Theory of Open Quantum Systems

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Contents

Chapter 1

Fundamentals

1.1 Hilbert Space

For every quantum system there is an associated vector space, the so-called Hilbert space $\mathcal{H}$. For our purpose a vector space is defined by a set of d- dimensional vectors

$$\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix} \qquad \text{with } v_k \in \mathbb{C} \ .$$

- Dirac notation:

  standard basis: $|k\rangle \ = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \longleftarrow k^{th}$ position

  Any vector can be expanded as

  $$|v\rangle \ = \sum_{k=1}^{d} v_k \, |k\rangle \qquad \text{with } u_k \in \mathbb{C} \ .$$

  The coefficients v_k are called probability amplitudes, since the probability for being in the state $|k\rangle$ is $|v_k|^2$.

- Scalar product (inner product):

  $$\text{“bra” vector} : \ \langle v| \ = |v\rangle^\dagger$$

  with $|v\rangle^\dagger$ being the conjugate transpose (complex conjugate and transpose) of $|v\rangle$.

$$\langle v|w\rangle = \begin{pmatrix} v_1^*, & \dots & , v_d^* \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_d \end{pmatrix} = \sum_{k=1}^{d} v_k^* w_k$$

- Normalization condition:

$$\underbrace{1 = \sum_{k=1}^{d} |v_k|^2}_{\substack{\text{probabilities} \\ \text{sum up to 1,} \\ \text{i.e. every vector in } \mathcal{H} \text{ shall be normalized}}} = \sum_{k=1}^{d} v_k^* v_k = \langle v|v\rangle = \| \, |v\rangle \, \|^2$$

- Outer product:

  The outer product of the two vectors in $\mathcal{H}$ is defined as

$$|v\rangle\langle w| = \begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix} \begin{pmatrix} w_1^* & \dots & w_d^* \end{pmatrix} = \begin{pmatrix} v_1 w_1^* & \dots & v_1 w_d^* \\ \vdots & \ddots & \\ v_d w_1^* & \dots & v_d w_d^* \end{pmatrix}$$

1.2 Time Evolution

The state vector of a quantum system satisfies the Schrödinger equation

$$\frac{\partial}{\partial t} |\psi(t)\rangle = -iH \, |\psi(t)\rangle \ ,$$

where H is the Hamiltonian of the system. Note that we will omit the $\hbar$ throughout this lecture.

There exists a unitary operator $U(t)$ which evolves the state in time:

$$|\psi(t)\rangle = U(t) \, |\psi(0)\rangle$$

For time independent Hamiltonians we find

$$U(t) = e^{-iHt} = \sum_{m=0}^{\infty} \frac{(-iHt)^m}{m!} \ .$$

In general for time dependent Hamiltonians this expression becomes

$$U(t) = \mathcal{T} e^{-i \int_0^t d\tau H(\tau)} \ .$$

where $\mathcal{T}$ is the time-ordering operator.

1.3 Spectral Decomposition

Often we are interested in the spectral decomposition of operators. For this purpose the *Spectral Theorem* turns out to be useful.

- Spectral Theorem:
  A linear operator $A : \mathcal{H} \to \mathcal{H}$ obeys $A^\dagger A = AA^\dagger$ i.e. it is a **normal operator**, if and only if $A = \sum_{a=1}^{d} \lambda_a |a\rangle\langle a|$ for a set of orthonormal basis vectors $\{|a\rangle\}$ for $\mathcal{H}$, which are also eigenvectors of A with respective eigenvalues $\{\lambda_a\}$.

  One easily sees that hermitian operators $H = H^\dagger$ are normal $HH^\dagger = H^\dagger H$ and can therefore be spectrally decomposed.

  As the Hamiltonian is hermitian, we find:

$$H = \sum_{n=1}^{d} E_n |E_n\rangle\langle E_n|$$

  where E_n are the eigenenergies and $|E_n\rangle\langle E_n|$ are the corresponding eigenstates.

- Functions of normal operators:
  Functions of normal operators are defined as

$$f(\hat{A}) = \sum_{a=1}^{d} f(\lambda_a) |a\rangle\langle a|$$

  This can easily be proven by regarding the taylor expansion of a function f.

Example: Time-evolution operator

$$U = e^{-itH} = \sum_{n=1}^{d} e^{-itE_n} |n\rangle\langle n|$$

1.4 Measurement Postulate

The measurement postulate consists of two parts:

i) Measuring states:
   Quantum measurements are described by a set $\{M_k\}_{k=1}^{N}$ of measurement operators, satisfying the constraint,

$$\sum_{k=1}^{N} M_k^\dagger M_k = \mathbb{1}$$

   Note that from now on we will omit the summation boundaries. The $\mathbb{1}$ on the right hand side of the equation denotes the $N \times N$ dimensional identity matrix.

Given a state $|\psi\rangle \in \mathcal{H}$, it instantaneously after the measurement becomes

$$|\psi\rangle \longrightarrow \frac{M_k |\psi\rangle}{\sqrt{p_k}} \equiv |\psi_k\rangle$$

with the probability

$$p_k = \langle\psi| M_k^\dagger M_k |\psi\rangle = \| M_k |\psi\rangle \|^2 \ .$$

Note, that probability/norm conservation is ensured by the above constraint

$$1 = \sum_k p_k = \sum_k \langle\psi| M_k^\dagger M_k |\psi\rangle = \langle\psi| \underbrace{\sum_k M_k^\dagger M_k}_{=\mathbb{1}} |\psi\rangle = \langle\psi|\psi\rangle$$

ii) Observables:
Every measurable physical quantity is associated with an observable, i.e. a hermitian operator A. Since A is hermitian and thus normal it has a spectral decomposition

$$A = \sum_a \lambda_a \overset{\text{orthonormal basis states}}{\overset{\downarrow}{|a\rangle\langle a|}}, \text{ with } \lambda_a \in \mathbb{R} \ .$$

The eigenvalues λ_a are the outcomes of the measurements.

- Projective measurement:
The measurement operators M_k are projectors (onto the eigenstates of A),

$$M_k = |k\rangle\langle k| \qquad , \qquad M_k^2 = |k\rangle \underbrace{\langle k|k\rangle}_{\mathbb{1}} \langle k| = |k\rangle\langle k| = M_k$$

They satisfy the constraint,

$$\mathbb{1} = \sum_k M_k = \sum_k |k\rangle\langle k|$$

If a system is in the state $|\psi\rangle$ before the observable A is measured, the probability of the outcome λ_a is

$$p_a = \langle\psi| M_a^\dagger M_a |\psi\rangle = \langle\psi| a\rangle\langle a |\psi\rangle = |\langle\psi|a\rangle|^2$$

Moreover, the state after the measurement with outcome λ_a is performed, is

$$|\psi_a\rangle = \frac{M_a |\psi\rangle}{\sqrt{p_a}} = \frac{\langle a|\psi\rangle}{|\langle a|\psi\rangle|} |a\rangle = e^{i\theta} |a\rangle \, ,$$

where $e^{i\theta}$ is the phase of the complex number $\langle a|\psi\rangle$.

Let us now consider two examples.

Example: Qubit

A Qubit (spin $\frac{1}{2}$ particle) in state

$$|\psi\rangle = c_0 |0\rangle + c_1 |1\rangle$$

The Measurement Operators are:
$M_0 = |0\rangle\langle 0|$, $M_1 = |1\rangle\langle 1|$

$$|\psi\rangle = \begin{cases} \frac{M_0|\psi\rangle}{|c_0|}, & \text{with probability } \; p_0 = |c_0|^2 \\ \\ \frac{M_1|\psi\rangle}{|c_1|}, & \text{with probability } \; p_1 = |c_1|^2 \end{cases}$$

This example corresponds to measuring the magnetization of the spin 1 Qubit in z-direction. The corresponding observable is the Pauli z-matrix,

$$\mathcal{Z} = \underset{\lambda_0}{\overset{\uparrow}{(+1)}} \underbrace{|0\rangle\langle 0|}_{M_0} + \underset{\lambda_1}{\overset{\uparrow}{(-1)}} \underbrace{|1\rangle\langle 1|}_{M_1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_z$$

We measure a magnetization of $+1$ (-1) with probability $|c_0|^2$ $\left(|c_1|^2\right)$.

Example: Energy measurement under Hamiltonian

Now we consider an energy measuremt unter a Hamiltionian that is not orthonormal in our basis:

$$\begin{aligned} H &= \omega_x \overbrace{\sigma_x}^{\text{Pauli } x\text{-matrix}} + \omega_z \sigma_z \\ &= E_- |E_-\rangle\langle E_-| + E_+ \underbrace{|E_+\rangle\langle E_+|}_{\substack{\text{projector on} \\ \text{energy eigenstate}}} \end{aligned}$$

with

$$E_\pm = \pm\sqrt{\omega_x^2 + \omega_z^2} = \pm|\vec{\omega}|^2$$

and

$$|E_+\rangle = \begin{pmatrix} \cos\alpha \\ -\sin\alpha \end{pmatrix} \quad , \quad |E_-\rangle = \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix}$$

with, $\alpha = \frac{1}{2}\,\text{atan}\left(\frac{\omega_x}{\omega_z}\right)$.
The probability for measuring spin with state $|\psi\rangle$ in the ground state is

$$p_- = |\,\langle E_-|\psi\rangle\,|^2 = \underbrace{|c_0^* \sin\alpha + c_1^* \cos\alpha|^2}_{\substack{\text{can show non-trivial} \\ \text{behavior as } \alpha \text{ is} \\ \text{varied} \rightarrow \text{interference}}}$$

1.5 The density operator

We motivate the introduction of the density operator by considering ensembles of pure states. Instead of having a single pure state, we only know that the system is in the pure state $|\psi_i\rangle$ with a probability q_i:

$$\text{pure state ensemble} : \{q_i, |\psi_i\rangle\}_{i=1}^N$$

When measuring with a set of measurement operator $\{M_k\}$ we find for each state

$$|\psi_i\rangle \to \frac{M_k |\psi_i\rangle}{p_{k|i}} = \left|\psi_i^k\right\rangle$$

where, $p_{k|i} = \langle\psi_i|M_k^\dagger M_k|\psi_i\rangle$ is the probability to obtain outcome k given a state $|\psi\rangle$. Let us now assume that we didn't know what the state was but only that it came from the ensemble $\{q_i |\psi_i\rangle\}_{i=1}^N$. The probability to obtain measurement outcome k is then

$$\begin{aligned} p_k &= \sum_{i=1}^N p_{k|i} q_i = \sum_{i=1}^N q_i \langle\psi_i| M_k^\dagger M_k |\psi_i\rangle = \sum_{i=1}^N q_i \langle\psi_i| \underbrace{\sum_\alpha |\alpha\rangle\langle\alpha|}_{\text{complete set of states}} M_k^\dagger M_k |\psi\rangle \\ &= \sum_\alpha \sum_{i=1}^N q_i \langle\alpha| M_k^\dagger M_k |\psi_i\rangle \langle\psi_i|\alpha\rangle = \mathrm{tr}\big[M_k^\dagger M_k \underbrace{\sum_{i=1}^N q_i |\psi_i\rangle\langle\psi_i|}_{\text{density operator matrix}}\big] \end{aligned}$$

Using the density matrix $\rho = \sum_{i=1}^N q_i |\psi_i\rangle \langle\psi_i|$ is equivalent to using the pure state ensemble $\{q_i |\psi_i\rangle\}_{i=1}^N$, but has the advantage of being directly useful for calculations.

State after a measurement

Suppose that outcome k is observed when measuring the initial state

$$\rho = \sum_i q_i |\psi_i\rangle\langle\psi_i| \ .$$

We need to sum over all possible states compatible with the outcome.

$$\begin{aligned} \rho \to \rho_k &= \sum_i p_{i|k} \left|\psi_i^k\right\rangle\left\langle\psi_i^k\right| = \sum_i p_{i|k} \frac{M_k |\psi_i\rangle\langle\psi_i| M_k^\dagger}{p_{k|i}} \\ &\overset{(*)}{=} \sum_i \frac{q_i}{p_k} M_k |\psi_i\rangle\langle\psi_i| M_k^\dagger \\ &= \frac{M_k \rho M_k^\dagger}{p_k} = \frac{M_k \rho M_k^\dagger}{\mathrm{tr}\Big(\rho M_k^\dagger M_k\Big)} \end{aligned} \tag{1.1}$$

We briefly motivate the rule we used in aboves calculation.

Calculation: Bayes' rule (*)

In the calculation above **Bayes' rule** as been used. It states that:

$$p_{i|k}p_k = p_{k|i}q_i$$

where

$$\begin{aligned} p_{i|k} &\equiv \text{probability of being in state } |\psi_i\rangle \text{ given the measurement outcome is } k. \\ p_k &\equiv \text{probability of having measurement outcome } k. \\ p_{k|i} &\equiv \text{probability of having measurement outcome } k \text{ given the state is } |\psi_i\rangle. \\ q_i &\equiv \text{probability of having state } |\psi_i\rangle. \end{aligned}$$

The change of states after a measurement can be summarized as follows.

Summary: State after measurment

$$\text{pure state: } |\psi\rangle \to \frac{M_k |\psi\rangle}{\sqrt{p_k}}$$

$$\text{density matrix: } \rho \to \frac{M_k \rho M_k^\dagger}{p_k}$$

Properties of the density operator

- unit trace:
$$\mathrm{tr}(\rho) = \sum_i q_i \,\mathrm{tr}(|\psi_i\rangle\langle\psi_i|) = \sum_i q_i = 1$$

- Hermiticity:
$$\rho^\dagger = \sum_i q_i^* \left(|\psi_i\rangle\langle\psi_i|\right)^\dagger = \sum_i q_i \,|\psi_i\rangle\langle\psi_i| = \rho$$
since the probabilities $q_i \in \mathbb{R}$ and $|\psi\rangle^\dagger = \langle\psi|$.

- positive semi-definite:
For all vectors $|v\rangle \in \mathcal{H}$ the density matrix ρ has the property
$$\langle v|\rho|v\rangle \geq 0 \quad : \quad \langle v|\rho|v\rangle = \sum_i \underbrace{q_i}_{\geq 0} \underbrace{\langle v|\psi_i\rangle \langle\psi_i|v\rangle}_{\geq 0} \geq 0$$
Therefore all eigenvalues of ρ are either positive or 0.

Dynamics of the density operator

The evolution of pure states is given by the Schrödinger equation

$$\frac{\partial}{\partial t}\left|\psi(t)\right\rangle = -iH\left|\psi(t)\right\rangle \longleftrightarrow \left|\psi(t)\right\rangle = U(t)\left|\psi(0)\right\rangle$$

For a density matrix we thus find,

$$\begin{aligned}\rho(t) &= \sum_i q_i \left|\psi_i(t)\right\rangle\!\left\langle\psi_i(t)\right| = \sum_i q_i U(t)\left|\psi_i(0)\right\rangle\!\left\langle\psi_i(0)\right| U^\dagger(t) \\ &= U(t)\sum_i q_i \left|\psi_i(0)\right\rangle\!\left\langle\psi_i(0)\right| U^\dagger(t) = U(t)\rho(0)U^\dagger(t)\end{aligned}$$

or in differential form,

$$\begin{aligned}\frac{\partial}{\partial t}\rho(t) &= \sum_i q_i \left[\left(\frac{\partial}{\partial t}\left|\psi_i(t)\right\rangle\right)\left\langle\psi_i(t)\right| + \left|\psi_i(t)\right\rangle\left(\frac{\partial}{\partial t}\left\langle\psi_i(t)\right|\right)\right] \\ &= \sum_i q_i \left[-iH\left|\psi_i(t)\right\rangle\!\left\langle\psi_i(t)\right| + i\left|\psi_i(t)\right\rangle\!\left\langle\psi_i(t)\right| H\right] \\ &= -i\left(H\rho(t) - \rho(t)H\right) = \underbrace{-i\left[H,\rho(t)\right]}_{\text{Von Neumann equation}}\end{aligned}$$

- Calculation of expectation values:
  The expectation values of an observable A is calculated as

$$\langle A\rangle = \mathrm{tr}(A\rho)\ .$$

- Pure vs. mixed states:
  The purity P of a state is defined as

$$P = \mathrm{tr}\left(\rho^2\right)\ . \tag{1.2}$$

  If $P = 1$ ρ is pure and can be represented as $\rho = \left|\psi\right\rangle\!\left\langle\psi\right|$. In general $P \leq 1$. States with $P < 1$ are called mixed states.

- unitary equivalence of ensembles:
  In order to see the equivalence and the explicit construction of a density matrix out of a pure state ensemble, we consider two examples.

Example: Eqivalence of ensembles

Firstly, we consider a biased classical coin:

i) $\{q_i, |\psi_i\rangle\} = \left\{\left(\frac{3}{4}, \frac{1}{4}\right), (|0\rangle, |1\rangle)\right\}$ (biased classical coin)
Calculating the density matrix leads to

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|$$

Secoundly, we study the superposition of two states:

ii) $\{q_i, |\psi_i\rangle\} = \left\{\left(\frac{1}{2}, \frac{1}{2}\right), \left(\sqrt{\frac{3}{4}}|0\rangle + \sqrt{\frac{1}{4}}|1\rangle, \sqrt{\frac{3}{4}}|0\rangle - \sqrt{\frac{1}{4}}|1\rangle\right)\right\}$
Again the density matrix is given by:

$$\begin{aligned}
\rho &= \frac{1}{2}\left(\sqrt{\frac{3}{4}}|0\rangle + \sqrt{\frac{1}{4}}|1\rangle\right)\left(\sqrt{\frac{3}{4}}\langle 0| + \sqrt{\frac{1}{4}}\langle 1|\right) \\
&+ \frac{1}{2}\left(\sqrt{\frac{3}{4}}|0\rangle - \sqrt{\frac{1}{4}}|1\rangle\right)\left(\sqrt{\frac{3}{4}}\langle 0| - \sqrt{\frac{1}{4}}\langle 1|\right) \\
&= \frac{1}{2}\left(\frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1| + \sqrt{\frac{3}{4}}(|0\rangle\langle 1| + |1\rangle\langle 0|)\right) \\
&+ \frac{1}{2}\left(\frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1| - \sqrt{\frac{3}{4}}(|0\rangle\langle 1| + |1\rangle\langle 0|)\right) \\
&= \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|
\end{aligned}$$

Both represent, in fact, the same density matrix. I.e. a density matrix can be "unravelled" in different ways.

In general two ensembles of pure states $\{q_i|\psi_i\rangle\}_{i=1}^N, \{r_j|\phi_j\rangle\}_{j=1}^N$ correspond to the same density operator if and only if there exists a unitary U with entries U_{ij} such that

$$\sqrt{q_i}|\psi_j\rangle = \sum_i U_{ij}\sqrt{r_j}|\phi_j\rangle \ . \tag{1.3}$$

This can be proven by a short calculation:

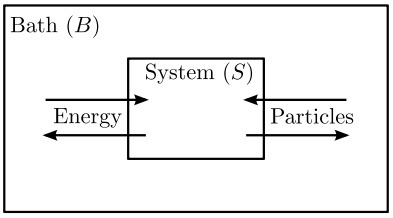
Proof

$$
\begin{aligned}
\rho &= \sum_i q_i \left|\psi_i\right\rangle\!\left\langle\psi_i\right| = \sum_i \sqrt{q_i}\left|\psi_i\right\rangle\!\left\langle\psi_i\right|\sqrt{q_i} \\
&\overset{(1.3)}{=} \sum_i \sum_j \sum_k U_{ij}\sqrt{r_j}\left|\phi_j\right\rangle\!\left\langle\phi_k\right|\sqrt{r_k}U_{ik}^* \\
&= \sum_j \sum_k \sqrt{r_j}\left|\phi_j\right\rangle\!\left\langle\phi_k\right|\sqrt{r_k} \underbrace{\sum_i U_{ik}^* U_{ij}}_{\sum_i \left(U^\dagger\right)_{ki} U_{ij} = \delta_{ki}} \\
&= \sum_j \sum_k \sqrt{r_j}\left|\phi_j\right\rangle\!\left\langle\phi_k\right|\sqrt{r_k}\delta_{kj} \\
&= \sum_j r_j \left|\phi_j\right\rangle\!\left\langle\phi_j\right|
\end{aligned}
$$

Chapter 2

Composite systems

2.1 Combining a system and a bath



Consider a two component system where we have the subsystem of interest, S, and another subsystem, B, which we call a bath. The total system evolved according to the Schrödinger equation. The Hilbert spaces of the system and the bath are, respectively,

$$\begin{aligned} \mathcal{H}_{\mathrm{S}} &= \mathrm{span}\{|i\rangle_{\mathrm{S}}\}, & i &= 1, \ldots, d_{\mathrm{S}}, \\ \mathcal{H}_{\mathrm{B}} &= \mathrm{span}\{|\mu\rangle_{\mathrm{B}}\}, & \mu &= 1, \ldots, d_{\mathrm{B}} \ . \end{aligned}$$

Usually the dimension of the bath, d_{B}, is much larger than d_{S}.
The Hilbert space of the two systems combined is the tensor product of the individual spaces,

$$\mathcal{H} = \mathcal{H}_{\mathrm{S}} \otimes \mathcal{H}_{\mathrm{B}} = \mathrm{span}\{|i\rangle_{\mathrm{S}} \otimes |\mu\rangle_{\mathrm{B}}\}$$

2.2 Partial Trace

Our primary interest is in the System S. Ideally we would thus like to find a way to remove the bath B from our description. This is achieved by the partial trace which averages out the components of B from the combined system-bath state.

Consider a (simple) operator $O = M_{\mathrm{S}} \otimes N_{\mathrm{B}}$ which acts on $\mathcal{H} = \mathcal{H}_{\mathrm{S}} \otimes \mathcal{H}_{\mathrm{B}}$.

Then,

$$\begin{aligned}\mathrm{tr_B}(M_\mathrm{S}\otimes N_\mathrm{B}) \equiv M_\mathrm{S}\,\mathrm{tr}(N_\mathrm{B}) &= M_\mathrm{S}\sum_\mu \langle\mu|\, N_\mathrm{B}\,|\mu\rangle \\ &= \sum_\mu \langle\mu|\, M_\mathrm{S}\otimes N_\mathrm{B}\,|\mu\rangle \ .\end{aligned}$$

For a general operator $O=\sum_{kj} l_{kj} M_S^k \otimes N_B^j$ we find due to linearity,

$$\mathrm{tr_B}(\hat{O}) = \sum_{kj} l_{kj}\,\mathrm{tr_B}(M_S^k\otimes N_B^j) = \sum_{kj} l_{kj} M_\mathrm{S}^k \sum_\mu \langle\mu|\, N_B^j\,|\mu\rangle$$

2.3 State of a quantum subsystem

The density matrix of the subsystem S is given by taking the partial trace of the total density matrix 'ρ' ,

$$\rho_\mathrm{S} = \mathrm{tr_B}(\rho)$$

Let us exploit this by two examples:

Example: product state

$$\rho = \rho_\mathrm{S}\otimes\rho_\mathrm{B}$$

Taking now the partial trace leads to

$$\mathrm{tr_B}\,\rho = \rho_\mathrm{S}\underbrace{\mathrm{tr_B}\,\rho_\mathrm{B}}_{=1} = \rho_\mathrm{S}\ .$$

Example: maximally entangled state ("Bell - state")

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_\mathrm{S}\,|0\rangle_\mathrm{B} + |1\rangle_\mathrm{S}\,|1\rangle_\mathrm{B}\right)$$

"Tracing" out the environment leads to

$$\begin{aligned}\mathrm{tr_B}\,|\psi\rangle\langle\psi| &= \frac{1}{2}\,\mathrm{tr_B}\left[\underset{S}{|0\rangle\langle 0|}\otimes\underset{B}{|0\rangle\langle 0|} + \underset{S}{|0\rangle\langle 1|}\otimes\underset{B}{|0\rangle\langle 1|} + \underset{S}{|1\rangle\langle 0|}\otimes\underset{B}{|1\rangle\langle 0|} + \underset{S}{|1\rangle\langle 1|}\otimes\underset{B}{|1\rangle\langle 1|}\right] \\ &= \frac{1}{2}\left[\underset{S}{|0\rangle\langle 0|}\cdot 1 + \underset{S}{|0\rangle\langle 1|}\cdot 0 + \underset{S}{|1\rangle\langle 0|}\cdot 0 + \underset{S}{|1\rangle\langle 1|}\cdot 1\right] \\ &= \frac{1}{2}\left[\underset{S}{|0\rangle\langle 0|} + \underset{S}{|1\rangle\langle 1|}\right] = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\end{aligned}$$

The reduced state of the system S is completely mixed $\left\{\left(\frac{1}{2},\frac{1}{2}\right),\left(|0\rangle,|1\rangle\right)\right\}$.

Chapter 3

Open system dynamics

3.1 Kraus operator representation

The joint unitary evolution of a system S plus bath B is given by $U(t) = e^{-iHt}$, where H is the total Hamiltonian. The total density matrix thus evolves as

$$\rho(t) = U(t)\underbrace{\rho(0)}_{\substack{\uparrow \\ \text{initial} \\ \text{state}}}U^\dagger(t) \ .$$

The reduced density matrix of S is thus

$$\rho_{\rm S}(t) = {\rm tr}_{\rm B}\,\rho(t) = \sum_\mu \langle\mu|U(t)\rho(0)U^\dagger(t)|\mu\rangle \ .$$

To simplify things we assume that initially the system and the bath are decoupled, that is $\rho(0) = \rho_{\rm S}(0) \otimes \rho_{\rm B}(0)$.

We further make use of the fact that the density matrix of the bath can be written in its spectral decomposition

$$\rho_{\rm B}(0) = \sum_\nu \lambda_\nu \, |\nu\rangle\langle\nu| \ .$$

Then

$$\begin{aligned}
\rho_{\rm S}(t) &= \sum_\mu \langle\mu|\, U(t)\rho_{\rm S}(0) \otimes \sum_\nu \lambda_\nu \, |\nu\rangle\langle\nu|\, U^\dagger(t)\, |\mu\rangle \\
&= \sum_{\mu\nu} \sqrt{\lambda_\mu}\, \langle\mu|\, U(t)\, |\nu\rangle_{\rm B}\, \rho_{\rm S}(0) \sqrt{\lambda_\nu}\, \langle\nu|\, U^\dagger(t)\, |\mu\rangle_{\rm B} \\
&= \sum_{\mu\nu} K_{\mu\nu}(t)\rho_{\rm S}(0)K^\dagger_{\mu\nu}(t)
\end{aligned}$$

In the last step we have introduced the **Kraus** operators,

$$K_{\mu\nu}(t) = \sqrt{\lambda_\nu}\, \langle\mu|\, U(t)\, |\nu\rangle \,,$$

which only act on the system.

Definition: Kraus operator sum representation (OSR)

$$\rho_S(t) = \sum_\alpha K_\alpha(t)\rho_S(0)K_\alpha^\dagger(t) \tag{3.1}$$

Since the trace of any density matrix $\rho_S(0)$ should be 1, we find

$$1 = \mathrm{tr}\big(\rho_S(t)\big) = \mathrm{tr}\Big[\sum K_\alpha(t)\rho_S(0)K_\alpha^\dagger(t)\Big] = \mathrm{tr}\Big[\sum K_\alpha^\dagger(t)K_\alpha(t)\rho_S(0)\Big] \ .$$

Therefore, the Kraus operators satisfy the normalization condition

$$\sum_\alpha K_\alpha^\dagger(t)K_\alpha(t) = \mathbb{1} \ . \tag{3.2}$$

In this representation we get Schrödinger evolution under the unitary operator U as a special case.

Example: Schrödinger evolution under unitary operator U

$$\rho_S(t) = U\rho_S(0)U^\dagger$$

So we see that in this case our Kraus operators become $K = U$.

3.2 Connection to measurement

The OSR does not only represent dynamics, but also captures measurements. Consider the measurement operator set $\{M_k\}$ with $\sum_k M_k^\dagger M_k = \mathbb{1}$. Provided that we measure outcome k, the density matrix becomes according to (1.1)

$$\rho \longrightarrow \rho_k = \frac{M_k\rho M_k^\dagger}{\mathrm{tr}\Big(M_k^\dagger M_k\rho\Big)} \ .$$

Now suppose we measure, but do not learn the outcome. In this case the density matrix will be the sum over all outcomes weighted by the corresponding probability.

$$\rho \longrightarrow \sum_k p_k\rho_k = \sum_k M_k\rho M_k^\dagger = \sum_k \underbrace{\mathrm{tr}\Big(M_k^\dagger M_k\rho\Big)}_{p_k} \underbrace{\frac{M_k\rho M_k^\dagger}{\mathrm{tr}\Big(M_k^\dagger M_k\rho\Big)}}_{\rho_k} \ .$$

By comparing to (3.1) we see that such selective measurement is represented by a OSR.

3.3 OSR as a map

It is useful to think of OSR as a map (synonym often used are "process" or "channel"), i.e.

$$\rho(t) = \Phi\big[\rho(0)\big] \quad \text{or} \quad \Phi : \rho(0) \to \rho(t)$$

with $\Phi[X] \equiv \sum_\alpha K_\alpha X K_\alpha^\dagger$. The operator Φ is often called *superoperator* as it acts in operators.

We find the following properties of the (Kraus)map:

1. Trace preservation:

$$\mathrm{tr}(\Phi[\rho]) = \mathrm{tr}\left(\sum_\alpha K_\alpha \rho K_\alpha^\dagger\right) = \mathrm{tr}\Big(\underbrace{\sum_\alpha K_\alpha^\dagger K_\alpha}_{=1}\rho\Big) = \mathrm{tr}(\rho)$$

2. Linearity:

$$\Phi\left[a\rho_1 + b\rho_2\right] = a\Phi[\rho_1] + b\Phi[\rho_2]$$

3. Positivity:

   (a) Using the spectral decomposition

$$\rho = \sum_i \lambda_i \left|i\right\rangle\!\left\langle i\right| \qquad \text{with} \qquad \lambda_i \geq 0 \ ,$$

   we find

$$\begin{aligned}\left\langle\nu\right| \Phi[\rho] \left|\nu\right\rangle &= \sum_\alpha \left\langle\nu\right| K_\alpha \rho K_\alpha^\dagger \left|\nu\right\rangle \\ &= \sum_{\alpha,i} \lambda_i \left\langle\nu\right| K_\alpha \left|i\right\rangle \left\langle i\right| K_\alpha^\dagger \left|\nu\right\rangle \\ &= \sum_{\alpha,i} \lambda_i \left|\left\langle\nu\right| K_\alpha \left|i\right\rangle\right|^2 \geq 0\end{aligned}$$

   Therefore $\Phi[\rho]$ is positive i.e. it maps positive operators on positive operators.

   (b) The Kraus map Φ is in fact a **completely positive (CP)** map. This means that it does not only map positive operators into positive operators, but it maintains positivity also when acting on a subspace of a larger Hilbert space.

   Consider the map $\Phi \otimes \mathbb{1}_\mathrm{X}$ which acts on the Hilbert space $\mathcal{H}_\mathrm{S} \otimes \mathcal{H}_\mathrm{X}$ then,

$$\begin{aligned}\left\langle\nu\right| \Phi \otimes \mathbb{1}_\mathrm{X}[\rho] \left|\nu\right\rangle &= \sum_{\alpha,i} \lambda_i \left\langle\nu\right| K_\alpha \otimes \mathbb{1}_\mathrm{X} \left|i\right\rangle\!\left\langle i\right| K_\alpha^\dagger \otimes \mathbb{1}_\mathrm{X} \left|\nu\right\rangle \\ &= \sum_{\alpha,i} \lambda_i \left|\left\langle\nu\right| K_\alpha \otimes \mathbb{1}_\mathrm{X} \left|i\right\rangle\right|^2 \geq 0\end{aligned}$$

   Therefore the Krausmap Φ is completely positive.

But why is it so important to demand positivity? As we will see in the following example, not all positive maps are completely positive.

Example: Transposition T

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

Eigenvalues of a matrix are invariant under transposition. Therefore, the transpose of ta positive matrix is positive. Now lets extend T in the sense that it acts on a subspace of a larger system. This is called partial transpose.

$$T^P = T \otimes \mathbb{1}_X \quad , \quad T^P\left(|i\rangle\langle j| \otimes |\mu\rangle\langle\nu|\right) = |j\rangle\langle i| \otimes |\mu\rangle\langle\nu|$$

Consider now a 2 qubit system in the state $|\psi\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle_S |0\rangle_X + |1\rangle_S |1\rangle_X\right]$. Then

$$\begin{aligned} T^P\left(|\psi\rangle\langle\psi|\right) &= \frac{1}{2}\left(T \otimes \mathbb{1}\right)\left[\overbrace{|00\rangle\langle 00|}^{=|0\rangle_S|0\rangle_X\langle 0|_S\langle 0|_X} + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right] \\ &= \left[|00\rangle\langle 00| + |10\rangle\langle 01| + |01\rangle\langle 10| + |11\rangle\langle 11|\right] \\ &= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

The eigenvalues of the partially transposed densitiy matrix are $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$.
We see that this is not a positive operator and thus T^P is not CP. Remark: T^P can be used as entanglement measure.

Since the properties of a quantum map are important, we briefly summarize those.

Summary: A quantum map

A map that is:

1. Trace preserving
2. Linear
3. Completely positive

3.4 Quantum maps of a qubit

3.4.1 Bloch sphere

The density matrix of a qubit can be parameterized as follows,

$$\rho = \frac{1}{2}(\mathbb{1} + \sum_i v_i \sigma_i) = \frac{1}{2}(\mathbb{1} + \underbrace{\vec{v}}_{\text{Bloch vector}} \cdot \vec{\sigma}) \tag{3.3}$$

Here we have used the Pauli's matrices,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ,$$

so that,

$$\rho = \frac{1}{2}\begin{pmatrix} 1+v_z & v_x - iv_y \\ v_x + iv_y & 1 - v_z \end{pmatrix} .$$

The trace of ρ is by construction 1, but we need to require that $||\vec{v}|| \leq 1$, for ρ to be positive. To calculate the purity, we use that

$$\rho^2 = \frac{1}{4}(\mathbb{1} + \vec{v}\cdot\vec{\sigma})(\mathbb{1} + \vec{v}\cdot\vec{\sigma}) = \frac{1}{4}(\mathbb{1} + 2\vec{v}\cdot\vec{\sigma} + (\vec{v}\cdot\vec{\sigma})^2) .$$

Following the definition of the purity (1.2) we find

$$\begin{aligned} P = \mathrm{tr}\left(\rho^2\right) &= \frac{1}{4}(\underbrace{\mathrm{tr}(\mathbb{1})}_{=2} + 2\sum_k v_k \underbrace{\mathrm{tr}(\sigma_k)}_{=0} + \sum_{km} v_k v_m \underbrace{\mathrm{tr}(\sigma_k\sigma_m)}_{\overset{(*)}{=} 2\delta_{km}}) \\ &= \frac{1}{4}(2 + 2\sum_k v_k^2) = \frac{1}{2}(1 + ||\vec{v}||^2) . \end{aligned}$$

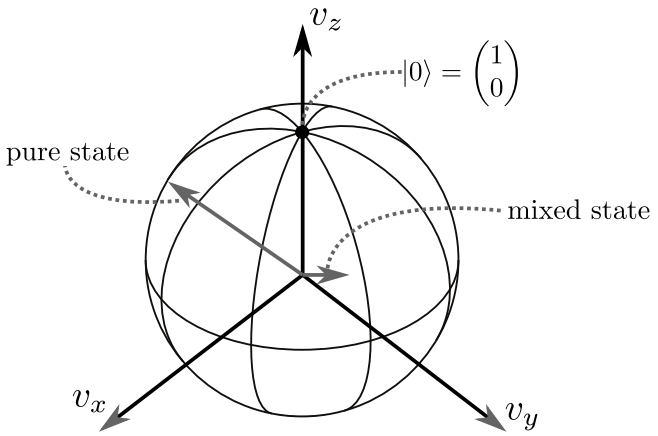
Hence, ρ is only pure if $||\vec{v}|| = 1$.

Property (*)

Note that in (*) we used a property of the Pauli matrices

$$\sigma_i\sigma_j = \delta_{i,j}\mathbb{1} + i\sum_{k=1}^{3}\epsilon_{ijk}\sigma_k .$$

The vector $\vec{v}$ is called the **Bloch vector** and allows the visualization of the density matrix on the **Bloch sphere**. This could be pictured as follows: For pure states the Blochvector reaches the surface of the Blochsphere. All non-pure (mixed) states lie inside the sphere. The northpole represents the state $|0\rangle$ and the southpole $|1\rangle$. At the origin one finds the fully mixed state $\frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$.



Fully mixed state $= \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)$

3.4.2 Phase damping map

We now come to our first example a map: the phase damping map.

$$\Phi(\rho) = p\rho + (1-p)\sigma_z\rho\sigma_z = \sum_{\alpha=0}^{1} K_\alpha \rho K_\alpha^\dagger \tag{3.4}$$

Following (3.4) we find the two Kraus operator as

$$K_0 = \sqrt{p}\mathbb{1} \qquad , \qquad K_1 = \sqrt{(1-p)}\sigma_z \ .$$

This map can be understood as follows:

$$\rho \longrightarrow \rho' = \begin{cases} \rho, & \text{with probability p} \\ \sigma_z\rho\sigma_z, & \text{with probability 1-p} \end{cases}$$

So with probability p it maps ρ on ρ and with probability $(1-p)$ it maps ρ on $\sigma_z\rho\sigma_z$. In order to see what happens to the Bloch vector under this transformation we write for a general Kraus map

$$\begin{aligned}\rho' &= \sum_\alpha K_\alpha \rho K_\alpha^\dagger = \frac{1}{2}\sum_\alpha K_\alpha \left(\mathbb{1} + \vec{v}\cdot\vec{\sigma}\right) K_\alpha^\dagger \\ &= \frac{1}{2}\left(\sum_\alpha K_\alpha K_\alpha^\dagger + \sum_{\alpha j} v_j K_\alpha \sigma_j K_\alpha^\dagger\right) \overset{!}{=} \frac{1}{2}\left(\mathbb{1} + \vec{v'}\cdot\vec{\sigma}\right)\end{aligned}$$

To get the components of $\vec{v'}$ we use

$$\mathrm{tr}(\rho'\sigma_i) = \frac{1}{2}\left(\mathrm{tr}(\sigma_i) + \sum_j v'_j \,\mathrm{tr}(\sigma_i\sigma_j)\right) = v'_i \tag{3.5}$$

and

$$\mathrm{tr}(\rho'\sigma_i) = \underbrace{\frac{1}{2}\,\mathrm{tr}\left(\sum_\alpha K_\alpha K_\alpha^\dagger \sigma_i\right)}_{C_i} + \sum_j v_j \underbrace{\frac{1}{2}\,\mathrm{tr}\big(K_\alpha \sigma_j K_\alpha^\dagger \sigma_i\big)}_{M_{ij}} \,. \tag{3.6}$$

Using the equality of equations (3.5) and (3.6) we find that the Kraus map performs an affine transformation of the Bloch vector

$$\vec{v} \to \vec{v'} = M\vec{v} + \vec{c}\,. \tag{3.7}$$

In our current example, the dephasing map we find

$$\vec{c} = 0 \qquad ; \qquad M = \begin{pmatrix} 2p-1 & 0 & 0 \\ 0 & 2p-1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence we see that the Bloch sphere is rescaled in x and y direction whereas in z it remains unchanged. Additionally, the pure states $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$ are fixed points.

3.4.3 Amplitude damping map (spontaneous emission)

Let us now consider a map in which we want to describe spontaneous emission.

Therefore, we want to find a map that accomplishes: $\begin{cases} |0\rangle \longrightarrow |0\rangle\,, & \text{with probability 1} \\ |1\rangle \longrightarrow |0\rangle\,, & \text{with probability } p \end{cases}$

One Kraus operator is obviously

$$K = \sqrt{p}\,|0\rangle\langle 1| = \sqrt{p}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\,.$$

The other Kraus operator K_0 is constrained by the normalization condition of Kraus operators (3.2). Therefore, it should satisfy $K_0^\dagger K_0 + K_1^\dagger K_1 = \mathbb{1}$. We hence make the ansatz:

$$K_0 = \begin{pmatrix} 1 & a \\ b & c \end{pmatrix}$$

This leads to

$$K_0^\dagger K_0 + K_1^\dagger K_1 = \begin{pmatrix} 1+|b|^2 & a+b^*c \\ a^*+bc^* & |a|^2+|c|^2 \end{pmatrix} + p\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \overset{!}{=} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

For the parameters $a = b = 0$ and $c = \sqrt{1-p}$ this equation is satisfied. Thus, we find the secound Kraus operator

$$K_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}$$

and the corresponding map

$$\Phi(\rho) = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix} \rho \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}}_{\text{“no-jump evolution”}} + \underbrace{p \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rho \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{\text{spin flip/spontaneous emission}} .$$

Looking at our map we see that it can be decomposed into a “no-jump” evolution and a “jump” term. This decomposition is characteristic for quantum maps. We will see more later when discussing quantum jump trajectories.

Regarding the affine transformation of the Bloch vector (3.7) we see that it transforms as

$$M = \begin{pmatrix} \sqrt{1-p} & 0 & 0 \\ 0 & \sqrt{1-p} & 0 \\ 0 & 0 & 1-p \end{pmatrix} \quad , \quad \vec{c} = \begin{pmatrix} 0 \\ 0 \\ p \end{pmatrix} .$$

Again the sphere of pure states is compressed along x and y direction. Now, unlike in the case of the dephasing map, the centre of sphere is shifted. When $p = 1$ the Bloch sphere is compressed to a single point, which is the north pole and represents a fixed point of the map:

$$\Phi(|0\rangle\langle 0|) = \underbrace{K_0 |0\rangle\langle 0| K_0^\dagger}_{|0\rangle\langle 0|} + \underbrace{K_1 |0\rangle\langle 0| K_1^\dagger}_{0} = |0\rangle\langle 0| .$$

Chapter 4

A qubit coupled to bath - exactly solvable models

4.1 Phase Damping

Let's now illustrate the rather abstract preceding discussions with a specific model. The model will describe a qubit whose coherences decay. It can be regraded as a physical realization of phase damping and allows us to understand, e.g, the involved time scales. The Hamiltonian of the system is found to be:

$$H = \underbrace{\frac{\omega_0}{2}\sigma_z}_{\substack{\text{spin}\\ \text{Hamiltonian}\\ H_{\mathrm{S}}}} + \underbrace{\sum_k \omega_k b_k^\dagger b_k}_{\substack{\text{free Hamiltonian}\\ \text{of the bath}\\ H_{\mathrm{B}}}} + \underbrace{\sum_k \sigma_z\left(g_k b_k^\dagger + g_k^* b_k\right)}_{\substack{\text{spin-bath coupling}\\ \text{Hamiltonian}\\ H_{\mathrm{SB}}}}$$

where,

$\omega_0 \rightarrow$ level spacing of qubit $\left.\begin{matrix}|1\rangle\\ |0\rangle\end{matrix}\right\}\omega_0$

$b_k^\dagger \rightarrow$ creation operator for bosonic bath mode k, satisfying $\left[b_k, b_{k'}^\dagger\right] = \delta_{kk'}$

$\omega_k \rightarrow$ frequency of the k^{th} bath mode

$g_k \rightarrow$ spin-bath mode coupling constants .

The first step is to transform in the interaction picture, by applying the unitary transformation,

$$U(t) = e^{-it(H_{\mathrm{S}}+H_{\mathrm{B}})}$$

to the Schrödinger equation:

$$i\frac{\partial}{\partial t}\left|\psi\right\rangle = \left(H_{\mathrm{S}} + H_{\mathrm{B}} + H_{\mathrm{SB}}\right)\left|\psi\right\rangle$$

with $|\psi\rangle = U(t)\,|\varphi\rangle$ we have,

$$iU(t)^\dagger \frac{\partial}{\partial t} U(t)\,|\varphi\rangle = U(t)^\dagger \big[H_\mathrm{S} + H_\mathrm{B} + H_\mathrm{SB}\big] U(t)\,|\varphi\rangle$$
$$\underbrace{iU(t)^\dagger (\frac{\partial}{\partial t} U)\,|\varphi\rangle}_{(H_\mathrm{S}+H_\mathrm{B})\,|\varphi\rangle} + i\frac{\partial}{\partial t}\,|\varphi\rangle = \big[H_\mathrm{S} + H_\mathrm{B} + U^\dagger H_\mathrm{SB} U\big]\,|\varphi\rangle$$

Therefore, we find a Schrödinger equation in the interaction picture

$$i\frac{\partial}{\partial t}\,|\varphi\rangle = U^\dagger H_\mathrm{SB} U\,|\varphi\rangle \ .$$

In order to see the systems dynamics, we start by calculating the *interaction picture Hamiltonian*:

$$H_\mathrm{I} = U^\dagger H_\mathrm{SB} U = \sum_k \underbrace{U^\dagger \sigma_z U}_{=\sigma_z} \big(g_k U^\dagger b_k^\dagger U + g_k^* U^\dagger b_k U\big)$$

using,

$$\begin{aligned} U^\dagger b_k^\dagger U &= e^{it(H_\mathrm{S}+H_\mathrm{B})} b_k^\dagger e^{-it(H_\mathrm{S}+H_\mathrm{B})} = e^{itH_\mathrm{B}} b_k^\dagger e^{-itH_\mathrm{B}} \\ &= e^{it\sum_m \omega_m b_k^\dagger b_m} b_k^\dagger e^{-it\sum_m \omega_m b_m^\dagger b_m} = e^{it\omega_k b_k^\dagger b_k} b_k^\dagger e^{it\omega_k b_k^\dagger b_k} \overset{(*)}{=} b_k^\dagger e^{i\omega_k t} \ . \end{aligned}$$

Calculation: (*)

Note that we used in the last step (*) the fact, that

$$\big[b^\dagger b, b^\dagger\big] = b^\dagger \underbrace{\big[b, b^\dagger\big]}_{=1} + \underbrace{\big[b^\dagger, b^\dagger\big]}_{=0} b = b^\dagger \ ,$$

which ultimately leads to

$$\begin{aligned} e^{it\omega b^\dagger b} b^\dagger e^{-it\omega b^\dagger b} &= b^\dagger + \underbrace{\big[it\omega b^\dagger b, b^\dagger\big]}_{=it\omega b^\dagger} + \frac{1}{2!} \underbrace{\Big[it\omega b^\dagger b, \underbrace{\big[it\omega b^\dagger b, b^\dagger\big]}_{=it\omega b^\dagger} \Big]}_{=(it\omega)^2 b^\dagger} \\ &\quad + \frac{1}{3!} \underbrace{\Big[it\omega b^\dagger b, \underbrace{\Big[it\omega b^\dagger b, \underbrace{\big[it\omega b^\dagger b, b^\dagger\big]}_{=it\omega b^\dagger} \Big]}_{=(it\omega)^2 b^\dagger} \Big]}_{=(it\omega)^3 b^\dagger} + \dots \\ &= \sum_{n=0}^{\infty} \frac{(it\omega)^n}{n!} b^\dagger = b^\dagger \sum_{n=0}^{\infty} \frac{(it\omega)^n}{n!} = b^\dagger e^{it\omega} \ . \end{aligned}$$

We find for the interaction Hamiltonian

$$H_{\mathrm{I}} = \sum_k \sigma_z \left(g_k b_k^\dagger e^{i\omega_k t} + g_k^* b_k e^{-i\omega_k t} \right) = H_{\mathrm{I}}(t)$$

To find the time evolution operator we use the **Magnus** expansion:

$$U_{\mathrm{I}}(t) = e^{\sum_{k=1}^{\infty} \Omega_k(t)} \tag{4.1}$$

with

$$\begin{aligned}
\Omega_1(t) &= \int_0^t d\tau [-iH_{\mathrm{I}}(\tau)] \\
\Omega_2(t) &= \frac{1}{2} \int_0^t d\tau \int_0^\tau d\tau' [-iH_{\mathrm{I}}(\tau), -iH_{\mathrm{I}}(\tau')] \\
\Omega_3(t) &= \frac{1}{6} \int_0^t d\tau \int_0^\tau d\tau' \int_0^{\tau'} d\tau'' \left([-iH_{\mathrm{I}}(\tau), [-H_{\mathrm{I}}(\tau'), -iH_{\mathrm{I}}(\tau'')]] \right. \\
&\quad \left. + [-iH_{\mathrm{I}}(\tau''), [-H_{\mathrm{I}}(\tau'), -iH_{\mathrm{I}}(\tau)]] \right) \\
&\vdots
\end{aligned}$$

Since

$$\begin{aligned}
[H_{\mathrm{I}}(\tau), H_{\mathrm{I}}(\tau')] &= \sum_{km} \left(g_k g_m^* e^{i\omega_k \tau - i\omega_m \tau'} \overbrace{[b_k^\dagger, b_m]}^{=-\delta_{km}} + g_k^* g_m e^{-i\omega_k \tau + i\omega_m \tau'} \overbrace{[b_k, b_k^\dagger]}^{=\delta_{km}} \right) \\
&= \sum_k \left(-|g_k|^2 e^{i\omega_k(\tau - \tau')} + |g_k|^2 e^{-i\omega_k(\tau - \tau')} \right) \\
&= -2i \sum_k |g_k|^2 \sin[\omega_k(\tau - \tau')]
\end{aligned}$$

is a complex number, $\Omega_3(t)$ and higher terms vanish. One thus finds for the interaction time evolution operator (4.1)

$$U_{\mathrm{I}}(t) = \underbrace{\exp\left(i \int_0^t d\tau \int_0^\tau d\tau' \sum_k |g_k|^2 \sin[\omega_k(\tau - \tau')] \right)}_{\equiv \exp(i\varphi(t))} \underbrace{\exp\left(-i \int_0^t d\tau H_{\mathrm{I}}(\tau) \right)}_{\equiv V_{\mathrm{I}}(t)}$$

with

$$\begin{aligned}
V_{\mathrm{I}}(t) &= \exp\left(-i\sigma_z \sum_k \left(g_k \frac{i(e^{i\omega_k t} - 1)}{\omega_k} b_k^\dagger + g_k^* \frac{(-i)(e^{-i\omega_k t} - 1)}{\omega_k} b_k \right) \right) \\
&= \exp\left(\sigma_z \sum_k \left(\alpha_k b_k^\dagger - \alpha_k^* b_k \right) \right) = \prod_k D(\sigma_z \alpha_k) \ .
\end{aligned}$$

Here $\alpha_k = \frac{g_k}{\omega_k}(e^{i\omega_k t} - 1)$ and the displacement operator $D(x) = e^{xb^\dagger - x^* b}$ were introduced.

Furthermore, one finds for $V_{\mathrm{I}}(t)$

$$\begin{aligned} V_{\mathrm{I}}(t) &= |0\rangle\langle 0| \prod_k D(-\alpha_k) + |1\rangle\langle 1| \prod_k D(\alpha_k) \\ &= |0\rangle\langle 0| \prod_k D(\alpha_k)^\dagger + |1\rangle\langle 1| \prod_k D(\alpha_k) \ . \end{aligned}$$

Having now obtained an explicit form of the Hamiltonian in the interaction picture, we want to study the decay of coherences. Let us suppose that the initial state of the total system is given by,

$$\rho(0) = \rho_{\mathrm{S}}(0) \otimes \rho_{\mathrm{B}} \quad \text{with} \quad \overbrace{\rho_{\mathrm{B}} = \underbrace{\frac{1}{Z_{\mathrm{B}}}}_{\text{partition function of the bath}} e^{-\beta H_{\mathrm{B}}}}^{\text{the bath is in thermal state with temperature } k_{\mathrm{B}}T = \frac{1}{\beta}} \ .$$

The spin density matrix we write with the help of the Bloch vector (3.3) as

$$\rho_{\mathrm{S}}(0) = \frac{1}{2}(\mathbb{1} + \vec{v}(0) \cdot \vec{\sigma}) \ .$$

Our goal is now to calculate the reduced density of the spin at time t: (note, that we are still in the interaction picture)

$$\begin{aligned} \rho_{\mathrm{S}}(0) &= \mathrm{tr}_{\mathrm{B}}\big[V_{\mathrm{I}}(t)\rho(0)V_{\mathrm{I}}(t)^\dagger\big] \\ &= \frac{1}{2}\Big[\overbrace{\mathrm{tr}_{\mathrm{B}}\big(V_{\mathrm{I}}(t)\rho_{\mathrm{B}}V_{\mathrm{I}}(t)^\dagger\big)}^{=\mathrm{tr}_{\mathrm{B}}\,\rho_{\mathrm{B}}\,=\,\mathbb{1}} + v_z \overbrace{\mathrm{tr}_{\mathrm{B}}\big(V_{\mathrm{I}}(t)\sigma_z\rho_{\mathrm{B}}V_{\mathrm{I}}(t)^\dagger\big)}^{=\mathrm{tr}_{\mathrm{B}}(V_{\mathrm{I}}(t)\rho_{\mathrm{B}}V_{\mathrm{I}}(t)^\dagger)\sigma_z\,=\,\sigma_z} \\ &+ v_y\,\mathrm{tr}_{\mathrm{B}}\left(V_{\mathrm{I}}(t)\sigma_y\rho_{\mathrm{B}}V_{\mathrm{I}}(t)^\dagger\right) + v_x\,\mathrm{tr}_{\mathrm{B}}\big(V_{\mathrm{I}}(t)\sigma_x\rho_{\mathrm{B}}V_{\mathrm{I}}(t)^\dagger\big)\Big] \\ &= \frac{1}{2}\Big[\mathbb{1} + v_z\sigma_z + (v_x - iv_y)\,\mathrm{tr}_{\mathrm{B}}\big(V_{\mathrm{I}}(t)\sigma^+\rho_{\mathrm{B}}V_{\mathrm{I}}(t)^\dagger\big) + (v_x + iv_y)\,\mathrm{tr}_{\mathrm{B}}\big(V_{\mathrm{I}}(t)\sigma^-\rho_{\mathrm{B}}V_{\mathrm{I}}(t)^\dagger\big)\Big] \end{aligned}$$

We see that the z component of Bloch vector does not change. Therefore, the populations do not change.

Let us now investigate the evolution of the coherences through analyzing the term

$$\begin{aligned} \mathrm{tr}\Big(V_{\mathrm{I}}(t)\sigma^-\rho_{\mathrm{B}}V_{\mathrm{I}}^\dagger(t)\Big) &= \mathrm{tr}\left(\prod_k D^\dagger(\alpha_k)\rho_{\mathrm{B}}D^\dagger(\alpha_k)\right)\sigma^- \\ &= \sigma^- \prod_k \mathrm{tr}\left(D^\dagger(2\alpha_k)\frac{e^{-\beta\omega_k b_k^\dagger b_k}}{Z_k}\right) \ . \end{aligned} \tag{4.2}$$

Calculation:

It is possible to evaluate the trace. One finds

$$\begin{aligned}\mathrm{tr}\Big(D^\dagger(2\alpha)e^{-\beta\omega b^\dagger b}\Big) &= \sum_n e^{-\beta\omega_n} \underbrace{\langle n|D^\dagger(2\alpha)|n\rangle}_{=\exp\left(-\frac{|2\alpha|^2}{2}\right)L_n\left(|2\alpha|^2\right)} \\ &= \exp\left(-\frac{|2\alpha|^2}{2}\right)\sum_n e^{\beta\omega_n}L_n\left(|2\alpha|^2\right) \ .\end{aligned}$$

Using the generating function of Laguerre polynomials L_n

$$\sum_{n=0}^{\infty} t^n L_n(x) = \frac{1}{1-t}e^{-x\frac{t}{1-t}}$$

one further simplifies

$$\begin{aligned}\mathrm{tr}\Big(D^\dagger(2\alpha)e^{-\beta\omega b^\dagger b}\Big) &= e^{-\frac{|2\alpha|^2}{2}}\frac{1}{1-e^{-\beta\omega}}\exp\left(-|2\alpha|^2\frac{e^{-\beta\omega}}{1-e^{-\beta\omega}}\right) \\ &= \underbrace{\frac{1}{1-e^{-\beta\omega}}}_{=Z}\exp\Big(-|2\alpha|^2\underbrace{\left(\frac{1}{2}+\frac{e^{-\beta\omega}}{1-e^{-\beta\omega}}\right)}_{\substack{=\frac{1}{2}\frac{1+e^{-\beta\omega}}{1-e^{-\beta\omega}} \\ =\frac{1}{2}\coth\left(\frac{\beta\omega}{2}\right)}}\Big) \\ &= Z\cdot\exp\left(-2|\alpha|^2\coth\left(\frac{\beta\omega}{2}\right)\right) \ . \qquad (4.3)\end{aligned}$$

Following the definition for α, one finds for $|\alpha|^2$

$$|\alpha|^2 = 2\frac{|g|^2}{\omega^2}\left(1-\cos(\omega t)\right) \ . \qquad (4.4)$$

Coming back to (4.2) we find by using (4.3) and (4.4)

$$\mathrm{tr}\Big(V_{\mathrm{I}}(t)\sigma^-\rho_{\mathrm{B}}V_{\mathrm{I}}^\dagger(t)\Big) = \sigma^-\exp\left(-\sum_k\frac{4|g_k|^2}{\omega_k^2}\coth\left(\frac{\beta\omega_k}{2}\right)\left(1-\cos(\omega_k t)\right)\right) = \sigma^- e^{\Gamma(t)} \ .$$

In order to make progress we have to deal with the summation. To this end we introduce the density $f(\omega)$ of the oscillator modes such that

$$\sum_k\frac{|g_k|^2}{\omega_k^2} \to \int_0^\infty d\omega f(\omega)|g(\omega)|^2\frac{1}{\omega^2} \ .$$

Furthermore, we introduce the spectral mode density: $J(\omega) = 4f(\omega)|g(\omega)|^2$, which leads to

$$\Gamma(t) = -\int_0^\infty d(\omega)J(\omega)\coth\left(\frac{\beta\omega}{2}\right)\frac{1-\cos\omega t}{\omega^2} \ .$$

In order to finish the calculation we require an explicit expression for the spectral mode density. We use $J(\omega) = \omega \exp\{-\frac{\omega}{\Omega}\}$ where Ω is the cut-off frequency.
We then obtain,

$$\Gamma(t) = \underbrace{-\int_0^\infty d\omega\ \omega e^{-\frac{\omega}{\Omega}} \frac{1-\cos(\omega t)}{\omega^2}}_{\Gamma_{\rm vac}(t)\cdots\text{vacuum contribution}} + \underbrace{\int_0^\infty d\omega\ \omega e^{-\frac{\omega}{\Omega}} \left(1-\coth\frac{\beta\omega}{2}\right)\frac{1-\cos(\omega t)}{\omega^2}}_{\Gamma_{\rm th}(t)\ldots\text{thermal contribution}}$$

The temperature independent vacuum contribution yields $\Gamma_{\rm vac}(t) = -\frac{1}{2}\ln(1+\Omega^2 t^2)$. It describes how vacuum fluctuations of the bath affect coherence and depends on the cut off frequency Ω. In order to evaluate the thermal part we rewrite it as,

$$\Gamma_{\rm th}(t) = -\frac{1}{\beta}\int_0^t ds \int_0^\infty dx\ e^{-\frac{x}{\Omega\beta}} \left[\coth\left(\frac{x}{2}\right)-1\right]\sin\left(\frac{sx}{\beta}\right).$$

We assume now that he temperature is much smaller than the cut off frequency, i.e. $\frac{1}{\Omega\beta} = \frac{T}{\Omega} \ll 1$.
With this assumption we obtain,

$$\begin{aligned}\Gamma_{\rm th}(t) &\approx -\frac{1}{\beta}\int_0^t ds \underbrace{\int_0^\infty dx \left[\coth\left(\frac{x}{2}\right)-1\right]\sin\left(\frac{sx}{\beta}\right)}_{\pi\coth\left(\frac{s}{\beta}\pi\right)-\frac{\beta}{s}} \\ &= \ln(t) + \ln\left(\frac{\pi}{\beta}\right) - \ln\left(\sinh\frac{t\pi}{\beta}\right) \\ &= -\ln\left(\frac{\sinh\left(\frac{\pi}{\beta}t\right)}{\frac{\pi}{\beta}t}\right) = -\ln\left(\frac{\sinh\left(\frac{t}{\tau_{\rm th}}\right)}{\frac{t}{\tau_{\rm th}}}\right).\end{aligned}$$

Here we have introduced the thermal correlation time $\tau_{\rm th} = \frac{\beta}{\pi} = \frac{1}{\pi T} = 2.4\cdot 10^{-12}\ \frac{\rm s}{\rm T[K]}$. Our final result for the reduced density matrix of the spin is thus:

$$\begin{aligned}\rho_{\rm S} &= \frac{1}{2}\left[\mathbb{1} + v_z\sigma_z + (v_x+iv_y)\sigma^- e^{\Gamma(t)} + (v_x-iv_y)\sigma^+ e^{\Gamma^*(t)}\right] \\ &= \frac{1}{2}\left[\mathbb{1} + (v_x\sigma_x + v_y\sigma_y)e^{\Gamma(t)} + v_z\sigma_z\right]\end{aligned}$$

with

$$\Gamma(t) \simeq -\frac{1}{2}\ln\left(1+\Omega^2 t^2\right) - \ln\left(\frac{\sinh\left(\frac{t}{\tau_{\rm th}}\right)}{\frac{t}{\tau_{\rm th}}}\right).$$

While the populations are unchanged the coherences (off-diagonal terms) decay. The corresponding affine transformation of the Bloch vector is,

$$\vec{v} \longrightarrow \vec{v}' = \begin{pmatrix} e^{\Gamma(t)} & & \\ & e^{\Gamma(t)} & \\ & & 1 \end{pmatrix}\vec{v}$$

which corresponds to the phase damping map with $p = \frac{1+e^{\Gamma(t)}}{2}$.

Let us finally have a look at the time scale at which the coherences decay. We can identify three regimes:

1. Short time regime, i.e. $t \ll \Omega^{-1}$:

   Here $\Gamma(t) \approx -\frac{1}{2}\Omega^2 t^2$. This quadratic dependence is characteristic for quantum dynamics in the sense that one typically obtains this fully expanding the time-evolution operator.

2. vacuum regime, i.e. $\Omega^{-1} \ll t \ll \tau_{\text{th}}$

   Here $\Gamma(t) \approx -\ln(\Omega t)$ and decoherence effects are mainly due to the vacuum fluctuations of the bath.

3. thermal regime, i.e. $\tau_{\text{th}} \ll t$

   Here $\Gamma(t) \approx -\frac{t}{\tau_{\text{th}}}$. Due to the linear dependence with time one calls this also the Markovian regime. Coherences decay exponentially. This dynamics in this regime can be captured by a *Markovian Lindblad Master equation.*

4.2 Radiative decay-spontaneous emission

It is well-known that an atom in an excited state $|e\rangle$ is not stationary. It will eventually decay into the ground state $|g\rangle$ under the emission of a photon into the electromagnetic field. Lets assume that initially the atom- field system is in the state

$$|\psi(0)\rangle = |e, \{0\}\rangle = \overbrace{|e\rangle}^{\substack{\text{excited state}\\ \text{of atom}}} \otimes \underbrace{|\{0\}\rangle}_{\substack{\text{vacuum state of}\\ \text{the electromagnetic}\\ \text{field}}}$$

The state will evolve under the Hamiltonian

$$H = \underbrace{\omega_0 |e\rangle\langle e|}_{\substack{\text{Hamiltonian of a}\\ \text{two-level atom}\\ |e\rangle \\ \\ |g\rangle}} + \underbrace{\sum_{\vec{k},s} \omega_k a^\dagger_{\vec{k}s} a_{\vec{k},s}}_{\substack{\text{Hamiltonian of the}\\ \text{electromagnetic field}\\ \vec{k}, s \text{ label wavevector}\\ \text{and polarization}}} \underbrace{- \sum_{\vec{k},s} \left(g_{\vec{k},s} |e\rangle\langle g| a_{\vec{k},s} + g^*_{\vec{k},s} |g\rangle\langle e| a^\dagger_{\vec{k},s} \right)}_{\substack{\text{coupling between elecromag. field}\\ \text{and atom (de-excitation of atom}\\ \text{creates photon)}}}$$

Here we have used

- $g_{\vec{k},s} = i\sqrt{\frac{\omega_k}{2\varepsilon V}} \left(\vec{d} \cdot \vec{\epsilon}_{\vec{k},s} \right) \ldots$ coupling constant
- $\vec{d} \ldots$ dipole moment of the atomic transition
- $\vec{\epsilon}_{\vec{k},s} \ldots$ polarization vector of photons

- V ... volume of space (result will not depend on that)

Under the action of the Hamiltonian the state $|\psi(0)\rangle$ will evolve into

$$|\psi(t)\rangle = \alpha(t)e^{-i\omega_0 t}\underbrace{|e,\{0\}\rangle}_{\text{initial state}} + \underbrace{\sum_{\vec{k},s}\beta_{\vec{k},s}(t)e^{-i\omega_k t}\left|g,1_{\vec{k},s}\right\rangle}_{\substack{\text{final state can only contain}\\ \text{one photon with wave}\\ \text{vectors } \vec{k} \text{ and polarization } s}}$$

In the next step we plug this wave function into the Schrödinger equation $i\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle$:

$$\begin{aligned} H|\psi(t)\rangle &= \alpha(t)e^{-i\omega_0 t}\omega_0|e,\{0\}\rangle + \sum_{\vec{k},s}\beta_{\vec{k},s}(t)e^{-i\omega_k t}\omega_k\left|g,1_{\vec{k},s}\right\rangle \\ &\quad - \sum_{\vec{k},s}\beta_{\vec{k},s}(t)g_{\vec{k},s}e^{-i\omega_k t}|e,\{0\}\rangle - \alpha(t)e^{-i\omega_0 t}\sum_{\vec{k},s}g^*_{\vec{k},s}\left|g,1_{\vec{k},s}\right\rangle \\ i\frac{\partial}{\partial t}|\psi(t)\rangle &= i\left(\dot{\alpha}(t) - i\omega_0\alpha(t)\right)e^{-i\omega_0 t})|e,\{0\}\rangle + i\sum_{\vec{k},s}\left(\dot{\beta}(t)_{\vec{k},s} - i\omega_k\beta_{\vec{k},s}(t)\right)\left|g,1_{\vec{k},s}\right\rangle \end{aligned}$$

Comparing coefficients leads to

$$\begin{aligned} \dot{\alpha}(t) &= i\sum_{\vec{k},s}g_{\vec{k},s}e^{-i(\omega_k-\omega_0)t}\beta_{\vec{k},s} \\ \dot{\beta}_{\vec{k},s}(t) &= ig^*_{\vec{k},s}e^{i(\omega_k-\omega_0)t}\alpha(t)\ . \end{aligned}$$

We formally integrate the equation for the $\beta_{\vec{k},s}$ using the initial condition $\beta_{\vec{k},s}(0) = 0$

$$\beta_{\vec{k},s} = ig^*_{\vec{k},s}\int_0^t dt' e^{-i(\omega_k-\omega_0)(t')}\alpha(t')$$

Plugging this back into the equation for $\alpha(t)$, yields

$$\dot{\alpha}(t) = -\sum_{\vec{k},s}\left|g_{\vec{k},s}\right|^2\overbrace{\int_0^t dt' e^{-i(\omega_k-\omega_0)(t-t')}\alpha(t')}^{\text{memory kernel}}$$

This type of integro-differential equation is very typical when dealing with open quantum systems. In order to solve it we need to make some simplifications/approximations. The first step is to go from the sum over $\vec{k}$ to an integral.

$$\sum_{\vec{k},s} \longrightarrow \sum_{s=1}^{2}\frac{V}{(2\pi)^3}\int_0^\infty dk k^2\int_0^\pi d\theta\sin\theta\int_0^{2\pi}d\varphi$$

One then has,

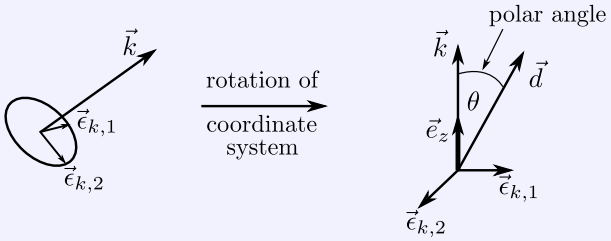
$$\sum_{\vec{k},s} \left| g_{\vec{k},s} \right|^2 = \int_0^\infty dk k^2 \frac{\omega_k}{2(2\pi)^3 \epsilon_0} \left[\sum_{s=1}^{2} \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\varphi (\vec{d} \cdot \vec{\epsilon}_{\vec{k},s})^2 \right]$$

Using

$$\left(\vec{d} \cdot \vec{\epsilon}_{\vec{k}1}\right)^2 + \left(\vec{d} \cdot \vec{\epsilon}_{\vec{k}2}\right)^2 = |\vec{d}|^2 (1 - \cos^2\theta) = |\vec{d}|^2 \sin^2\theta$$

Calculation:

One can see this by rotating the coordinate system in such a manner, that $\vec{k}$ is parallel to the z axis.



Therefore,

$$\left(\vec{d} \cdot \vec{e_z}\right)^2 + \left(\vec{d} \cdot \vec{\epsilon_{k,1}}\right)^2 + \left(\vec{d} \cdot \vec{\epsilon_{k,2}}\right)^2 = |\vec{d}|^2$$

with $\vec{d} \cdot \vec{e_z} = |\vec{d}| \cos\theta$ one sees

$$\left(\vec{d} \cdot \vec{\epsilon_{k,1}}\right)^2 + \left(\vec{d} \cdot \vec{\epsilon_{k,2}}\right)^2 = |\vec{d}|^2 \left(1 - \cos^2\theta\right)$$

one finds

$$\sum_{s=1}^{2} \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\varphi (\vec{d} \cdot \vec{\epsilon}_{\vec{k},s})^2 = \frac{8\pi}{3} |\vec{d}|^2 \ .$$

Which leads to

$$\sum_{\vec{k},s} |g_{\vec{k},s}|^2 = \frac{|\vec{d}|^2}{6\pi^2 \epsilon_0 c^3} \int_0^\infty d\omega_k \omega_k^3$$

where we used $\omega_k = c|\vec{k}|$. With this we can write for the excited state amplitude:

$$\dot{\alpha}(t) = -\frac{|\vec{d}|^2}{6\pi^2 \epsilon_0 c^3} \int_0^\infty d\omega_k \omega_k^3 \int_0^t dt' e^{-i(\omega_k - \omega_0)(t-t')} \alpha(t')$$

Using the method of substitution one can rewrite the last integral by introducing $\tau = t - t'$

$$\begin{aligned}\int_0^t dt' e^{-i(\omega_k-\omega_0)(t-t')}\alpha(t') &= -\int_t^0 d\tau e^{-i(\omega_k-\omega_0)\tau}\alpha(t-\tau) \\ &= \int_0^t d\tau e^{-i(\omega_k-\omega_0)\tau}\alpha(t-\tau)\ .\end{aligned}$$

This leads to

$$\dot{\alpha}(t) = -\frac{|\vec{d}|^2}{6\pi^2\epsilon_0 c^3}\int_0^\infty d\omega_k\omega_k^3\int_0^t d\tau e^{-i(\omega_k-\omega_0)\tau}\alpha(t-\tau)\ .$$

In the next step we assume that $\alpha(t)$ evolves on a time scale that is much slower than the inverse of the frequency ω_0. Therefore, $\alpha(t')$ is hardly changing its value over the time before the integral over the the exponential is averaging to zero.

$$\dot{\alpha}(t) \approx -\frac{|\vec{d}|^2}{6\pi^2\epsilon_0 c^3}\left[\int_0^\infty d\omega_k\omega_k^3\int_0^t d\tau' e^{-i(\omega_k-\omega_0)\tau}\right]\alpha(t)$$

This makes the equation local in time, i.e. $\dot{\alpha}(t)$ does not depend on the past, but only on $\alpha(t)$.

The final approximation is to extend the upper limit of the integration to infinity. This is justified because $\alpha(t)$ varies on a time scale $t \gg \frac{1}{\omega_0}$. Hence for the times of interest we can as well assume $t \to \infty$. Therefore,

$$\int_0^t dt' e^{-i(\omega_k-\omega_0)(t-t')} \to \int_0^\infty d\tau e^{-i(\omega_k-\omega_0)\tau} = \pi\delta(\omega_k-\omega_0) - i\mathcal{P}\left\{\frac{1}{\omega_k-\omega_0}\right\}$$

where $\mathcal{P}$ stands for Cauchy principal part.

Calculation:

In order to solve the integral we used:

$$\begin{aligned}\int_0^\infty dt e^{-at}e^{-i\omega t} &= \frac{1}{a+i\omega} \\ &= \frac{a}{a^2+\omega^2} - i\frac{\omega}{a^2+\omega^2}\end{aligned}$$

With

$$\begin{aligned}\lim_{a\to 0}\frac{a}{a^2+\omega^2} &= \pi\delta(\omega) \\ \lim_{a\to 0}\frac{\omega}{a^2+\omega^2} &= \frac{1}{\omega} \to \mathcal{P}\left\{\frac{1}{\omega}\right\}\ .\end{aligned}$$

Both of the above expressions are distributions that appear under an integral.

We thus obtain,

$$\dot{\alpha}(t) = \left[i\Delta\omega - \frac{\Gamma}{2} \right] \alpha(t)$$

where $\Gamma = \frac{\omega_0^3 |\vec{d}|^2}{3\pi\epsilon_0 c^3}$ is the decay rate and $\Delta\omega = \frac{|\vec{d}|^2}{6\pi^2\epsilon_0 c^3} \mathcal{P} \left\{ \int_0^{\cdots} d\omega_k \frac{\omega_k^3}{\omega_k - \omega_0} \right\}$ is the so-called Lamb shift.

Within this treatment the Lamb-shift diverges, which we ignore here (the integral must be properly regularized). Nevertheless, we can calculate for the atom being in the excited state, which is given by $|\alpha(t)|^2$. Since, $\alpha(t) = \alpha(0) e^{[i\Delta\omega - \frac{\Gamma}{2}]t}$ we find

$$|\alpha(t)|^2 = |\alpha(0)|^2 e^{-\Gamma t} \ .$$

At the level of the Bloch vector $\vec{v}$ this dynamics corresponds to the affine map

$$\vec{v'} = M\vec{v} + \vec{c}$$

with

$$M = \begin{pmatrix} e^{-\frac{\Gamma}{2}t} & 0 & 0 \\ 0 & e^{-\frac{\Gamma}{2}t} & 0 \\ 0 & 0 & e^{-\frac{\Gamma}{2}t} \end{pmatrix} \qquad , \qquad \vec{c} = \begin{pmatrix} 0 \\ 0 \\ 1 - e^{-\Gamma t} \end{pmatrix}$$

Note that we here assumed $\Delta\omega = 0$, i.e. we neglect the Lamb-shift.
Further we see that:

- This is amplitude damping.
- The stationary state is given by

  $$\lim_{t\to\infty} \vec{v'} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

  which corresponds to the stationary density matrix $\rho_{\text{ss}} = |g\rangle\langle g|$.

Chapter 5

The Lindblad master equation

Our goal is to find a more general procedure that allows us to obtain an equation of motion for the density matrix of the system.

We consider a situation where the system is weakly coupled to the bath. The total Hamiltonian is assumed to have the form:

$$H = \underbrace{H_{\mathrm{S}}}_{\text{system}} + \underbrace{H_{\mathrm{B}}}_{\text{bath}} + \underbrace{H_{\mathrm{SB}}}_{\text{interaction}}$$

In the next step we transform into the interaction picture, using the unitary transformation

$$U_{\mathrm{I}} = \exp\{-it(H_{\mathrm{S}} + H_{\mathrm{B}})\} \ .$$

The density matrix ρ of the total system, i.e. system + bath, then evolves in time according to the von Neumann equation

$$\frac{\partial}{\partial t}\rho(t) = -i\,[H_{\mathrm{I}}(t), \rho(t)]$$

with $H_{\mathrm{I}}(t) = U^{\dagger} H_{\mathrm{SB}} U$. Formally integrating this equation yields

$$\rho(t) = \rho(0) - i\int_0^t ds\big[H_{\mathrm{I}}(s), \rho(s)\big] \ ,$$

and after inserting this result into the differential equation, we obtain:

$$\frac{\partial}{\partial t}\rho(t) = -i\big[H_{\mathrm{I}}(t), \rho(0)\big] - \int_0^t ds\Big[H_{\mathrm{I}}(t), \big[H_{\mathrm{I}}(s), \rho(s)\big]\Big] \ .$$

In the next step we take the trace over the bath degrees of freedom and assume that $\mathrm{tr}_{\mathrm{B}}[H_{\mathrm{I}}(t), \rho(0)] = 0$ (this can be achieved when the state of the bath and system factorize at $t = 0$, which we assume see below).

The density matrix of system evolves according to

$$\frac{\partial}{\partial t}\rho_{\mathrm{S}}(t) = \frac{\partial}{\partial t}\,\mathrm{tr}_{\mathrm{B}}(\rho(t)) = -\int_0^t ds\,\mathrm{tr}_{\mathrm{B}}\Big\{\Big[H_{\mathrm{I}}(t),\big[H_{\mathrm{I}}(s),\rho(s)\big]\Big]\Big\}\ .$$

- Born Approximation:

  We assume that that the coupling between the system and the reservoir is weak. Thus the density matrix of the reservoir is only negligibly affected by the interaction. The total density matrix at time t may then be approximated by a tensor product

  $$\rho(t) = \rho_{\mathrm{S}}(t)\otimes\rho_{\mathrm{B}}\ .$$

  This yields a closed integro-differential equation.

  $$\frac{\partial}{\partial t}\rho_{\mathrm{S}}(t) = \frac{\partial}{\partial t}\,\mathrm{tr}_{\mathrm{B}}(\rho(t)) = -\int_0^t ds\,\mathrm{tr}_{\mathrm{B}}\Big\{\Big[H_{\mathrm{I}}(t),\big[H_{\mathrm{I}}(s),\rho_{\mathrm{S}}(s)\otimes\rho_{\mathrm{B}}\big]\Big]\Big\}$$

- Markov Approximaion:

  We replace $\rho_{\mathrm{S}}(s)$ by $\rho_{\mathrm{S}}(t)$, which will be justified by the fact that bath correlations/ excitations decay on a much shorter time-scale than that of any relevant dynamics of the system. In some sense this is equivalent to a coarse-graining over a time-scale which is long for the bath, but short for the system.

  This yields the *Redfield equation*:

  $$\frac{\partial}{\partial t}\rho_{\mathrm{S}}(t) = -\int_0^t ds\,\mathrm{tr}_{\mathrm{B}}\Big\{\big[H_{\mathrm{I}}(t),[H_{\mathrm{I}}(s),\rho_{\mathrm{S}}(t)\otimes\rho_{\mathrm{B}}]\big]\Big\}\ .$$

  The final step is to perform the change of variable $s\to t-s$ and to shift the upper integration boundary to infinity. This is justified when the integrand vanishes sufficiently fast (we justify the a posteriori). This leads to the *Markovian quantum master equation*:

  $$\frac{\partial}{\partial t}\rho_{\mathrm{S}}(t) = -\int_0^\infty ds\,\mathrm{tr}_{\mathrm{B}}\Big\{\big[H_{\mathrm{I}}(t),[H_{\mathrm{I}}(t-s),\rho_{\mathrm{S}}(t)\otimes\rho_{\mathrm{B}}]\big]\Big\}\ .$$

- Secular approximation:

  Ultimately, this approximation entails an averaging over rapidly oscillating terms. It will allow us to bring the master equation into a convenient shape, which is called the *Lindblad form*.

  The starting point is the following general decomposition of the system-bath interaction

  $$H_{\mathrm{SB}} = \sum_\alpha \overbrace{A_\alpha}^{\text{system operator}}\otimes\underbrace{B_\alpha}_{\text{bath operator}}$$

  with $A_\alpha = A_\alpha^\dagger$ and $B_\alpha = B_\alpha^\dagger$. As a first step towards the envisaged averaging procedure

we introduce the operators

$$A_\alpha(\omega) = \sum_{\epsilon'-\epsilon=\omega} \Pi(\epsilon) A_\alpha \Pi(\epsilon'),$$

where the $\Pi(\epsilon)$ are projectors onto the energy eigenspaces (with energy ϵ) of $H_{\rm S}$. They obey $\sum_\epsilon \Pi(\epsilon) = \mathbb{1}$, such that

$$H_{\rm S} = \sum_{\epsilon\epsilon'} \Pi(\epsilon) H_{\rm S} \Pi(\epsilon') = \sum_\epsilon \epsilon \Pi(\epsilon)$$

and hence

$$A_\alpha = \sum_{\epsilon\epsilon'} \Pi(\epsilon) A_\alpha \Pi(\epsilon') = \sum_\omega \sum_{\epsilon'-\epsilon=\omega} \Pi(\epsilon) A_\alpha \Pi(\epsilon') = \sum_\omega A_\alpha(\omega) \ .$$

We thus find that

$$\big[H_{\rm S}, A_\alpha(\omega)\big] = -\omega A_\alpha(\omega)$$

and

$$\big[H_{\rm S}, A_\alpha^\dagger(\omega)\big] = \omega A_\alpha^\dagger(\omega)$$

The $A_\alpha(\omega)$ and $A_\alpha^\dagger(\omega)$ are eigenoperators of $H_{\rm S}$ and as a consequence their time evolution in the interaction picture is

$$\begin{aligned} e^{iH_{\rm S}t} A_\alpha(\omega) e^{-iH_{\rm S}t} &= e^{-i\omega t} A_\alpha(\omega) \\ e^{iH_{\rm S}t} A_\alpha^\dagger(\omega) e^{-iH_{\rm S}t} &= e^{i\omega t} A_\alpha^\dagger(\omega) \ . \end{aligned}$$

The interaction picture representation of the system-bath interaction then becomes:

$$H_{\rm I} = \sum_{\alpha\omega} e^{-i\omega t} A_\alpha(\omega) \otimes B_\alpha(t)$$

with $B_\alpha(t) = e^{iH_{\rm B}t} B_\alpha e^{-iH_{\rm B}t}$. Note, that with this decomposition the condition $\mathrm{tr}_{\rm B}[H_{\rm I}(t), \rho(0)] = 0$ can be reformulated as

$$\langle B_\alpha(t) \rangle = \mathrm{tr}\{B_\alpha(t)\rho_{\rm B}\} = 0 \ ,$$

i.e the average $B_\alpha(t)$ is zero. This can be achieved be a redefinition of the system bath coupling (a possible constant part will simply be absorbed in the system Hamiltonian).

The master equation becomes now

$$\begin{aligned}\frac{\partial}{\partial t}\rho_{\mathrm{S}}(t) &= \int_0^\infty ds\,\mathrm{tr}_{\mathrm{B}}\Big\{H_{\mathrm{I}}(t-s)\rho_{\mathrm{S}}(t)\rho_{\mathrm{B}}H_{\mathrm{I}}(t) - H_{\mathrm{I}}(t)H_{\mathrm{I}}(t-s)\rho_{\mathrm{S}}(t)\rho_{\mathrm{B}}\Big\} + \text{ h.c} \\ &= \sum_{\omega\omega'}\sum_{\alpha\beta}\Gamma_{\alpha\beta}e^{i(\omega'-\omega)t}\left(A_\beta(\omega)\rho_{\mathrm{S}}(t)A_\alpha^\dagger(\omega') - A_\alpha^\dagger(\omega')A_\beta(\omega)\rho_{\mathrm{S}}(t)\right) + \text{ h.c}\end{aligned}$$

with

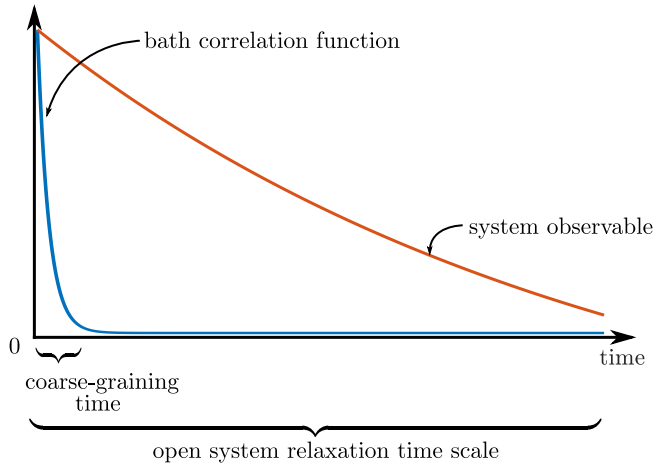
$$\Gamma_{\alpha\beta}(\omega) = \int_0^\infty ds e^{i\omega s}\,\mathrm{tr}_{\mathrm{B}}\left\{B_\alpha^\dagger(t)B_\beta(t-s)\rho_{\mathrm{B}}\right\} = \int_0^\infty ds e^{i\omega s}\left\langle B_\alpha^\dagger(t)B_\beta(t-s)\right\rangle \ ,$$

which is the Fourier transform of the bath two-time correlation function.

When the bath is in a stationary state (which we assume), this correlation function does not depend on the absolute time t, but only on the time differences

$$\left\langle B_\alpha^\dagger(t)B_\beta(t-s)\right\rangle = \left\langle B_\alpha^\dagger(s)B_\beta(0)\right\rangle \ .$$

The assumption underlying the Markov approximation was that this correlation function decays sufficiently fast.



When the relaxation time scale of the open system is much longer than the intrinsic time-scale (typical values of $|\omega-\omega'|^{-1}$) we can perform the secular approximation: "$e^{i(\omega-\omega')t} \to \delta_{\omega\omega'}$" (in quantum optics $|\omega-\omega'| \approx 10^{15}$ Hz, decay rate $\approx 10^9$ Hz)

$$\frac{\partial}{\partial t}\rho_s(t) = \sum_{\omega\alpha\beta}\Gamma_{\alpha\beta}(\omega)\Big[A_\beta(\omega)\rho_{\mathrm{S}}(t)A_\alpha^\dagger(\omega) - A_\alpha^\dagger(\omega)A_\beta(\omega)\rho_{\mathrm{S}}(t)\Big] + \text{h.c}$$

In the final step we decompose

$$\Gamma_{\alpha\beta}(\omega) = \frac{1}{2}\gamma_{\alpha\beta}(\omega) + \underbrace{iS_{\alpha\beta}(\omega)}_{\text{energy shifts (lamb shift)}}$$

using the decay rates (*Kossakowski* matrix)

$$\gamma_{\alpha\beta}(\omega) = \Gamma_{\alpha\beta}(\omega) + \Gamma^*_{\beta\alpha} = \underbrace{\int_{-\infty}^{\infty} ds e^{i\omega s} \left\langle B^\dagger_\alpha(s) B_\beta(0) \right\rangle}_{\text{Fourier transform of the bath correlation function}} .$$

The final result is the *Lindblad Kossakowski master equation*:

$$\frac{\partial}{\partial t}\rho_{\rm S}(t) = -i\big[H_{LS}, \rho_{\rm S}(t)\big] + \mathcal{D}(\rho_{\rm S}(t))$$

with the Lamb-shift Hamiltonian

$$H_{LS} = \sum_{\omega\alpha\beta} S_{\alpha\beta}(\omega) A^\dagger_\alpha(\omega) A_\beta(\omega)$$

and the dissipator

$$\mathcal{D}(\rho_{\rm S}) = \sum_{\omega\alpha\beta} \gamma_{\alpha\beta}(\omega)\big(A_\beta(\omega)\rho_{\rm S}A^\dagger_\alpha(\omega) - \frac{1}{2}\{A^\dagger_\alpha(\omega)A_\beta(\omega), \rho_{\rm S}\}\big) .$$

In the final step let us make one further simplification of the dissipator. We introduce the eigenvalues and eigenvectors of the Kossakowski matrix,

$$\gamma(\omega) = \sum_\alpha \gamma_\alpha(\omega) v_\alpha(\omega) v^\dagger_\alpha(\omega)$$

This allows us to write,

$$\mathcal{D}(\rho_{\rm S}) = \sum_{\omega\alpha} \gamma_\alpha(\omega) \left(L_\alpha(\omega)\rho_{\rm S}L^\dagger_\alpha(\omega) - \frac{1}{2}\{L^\dagger_\alpha(\omega)L_\alpha(\omega), \rho_{\rm S}\}\right) .$$

The $\gamma_\alpha(\omega)$ are the decay rates and the

$$L_\alpha(\omega) = \sum_j v_{\alpha j}(\omega) A_j(\omega)$$

are the so-called jump operators. This takes the master equation to the Lindblad form.

Summary: Master equation in Lindblad form

$$\frac{\partial}{\partial t}\rho = -i[H,\rho] + \sum_j \gamma_j\big(L_j\rho L^\dagger_j - \frac{1}{2}\{L^\dagger_j L_j, \rho\}\big)$$

Chapter 6

Lindblad master equation and quantum maps

In the following we will see that the master equation can be derived through a short-time expansion of a quantum map.

To see this we note that the evolution of an initial state $\rho(0)$ over an infinitesimally small time interval dt is given by

$$\rho(dt) = \rho(0) + \dot{\rho}(0)dt + \mathcal{O}(dt^2) \ .$$

On the other hand we should have

$$\rho(dt) = \sum_\alpha K_\alpha(dt)\rho(0)K_\alpha^\dagger(dt) \ .$$

Let us now try to find Kraus operators that make those two equations agree up to order dt. Clearly, in order to get the $\rho(0)$ term one Kraus operator has to contain the identity. We thus attempt,

$$K_0 = \mathbb{1} + L_0 dt \ ,$$

so that

$$K_0\rho(0)K_0^\dagger = \rho(0) + \Big[L_0\rho(0) + \rho(0)L_0^\dagger\Big]\, dt + \mathcal{O}(dt^2) \ .$$

There must be further Kraus operators, since having only one would correspond to unitary evolution. So we choose

$$K_\alpha = \sqrt{dt}L_\alpha \qquad\qquad \alpha \geq 1$$

such that

$$K_\alpha\rho(0)K_\alpha^\dagger = L_\alpha\rho(0)L_\alpha^\dagger dt \ .$$

Enforcing the normalization condition yields

$$\mathbb{1} = K_0^\dagger K_0 + \sum_{\alpha\geq 1} K_\alpha^\dagger K_\alpha = \mathbb{1} + dt\big(L_0 + L_0^\dagger + \sum_{\alpha\geq 1} L_\alpha^\dagger L_\alpha\big) + \mathcal{O}(dt^2) \ .$$

We can now decompose into an hermitean and an anti-hermitean part: $L_0 = A - iH$, with $A = A^\dagger$ and $H = H^\dagger$. Thus, in order to fulfill the normalization condition up to order dt, we need to choose

$$A = -\frac{1}{2}\sum_{\alpha\geq 1} L_\alpha^\dagger L_\alpha \ .$$

We thus have

$$\begin{aligned}
\rho(dt) &= K_0\rho_0 K_0^\dagger + \sum_{\alpha\geq 1} K_\alpha \rho(0) K_\alpha^\dagger \\
&= \rho(0) + (A - iH)\rho(0)dt + \rho(0)(A + iH)dt + \sum_{\alpha\geq 1} L_\alpha \rho(0) L_\alpha^\dagger + \mathcal{O}(dt^2) \\
&= \rho(0) - i\big[H, \rho(0)\big]dt + \big\{A, \rho(0)\big\}dt + \sum_{\alpha\geq 1} L_\alpha \rho L_\alpha^\dagger + \mathcal{O}(dt^2) \\
&= \rho(0) - i\big[H, \rho(0)\big]dt + \sum_{\alpha\geq 1}\Big(L_\alpha \rho L_\alpha^\dagger - \frac{1}{2}\big\{L_\alpha^\dagger L_\alpha, \rho(0)\big\}\Big)dt + \mathcal{O}(dt^2) \ .
\end{aligned}$$

Finally, we obtain:

$$\begin{aligned}
\dot\rho(t) &= \lim_{dt\to 0}\frac{\rho(dt) - \rho(0)}{dt} = -i\big[H, \rho(0)\big] + \sum_{\alpha\geq 1}\Big(L_\alpha \rho(0) L_\alpha^\dagger - \frac{1}{2}\big\{L_\alpha^\dagger L_\alpha, \rho(0)\big\}\Big) \\
&= \mathcal{L}(\rho)
\end{aligned}$$

which is the Lindblad master equation.

- Semi-group property:

  The formal solution of the master equation $\dot\rho(t) = \mathcal{L}(\rho)$ is,

  $$\rho(t) = e^{\mathcal{L}t}\rho(0) = \Lambda_t \rho(0) \ ,$$

  where Λ_t is called the Markovian evolution operator. The set $\{\Lambda_t\}_{t\geq 0}$ forms a one-parameter semi-group, because it satisfies three out of four properties of a group:

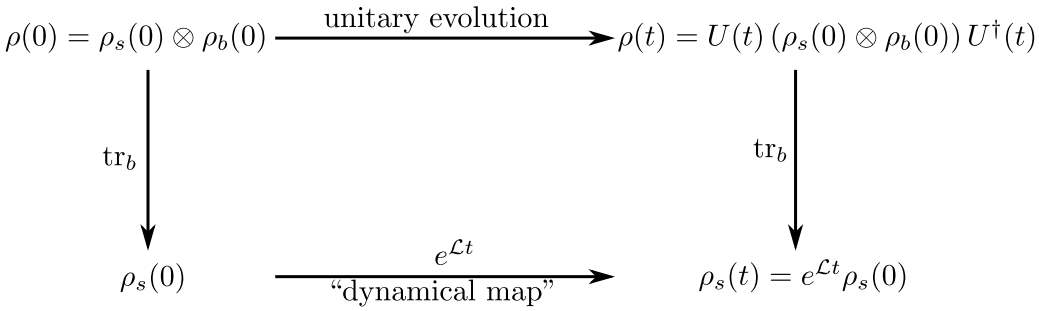
  1. Identity Operator: $\Lambda_0 = \mathbb{1}$
  2. Closed under multiplication:

     $$\Lambda_t\Lambda_s = e^{\mathcal{L}t}e^{\mathcal{L}s} = e^{\mathcal{L}(t+s)} = \Lambda_{t+s}$$

  3. Associativity: $(\Lambda_t\Lambda_s)\Lambda_r = \Lambda_t(\Lambda_s\Lambda_r)$

  However, not every element has an inverse. The map Λ_t is contractive, corresponding to exponential decay. I.e. Λ_∞ has at least one zero eigenvalue, so that it does not possess an inverse.

  The relation between state changes of the open system and the unitary dynamics of the total system, including the bath, can be visualized as follows:


$\rho(0) = \rho_s(0) \otimes \rho_b(0)$
unitary evolution
$\rho(t) = U(t)\,(\rho_s(0) \otimes \rho_b(0))\,U^\dagger(t)$
tr_b
tr_b
$\rho_s(0)$
$e^{\mathcal{L}t}$
"dynamical map"
$\rho_s(t) = e^{\mathcal{L}t}\rho_s(0)$

Chapter 7

The Choi–Jamiołkowski isomorphism

The master equation $\dot{\rho} = \mathcal{L}\rho$ is a linear equation, which has the solution $\rho(t) = e^{\mathcal{L}t}\rho(0)$ (provided that $\mathcal{L}$ is not explicitly time-dependent). This is formally equivalent to solving, e.g. the Schrödinger equation. However, a complication arises since $\mathcal{L}$ involves terms in which operators are multiplied to both sides of the density matrix.

The idea behind the Choi-Jamiołkowski isomorphism is to bring the master equation into a form, where $\mathcal{L}$ is an $N^2 \times N^2$ matrix (assuming the dimension of the system Hilbert space is N) and ρ is represented by an N^2- dimensional vector.

The procedure is often referred to "vectorisation" and becomes tremendously helpful when solving master equations in practice, e.g. on a computer.

The isomorphism is given by the relation $|i\rangle\langle j| \to |j\rangle \otimes |i\rangle$, i.e. the outer product is turned into a direct product. The density matrix $\rho = \sum_{ij} \rho_{ij} |i\rangle\langle j|$ then becomes a vector in a "doubled space", which has dimension N^2

$$\rho \to \underbrace{\vec{\rho} = \sum_{ij} \rho_{ij} |j\rangle \otimes |i\rangle}_{\text{vectorised density matrix}}$$

For example, if

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow \vec{M} = \begin{pmatrix} a \\ c \\ b \\ d \end{pmatrix}$$

A first convenient property of the isomorphism is that it turns the Hilbert-Schmidt inner product of operators into the standard inner product for its vectorised forms:

$$\mathrm{tr}\big(A^\dagger B\big) = \vec{A}^\dagger \cdot \vec{B}$$

With the vectorization of the identity operator

$$\mathbb{1} = \sum_i |i\rangle\langle i| \to \vec{\mathbb{1}} = \sum_i |i\rangle \otimes |i\rangle \quad .$$

We can thus write the normalisation condition of the density matrix as

$$\mathrm{tr}(\rho) = \mathrm{tr}\big(\mathbb{1}^\dagger \rho\big) = \vec{\mathbb{1}}^\dagger \vec{\rho} = 1 \ .$$

A second useful property of the isomorphism concerns the product of three matrices

$$\overrightarrow{ABC} = (C^T \otimes A)\vec{B} \ .$$

Calculation:

In order to see that this is true we write both sides explicitly

$$ABC = \sum_{ijklmn} \left(A_{ij} \left|i\right\rangle\!\left\langle j\right|\right) \left(B_{kl} \left|k\right\rangle\!\left\langle l\right|\right) \left(C_{mn} \left|m\right\rangle\!\left\langle n\right|\right) = \sum_{ijmn} A_{ij} B_{jm} C_{mn} \left|i\right\rangle\!\left\langle n\right| \ .$$

So the vecorised form is

$$\overrightarrow{ABC} = \sum_{ijmn} A_{ij} B_{jm} C_{mn} \left|n\right\rangle \otimes \left|i\right\rangle \ .$$

Now looking at the right hand side of the equation one finds

$$\begin{aligned}(C^T \otimes A)\vec{B} &= \left(\sum_{mnij} C_{mn} A_{ij} \left|n\right\rangle\!\left\langle m\right| \otimes \left|i\right\rangle\!\left\langle j\right|\right) \sum_{kl} B_{kl} \left|l\right\rangle \otimes \left|k\right\rangle \\ &= \sum_{mnij} C_{mn} A_{ij} B_{jm} \left|n\right\rangle \otimes \left|i\right\rangle = \overrightarrow{ABC} \ .\end{aligned}$$

Let us apply these new rules to the master equation in Lindblad form

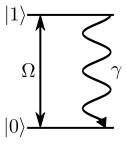
$$\dot{\rho} = -i\big[H, \rho\big] + \sum_\alpha \Big(L_\alpha \rho L_\alpha^\dagger - \frac{1}{2}\{L_\alpha^\dagger L_\alpha, \rho\}\Big) \ .$$

We find

$$\begin{aligned}\frac{\partial}{\partial t}\vec{\rho} &= -i\overrightarrow{H\rho\mathbb{1}} + i\overrightarrow{\mathbb{1}\rho H} + \sum_\alpha \Big(\overrightarrow{L_\alpha \rho L_\alpha^\dagger} - \frac{1}{2}\overrightarrow{L_\alpha^\dagger L_\alpha \rho \mathbb{1}} - \frac{1}{2}\overrightarrow{\mathbb{1}\rho L_\alpha^\dagger L_\alpha}\Big) \\ &= \underbrace{\Big[-i\mathbb{1}\otimes H + iH^T \otimes \mathbb{1} + \sum_\alpha \Big(L_\alpha^* \otimes L_\alpha - \frac{1}{2}\mathbb{1}\otimes L_\alpha^\dagger L_\alpha - \frac{1}{2}(L_\alpha^\dagger L_\alpha)^T \otimes \mathbb{1}\Big)\Big]}_{\hat{\mathcal{L}}} \vec{\rho} \ .\end{aligned}$$

The master operator $\hat{\mathcal{L}}$ has now been turned into a matrix $\hat{\mathcal{L}}$, which can be analysed and manipulated with the standard tools (spectral decomposition, etc.).

7.1 Resonantly driven two-level atom with decay



- Hamiltonian: $H = \Omega\sigma_x$, with Ω being the so-called Rabi frequency (α electric field strength of the laser)
- Decay: $L = \sqrt{\gamma}\sigma^-$, with γ being the decay rate.

$$\hat{\mathcal{L}} = -i\Omega \overbrace{\mathbb{1}\otimes\sigma_x}^{\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}} + i\Omega \overbrace{\sigma_x\otimes\mathbb{1}}^{\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}} + \gamma \overbrace{\sigma^-\otimes\sigma^-}^{\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{pmatrix}} - \frac{1}{2}\overbrace{\mathbb{1}\otimes\underbrace{(\sigma^+\sigma^-)}_{\begin{pmatrix}1&0\\0&0\end{pmatrix}}}^{\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}} - \frac{1}{2}\overbrace{\underbrace{(\sigma^+\sigma^-)}_{\begin{pmatrix}1&0\\0&0\end{pmatrix}}\otimes\mathbb{1}}^{\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}}$$

$$= \begin{pmatrix} -\gamma & -i\Omega & i\Omega & 0 \\ -i\Omega & -\frac{\gamma}{2} & 0 & i\Omega \\ i\Omega & 0 & -\frac{\gamma}{2} & -i\Omega \\ \gamma & i\Omega & -i\Omega & 0 \end{pmatrix}$$

This is a *non-hermitian* matrix (opposed to a Hamiltonian). In order to proceed we decompose it into its spectral representation:

$$\hat{\mathcal{L}} = \lambda_0 \vec{r_0}\vec{l_0}^\dagger + \sum_{\alpha=1}^{3} \lambda_\alpha \vec{r_\alpha}\vec{l_\alpha} \qquad \text{, with } \vec{l_\alpha}^\dagger \vec{r_\beta} = \delta_{\alpha\beta}$$

Here, λ_α are the eigenvalues and $\vec{r_\alpha}$ and $\vec{l_\alpha}$ are the corresponding right and left eigenvectors, respectively. Note, that $\operatorname{tr} r_0 = 1$ and $\operatorname{tr} r_{\alpha>0} = 0$. For all master operators the following holds:

1. $\lambda_0 = 0$, is the eigenvalue with the largest real part (unless these are oscillating solutions or degeneracies). All other eigenvalues have a real part that is negative.

2. $\vec{r_0}$ is the stationary state :

$$\hat{\mathcal{L}}\vec{r_0} = \lambda_0 \vec{r_0} \underbrace{(\vec{l_0}^\dagger \cdot \vec{r_0})}_{1} + \sum_\alpha \lambda_\alpha \vec{r_\alpha} \underbrace{(\vec{l_\alpha}^\dagger \vec{r_0})}_{\delta_{\alpha 0}} = \lambda_0 \vec{r_0} = 0$$

3. $\vec{l_0}$ is the identity operator

$$1 = \vec{l_0}^\dagger \vec{r_0} = \operatorname{tr}\big(\underbrace{l_0^\dagger r_0}_{\substack{\text{stationary state}\\ \text{density matrix}\\ \text{(not vectorized)}}}\big) = \operatorname{tr}(r_0)$$

4. Time evolution operator

$$e^{\hat{\mathcal{L}}t} = e^{\lambda_0 t}\vec{r_0}\vec{l_0}^{\dagger} + \sum_{\alpha} e^{\lambda_\alpha t}\vec{r_\alpha}\vec{l_\alpha}^{\dagger}$$

$$\to \lim_{t\to\infty} e^{\hat{\mathcal{L}}t} = \underbrace{\vec{r_0}\vec{l_0}^{\dagger}}_{\text{projector on stationary state}}$$

Let us come back to the two-level atom and set for simplicity $\Omega = 0$ (pure decay). We then have

$$\hat{\mathcal{L}} = 0 \cdot \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\begin{pmatrix}1&0&0&1\end{pmatrix} - \frac{\gamma}{2}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\begin{pmatrix}0&1&0&0\end{pmatrix} - \frac{\gamma}{2}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\begin{pmatrix}0&0&1&0\end{pmatrix}$$
$$- \gamma\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}\begin{pmatrix}1&0&0&0\end{pmatrix} = \begin{pmatrix}-\gamma&0&0&0\\0&-\gamma&0&0\\0&0&-\gamma&0\\\gamma&0&0&0\end{pmatrix}$$

The time evolution is hence given by the operator

$$e^{\hat{\mathcal{L}}t} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\begin{pmatrix}1&0&0&1\end{pmatrix} + e^{-\frac{\gamma}{2}t}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}\begin{pmatrix}0&1&0&0\end{pmatrix} + e^{-\frac{\gamma}{2}t}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}\begin{pmatrix}0&0&1&0\end{pmatrix}$$
$$+ e^{-\gamma t}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}\begin{pmatrix}1&0&0&0\end{pmatrix} .$$

This allows us to calculate the evolution of an atom, that is initially in the excited state. Its density matrix is:

$$\rho(0) = \begin{pmatrix}1&0\\0&0\end{pmatrix} \quad \longrightarrow \quad \vec{\rho(0)} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$e^{\mathcal{L}t}\vec{\rho}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + e^{-\gamma t}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} e^{\gamma t} \\ 0 \\ 0 \\ 1 - e^{-\gamma t} \end{pmatrix}$$

$$\rightarrow \rho(t) = \begin{pmatrix} e^{-\gamma t} & 0 \\ 0 & 1 - e^{-\gamma t} \end{pmatrix} = \frac{1}{2}\left[\mathbb{1} - (1 - 2e^{-\gamma t})\sigma_z\right] \ .$$

The stationary state is $|0\rangle$, with Bloch vector $\vec{v}_\infty = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$.

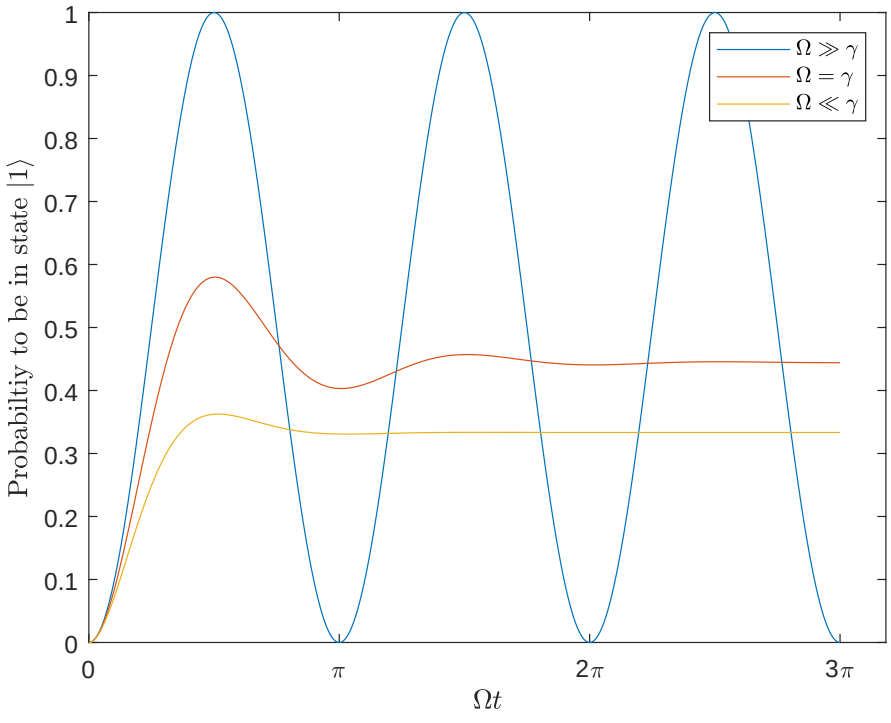
Performing the calculation for $\Omega \neq 0$ is significantly more involved. The stationary state is,

$$\rho(\infty) = \frac{1}{\gamma^2 + 8\Omega^2}\begin{pmatrix} 4\Omega^2 & 2i\gamma\Omega \\ -2i\gamma\Omega & \gamma^2 + 4\Omega^2 \end{pmatrix} \ .$$

For the time-dependent state, starting from $\rho(0) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, we obtain

$$\rho(t) = \begin{pmatrix} \frac{1}{2}\frac{\gamma^2-\Gamma^2}{\Gamma^2-9\gamma^2}\left(1 + \left\{\cosh\left(\frac{\Gamma t}{4}\right) + 3\frac{\gamma}{\Gamma}\sinh\left(\frac{\Gamma t}{4}\right)\right\}e^{-\frac{3}{4}\gamma t}\right) & * \\ \frac{-i}{4}\frac{\sqrt{\gamma^2-\Gamma^2}}{\Gamma}\left(\frac{8\gamma\Gamma}{9\gamma^2-\Gamma^2} + e^{-\frac{3}{4}\gamma t}\left\{\frac{(\gamma+\Gamma)e^{\frac{\Gamma}{4}t}}{\Gamma-3\gamma} + \frac{(\gamma-\Gamma)e^{\frac{\Gamma}{4}t}}{\Gamma-3\gamma}\right\}\right) & * \end{pmatrix}$$

with $\Gamma = \sqrt{\gamma^2 - 64\omega^2}$.



- When $\Omega \gg \gamma$, the probability of being in the excited state is oscillating in time with a

frequency $2\pi\Omega$ (Rabi Oscillator).

- When $\Omega \approx \gamma$ the oscillations are damped.
- For $\Omega < 8\gamma$ the oscillations are overdamped.

Chapter 8

Quantum jump trajectories

Generally, we can write the Markovian evolution operator $\mathcal{L}$ as a sum of two parts:

$$\mathcal{L} = \mathcal{L}_0 + J$$

where

$$\mathcal{L}_0\rho = -i[H,\rho] - \frac{1}{2}\sum_k^N \gamma_k\{L_k^\dagger L_k, \rho\} \text{ and}$$
$$J\rho = \sum_k^N \gamma_k L_k \rho L_k^\dagger \ .$$

The first term we can rewrite as

$$\begin{aligned}\mathcal{L}_0\rho &= -i\left(H - \frac{i}{2}\sum_k^N \gamma_k L_k^\dagger L_k\right)\rho + i\rho\big(H + \frac{i}{2}\sum_k^N \gamma_k L_k^\dagger L_k\big) \\ &= -iH_{\text{eff}}\rho + i\rho H_{\text{eff}}^\dagger\end{aligned}$$

with

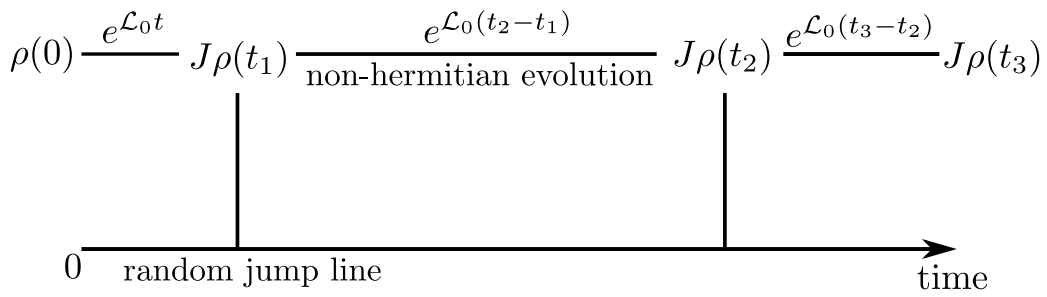
$$H_{\text{eff}} = H - \frac{i}{2}\sum_k^N \gamma_k L_k^\dagger L_k$$

being the so-called effective Hamiltonian. Note, that H_{eff} is not hermitian. However, it evolves the quantum state ρ in a similar way as an ordinary Hamiltonian.

$$\frac{\partial}{\partial t}\rho = \mathcal{L}\rho \quad \longrightarrow \quad \rho(t) = e^{-itH_{\text{eff}}}\rho(0)e^{itH_{\text{eff}}} = e^{\mathcal{L}_0 t}\rho(0)$$

The term $J\rho$ can be interpreted as a "jump term" as its action on the quantum state causes transitions of the kind $\rho \to L_k\rho L_k^\dagger$ at a rate γ_k.

The evolution of an open quantum system can indeed be regarded as proceeding via a non-hermitian dynamics under H_{eff}, which is interspersed with quantum jumps.



This picture can be formalized by expanding the master equation evolution into a Dyson series: Our starting point is to consider the evolution of the object.

$$\bar{\rho}(t) = e^{-\mathcal{L}_0 t}\rho(t)$$

Where $\rho(t)$ is the density matrix obeying $\frac{\partial}{\partial t}\rho(t) = (\mathcal{L}_0 + J)\rho(t)$.

We find that,

$$\begin{aligned}\frac{\partial}{\partial t}\bar{\rho}(t) &= -\mathcal{L}_0 e^{-\mathcal{L}_0 t}\rho(t) + e^{-\mathcal{L}_0 t}\frac{\partial}{\partial t}\rho(t) \\ &= -\mathcal{L}_0 e^{-\mathcal{L}_0 t}\rho(t) + e^{-\mathcal{L}_0 t}(\mathcal{L}_0 + J)\frac{\partial}{\partial t}\rho(t) = e^{-\mathcal{L}_0 t}J\rho(t)\end{aligned}$$

Integrating this equation yields,

$$\bar{\rho}(t) = \bar{\rho}(0) + \int_0^t d\tau e^{-\mathcal{L}_0\tau}J\rho(\tau)$$

and hence (using $\bar{\rho}(t) = e^{-\mathcal{L}_0 t}\rho(t)$ and realising that $\bar{\rho}(0) = \rho(0)$)

$$\rho(t) = e^{\mathcal{L}_0 t}\rho(0) + \int_0^t d\tau e^{\mathcal{L}_0(t-\tau)}J\rho(\tau)\ .$$

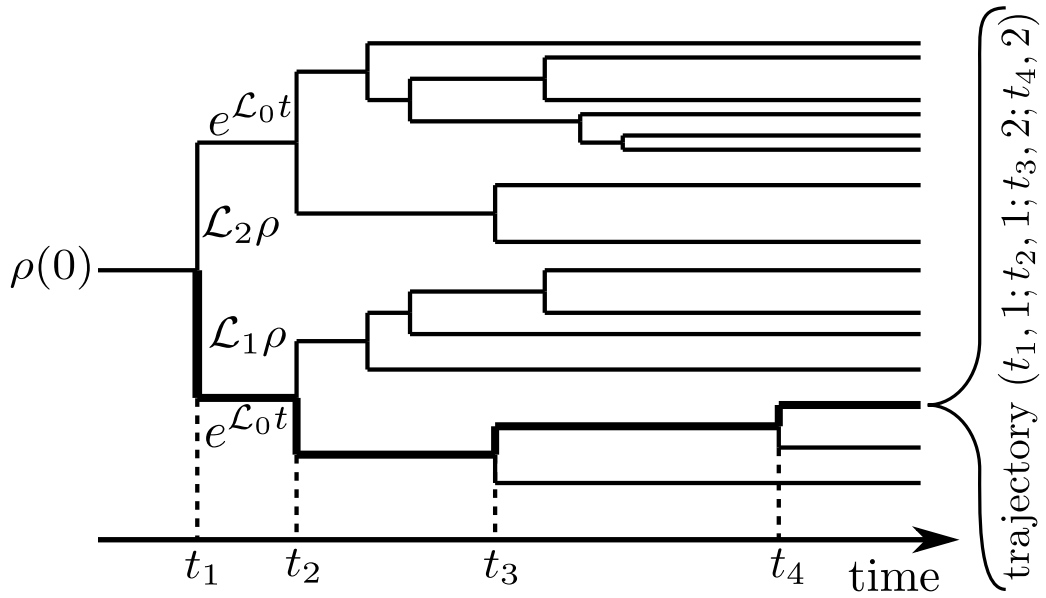
We can iterate this expression, which yields

$$\begin{aligned}\rho(t) &= e^{\mathcal{L}_0 t}\rho(0) + \int_0^t d\tau e^{\mathcal{L}_0(t-\tau)}Je^{\mathcal{L}_0\tau}\rho(0) + \int_0^t d\tau\int_0^\tau d\tau' e^{\mathcal{L}_0(t-\tau)}Je^{-\mathcal{L}_0(\tau-\tau')}J\rho(\tau') \\ &= e^{\mathcal{L}_0 t}\rho(0) + \sum_{n=1}^{\infty}\int_0^t dt_n\int_0^{t_n} dt_{n-1}\cdots\int_0^{t_2} dt_1 e^{\mathcal{L}_0(t-t_n)}Je^{\mathcal{L}_0(t_n-t_{n-1})}J\cdots e^{\mathcal{L}_0(t_2-t_1)}Je^{\mathcal{L}_0 t_1}\rho(0)\end{aligned}$$

The latter expression is the Dyson expansion and indeed we see here that the state at time t is given by a sum of "trajectories" which correspond to repeated actions of operator J and a time evolution operator H_{eff} in between. We now rewrite the operator J as

$$J\rho = \sum_k \gamma_k L_k \rho L_k^\dagger = \sum_k \mathcal{L}_k\rho$$

in order to explicitly account for the different possible jump events.



A particular realization of n jump levels is specified by a sequence of the form

$$R_n^t = (\underbrace{t_1}_{\text{jump time}}, \overbrace{k_1}^{\text{jump type}}, t_2, k_2, \cdots, t_n, k_n) \ .$$

The jump times satisfy $0 < t_1 < t_2 < \cdots < t_n$ and $k_j \in \{1, \cdots, N\}$. We call R_n^t a quantum jump trajectory. The time evolution of the quantum state can now be written as

$$\rho(t) = e^{\mathcal{L}_0 t}\rho(0) + \underbrace{\sum_{k=1}^{\infty}}_{\text{sum over jumps}} \overbrace{\int_0^t dt_n \cdots \int_0^{t_2} dt_1}^{\text{integral over jump times}} \underbrace{\sum_{\{R_n^t\}}}_{\text{sum over all possible quantum trajectories}} \overbrace{e^{\mathcal{L}_0(t-t_n)}\mathcal{L}_{k_n} e^{\mathcal{L}_0(t_n - t_{(n-1)})}\mathcal{L}_{k_{n-1}} \cdots \mathcal{L}_{k_1} e^{\mathcal{L}_0 t_1}}^{K_{R_n} t} \rho(0)$$

The probability of no jumps occurring over a time period t is given by the trace of the first term, which represents the quantum jump trajectory containing zero jumps

$$P(R_0^\dagger|\rho(0)) = \mathrm{tr}\big(e^{\mathcal{L}_0 t}\rho(0)\big) \ .$$

Analogously the probability for a particular trajectory R_n^t to occur, given the initial state $\rho(0)$, is

$$P(R_n^t, \rho(0)) = \mathrm{tr}\big(K_{R_n^t}\rho(0)\big) \ ,$$

with

$$K_{R_n^t} = e^{\mathcal{L}_0(t-t_n}\mathcal{L}_{K_n} e^{\mathcal{L}_0(t_n - t_{n-1})} \cdots \mathcal{L}_{k_1} e^{\mathcal{L}_0 t_1} \ .$$

The quantum state conditioned to a specific trajectory R_n^t is given as

$$\rho_{R_n^t} = \frac{K_{R_n^t}}{\mathrm{tr}\big(K_{R_n^t}\rho(0)\big)} \ .$$

However, since we can write

$$K_{R_n^t}\rho = \underbrace{e^{-iH_{\rm eff}(t-t_n)}L_{k_n}e^{-iH_{\rm eff}(t_n-t_{n-1})}L_{k_{n-1}}\cdots .L_{k_1}e^{-iH_{\rm eff}t_1}}_{M_{R_n^t}}\rho\,\underbrace{e^{iH_{\rm eff}t_1}L_{k_1}^\dagger\cdots .L_{k_n}^\dagger e^{+iH_{\rm eff}(t-t_n)}}_{M_{R_n^t}^\dagger},$$

this establishes immediately a connection to the transformation of a quantum state under a measurement:

$$\rho_{R_n^t} = \frac{M_{R_n^t}\rho(0)M_{R_n^t}^\dagger}{\mathrm{tr}\Big(M_{R_n^t}\rho(0)M_{R_n^t}^\dagger\Big)}$$

with the compound measurement operator

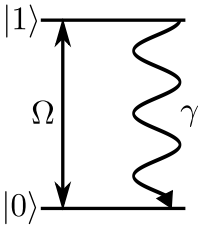
$$M_{R_n^t} = e^{-H_{\rm eff}(t-t_n)}L_{k_n}\cdots L_{k_2}e^{-iH_{\rm eff}(t_2-t_1)}L_{k_1}e^{-iH_{\rm eff}t}\ .$$

This shows that one can view the open system dynamics generated by $\mathcal{L}$ to emerge from a continuous monitoring of the system by the environment.

The quantum state at time t can then be written as

$$\rho(t) = \overbrace{\mathrm{tr}\big(K_{R_0^t}\rho(0)\big)}^{\text{probability to have zero jumps}}\underbrace{\rho_{R_0^t}}_{\substack{\text{state, conditioned}\\ \text{on zero jumps}}} + \sum_{n=1}^{\infty}\int_0^t dt_n\cdots dt_1\sum_{R_n}\mathrm{tr}\big(K_{R_n^t}\rho(0)\big)\overbrace{\rho_{R_n^t}}^{\substack{\text{state,conditioned on a specific quantum jump trajectory}}}$$

8.1 Example: two-level system



The Master equation is given by

$$\dot\rho = -i\Omega[\sigma_x,\rho] + \gamma\sigma^-\rho\sigma^+ - \frac{\gamma}{2}\{\sigma^+\sigma^-,\rho\},$$

and we read off the effective Hamiltonian

$$H_{\rm eff} = \Omega\sigma_x - i\frac{\gamma}{2}\sigma^+\sigma^- = \begin{pmatrix} -i\frac{\gamma}{2} & \Omega \\ \Omega & 0\end{pmatrix}\ .$$

We also can see that

$$J\rho = \gamma\sigma^-\rho\sigma^+ = \gamma\begin{pmatrix}0&0\\1&0\end{pmatrix}\rho\begin{pmatrix}0&1\\0&0\end{pmatrix}\ .$$

Let us now assume that the atom is initially prepared in the ground state $|0\rangle$. We can now use the formalism developed before in order to calculate how much time one needs to wait on average before a photon is emitted, i.e. a jump event takes place. To this we first start by calculating the non-jump probability.

This is given by

$$p_{\text{no jump}}(t) = \text{tr}\big(e^{\mathcal{L}_0 t}\rho(0)\big)$$

with

$$\mathcal{L}_0\rho = -iH_{\text{eff}}\rho + i\rho H_{\text{eff}}^\dagger \ .$$

Since $\rho(0) = |0\rangle\langle 0|$, we have

$$e^{\mathcal{L}_0 t}\rho(0) = e^{-iH_{\text{eff}}t}\,|0\rangle\langle 0|\,e^{iH_{\text{eff}}^\dagger t} \ .$$

Using,

$$e^{-iH_{\text{eff}}t}\,|0\rangle = \frac{-i4\Omega}{\bar{\gamma}}e^{-\frac{\gamma}{4}t}\sinh\left(\frac{\bar{\gamma}t}{4}\right)|0\rangle + e^{-\frac{\gamma}{4}t}\left[\cosh\left(\frac{\bar{\gamma}t}{4}\right) + \frac{\gamma}{\bar{\gamma}}\sinh\left(\frac{\bar{\gamma}t}{4}\right)\right]|1\rangle$$

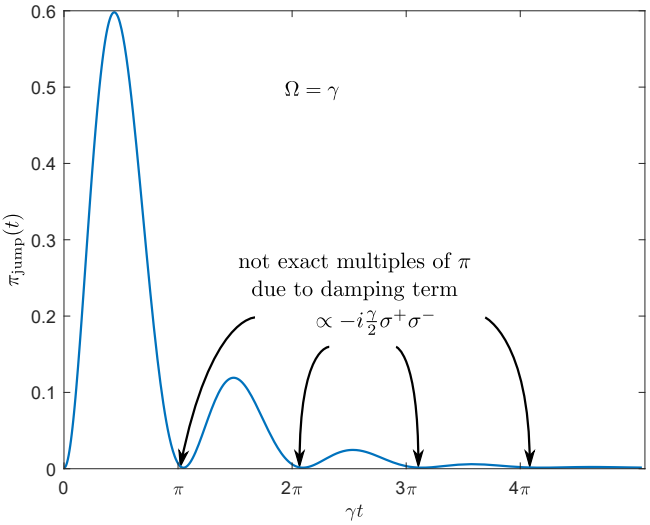
with $\bar{\gamma} = \sqrt{\gamma^2 - 16\Omega^2}$, we obtain

$$\begin{aligned} p_{\text{no jump}}(t) &= e^{-\frac{\gamma}{2}t}\left(\frac{16\Omega^2}{\bar{\gamma}^2}\sinh^2\left(\frac{\bar{\gamma}t}{4}\right) + \left[\cosh\left(\frac{\bar{\gamma}t}{4}\right) + \frac{\gamma}{\bar{\gamma}}\sinh\left(\frac{\bar{\gamma}t}{4}\right)\right]^2\right) \\ &= e^{-\frac{\gamma}{2}t}\left(\frac{\gamma^2}{\bar{\gamma}^2}\cosh\left(\frac{\bar{\gamma}t}{2}\right) + \frac{\gamma}{\bar{\gamma}}\sinh\left(\frac{\bar{\gamma}t}{2}\right) - \frac{16\Omega^2}{\bar{\gamma}^2}\right) \ . \end{aligned}$$

Hence, the probability for a jump event to take place before time t is given by $p_{\text{jump}}(t) = 1 - p_{\text{no jump}}(t)$, which can be written also in terms of a probability density

$$p_{\text{jump}}(t) = \int_0^t d\tau \Pi_{\text{jump}}(\tau)$$

with, $\Pi_{\text{jump}}(t) = 16\gamma\frac{\Omega^2}{\bar{\gamma}^2}e^{-\frac{\gamma}{2}t}\sinh^2\left(\frac{\bar{\gamma}t}{4}\right)$. This probability density is oscillating, which is a consequence of the fact that the atom is performing "Rabi oscillations" under the effective Hamiltonian, and that there cannot be an emission/jump event, when the atom was rotated back to it's ground state ,$|0\rangle$.



The average time to the jump event is calculated as

$$\langle t\rangle = \int_0^\infty d\tau \Pi_{\text{jump}}(\tau)\tau = \frac{2}{\gamma} + \frac{\gamma}{4\Omega^2} \ ,$$

and the fluctuations of the jump time are

$$\Delta t = \sqrt{\langle t^2\rangle - \langle t\rangle^2} = \sqrt{\frac{4}{\gamma^2} - \frac{1}{2\Omega^2} + \frac{\gamma^2}{16\Omega^2}} \ .$$

In the limit of the large decay rate γ, we find $\langle t\rangle \to \frac{\gamma}{4\omega^2}$ and $\Delta t \to \frac{\gamma}{4\Omega^2}$, i.e. the fluctuations become equal to the mean. The probability density of the jump time assumes a particularly simple form,

$$\Pi_{\text{jump}}(t) \overset{\gamma\gg\Omega}{=} \frac{4\Omega^2}{\gamma} e^{-\frac{4\Omega^2}{\gamma}t} \ ,$$

and so does the no-jump probability

$$p_{\text{no jump}}(t) = e^{-\frac{4\Omega^2}{\gamma}t} \ .$$

Let us now investigate more generally the statistics of the jump events. The probability for the occurrence of n events before time t is given by

$$p_n(t) = \text{tr} \int_0^t dt_n \cdots \int_0^{t_2} dt_1 e^{\mathcal{L}_0(t-t_n)} J e^{\mathcal{L}_0(t_n-t_{n-1})} J \cdots J e^{\mathcal{L}_0 t_1}\rho(0) \ .$$

Using the fact that $\rho(0) = |0\rangle\langle 0|$, and that J can be written as

$$J\rho = \gamma\sigma^-\rho\sigma^+ = \gamma\,|0\rangle\langle 1|\,\rho\,|1\rangle\langle 0| \ ,$$

this expression simplifies to

$$p_n(t) = \text{tr} \int_0^t dt_n \cdots \int_0^{t_2} dt_1 \gamma^n e^{-iH_{\text{eff}}(t-t_n)}\,|0\rangle\langle 1|\cdots|0\rangle \underbrace{\langle 1|\,e^{-iH_{\text{eff}}t_1}\,|0\rangle\langle 0|\,e^{iH_{\text{eff}}^\dagger t_1}\,|1\rangle}_{q(t_1)/\gamma} \langle 0|\cdots|1\rangle\langle 0|\,e^{iH_{\text{eff}}^\dagger(t-t_n)}$$

Introducing $q(t) = \gamma|\,\langle 1|\,e^{-iH_{\text{eff}}t}\,|0\rangle\,|^2$, we can write

$$\begin{aligned} p_n(t) &= \text{tr} \int_0^t dt_n \cdots dt_1 e^{-iH_{\text{eff}}(t-t_n)}\,|0\rangle\,q(t_n-t_{n-1})q(t_{n-1}-t_{n-2})\cdots q(t_1)\,\langle 0|\,e^{iH_{\text{eff}}^\dagger(t-t_n)} \\ &= \int_0^t dt_n \cdots dt_1 \underbrace{\text{tr}\left[e^{-iH_{\text{eff}}(t-t_n)}\,|0\rangle\langle 0|\,e^{iH_{\text{eff}}^\dagger(t-t_n)}\right]}_{p_{\text{no jump}}(t-t_n)} q(t_n-t_{n-1})q(t_{n-1}-t_{n-2})\cdots q(t_1) \\ &= \int_0^t dt_n \cdots \int_0^{t_2} dt_1 p_{\text{no jump}}(t-t_n)q(t_n-t_{n-1})q(t_{n-1}-t_{n-2})\cdots q(t_1) \end{aligned}$$

An explicit calculation shows, that in fact

$$q(t) = \Pi_{\text{jump}}(t) \quad \longleftrightarrow \quad p_{\text{no jump}}(t) = 1 - \gamma \int_0^t dt \left| \langle 1 | \, e^{-iH_{\text{eff}}t} \, |0\rangle \right|^2,$$

so that

$$p_n(t) = \int_0^t dt_n \cdots \int_0^{t_2} dt_1 p_{\text{no jump}}(t - t_n)\Pi_{\text{jump}}(t_n - t_{n-1})\Pi_{\text{jump}}(t_{n-1} - t_{n-2}) \cdots \Pi_{\text{jump}}(t_1) \ .$$

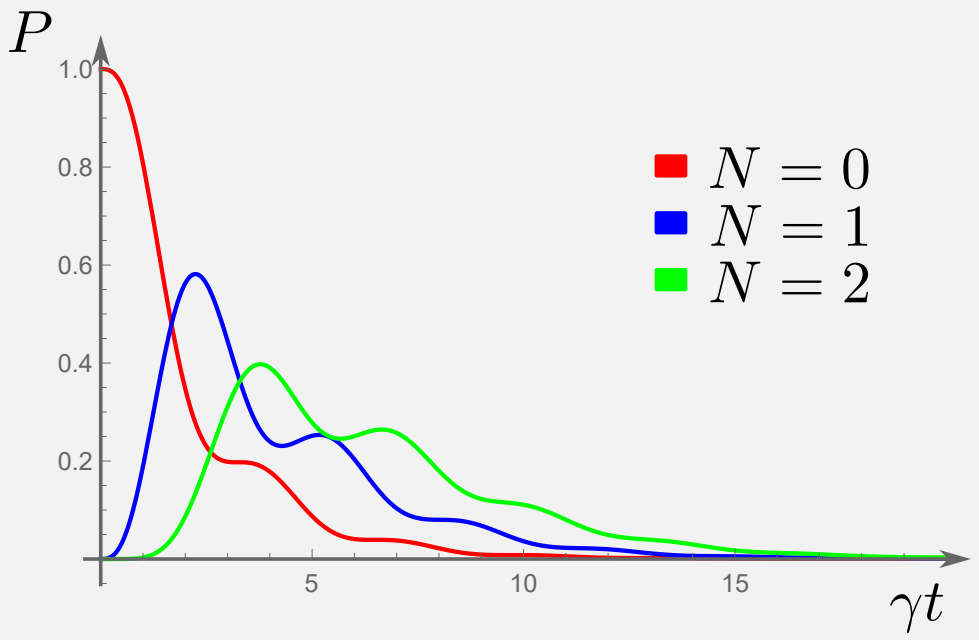
The interpretation of this equation is rather straight-forward, e.g.

$$\underbrace{p_1(t) = \int_0^t dt_1 p_{\text{no jump}}(t - t_1)\Pi_{\text{jump}}(t_1)}_{\substack{\text{probability for the occurrence} \\ \text{of only one jump event up to time } t}}$$

is the integral over the probability of all trajectories that contains one jump event at time t, that then is followed by an evolution without jumps up to time t.

Example: 2-level system

If we consider a 2-level system with $\gamma = \Omega$, we can plot the time-dependence of the probability for N jumps to occur.



In general, there is no closed form for $p_n(t)$, but we can obtain one in the case where $\gamma \gg \Omega$ and Ω also being large. Here we have

$$p_{\text{no jump}}(t) = e^{-\frac{4\Omega^2}{\gamma}t}$$

and

$$\Pi_{\text{jump}}(t) = \frac{4\Omega^2}{\gamma} e^{-\frac{4\Omega^2}{\gamma}t} \ .$$

Taking the Laplace transform of $p_n(t)$, yields

$$\begin{aligned}\bar{p}_n(s) &\underset{\substack{\uparrow\\ \text{convolution}\\ \text{theorem}}}{=} \mathscr{L}\{p_n(t)(s)\} = \int_0^\infty e^{-st} p_n(t) dt \\ &\overset{!}{=} \mathscr{L}\{p_{\text{no jump}}(t)\}(s)\big[\mathscr{L}\{\Pi_{\text{jump}}(t)\}\big]^n \\ &= \frac{1}{s+\frac{4\Omega^2}{\gamma}}\left(\frac{4\Omega^2}{\gamma}\right)^n\left[\frac{1}{s+\frac{4\Omega^2}{\gamma}}\right]^n = \left(\frac{4\Omega^2}{\gamma}\right)^n\left[\frac{1}{s+\frac{4\Omega^2}{\gamma}}\right]^{n+1}\end{aligned}$$

Note, that we have denoted the Laplace transform with $\mathscr{L}$. Inverting the Laplace transform results in

$$\begin{aligned}p_n(t) &= \mathscr{L}^{-1}\{\bar{p}_n(s)\}(t) = \left(\frac{4\Omega^2}{\gamma}\right)^n \mathscr{L}^{-1}\left\{\left(\frac{1}{s+\frac{4\Omega^2}{\gamma}}\right)^{n+1}\right\}(t) \\ &= \left(\frac{4\Omega^2}{\gamma}\right)^n \frac{t^n}{n!} e^{-\frac{4\Omega^2}{\gamma}t} = \frac{\left(\frac{4\Omega^2}{\gamma}t\right)^n}{n!} e^{-\frac{4\Omega^2}{\gamma}t} \ .\end{aligned}$$

This is a Poisson distribution $P_\lambda = \frac{\lambda^n}{n!}e^{-\lambda}$ with the parameter $\lambda = \frac{4\Omega^2}{\gamma}t$. Since λ is the mean and also the variance, the average number of jump events, but also the numbers fluctuations, increase linearly with observation time t.

8.2 Quantum Jump Monte Carlo

The trajectory picture allows to derive an elegant approach for simulating the dynamics of an open quantum system, via so-called *quantum jump Monte Carlo*.

To see this let us come back to the integral equation

$$\rho(t) = e^{\mathcal{L}_0 t}\rho(0) + \int_0^t d\tau e^{\mathcal{L}_0(t-\tau)} J\rho(\tau) \ .$$

This equation tells us that the density matrix at time t is given by the no-jump evolution plus all other evolution paths that contain at last one jump. Now, let us assume the trajectory view point, and instead of looking at the density matrix, which is the "superposition" of all paths, we consider the instantaneous stack of the system.

To this end, let us assume that at time $t = 0$, the state is pure: $|\psi_0\rangle$.
The probability to undergo a quantum jump up to a time t_{jump} is then given by:

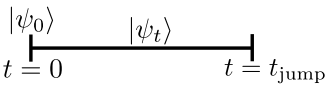
$$\begin{aligned}p_{\text{jump}}(t_{\text{jump}}) &= 1 - p_{\text{no jump}}(t_{\text{jump}}) \\ &= 1 - \text{tr}\big(e^{\mathcal{L}_0 t_{\text{jump}}}|\psi_0\rangle\langle\psi_0|\big) \\ &= 1 - \big\|e^{-iH_{\text{eff}}t_{\text{jump}}}|\psi_0\rangle\big\|^2\end{aligned}$$

The jump time is of course random, and in order to obtain a value for a single realization, we

have to draw a uniformly random number $r \in [0,1]$ and solve the equation

$$r = 1 - p_{\text{no jump}}(t_{\text{jump}}) \ .$$

Up to this point in time the state is $|\psi_t\rangle = \frac{e^{-iH_{\text{eff}}t}}{\left\|e^{-iH_{\text{eff}}t}|\psi_0\rangle\right\|}|\psi_0\rangle$



At $t = t_{\text{jump}}$ the state becomes

$$\rho(t_{\text{jump}}) = \sum_k p_k \frac{L_k \left|\psi_{t_{\text{jump}}}\right\rangle\!\left\langle\psi_{t_{\text{jump}}}\right| L_k^\dagger}{\sqrt{p_k}\sqrt{p_k}}$$

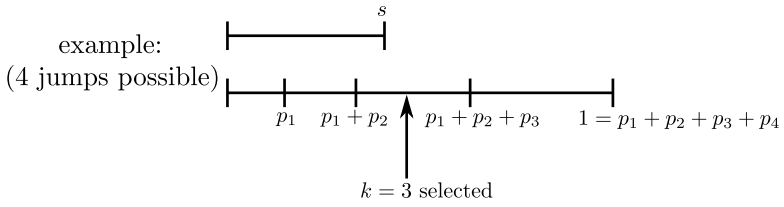
with

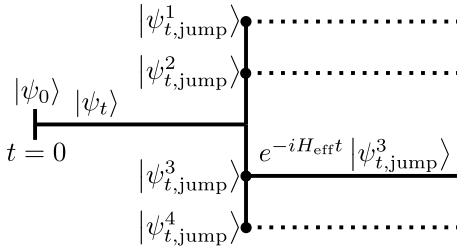
$$p_k = \left\langle\psi_{t_{\text{jump}}}\right| L_k^\dagger L_k \left|\psi_{t_{\text{jump}}}\right\rangle \ .$$

Therefore, with probability p_k, the system in the state

$$\left|\psi^k_{t_{\text{jump}}}\right\rangle = \frac{L_k \left|\psi_{t_{\text{jump}}}\right\rangle}{\sqrt{p_k}} \ .$$

In a situation one selects the state $\left|\psi^k_{t_{\text{jump}}}\right\rangle$ by drawing a random number $s \in [0,1]$ and selects the state for which $\max_{k_{\text{jump}}} \sum_{m=1}^{k_{\text{jump}}} p_m < s$.

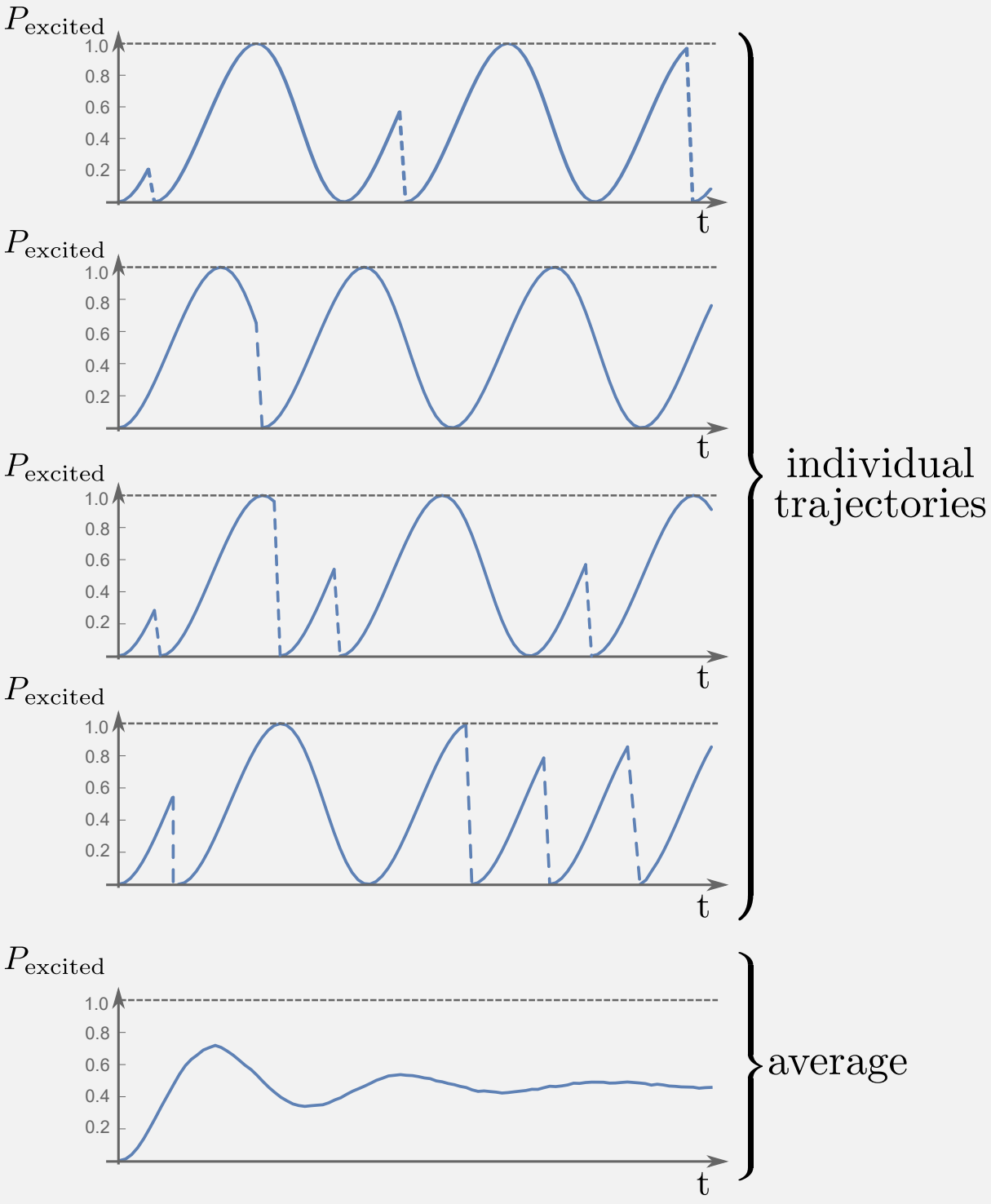




The state $\left|\psi^{k_{\text{jump}}}_{t_{\text{jump}}}\right\rangle$ is selected as new initial state and the procedure starts over. In order to calculate expectation values of observables, one calculates the corresponding expectation value for each trajectory and then averages over many trajectories. For a two-level atom with decay we find:

Example: two-level atom with decay

Let us consider a two-level atom with decay and with a driving Rabi-frequency of $\Omega = 2\gamma$.



In the last picture, we averaged over 1000 trajectories.