Computing cone-constrained singular values of matrices

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Class of Computational Complexity

Cones and Notation

P and Q are convex, closed and finitely generated cones

$$P = G(\mathbb{R}^p_+) = \{Gx \mid x \geq 0\}$$

$$Q = H(\mathbb{R}^q_+) = \{Hy \mid y \geq 0\}$$

where the generators $G = [g_1, \ldots, g_p]$, $H = [h_1, \ldots, h_q]$ have nonnegative independent unit columns

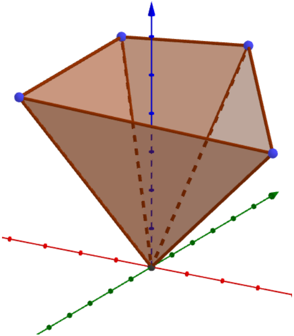
The ***faces*** of P are the polyhedral cones $F \subseteq P$ such that

$$v_1 + v_2 \in F, \quad v_1, v_2 \in P \implies v_1, v_2 \in F$$

The dimension of a face is equal to the dimension of the subspace spanned

A ***vertex*** of P is any nonzero vector of a 1-dimensional face of P, or any positive multiple of the columns of G

The ***facets*** of the cone P are the faces of maximal dimension different from P



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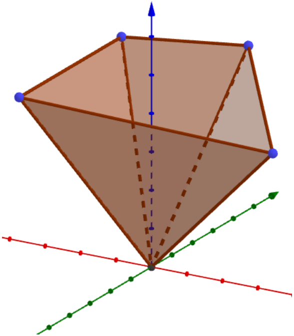

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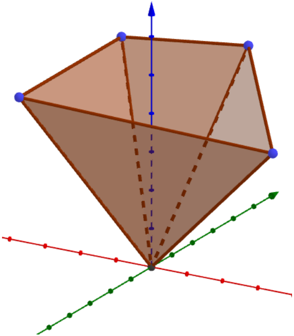

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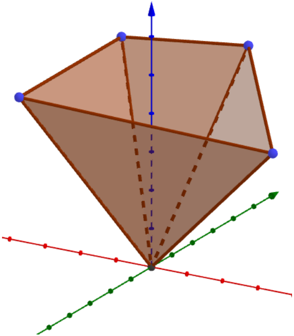

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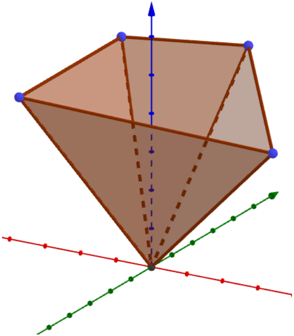

Conical Singular Values

$$\min_{\substack{u \in P,\ \|u\| = 1, \\ v \in Q,\ \|v\| = 1,}} \quad u^\top A v \qquad P, Q \quad \begin{array}{l}\text{closed convex cones} \\ \text{finitely generated}\end{array}$$

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Pareto Singular Values

$$\min_{\substack{u \geq 0,\ \|u\| = 1,\\ v \geq 0,\ \|v\| = 1,}} \quad u^\top A v$$

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Conic Angles

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Singular Values

$$\min_{\|u\|=\|v\|=1} u^\top A v$$

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Polynomial Time
$O(mn^2)$ to compute
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Pareto Singular Values

$$\min_{\substack{u \geq 0,\ \|u\| = 1,\\ v \geq 0,\ \|v\| = 1,}} u^\top A v \qquad A \in \mathbb{R}^{m \times n} \qquad \Xi(A) := \{\text{stationary values of the o.p.}\}$$

"Simple" Case: $A \geq 0$

Theorem (Seeger, S. 2023)

$$A \geq 0 \iff \Xi(A) = \{\|B\| : B \trianglelefteq A\}$$

Theorem

The set of matrices $A \in \mathbb{R}_+^{m \times n}$ for which all the submatrices have different norms, is dense, open and its complementary has measure zero on the space of nonnegative matrices

$\implies$ A generic nonnegative matrix has **exponentially** many Pareto Singular Values

Still it does say nothing on the complexity of recovering the minimum, since

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It is possible to extend the above results to any **Nash Pair**:

- (u^*, v^*) is a Nash Pair of the problem if

  $$\langle u, Av^* \rangle \geq \langle u^*, Av^* \rangle, \qquad \langle u^*, Av \rangle \geq \langle u^*, Av^* \rangle, \qquad \forall\, (u, v) \in P \times Q$$

  where u, v, u^*, v^* have all unit norm

- If (u^*, v^*) is a Nash Pair of the problem, then $u = g_i$, $v = h_j$ for some i, j

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$$\min_{\substack{u \in P,\ \|u\| = 1, \\ v \in Q,\ \|v\| = 1,}} u^\top v \qquad P, Q \subseteq \mathbb{R}^n \text{ non trivial polyhedral cones}$$

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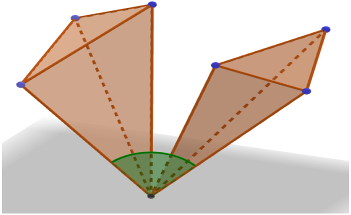

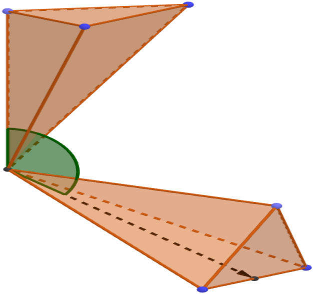

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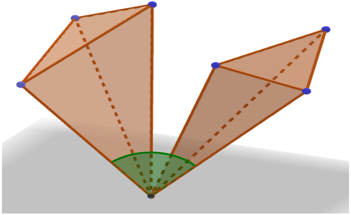

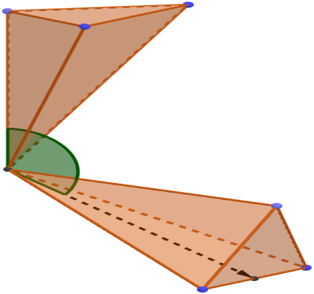

Conic Angles

If one of u, v in the antipodal pair is a vertex then the problem is **Polynomial** in n and the number of generators of P, Q

$$\min_{v \in Q,\, \|v\|=1} u^\top v = -\max_{v \in -Q,\, \|v\|=1} u^\top v \implies v = -\frac{Proj(u,-Q)}{\|Proj(u,-Q)\|}$$

$$Proj(u,-Q) \equiv \min_{y \geq 0} \|u - (-H)y\| \qquad \text{NNLS, convex}$$

if $\min_{\substack{u \in P,\ \|u\|=1, \\ v \in Q,\ \|v\|=1,}} u^\top v < 0$, when is it that one among u, v is a vertex?

Theorem (B., G., S. 2024)

Let (u, v) be a stationary point and let $u \in int(F_u)$, $v \in int(F_v)$ where F_u, F_v are facets of P, Q. If $\dim(F_u) + \dim(F_v) > n$ and $v \neq \pm u$, then (u, v) is a saddle point

Corollary (B., G., S. 2024)

If (u, v) is a local minimum in dimension $n \leq 3$ with $u \neq -v$, then at least one among u, v is a vertex

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- $\dim(F_u) + \dim(F_v) > n$ then (u, v) is a saddle point
- (u, v) local minimum, $n \leq 3$, then u or v is a vertex

Idea of proof:

- If $\dim(F_u) + \dim(F_v) > n$ then $0 \neq z \in Span(F_u) \cap Span(F_v)$
- u, v are in the interior part of F_u, F_v, so they can move in $Span(u, z)$, $Span(v, z)$

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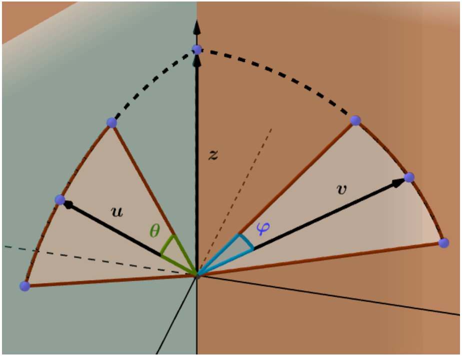

$$H_{\theta,\varphi}(u_\theta, v_\varphi)\Big|_{\theta=\varphi=0} = \begin{pmatrix} -\langle u, v\rangle & \pm 1 \\ \pm 1 & -\langle u, v\rangle \end{pmatrix}$$

Hessian has negative determinant

since $u \neq \pm v$

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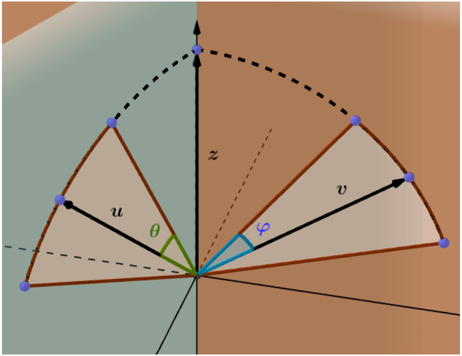

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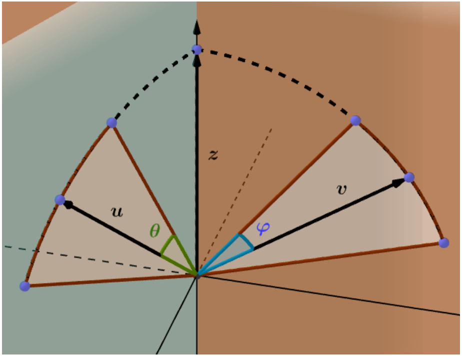

$$H_{\theta,\varphi}(u_\theta, v_\varphi)\Big|_{\theta=\varphi=0} = \begin{pmatrix} -\langle u, v\rangle & \pm 1 \\ \pm 1 & -\langle u, v\rangle \end{pmatrix}$$

Hessian has negative determinant

since $u \neq \pm v$

$\downarrow$

(u, v) is a saddle point

Conic Angles

- $\dim(F_u) + \dim(F_v) > n$ then (u, v) is a saddle point
- (u, v) local minimum, $n \leq 3$, then u or v is a vertex

Idea of proof:

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Counterexample for n $\geq$ 4**:**

$$P = \left\langle \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \right\rangle \qquad Q = \left\langle \begin{pmatrix} -1 & -1 \\ 0 & 0 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} \right\rangle \qquad u = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad v = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\dim(F_u) = \dim(F_v) = 2 \qquad u^\top v = -\frac{1}{\sqrt{2}} < 0$$

(u, v) is an antipodal pair in the interior part of P and Q

Conical Singular Values

$$\min_{\substack{u \in P,\ \|u\| = 1,\\ v \in Q,\ \|v\| = 1,}} u^\top A v \qquad P, Q \quad \text{closed convex cones finitely generated}$$

Pareto Singular Values

$$\min_{\substack{u \geq 0,\ \|u\| = 1,\\ v \geq 0,\ \|v\| = 1,}} u^\top A v$$

Polynomial Time for $A \geq 0$ or $A \leq 0$

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Singular Values

$$\min_{\|u\|=\|v\|=1} u^\top A v$$

Polynomial Time $O(mn^2)$ to compute all Singular Values

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Hardest Problem

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Lemma (B., G., S. 2024)

Any matrix $A \in \mathbb{R}^{m \times n}$ of spectral norm 1 and $m \geq n$ can be decomposed as $A = U^T V$ where $U \in \mathbb{R}^{(m+n) \times m}$, $V \in \mathbb{R}^{(m+n) \times n}$ have orthonormal columns

Proof: Let

$$A = U^T V \qquad U = \begin{pmatrix} I \\ 0 \end{pmatrix} \in \mathbb{R}^{(m+n) \times m} \qquad V := \begin{pmatrix} A \\ C \end{pmatrix} \in \mathbb{R}^{(m+n) \times n}$$

Given the SVD $A = W \Sigma Z^\top$ let $C = (I - \Sigma^\top \Sigma)^{1/2} Z^\top$ so that

$$V^\top V = A^\top A + C^\top C = A^\top A + Z(I - \Sigma^\top \Sigma) Z^\top = ZZ^\top = I$$

i.e. all columns of V are orthogonal to each other and with unitary norm

Notice:

The columns of U are a subset of the canonical basis, A is the projection of V on $\mathbb{R}^m$

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$$\min_{u,v} u^\top A v : \quad \|u\| = \|v\| = 1 \qquad u \in P = \langle G \rangle \subseteq \mathbb{R}^n \quad v \in Q = \langle H \rangle \subseteq \mathbb{R}^n$$

$$\min_{x,y} (Gx)^\top A (Hy) : \quad \|Gx\| = \|Hy\| = 1 \qquad x, y \geq 0$$

$$\|A\| \min_{x,y} x^\top G^\top U^\top V H y : \quad \|Gx\| = \|Hy\| = 1 \qquad x, y \geq 0$$

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$$\|A\| \min_{\widetilde{u},\widetilde{v}} \widetilde{u}^\top \widetilde{v} : \quad \|\widetilde{u}\| = \|\widetilde{v}\| = 1 \quad \widetilde{u} \in P' = \langle UG \rangle \subseteq \mathbb{R}^{n+m} \quad \widetilde{v} \in Q' = \langle VH \rangle \subseteq \mathbb{R}^{n+m}$$

From Conical SV to Conic Angles

The minimum conical singular value of dimension n with number of generators p, q for G, H reduces polynomially to the maximum angle between cones of dimension $n + m$ with same number of generators for UG, VH

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How difficult is the Pareto SV problem?

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Reduction from Maximum Edge Biclique to Minimal Pareto Singular Value

Theorem (Seeger, S. 2023)

Let (σ_0, u, v) be the optimal solution of

$$\sigma_0 = \min_{u,v \geq 0} u^\top A v \quad : \quad \|u\| = \|v\| = 1$$

If A has at least one negative entry then $(x, y) = \sqrt{-\sigma_0}(u, v)$ is optimal for

$$\min_{x,y \geq 0} \| - A - xy^\top \|_F^2$$

This shows that the Minimal Pareto Singular Value is **at least as hard** as the **Nonnegative Rank 1 Approximation** problem

Theorem (G., Glineur 2013)

Let $B \in \{0, 1\}^{m \times n}$ be the bi-adjacency matrix of a bipartite graph (N_1, N_2, E) where $B_{i,j} = 1$ iff node i in N_1 and node j in N_2 are connected and $d \geq \max\{m, n\}$.

$$\min_{x,y \geq 0} \|B - d(1 - B) - xy^\top\|_F^2$$

is solved by binary vectors x, y that identify the fully connected subsets $S_1 \subseteq N_1$ and $S_2 \subseteq N_2$ corrisponding to the **Maximum Edge Biclique**, i.e they maximise $|S_1| \cdot |S_2|$

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- $\min_{x,y\geq 0} \|B - d(1-B) - xy^\top\|_F^2$ identifies the Maximum Edge Biclique

$$M = B - d(1-B) = \begin{pmatrix} -d & 1 & 1 & -d & 1 & 1 & -d \\ 1 & 1 & -d & -d & 1 & 1 & 1 \\ -d & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -d & 1 & 1 & 1 & -d \\ 1 & -d & 1 & -d & -d & 1 & 1 \end{pmatrix}$$

Idea of Proof:

All the Maximal Bicliques (S_1, S_2) are local minima of $\|M - xy^\top\|_F^2$ where $x = \chi(S_1)$, $y = \chi(S_2)$ because any extension of $S_1 \times S_2$ gets a $-d$ and

$$(-d-\epsilon)^2 = d^2 + 2d\epsilon + \epsilon^2$$

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Everything is Hard

Theorem (Peeters 2003)

The Maximal Edge Biclique problem is NP-hard

Maximal Edge Biclique

Maximum Number of Edges in a Bipartite Connected Subgraph

NP-hard

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Nonnegative Rank 1

$$\min_{x,y \geq 0} \|M - xy^\top\|$$

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Theorem (B., G., S. 2024)

The maximum angle between convex closed cones problem

$$\min_{\substack{u \in P,\ \|u\| = 1,\\ v \in Q,\ \|v\| = 1}} \langle u, v\rangle$$

with P being generated by a subset of the canonical basis is NP-hard

Conjecture (B., G., S. 2024)

The maximum angle between the positive orthant and another convex closed cone

$$\min_{\substack{x \geq 0,\ \|x\| = 1,\\ v \in Q,\ \|v\| = 1}} \langle x, v\rangle = -\max_{v \in Q,\ \|v\| = 1} \|v^-\|$$

is NP-hard

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$$\min_{\substack{u \in P,\ \|u\| = 1, \\ v \in Q,\ \|v\| = 1}} \langle u, v \rangle$$

with P being generated by a subset of the canonical basis is NP-hard

Conjecture (B., G., S. 2024)

The maximum angle between the positive orthant and another convex closed cone

$$\min_{\substack{x \geq 0,\ \|x\| = 1, \\ v \in Q,\ \|v\| = 1}} \langle x, v \rangle = - \max_{v \in Q,\ \|v\| = 1} \|v^-\|$$

is NP-hard

Recall: from $A = \|A\| U^\top V$

$$\min_{\substack{x \geq 0,\ \|x\| = 1,\\ y \geq 0,\ \|y\| = 1}} \langle x, Ay \rangle = \|A\| \min_{\substack{x \geq 0,\ \|Ux\| = 1,\\ y \geq 0,\ \|Vy\| = 1}} \langle Ux, Vy \rangle = \|A\| \min_{\substack{u \in P,\ \|u\| = 1,\\ v \in Q,\ \|v\| = 1}} \langle u, v \rangle,$$

with $U^\top = \begin{pmatrix} I & 0 \end{pmatrix}$

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Algorithms

Active Set

$$\lambda^* = \min_{\substack{u \in P,\ \|u\| = 1, \\ v \in Q,\ \|v\| = 1,}} u^\top A v = \min_{\substack{x \geq 0,\ \|Gx\| = 1, \\ y \geq 0,\ \|Hy\| = 1,}} x^\top G^\top A H y$$

Idea: If we know the sets $\mathcal{I}$, $\mathcal{J}$ of indices for which $x_i^*, y_j^* > 0$, called **Active Sets**, then a direct gradient computation solves the problem

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For the optimal solution $(u^*, v^*) = (Gx^*, Hy^*) = (\overline{G}\overline{x}, \overline{H}\overline{y})$, and $\lambda^* = (u^*)^\top A\, v^*$ we have that

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Input: $A \in \mathbb{R}^{m\times n}$, $G \in \mathbb{R}^{m\times p}$, $H \in \mathbb{R}^{n\times q}$, $P = \langle G\rangle$, $Q = \langle H\rangle$

Output: $\lambda = \min u^\top A v$ such that $\|u\| = \|v\| = 1$, $u \in P$, $v \in Q$

$\lambda = g_i^\top A h_j = \min_{k,\ell} (G^\top A H)_{k,\ell}$, $u = g_i$, $v = h_j$, $r = \mathrm{Null}(A^\top A - \|A\|^2 I_n)$

$\mathscr{I} := \{(\mathcal{I}, \mathcal{J}) : 2 < |\mathcal{I}| + |\mathcal{J}| \le m + n - r$, $\overline{G} := G_{:,\mathcal{I}}$ and $\overline{H} := H_{:,\mathcal{J}}$ full column rank$\}$

for $(\mathcal{I}, \mathcal{J}) \in \mathscr{I}$, **do**

  $A_x = \overline{G}^\dagger A^\top \overline{H}$, $A_y = \overline{H}^\dagger A \overline{G}$

  $A_\lambda = A_y A_x$, $\widetilde{A}_\lambda = A_x$ (or $A_\lambda = A_x A_y$, $\widetilde{A}_\lambda = A_y$ if $|\mathcal{I}| > |\mathcal{J}|$)

  if $\rho(A_\lambda) \le \lambda^2$ **then** Skip to the next $(\mathcal{I}, \mathcal{J}) \in \mathscr{I}$

  U right eigenspace of $\rho(A_\lambda)$ in A_λ, $\mu = -\sqrt{\rho(A_\lambda)}$, $W = \begin{pmatrix} \widetilde{A}_\lambda U/\mu \\ U \end{pmatrix}$

  Compute the reduced QR of $W = VR$

  if $(VV^\top - I)z = 0$, $z \ge 0$, $e^\top z = 1$ admits a solution **then**

    $\lambda = \mu$, $z = [y^\top \ x^\top]^\top$ (or $z = [x^\top \ y^\top]^\top$ if $|\mathcal{I}| > |\mathcal{J}|$)

    $u = \overline{G}x/\|\overline{G}x\|$, $v = \overline{H}y/\|\overline{H}y\|$

  end if

end for

(Not) a Gurobi AD

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GUROBI OPTIMIZATION

minimize	$x^TQx + c^Tx + \mathrm{alpha}$	
subject to	$Ax = b$	(linear constraints)
	$\ell \le x \le u$	(bound constraints)
	some x_j integral	(integrality constraints)
	$x^TQc\,x + q^Tx \le \mathrm{beta}$	(quadratic constraints)
	some x_i in SOS	(special ordered set constraints)
	min, max, abs, or, ...	(general constraints)

- ✓ Interfaces: C, C++, Python, Java, Matlab, .NET, R
- × Proprietary, not Open Source
- ✓ Free for Academic use
- \- Slower than approximating iterative solvers, but has decent Heuristics
- ✓ Solve the problem exactly even in the indefinite case

Uses **McCormick Envelope Relaxation**:

$$\min_{(u,v)\in K}\langle u, Av\rangle = \min_{(u,v)\in K}\sum_{i,j} A_{i,j}u_iv_j = \min_{(u,v)\in K,\, u_iv_j=w_{i,j}} \sum_{i,j} A_{i,j}w_{i,j} \ge \min_{(u_i,v_j,w_{i,j})\in K_{i,j}} \sum_{i,j} A_{i,j}w_{i,j}$$

where $K_{i,j} = \mathrm{Conv}((u_i, v_j, w_{i,j}) : w_{i,j} = u_iv_j,\ \underline{u_i} \le u_i \le \overline{u_i},\ \underline{v_j} \le v_j \le \overline{v_j},\ u, v \in K)$

Behaves great when A is Sparse since it has less slack variables $w_{i,j}$

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$$\min_{(u,v)\in K} \langle u, Av\rangle = \min_{(u,v)\in K} \sum_{i,j} A_{i,j} u_i v_j = \min_{(u,v)\in K, u_i v_j = w_{i,j}} \sum_{i,j} A_{i,j} w_{i,j} \geq \min_{(u_i, v_j, w_{i,j})\in K_{i,j}} \sum_{i,j} A_{i,j} w_{i,j}$$

where $K_{i,j} = \mathrm{Conv}((u_i, v_j, w_{i,j}) : w_{i,j} = u_i v_j,\ \underline{u_i} \leq u_i \leq \overline{u_i},\ \underline{v_j} \leq v_j \leq \overline{v_j}, u, v \in K)$

Behaves great when A is Sparse since it has less slack variables $w_{i,j}$

(Not) a Gurobi AD

GUROBI OPTIMIZATION

minimize	$x^T Q x + c^T x + \mathrm{alpha}$	
subject to	$Ax = b$	(linear constraints)
	$\ell \le x \le u$	(bound constraints)
	some x_j integral	(integrality constraints)
	$x^T Qc\, x + q^T x \le \mathrm{beta}$	(quadratic constraints)
	some x_i in SOS	(special ordered set constraints)
	min, max, abs, or, ...	(general constraints)

- ✓ Interfaces: C, C++, Python, Java, Matlab, .NET, R
- × Proprietary, not Open Source
- ✓ Free for Academic use
- \- Slower than approximating iterative solvers, but has decent Heuristics
- ✓ Solve the problem exactly even in the indefinite case

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Behaves great when A is Sparse since it has less slack variables $w_{i,j}$

Alternating projection with extrapolation

$$\lambda^* = \min_{\substack{u \in P,\ \|u\| = 1,\\ v \in Q,\ \|v\| = 1,}} u^\top A v$$

Idea: We have seen that if we know u^* or v^*, then finding the other is equivalent to solve an easy convex problem

Alternate Projection: starting from an initial feasible point (u_0, v_0) and $k = 0$

- $u_{k+1} = \arg\min_{x \in P} x^\top A v_k$ such that $\|x\|_2 = 1$
- $v_{k+1} = \arg\min_{y \in Q} u_{k+1}^\top A y$ such that $\|y\|_2 = 1$
- $k = k + 1$

To accelerate the convergence, we add an **Extrapolation step** after each update

- $u_{k+1} = u_{k+1} + \beta(u_{k+1} - u_k)$
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The method **converges** to a stationary point, that **may not be optimal**

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Alternating projection with extrapolation

Input: $A \in \mathbb{R}^{m \times n}$, cones $P \subseteq \mathbb{R}^m$ and $Q \subseteq \mathbb{R}^n$

Output: An approximate solution to $\min_{u \in P, v \in Q} u^\top A v$ such that $\|u\|_2 = \|v\|_2 = 1$.

$u = 0,\ v = 0,\ v_e = v_0,\ k = 1.$
while $k \leq K$ and $(\|u - u_p\|_2 \geq \delta$ or $\|v - v_p\|_2 \geq \delta)$ **do**
  $u_p = u$. *% Keep previous iterate in memory*
  $u = \arg\min_{x \in P} x^\top A v_e$ such that $\|x\|_2 = 1$.
  $u_e = u + \beta(u - u_p)$. *% Extrapolated point*
  $v_p = v$. *% Keep previous iterate in memory*
  $v = \arg\min_{y \in Q} u_e^\top A y$ such that $\|y\|_2 = 1$.
  $v_e = v + \beta(v - v_p)$. *% Extrapolated point*
  $e_k \leftarrow u^\top A v$.
  if $k \geq 2$ and $e_k > e_{k-1}$ **then**
    $u = u_p,\ v = v_p,\ \beta = \frac{\beta}{\eta}$.
  else
    $\beta \leftarrow \min(1, \gamma\beta)$.
  end if
  $k \leftarrow k + 1$.
end while

Sequential Regularized Partial Linearization

$$\lambda^* = \min_{\substack{u \in P,\ \|u\| = 1, \\ v \in Q,\ \|v\| = 1,}} u^\top A v = \min_{\substack{u \in P,\ u \neq 0, \\ v \in Q,\ v \neq 0,}} \frac{u^\top A v}{\|u\|\|v\|} = \min_{\substack{e^\top x = 1,\ x \geq 0, \\ e^\top y = 1,\ y \geq 0,}} \frac{x^\top G^\top A H y}{\|Gx\|\|Hy\|}$$

Idea: If the minimum of $f_\delta(x, y) := x^\top G^\top A H y - \delta \|Gx\|\|Hy\|$ over $(x, y) \in \Delta_p \times \Delta_q$ is $\mu < 0$ then we get a decrease in the objective function

$$\frac{x^\top G^\top A H y}{\|Gx\|\|Hy\|} = \delta + \frac{\mu}{\|Gx\|\|Hy\|} < \delta$$

Partial Linearization: starting from an initial feasible point (x_0, y_0) and $k = 0$,

- $\delta = \frac{x_k^\top G^\top A H y_k}{\|Gx_k\|\|Hy_k\|}$
- Linearize wrt x the function $f_\delta(x, y_k)$, penalize it with $\|x - x_k\|^2$ and minimize it to obtain x_{k+1}
- Linearize wrt y the function $f_\delta(x_{k+1}, y)$, penalize it with $\|y - y_k\|^2$ and minimize it to obtain y_{k+1}
- $k = k + 1$

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Output: An approximate solution to $\min_{u \in P, v \in Q} \langle u, Av \rangle$ such that $\|u\| = \|v\| = 1$

1: Set

$$\delta_k := \frac{\langle Gx^k, AHy^k \rangle}{\|Gx^k\| \|Hy^k\|}$$

2: Let $L_1^k(x) := \left\langle Gx, AHy^k - \delta_k \|Gx^k\|^{-1} \|Hy^k\| Gx^k \right\rangle$
Compute a solution $\tilde{x}^k$ to the convex program

$$\min L_1^k(x) + \tfrac{\mu_1}{2} \|x - x^k\|^2 \quad \text{such that } x \in \Delta_p$$

3: Let $L_2^k(y) := \left\langle Hy, A^\top Gx^k - \delta_k \|Gx^k\| \|Hy^k\|^{-1} Hy^k \right\rangle$
Compute a solution $\tilde{y}^k$ to the convex program

$$\min L_2^k(y) + \tfrac{\mu_2}{2} \|y - y^k\|^2 \quad \text{such that } y \in \Delta_q$$

4: Let $d_1^k := \tilde{x}^k - x^k$ and $d_2^k := \tilde{y}^k - y^k$

5: If $(|L_1^k(d_1^k)| < \delta$ and $|L_2^k(d_2^k)| < \delta)$ or $k \geq K$ terminate
Otherwise, let $t_k := \beta \rho^{\ell_k}$, where ℓ_k is the smallest nonnegative integer ℓ such that

$$\Phi(x^k + t^k d_1^k, y^k + t^k d_2^k) \leq \Phi(x^k, y^k) + \alpha t_k \frac{L_1^k(d_1^k) + L_2^k(d_2^k)}{\|Gx^k\| \|Hy^k\|}$$

Set $\left(x^{k+1}, y^{k+1}\right) := (x^k, y^k) + t_k(d_1^k, d_2^k)$ and $k = k + 1$. Go to step 1

Experiments

An Example: Schur Cone

The **Schur Cone** is generated by the matrix

$$H = \begin{pmatrix} 1 & 0 & \dots & 0 \\ -1 & 1 & \dots & 0 \\ 0 & -1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & -1 \end{pmatrix} \in \mathbb{R}^{n \times n-1} \qquad \langle H \rangle \subseteq e^{\perp}$$

One can prove that the maximum angle between the Schur cone Q and the positive orthant P is achieved by

$$y = e_n \in P \quad x = (a\, a\, \dots\, a\, b) \in Q \quad a = \sqrt{\frac{1}{n(n-1)}} \quad b = -\sqrt{1 - \frac{1}{n}} = x^\top y$$

that can be proved being the maximum angle as

$$\min_{\substack{x \in Q, y \in P \\ \|x\| = \|y\| = 1}} x^\top y \geq \min_{\substack{x \in e^{\perp}, y \geq 0 \\ \|x\| = \|y\| = 1}} x^\top y = \min_{y \geq 0, \|y\|=1} -\|P_{e^{\perp}}(y)\| = -\sqrt{1 - \frac{1}{n}}$$

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Table 1: Numerical comparison for Gur and BFAS for different dimensions for the problem of finding the maximum angle between the Schur cone and the positive orthant cone. The table reports the optimal objective functions values found in the timelimit (60 seconds) and the actual elapsed time. We also report the exact value for each problem.

n	5	10	20	50
exact	0.852416π	0.897584π	0.928217π	0.954833π
Gur	0.852416π	0.897584π	0.928218π	0.954833π
	0.1134 s	0.2016 s	20.1493 s	60^* s
BFAS	0.852416π	0.897584π	0.750000π	0.750000π
	0.3310 s	48.3153 s	60^* s	60^* s

n	100	200	500
exact	0.968116π	0.977473π	0.985760π
Gur	0.968116π	0.977473π	0.985756π
	60^* s	60^* s	60^* s
BFAS	0.750000π	0.750000π	0.750000π
	60^* s	60^* s	60^* s

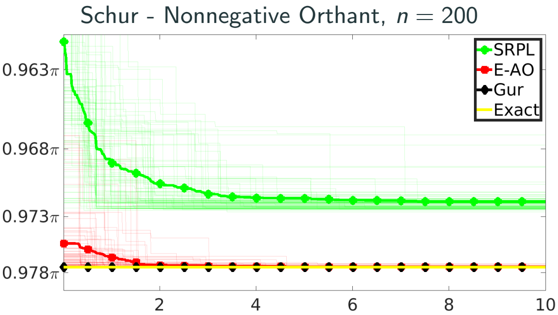
Schur Cone and Positive Orthant

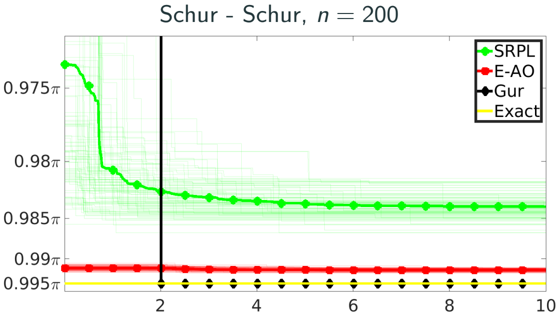
Table 1: Numerical comparison for Gurobi and BFAS for different dimensions for the problem of finding the maximum angle between the Schur cone and itself. The table reports the optimal objective functions values found in the timelimit (60 seconds) and the actual elapsed time. We also report the exact value for each problem.

n	5	10	20	50
exact	0.800000π	0.900000π	0.950000π	0.980000π
Gur	0.800001π	0.900000π	0.950000π	0.980000π
	0.2508 s	60^* s	60^* s	60^* s
BFAS	0.800000π	0.900000π	0.859157π	0.804087π
	0.3856 s	60^* s	60^* s	60^* s

n	100	200	500
exact	0.990000π	0.995000π	0.998000π
Gur	0.936315π	0.994996π	0.998011π
	60^* s	60^* s	60^* s
BFAS	0.750000π	0.750000π	0.750000π
	60^* s	60^* s	60^* s

Schur Cone and Positive Orthant





Maximum Edge Biclique Problem

Recall that solving the Pareto singular value problem is equivalent to solve the maximum edge biclique problem.

Here we thus test all four algorithms on four bipartite graphs taken from a benchmark dataset[1]. All graphs have been randomly generated with a fixed edge density, and then a biclique has been added to them. In particular,

- the first graph is a 100×100 graph with density 0.2 and planted biclique of size $50 \times 50 = 2500$,
- the second graph is a 300×300 graph with density 0.3 and planted biclique of size $2 \times 55 = 110$,
- the third graph is a 100×100 graph with density 0.71 and planted biclique of size $80 \times 80 = 6400$,
- the fourth graph is a 10000×300 graph with density 0.03 and planted biclique of size $22 \times 2 = 44$.

[1] Shaham, E.: maximum biclique benchmark. https://github.com/shahamer/maximum-biclique-benchmark (2019)

Maximum Edge Biclique Problem

Table 1: Numerical comparison for Gurobi, BFAS, E-AO and SRPL for the problem of finding the maximum edge biclique in four different bipartite graphs. The table reports the maximum edgee biclique found in the timelimit (10 seconds) for Gurobi and BFAS. The reported number for E-AO and SRPL are instead the average value found at 10 seconds for 100 runs, and in parentheses the best value found throughout all 100 runs when it differs from the average one. Gurobi cannot be executed on the last graph due to its excessive size.

n	100×100	300×300	100×100	10000×300
Gur	2500	0	310	NA
BFAS	3	2	2	2
E-AO	66	114	87	12
SRPL	2500	114	6400	46(358)

Maximum Angle between PSD and Nonnegative Symmetric Matrices

Given the inner product $\langle A, B\rangle = Tr(A^\top B)$ on the space of $n \times n$ real symmetric matrices $\mathcal{S}^n$ an open question is the maximum angle between the cone of PSD matrices $\mathcal{P}^n$ and the cone of nonnegative symmetric matrices $\mathcal{N}^n$ for $n \geq 5$

$$\gamma_n := \min_{\substack{A \in \mathcal{P}^n, B \in \mathcal{N}^n \\ \|A\|_F = \|B\|_F = 1}} \langle A, B\rangle = -\max_{A \in \mathcal{P}^n, \|A\|_F = 1} \|A^-\|_F = -\frac{1}{2} \max_{B \in \mathcal{N}^n, \|B\|_F = 1} \|B - \sqrt{B^2}\|_F$$

It is known that

$$n = 2, 3, 4 \implies \gamma_n = -\frac{1}{\sqrt{2}} = \cos\left(\frac{3}{4}\pi\right) \qquad \lim_{n\to\infty} \gamma_n \downarrow -1 = \cos(\pi)$$

This is a lower bound on the maximum angle in the cone of copositive matrices

$$\mathcal{C}^n := \{A \in \mathcal{S}^n \,:\, x^\top A x \geq 0 \;\forall x \geq 0\}$$

All the algorithms to compute γ_n are iteratively converging to a critical angle, i.e. a stationary point of the optimization problem

For $n \geq 5$ **we only have lower bounds** on the exact angle

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Maximum Angle between PSD and Nonnegative Symmetric Matrices

Given the inner product $\langle A, B\rangle = Tr(A^\top B)$ on the space of $n \times n$ real symmetric matrices $\mathcal{S}^n$ an open question is the maximum angle between the cone of PSD matrices $\mathcal{P}^n$ and the cone of nonnegative symmetric matrices $\mathcal{N}^n$ for $n \geq 5$

$$\gamma_n := \min_{\substack{A \in \mathcal{P}^n,\, B \in \mathcal{N}^n \\ \|A\|_F = \|B\|_F = 1}} \langle A, B\rangle = -\max_{A\in\mathcal{P}^n, \|A\|_F=1} \|A^-\|_F = -\frac{1}{2} \max_{B\in\mathcal{N}^n, \|B\|_F=1} \|B - \sqrt{B^2}\|_F$$

It is known that

$$n = 2, 3, 4 \implies \gamma_n = -\frac{1}{\sqrt{2}} = \cos\left(\frac{3}{4}\pi\right) \qquad \lim_{n\to\infty} \gamma_n \downarrow -1 = \cos(\pi)$$

This is a lower bound on the maximum angle in the cone of copositive matrices

$$\mathcal{C}^n := \{A \in \mathcal{S}^n \,:\, x^\top Ax \geq 0 \;\forall x \geq 0\}$$

All the algorithms to compute γ_n are iteratively converging to a critical angle, i.e. a stationary point of the optimization problem

For $n \geq 5$ **we only have lower bounds** on the exact angle

A Conjecture

Known Antipodal Couples:

$n = 1$: $A_1 = B_1 = 1$

$n = 2$: $A_2 = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad B_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$n = 3, 4$: $A_n = \begin{pmatrix} A_2 & 0 \\ 0 & 0 \end{pmatrix} \quad B_n = \begin{pmatrix} B_2 & 0 \\ 0 & 0 \end{pmatrix}$

$n = 5$: (Best Known Stationary Point) F_5 is the Fourier matrix

$$A_n = F_5 \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & & \frac{1}{\sqrt{2}} & & \\ & & & \frac{1}{\sqrt{2}} & \\ & & & & 0 \end{pmatrix} F_5^H \qquad B_n = \frac{1}{\sqrt{10}}\begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

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Every antipodal pair is block circulant

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Circulant Symmetric Matrices

The algebra of circulant real symmetric matrices $\mathcal{SC}^n$ is the set of

$$\begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_1 \\ a_1 & a_0 & a_1 & \ddots & \vdots \\ a_2 & a_1 & \ddots & \ddots & a_2 \\ \vdots & \ddots & \ddots & a_0 & a_1 \\ a_1 & \dots & a_2 & a_1 & a_0 \end{pmatrix} = a_0 I_n + F_n \,\mathrm{diag}\left(\sum_{j>0} 2a_j \cos(2\pi ij/n)\right)_{i=0:n-1} F_n^H$$

Properties:

- Both $\mathcal{SC}^n \cap \mathcal{P}^n$ and $\mathcal{SC}^n \cap \mathcal{N}^n$ are finitely generated cones with $\lceil \frac{n+1}{2} \rceil$ generators
- Given $C \in \mathcal{SC}^n \cap \mathcal{P}^n$ its projection C' onto $-\mathcal{N}^n$ is still in $\mathcal{SC}^n$, and the angle between $C, -C'$ is the maximum angle between C and $\mathcal{N}^n$
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Check with Gurobi

If n is odd and $n = 1 + 2m$

$$\min_{\substack{A \in \mathcal{SC}^n \cap \mathcal{P}^n,\ \|A\|_F = 1 \\ B \in \mathcal{SC}^n \cap \mathcal{N}^n,\ \|B\|_F = 1}} \langle A, B \rangle = \min_{\substack{x \geq 0,\ \|x\|_F = 1 \\ y \geq 0,\ \|y\|_F = 1}} \langle x, My \rangle \qquad M = \frac{2}{\sqrt{n}} \left[\cos\left(\frac{2\pi}{n} ij \right) \right]_{i,j=1:m}$$

A similar reduction holds for n even

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5	$0.7575\ \pi$	$0.7575\ \pi$	18	$0.7699\ \pi$	$0.7670\ \pi$
6	$0.7575\ \pi$	$0.7575\ \pi$	19	$0.7703\ \pi$	$0.7681\ \pi$
7	$0.7575\ \pi$	$0.7575\ \pi$	20	$0.7719\ \pi$	$0.7719\ \pi$
8	$0.7608\ \pi$	$0.7608\ \pi$	21	$0.7719\ \pi$	$0.7719\ \pi$
9	$0.7608\ \pi$	$0.7608\ \pi$	22	$0.7719\ \pi$	$0.7719\ \pi$
10	$0.7609\ \pi$	$0.7608\ \pi$	23	$0.7722\ \pi$	$0.7719\ \pi$
11	$0.7627\ \pi$	$0.7627\ \pi$	24	$0.7735\ \pi$	$0.7730\ \pi$
12	$0.7649\ \pi$	$0.7649\ \pi$	25	$0.7735\ \pi$	$0.7730\ \pi$
13	$0.7649\ \pi$	$0.7649\ \pi$	26	$0.7735\ \pi$	$0.7730\ \pi$
14	$0.7659\ \pi$	$0.7649\ \pi$	27	$0.7739\ \pi$	$0.7730\ \pi$
15	$0.7678\ \pi$	$0.7649\ \pi$	28	$0.7750\ \pi$	$0.7730\ \pi$
16	$0.7699\ \pi$	$0.7670\ \pi$	29	$0.7750\ \pi$	$0.7741\ \pi$
17	$0.7699\ \pi$	$0.7670\ \pi$	30	$0.7757\ \pi$	$0.7741\ \pi$

Left: Lower bounds on γ_n

Right:

- In **black** the exact angle $\mathcal{SC}^n \cap \mathcal{P}^n \angle \mathcal{SC}^n \cap \mathcal{N}^n$
- In blue if a previous angle was bigger then the exact solution
- In red if it is a lower bound

Table 2: Numerical comparison of Gur and BFAS for different dimensions for the problem of finding the maximum angle between the PSD cone and the nonnegative symmetric cone, both restricted to the subalgebra of circulant matrices. Timelimit: 60 seconds

n	13	15	17	19	21	23
exact	0.762950π	0.757765π	0.764971π	0.768062π	0.768769π	0.766370π
Gur	0.762950π	0.757765π	0.764971π	0.767876π	0.765409π	0.766370π
	0.854 s	25.061 s	60^* s	60^* s	60^* s	60^* s
BFAS	0.762950π	0.757765π	0.764971π	0.768062π	0.768768π	0.766370π
	0.333 s	0.356 s	1.114 s	4.418 s	19.953 s	60^* s

Table 3: Numerical comparison of Gur, BFAS, E-AO and SRPL for the same problem. Timelimit: 10 seconds. We also report the exact value for each problem, and the best known lower bound when the exact value is not available, indicated with an asterisk

n	17	19	21	23	25	27
exact	0.764971π	0.768062π	0.768769π	0.766370π	$0.767385\pi^*$	$0.768258\pi^*$
Gur	0.764971π	0.759309π	0.765409π	0.766370π	0.767385π	0.760879π
BFAS	0.764971π	0.768062π	0.768768π	0.766370π	0.762620π	0.756841π
E-AO	0.764971π	0.768062π	0.768768π	0.766370π	0.767385π	0.768258π
SRPL	0.764970π	0.768062π	0.768768π	0.766369π	0.767384π	0.768257π

PSD and SNN matrices

Since E-AO and SRPL main steps are projections, they can be adapted to the case of NON-polyhedral cones, as long as we know how to compute the projection on such cones

We can thus test them on the task to find the maximum angle between the cone of Positive Semi-Definite matrices and the con of Symmetric Nonnegative matrices

Table 4: Numerical comparison for E-AO and SRPL for different dimensions for the problem of finding the maximum angle between the PSD cone and the nonnegative symmetric cone. The table reports the best and average value found over 10000 random initializations, together with the average elapsed time. We also report the best known value for each dimension.

n	30	40	50	60
best known	0.7757π	0.7789π	0.7812π	0.7837π
EAO_b	0.7755π	0.7782π	0.7802π	0.7811π
EAO_a	0.7717π	0.7734π	0.7745π	0.7760π
	0.006 ± 0.002 s	0.056 ± 0.019 s	0.030 ± 0.011 s	0.057 ± 0.019 s
SRPL_b	0.7757π	0.7789π	0.7812π	0.7837π
SRPL_a	0.7739π	0.7766π	0.7787π	0.7802π
	0.062 ± 0.025 s	0.155 ± 0.060 s	0.319 ± 0.130 s	0.565 ± 0.229 s

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Workshop on Low-Rank Models and Applications (LRMA)

11-12 September 2025, Mons, Belgium

Plenary speakers: Stanislav Budzinskiy, Luca Calatroni, Alice Cortinovis, Mariya Ishteva, Paul Magron, Margherita Porcelli, Bertrand Rivet, and Lawrence Saul.

https://sites.google.com/view/lrma25

Thank You!