# Ergodic Estimations for Toeplitz Sequences

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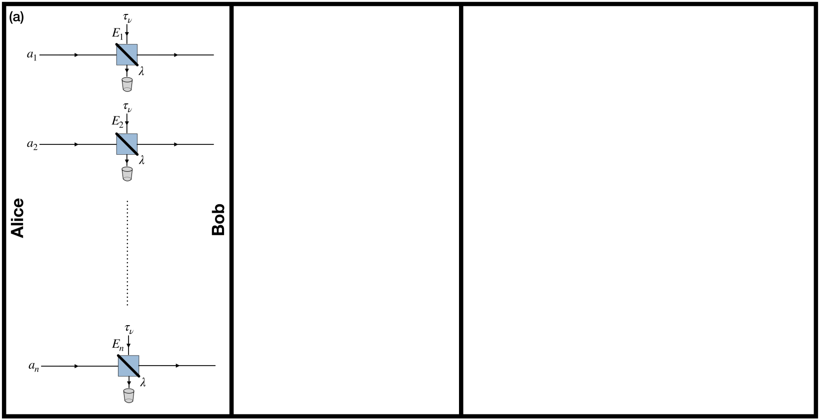
# Application

# Long-Distance Quantum Communication

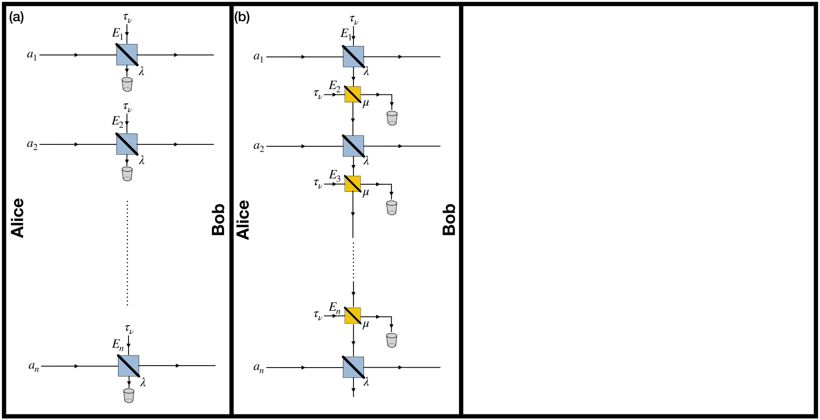
Quantum communication over long distance ($>$ 21 km on optical fibres) is very vulnerable to noise

# Long-Distance Quantum Communication




Sending several signals does not improve the transmitted quantity of information

# Long-Distance Quantum Communication




If the signal is transmitted in short time intervals the *memory effects* influence locally the noise in controlled ways

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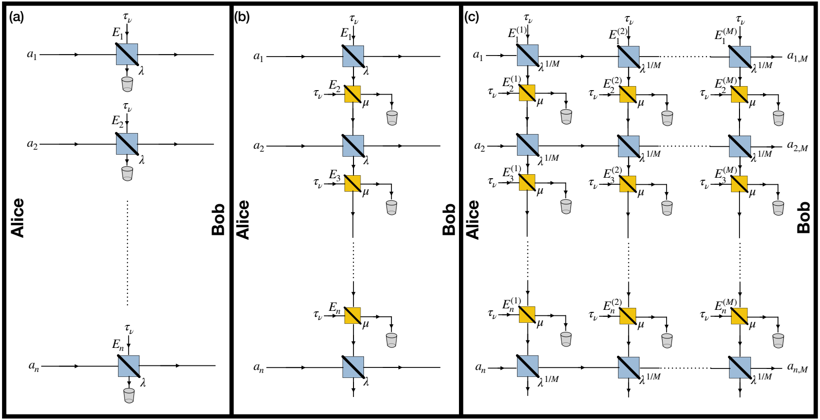


The *Delocalised Interaction Model* takes account of memory effects through the whole length of the optical fibre

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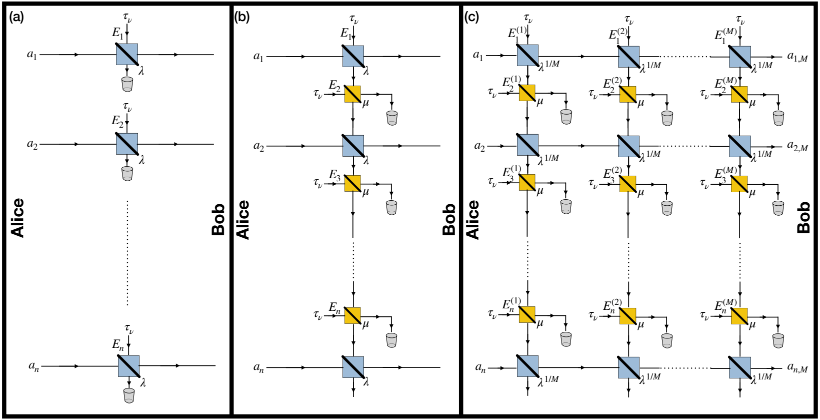


DIM Model : Quantum Channel $\Phi^{(n)}_{\lambda,\mu}$

- $\lambda \in (0, 1]$ signal trasmissivity (length, material, ...)
- $\mu \in (0, 1]$ memory parameter (time interval, ...)
- $n$ **number of input signals**

# Capacity of the Quantum Channel

**Quantum Capacity :** $Q(\{\Phi^{(n)}_{\lambda,\mu}\}_n)$ is the maximum rate of qubits correctly transmitted *at steady state*

**Theorem (F.A.M., M.F. et al, 2023)**

There exists a function $F \in C_c(\mathbb{R})$ such that

$$Q(\{\Phi^{(n)}_{\lambda,\mu}\}_n) = \lim_{n\to\infty} \frac{1}{n} F(\sigma_i(T_n(\phi(\theta)))^2) = \frac{1}{2\pi}\int_0^{2\pi} F(|\phi(\theta)|^2)d\theta$$

where

$$\phi(\theta) = \lambda^{-\frac{1}{2}+\frac{1}{1-\sqrt{\mu}\exp(\mathbf{i}\theta)}} \in C^{\infty}_{per}(\mathbb{R})$$

The quantity $\frac{1}{n}F(\sigma_i(T_n(\phi(\theta)))^2)$ represents the capacity of the quantum channel for a finite number of input signals and it converges when a stable flow of informations is achieved

Studying how well $\sigma_i(T_n(\phi(\theta)))$ approximate $|\phi(\theta)|$ gives us an explicit estimation on **how many signals we need to ensure that the Quantum Channel operates near full capacity**

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# Widom Approximation

# Widom Asymptotic Result

**Theorem (Widom, 1989)**

For any function $\phi(\theta) \in L^\infty[-\pi,\pi]$ with $|||\phi|||^2 := \sum_{k=-\infty}^{\infty} |k||\phi_k|^2 < \infty$ and for any $C_c^3$ function $F(x)$ we have

$$\lim_{n\to\infty} \left\{ \sum_{j=1}^{n} F(\sigma_j(T_n(\phi))^2) - \frac{n}{2\pi}\int_{-\pi}^{\pi} F(|\phi(\theta)|^2)d\theta \right\}$$
$$= Tr[F(T(\overline{\phi})T(\phi)) + F(T(\phi)T(\overline{\phi})) - 2T(F(|\phi|^2))]$$

The hypotheses are satisfied already for $\phi \in C^1_{per}[-\pi,\pi]$, so

$$\left| \frac{1}{n}\sum_{j=1}^{n} F(\sigma_j(T_n(\phi))^2) - \frac{1}{2\pi}\int_{-\pi}^{\pi} F(|\phi(\theta)|^2)d\theta \right| = \Theta\left(\frac{1}{n}\right)$$

**Problems :**

- All available estimations are asymptotic formulae, instead we want explicit bounds for any finite $n$
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**Theorem (G.B., 2023)**

Let $\phi \in L^\infty([-\pi,\pi])$ with $|||\phi|||^2 < \infty$ and let $F \in C_c^3(\mathbb{R})$. Then, for every $n \geq 1$,

$$\left|\sum_{j=1}^{n} F(\sigma_j(T_n(\phi))^2) - \frac{n}{2\pi}\int_{-\pi}^{\pi} F(|\phi(\theta)|^2)d\theta\right| \leq 2|||\phi|||^2 \left[c_1 + 2c_2\|\phi\|_\infty^2\right]$$

where $c_1 = 2\|F'\|_1 + \sqrt{2}\|F''\|_2$ and $c_2 = 2\|F''\|_1 + \sqrt{2}\|F'''\|_2$.

**Problem :**

- For some communication protocol,

$$Q(\{\Phi_{\lambda,\mu}^{(n)}\}_n) = \lim_{n\to\infty} \frac{1}{n} F(\sigma_i(T_n(\phi(\theta)))^2)$$

holds for $F(x) = \max\left\{0, \log_2\left(\frac{x}{1-x}\right)\right\} \notin C^1(\mathbb{R})$

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# Circulant Approximation

## Step-by-Step Approximation

$$\textbf{Idea :}\quad T_n(\phi) \to T_n(\phi_N) \to C_n(\phi_N) \to \phi_N \to \phi$$

$$\left| \frac{1}{n}\sum_{j=1}^{n} F(\sigma_j(T_n(\phi))) - \frac{1}{2\pi}\int_{-\pi}^{\pi} F(|\phi(\theta)|)d\theta \right| \leq$$

$$\left| \frac{1}{n}\sum_{j=1}^{n} F(\sigma_j(T_n(\phi))) - \frac{1}{n}\sum_{j=1}^{n} F(\sigma_j(T_n(\phi_N))) \right| \qquad \leq \frac{L}{\sqrt{2\pi}}\|\phi - \phi_N\|_2 \quad (F \in Lip(L),\ \phi \in L^2[-\pi,\pi])$$

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# Explicit Bounds

**Theorem (G.B., 2023)**

Let $\phi \in L^2([-\pi,\pi])$ and $F \in C_c(\mathbb{R}) \cap Lip(L)$. Then for every $n \geq 4$

$$\left| \frac{1}{n}\sum_{j=1}^{n} F(\sigma_j(T_n(\phi))) - \frac{1}{2\pi}\int_{-\pi}^{\pi} F(|\phi(\theta)|)d\theta \right| \leq$$

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Using $N = \lfloor n^{\frac{1}{1+k}} \rfloor$ and

$$\phi \in C^k_{per}[-\pi,\pi] \implies \|\phi - \phi_N\|_2 \leq \frac{\|\phi^{(k)}\|_2}{N^k} \qquad k \geq 1 \implies \|\phi_N'\|_\infty \leq \sqrt{2N}\|\phi'\|_2$$

we get for $n > 4$

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# Conclusions and Future Works

For $F \in C_c^3(\mathbb{R})$ and $\phi \in L^\infty$, $|||\phi||| < \infty$ Widom's approximation gives us an explicit constant $C_W$ such that for any $n$

$$\left| \frac{1}{n} \sum_{j=1}^{n} F(\sigma_j(T_n(\phi))^2) - \frac{1}{2\pi} \int_{-\pi}^{\pi} F(|\phi(\theta)|^2) d\theta \right| \leq C_W \frac{1}{n}$$

For $F \in C_c(\mathbb{R}) \cup Lip(L)$ and $\phi \in C^k_{per}[-\pi, \pi]$, the circulant algebra gives us explicit constants $C_1$, $C_2$ such that for any $n > 4$

$$\left| \frac{1}{n} \sum_{j=1}^{n} F(\sigma_j(T_n(\phi))) - \frac{1}{2\pi} \int_{-\pi}^{\pi} F(|\phi(\theta)|) d\theta \right| \leq C_1 \frac{1}{n^{\frac{k}{1+k}}} + C_2 \frac{1}{n^{\frac{1/2+k}{1+k}}}$$

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**Still to do:**

- There exists a $O(1/n)$ bound for the second case?
- Can we say more if we impose $\phi$ smooth or analytic?
- Can something be said when $F$ is an indicator function?

# Thank You!


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