbol{n},\alpha\beta}\Big],\tag{4.3}$$

where, for every $\alpha,\beta=1,\ldots,d$ with $\alpha\neq\beta$,

$$D^+_{\boldsymbol{n},\alpha\alpha}=\operatorname*{diag}_{\boldsymbol{j}=\boldsymbol{1},\ldots,\boldsymbol{n}}a_{\alpha\alpha}(\boldsymbol{x}_{\boldsymbol{j}+\frac12\boldsymbol{e}_\alpha}),\qquad D^-_{\boldsymbol{n},\alpha\alpha}=\operatorname*{diag}_{\boldsymbol{j}=\boldsymbol{1},\ldots,\boldsymbol{n}}a_{\alpha\alpha}(\boldsymbol{x}_{\boldsymbol{j}-\frac12\boldsymbol{e}_\alpha}),\tag{4.4}$$

$$D^+_{\boldsymbol{n},\alpha\beta}=\operatorname*{diag}_{\boldsymbol{j}=\boldsymbol{1},\ldots,\boldsymbol{n}}a_{\alpha\beta}(\boldsymbol{x}_{\boldsymbol{j}+\boldsymbol{e}_\alpha}),\qquad D^-_{\boldsymbol{n},\alpha\beta}=\operatorname*{diag}_{\boldsymbol{j}=\boldsymbol{1},\ldots,\boldsymbol{n}}a_{\alpha\beta}(\boldsymbol{x}_{\boldsymbol{j}-\boldsymbol{e}_\alpha}),\tag{4.5}$$

and the matrices $K^\pm_{\boldsymbol{n},\alpha\alpha}$, $K^\pm_{\boldsymbol{n},\alpha\beta}$ are defined by their action on a generic vector $\boldsymbol{u}=[u_{\boldsymbol{\ell}}]_{\boldsymbol{\ell}=\boldsymbol{1}}^{\boldsymbol{n}}\in\mathbb{R}^{N(\boldsymbol{n})}$, as follows:

$$(K^+_{\boldsymbol{n},\alpha\alpha}\boldsymbol{u})_{\boldsymbol{j}}=u_{\boldsymbol{j}}-u_{\boldsymbol{j}+\boldsymbol{e}_\alpha},\qquad (K^-_{\boldsymbol{n},\alpha\alpha}\boldsymbol{u})_{\boldsymbol{j}}=u_{\boldsymbol{j}}-u_{\boldsymbol{j}-\boldsymbol{e}_\alpha},\tag{4.6}$$

$$(K^+_{\boldsymbol{n},\alpha\beta}\boldsymbol{u})_{\boldsymbol{j}}=\frac{u_{\boldsymbol{j}+\boldsymbol{e}_\alpha-\boldsymbol{e}_\beta}-u_{\boldsymbol{j}+\boldsymbol{e}_\alpha+\boldsymbol{e}_\beta}}{4},\qquad (K^-_{\boldsymbol{n},\alpha\beta}\boldsymbol{u})_{\boldsymbol{j}}=\frac{u_{\boldsymbol{j}-\boldsymbol{e}_\alpha+\boldsymbol{e}_\beta}-u_{\boldsymbol{j}-\boldsymbol{e}_\alpha-\boldsymbol{e}_\beta}}{4},\tag{4.7}$$

for $\boldsymbol{j} = \boldsymbol{1}, \ldots, \boldsymbol{n}$. We remark that in (4.6)–(4.7) we use the convention $u_{\boldsymbol{i}} = 0$ for $\boldsymbol{i} \notin \{\boldsymbol{1}, \ldots, \boldsymbol{n}\}$. The next theorem is the d-dimensional version of Theorem 4.1. We denote by $\circ$ the componentwise (Hadamard) product of matrices.

Theorem 4.2. *Suppose that $A(\boldsymbol{x})$ is symmetric for every $\boldsymbol{x} \in (0,1)^d$ and the coefficients $a_{\alpha\beta} : [0,1]^d \to \mathbb{R}$ are continuous a.e. in $[0,1]^d$ for all $\alpha, \beta = 1, \ldots, d$. Let $\boldsymbol{\nu} \in \mathbb{Q}^d$ be a row vector with positive components and assume that $\boldsymbol{n} + \boldsymbol{1} = n\boldsymbol{\nu}$ (it is understood that n varies in the infinite subset of $\mathbb{N}$ such that $\boldsymbol{n} + \boldsymbol{1} = n\boldsymbol{\nu} \in \mathbb{N}^d$). Then*

$$\{n^{-2} A_{\boldsymbol{n}}\}_n \sim_{\mathrm{GLT},\sigma,\lambda} \sum_{\alpha,\beta=1}^{d} \nu_\alpha \nu_\beta a_{\alpha\beta}(\boldsymbol{x}) H_{\alpha\beta}(\boldsymbol{\theta}) = \boldsymbol{\nu}(A(\boldsymbol{x}) \circ H(\boldsymbol{\theta}))\boldsymbol{\nu}^T,$$

where $H(\boldsymbol{\theta})$ is the $d \times d$ symmetric matrix defined as follows:

$$H_{\alpha\beta}(\boldsymbol{\theta}) = \begin{cases} 2 - 2\cos\theta_\alpha, & \text{if } \alpha = \beta, \\ \sin\theta_\alpha \sin\theta_\beta, & \text{if } \alpha \neq \beta. \end{cases}$$

When the coefficients $a_{\alpha\beta}$ are continuous on $[0,1]^d$, Theorem 4.2 can be proved by standard GLT arguments without resorting to Theorem 3.1; see, e.g., [14, Section 7.3]. When the coefficients are only continuous a.e., we need Theorem 3.1 and the following lemmas. In what follows, for every $n \in \mathbb{N}$, we denote by K_n^+ and K_n^- the matrices in (4.1), by $I_n = T_n(1)$ the identity matrix of size n, by J_n^+ and J_n^- the Jordan matrices defined as $J_n^+ = I_n - K_n^+ = T_n(\mathrm{e}^{-\mathrm{i}\theta})$ and $J_n^- = I_n - K_n^- = T_n(\mathrm{e}^{\mathrm{i}\theta})$, and by H_n the $n \times n$ matrix

$$H_n = \frac{1}{2}\begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ & & & -1 & 0 \end{bmatrix} = -\mathrm{i}\, T_n(\sin\theta).$$

Lemma 4.1. *For every $\alpha, \beta = 1, \ldots, d$ with $\alpha \neq \beta$, we have*

$$\begin{aligned} K^+_{\boldsymbol{n},\alpha\alpha} &= \left(\textstyle\bigotimes_{r=1}^{\alpha-1} I_{n_r}\right) \otimes K^+_{n_\alpha} \otimes \left(\textstyle\bigotimes_{r=\alpha+1}^{d} I_{n_r}\right) \\ &= T_{\boldsymbol{n}}(1 - \mathrm{e}^{-\mathrm{i}\theta_\alpha}), \end{aligned} \tag{4.8}$$

$$\begin{aligned} K^-_{\boldsymbol{n},\alpha\alpha} &= \left(\textstyle\bigotimes_{r=1}^{\alpha-1} I_{n_r}\right) \otimes K^-_{n_\alpha} \otimes \left(\textstyle\bigotimes_{r=\alpha+1}^{d} I_{n_r}\right) \\ &= T_{\boldsymbol{n}}(1 - \mathrm{e}^{\mathrm{i}\theta_\alpha}), \end{aligned} \tag{4.9}$$

$$\begin{aligned} K^+_{\boldsymbol{n},\alpha\beta} &= \begin{cases} -\frac{1}{2}\left(\bigotimes_{r=1}^{\alpha-1} I_{n_r}\right) \otimes J^+_{n_\alpha} \otimes \left(\bigotimes_{r=\alpha+1}^{\beta-1} I_{n_r}\right) \otimes H_{n_\beta} \otimes \left(\bigotimes_{r=\beta+1}^{d} I_{n_r}\right), & \text{if } \alpha < \beta, \\ -\frac{1}{2}\left(\bigotimes_{r=1}^{\beta-1} I_{n_r}\right) \otimes H_{n_\beta} \otimes \left(\bigotimes_{r=\beta+1}^{\alpha-1} I_{n_r}\right) \otimes J^+_{n_\alpha} \otimes \left(\bigotimes_{r=\alpha+1}^{d} I_{n_r}\right), & \text{if } \alpha > \beta, \end{cases} \\ &= \frac{\mathrm{i}}{2}\, T_{\boldsymbol{n}}(\mathrm{e}^{-\mathrm{i}\theta_\alpha} \sin\theta_\beta), \end{aligned} \tag{4.10}$$

$$\begin{aligned} K^-_{\boldsymbol{n},\alpha\beta} &= \begin{cases} \frac{1}{2}\left(\bigotimes_{r=1}^{\alpha-1} I_{n_r}\right) \otimes J^-_{n_\alpha} \otimes \left(\bigotimes_{r=\alpha+1}^{\beta-1} I_{n_r}\right) \otimes H_{n_\beta} \otimes \left(\bigotimes_{r=\beta+1}^{d} I_{n_r}\right), & \text{if } \alpha < \beta, \\ \frac{1}{2}\left(\bigotimes_{r=1}^{\beta-1} I_{n_r}\right) \otimes H_{n_\beta} \otimes \left(\bigotimes_{r=\beta+1}^{\alpha-1} I_{n_r}\right) \otimes J^-_{n_\alpha} \otimes \left(\bigotimes_{r=\alpha+1}^{d} I_{n_r}\right), & \text{if } \alpha > \beta, \end{cases} \\ &= -\frac{\mathrm{i}}{2}\, T_{\boldsymbol{n}}(\mathrm{e}^{\mathrm{i}\theta_\alpha} \sin\theta_\beta). \end{aligned} \tag{4.11}$$

Proof. We only prove (4.8) as the proofs of (4.9)–(4.11) are completely analogous. The second equality in (4.8) follows directly from **T2** and the equations $I_n = T_n(1)$ and $K_n^+ = T_n(1 - \mathrm{e}^{-\mathrm{i}\theta})$. We prove the first equality by showing that the matrices in the left- and right-hand side act in the same way on a generic vector $\boldsymbol{u} \in \mathbb{R}^{N(\boldsymbol{n})}$. Using **P4** and the convention $u_{\boldsymbol{i}} = 0$ for $\boldsymbol{i} \notin \{\boldsymbol{1}, \ldots, \boldsymbol{n}\}$, for every $\boldsymbol{u} = [u_{\boldsymbol{\ell}}]_{\boldsymbol{\ell}=\boldsymbol{1}}^{\boldsymbol{n}} \in \mathbb{R}^{N(\boldsymbol{n})}$ and every $\boldsymbol{j} = \boldsymbol{1}, \ldots, \boldsymbol{n}$ we obtain

$$\begin{aligned} \left[\left(\left(\bigotimes_{r=1}^{\alpha-1} I_{n_r}\right) \otimes K^+_{n_\alpha} \otimes \left(\bigotimes_{r=\alpha+1}^{d} I_{n_r}\right)\right)\boldsymbol{u}\right]_{\boldsymbol{j}} &= \sum_{\boldsymbol{\ell}=\boldsymbol{1}}^{\boldsymbol{n}} \left[\left(\bigotimes_{r=1}^{\alpha-1} I_{n_r}\right) \otimes K^+_{n_\alpha} \otimes \left(\bigotimes_{r=\alpha+1}^{d} I_{n_r}\right)\right]_{\boldsymbol{j\ell}} u_{\boldsymbol{\ell}} \\ &= \sum_{\boldsymbol{\ell}=\boldsymbol{1}}^{\boldsymbol{n}} u_{\boldsymbol{\ell}}\, (K^+_{n_\alpha})_{j_\alpha \ell_\alpha} \prod_{\substack{r=1 \\ r\neq\alpha}}^{d} (I_{n_r})_{j_r \ell_r} = u_{\boldsymbol{j}} - u_{\boldsymbol{j}+\boldsymbol{e}_\alpha} = (K^+_{\boldsymbol{n},\alpha\alpha}\mathbf{u})_{\boldsymbol{j}}, \end{aligned}$$

where the second-to-last equality follows from the fact that, when $\boldsymbol{\ell}$ varies from $\mathbf{1}$ to $\boldsymbol{n}$,

$$(K_{n_\alpha}^+)_{j_\alpha\ell_\alpha}\prod_{\substack{r=1\\r\neq\alpha}}^d (I_{n_r})_{j_r\ell_r}=\begin{cases}1,\ -1, & \text{if } \boldsymbol{\ell}=\boldsymbol{j},\ \boldsymbol{j}+\boldsymbol{e}_\alpha, \text{ respectively},\\ 0, & \text{otherwise.}\end{cases}\qquad\square$$

Lemma 4.2. *If $A(\boldsymbol{x})$ is symmetric for every $\boldsymbol{x}\in(0,1)^d$ then the matrix $A_{\boldsymbol{n}}$ is symmetric.*

Proof. In view of (4.3), it suffices to prove that, for every $\alpha,\beta=1,\ldots,d$ with $\alpha\neq\beta$,

$$(D_{\boldsymbol{n},\alpha\alpha}^+K_{\boldsymbol{n},\alpha\alpha}^+ + D_{\boldsymbol{n},\alpha\alpha}^-K_{\boldsymbol{n},\alpha\alpha}^-)^T = D_{\boldsymbol{n},\alpha\alpha}^+K_{\boldsymbol{n},\alpha\alpha}^+ + D_{\boldsymbol{n},\alpha\alpha}^-K_{\boldsymbol{n},\alpha\alpha}^-, \tag{4.12}$$

$$(D_{\boldsymbol{n},\alpha\beta}^+K_{\boldsymbol{n},\alpha\beta}^+ + D_{\boldsymbol{n},\alpha\beta}^-K_{\boldsymbol{n},\alpha\beta}^-)^T = D_{\boldsymbol{n},\beta\alpha}^+K_{\boldsymbol{n},\beta\alpha}^+ + D_{\boldsymbol{n},\beta\alpha}^-K_{\boldsymbol{n},\beta\alpha}^-. \tag{4.13}$$

Actually, we only prove (4.13) as the proof of (4.12) is completely analogous. We show that the matrices in the left- and right-hand side of (4.13) act in the same way on a generic vector $\boldsymbol{u}\in\mathbb{R}^{N(\boldsymbol{n})}$. By Lemma 4.1, **P3** and the equations $(J_n^+)^T=J_n^-$ and $H_n^T=-H_n$, we have

$$(D_{\boldsymbol{n},\alpha\beta}^+K_{\boldsymbol{n},\alpha\beta}^+ + D_{\boldsymbol{n},\alpha\beta}^-K_{\boldsymbol{n},\alpha\beta}^-)^T = K_{\boldsymbol{n},\alpha\beta}^-D_{\boldsymbol{n},\alpha\beta}^+ + K_{\boldsymbol{n},\alpha\beta}^+D_{\boldsymbol{n},\alpha\beta}^-.$$

Keeping in mind (4.4)–(4.7) and the usual convention $u_{\boldsymbol{i}}=0$ for $\boldsymbol{i}\notin\{\mathbf{1},\ldots,\boldsymbol{n}\}$, for every $\boldsymbol{u}=[u_{\boldsymbol{\ell}}]_{\boldsymbol{\ell}=\mathbf{1}}^{\boldsymbol{n}}\in\mathbb{R}^{N(\boldsymbol{n})}$ and every $\boldsymbol{j}=\mathbf{1},\ldots,\boldsymbol{n}$ we obtain

$$\begin{aligned}\big[(D_{\boldsymbol{n},\alpha\beta}^+K_{\boldsymbol{n},\alpha\beta}^+ + D_{\boldsymbol{n},\alpha\beta}^-K_{\boldsymbol{n},\alpha\beta}^-)^T\boldsymbol{u}\big]_{\boldsymbol{j}} &= \big[K_{\boldsymbol{n},\alpha\beta}^-D_{\boldsymbol{n},\alpha\beta}^+\boldsymbol{u} + K_{\boldsymbol{n},\alpha\beta}^+D_{\boldsymbol{n},\alpha\beta}^-\boldsymbol{u}\big]_{\boldsymbol{j}}\\ &= \big[K_{\boldsymbol{n},\alpha\beta}^-[a_{\alpha\beta}(\boldsymbol{x}_{\boldsymbol{\ell}+\boldsymbol{e}_\alpha})u_{\boldsymbol{\ell}}]_{\boldsymbol{\ell}=\mathbf{1}}^{\boldsymbol{n}} + K_{\boldsymbol{n},\alpha\beta}^+[a_{\alpha\beta}(\boldsymbol{x}_{\boldsymbol{\ell}-\boldsymbol{e}_\alpha})u_{\boldsymbol{\ell}}]_{\boldsymbol{\ell}=\mathbf{1}}^{\boldsymbol{n}}\big]_{\boldsymbol{j}}\\ &= \frac{a_{\alpha\beta}(\boldsymbol{x}_{\boldsymbol{j}+\boldsymbol{e}_\beta})u_{\boldsymbol{j}-\boldsymbol{e}_\alpha+\boldsymbol{e}_\beta} - a_{\alpha\beta}(\boldsymbol{x}_{\boldsymbol{j}-\boldsymbol{e}_\beta})u_{\boldsymbol{j}-\boldsymbol{e}_\alpha-\boldsymbol{e}_\beta}}{4}\\ &\quad + \frac{a_{\alpha\beta}(\boldsymbol{x}_{\boldsymbol{j}-\boldsymbol{e}_\beta})u_{\boldsymbol{j}+\boldsymbol{e}_\alpha-\boldsymbol{e}_\beta} - a_{\alpha\beta}(\boldsymbol{x}_{\boldsymbol{j}+\boldsymbol{e}_\beta})u_{\boldsymbol{j}+\boldsymbol{e}_\alpha+\boldsymbol{e}_\beta}}{4}.\end{aligned}$$

Similarly,

$$\begin{aligned}\big[(D_{\boldsymbol{n},\beta\alpha}^+K_{\boldsymbol{n},\beta\alpha}^+ + D_{\boldsymbol{n},\beta\alpha}^-K_{\boldsymbol{n},\beta\alpha}^-)\boldsymbol{u}\big]_{\boldsymbol{j}} &= \frac{a_{\beta\alpha}(\boldsymbol{x}_{\boldsymbol{j}+\boldsymbol{e}_\beta})u_{\boldsymbol{j}+\boldsymbol{e}_\beta-\boldsymbol{e}_\alpha} - a_{\beta\alpha}(\boldsymbol{x}_{\boldsymbol{j}+\boldsymbol{e}_\beta})u_{\boldsymbol{j}+\boldsymbol{e}_\beta+\boldsymbol{e}_\alpha}}{4}\\ &\quad + \frac{a_{\beta\alpha}(\boldsymbol{x}_{\boldsymbol{j}-\boldsymbol{e}_\beta})u_{\boldsymbol{j}-\boldsymbol{e}_\beta+\boldsymbol{e}_\alpha} - a_{\beta\alpha}(\boldsymbol{x}_{\boldsymbol{j}-\boldsymbol{e}_\beta})u_{\boldsymbol{j}-\boldsymbol{e}_\beta-\boldsymbol{e}_\alpha}}{4}.\end{aligned}$$

Since $a_{\beta\alpha}(\boldsymbol{x})=a_{\alpha\beta}(\boldsymbol{x})$ for all $\boldsymbol{x}\in(0,1)^d$ by the symmetry assumption on $A(\boldsymbol{x})$, we conclude that

$$\big[(D_{\boldsymbol{n},\alpha\beta}^+K_{\boldsymbol{n},\alpha\beta}^+ + D_{\boldsymbol{n},\alpha\beta}^-K_{\boldsymbol{n},\alpha\beta}^-)^T\boldsymbol{u}\big]_{\boldsymbol{j}} = \big[(D_{\boldsymbol{n},\beta\alpha}^+K_{\boldsymbol{n},\beta\alpha}^+ + D_{\boldsymbol{n},\beta\alpha}^-K_{\boldsymbol{n},\beta\alpha}^-)\boldsymbol{u}\big]_{\boldsymbol{j}}$$

for $\boldsymbol{j}=\mathbf{1},\ldots,\boldsymbol{n}$ and $\boldsymbol{u}\in\mathbb{R}^{N(\boldsymbol{n})}$, which immediately gives (4.13). $\square$

Proof of Theorem 4.2. By the decomposition (4.3) and the equation $\boldsymbol{n}+\mathbf{1}=n\boldsymbol{\nu}$, we have

$$n^{-2}A_{\boldsymbol{n}}=\sum_{\alpha=1}^d\nu_\alpha^2\Big[D_{\boldsymbol{n},\alpha\alpha}^+K_{\boldsymbol{n},\alpha\alpha}^+ + D_{\boldsymbol{n},\alpha\alpha}^-K_{\boldsymbol{n},\alpha\alpha}^-\Big]+\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^d\nu_\alpha\nu_\beta\Big[D_{\boldsymbol{n},\alpha\beta}^+K_{\boldsymbol{n},\alpha\beta}^+ + D_{\boldsymbol{n},\alpha\beta}^-K_{\boldsymbol{n},\alpha\beta}^-\Big].$$

It is easy to check that the grids

$$\begin{aligned}\mathcal{G}_{\boldsymbol{n},\alpha}^+&=\{\boldsymbol{x}_{\boldsymbol{j}+\frac12\boldsymbol{e}_\alpha}\}_{\boldsymbol{j}=\mathbf{1},\ldots,\boldsymbol{n}}, &\qquad \mathcal{G}_{\boldsymbol{n},\alpha}^-&=\{\boldsymbol{x}_{\boldsymbol{j}-\frac12\boldsymbol{e}_\alpha}\}_{\boldsymbol{j}=\mathbf{1},\ldots,\boldsymbol{n}},\\ \mathcal{H}_{\boldsymbol{n},\alpha}^+&=\{\boldsymbol{x}_{\boldsymbol{j}+\boldsymbol{e}_\alpha}\}_{\boldsymbol{j}=\mathbf{1},\ldots,\boldsymbol{n}}, &\qquad \mathcal{H}_{\boldsymbol{n},\alpha}^-&=\{\boldsymbol{x}_{\boldsymbol{j}-\boldsymbol{e}_\alpha}\}_{\boldsymbol{j}=\mathbf{1},\ldots,\boldsymbol{n}},\end{aligned}$$

are a.u. in $[0,1]^d$. Hence, by Theorem 3.1,

$$\begin{aligned}\{D_{\boldsymbol{n},\alpha\alpha}^+\}_n&\sim_{\rm GLT}a(\boldsymbol{x}), &\qquad \{D_{\boldsymbol{n},\alpha\alpha}^-\}_n&\sim_{\rm GLT}a(\boldsymbol{x}),\\ \{D_{\boldsymbol{n},\alpha\beta}^+\}_n&\sim_{\rm GLT}a(\boldsymbol{x}), &\qquad \{D_{\boldsymbol{n},\alpha\beta}^-\}_n&\sim_{\rm GLT}a(\boldsymbol{x}).\end{aligned}$$

We then infer from Lemma 4.1 and **GLT3–GLT4** that

$$\begin{aligned}\{n^{-2}A_{\boldsymbol{n}}\}_n &\sim_{\mathrm{GLT}} \sum_{\alpha=1}^{d}\nu_\alpha^2\Big[a(\boldsymbol{x})(1-\mathrm{e}^{-\mathrm{i}\theta_\alpha})+a(\boldsymbol{x})(1-\mathrm{e}^{\mathrm{i}\theta_\alpha})\Big]+\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{d}\nu_\alpha\nu_\beta\Big[a(\boldsymbol{x})\frac{\mathrm{i}}{2}\,\mathrm{e}^{-\mathrm{i}\theta_\alpha}\sin\theta_\beta-a(\boldsymbol{x})\frac{\mathrm{i}}{2}\,\mathrm{e}^{\mathrm{i}\theta_\alpha}\sin\theta_\beta\Big]\\
&=\sum_{\alpha=1}^{d}\nu_\alpha^2\,a(\boldsymbol{x})(2-2\cos\theta_\alpha)+\sum_{\substack{\alpha,\beta=1\\ \alpha\neq\beta}}^{d}\nu_\alpha\nu_\beta\,a(\boldsymbol{x})\sin\theta_\alpha\sin\theta_\beta\\
&=\sum_{\alpha,\beta=1}^{d}\nu_\alpha\nu_\beta\,a(\boldsymbol{x})H_{\alpha\beta}(\boldsymbol{\theta})=\boldsymbol{\nu}(A(\boldsymbol{x})\circ H(\boldsymbol{\theta}))\boldsymbol{\nu}^T,\end{aligned}$$

and we finally obtain $\{n^{-2}A_{\boldsymbol{n}}\}_n\sim_{\sigma,\lambda}\boldsymbol{\nu}(A(\boldsymbol{x})\circ H(\boldsymbol{\theta}))\boldsymbol{\nu}^T$ by **GLT1** and the symmetry of $A_{\boldsymbol{n}}$, which is a consequence of Lemma 4.2 and the symmetry assumption on $A(\boldsymbol{x})$. ☐

5 Conclusions

We have extended the theory of GLT sequences by showing in Theorem 3.1 that any sequence of diagonal sampling matrices constructed from a.u. samples of an a.e. continuous function falls in the class of GLT sequences. We have also detailed a few representative applications of Theorem 3.1 in the context of FD discretizations of DEs with coefficients that are only supposed to be continuous a.e. We conclude by highlighting that the applicability of Theorem 3.1 is not confined to FD discretizations. In this regard, it is worth pointing out the following.

- The present paper was inspired by a work in progress regarding the GLT spectral analysis of matrices arising from isogeometric Galerkin immersed methods [17]. Theorem 3.1 well applies in this framework, where one has to deal with grids $\mathcal{G}_{\boldsymbol{n}}=\{\boldsymbol{x}_{\boldsymbol{i},\boldsymbol{n}}\}_{\boldsymbol{i}=\boldsymbol{1},\ldots,\boldsymbol{n}}$ that are a.u. in $[0,1]^d$ but with some of the grid points $\boldsymbol{x}_{\boldsymbol{i},\boldsymbol{n}}$ lying outside $[0,1]^d$. We remark that these "outliers" belong to the computational domain where the physical domain is immersed.
- When dealing with B-spline isogeometric collocation methods [1], a common choice for the collocation points is given by the Greville abscissae associated with the considered B-splines. In this context, diagonal sampling matrices of the form $D_{\boldsymbol{n}}^{\mathcal{G}_{\boldsymbol{n}}}(a)$, with $\mathcal{G}_{\boldsymbol{n}}=\{\boldsymbol{\xi}_{\boldsymbol{i},\boldsymbol{n}}\}_{\boldsymbol{i}=\boldsymbol{1},\ldots,\boldsymbol{n}}$ and $\boldsymbol{\xi}_{\boldsymbol{1},\boldsymbol{n}},\ldots,\boldsymbol{\xi}_{\boldsymbol{n},\boldsymbol{n}}$ being the used Greville abscissae, naturally arise. The grid $\mathcal{G}_{\boldsymbol{n}}$ is not uniform but it is a.u. in $[0,1]^d$ according to our definition; see [13, Section 10.7.1] for the unidimensional case $d=1$ and [14, Section 7.5] for the multidimensional case $d>1$. It is then clear that Theorem 3.1 applies even in this framework.

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References

[1] Auricchio F., Beirão da Veiga L., Hughes T. J. R., Reali A., Sangalli G. *Isogeometric collocation methods.* Math. Models Methods Appl. Sci. 20 (2010) 2075–2107.

[2] Barbarino G. *Equivalence between GLT sequences and measurable functions.* Linear Algebra Appl. 529 (2017) 397–412.

[3] Barbarino G. *Normal form for GLT sequences.* arXiv:1805.08708 (2018).

[4] Barbarino G. *A systematic approach to reduced GLT.* BIT Numer. Math. (2021) https://doi.org/10.1007/s10543-021-00896-7.

[5] Barbarino G., Garoni C. *From convergence in measure to convergence of matrix-sequences through concave functions and singular values.* Electron. J. Linear Algebra 32 (2017) 500–513.

[6] Barbarino G., Garoni C., Serra-Capizzano S. *Block generalized locally Toeplitz sequences: theory and applications in the unidimensional case.* Electron. Trans. Numer. Anal. 53 (2020) 28–112.

[7] Barbarino G., Garoni C., Serra-Capizzano S. *Block generalized locally Toeplitz sequences: theory and applications in the multidimensional case.* Electron. Trans. Numer. Anal. 53 (2020) 113–216.

[8] Barbarino G., Serra-Capizzano S. *Non-Hermitian perturbations of Hermitian matrix-sequences and applications to the spectral analysis of the numerical approximation of partial differential equations.* Numer. Linear Algebra Appl. 27 (2020) e2286.

[9] Bhatia R. *Matrix Analysis.* Springer (1997).

[10] Bianchi D. *Analysis of the spectral symbol associated to discretization schemes of linear self-adjoint differential operators.* Calcolo 58 (2021) 38.

[11] Böttcher A., Garoni C., Serra-Capizzano S. *Exploration of Toeplitz-like matrices with unbounded symbols is not a purely academic journey.* Sb. Math. 208 (2017) 1602–1627.

[12] Garoni C., Mazza M., Serra-Capizzano S. *Block generalized locally Toeplitz sequences: from the theory to the applications.* Axioms 7 (2018) 49.

[13] Garoni C., Serra-Capizzano S. *Generalized Locally Toeplitz Sequences: Theory and Applications (Volume I).* Springer (2017).

[14] Garoni C., Serra-Capizzano S. *Generalized Locally Toeplitz Sequences: Theory and Applications (Volume II).* Springer (2018).

[15] Garoni C., Serra-Capizzano S. *Generalized locally Toeplitz sequences: a spectral analysis tool for discretized differential equations.* Lect. Notes Math. 2219 (2018) 161–236.

[16] Garoni C., Serra-Capizzano S. *Multilevel generalized locally Toeplitz sequences: an overview and an example of application.* AIP Conf. Proc. 2116 (2019) 020003.

[17] Pelosi F., Giannelli C., Manni C., Sampoli M. L., Speleers H. *Splines over regular triangulations in numerical simulation.* Comput. Aided Des. 82 (2017) 100–111.